1 Results

Definition 1. (**Interference**) Given two sets U, W, we say U $k-$*interferes* with W if

$$|U \cap W| \geq \frac{|W|}{k} \tag{1}$$

for some $k \in (0, |W|]$

Note that if $|U| = |W|$, then U k-interferes with W if and only if W k-interferes with U. We restrict the upper range of k to $|W|$ for convenience, as beyond that all values of $\frac{|W|}{k}$ will be less than 1.

Definition 2. (**(r,T,k)-Subgraph capacity**) Given a vertex set $V = \{v_1, ..., v_n\}$, the (r, T, k)-*subgraph capacity* of V is the maximum number of subgraphs that can be generated randomly with size r subject to the constraint that for any randomly selected subgraph U,

$$\mathbb{E}[|X_U^k|] \leq T \tag{2}$$

where $X_U^k = \{W \in V^r \mid U \text{ } k\text{-interferes with } W\}$ and V^r is the set of all currently generated subgraphs with size r.

Intuitively, $\mathbb{E}[|X_U^k|]$ represents the expected number of k-interferences caused by randomly picking a subgraph U.

Lemma 3. *Given a vertex set V with n vertices and two subsets U, W of size r, the probability that U k-interferes with W is*

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$$

Proof. if $V = \{v_1, ..., v_n\}$, we can represent a subset U as a vector of length n, u defined by

$$u_i = \begin{cases} 1 & \text{if } v_i \in U \\ 0 & \text{if } v_i \notin U \end{cases}$$

With this representation, U, W intersect at the indices where both vectors u, w have a 1. Let Y be a random variable denoting the number of indices where both u, w have a 1. Then

$$P(Y = y) = \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}} \tag{3}$$

Then the probability that U k-interferes with W is

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}} \tag{4}$$

□

Theorem 4. *Given a vertex set V with n vertices, the (r, T, k)-subgraph capacity of V is*

$$\frac{T}{\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}} + 1$$

Proof. Let M be the number of subgraphs of size r currently generated. Pick an arbitrary subgraph U of size r. From lemma 3., we know that the probability of U k-interfering with another subgraph is $\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$. Since there are $M - 1$ such subgraphs, the expected number of k-interferences by picking U is

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}(M-1) \tag{5}$$

From equation (2), we have

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}(M-1) < T \implies M < \frac{T}{\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}} + 1 \tag{6}$$

The (r, T, k)-subgraph capacity of V is the largest M that satisfies equation (6). $\square$