Introduction to Axiomatic Semantics

Lecture 7-8
CS263

Review

- Operational semantics
  - relatively simple
  - many flavors
  - adequate guide for an implementation of the language
  - not compositional
- Denotational semantics
  - mathematical
  - canonical
  - compositional
- Operational ⇔ denotational
- We would also like a semantic that is appropriate for arguing program correctness

Axiomatic Semantics

- An axiomatic semantics consists of:
  - A language for stating assertions about programs,
  - Rules for establishing the truth of assertions
- Some typical kinds of assertions:
  - This program terminates
  - If this program terminates, the variables x and y have the same value throughout the execution of the program,
  - The array accesses are within the array bounds
- Some typical languages of assertions
  - First-order logic
  - Other logics (temporal, linear)
  - Special-purpose specification languages (Z, Larch, JML)

History

- Program verification is almost as old as programming (e.g., "Checking a Large Routine", Turing 1949)
- In the late '60s, Floyd had rules for flow-charts and Hoare had rules for structured languages
- Since then, there have been axiomatic semantics for substantial languages, and many applications

Hoare Said

- "Thus the practice of proving programs would seem to lead to solution of three of the most pressing problems in software and programming, namely, reliability, documentation, and compatibility. However, program proving, certainly at present, will be difficult even for programmers of high caliber; and may be applicable only to quite simple program designs."

C.A.R Hoare,
"An Axiomatic Basis for
Computer Programming",
1969

Dijkstra Said

- "Program testing can be used to show the presence of bugs, but never to show their absence!"

Hoare Also Said

- “It has been found a serious problem to define these languages [ALGOL, FORTRAN, COBOL] with sufficient rigor to ensure compatibility among all implementations. ... one way to achieve this would be to insist that all implementations of the language shall satisfy the axioms and rules of inference which underlie proofs of properties of programs expressed in the language. **In effect, this is equivalent to accepting the axioms and rules of inference as the ultimately definitive specification of the meaning of the language.”**

Other Applications of Axiomatic Semantics

- The project of defining and proving everything formally has not succeeded (at least not yet)

- Proving has not replaced testing and debugging (and praying)

- Applications of axiomatic semantics:
  - Proving the correctness of algorithms (or finding bugs)
  - Proving the correctness of hardware descriptions (or finding bugs)
  - “extended static checking” (e.g., checking array bounds)
  - Documentation of programs and interfaces

Assertions for IMP

- The assertions we make about IMP programs are of the form:

  $\{A\}\ c\ \{B\}$

  with the meaning that:
  - If A holds in state σ and $<c, \sigma> \Downarrow \sigma'$
  - then B holds in σ'
- A is called <u>precondition</u> and B is called <u>postcondition</u>
- For example:

  $\{ y \leq x \}\ z := x;\ z := z +1\ \{ y < z \}$

  is a valid assertion
- These are called <u>Hoare triple</u> or <u>Hoare assertions</u>

Assertions for IMP (II)

- $\{A\}\ c\ \{B\}$ is a <u>partial correctness assertion</u>. It does not imply termination
- $[A]\ c\ [B]$ is a <u>total correctness assertion</u> meaning that

  If A holds in state σ

  then there exists σ' such that $<c, \sigma> \Downarrow \sigma'$

  and B holds in state σ'

- Now let’s be more formal
  - Formalize the language of assertions, A and B
  - Say when an assertion holds in a state
  - Give rules for deriving Hoare triples

The Assertion Language

- We use first-order predicate logic with IMP expressions

  $$\begin{aligned} A ::= &\ \text{true} \mid \text{false} \mid e_1 = e_2 \mid e_1 \geq e_2 \\ \mid &\ A_1 \wedge A_2 \mid A_1 \vee A_2 \mid A_1 \Rightarrow A_2 \mid \forall x.A \mid \exists x.A \end{aligned}$$

- Note that we are somewhat sloppy and mix the logical variables and the program variables

- Implicitly, for us all IMP variables range over integers

- All IMP boolean expressions are also assertions

Semantics of Assertions

- We introduced a language of assertions, we need to assign meanings to assertions.

- Notation $\sigma \models A$ to say that an assertion holds in a given state .
  - This is well-defined when σ is defined on all variables occurring in A.

- The $\models$ judgment is defined inductively on the structure of assertions.

- It relies on the denotational semantics of arithmetic expressions from IMP

Semantics of Assertions

- Formal definition:

$\sigma \models \text{true}$	always
$\sigma \models e_1 = e_2$	iff $[\![e_1]\!]\,\sigma = [\![e_2]\!]\sigma$
$\sigma \models e_1 \geq e_2$	iff $[\![e_1]\!]\,\sigma \geq [\![e_2]\!]\sigma$
$\sigma \models A_1 \wedge A_2$	iff $\sigma \models A_1$ and $\sigma \models A_2$
$\sigma \models A_1 \vee A_2$	iff $\sigma \models A_1$ or $\sigma \models A_2$
$\sigma \models A_1 \Rightarrow A_2$	iff $\sigma \models A_1$ implies $\sigma \models A_2$
$\sigma \models \forall x.A$	iff $\forall n \in \mathbb{Z}.\sigma[x{:=}n] \models A$
$\sigma \models \exists x.A$	iff $\exists n \in \mathbb{Z}.\sigma[x{:=}n] \models A$

Semantics of Assertions

- Now we can define formally the meaning of a partial correctness assertion
  $\models \{ A\}\ c\ \{ B \}$:
  $$\forall\sigma\in\Sigma.\forall\sigma'\in\Sigma.(\sigma \models A \wedge \langle c,\sigma\rangle \Downarrow \sigma') \Rightarrow \sigma' \models B$$
- … and the meaning of a total correctness assertion $\models [A]\ c\ [B]$ iff
  $$\forall\sigma\in\Sigma.\sigma \models A \Rightarrow \exists\sigma'\in\Sigma.\ \langle c,\sigma\rangle \Downarrow \sigma' \wedge \sigma' \models B$$
- or alternatively:
  $$\forall\sigma\in\Sigma.\forall\sigma'\in\Sigma.(\sigma \models A \wedge \langle c,\sigma\rangle \Downarrow \sigma') \Rightarrow \sigma' \models B$$
  $$\wedge$$
  $$\forall\sigma\in\Sigma.\sigma \models A \Rightarrow \exists\sigma'\in\Sigma.\ \langle c,\sigma\rangle \Downarrow \sigma'$$

Deriving Assertions

- Now we have the formal mechanism to decide when $\{A\}\ c\ \{B\}$
  - But it is not satisfactory
  - Because $\models \{A\}\ c\ \{B\}$ is defined in terms of the operational semantics, we practically have to run the program to verify an assertion
  - And also it is impossible to effectively verify the truth of a $\forall x.\ A$ assertion (by using the definition of validity)
- So we define a symbolic technique for deriving valid assertions from others that are known to be valid
  - We start with validity of first-order formulas

Derivation Rules

- We write $\vdash A$ when A can be derived from basic axioms
- The derivation rules for $\vdash A$ are the usual ones from first-order logic with arithmetic:
- Natural deduction style axioms:

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B} \qquad \frac{\vdash [a/x]A \quad (\text{a is fresh})}{\vdash \forall x.A} \qquad \frac{\vdash \forall x.A}{\vdash [e/x]A}$$

$$\frac{\begin{array}{c}\vdash A\\ \dots\\ \vdash B\end{array}}{\vdash A \Rightarrow B} \qquad \frac{\vdash A \Rightarrow B \quad \vdash A}{\vdash B} \qquad \frac{\vdash [e/x]A}{\vdash \exists x.A} \qquad \frac{\vdash \exists x.A \qquad \begin{array}{c}\vdash [a/x]A\\ \dots\\ \vdash B\end{array}}{\vdash B}$$

Derivation Rules for Hoare Triples

- Similarly we write $\vdash \{A\}\ c\ \{ B\}$ when we can derive the triple using derivation rules
- There is one derivation rule for each command in the language
- Plus, the rule of <u>consequence</u>

$$\frac{\vdash A' \Rightarrow A \quad \vdash \{A\}\ c\ \{B\} \quad \vdash B \Rightarrow B'}{\vdash \{A'\}\ c\ \{B'\}}$$

Derivation Rules for Hoare Logic

- One rule for each syntactic construct:

$$\frac{}{\vdash \{A\}\ \text{skip}\ \{A\}} \qquad \frac{}{\vdash \{[e/x]A\}\ x := e\ \{A\}}$$

$$\frac{\vdash \{A\}\ c_1\ \{B\} \quad \vdash \{B\}\ c_2\ \{C\}}{\vdash \{A\}\ c_1;\ c_2\ \{C\}}$$

$$\frac{\vdash \{A \wedge b\}\ c_1\ \{B\} \quad \vdash \{A \wedge \neg\, b\}\ c_2\ \{B\}}{\vdash \{A\}\ \text{if } b \text{ then } c_1 \text{ else } c_2\ \{B\}}$$

$$\frac{\vdash \{A \wedge b\}\ c\ \{A\}}{\vdash \{A\}\ \text{while } b \text{ do } c\ \{A \wedge \neg\, b\}}$$

Hoare Rules

- For some constructs multiple rules are possible:

$$\frac{}{\vdash \{A\}\ x := e\ \{\exists x_0 . [x_0/x]A \wedge x = [x_0/x]e\}}$$

(This was the "forward" axiom for assignment)

$$\frac{\vdash A \wedge b \Rightarrow C \quad \vdash \{C\}\ c\ \{A\} \quad \vdash A \wedge \neg\, b \Rightarrow B}{\vdash \{A\}\ \text{while } b \text{ do } c\ \{B\}}$$

- Exercise: these rules can be derived from the previous ones using the consequence rules

Example: Assignment

- Assume that x does not appear in e

  Prove that {true} x := e { x = e }
- But

$$\frac{}{\vdash \{e = e\}\ x := e\ \{x = e\}}$$

because $[e/x](x = e) \equiv e = [e/x]e \equiv e = e$

- Assignment + consequence:

$$\frac{\vdash \text{true} \Rightarrow e = e \qquad \dfrac{}{\vdash \{e = e\}\ x := e\ \{x = e\}}}{\vdash \{\text{true}\}\ x := e\ \{x = e\}}$$

The Assignment Axiom (Cont.)

- Hoare said: "Assignment is undoubtedly the most characteristic feature of programming a digital computer, and one that most clearly distinguishes it from other branches of mathematics. It is surprising therefore that the axiom governing our reasoning about assignment is quite as simple as any to be found in elementary logic."
- Caveats are sometimes needed for languages with aliasing:
  - If x and y are aliased then

    { true } x := 5 { x + y = 10}

    is true

Example: Conditional

$$\frac{\begin{array}{c} D_1 :: \vdash \{\text{true} \wedge y \leq 0\}\ x := 1\ \{x > 0\} \\ D_2 :: \vdash \{\text{true} \wedge y > 0\}\ x := y\ \{x > 0\} \end{array}}{\vdash \{\text{true}\}\ \text{if } y \leq 0 \text{ then } x := 1 \text{ else } x := y\ \{x > 0\}}$$

- D_1 is obtained by consequence and assignment

$$\frac{\begin{array}{c} \vdash \{1 > 0\}\ x := 1\ \{x > 0\} \\ \vdash \text{true} \wedge y \leq 0 \Rightarrow 1 > 0 \end{array}}{\vdash \{\text{true} \wedge y \leq 0\}\ x := 1\ \{x \geq 0\}}$$

- D_2 is also obtained by consequence and assignment

$$\frac{\begin{array}{c} \vdash \{y > 0\}\ x := y\ \{x > 0\} \\ \vdash \text{true} \wedge y > 0 \Rightarrow y > 0 \end{array}}{\vdash \{\text{true} \wedge y > 0\}\ x := y\ \{x > 0\}}$$

Example: Loop

- We want to derive that

  $\vdash \{x \leq 0\}$ while $x \leq 5$ do $x := x + 1$ $\{ x = 6\}$
- Use the rule for while with invariant $x \leq 6$

$$\frac{\dfrac{\vdash x \leq 6 \wedge x \leq 5 \Rightarrow x + 1 \leq 6 \qquad \dfrac{}{\vdash \{x + 1 \leq 6\}\ x := x + 1\ \{ x \leq 6 \}}}{\vdash \{x \leq 6 \wedge x \leq 5 \}\ x := x + 1\ \{x \leq 6\}}}{\vdash \{x \leq 6\}\ \text{while } x \leq 5 \text{ do } x := x + 1\ \{ x \leq 6 \wedge x > 5\}}$$

- Then finish-off with consequence

$$\frac{\begin{array}{l} \vdash x \leq 0 \Rightarrow x \leq 6 \\ \vdash x \leq 6 \wedge x > 5 \Rightarrow x\ \ =6 \qquad \vdash \{x \leq 6\}\ \text{while} \ldots \{ x \leq 6 \wedge x > 5\} \end{array}}{\vdash \{x \leq 0\}\ \text{while} \ldots \{x = 6\}}$$

Another Example

- Verify that

  $\vdash \{A \}$ while true do c $\{ B\}$

  holds for any A, B and c
- We must construct a derivation tree

$$\frac{\begin{array}{c} \vdash A \Rightarrow \text{true} \\ \vdash \text{true} \wedge \text{false} \Rightarrow B \end{array} \qquad \dfrac{\vdash \{\text{true} \wedge \text{true}\}\ c\ \{ \text{true} \}}{\{\text{true}\}\ \text{while true do } c\ \{\text{true} \wedge \text{false}\}}}{\vdash \{A\}\ \text{while true do } c\ \{ B\}}$$

- We need an additional lemma:

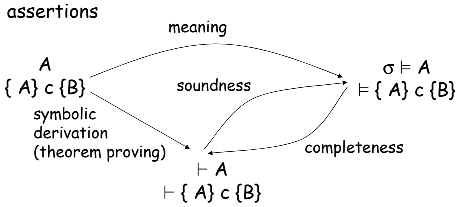
  $\forall A . \forall c . \vdash \{ A \}\ c\ \{\text{true}\}$
  - How do you prove this one?

Using Hoare Rules. Notes

- Hoare rules are mostly syntax directed
- There are three wrinkles:
  - When to apply the rule of consequence ?
  - What invariant to use for while ?
  - How do you prove the implications involved in consequence ?
- The last one is how theorem proving gets in the picture
  - This turns out to be doable !
  - The loop invariants turn out to be the hardest problem ! (Should the programmer give them? See Dijkstra.)

Where Do We Stand?

- We have a language for asserting properties of programs
- We know when such an assertion is true
- We also have a symbolic method for deriving assertions



meaning

A
{ A} c {B}

soundness

σ ⊨ A
⊨ { A} c {B}

symbolic
derivation
(theorem proving)

completeness

⊢ A
⊢ { A} c {B}

Soundness of Axiomatic Semantics

- Formal statement

  If ⊢ { A } c { B} then ⊨ { A} c { B}

  or, equivalently

  For all σ, if σ ⊨ A and D :: <c, σ> ⇓ σ' and H :: ⊢ { A } c { B} then σ' ⊨ B
- How can we prove this?
  - By induction on the structure of c?
    - No, problems with while and rule of consequence
  - By induction on the structure of D?
    - No, problems with rule of consequence
  - By induction on the structure of H?
    - No, problems with while
  - By simultaneous induction on the structure of D and H

Simultaneous Induction

- Consider two structures D and H
  - Assume that x < y iff x is a substructure of y
- Define the ordering

  (d, h) < (d', h') iff d < d' or d = d' and h < h'
  - Called lexicographic ordering
  - Just like the ordering in a dictionary
- This is a well founded order and leads to simultaneous induction
- If d < d' then h can actually be larger than h'!
- It can even be unrelated to h' !

Soundness of the Consequence Rule

- Case: last rule used in H :: ⊢ { A} c { B} is the consequence rule:

$$\frac{\vdash A \Rightarrow A' \qquad H_1 :: \vdash \{A'\}\ c\ \{B'\} \qquad \vdash B' \Rightarrow B}{\vdash \{A\}\ c\ \{B\}}$$

- From soundness of the first-order logic derivations we have σ ⊨ A ⇒ A', hence σ ⊨ A'
- From IH with H_1 and **D** we get that σ' ⊨ B'
- From soundness of the first-order logic derivations we have that σ' ⊨ B' ⇒ B, hence σ' ⊨ B, q.e.d.

Soundness of the Assignment Axiom

- Case: the last rule used in H :: ⊢ { A } c { B} is the assignment rule

$$\frac{}{\vdash \{[e/x]B\}\ x := e\ \{B\}}$$

- The last rule used in D :: <x := e, σ> ⇓ σ' must be

$$\frac{D_1 :: \langle e, \sigma \rangle \Downarrow n}{\langle x := e, \sigma \rangle \Downarrow \sigma[x := n]}$$

- We must prove the substitution lemma:

  If σ ⊨ [e/x]B and <σ, e> ⇓ n then σ[x := n] ⊨ B

Soundness of the While Rule

- Case: last rule used in $H : \vdash \{ A \}\ c\ \{ B\}$ was the while rule:

$$\frac{H_1 :: \vdash \{A \wedge b\}\ c\ \{A\}}{\vdash \{A\}\ \text{while } b \text{ do } c\ \{A \wedge \neg b\}}$$

- There are two possible rules at the root of D.
  - We do only the complicated case

$$\frac{D_1 :: \langle b, \sigma\rangle \Downarrow \text{true} \qquad D_2 :: \langle c,\sigma\rangle \Downarrow \sigma' \qquad D_3 :: \langle \text{while } b \text{ do } c, \sigma' \rangle \Downarrow \sigma''}{\langle \text{while } b \text{ do } c, \sigma \rangle \Downarrow \sigma''}$$

Soundness of the While Rule (Cont.)

Assume that $\sigma \vDash A$

To show that $\sigma'' \vDash A \wedge \neg b$

- By property of booleans and D_1 we get $\sigma \vDash b$
  - Hence $\sigma \vDash A \wedge b$
- By IH on H_1 and D_2 we get $\sigma' \vDash A$
- By IH on H and D_3 we get $\sigma'' \vDash A \wedge \neg b$, q.e.d.

- Note that in the last use of IH the derivation H did not decrease
- See Winskel, Chapter 6.5 for a soundness proof with denotational semantics

Completeness of Axiomatic Semantics
Weakest Preconditions

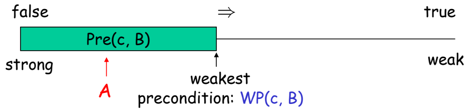
Completeness of Axiomatic Semantics

- Is it true that whenever $\vDash \{A\}\ c\ \{B\}$ we can also derive $\vdash \{A\}\ c\ \{B\}$?
- If it isn't then it means that there are valid properties of programs that we cannot verify with Hoare rules

- Good news: for our language the Hoare triples are complete
- Bad news: only if the underlying logic is complete
  (whenever $\vDash A$ we also have $\vdash A$)
  - this is called relative completeness

Proof Idea

- Dijkstra's idea: To verify that $\{ A \}\ c\ \{ B\}$
  a) Find out all predicates A' such that $\vDash \{ A'\}\ c\ \{ B\}$
    - call this set Pre(c, B)
  b) Verify for one $A' \in \text{Pre}(c, B)$ that $A \Rightarrow A'$
- Assertions can be ordered:


false ⇒ true

Pre(c, B)

strong weak

A

weakest precondition: WP(c, B)

- Thus: compute WP(c, B) and prove $A \Rightarrow \text{WP}(c, B)$

Proof Idea (Cont.)

- Completeness of axiomatic semantics:
  If $\vDash \{ A \}\ c\ \{ B \}$ then $\vdash \{ A \}\ c\ \{ B\}$
- Assuming that we can compute wp(c, B) with the following properties:
  1. wp is a precondition (according to the Hoare rules)
     $\vdash \{ \text{wp}(c, B) \}\ c\ \{ B\}$
  2. wp is the weakest precondition
     If $\vDash \{ A \}\ c\ \{ B\}$ then $\vDash A \Rightarrow \text{wp}(c, B)$

$$\frac{\vdash A \Rightarrow \text{wp}(c, B) \qquad \vdash \{\text{wp}(c, B)\}\ c\ \{B\}}{\vdash \{A\}\ c\ \{B\}}$$

- We also need that whenever $\vDash A$ then $\vdash A$!

Weakest Preconditions

- Define wp(c, B) inductively on c, following Hoare rules:

$$\frac{\{A\}\ c_1\ \{C\} \qquad \{C\}\ c_2\ \{B\}}{\{\ A\ \}\ c_1;\ c_2\ \{B\}}$$

$wp(c_1;\ c_2, B) = wp(c_1, wp(c_2, B))$

$$\frac{}{\{\ [e/x]B\ \}\ x := E\ \{B\}}$$

$wp(x := e, B) = [e/x]B$

$$\frac{\{A_1\}\ c_1\ \{B\} \qquad \{A_2\}\ c_2\ \{B\}}{\{\ E \Rightarrow A_1 \wedge \neg\ E \Rightarrow A_2\}\ \text{if } E \text{ then } c_1 \text{ else } c_2\ \{B\}}$$

$wp(\text{if } E \text{ then } c_1 \text{ else } c_2, B) = E \Rightarrow wp(c_1, B) \wedge \neg\ E \Rightarrow wp(c_2, B)$

Weakest Preconditions for Loops

- We start from the equivalence
  while b do c = if b then c; while b do c else skip
- Let w = while b do c and W = wp(w, B)
- We have that
  $W = b \Rightarrow wp(c, W) \wedge \neg\ b \Rightarrow B$
- But this is a recursive equation !
  - We know how to solve these using domain theory
- We need a domain for assertions

A Partial-Order for Assertions

- What is the assertion that contains least information?
  - true – does not say anything about the state
- What is an appropriate information ordering ?
  $A \sqsubseteq A'$ iff $\models A' \Rightarrow A$
- Is this partial order complete?
  - Take a chain $A_1 \sqsubseteq A_2 \sqsubseteq \ldots$
  - Let $\wedge A_i$ be the infinite conjunction of A_i
    $\sigma \models \wedge A_i$ iff for all i we have that $\sigma \models A_i$
  - Verify that $\wedge A_i$ is the least upper bound
- Can $\wedge A_i$ be expressed in our language of assertions?
  - In many cases yes (see Winskel), we'll assume yes for now

Weakest Precondition for WHILE

- Use the fixed-point theorem
  $F(A) = b \Rightarrow wp(c, A) \wedge \neg\ b \Rightarrow B$
  - Verify that F is both monotonic and continuous
- The least-fixed point (i.e. the weakest fixed point) is
  $wp(w, B) = \wedge F^i(\text{true})$
- Notice that unlike for denotational semantics of IMP we are not working on a flat domain !

Weakest Preconditions (Cont.)

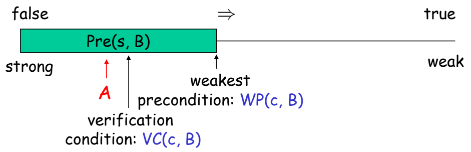
- Define a family of wp's
  - wp_k(while e do c, B) = weakest precondition on which the loop if it terminates in k or fewer iterations, it terminates in B

  $wp_0 = \neg\ E \Rightarrow B$
  $wp_1 = E \Rightarrow wp(c, wp_0) \wedge \neg\ E \Rightarrow B$
  …
- wp(while e do c, B) = $\wedge_{k \geq 0} wp_k$ = lub $\{wp_k \mid k \geq 0\}$
- See document on the home page for the proof of completeness with weakest preconditions
- Weakest preconditions are
  - Impossible to compute (in general)
  - Can we find something easier to compute yet sufficient ?

Verification Conditions

Not Quite Weakest Preconditions

- Recall what we are trying to do:



false ⇒ true

Pre(s, B)

strong

weak

A

weakest precondition: WP(c, B)

verification condition: VC(c, B)

- We shall construct a <u>verification condition</u>: VC(c, B)
  - The loops are annotated with loop invariants !
  - VC is guaranteed stronger than WP
  - But hopefully still weaker than A: A ⇒ VC(c, B) ⇒ WP(c, B)

Verification Conditions

- Factor out the hard work
  - Loop invariants
  - Function specifications
- Assume programs are annotated with such specs.
  - Good software engineering practice anyway
- We will assume that the new form of the while construct includes an invariant:

  $while_I$ b do c

  - The invariant formula must hold every time before b is evaluated

Verification Condition Generation (1)

- Mostly follows the definition of the wp function

VC(skip, B) = B

VC(c_1; c_2, B) = VC(c_1, VC(c_2, B))

VC(if b then c_1 else c_2, B) = b ⇒ VC(c_1, B) ¬b ⇒ VC(c_2, B)

VC(x := e, B) = [e/x]B

VC(let x = e in c, B) = [e/x] VC(c, B)

VC(while b do c, B) = ?

Verification Condition Generation for WHILE

$$VC(\text{while}_I\ e\ \text{do}\ c, B) = \underbrace{I}_{\text{I holds on entry}} \wedge (\forall x_1 \ldots x_n.\ \underbrace{I \Rightarrow (e \Rightarrow VC(c, I)}_{\text{I is preserved in an arbitrary iteration}} \wedge \underbrace{\neg e \Rightarrow B)}_{\text{B holds when the loop terminates in an arbitrary iteration}}\ )$$

- I is the loop invariant (provided externally)
- $x_1, \ldots, x_n$ are all the variables modified in c
- The ∀ is similar to the ∀ in mathematical induction:

  $P(0) \wedge \forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

Example of VC

- Compute the VC for the following subprogram

```
x = x0; y = y0;
while x != y do
  if x < y then
     y := y - x
  else
     x := x - y
inv ???
```

Soundness of VCGen

- Simple form

  ⊨ {VC(c,B)} c {B}

- Or equivalently that

  ⊨ VC(c, B) ⇒ wp(c, B)

- Proof is by induction on the structure of c
  - Try it!

- Soundness holds for <u>any</u> choice of invariants !
- We'll look now at properties and extensions of VCs

VC and Invariants

- Consider the Hoare triple:

  $\{x \leq 0\}\ \text{while}_I\ x \leq 5\ \text{do}\ x := x + 1\ \{x = 6\}$

- The VC for this is:

  $x \leq 0 \Rightarrow I(x) \wedge \forall x. (I(x) \Rightarrow (x > 5 \Rightarrow x = 6 \wedge$
  $x \leq 5 \Rightarrow I(x{+}1)\ ))$

- Requirements on the invariant:
  - Holds on entry $\forall x.\ x \leq 0 \Rightarrow I(x)$
  - Preserved by the body $\forall x.\ I(x) \wedge x \leq 5 \Rightarrow I(x{+}1)$
  - Useful $\forall x.\ I(x) \wedge x > 5 \Rightarrow x = 6$

- Check that $I(x) = x \leq 6$ satisfies all constraints

Forward Verification Condition Generation

- Traditionally VC is computed backwards
  - Works well for structured code

- But it can also be computed in a forward direction
  - Works even for un-structured languages (e.g., assembly language)
  - Uses symbolic evaluation, a technique that has broad applications in program analysis
    - e.g. the PREfix tool (Intrinsa, Microsoft) works this way

Forward VC Gen. Idea

- Consider the sequence of assignments

  $x_1 := e_1;\ x_2 := e_2$

- The $VC(c, B) = [e_1/x_1]([e_2/x_2]B)$

  $= [e_1/x_1,\ e_2[e_1/x_1]/x_2]\ B$

- We can compute the substitution in a forward way using symbolic evaluation
  - Keep a symbolic state that maps variables to expressions
  - Initially, $\Sigma_0 = \{\ \}$
  - After $x_1 := e_1$, $\Sigma_1 = \{\ x_1 \rightarrow e_1\ \}$
  - After $x_2 := e_2$, $\Sigma_2 = \{x_1 \rightarrow e_1,\ x_2 \rightarrow e_2[e_1/x_1]\ \}$
  - Note that we have applied Σ_1 as a substitution to right-hand side of assignment $x_2 := e_2$

Forward VC Generation by Symbolic Evaluation

Details

Symbolic Evaluation

- Consider the language of instructions:

  x := e | f() | if e goto L | goto L | L: | return | inv e

- The "inv e" instruction is an annotation
  - Says that boolean expression e holds at that point

- Notation: I_k is the instruction at address k

Symbolic Evaluation. The State.

- We set up a symbolic evaluation state:

$\Sigma : \text{Var} \rightarrow \text{SymbolicExpressions}$

$\Sigma(x)$ = the symbolic value of x in state Σ

$\Sigma[x{:=}e]$ = a new state in which x's value is e

We shall use states also as substitutions:

$\Sigma(e)$ - obtained from e by replacing x with $\Sigma(x)$

So far this is pretty much like the operational semantics

Symbolic Evaluation. The Invariants.

- The symbolic evaluator keeps track of the encountered invariants
- A new element of execution state: $Inv \subseteq \{1\ldots n\}$
- If $k \in Inv$ then
  - I_k is an invariant instruction that we have already executed
- Basic idea: execute an inv instruction only twice:
  - The first time it is encountered
  - And one more time around an arbitrary iteration

Symbolic Evaluation. Rules.

- Define a VC function as an interpreter:

  $VC : 1..n \times SymbolicState \times InvariantState \rightarrow Assertion$

$VC(k, \Sigma, Inv) =$

$VC(L, \Sigma, Inv)$	if I_k = goto L
$e \Rightarrow VC(L, \Sigma, Inv) \quad \wedge$ $\neg e \Rightarrow VC(k+1, \Sigma, Inv)$	if I_k = if e goto L
$VC(k+1, \Sigma[x:=\Sigma(e)], Inv)$	if I_k = x := e
$\Sigma(Post_{current\text{-}function})$	if I_k = return
$\Sigma(Pre_f) \quad \wedge$ $\forall a_1..a_m.\Sigma'(Post_f) \Rightarrow VC(k+1, \Sigma', Inv)$ (where $y_1, \ldots, y_m$ are modified by f) and $a_1, \ldots, a_m$ are fresh parameters and $\Sigma' = \Sigma[y_1 := a_1, \ldots, y_m := a_m]$	if I_k = f()

Symbolic Evaluation. Invariants.

Two cases when seeing an invariant instruction:

1. We see the invariant for the first time
   - I_k = inv e.
   - $k \not\in Inv$
   - Let $\{y_1, \ldots, y_m\}$ = the variables that could be modified on a path from the invariant back to itself
   - Let $a_1, \ldots, a_m$ be fresh new symbolic parameters

$VC(k, \Sigma, Inv) =$

$$\Sigma(e) \wedge \forall a_1 \ldots a_m.\ \Sigma'(e) \Rightarrow VC(k+1, \Sigma', Inv \cup \{k\})$$

with $\Sigma' = \Sigma[y_1 := a_1, \ldots, y_m := a_m]$

(like a function call)

Symbolic Evaluation. Invariants.

2. We see the invariant for the second time
   - I_k = inv E
   - $k \in Inv$

$VC(k, \Sigma, Inv) = \Sigma(e)$

(like a function return)

- Some tools take a more simplistic approach
  - Do not require invariants
  - Iterate through the loop a fixed number of times
  - PREfix, versions of ESC (Compaq SRC)
  - Sacrifice completeness for usability

Symbolic Evaluation. Putting it all together

- Let
  - $x_1, \ldots, x_n$ be all the variables and $a_1, \ldots, a_n$ fresh parameters
  - Σ_0 be the state $[x_1 := a_1, \ldots, x_n := a_n]$
  - $\emptyset$ be the empty Inv set
- For all functions f in your program, compute

  $$\forall a_1 \ldots a_n.\ \Sigma_0(Pre_f) \Rightarrow VC(f_{entry}, \Sigma_0, \emptyset)$$

- If all of these predicates are valid then:
  - If you start the program by invoking any f in a state that satisfies Pre_f the program will execute such that
    - At all “inv e” the e holds, and
    - If the function returns then $Post_f$ holds
  - Can be proved w.r.t. a real interpreter (operational semantics)
  - Proof technique called co-induction (or, assume-guarantee)

VC Generation Example

- Consider the program

```
                    Precondition: x ≤ 0
Loop: inv x ≤ 6
      if x > 5 goto End
      x := x + 1
      goto Loop
End:  return        Postconditon: x = 6
```

VC Generation Example (cont.)

$$
\begin{array}{l}
\forall x. \\
\quad x \leq 0 \Rightarrow \\
\qquad x \leq 6 \wedge \\
\qquad\quad \forall x'. \\
\qquad\qquad (x' \leq 6 \Rightarrow \\
\qquad\qquad\quad x' > 5 \Rightarrow x' = 6 \\
\qquad\qquad\qquad \wedge \\
\qquad\qquad\quad x' \leq 5 \Rightarrow x' + 1 \leq 6 \;)
\end{array}
$$

- VC contains both proof obligations and assumptions about the control flow

VC Can Be Large

- Consider the sequence of conditionals

  (if $x < 0$ then $x := -x$); (if $x \leq 3$ then x += 3)
  - With the postcondition $P(x)$
- The VC is

$$
\begin{array}{l}
x < 0 \wedge -x \leq 3 \Rightarrow P(-x + 3) \;\; \wedge \\
x < 0 \wedge -x > 3 \Rightarrow P(-x) \quad \wedge \\
x \geq 0 \wedge x \leq 3 \Rightarrow P(x + 3) \quad \wedge \\
x \geq 0 \wedge x > 3 \Rightarrow P(x)
\end{array}
$$

- There is one conjunct for each path

  => exponential number of paths !
  - Conjuncts for non-feasible paths have un-satisfiable guard !
- Try with $P(x) = x \geq 3$

VC Can Be Large (2)

- VCs are exponential in the size of the source because they attempt relative completeness:
  - To handle the case then the correctness of the program must be argued independently for each path
- Remark:
  - It is unlikely that the programmer could write a program by considering an exponential number of cases
  - But possible. Any examples?
- Solutions:
  - Allow invariants even in straight-line code
  - Thus do not consider all paths independently !

Invariants in Straight-Line Code

- Purpose: modularize the verification task
- Add the command "after c establish I"
  - Same semantics as c (I is only for verification purposes)

  $\mathrm{VC}(\text{after } c \text{ establish } I, P) =_{def} \mathrm{VC}(c, I) \wedge \forall x_i.\ I \Rightarrow P$
    - where x_i are the ModifiedVars(c)
- Use when c contains many paths

  after if $x < 0$ then $x := -x$ establish $x \geq 0$;

  if $x \leq 3$ then x += 3 { P(x) }
- VC now is (for $P(x) = x \geq 3$)

$$
\begin{array}{l}
(x < 0 \Rightarrow -x \geq 0) \wedge (x \geq 0 \Rightarrow x \geq 0) \wedge \\
\forall x.\ x \geq 0 \Rightarrow (x \leq 3 \Rightarrow P(x+3) \wedge x > 3 \Rightarrow P(x))
\end{array}
$$

Dropping Paths

- In absence of annotations drop some paths
- VC(if E then c_1 else c_2, P) = choose one of
  - $E \Rightarrow \mathrm{VC}(c_1, P) \wedge \neg E \Rightarrow \mathrm{VC}(c_2, P)$
  - $E \Rightarrow \mathrm{VC}(c_1, P)$
  - $\neg E \Rightarrow \mathrm{VC}(c_2, P)$
- We sacrifice soundness !
  - No more guarantees but possibly still a good debugging aid
- Remarks:
  - A recent trend is to sacrifice soundness to increase usability
  - The PREfix tool considers only 50 non-cyclic paths through a function (almost at random)

VCGen for Exceptions

- We extend the source language with exceptions without arguments:

– throw	throws an exception
– try c_1 handle c_2	executes c_2 if c_1 throws

- Problem:
  - We have non-local transfer of control
  - What is VC(throw, P) ?
- Solution: use 2 postconditions
  - One for normal termination
  - One for exceptional termination

VCGen for Exceptions (2)

- Define: VC(c, P, Q) is a precondition that makes c either not terminate, or terminate normally with P or throw an exception with Q
- Rules

  $VC(skip, P, Q) = P$
  $VC(c_1; c_2, P, Q) = VC(c_1, VC(c_2, P, Q), Q)$
  $VC(throw, P, Q) = Q$
  $VC(try\ c_1\ handle\ c_2, P, Q) = VC(c_1, P, VC(c_2, Q, Q))$
  $VC(try\ c_1\ finally\ c_2, P, Q) = ?$

Handling Program State

- We cannot have side-effects in assertions
  - While creating the VC we must remove side-effects !
  - But how to do that when lacking precise aliasing information ?
- Important technique: Postpone alias analysis
- Model the state of memory as a symbolic mapping from addresses to values:
  - If E denotes an address and M a memory state then:
  - sel(M,E) denotes the contents of the memory cell
  - upd(M,E,V) denotes a new memory state obtained from M by writing V at address E

More on Memory

- We allow variables to range over memory states
  - So we can quantify over all possible memory states
- And we use the special pseudo-variable μ in assertions to refer to the current state of memory
- Example:

  "$\forall i.\ i \geq 0 \wedge i < 5 \Rightarrow sel(\mu, A + i) > 0$" = $allpositive(\mu, A, 0, 5)$

  says that entries 0..4 in array A are positive

Hoare Rules: Assignment

- When is the following Hoare triple valid?

  { A } *x = 5 { *x + *y = 10 }

- A ought to be "*y = 5 or x = y"
- The Hoare rule for assignment would give us:

  [5/*x](*x + *y = 10)
  = 5 + *y = 10
  = *y = 5 (we lost one case)

- How come the rule does not work?

Hoare Rules: Side-Effects

- To model writes correctly we use memory expressions
  - A memory write changes the value of memory

$$\frac{}{\{ B[upd(\mu, E_1, E_2)/\mu] \}\ {*}E_1 := E_2\ \{B\}}$$

- Important technique: treat memory as a whole
- And reason later about memory expressions with inference rules such as (McCarthy):

$$sel(upd(M, E_1, E_2), E_3) = \begin{cases} E_2 & \text{if } E_1 = E_3 \\ sel(M, E_3) & \text{if } E_1 \neq E_3 \end{cases}$$

Memory Aliasing

- Consider again: { A } *x := 5 { *x + *y = 10 }
- We obtain:

  A = [upd(μ, x, 5)/μ] (*x + *y = 10)
  = [upd(μ, x, 5)/μ] (sel(μ, x) + sel(μ, y) = 10)
  = sel(upd(μ, x, 5), x) + sel(upd(μ, x, 5), y) = 10 (*)
  = 5 + sel(upd(μ, x, 5), y) = 10
  = if x = y then 5 + 5 = 10 else 5 + sel(μ, y) = 10
  = x = y or *y = 5 (**)

- To (*) is theorem generation
- From (*) to (**) is theorem proving

Alternative Handling for Memory

- Reasoning about aliasing can be expensive (NP-hard)
- Sometimes completeness is sacrificed with the following (approximate) rule:

$$\text{sel}(\text{upd}(M, E_1, E_2), E_3) = \begin{cases} E_2 & \text{if } E_1 = \text{(obviously) } E_3 \\ \text{sel}(M, E_3) & \text{if } E_1 \neq \text{(obviously) } E_3 \\ p & \text{otherwise (p is a fresh new parameter)} \end{cases}$$

- The meaning of "obvious" varies:
  - The addresses of two distinct globals are ≠
  - The address of a global and one of a local are ≠
- "PREfix" and GCC use such schemes

VC Generation Example

- Consider the program

```
1: I := 0            Precondition: B : bool ∧ A : array(bool, L)
   R := B
3: inv I ≥ 0 ∧ R : bool
   if I ≥  L goto 9
   assert saferd(A + I)
   T := *(A + I)
   I := I + 1
   R := T
   goto 3
9: return R           Postconditon: R : bool
```

VC Generation Example (cont.)

$$\begin{array}{l} \forall A.\ \forall B.\ \forall L.\ \forall \mu \\ \quad B : \text{bool} \wedge A : \text{array}(\text{bool}, L) \Rightarrow \\ \quad\quad 0 \geq 0 \wedge B : \text{bool} \wedge \\ \quad\quad\quad \forall I.\ \forall R. \\ \quad\quad\quad\quad I \geq 0 \wedge R : \text{bool} \Rightarrow \\ \quad\quad\quad\quad\quad I \geq L \Rightarrow R : \text{bool} \\ \quad\quad\quad\quad\quad\quad \wedge \\ \quad\quad\quad\quad\quad I < L \Rightarrow \text{saferd}(A + I) \wedge \\ \quad\quad\quad\quad\quad\quad\quad I + 1 \geq 0 \wedge \\ \quad\quad\quad\quad\quad\quad \text{sel}(\mu, A + I) : \text{bool} \end{array}$$

- VC contains both proof obligations and assumptions about the control flow

Mutable Records - Two Models

- Let r : RECORD f1 : T1; f2 : T2 END
- Records are reference types
- Method 1
  - One "memory" for each record
  - One index constant for each field. We postulate f1 ≠ f2
  - r.f1 is sel(r,f1) and r.f1 := E is r := upd(r,f1,E)
- Method 2
  - One "memory" for each field
  - The record address is the index
  - r.f1 is sel(f1,r) and r.f1 := E is f1 := upd(f1,r,E)

VC as a "Semantic Checksum"

- Weakest preconditions are an expression of the program's semantics:
  - Two equivalent programs have logically equivalent WP
  - No matter how similar their syntax is !
- VC are almost as powerful

VC as a "Semantic Checksum" (2)

- Consider the program below
  - In the context of type checking:

```
x := 4
x := x == 5
  assert x : bool
x := not x
  assert x
```

- High-level type checking is not appropriate here
- The VC is: 4 == 5 : bool ∧ not (4 == 5)
- No confusion because reuse of x with different types

Invariance of VC Across Optimizations

- VC is so good at abstracting syntactic details that it is syntactically preserved by many common optimizations
  - Register allocation, instruction scheduling
  - Common-subexpression elimination, constant and copy prop.
  - Dead code elimination
- We have identical VC whether or not an optimization has been performed
  - Preserves syntactic form, not just semantic meaning !
- This can be used to verify correctness of compiler optimizations (Translation Validation)

VC Characterize a Safe Interpreter

- Consider a fictitious "safe" interpreter
  - As it goes along it performs checks (e.g. saferd, validString)
  - Some of these would actually be hard to implement
- The VC describes <u>all</u> of the checks to be performed
  - Along with their context (assumptions from conditionals)
  - Invariants and pre/postconditions are used to obtain a finite expression (through induction)
- VC is valid $\Rightarrow$ interpreter never fails
  - We enforce same level of "correctness"
  - But better (static + more powerful checks)

Review

- Verification conditions
  - Capture the semantics of code + specifications
  - Language independent
  - Can be computed backward/forward on structured/unstructured code
  - Can be computed on high-level/low-level code

Invariants Are Not Easy

- Consider the following code from QuickSort

```
int partition(int *a, int L_0, int H_0, int pivot) {
     int L = L_0, H = H_0;
     while(L < H) {
           while(a[L] < pivot) L ++;
           while(a[H] > pivot) H --;
           if(L < H) { swap a[L] and a[H] }
     }
     return L
}
```

- Consider verifying only memory safety
- What is the loop invariant for the outer loop ?