Uncertainty
Modelling

Introduction to Artificial Intelligence

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Our goal is to quantify the uncertainty in an uncertain situation. One way to do this as follows:

Definition (Relative Frequency)

Suppose we have an experiment whose possible outcomes are given by the set, Ω. sample space

The **probability** of any one outcome ω is given by

number of times ω occurs

$$\Pr\{\omega\} := \lim_{N\to\infty} \frac{N(\omega)}{N}.$$

number of trials

probability of ω

Definition (Event)

An **event**, $A \subseteq \Omega$ is a subset of outcomes. A occurs if any one of its outcomes, $\omega \in A$, occurs.

The set of all events is the power-set of the sample space, i.e., 2^Ω.

Definition (Marginal Probabilities)

The **marginal probability** of the event $A \subseteq \Omega$ is the sum of the probabilities of each outcome $\omega \in A$, i.e.,

$$\Pr\{A\} = \sum_{\omega \in A} \Pr\{\omega\}.$$

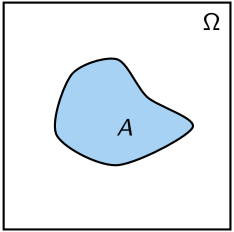


Figure 1: The probability of an event, $A \subset \Omega$, is the area taken up by A, relative to that of Ω.

Definition (Joint Probabilities)

The **joint probability** of the events A and B is the probability of the event $A \cap B$, i.e.,

$$\Pr\{A \text{ and } B\} = \sum_{\omega \in A \cap B} \Pr\{\omega\}$$

In general, we cannot determine $\Pr\{A \cap B\}$ from $\Pr\{A\}$ and $\Pr\{B\}$.

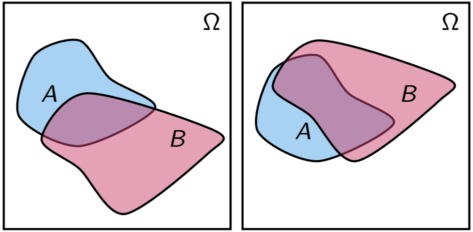


Figure 2: The area of A and B are the same in both cases, but that of $A \cap B$ is different.

Definition (Partition)

A **partition** of Ω, is a disjoint set of events, $B_1, \ldots, B_n$, such that $\bigcup_{i=1}^{n} B_i = \Omega$.

Theorem (Marginalization)

For any event A and partition, $B_1, \ldots, B_n$,

$$\Pr\{A\} = \sum_{i=1}^{n} \Pr\{A \text{ and } B_i\},$$

This is called **marginalization**.

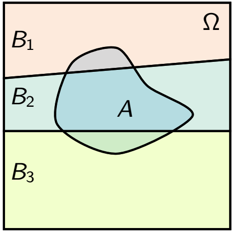


Figure 3: $A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$

Definition (Conditional Probabilities)

The **conditional probability** of the event A given that the event B has occurred is the probability of A relative to B, i.e.,

$$\Pr\{A \text{ given } B\} = \frac{\Pr\{A \text{ and } B\}}{\Pr\{B\}}$$

Of course, repeating an experiment infinitely many times to compute probabilities is not actually possible, and so, we must assign the values ourselves.

The exact values are irrelevant, but they must satisfy a few axioms.

Definition (Probability Measure)

A **probability measure** is a function, $P : 2^{\Omega} \to [0, 1]$, such that for each event, A,

$$\Pr\{A\} := P(A),$$

and that is:

1. **non-negative**: $P(A) \geq 0$, for any event, A
2. **additive**: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \varnothing$
3. **normalized**: $P(\Omega) = 1, P(\varnothing) = 0$
4. **complimentary**: $P(\neg A) = 1 - P(A)$

We call $(\Omega, 2^{\Omega}, P)$ a **probability space**.

We would like to know exactly how the knowledge of an event influences the probability of another.

Theorem (Bayes' Theorem)

For any two events, A and B,

$$\underbrace{P(A|B)}_{\text{posterior of } A \text{ given } B} = \underbrace{P(A)}_{\text{prior of } A}\overbrace{\frac{P(B|A)}{P(B)}}^{\text{factor by which } B \text{ reinforces } A},$$

provided $P(B) \neq 0$.

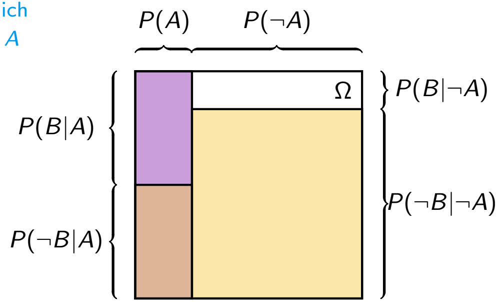


Figure 4: A visual representation of the various conditional probabilities of B given A.

Proof :
By definition,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

But also, by definition, $P(A \cap B) = P(A)P(B|A)$. Therefore,

$$P(A|B) = P(A)\frac{P(B|A)}{P(B)}.$$

■

We need a systematic way to represent dependence relationships.

Definition (Bayesian Network)

A **Bayesian network** is a directed acyclic graph, $(\mathcal{V}, \mathcal{E})$, in which the vertices represent the related random variables, and the edges represent the dependence relationships between them:

- If $(V_1, V_2) \in \mathcal{E}$, where $V_1, V_2 \in \mathcal{V}$, then $V_1 \not\perp V_2$ (but the converse need not be true).
- The directions of the arcs is technically irrelevant since dependence is commutative, but they typically indicate the direction of causality.

Figure 5: Both networks above indicate that X_1 and X_2 are dependent, but the left suggests that X_1 causes X_2, while the right suggests that X_2 causes X_1.

Given a Bayesian network, $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, our task is to determine if $V_1 \perp V_2 | \mathcal{K}$, where $\mathcal{K} \subset \mathcal{V} \setminus \{V_1, V_2\}$.

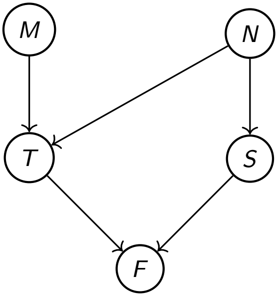
Example (Modelling Uncertain Situations)

The variables of interest are:

- whether we catch the flight or not, F
- when we get to the airport, T
- how long it takes to get through security, S
- how we get to the airport, M
- how many bags we have, N

The causal relationships are that:

- F is directly influenced by T and S
- T is directly influenced by M and N
- S is directly influenced by N



Our goal now is to determine whether any two variables in a Bayesian network are dependent or not.

We first consider the simplest non-trivial case.

Definition (Junction Network)

A **junction**, $\mathcal{J} = (\{X_1, X_2, X_3\}, \{(e_1, e_2)\})$ is a Bayesian network that consists of three random variables, X_1, X_2, X_3, connected by two arcs, e_1 and e_2.

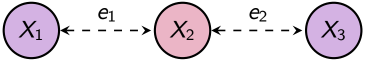


Figure 6: There are three distinct types of junctions.

We call X_1 and X_3, the **outer variables**, and X_2 the **central variable**.

In a junction network, the central variable and each outer variable are always dependent.

We want to know when the outer variables are dependent.

Definition (Causal Chain Junction)

In a **causal chain** junction, X_1 directly influences X_2, which in turn, directly influences X_3.

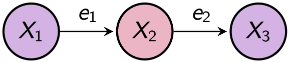


Figure 7: A causal chain junction.

In this case, $X_1 \not\perp X_3$ and $X_1 \perp X_3 | X_2$.

We say that $\mathcal{J}$ is closed given X_2, and open given no evidence.

Example (Causal Chain Relationship)

Rain causes the grass to get wet, which then causes worms to come out.

Definition (Common Cause Junction)

In a **common cause** junction, X_2 directly influences X_1 and X_3.

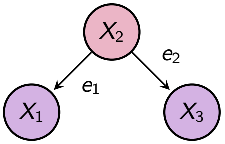


Figure 8: A common cause junction.

In this case, $X_1 \not\perp X_3$ and $X_1 \perp X_3 | X_2$.

We say $\mathcal{J}$ is closed given X_2, and open given no evidence.

Example (Common Cause Relationship)

Rain causes the grass to get wet, and the road to be slippery.

Definition (Common Effect Junction)

In a **common effect** junction, X_1 and X_3 both directly influence X_2.

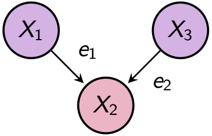


Figure 9: A common effect junction.

In this case, $X_1 \perp X_3$ and $X_1 \perp X_3 | K$, where $K \subseteq \{X_2\} \cup \mathrm{des}(X_2)$.

We say $\mathcal{J}$ is open given X_2 and/or any of its descendants, and closed otherwise.

Example (Common Effect Relationship)

Both rain and watering the garden can cause the grass to be wet.

Theorem (Dependence Separation)

Let $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ be some Bayesian network and $\mathcal{K} \subseteq \mathcal{V}$.

Let a path, p, in $\mathcal{B}$ be an ordered list of junctions,

$$\mathcal{J}^{(i)} = \left(\left\{ X_1^{(i)}, X_2^{(i)}, X_3^{(i)} \right\}, \left\{ (e_1^{(i)}, e_2^{(i)}) \right\} \right)$$

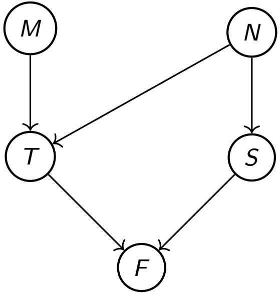
such that $X_1^{(i+1)} = X_2^{(i)}$ and $X_2^{(i+1)} = X_3^{(i)}$, or equivalently, $e_1^{(i+1)} = e_2^{(i)}$.

The path, p, is **blocked** under K if $\mathcal{J}^{(i)}$ is blocked under $\mathcal{K}$ for some i.

For any $V_1, V_2 \in \mathcal{V}$ and $\mathcal{K} \subseteq \mathcal{V} \setminus \{V_1, V_2\}$, we have $V_1 V_2 | \mathcal{K}$ if and only if every path between V_1 and V_2 is blocked under $\mathcal{K}$.

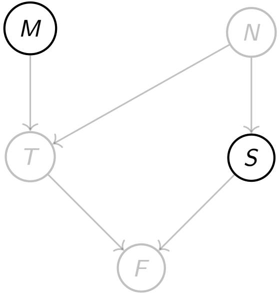
Example (Dependence Separation)

Determine whether S and M are independent or not, given no evidence.



Example (Dependence Separation)

Suppose we want to know whether S and M are independent or not given no evidence.



There are two paths from S to M; $S \perp M$ if and only if both are closed.

Example (Dependence Separation)

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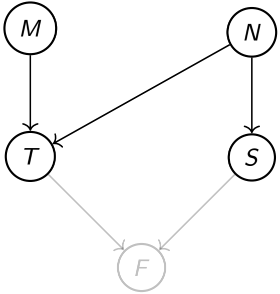

One path from M to S is $p_1 = (\{M, T, N\}, \{T, N, S\})$.

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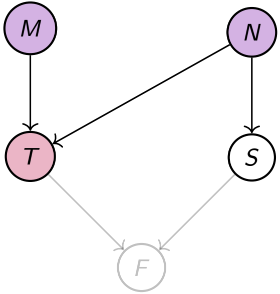

$\{M, T, N\}$ is a common effect junction and is **closed** given no evidence; thus p_1 is **closed**.

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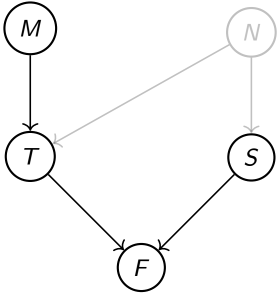

The other path from M to S is $p_2 = (\{M, T, F\}, \{T, F, S\})$.

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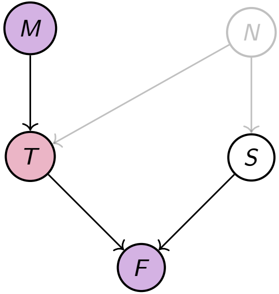

$\{M, T, F\}$ is a causal chain junction and is **open** given no evidence.

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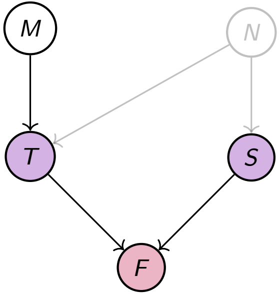
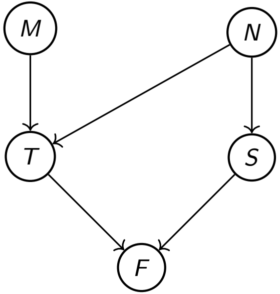


$\{T, F, S\}$ is a common effect junction and is **closed** given no evidence; thus p_2 is closed.

Example (Dependence Separation)

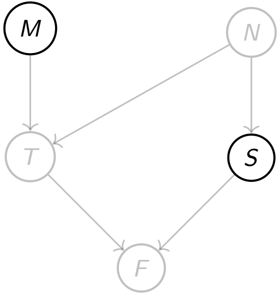
Suppose we want to know whether S and M are independent or not given no evidence.



Since p_1 and p_2 are both closed, it follows that $S \perp M$.

Example (Dependence Separation)

Suppose we want to know whether S and M are independent or not given F.



There are two paths from S to M; $S \perp M|F$ if and only if both are closed given F.

Example (Dependence Separation)

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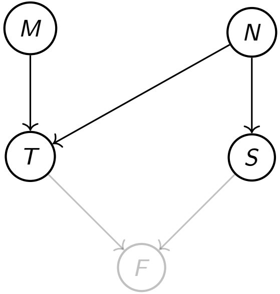

One path from M to S is $p_1 = (\{M, T, N\}, \{T, N, S\})$.

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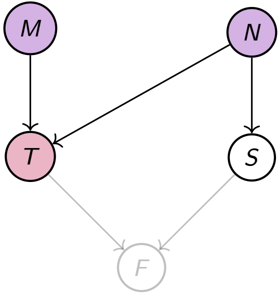

$\{M, T, N\}$ is a common effect junction and is **open** given F.

Example (Dependence Separation)

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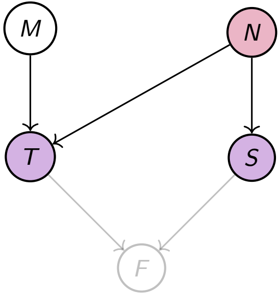
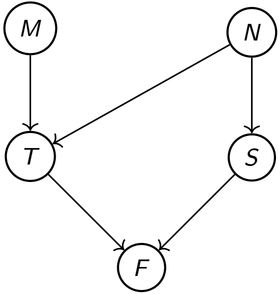


$\{T, N, S\}$ is a common cause junction and is **open** given F; thus p_1 is open.

Example (Dependence Separation)

Suppose we want to know whether S and M are independent or not given no evidence.



Since p_1 is open, it follows that $S \not\perp M | F$.

Dependence separation allows us to prove an even more important result.

Theorem (Independence Relationships)

If X is a variable in a Bayesian network, $(\mathcal{V}, \mathcal{E})$, then

$$P(X|\mathcal{S} \cup \text{pts}(X)) = P(X|\text{pts}(X))$$

X is independent of its non-descendants given its parents.

where $\mathcal{S} \subseteq \mathcal{V} \setminus \text{des}(X)$.

Proof :
Observe that every path from X to any of its *non-descendants* must contain either:

- a causal chain junction through one of X's parents
- a common cause junction through one of X's parents
- a common effect junction through one of X's descendants

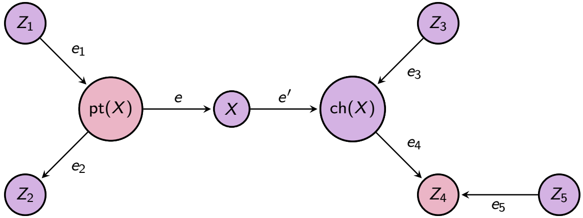


Figure 10: The various paths between X and its non-descendants.

Thus, all paths to X from any of its non-descendants are blocked given pts(X). ■