Chapter 5

Search: Constraint Satisfaction Problems

Introduction to Artificial Intelligence

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- The following is based on material developed by many individuals, including (but not limited to):
    - Sheila McIlraith
    - Fahiem Bacchus
    - Sonya Allin
    - Craig Boutilier
    - Hojjat Ghaderi
    - Rich Zemel
    - Elliot Creager

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  - In the Slider puzzle, we already know the goal, but want to find to get there.
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- To see why, we need to formalized the notion of a CSP.

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  - the states can be defined through a fixed set of variables, $\mathcal{V} = \{ V_1, \ldots, V_{|\mathcal{V}|} \}$.
  - the goals can be defined through a fixed set of constraints, $\mathcal{C} = \{ C_1, \ldots, C_{|\mathcal{C}|} \}$.

- Each state, $s \in \mathcal{S}$, is represented by assigning each variable, $V \in \mathcal{V}$, a unique value, $v \in \mathrm{dom}\, V$, which we represent using a set $\{V = v, V \in \mathcal{V}\}$.

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- Every state must be representable as an assignment of the variables, but not every assignment needs to represent a state, i.e., $\mathcal{S}$ is a subset of $\mathrm{dom}\mathcal{V}$.

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- As we will see later, the ability to form partial assignments is critical to developing efficient algorithms for CSPs.

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  - The constraints are:
    - **row/column constraints**: $V_i \neq V_j, \forall i \neq j$
    - **diagonal constraints**: $|n(V_i) - n(V_j)| \neq |i - j|, \forall i \neq j$, where $n(V)$ denotes the alphabetic position of V (e.g., $n(b) = 2$).

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  - The feasible actions involve modifying the value of any variable.
  - The goal test function checks that the assignment satisfies all the constraints.
- As stated before, this is inefficient. There are two main reasons why.

1. We must assign all variables simultaneously before checking the constraints, but constraints are often violated much earlier.

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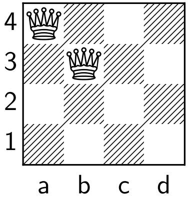
Example: Invalid Partial Assignment in *N*-Queens

- Suppose we place queens on a4 and b3. Then, no matter where we place the remaining queens, the resulting board configuration will be invalid.

- Still, because we must assign all variables simultaneously, we need to check all 16 board configurations that include queens on a4 and b3.



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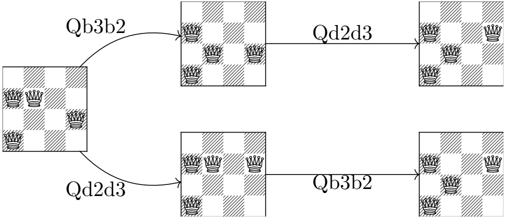
- In the tree below, the order in which the moves Qb3b2 and Qd2d3 are played is irrelevant, as both result in the same board configuration.



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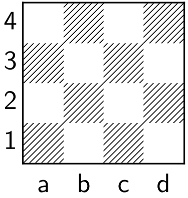
- **Procedure**: Backtracking Search
  1. Start with the empty assignment, $\{\}$.
  2. If all variables are assigned, a solution has been found. Otherwise, pick any unassigned variable, $V \in \mathcal{V}$.
  3. For each value, $v \in \operatorname{dom} V$: If every bound constraint is satisfied, continue searching recursively. If every value in $\operatorname{dom} V$ has been exhausted and none satisfied the bound constraints, backtrack.

Backtracking Search: Pseudo-code

```
procedure Search()
    if Assigned(V_1, ..., V_|𝒱|) then                          ▷ all variables are assigned
        𝒢.Append(([V_1], ..., [V_|𝒱|]))
    else
        V ← SelectUnassigned(𝒱)                              ▷ choose an unassigned variable
        for v ∈ dom V do
            Assign(v, V)
            γ ← false                                        ▷ flag for if constraints are violated
            for C ∈ 𝒞 : Bound(C) do                          ▷ for each bound constraint
                if Violated(C) then
                    γ ← true                                 ▷ the constraint is violated
            if γ = false then
                Search()                                     ▷ search extensions
        Unassign(V)
```

- In this example, we perform backtracking search on the 4-Queens puzzle.

Itr.	V_1	V_2	V_3	V_4
0				
1	a			
2	a	a		
3	a	b		
4	a	c		
5	a	c	a	
6	a	c	b	
7	a	c	c	
8	a	c	d	
9	a	d		
10	a	d	a	
11	a	d	b	
12	a	d	b	a
13	a	d	b	b
14	a	d	b	c
15	a	d	b	d
16	a	d	c	
17	a	d	d	
18	b			
19	b	a		
20	b	b		
21	b	c		
22	b	d		
23	b	d	a	
24	b	d	a	a
25	b	d	a	b
26	b	d	a	c



Heuristics for Variable Selection: Most Constraints

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- If a variable is involved in more constraints, it is more likely to violate one of them, and thus should be checked early.
- This heuristic is called **most constraints**.

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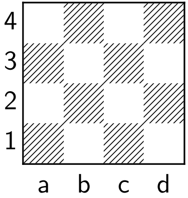
    - This is because the aforementioned choices would violate at least one constraint.
- After each assignment, we could iterate through each constraint, C, with exactly one unassigned variable, V, (so-called "almost bound" constraints) and prune dom V of any values that when assigned to V, violate C.

```
procedure SearchWithFC()
    if Assigned(V_1, ..., V_|𝒱|) then                          ▷ all variables are assigned
        𝒢.Append(([V_1], ..., [V_|𝒱|]))
    else
        V ← SelectUnassigned(𝒱)                               ▷ choose an unassigned variable
        for v ∈ dom V do
            Assign(v, V)
            γ ← false                                         ▷ flag for if constraints are violated
            for C ∈ 𝒞 : AlmostBound(C) do                     ▷ for each almost bound constraint
                if PruneWithFC(C) = DWO then
                    γ ← true                                  ▷ a domain wipe-out occurred
            if γ = false then
                SearchWithFC()                                ▷ search extensions
            RestorePruned()                                   ▷ restore pruned domains
        Unassign(V)
```

procedure PruneWithFC()
    $V \leftarrow$ FindUnassigned($\mathcal{V}_C$) ▷ find an unassigned variable in C's scope
    for $v \in \text{dom}\, V$ **do**
        Assign(v, V)
        if Violated(C) **then**
            Remove($v, \text{dom}\, V$)
            if $\text{dom}\, V = \emptyset$ **then** ▷ domain wipe-out occurred
                return `DWO`
    return `true`

- In this example, we perform backtracking search with forward-checking on the 4-Queens puzzle.

Itr.	V_1	V_2	V_3	V_4	dom V_1	dom V_2	dom V_3	dom V_4
0					$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$
1	a					$\{c, d\}$	$\{b, d\}$	$\{b, c\}$
2	a	c					$\varnothing$	
3	a	d				$\{b\}$	$\{c\}$	
4	a	d	b					$\varnothing$
5	b					$\{d\}$	$\{a, c\}$	$\{a, c, d\}$
6	b	d					$\{a\}$	$\{a, c\}$
7	b	d	a					$\{c\}$
8	b	d	a	c				



- Note that domains are restored whenever a domain wipe-out occurs.

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- Forward checking gives us, for free, a very powerful heuristic for which variables to try next:
  - Always choose the variable with the fewest values remaining in its domain.
- If a variable only has one value left, that value is forced, so we should propagate its consequences immediately.
- This heuristic is called **fewest remaining values** (we actually employ it in the previous example).

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- Of course, in theory, we could check constraints whose scopes contain multiple unassigned variables...though this would take longer to enforce.
- Alternatively, we could observe that some partial assignments will violate an unbound constraint regardless of what we assign to the other variables in its scope.

- **Example**: Pruning Inconsistent Domains

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    - If we let $Y = 15$, the constraint is not satisfiable since there is no $x \in \operatorname{dom} X$ such that $x > 15$.

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  - Consider the constraint, $X > Y$.
  - If we let $Y = 15$, the constraint is not satisfiable since there is no $x \in \operatorname{dom} X$ such that $x > 15$.
  - If we let $X = 1$, the constraint is not satisfiable since there is no $y \in \operatorname{dom} Y$ such that $1 > Y$.

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  - Consider the constraint, $X > Y$.
  - If we let $Y = 15$, the constraint is not satisfiable since there is no $x \in \operatorname{dom} X$ such that $x > 15$.
  - If we let $X = 1$, the constraint is not satisfiable since there is no $y \in \operatorname{dom} Y$ such that $1 > Y$.
  - Thus, we can prune '1' from $\operatorname{dom} X$ and '15' from $\operatorname{dom} Y$.

Backtracking Search with Generalized Arc Consistency: Pseudo-code

```
procedure SearchWithGAC()
    if Assigned(V_1, ..., V_|𝒱|) then                          ▷ all variables are assigned
        𝒢.Append(([V_1], ..., [V_|𝒱|]))
    else
        V ← SelectUnassigned(𝒱)                                ▷ choose an unassigned variable
        for v ∈ dom V do
            Assign(v, V)
            𝒬 = ∅
            for C ∈ 𝒞 : V ∈ 𝒱_C do                ▷ for each constraint whose scope contains V
                𝒬.Append(C)
            if not PruneWithGAC(𝒬) = DWO then                   ▷ no domain wipe-out occurred
                SearchWithGAC()
            RestorePruned()                                     ▷ restore pruned domains
        Unassign(V)
```

Backtracking Search with Generalized Arc Consistency: Pseudo-code

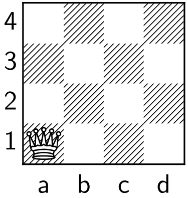
```
procedure PruneWithGAC(𝒬)
    while 𝒬 ≠ ∅ do
        C ← 𝒬.Next()
        for V ∈ 𝒱_C do
            for v ∈ dom V do
                if not Satisfiable(C, V, v) then
                    Remove(v, dom V)
                    if dom V = ∅ then
                        return DWO
                    else
                        for C ∈ 𝒞 do
                            if V ∈ 𝒱_C  and  C ∉ 𝒬 then
                                𝒬.Append(C)
```

Backtracking Search with Generalized Arc Consistency: Example

- In this example, we perform backtracking search while enforcing arc consistency on the 4-Queens puzzle.

Itr. 1:

dom V_1	dom V_2	dom V_3	dom V_4	$\mathcal{Q}$
$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{\}$
$\{a\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{C_1^2, C_1^3, C_1^4\}$
$\{a\}$	$\{c, d\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$	$\{C_1^3, C_1^4, C_2^3, C_2^4\}$
$\{a\}$	$\{c, d\}$	$\{b, d\}$	$\{a, b, c, d\}$	$\{C_1^4 C_2^3, C_2^4, C_3^4\}$
$\{a\}$	$\{c, d\}$	$\{b, d\}$	$\{b, c\}$	$\{C_2^3, C_2^4, C_3^4\}$
$\{a\}$	$\{d\}$	$\{b\}$	$\{b, c\}$	$\{C_2^4, C_3^4, C_2^1, C_2^4, C_3^1\}$
$\{a\}$	$\{d\}$	$\{b\}$	$\{c\}$	$\{C_3^4, C_2^1, C_2^4, C_3^1, C_4^1\}$
$\{a\}$	$\{d\}$	$\emptyset$	$\{c\}$	$\{C_2^1, C_2^4, C_3^1, C_4^1\}$

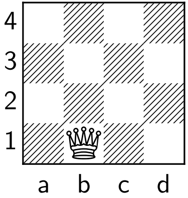


Backtracking Search with Generalized Arc Consistency: Example

- In this example, we perform backtracking search while enforcing arc consistency on the 4-Queens puzzle.

Itr. 2:

dom V_1	dom V_2	dom V_3	dom V_4	$\mathcal{Q}$
$\{b\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{C_1^2, C_1^3, C_1^4\}$
$\{b\}$	$\{d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{C_1^3, C_1^4, C_2^3, C_2^4\}$
$\{b\}$	$\{d\}$	$\{a,c\}$	$\{a,b,c,d\}$	$\{C_1^4, C_2^3, C_2^4, C_3^4\}$
$\{b\}$	$\{d\}$	$\{a,c\}$	$\{a,c,d\}$	$\{C_2^3, C_2^4, C_3^4\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{a,c,d\}$	$\{C_2^4, C_3^4, C_2^1\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{a,c\}$	$\{C_3^4, C_2^1, C_4^1\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_2^1, C_4^1, C_4^2\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_4^1, C_4^2\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_4^2\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{\}$



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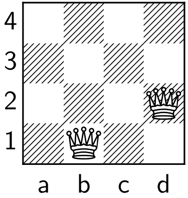
Itr. 3:

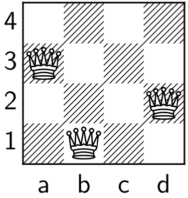
dom V_1	dom V_2	dom V_3	dom V_4	$\mathcal{Q}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_2^1, C_2^3, C_2^4\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_2^3, C_2^4\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_2^4\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{\}$



- In this example, we perform backtracking search while enforcing arc consistency on the 4-Queens puzzle.

Itr. 4:

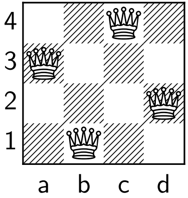
dom V_1	dom V_2	dom V_3	dom V_4	$\mathcal{Q}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_3^1, C_3^2, C_3^4\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_3^2, C_3^4\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{C_3^4\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{\}$



- In this example, we perform backtracking search while enforcing arc consistency on the 4-Queens puzzle.

Itr. 5:

dom V_1	dom V_2	dom V_3	dom V_4	$\mathcal{Q}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\left\{C_4^1, C_4^2, C_4^3\right\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\left\{C_4^2, C_4^3\right\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\left\{C_4^3\right\}$
$\{b\}$	$\{d\}$	$\{a\}$	$\{c\}$	$\{\}$



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- This is true in general.