Partial Differential Equations

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Fall 2020

Contents

1 Introduction to Partial Differential Equations (PDEs)

1.1 The Definition of a PDE

We begin with a definition of a partial differential equation:

Definition 1.1. (Partial Differential Equation) A **partial differential equation** (PDE) is an equation that relates an unknown function, u, and some of its partial derivatives.

Typically, u is defined over the closure of an n-dimensional spatial domain $\Omega \subset \mathbb{R}^n$, and time interval, $I \subset \mathbb{R}_+$, i.e., $u : \overline{\Omega \times I} \to \mathbb{R}$.

However, the PDE governs u only within the interior of $\Omega \times I$, i.e.,

$$G\left(x_1, \ldots, x_n, t, u\right) = 0 \text{ on } \Omega \times I, \tag{1.1}$$

where $x_1, \ldots, x_n$ are the spatial coordinates, t represents time, and G is some arbitrary operator which computes at least one partial derivatives of u.

We now make a few quick remarks:

Remark 1.1. The right-hand side of eq. 1.1 can be assumed to be zero w/o loss of generality since any non-zero right-hand side may be absorbed into G itself.

Remark 1.2. It is sometimes more convenient to view time as the 0^{th} spatial coordinate, x_0, and we will henceforth use t and x_0 interchangeably.

Here we explore methods to find solutions to the PDE described by eq. 1.1.

Definition 1.2. (Solution of a PDE) A **solution** to the PDE described by eq. 1.1 is a function, $u^* : \overline{\Omega \times I} \to \mathbb{R}$ such that $G(x_1, \ldots, x_n, t, u^*) = 0$ on $\Omega \times I$.

Obviously, the solution to a PDE is not necessarily unique since its behaviour on the space-time boundary, $\Gamma = \partial\Omega \times I \cup \overline{\Omega} \times \{t = 0\}$ is undefined.

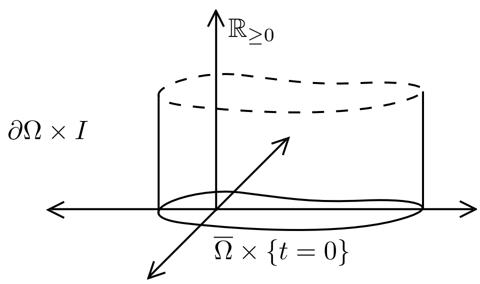

Figure 1: Space-time boundary, Γ.

To ensure uniqueness in the solution, so-called initial and boundary conditions need to be applied.

Definition 1.3. (Initial Conditions) Initial conditions govern the behaviour of u on the temporal boundary, $\overline{\Omega} \times \{t = 0\}$.

Definition 1.4. (Boundary Conditions) Boundary conditions govern the behaviour of u on the spatial boundary, i.e., $\partial\Omega \times I$.

In any case, the soluion obviously depends on the properties of G itself. A property of particular interest is the order.

Definition 1.5. (Order of a PDE) The **order** of the PDE described by eq. 1.1 is that of the highest order derivative of u computed by G.

We will focus on 2^{nd}-order linear PDEs.

1.2 Linear PDEs

Definition 1.6. (Linear Operator) An operator, $\mathcal{L}$ is **linear** iff

$$\mathcal{L}(k_1 u_1 + k_2 u_2) = k_1 \mathcal{L} u_1 + k_2 \mathcal{L} u_2,$$

for every pair of functions, u_1 and u_2, and scalars, $k_1, k_2 \in \mathbb{R}$.

Definition 1.7. (Linear PDEs) A PDE is said to be **linear** iff it can be expressed in terms of a linear operator, $\mathcal{L}$, of the solution u, i.e., as

$$\mathcal{L}u = f, \tag{1.2}$$

where f is an arbitrary function independent of u called the **forcing function**.

Definition 1.8. (Homogeneous Linear PDE) A linear PDE is said to be **homogeneous** iff $f \equiv 0$, i.e., if it is of the form

$$\mathcal{L}u = 0. \tag{1.3}$$

Remark 1.3. We say that eq. 1.3 is the related homogeneous PDE of eq. 1.2.

Homogeneous linear PDEs obey the so-called principle of superposition:

Principle 1.1. (Superposition) If u_1 and u_2 are two solutions to eq. 1.3, then so too is the function $k_1 u_1 + k_2 u_2$ for any $k_1, k_2 \in \mathbb{R}$.

Proof. Since $\mathcal{L}$ is a linear operator, $\mathcal{L}(k_1 u_1 + k_2 u_2) = k_1 \mathcal{L} u_1 + k_2 \mathcal{L} u_2$. Since u_1 and u_2 are themselves solutions to eq. 1.3, $\mathcal{L}u_1 = \mathcal{L}u_2 = 0$. Consequently, $\mathcal{L}(k_1 u_1 + k_2 u_2) = 0$, which means $k_1 u_1 + k_2 u_2$ is also a solution to eq. 1.3. □

Furthermore, if u_p is a particular solution to the inhomogeneous linear PDE described by eq. 1.2, then so too is $u_\text{p} + k u_\text{h}$, where u_h is some solution to the related homogeneous PDE described by eq. 1.3, and $k \in \mathbb{R}$.

The principle of superposition will be exploited heavily in solving linear PDEs.

1.3 2^{nd}-Order Linear PDEs

Any 2^{nd}-order linear PDE can be represented as

$$\mathcal{L}u = \sum_{j=0}^{n}\sum_{k=0}^{n} a_{j,k}\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=0}^{n} b_j \frac{\partial u}{\partial x_j} + cu. \tag{1.4}$$

Definition 1.9. (Dirichlet Boundary Conditions) The condition $u = h$ on

As we will see later, the behaviour the solution(s) of eq. 1.4 are heavily dependant on the coefficients $a_{j,k}$, or more specifically, the eigenvalues, $\lambda_1, \dots, \lambda_{n+1}$, of a particular $(n+1)\times(n+1)$ matrix, $\mathbf{A}$, defined such that $\mathbf{A}_{j,k} = a_{j,k}$.

In particular, we say that that eq. 1.4 is

- elliptical, if all eigenvalues are non-zero and have the same sign,
- hyperbolic, if all eigenvalues are non-zero and exactly one of them has a sign opposite to the others, and;
- parabolic, if exactly one eigenvalue is zero, and all others have the same sign.

Remark 1.4. The names; "eliptical", "hyperbolic", and "parabolic", are those of conic sections that are traced by the quadratic form $\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0, \mathbf{x} \in \mathbb{R}^{n+1}$.

1.3.1 Sturm-Liouville Theory

We first study the properties of a particular 2^{nd} order BVP, called the Sturm-Louiville BVP that will come up often.

Definition 1.10. (Strum-Louiville Problem) The regular Sturm-Liouville problem is given by

$$\begin{aligned} -(p\phi')' + q\phi' &= \lambda w \phi \in (a,b) \\ \alpha_1\phi(a) + \alpha_2\phi'(a) &= 0 \\ \beta_1\phi(b) + \beta_2\phi'(b) &= 0 \end{aligned} \tag{1.5}$$

where $\phi, \phi', p, q : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$, $p, q > 0$, $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$.

Theorem 1.1. (Realness of Sturm-Louiville Eigenvalues) If λ is an eigenvalue of Eq. 1.5, then $\lambda \in \mathbb{R}$.

Theorem 1.2. (Orthogonality of Sturm-Louville Eigenfunctions) Let (ϕ_1, λ_1) and (ϕ_2, λ_2) be two eigenfunction-eigenvalue pairs associated with Eq. 1.5. with $\lambda_1 \neq \lambda_2$. Then ϕ_1 and ϕ_2 are orthogonal in the sense that

$$\frac{2}{b-a}\int_a^b w\phi_1\phi_2 dx = 0.$$

In fact, the eigenfunctions become orthonormal in the above metric.

Theorem 1.3. (Positiveness of Sturm-Louville Eigenvalues) If λ is an eigenvalue of Eq. 1.5 where $w > 0$ on $[a,b]$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, then $\lambda \geq 0$.

1.3.2 Duhamel's Principle

Theorem 1.4. (Duhamel's Principle) Consider the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}u = f \text{ in } \mathbb{R}^n \times \mathbb{R}_{>0} \\ u = g \text{ on } \mathbb{R}^n \times \{t = 0\}, \end{aligned} \tag{1.6}$$

where $\mathcal{L}$ is any linear differential operator, $f : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}$ is the source function, and $g : \mathbb{R}^n \to \mathbb{R}$ is the initial condition.

For any fixed $s \in \mathbb{R}_{>0}$, let $u_f^s : \mathbb{R}^n \times \mathbb{R}_{>s} \to \mathbb{R}$ be the solution to

$$\begin{aligned} \frac{\partial u^s}{\partial t} + \mathcal{L}u_f^s = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_{>s} \\ u^s = f(\cdot, s) \text{ on } \mathbb{R}^n \times \{t = s\}, \end{aligned}$$

and let $u_g : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}$ be the solution to

$$\begin{aligned} \frac{\partial u_g}{\partial t} + \mathcal{L}u_g = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_{>0} \\ u_g = g \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{aligned}$$

Then, the solution is

$$u(x,t) = u_g(x,t) + \int_{s=0}^{t} u_f^s(x,t) ds, x \in \mathbb{R}^n, t \in \mathbb{R}_{>0}. \tag{1.7}$$

Proof. We first consider the case of $f \neq 0$ and $g = 0$ (and hence $u_g = 0$). We differentiate Eq. **??** w.r.t. t to obtain

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial u}{\partial t}\left(\int_{s=0}^{t} u_f^s(x,t) ds\right)$$
$$= u_f^t(x,t) + \int_{s=0}^{t} \frac{\partial u_f^s}{\partial t}(x,t) ds = f(x,t) - \int_{s=0}^{t} \mathcal{L}u_f^s(x,t) ds.$$

Thus, the solution satisfies the ∎

2 Representing Solutions of PDEs

2.1 Representing Functions via Series

The principle of superposition can be exploited by representing the solution, u, as a linear combination of a set of known basis functions, $\{\phi_k\}$.

Since we are assuming $\Omega \subset \mathbb{R}$, we consider the representations of a function $f : \mathbb{R} \to \mathbb{R}$ in only one-dimension and write

$$f(x) = \sum_k \hat{f}_k \phi_k(x),$$

for the appropriately chosen $\hat{f}_k$. Of course, we require that $f \in \mathrm{Span}\{\phi_k\}$, which is not always the case, but for the moment, we will assume it to be.

While the basis is otherwise arbitrary, it is preferable to choose one that is orthogonal since it allows us to derive a relatively simple formula for the $\hat{f}_k$.

Definition 2.1. (Orthogonal Basis) A set of basis functions, $\{\phi_k\}$ is **orthogonal** if $\langle \phi_j, \phi_k \rangle = 0$ whenever $k \neq j$, where $\langle \cdot, \cdot \rangle$ denotes an inner-product.

In this case, $\hat{f}_k$ may be computed as

$$\frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} = \frac{1}{\langle \phi_k, \phi_k \rangle} \left\langle \sum_j \hat{f}_j \phi_j, \phi_k \right\rangle = \frac{1}{\langle \phi_k, \phi_k \rangle} \sum_j \langle \hat{f}_j \phi_j, \phi_k \rangle = \hat{f}_k.$$

If the basis is **orthonormal**, i.e., orthogonal and moreover $\langle \phi_k, \phi_k \rangle = 1, \forall k$, the above reduces to $\hat{f}_k = \langle f, \phi_k \rangle$.

2.2 The Fourier Series and its Properties

One such orthonormal basis which will be particularly useful in the context of PDEs is the so-called Fourier basis,

$$\left\{ \phi_k(x) := \exp\left(\frac{2\pi i k x}{T_0} \right), k \in \mathbb{Z} \right\},$$

where $T_0 \in \mathbb{R}$ is simply some arbitrary constant for now.

Claim 2.1. The Fourier basis is orthonormal w.r.t. the following inner-product:

$$\langle \phi_j, \phi_k \rangle := \frac{1}{T_0} \int_{x_0}^{x_0+T_0} \phi_j(x) \phi_k^*(x) dx.$$

Proof. We have

$$\langle \phi_j, \phi_k \rangle = \left\langle \exp\left(\frac{2\pi i j x}{T_0} \right), \exp\left(\frac{2\pi i k x}{T_0} \right) \right\rangle$$

$$= \frac{1}{T_0} \int_{x_0}^{x_0+T_0} \exp\left(\frac{2\pi i j x}{T_0} \right) \exp\left(-\frac{2\pi i k x}{T_0} \right) dx$$

$$= \frac{1}{T_0} \int_{x_0}^{x_0+T_0} \exp\left(\frac{2\pi i (j-k) x}{T_0} \right) dx.$$

For $j = k$, the above integral reduces to

$$\frac{1}{T_0}\int_{x_0}^{x_0+T_0} dx = \frac{1}{T_0} x|_{-T_0/2}^{T_0/2} = 1.$$

For $j \neq k$, we note that since $j - k \in \mathbb{Z}$, the integrand traces a whole number of revolutions around the unit-circle in the complex plane, and thus,

$$\frac{1}{T_0}\int_{x_0}^{x_0+T_0} \exp\left(\frac{2\pi i(j-k)x}{T_0}\right) dx = 0.$$

$\square$

Remark 2.1. It is often useful to choose $x_0 = -T_0/2$ so that the interval of integration is symmetric.

As we said earlier, we require $f \in \text{Span}\{\phi_k\}$, and for the Fourier basis above, this is only the case if f is periodic with T_0 being its fundamental period.

Definition 2.2. (Periodic Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** with a period of T, if for any $k \in \mathbb{Z}$,

$$f(x + kT) = f(x), \forall x \in \mathbb{R}.$$

The smallest T for which this relationship holds is called the **fundamental period**, often denoted with T_0.

In the case where f is not periodic, we can instead consider its so-called periodic extension.

Definition 2.3. (Periodic Extension of an Aperiodic Function) Given an aperiodic function, $f : [0, T_0) \to \mathbb{R}$, its **periodic extension** is defined as

$$\tilde{f}(x + kT_0) = f(x), \forall x \in [0, T_0),$$

for all $k \in \mathbb{Z}$. Note that $\tilde{f}$ has a fundamental period of T_0.

We may now define the Fourier series representation of a function.

Definition 2.4. (Complex Fourier Series) Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with a fundamental period of T_0. The **complex Fourier series expansion** of f is

$$f(x) = \sum_{=-\infty}^{\infty} c_k \exp\left(\frac{2\pi i k x}{T_0}\right),$$

where

$$c_k = \frac{1}{T_0}\int_{x_0}^{x_0+T_0} f(x) \exp\left(\frac{2\pi i k x}{T_0}\right) dx \in \mathbb{C},$$

are called the **complex Fourier coefficients**.

In the case where f is real-valued, i.e., the case we are considering, we can derive an alternate representation to that of Def. 2.4.

The derivation makes use of Euler's formula, i.e., $e^{ix} = cosx + i\sin x$, which can be used to show that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

To begin, we express c_k in Def. 2.4 as $c_k := \frac{1}{2}(a_k - ib_k)$ for some $a_k, b_k \in \mathbb{R}$. We then note that when f is real-valued, $c_{-k} = c_k^* = \frac{1}{2}(a_k + ib_k)$, and furthermore $c_0 \in \mathbb{R}$, i.e., $b_0 = 0$.

Thus, we may write,

$$f(x) = \frac{1}{2}\sum_{k=-\infty}^{\infty}(a_k - ib_k)\exp\left(\frac{2\pi ikx}{T_0}\right)$$

$$= \frac{1}{2}a_0 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k - ib_k)\exp\left(\frac{2\pi ikx}{T_0}\right) + (a_k + ib_k)\exp\left(-\frac{2\pi ikx}{T_0}\right)$$

$$= \frac{1}{2}a_0 + \sum_{k=0}^{\infty}\frac{a_k}{2}\left[\exp\left(\frac{2\pi ikx}{T_0}\right) + \exp\left(-\frac{2\pi ikx}{T_0}\right)\right]$$

$$- \frac{ib_k}{2}\left[\exp\left(\frac{2\pi ikx}{T_0}\right) - \exp\left(-\frac{2\pi ikx}{T_0}\right)\right]$$

$$= \frac{1}{2}a_0 + \sum_{k=1}^{\infty}a_k\cos\left(\frac{2\pi kx}{T_0}\right) + b_k\sin\left(\frac{2\pi kx}{T_0}\right),$$

where $a_k = 2\Re(c_k)$ and $b_k = 2\Im(c_k)$.

Definition 2.5. (Real Fourier Series) Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with a fundamental period of T_0. The **real Fourier series expansion** of f is

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}a_k\cos\left(\frac{2\pi kx}{T_0}\right) + b_k\sin\left(\frac{2\pi kx}{T_0}\right),$$

where

$$a_k = \frac{2}{T_0}\int_{x_0}^{x_0+T_0} f(x)\cos\left(\frac{2\pi kx}{T_0}\right)dx \text{ and } b_k = \frac{2}{T_0}\int_{x_0}^{x_0+T_0} f(x)\sin\left(\frac{2\pi kx}{T_0}\right)dx.$$

We now consider two special cases where f is either even or odd.

Definition 2.6. (Even Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is said to be **even** if

$$f(x) = f(-x), \forall x \in \mathbb{R}.$$

Definition 2.7. (Odd Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is said to be **odd** if

$$f(x) = -f(-x), \forall x \in \mathbb{R}.$$

We now state a few properties of even/odd functions which may be easily proved using definitions 2.6 and 2.7:

- **Summation**. The sum of two even functions is even. The sum of two odd functions is odd.

- **Multiplication**. The product of two even functions or two odd functions is even. The product of an even function and an odd function is odd.
- **Differentiation**. The derivative of an odd function is even, and the derivative of an even function is odd.
- **Anti-differentiation**. The anti-derivative of an even function is odd. The anti-derivative of an odd function is even.

The properties above can be used to easily show that if f is a periodic, even (resp. odd), then $b_k = 0$ (resp. $a_k = 0$) for all k. The resulting series is colloquially refereed to as the Fourier cosine (resp. sine) series.

As it turns out, in the case where f is aperiodic, we can define the periodic extension to be either even or odd.

Definition 2.8. (Even Periodic Extension of an Aperiodic Function) Given an aperiodic function, $f : [0, T_0] \to \mathbb{R}$, its even periodic extension is defined such that

$$\tilde{f}_{\mathrm{e}}(x + 2kT_0) = \begin{cases} f(x), x \in [0, T_0] \\ f(-x), x \in [-T_0, 0), \end{cases}$$

for all $k \in \mathbb{Z}$ and where $x \in [0, T_0]$. The fundamental period of $\tilde{f}_{\mathrm{e}}$ is $2T_0$.

Definition 2.9. (Odd Periodic Extension of an Aperiodic Function) Given an aperiodic function, $f : [0, T_0] \to \mathbb{R}$, its odd periodic extension is defined such that

$$\tilde{f}_{\mathrm{o}}(x + 2kT_0) = \begin{cases} f(x), x \in [0, T_0] \\ -f(-x), x \in [-T_0, 0), \end{cases}$$

for all $k \in \mathbb{Z}$ and where $x \in [0, T_0]$. The fundamental period of $\tilde{f}_{\mathrm{o}}$ is $2T_0$.

2.3 Regularity of Functions

Something we have yet to consider is if a series representation of a function actually converges.

To analyse this, we will need to characterize the regularity of the function.

Definition 2.10. (Continuous Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is **continuous** on the open-interval, (a, b), if and only if

$$\lim_{x \to x_0} f(x) = f(x_0), \forall x_0 \in (a, b).$$

For f to be continuous on the closed-interval $[a, b]$ we additionally require that

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b).$$

The space of continuous functions on an interval, I, is denoted $C^0(I)$.

Definition 2.11. (Differentiable Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** on the open-interval (a, b) if and only if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is defined for all $x \in (a, b)$. For f to be differentiable on the closed interval $[a, b]$ we additionally require that

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ and } \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

both also exist.

Remark 2.2. If f is differentiable on I it is also continuous on I.

Definition 2.12. (Continuously Differentiable Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is n-times **continuously differentiable** on the interval I if f is continuous, and $f^{(j)}$ exists and is continuous for $j = 1, \ldots, n$.

The space of n-times continuously differentiable functions on some interval I is denoted with $C^n(I)$.

Definition 2.13. (Piecewise Continuous Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is **piecewise continuous** on the open-interval (a, b) if there exists a set of points, $a = x_1 < \cdots < x_N = b$, such that for $i = 1, \ldots, N$,

- f is continuous on the sub-interval, (x_i, x_{i+1}), and
- the one-sided limits, $\lim_{x \to x_i^+} f(x)$ and $\lim_{x \to x_{i+1}^-} f(x)$ exist.

Definition 2.14. (Piecewise Continuously Differentiable Function) A function, $f : \mathbb{R} \to \mathbb{R}$ is n-times **piecewise continuously differentiable** on the open-interval, (a, b) if there exists a set of points, $a = x_1 < \cdots < x_N = b$, such that for $i = 1, \ldots, N$,

- f is n-times continuously differentiable on the sub-interval, (x_i, x_{i+1}), and
- the one-sided limits, $\lim_{x \to x_i^+} f^{(j)}(x)$ and $\lim_{x \to x_{i+1}^-} f^{(j)}(x)$, exist for $j = 1, \ldots, n$.

2.4 Convergence of the Fourier Series

To discuss the convergence of the Fourier series, we must first define its truncated approximation.

Definition 2.15. (Truncated Fourier Series) Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with a fundamental period of T_0. The **real truncated Fourier series expansion** of f is

$$f(x) \approx s_N(x) := \frac{1}{2} a_0 + \sum_{k=1}^{N} a_k \cos\left(\frac{2\pi k x}{T_0}\right) + b_k \sin\left(\frac{2\pi k x}{T_0}\right),$$

where a_k and b_k are defined exactly as before.

Theorem 2.1. (Pointwise Convergence of the Fourier Series) Let f be a piecewise continuously differentiable function with a period T_0, and $s(N)$ be the associated truncated Fourier series. Then for any fixed $x_0 \in \mathbb{R}$,

$$s_N(x_0) \to \frac{1}{2T_0}\left(\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)\right) \text{ as } N \to \infty.$$

Corollary 2.1. If f is continuous at x_0, then $s_N(x_0) \to f(x_0)$ as $N \to \infty$.

3 The Diffusion Equation

3.1 Definition and Problem Setup

Definition 3.1. (The Diffusion Equation) Consider an arbitrary spatial domain, $\Omega \subset \mathbb{R}^n$, and temporal interval, $I \subset \mathbb{R}_{\geq 0}$. A function, $u : \Omega \times I \to \mathbb{R}$ satisfies the **Diffusion Equation** iff

$$\frac{\partial u}{\partial t} - \nabla^2 u = f, \tag{3.1}$$

where $f : \Omega \times I \to \mathbb{R}$ is independent of u.

Remark 3.1. The Diffusion Equation is linear 2^{nd}-order parabolic PDE.

To form a well-posed problem, we introduce additional constraints. In particular, we will need:

- one boundary condition for each point on $\partial\Omega \times \mathbb{R}_{>0}$, i.e., the spatial boundary, and
- one initial condition for each point on $\Omega \times \{t = 0\}$, i.e., the temporal boundary.

Together, the initial and boundary conditions define the behaviour of u on the space-time boundary $\Gamma := \Omega \times \{t = 0\} \cup \partial\Omega \times I$, as shown below:

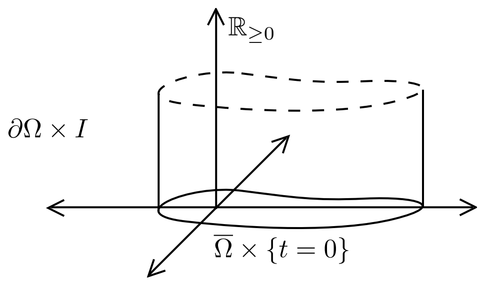

Figure 2: Space-time boundary, Γ.

Definition 3.2. (The Diffusion IBVP) Let $g_{\text{D}} : \Gamma_{\text{D}} \to \mathbb{R}$, $g_{\text{N}} : \Gamma_{\text{N}} \to \mathbb{R}$, and $h : \Omega \to \mathbb{R}$ where $\overline{\Gamma_{\text{D}} \cup \Gamma_{\text{N}}} = \partial\Omega$. We consider the following PDE IBVP associated with Eq. 3.1:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla^2 u = f \text{ in } \Omega \times \mathbb{R}_{>0} \\ u = g_{\text{D}} \text{ on } \Gamma_{\text{D}} \times \mathbb{R}_{>0} \\ \frac{\partial u}{\partial n} = g_{\text{N}} \text{ on } \Gamma_{\text{N}} \times \mathbb{R}_{>0} \\ u = h \text{ on } \Omega \times \{t = 0\}. \end{aligned} \tag{3.2}$$

where $\frac{\partial}{\partial n}$ is the directional derivative tangent to Γ_{N}.

3.2 Properties of Solutions

Theorem 3.1. (Non-Dimensionalization) The solution, u_{d}, to the dimensionalized heat-equation,

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial u^2}{\partial x^2} = 0 \text{ in } (0, L) \times \mathbb{R}_{>0},$$

is related to the solution, u_{n} to the non-dimensionalized analogue,

$$\frac{\partial u}{\partial t} - \frac{\partial u^2}{\partial x^2} = 0 \text{ in } (0, 1) \times \mathbb{R}_{>0},$$

through

$$u_{\mathrm{d}}(x, t) = \tilde{u}_{\mathrm{n}}(x/L, \kappa^2 t/L^2).$$

Proof. Choosing $\tilde{x} = x/L$ and $\tilde{t} = t/T$, where $T \equiv L^2/\kappa$, we have

$$\frac{\partial u}{\partial t} = \frac{1}{T^2} \frac{\partial u}{\partial \tilde{t}} \text{ and } \frac{\partial^2 u}{\partial x^2} = \frac{1}{L^2} \frac{\partial^2 u}{\partial \tilde{x}^2} = \frac{1}{T^2 c^2} \frac{\partial^2 u}{\partial \tilde{x}^2}.$$

Thus, u is the solution to

$$\frac{\partial u}{\partial \tilde{t}} - \frac{\partial u^2}{\partial \tilde{x}^2} = 0 \text{ in } (0, 1) \times \mathbb{R}_{>0}$$

in the variables, $\tilde{x}$ and $\tilde{t}$. □

We now consider two important properties of the heat equation relating to the extrema of its solutions. In particular, we can show that the extrema of u occur somewhere on the space-time boundary, Γ.

To begin, we consider the maximum of u.

Theorem 3.2. (Maximum Principle) Suppose u is a solution to Eq. 3.1. Then, for any interval, $I \subset \mathbb{R}_{\geq 0}$,

$$\max_{(\boldsymbol{x},t) \in \overline{\Omega \times I}} u(\boldsymbol{x}, t) = \max_{(\boldsymbol{x},t) \in \Gamma} u(\boldsymbol{x}, t),$$

where $\Gamma = \left(\overline{\Omega} \times \{t = 0\}\right) \cup \left(\partial\Omega \times \overline{I}\right)$ is the space-time boundary.

Proof. Let $(\boldsymbol{x}^*, t^*)$ be the maximizer. Assume towards a contraction that, $(\boldsymbol{x}^*, t^*) \notin \Gamma$. Then, it must be the case that $(\boldsymbol{x}^*, t^*) \in \Omega \times I$. □

Now for the minimum of u.

Theorem 3.3. (Minimum Principle) Suppose u is a solution to Eq. 3.1. Then, for any interval, $I \subset \mathbb{R}_{\geq 0}$,

$$\min_{(\boldsymbol{x},t) \in \overline{\Omega \times I}} u(\boldsymbol{x}, t) = \min_{(\boldsymbol{x},t) \in \Gamma} u(\boldsymbol{x}, t),$$

where $\Gamma = \left(\overline{\Omega} \times \{t = 0\}\right) \cup \left(\partial\Omega \times \overline{I}\right)$ is the space-time boundary.

Proof. Since the heat equation is linear, it follows that $-u$ is also a solution, and must satisfy the maximum principle, i.e.,

$$\max_{(\boldsymbol{x},t)\in\overline{\Omega\times I}} w(\boldsymbol{x},t) = \max_{(\boldsymbol{x},t)\in\partial\Omega} w(\boldsymbol{x},t).$$

On the left side, we have

$$\max_{(\boldsymbol{x},t)\in\overline{\Omega\times I}} w(\boldsymbol{x},t) = \max_{(\boldsymbol{x},t)\in\overline{\Omega\times I}} -u(\boldsymbol{x},t) = -\min_{(\boldsymbol{x},t)\in\overline{\Omega\times I}} u(\boldsymbol{x},t).$$

On the right side, we have

$$\max_{(\boldsymbol{x},t)\in\Gamma} w(\boldsymbol{x},t) = \max_{(\boldsymbol{x},t)\in\Gamma} -u(\boldsymbol{x},t) = -\min_{(\boldsymbol{x},t)\in\Gamma} u(\boldsymbol{x},t).$$

The theorem immediately follows by equating the above results. □

These results allow us to make general statements about the solutions without actually computing them.

3.3 Uniqueness and Stability

Before finding solutions to the heat equation, we first show that the IBVP associated with it is well-posed, i.e., its solutions are unique and stable.

We first show uniqueness.

Theorem 3.4. (Uniqueness of the Solution to the Heat Equation) Suppose u_1 and u_2 are two solutions to Eq. 3.2 with the same initial and boundary conditions. Then $u_1 \equiv u_2$.

Proof. Let u_1 and u_2 be two solutions to the heat equation, and define $w \equiv u_1 - u_2$. By linearity, w also satisfies the heat equation, i.e.,

$$\frac{\partial w}{\partial t} - \nabla^2 w = 0 \text{ in } \Omega \times I.$$

We define the energy in w (not to be confused with thermal energy) as

$$E(t) := \frac{1}{2}\int_\Omega w(\boldsymbol{x},t)^2 d\Omega, t \in I.$$

First, we notice that $E(t) \geq 0$. Now,

$$\frac{d}{dt}E = \int_\Omega w\frac{\partial w}{\partial t}d\Omega = \int_\Omega w\nabla^2 w d\Omega = -\underbrace{\int_\Omega \nabla w \cdot \nabla w d\Omega}_{I_1} + \underbrace{\int_{\partial\Omega} w\nabla w \cdot d\boldsymbol{s}}_{I_2}.$$

Since $\nabla w \cdot \nabla w = |\nabla w|^2 \geq 0$, it follows that $I_1 \geq 0$. Conversely, it can be shown that $I_2 \leq 0$, but the justification depends on the boundary condition used. In any case, the upshot is that

$$\frac{d}{dt}E \leq 0$$

and so E is a non-increasing function of time. Thus, $E(t) \leq E(0)$, meaning that

$$0 \leq \int_\Omega w(\boldsymbol{x}, t)^2 d\Omega \leq \int_\Omega w(\boldsymbol{x}, 0)^2 d\Omega = 0,$$

where the last equality follows from the fact that u_1 and u_2 are initially equivalent. Therefore, $w \equiv 0$ and u_1 and u_2 are always identical everywhere.

□

We now consider the stability.

Theorem 3.5. (L^2 Stability of Solutions to the Heat Equation) Suppose u_1 and u_2 are two solutions to Eq. 3.2 with the same boundary conditions but different initial conditions, h_1 and h_2 respectively. Then

$$\int_\Omega ((\boldsymbol{x}, t) - u_2(\boldsymbol{x}, t))^2 d\Omega \leq \int_\Omega (h_1(\boldsymbol{x}) - h_2(\boldsymbol{x}))^2 \, d\Omega,$$

i.e., the solutions are stable in the L^2 sense.

Proof. The proof immediately follows from the inequality

$$\int_\Omega w(\boldsymbol{x}, t)^2 d\Omega \leq \int_\Omega w(\boldsymbol{x}, 0)^2 d\Omega,$$

shown in the proof of Theorem 3.4. □

Corollary 3.1. (Uniform Stability of Solutions to the Heat Equation) The problem defined by Eq. 3.2 where $\Gamma_N \equiv \emptyset$, is stable.

In particular, suppose

$$\begin{aligned} \frac{\partial u_i}{\partial t} - \nabla^2 u &= 0 \text{ in } \Omega \times \mathbb{R}_{>0} \\ u &= g_i \text{ on } \partial\Omega \times \mathbb{R}_{>0} \\ u &= h_i \text{ on } \Omega \times \{t = 0\} \end{aligned}$$

for $i = 1, 2$. Then, u_1 and u_2 are stable in the sense that

$$\max_{\Omega \times I} |u_1 - u_2| \leq \max \left\{ \max_{\partial\Omega \times \bar{I}} |h_1 - h_2|, \max_{\bar{\Omega}} |g_1 - g_2| \right\}.$$

Proof. Omitted. □

3.4 Solutions on Bounded Domains

We now find solutions to Eq. 3.2 where $\Omega \subset \mathbb{R}$ is bounded. However, we will only consider the case where $\Omega = (0, 1)$ since the more general solution can be readily derived from Thm. **??**.

3.4.1 with Homogeneous Boundary Conditions

We first consider the case with no source term and homogeneous boundary conditions, i.e.,

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 \text{ in } \Omega \times \mathbb{R}_{>0} \\ u &= 0 \text{ on } \Gamma_{\mathrm{D}} \times \mathbb{R}_{>0} \\ \frac{\partial u}{\partial x} &= 0 \text{ on } \Gamma_{\mathrm{N}} \times \mathbb{R}_{>0} \\ u &= h \text{ on } \partial\Omega \times \{t = 0\}. \end{aligned} \tag{3.3}$$

To begin, we seek a family of solutions of the separable form,

$$u_n(x,t) = \phi_n(x) T_n(x), n = 0, 1, 2, \ldots,$$

that satisfy the PDE and boundary conditions, but not necessarily the initial conditions.

Substituting u_n into the PDE, we have

$$\phi_n T_n' - \phi_n'' T_n = 0.$$

Assuming $\phi_n \neq 0$ and $T_n \neq 0$, we rearrange the expression to obtain

$$\frac{\phi_n''}{\phi_n} = \frac{T_n'}{T_n} = -\alpha_n,$$

where α_n is some constant.

From here, the procedure differs slightly depending on the boundary conditions imposed. We consider each case separately:

- **Neumman**: $\phi_n'(0) = \phi_n'(1) = 0$. Imposing the boundary conditions, we have

  $$\frac{\partial u_n}{\partial x}(x = 0, t) = \phi_n'(0) T_n(t) = 0,$$

  and

  $$\frac{\partial u_n}{\partial x}(x = 1, t) = \phi_n'(1) T_n(t) = 0.$$

  We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

  Thus, ϕ_n is a solution to the spatial eigenproblem

  $$\begin{aligned} -\phi_n'' &= \alpha_n \phi_n \text{ in } (0,1) \\ \phi_n'(0) &= 0 \\ \phi_n'(1) &= 0 \end{aligned} \quad .$$

  We can readily verify that this is a Sturm-Liouville problem in $(0,1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_1 = 0$, and $\alpha_2 = \beta_2 = 1$.

  It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$ and $\alpha_n > 0, \forall n \neq 0$.

Thus, the general the solution to the spatial eigenproblem is

$$\phi_n(x) = \begin{cases} A_0 + B_0 x, n = 0 \\ A_n \cos\left(\sqrt{\alpha_n} x\right) + B_n \sin\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots. \end{cases} \tag{3.4}$$

We can now compute

$$\phi_n'(x) = \begin{cases} B_0, n = 0 \\ -A_n \sqrt{\alpha_n} \sin\left(\sqrt{\alpha_n} x\right) + B_n \sqrt{\alpha_n} \cos\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots. \end{cases}$$

Requiring that $\phi_n'(0) = \phi_n'(1) = 0$, we compute that

$$\sqrt{\alpha_n} = n\pi, \text{ and } B_n = 0, \forall n.$$

Thus, the general solution reduces to

$$\phi_n(x) = \begin{cases} 1, n = 0 \\ \cos\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots. \end{cases}$$

- **Dirichlet**: $\phi_n(0) = \phi_n(1) = 0$. Imposing the boundary conditions, we have

$$u_n(x = 0, t) = \phi_n(0) T_n(t) = 0,$$

and

$$u_n(x = 1, t) = \phi_n(1) T_n(t) = 0.$$

We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

Thus, ϕ_n is a solution to the spatial eigenproblem

$$\begin{aligned} -\phi_n'' &= \alpha_n \phi_n \text{ in } (0,1) \\ \phi_n(0) &= 0 \\ \phi_n(1) &= 0 \end{aligned}.$$

We can readily verify that this is a Sturm-Liouville problem in $(0,1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = 0$.

It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$ and $\alpha_n > 0, \forall n \neq 0$.

The general solution again given by Eq. 6.2. Requiring that $\phi_n(0) = \phi_n(1) = 0$, we compute that

$$\sqrt{\alpha_n} = n\pi, A_n = 0, \forall n, \text{ and } B_0 = 0.$$

Thus, the general solution reduces to

$$\phi_n(x) = \begin{cases} 0, n = 0 \\ \sin\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots. \end{cases}$$

- **Mixed (Type 1)**: $\phi_n(0) = \phi_n'(1) = 0$. Imposing the boundary conditions, we have
$$u_n(x = 0, t) = \phi_n(0)T_n(t) = 0,$$
and
$$\frac{\partial u_n}{\partial x}(x = 1, t) = \phi_n'(1)T_n(t) = 0.$$
We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

Thus, ϕ_n is a solution to the spatial eigenproblem
$$\begin{aligned} -\phi_n'' &= \alpha_n \phi_n \text{ in } (0,1) \\ \phi_n(0) &= 0 \\ \phi_n'(1) &= 0 \end{aligned} .$$
We can readily verify that this is a Sturm-Liouville problem in $(0, 1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_2 = 1$, and $\alpha_2 = \beta_1 = 1$.

It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$ and $\alpha_n > 0, \forall n \neq 0$.

The general solution again given by Eq. 6.2. Requiring that $\phi_n(0) = \phi_n'(1) = 0$, we compute that
$$\sqrt{\alpha_n} = \left(n - \frac{1}{2}\right)\pi, A_n = 0, \forall n, \text{ and } B_0 = 0.$$
Thus, the general solution is then
$$\phi_n(x) = \begin{cases} 0, n = 0 \\ \sin\left(\sqrt{\alpha_n}x\right), n = 1, 2, \ldots. \end{cases} .$$

- **Mixed (Type 2)**: $\phi_n'(0) = \phi_n(1) = 0$. Imposing the boundary conditions, we have
$$\frac{\partial u_n}{\partial x}(x = 0, t) = \phi_n'(0)T_n(t) = 0,$$
and
$$u_n(x = 1, t) = \phi_n(1)T_n(t) = 0.$$
We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

Thus, ϕ_n is a solution to the spatial eigenproblem
$$\begin{aligned} -\phi_n'' &= \alpha_n \phi_n \text{ in } (0,1) \\ \phi_n'(0) &= 0 \\ \phi_n(1) &= 0 \end{aligned} .$$
We can readily verify that this is a Sturm-Liouville problem in $(0, 1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_2 = 0$, and $\alpha_2 = \beta_1 = 1$.

It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$

and $\alpha_n > 0, \forall n \neq 0$.

The general solution again given by Eq. 6.2. Requiring that $\phi_n'(0) = \phi_n(1) = 0$, we compute that

$$\sqrt{\alpha_n} = \left(n - \frac{1}{2}\right)\pi, A_0 = 0, \text{ and } B_n = 0, \forall n.$$

Thus, the general solution reduces to

$$\phi_n(x) = \begin{cases} 0, n = 0 \\ \cos\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots. \end{cases}.$$

In any case, we note that T_n must satisfy the temporal eigenproblem

$$-T_n' = \alpha_n T_n.$$

The general solution to the temporal eigenproblem is

$$T_n(t) = \begin{cases} A_0, n = 0 \\ A_n \exp\left(-\alpha_n t\right) n = 1, 2, \ldots \end{cases}$$

Therefore, the solution to the PDE IBVP is of the form

$$u(x,t) = \frac{1}{2}A_0\phi_0(x) + \sum_{n=1}^{\infty} A_n \exp\left(-\alpha_n t\right)\phi_n(x), \tag{3.5}$$

where A_n and B_n are chosen so that u satisfies the initial conditions.

Imposing the initial condition, we have

$$u(x, t = 0) = \frac{1}{2}A_0\phi_0(x) + \sum_{n=1}^{\infty} A_n\phi_n(x) = h(x).$$

Therefore, $A_n = \langle \phi_n(x), h(x) \rangle$.

3.4.2 with Non-Homogeneous Source Term

We now consider the case with a source term, i.e.,

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f \text{ in } \Omega \times \mathbb{R}_{>0} \\ u &= 0 \text{ on } \Gamma_{\mathrm{D}} \times \mathbb{R}_{>0} \\ \frac{\partial u}{\partial x} &= 0 \text{ on } \Gamma_{\mathrm{N}} \times \mathbb{R}_{>0} \\ u &= h \text{ on } \partial\Omega \times \{t = 0\}. \end{aligned} \tag{3.6}$$

Let u_{h} be the solution to the homogeneous analogue, i.e., Eq. 3.3 and let u_{p} be the solution to Eq. **??** with $h \equiv 0$. The solution to Eq. 3.6 with $h \not\equiv 0$ is then

$$u := u_{\mathrm{h}} + u_{\mathrm{p}}.$$

To find u_{p}, we assume u_{p} is of the form

$$u_{\mathrm{p}}(x,t) = \sum_{n=0}^{\infty} \hat{T}_n(t)\phi_n(x),$$

for some set of time-dependent coefficients $\hat{T}_n(t)$.

Substituting Eq. **??** into Eq. 3.1, and employing the fact that $\phi_n'' = -\alpha_n\phi_n$, we obtain the relation

$$\sum_{n=0}^{\infty} \left(\hat{T}_n'(t) + \alpha_n \hat{T}_n(t)\right)\phi_n(x) = f(x,t).$$

Appealing to the orthogonality of ϕ_n, we get

$$\hat{T}_n' + \alpha_n \hat{T}_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}.$$

Imposing the initial condition, we get that

$$u_{\mathrm{p}}(x,0) = \sum_{n=0}^{\infty} \hat{T}_n(0)\phi_n(x) = 0, \forall x.$$

Therefore, $\hat{T}_n(0) = 0, \forall n$ and $\hat{T}_n$ satisfies

$$\begin{aligned} \hat{T}_n' + \alpha_n \hat{T}_n &= \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} \\ \hat{T}_n(0) &= 0. \end{aligned} \tag{3.7}$$

3.4.3 with Non-Homogeneous Boundary Conditions

3.5 Solutions on Unbounded Domains

We now consider the case where Ω is unbounded.

3.5.1 on the Entire Real Line, $\mathbb{R}$

We first consider the case where $\Omega = \mathbb{R}$. The boundary condition is replaced with a boundedness condition, $|u(x,t)| < \infty$ as $x \to \pm\infty$.

More specifically, we wish to solve the problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 \text{ in } \mathbb{R} \times \mathbb{R}_{>0} \\ |u(x,t)| &< \infty \text{ as } x \to \pm\infty \\ u &= h \text{ on } \mathbb{R} \times \{t = 0\}. \end{aligned} \tag{3.8}$$

To begin, we seek a family of solutions of the separable form,

$$u_\lambda(x,t) = \phi_\lambda(x) T_n(x),$$

that satisfy the PDE and boundedness condition, but not necessarily the initial conditions.

Substituting u_λ into the PDE, we have

$$\phi_\lambda T_\lambda' - \phi_\lambda'' T_\lambda = 0.$$

Assuming $\phi_\lambda \neq 0$ and $T_\lambda \neq 0$, we rearrange the expression to obtain

$$\frac{\phi_\lambda''}{\phi_\lambda} = \frac{T_\lambda'}{T_\lambda} = -\alpha_\lambda,$$

where α_λ is some constant. Without loss of generality, we may let $\alpha_\lambda = \lambda^2$.

Thus, ϕ_λ satisfies the eigenproblem

$$\begin{gathered} \phi_\lambda'' = -\lambda^2 \phi_\lambda \text{ in } \mathbb{R} \\ |\phi_\lambda(x)| < \infty \text{ as } x \to \pm\infty. \end{gathered}$$

It can readily be verified then that

$$\phi_\lambda(x) = A_\lambda \cos(\lambda x) + B_\lambda \sin(\lambda x).$$

Likewise, T_λ satisfies the temporal differential equation,

$$T_\lambda' = -\lambda^2 T_\lambda.$$

It can readily be verified then that

$$T_\lambda(t) = \exp(-\lambda t).$$

We therefore conclude that

$$u_\lambda(x,t) = \exp(-\lambda t) \left(A_\lambda \cos(\lambda x) + B_\lambda \sin(\lambda x) \right).$$

We assume the solution is a linear combination of such basis functions, i.e.,

$$u(x,t) = \int_0^\infty \exp(-\lambda t)\left(A_\lambda \cos(\lambda x) + B_\lambda \sin(\lambda x)\right) d\lambda.$$

Imposing the initial conditions, we see that

$$A_\lambda = \frac{1}{\pi}\langle \cos(\lambda x), h(x)\rangle_{-\infty}^{\infty} \text{ and } B_\lambda = \frac{1}{\pi}\langle \sin(\lambda x), h(x)\rangle_{-\infty}^{\infty}.$$

3.5.2 on the Half Real Line, $\mathbb{R}_{\geq 0}$

We now consider the case where $\mathbb{R}_{\geq 0}$. As such, we will need to introduce a boundary condition at $x = 0$.

More specifically, we wish to solve the problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 \text{ in } \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \\ |u(x,t)| &< \infty \text{ as } x \to \infty \\ u &= h \text{ on } \mathbb{R}_{\geq 0} \times \{t = 0\}. \end{aligned} \tag{3.9}$$

such that either

- a Dirichlet boundary condition is satisfied, i.e.,

$$u(x = 0, t) = 0 \text{ for } t \in \mathbb{R}_{>0},$$

- a Neumman boundary condition is satisfied, i.e.,

$$\frac{\partial u}{\partial x}(x = 0, t) = 0 \text{ for } t \in \mathbb{R}_{>0}.$$

To do so, we shall solve a problem on $\Omega = \mathbb{R}$ whose solution restricted to $\mathbb{R}_{\geq 0}$ is the one we seek.

We shall consider both cases seperately:

- In the Dirichlet case, we begin by noting that the odd extension of a continuous function, $f : \mathbb{R} \to \mathbb{R}$ satisfies the Dirichlet condition,

$$f^{\mathrm{o}}(0) = 0.$$

  Thus, the solution to Eq. 3.9 can be found from that of Eq. 3.8 by using h^{o} as the initial conditions and then restricting the solution to $\mathbb{R}_{\geq 0}$..

- In the Neumman case, we note that the even periodic extension of a continuous function, $f : \mathbb{R} \to \mathbb{R}$ satisfies the Neumman condition,

$$\frac{\partial f^{\mathrm{e}}}{\partial x}(0) = 0.$$

  Thus, the solution to Eq. 3.9 can be found from that of Eq. 3.8 by using h^{e} as the initial conditions and then restricting the solution to $\mathbb{R}_{\geq 0}$.

3.6 The Fundamental Solution

Definition 3.3. (The Fundamental Solution) The **Fundamental Solution** to Eq. 3.8 is

$$\Phi(x,t) = \begin{cases} \frac{1}{4\pi t}\exp\left(\frac{-(\xi-x)^2}{4t}\right), t > 0 \\ 0, t < 0. \end{cases}$$

Theorem 3.6. (Solution to the Heat Equation) The solution to Eq. 3.8 is given by

$$u(x,t) = \int_0^\infty \Phi(\xi,t)g(\xi)d\xi.$$

Proof. Recall that the solution to Eq. 3.8 was shown to be

$$u(x,t) = \int_0^\infty \exp(-\lambda t)\left(A_\lambda \cos(\lambda x) + B_\lambda \sin(\lambda x)\right)d\lambda.$$

where

$$A_\lambda = \frac{1}{\pi}\langle\cos(\lambda x), h(x)\rangle_{-\infty}^{\infty} \text{ and } B_\lambda = \frac{1}{\pi}\langle\sin(\lambda x), h(x)\rangle_{-\infty}^{\infty}.$$

Substituting the expressions for A_λ and B_λ into that of u, we have

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \cos(\lambda\xi)h(\xi)d\xi\cos(\lambda x) + \int_{-\infty}^\infty \sin(\lambda\xi)h(\xi)d\xi\sin(\lambda x)e^{-\lambda t}d\lambda.$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \left(\cos(\lambda\xi)\cos(\lambda x) + \sin(\lambda\xi)\sin(\lambda x)\right)h(\xi)d\xi\exp(-\lambda t)d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty \cos(\lambda(\xi - x))h(\xi)d\xi\exp(-\lambda t)d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty \int_0^\infty \cos(\lambda(\xi - x))\exp(-\lambda t)d\lambda h(\xi)d\xi$$

$$= \frac{1}{4\sqrt{4\pi t}}\int_{-\infty}^\infty \exp\left(\frac{-(\xi-x)^2}{4t}\right)g(\xi)d\xi.$$

$\square$

4 Laplace's Equation

4.1 Definition and Problem Setup

Definition 4.1. (Poisson's Equation) Consider an arbitrary spatial domain, $\Omega \subset \mathbb{R}$. A function, $u : \Omega \to \mathbb{R}$ satisfies **Poissons's equation** iff

$$-\nabla^2 u = f \text{ in } \Omega,$$

where f is independent of u.

Remark 4.1. Poisson's equation is an elliptic equation.

Remark 4.2. Intuitively, Poisson's equation describes the steady-state behaviour of a diffusion process. Indeed, one can view Poissons's equation as a special case of the heat equation where where $\partial u/\partial t = 0$.

Definition 4.2. (Laplace's Equation) In the special case where $f \equiv 0$, Poisson's equation is also called Laplace's equation.

Remark 4.3. Solutions to Laplace's equation are called **harmonic functions**.

To form a well-posed problem, we introduce additional constraints. In particular, we will need:

- one boundary condition for each point on $\partial\Omega$.

Definition 4.3. (Poissons's IBVP) Let $g_{\mathrm{D}} : \Gamma_{\mathrm{D}} \to \mathbb{R}$, $g_{\mathrm{N}} : \Gamma_{\mathrm{N}} \to \mathbb{R}$, and $h_1, h_2 : \Omega \to \mathbb{R}$, where $\overline{\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}} = \partial\Omega$. We consider the following PDE IBVP associated with Eq. 4.2:

$$\begin{aligned} -\nabla^2 u &= f \text{ in } \Omega \\ u &= g_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}} \\ \frac{\partial u}{\partial n} &= g_{\mathrm{N}} \text{ on } \Gamma_{\mathrm{N}}, \end{aligned} \tag{4.1}$$

where $\frac{\partial}{\partial n}$ is the directional derivative tangent to Γ_{N}.

4.2 Properties of Solutions

We now establish some general properties of any harmonic function without deriving any explicit representation.

Theorem 4.1. (Non-Dimensionalization) The solution, u_{d} to the dimensionalized equation,

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } (0, L_x) \times (0, L_y),$$

is related to the solution, u_{n} to the non-dimensionalized analouge,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 y}{\partial y^2} = 0 \text{ in } (0, 1)^2,$$

through

$$u_{\mathrm{d}}(x, y) = \tilde{u}_{\mathrm{n}}(x/L_x, y/L_y).$$

Proof. Omitted. $\square$

We begin with the so-called mean-value theorem, which describes the average value of a harmonic function on or within some ball.

Theorem 4.2. (Mean-Value Theorem) Consider a ball of radius, r, centred at $\boldsymbol{x}_0 \in \mathbb{R}^n$, i.e., the open set,

$$\mathcal{B}(\boldsymbol{x}_0, r) = \{\boldsymbol{x} \in \mathbb{R}^n \text{ s. t. } \|\boldsymbol{x} - \boldsymbol{x}_0\|_2 < r\}.$$

If u is harmonic in $\mathcal{B}(\boldsymbol{x}_0, r) \subset \Omega$, then

$$u(\boldsymbol{x}_0) = \underbrace{\frac{1}{|\mathcal{B}(\boldsymbol{x}_0, r)|} \int_{\mathcal{B}(\boldsymbol{x}_0,r)} u dV}_{\substack{\text{the average value of} \\ u \text{ in } \mathcal{B}(\boldsymbol{x}_0,r)}} = \underbrace{\frac{1}{|\partial\mathcal{B}(\boldsymbol{x}_0, r)|} \int_{\partial\mathcal{B}(\boldsymbol{x}_0,r)} u ds}_{\substack{\text{the average value of} \\ u \text{ on } \partial\mathcal{B}(\boldsymbol{x}_0,r)}}.$$

Proof. Omitted. □

Theorem 4.3. (Converse of Mean-Value Theorem) If $u \in C^2(\Omega)$ satisfies

$$u(\boldsymbol{x}_0) = \frac{1}{|\mathcal{B}(\boldsymbol{x}_0, r)|} \int_{\mathcal{B}(\boldsymbol{x}_0,r)} u d\boldsymbol{x},$$

for every $\mathcal{B}(\boldsymbol{x}_0, r) \subset \Omega$, then u is harmonic in $\mathcal{B}(\boldsymbol{x}_0, r)$.

Proof. Omitted. □

We continue with a theorems concerning the extrema of harmonic functions.

Theorem 4.4. (Strong Extrema Principle) Any function u which is harmonic in some closed domain $\overline{\Omega}$ which attains an extrema in Ω, must be constant in $\overline{\Omega}$. Alternatively, if u is not constant in $\overline{\Omega}$, then

$$\min_{\partial\Omega} u < u < \max_{\partial\Omega} u \text{ in } \overline{\Omega}.$$

Proof. Omitted. □

Corollary 4.1. (Weak Extrema Principle) Let u be a harmonic function in a connected open domain, $\Omega \subset \mathbb{R}^n$. Then,

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \text{ and } \min_{\overline{\Omega}} u = \min_{\partial\Omega} u,$$

i.e., u attains its extrema on $\partial\Omega$.

Proof. We only prove the maximum principle since the proof of the minimum principle is analogous.

Suppose $\boldsymbol{x}^* \in \overline{\Omega}$ is the maximizer. We have two cases:

1. If $\boldsymbol{x}^*$ is an interior point, i.e., $\boldsymbol{x}^* \in \Omega$, then by the extrema principle, u is constant in $\overline{\Omega}$ and hence $\max_{\overline{\Omega}} u = \max_{\Omega} u$.

2. If $\boldsymbol{x}^*$ is a boundary point, i.e., $\boldsymbol{x}^* \in \partial\Omega$, the result immediately follows.

□

4.3 Uniqueness and Stability

Theorem 4.5. (Uniquness) Suppose u_1, and u_2 are two solutions to the Eq. 4.1. Then, their difference $w \equiv u_1 - u_2$ satisfies

Proof. We define the energy in w as

$$E := \frac{1}{2} \int_{\Omega} w(\boldsymbol{x})^2 d\Omega.$$

First, we notice that $E \geq 0$. Now,

$$\frac{d}{dt} E = \int_{\Omega} w \frac{\partial w}{\partial t} d\Omega = \int_{\Omega} w \nabla^2 w d\Omega = -\underbrace{\int_{\Omega} \nabla w \cdot \nabla w d\Omega}_{I_1} + \underbrace{\int_{\partial\Omega} w \nabla w \cdot d\boldsymbol{s}}_{I_2}.$$

Since $\nabla w \cdot \nabla w = |\nabla w|^2 \geq 0$, it follows that $I_1 \geq 0$. Conversely, it can be shown that $I_2 \leq 0$, but the justification depends on the boundary condition used. In any case, the upshot is that

$$\frac{d}{dt} E \leq 0$$

and so E is a non-increasing function of time. Thus, $E(t) \leq E(0)$, meaning that

$$0 \leq \int_{\Omega} w(\boldsymbol{x}, t)^2 d\Omega \leq \int_{\Omega} w(\boldsymbol{x}, 0)^2 d\Omega = 0,$$

where the last equality follows from the fact that u_1 and u_2 are initially equivalent. Therefore, $w \equiv 0$ and u_1 and u_2 are always identical everywhere.

$\square$

Theorem 4.6. (Stability) The problem defined by Eq. 4.1 where $\Gamma_N \equiv \emptyset$, is stable.

In particular, suppose

$$\begin{aligned} -\nabla^2 u &= 0 \text{ in } \Omega \\ u &= g_i \text{ on } \partial\Omega \end{aligned}$$

for $i = 1, 2$. Then, u_1 and u_2 are stable in the sense that

$$\max_{\overline{\Omega}} |u_1 - u_2| \leq \max_{\partial\Omega} |g_1 - g_2|.$$

Proof. Omitted. $\square$

4.4 Solutions on Bounded Domains

4.4.1 On an Annulus

First, we find a set of separable solutions, $u_n(r,\theta) = R_n(r)\Theta_n(\theta)$, which satisfy the PDE. Substituting u_n into the PDE, we have

$$-\nabla^2 u_n = -\frac{\partial^2 u_n}{\partial r^2} - \frac{1}{r}\frac{\partial u_n}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u_n}{\partial \theta^2} = 0$$

$$\Rightarrow -R_n''(r)\Theta_n(\theta) - \frac{1}{r}R_n'(r)\Theta_n(\theta) - \frac{1}{r^2}R_n(r)\Theta_n''(\theta) = 0.$$

Multiplying both sides by $-r^2/(R_n(r)\Theta_n(\theta))$, we have

$$r^2\frac{R_n''(r)}{R_n(r)} + r\frac{R_n'(r)}{R_n(r)} + \frac{\Theta_n''(\theta)}{\Theta_n(\theta)} = 0.$$

Re-arranging and appealing to the fact that the result holds for every r and θ, we get that

$$r^2\frac{R_n''(r)}{R_n(r)} + r\frac{R_n'(r)}{R_n(r)} = -\frac{\Theta_n''(\theta)}{\Theta_n(\theta)} = \kappa_n,$$

where κ_n is some constant.

From this, we conclude that Θ_n satisfies the ODE

$$-\Theta_n''(\theta) = \alpha_n\Theta_n(\theta) \text{ on } (0, 2\pi)$$

whose solutions are of the form

$$\Theta_n(\theta) = \alpha_n \sin(\sqrt{\kappa_n}\theta) + \beta_n \cos(\sqrt{\kappa_n}\theta)$$

for some constants α_n and β_n. If we require that Θ_n be cyclically continuous, i.e., $\Theta_n(0) = \Theta_n(2\pi)$, it immediately follows that $\sqrt{\kappa_n} = n$.

Thus, we conclude that $\Theta_n(\theta) = \alpha_n \sin(n\theta) + \beta_n \cos(n\theta)$.

Using the fact that $\sqrt{\kappa_n} = n \Rightarrow \kappa_n = n^2$, it then follows that R_n satisfies the ODE

$$r^2 R_n''(r) + rR_n'(r) - n^2 R_n(r) = 0 \text{ on } (r_1, r_2).$$

For $n = 0$, the above reduces to

$$r^2 R_0''(r) + rR_0'(r) = 0 \text{ on } (r_1, r_2)$$

and we can readily verify that $R_0(r) = \gamma_0 + \zeta_0 \log(r)$, for some constants γ_0 and ζ_0, is indeed a solution.

For $n \neq 0$, we can readily verify that $R_n(r) = a_n r^n + b_n r^{-n}$, for some constants a_n and b_n, is indeed a solution.

Combining these results, we see that

$$u_n(r,\theta) = \begin{cases} (a_n r^n + b_n r^{-n})\left(\alpha_n \sin(n\theta) + \beta_n \cos(n\theta)\right), n \neq 0 \\ (a_0 + b_0 \log(r))\,\beta_0, n = 0 \end{cases}.$$

Redefining α_n, β_n, a_n, and b_n appropriately, we may equivalently write

$$u_n(r,\theta) = \begin{cases} (\alpha_n r^n + a_n r^{-n}) \sin(n\theta) + (\beta_n r^n + b_n r^{-n}) \cos(n\theta), n \neq 0 \\ \beta_0 + b_0 \log(r), n = 0. \end{cases}$$

Finally, for notational convenience, we define

$$A_n(r) = \begin{cases} \alpha_n r^n + a_n r^{-n}, n \neq 0 \\ 0, n = 0 \end{cases}$$

and

$$B_n(r) = \begin{cases} \beta_n r^n + b_n r^{-n}, n \neq 0 \\ 2(\beta_0 + b_0 \log(r)), n = 0 \end{cases}$$

so that

$$u_n(r,\theta) = A_n(r) \sin(n\theta) + B_n(r) \cos(n\theta), n = 0, \ldots.$$

Appealing to the principle of superposition, we have

$$u(r,\theta) = \frac{1}{2} B_0(r) + \sum_{n=1}^{\infty} A_n(r) \sin(n\theta) + B_n(r) \cos(n\theta).$$

We now impose the boundary conditions. We will first consider Dirichlet boundary conditions, $u(r_1,\theta) = g_1$ and $u(r_2,\theta) = g_2$.

Imposing these conditions, we conclude that $A_n(r_1)$ and $B_n(r_1)$ are the Fourier coefficients of g_1 and $A_n(r_2)$ and $B_n(r_2)$ are the Fourier coefficients of g_2.

For a fixed n, the coefficients α_n, β_n, a_n, and b_n then satisfy a system of linear equations which can readily be solved.

4.5 Solutions on Unbounded Domains

Definition 4.4. (Fundamental Solution to Laplace's Equation) The **Fundamental Solution**, $\Phi : \mathbb{R}^n \to \mathbb{R}$ to Eq. 4.1 is such that

$$-\nabla\Phi = \delta \text{ in } \mathbb{R}^n.$$

Theorem 4.7. (Solution to Poisson's Problem) The solution to Eq. 4.1 is

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-\xi) f(\xi) d\xi.$$

Proof. Omitted. □

Definition 4.5. (Green's Function) A **Green's Function** is a function, $G : \Omega \to \mathbb{R}$ such that

$$-\nabla^2 G(x,\xi) = \delta(x-\xi), \forall x \in \Omega$$
$$G(x,\xi) = 0, \forall x \in \partial\Omega.$$

Theorem 4.8. (Symmetry of Green's Functions) It can be shown that

$$G(x,\xi) = G(\xi,x).$$

Proof. Omitted. □

Theorem 4.9. (Green's Representation Formula) Let $G : \Omega \to \mathbb{R}$ be the Green's function associated with the domain Ω. Then the solution is

$$u(x) = \int_{\Omega} G(x,\xi) f(\xi) d\xi - \int_{\partial\Omega} \nu \cdot \nabla_\xi G(x,\xi) g(\xi) d\xi. \tag{4.2}$$

Proof. Omitted. □

4.5.1 on the Upper Half-Plane, $\mathbb{R} \times \mathbb{R}_{\geq 0}$

Here we consider the problem

$$-\nabla^2 u = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \mathbb{R} \times \{x_2 = 0\}.$$

Let G be the Green's function associated with $\Omega = \mathbb{R} \times \mathbb{R}_{\geq 0}$.

Take an arbitrary point $\xi = (\xi_1, \xi_2) \in \Omega$. We require,

$$-\nabla^2 G(x,\xi) = \delta(x-\xi).$$

We may achieve this by choosing $G(x,\xi) = \Phi(x,\xi) - \Phi(x,\tilde{\xi})$, where $\tilde{\xi} = (\xi_1 - \xi_2)$.

Indeed, by the definition of Φ, we can readily see that

$$\nabla^2 G(x,\xi) = \delta(x-\xi) - \delta(x-\tilde{\xi}) = \delta(x-\xi) \text{ in } \Omega,$$

where the last equality follows from the fact that $\tilde{\xi} \notin \Omega$.

Moreover, since for any $x \in \partial\Omega$, we have $\|x-\xi\| = \|x-\tilde{\xi}\|$, it follows that

$$G(x,\xi) = 0, \forall x \in \partial\Omega.$$

Thus, the Green's function for $\mathbb{R} \times \mathbb{R}_{\geq 0}$ is

$$G(x, \xi) = -\frac{1}{2\pi} \log \|x - \xi\| + \frac{1}{2\pi} \log \|x - \tilde{\xi}\|.$$

We can now find the solution, u, using Eq. 4.2. To that end, we first compute $\nu \cdot \nabla_\xi G(x, \xi)$, where $\nu = -\hat{x}_2$, and G is defined as above.

We have

$$\nu \cdot \nabla_\xi G(x, \xi) = -\frac{\partial G(x, \xi)}{\partial \xi_2} = \frac{1}{2\pi} \frac{\partial}{\partial \xi_2} \left(\log \|x - \xi\| - \log \|x - \tilde{\xi}\| \right).$$

Definition 4.6. (Poisson's Kernel for $\mathbb{R} \times \mathbb{R}_{\geq 0}$) **Poisson's Kernel** for $\mathbb{R} \times \mathbb{R}_{\geq 0}$ is

$$K(x, \xi) = \frac{1}{\pi} \frac{x_2}{\|x - \xi\|^2}.$$

5 The Transport Equation

5.1 Definition and Problem Setup

Definition 5.1. (The Transport Equation) Consider an arbitrary spatial domain, $\Omega \subset \mathbb{R}^n$. A function, $u : \Omega \times \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the **Transport equation** iff

$$\frac{\partial^2 u}{\partial t} + b \cdot \nabla u = f, \tag{5.1}$$

where f is independent of u and b is a function of x, t and u.

To form a well-posed problem, we introduce additional constraints. In particular, we will need:

- one initial condition for each in $\Omega \times \{t = 0\}$, i.e. the temporal boundary.

Remark 5.1. Note that we need no boundary conditions.

Definition 5.2. (The Transport Equation IVP) We consider the following IVP associated with the Transport equation,

$$\begin{aligned} \frac{\partial u}{\partial t} + b \cdot \nabla u = f \text{ in } \Omega \times \mathbb{R}_{>0} \\ u = g \text{ on } \Omega \times \{t = 0\}. \end{aligned} \tag{5.2}$$

5.2 Solutions on Unbounded Domains

We consider the special case where $\Omega = \mathbb{R}^n$ and assume that u is continuously differentiable, i.e., $u \in C^1(\mathbb{R}^n \times \mathbb{R}_{\geq 0})$.

Let $x_c : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be a time-parametrized contour in $\mathbb{R}^n$, and denote the solution u along it as

$$u_c(t) = u(x_c(t), t).$$

We find that the solution u_c evolves along the contour as follows:

$$\frac{d}{dt}(u_c(t)) = \frac{d}{dt} u(x_c(t), t) = \frac{\partial u}{\partial t} + \frac{dx_c}{dt} \cdot \nabla u.$$

If we choose a curve, x_c such that $\frac{dx_c}{dt} = b(x_c(t), t, u(x_c(t), t))$, and appeal to the fact that u satisfies the Eq. 5.2, we have

$$\frac{d}{dt}(u_c(t)) = \frac{d}{dt} u(x_c(t), t) = \frac{\partial u}{\partial t} + b(x_c(t), t, u(x_c(t), t)) \cdot \nabla u = f(x_c(t), t).$$

Definition 5.3. (Characteristic Equations for the Transport Equation) The chracteristic equations for Eq. 5.2 is a pair of ODEs,

$$\begin{aligned} \frac{dx_c}{dt}(t) &= b(x_c(t), t, u_c(t)), t \in \mathbb{R}_{>0}, \\ \frac{du_c}{dt}(t) &= f(x_c(t), t), t \in \mathbb{R}_{>0}, \end{aligned}$$

where $u_c : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is called the characteristic curve.

We can readily verify that

$$x_c(t) = \int_{\tau=0}^{t} b(x_c(\tau), \tau, u_c(\tau))d\tau + x_{c,0},$$

where $x_{c,0} \equiv x_c(t = 0)$, and

$$u_c(t) = u_c(0) + \int_{\tau=0}^{t} f(x_c(\tau), \tau)d\tau = g(x_{c,0}) + \int_{\tau=0}^{t} f(x_c(\tau), \tau)d\tau.$$

We see that the solution u, along the characteristic contour, x_c, is the integral of the forcing function, f along it, biased by $g(x_{c,0})$.

Remark 5.2. If $f \equiv 0$, it immediately follows that $u_c(t) = g(x_{c,0})$.

6 The Wave Equation

6.1 Definition and Problem Setup

Definition 6.1. (The Wave Equation) Consider an arbitrary spatial domain, $\Omega \subset \mathbb{R}^n$. A function, $u : \Omega \to \mathbb{R}$ satisfies the **Wave Equation** iff

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f \text{ in } \Omega \times \mathbb{R}_{>0},$$

where f is independent of u.

Remark 6.1. The wave equation is a hyperbolic equation.

To form a well-posed problem, we introduce additional constraints. In particular, we will need:

- one boundary condition for each point on $\partial\Omega \times \mathbb{R}_{>0}$, i.e., the spatial boundary, and
- two initial conditions for each point on $\operatorname{Int}\Omega \times \{t = 0\}$, i.e., the temporal boundary.

Definition 6.2. (The Wave IBVP) Let $g_\mathrm{D} : \Gamma_\mathrm{D} \to \mathbb{R}$, $g_\mathrm{N} : \Gamma_\mathrm{N} \to \mathbb{R}$, $h_1, h_2 : \Omega \to \mathbb{R}$ and assume $\overline{\Gamma_\mathrm{D} \cup \Gamma_\mathrm{N}} = \partial\Omega$. We consider the following PDE IBVP associated with the wave equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u &= f \text{ in } \Omega \times \mathbb{R}_{>0} \\ u &= g_\mathrm{D} \text{ on } \Gamma_\mathrm{D} \times \mathbb{R}_{>0} \\ \frac{\partial u}{\partial t} &= g_\mathrm{N} \text{ on } \Gamma_\mathrm{N} \times \mathbb{R}_{>0} \\ u &= h_1 \text{ on } \Omega \times \{t = 0\} \\ \frac{\partial u}{\partial t} &= h_2 \text{ on } \Omega \times \{t = 0\}. \end{aligned} \tag{6.1}$$

6.2 Properties of Solutions

We now establish some general properties of solutions to the wave equation without deriving any explicit representation.

Theorem 6.1. (Non-Dimensionalization) The solution, u_d, to the dimensionalized wave-equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial u^2}{\partial x^2} = 0 \text{ in } (0, L) \times \mathbb{R}_{>0},$$

is related to the solution, u_n to the non-dimensionalized analogue,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u^2}{\partial x^2} = 0 \text{ in } (0, 1) \times \mathbb{R}_{>0},$$

through

$$u_\mathrm{d}(x, t) = \tilde{u}_\mathrm{n}(x/L, ct/L).$$

Proof. Choosing $\tilde{x} = x/L$ and $\tilde{t} = t/T$, where $T \equiv L/c$, we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{T^2}\frac{\partial^2 u}{\partial \tilde{t}^2} \text{ and } \frac{\partial^2 u}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2 u}{\partial \tilde{x}^2} = \frac{1}{T^2 c^2}\frac{\partial^2 u}{\partial \tilde{x}^2}.$$

Thus, u is the solution to

$$\frac{\partial^2 u}{\partial \tilde{t}^2} - \frac{\partial u^2}{\partial \tilde{x}^2} = 0 \text{ in } (0,1) \times \mathbb{R}_{>0}$$

in the variables, $\tilde{x}$ and $\tilde{t}$. □

Theorem 6.2. (Conservation of Energy) Suppose u is the solution to the homogeneous wave equation with homogeneous boundary conditions, i.e.,

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u &= f \text{ in } \Omega \times \mathbb{R}_{>0} \\ u &= 0 \text{ on } \Gamma_{\mathrm{D}} \times \mathbb{R}_{>0} \\ \frac{\partial u}{\partial t} &= 0 \text{ on } \Gamma_{\mathrm{N}} \times \mathbb{R}_{>0}, \end{aligned}$$

where Γ_{D} and Γ_{N} are Dirichlet and Neumann boundaries, respectively, so that $\overline{\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}} = \partial\Omega$. Define the total energy in u, at time t, as

$$E(t) \equiv \frac{1}{2}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 + \nabla u \cdot \nabla u dx.$$

Then, the total energy is conserved, i.e.,

$$E(t) = E(0), \forall t \in \mathbb{R}_{>0}.$$

Proof. We split the energy into two components,

$$K \equiv \frac{1}{2}\int_\Omega \left(\frac{\partial u}{\partial t}^2\right) dx,$$

and

$$P \equiv \frac{1}{2}\int_\Omega \nabla u \cdot \nabla u dx,$$

so that $E = K + P$. We have

$$\frac{dK}{dt} = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 dx = \int_\Omega \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx = \int_\Omega \frac{\partial u}{\partial t}\nabla^2 u dx.$$

Using integration by parts,

$$-\int_\Omega \left(\frac{\partial}{\partial t}\nabla^2 u\right) \cdot \nabla^2 u dx$$ □

6.3 Uniqueness and Stability

We now show that the solution to Eq. 6.1, if it exists, is unique.

Theorem 6.3. (Uniqueness of the Solution) Suppose u_1, and u_2 are two solutions to the Eq. 6.1. Then, their difference $w \equiv u_1 - u_2$ satisfies

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} - \nabla^2 w &= 0 \text{ in } \Omega \times \mathbb{R}_{>0} \\
w &= 0 \text{ on } \Gamma_{\mathrm{D}} \times \mathbb{R}_{>0} \\
\frac{\partial w}{\partial n} &= 0 \text{ on } \Gamma_{\mathrm{N}} \times \mathbb{R}_{>0} \\
w &= 0 \text{ on } \Omega \times \{t = 0\} \\
\frac{\partial w}{\partial t} &= 0 \text{ on } \Omega \times \{t = 0\}.
\end{aligned}$$

The total energy of w is then

$$E(t) = \frac{1}{2} \int_\Omega \left(\left(\frac{\partial w}{\partial t} \right)^2 + \nabla w \cdot \nabla w dx \right) = \frac{1}{2} \int_\Omega \cancelto{0}{\left(\frac{\partial w}{\partial t} \right)^2} + \cancelto{0}{\nabla^2 w \cdot \nabla^2 w dx} \Bigg|_{t=0} = 0,$$

where the second equality follows from energy conservation, and the third equality follows from the fact that

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = 0,$$

by one of the initial condition and $\nabla^2 w(t = 0) = 0$ since $w(t = 0) \equiv 0$ by the other initial condition.

Noting that $E(t)$ is an integral of the sum of two non-negative terms, for all $t > 0$, we conclude that

$$\left.\frac{\partial w}{\partial t}\right|_{t} = 0.$$

Finally since $w(t = 0) = 0$, it follows that $w(t) = u_1(t) - u_2(t) = 0$ for all $t \geq 0$.

6.4 Solutions on Bounded Domains

We now find solutions to Eq. 6.1 where $\Omega \subset \mathbb{R}$ is bounded. However, we will only consider the case where $\Omega = (0,1)$ since the more general solution can be readily derived from Thm. **??**.

6.4.1 with Homogeneous Boundary Conditions

We first consider the case with no source term and homogeneous boundary conditions, i.e.,

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0 \text{ in } (0,1) \times \mathbb{R}_{>0} \\
u &= 0 \text{ on } \Gamma_{\mathrm{D}} \times \mathbb{R}_{>0} \\
\frac{\partial u}{\partial x} &= 0 \text{ on } \Gamma_{\mathrm{N}} \times \mathbb{R}_{>0} \\
u &= h_1 \text{ on } \Omega \times \{t = 0\} \\
\frac{\partial u}{\partial t} &= h_2 \text{ on } \Omega \times \{t = 0\}.
\end{aligned}$$

To begin, we seek a family of solutions of the separable form,

$$u_n(x,t) = \phi_n(x) T_n(x), n = 0, 1, 2, \ldots,$$

that satisfy the PDE and boundary conditions, but not necessarily the initial conditions.

Substituting u_n into the PDE, we have

$$\phi_n T_n'' - \phi_n'' T_n = 0.$$

Assuming $\phi_n \neq 0$ and $T_n \neq 0$, we rearrange the expression to obtain

$$\frac{\phi_n''}{\phi_n} = \frac{T_n''}{T_n} = -\alpha_n,$$

where α_n is some constant.

From here, the procedure differs slightly depending on the boundary conditions imposed. We consider each case separately:

- **Neumman**: $\phi_n'(0) = \phi_n'(1) = 0$. Imposing the boundary conditions, we have

  $$\frac{\partial u_n}{\partial x}(x = 0, t) = \phi_n'(0) T_n(t) = 0,$$

  and

  $$\frac{\partial u_n}{\partial x}(x = 1, t) = \phi_n'(1) T_n(t) = 0.$$

  We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

  Thus, ϕ_n is a solution to the spatial eigenproblem

  $$\begin{aligned}
  -\phi_n'' &= \alpha_n \phi_n \text{ in } (0,1) \\
  \phi_n'(0) &= 0 \\
  \phi_n'(1) &= 0
  \end{aligned} \quad .$$

We can readily verify that this is a Sturm-Liouville problem in $(0,1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_1 = 0$, and $\alpha_2 = \beta_2 = 1$.

It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$ and $\alpha_n > 0, \forall n \neq 0$.

Thus, the general the solution to the spatial eigenproblem is

$$\phi_n(x) = \begin{cases} A_0 + B_0 x, n = 0 \\ A_n \cos(\sqrt{\alpha_n} x) + B_n \sin(\sqrt{\alpha_n} x), n = 1, 2, \ldots \end{cases} \tag{6.2}$$

We can now compute

$$\phi_n'(x) = \begin{cases} B_0, n = 0 \\ -A_n \sqrt{\alpha_n} \sin(\sqrt{\alpha_n} x) + B_n \sqrt{\alpha_n} \cos(\sqrt{\alpha_n} x), n = 1, 2, \ldots \end{cases}$$

Requiring that $\phi_n'(0) = \phi_n'(1) = 0$, we compute that

$$\sqrt{\alpha_n} = n\pi, \text{ and } B_n = 0, \forall n.$$

Thus, the general solution reduces to

$$\phi_n(x) = \begin{cases} 1, n = 0 \\ \cos\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots \end{cases}$$

- **Dirichlet**: $\phi_n(0) = \phi_n(1) = 0$. Imposing the boundary conditions, we have

  $$u_n(x = 0, t) = \phi_n(0) T_n(t) = 0,$$

  and

  $$u_n(x = 1, t) = \phi_n(1) T_n(t) = 0.$$

  We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

  Thus, ϕ_n is a solution to the spatial eigenproblem

  $$\begin{aligned} -\phi_n'' &= \alpha_n \phi_n \text{ in } (0,1) \\ \phi_n(0) &= 0 \\ \phi_n(1) &= 0 \end{aligned} .$$

  We can readily verify that this is a Sturm-Liouville problem in $(0,1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = 0$.

  It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$ and $\alpha_n > 0, \forall n \neq 0$.

  The general solution again given by Eq. 6.2. Requiring that $\phi_n(0) = \phi_n(1) = 0$, we compute that

  $$\sqrt{\alpha_n} = n\pi, A_n = 0, \forall n, \text{ and } B_0 = 0.$$

  Thus, the general solution reduces to

  $$\phi_n(x) = \begin{cases} 0, n = 0 \\ \sin\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots \end{cases}$$

- **Mixed (Type 1)**: $\phi_n(0) = \phi_n'(1) = 0$. Imposing the boundary conditions, we have
$$u_n(x = 0, t) = \phi_n(0)T_n(t) = 0,$$
and
$$\frac{\partial u_n}{\partial x}(x = 1, t) = \phi_n'(1)T_n(t) = 0.$$
We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

Thus, ϕ_n is a solution to the spatial eigenproblem
$$\begin{aligned} -\phi_n'' &= \alpha_n \phi_n \text{ in } (0, 1) \\ \phi_n(0) &= 0 \\ \phi_n'(1) &= 0 \end{aligned} \quad .$$
We can readily verify that this is a Sturm-Liouville problem in $(0, 1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_2 = 1$, and $\alpha_2 = \beta_1 = 1$.

It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$ and $\alpha_n > 0, \forall n \neq 0$.

The general solution again given by Eq. 6.2. Requiring that $\phi_n(0) = \phi_n'(1) = 0$, we compute that
$$\sqrt{\alpha_n} = \left(n - \frac{1}{2}\right)\pi, A_n = 0, \forall n, \text{ and } B_0 = 0.$$
Thus, the general solution is then
$$\phi_n(x) = \begin{cases} 0, n = 0 \\ \sin\left(\sqrt{\alpha_n} x\right), n = 1, 2, \ldots. \end{cases} \quad .$$

- **Mixed (Type 2)**: $\phi_n'(0) = \phi_n(1) = 0$. Imposing the boundary conditions, we have
$$\frac{\partial u_n}{\partial x}(x = 0, t) = \phi_n'(0)T_n(t) = 0,$$
and
$$u_n(x = 1, t) = \phi_n(1)T_n(t) = 0.$$
We shall assume $T_n(t) \not\equiv 0$ to avoid the trivial solution.

Thus, ϕ_n is a solution to the spatial eigenproblem
$$\begin{aligned} -\phi_n'' &= \alpha_n \phi_n \text{ in } (0, 1) \\ \phi_n'(0) &= 0 \\ \phi_n(1) &= 0 \end{aligned} \quad .$$
We can readily verify that this is a Sturm-Liouville problem in $(0, 1)$, with $p \equiv 1$, $q \equiv 0$, $w \equiv 1$, $\alpha_1 = \beta_2 = 0$, and $\alpha_2 = \beta_1 = 1$.

It follows that $\alpha_n \geq 0$. Since the α_n are unique, we shall assume $\alpha_0 = 0$

and $\alpha_n > 0, \forall n \neq 0$.

The general solution again given by Eq. 6.2. Requiring that $\phi_n'(0) = \phi_n(1) = 0$, we compute that

$$\sqrt{\alpha_n} = \left(n - \frac{1}{2}\right)\pi, A_0 = 0, \text{ and } B_n = 0, \forall n.$$

Thus, the general solution reduces to

$$\phi_n(x) = \begin{cases} 0, n = 0 \\ \cos\left(\sqrt{\alpha_n}x\right), n = 1, 2, \ldots. \end{cases}.$$

In any case, we note that T_n must satisfy the temporal eigenproblem

$$-T_n'' = \alpha_n T_n.$$

The general solution to the temporal eigenproblem is

$$T_n(t) = \begin{cases} A_0 + B_0 t, n = 0 \\ A_n \cos\left(\sqrt{\alpha_n}t\right) + B_n \sin\left(\sqrt{\alpha_n}t\right), n = 1, 2, \ldots \end{cases}$$

Therefore, the solution to the PDE IBVP is of the form

$$u(x,t) = \frac{1}{2}\left(A_0 + B_0 t\right)\phi_0(x) + \sum_{n=1}^{\infty}\left(A_n \cos\left(\sqrt{\alpha_n}t\right) + B_n \sin\left(\sqrt{\alpha_n}t\right)\right)\phi_n(x), \tag{6.3}$$

where A_n and B_n are chosen so that u satisfies the initial conditions.

Imposing the first initial condition, we have

$$u(x, t = 0) = \frac{1}{2}A_0\phi_0(x) + \sum_{n=1}^{\infty} A_n\phi_n(x) = h_1(x).$$

Therefore, $A_n = \langle\phi_n(x), h_1(x)\rangle$.

Imposing the second boundary condition, we have

$$\frac{\partial u}{\partial t}(x, t = 0) = \frac{1}{2}B_0\phi_0(x) + \sum_{n=1}^{\infty} B_n\sqrt{\alpha_n}\phi_n(x) = h_2(x).$$

Therefore, $B_n\sqrt{\alpha_n} = \langle\phi_n(x), h_2(x)\rangle$.

6.5 d'Alembert's Formula

We observe that Eq. 6.3 can be interpreted as a sinusoidal wave whose amplitude also oscillates sinusoidally with time.

However, by slightly re-arrainging the formula, we can arrive at a different interpretation.

6.5.1 with Dirichlet Boundary Conditions

In this case, we saw that

$$\phi_n(x) = \sin(\sqrt{n\pi}x),$$

and so

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(n\pi t) + B_n \sin(n\pi t)) \sin(n\pi x).$$

We consider the solution for two cases and then take their superposition.

1. $h_1 \neq 0, h_2 = 0$. We set $B_n = 0, \forall n$ to obtain

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n\pi t) \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} A_n \sin(\sin(n\pi(x-t)) + \sin(n\pi(x+t))).$$

   Since $A_n = \langle h_1, \sin(n\pi x)\rangle$, it follows that

$$u(x,t) = \frac{1}{2}\left[h_1^{\text{o.p.}}(x-t) + h_1^{\text{o.p.}}(x+t)\right].$$

2. $h_1 = 0, h_2 \neq 0$. We set $A_n = 0, \forall n$ to obtain

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi t) \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} B_n \sin(\cos(n\pi(x-t)) + \cos(n\pi(x+t))).$$

6.6 Solutions on Unbounded Domains

We now consider the case of unbounded domains. We consider the one-dimensional wave equation in $\mathbb{R} \times \mathbb{R}_{>0}$:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0 \text{ in } \mathbb{R} \times \mathbb{R}_{>0} \\ u &= g \text{ on } \mathbb{R} \times \{t = 0\} \\ \frac{\partial u}{\partial t} &= h \text{ on } \mathbb{R} \times \{t = 0\}, \end{aligned} \tag{6.4}$$

for some functions, $g, h : \mathbb{R} \to \mathbb{R}$. We factor the second-order operator into a product of two first-order operators:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u}_{\equiv v}.$$

This factorization allows us to decompose the eave equation, which is a second-order PDE, into a pair of first-order PDEs:

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = v, \tag{6.5}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = 0. \tag{6.6}$$

We recognize these as transport equations. To begin, we solve for v.

Applying the method of characteristics to Eq. 6.6, we obtain

$$v(x, t) = v^0(x - t),$$

where $v^0(x) = v(x, t = 0), \forall x \in \mathbb{R}$ is the initial condition, which we specify later.

Substituting the expression for v into Eq. 6.5, we see that

$$\frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial x}(x, t) = v^0(x - t),$$

which we recognize as a non-homogeneous transport equation with $f(x, t) = v^0(x - t)$ and $b = -1$.

Applying the method of characteristics once more, we obtain

$$u(x, t) = u(x - bt, 0) + \int$$