

# 1 Results

**Definition 1. (Interference)** Given two sets  $U, W$ , we say  $U$   $k$ -interferes with  $W$  if

$$|U \cap W| \geq \frac{|W|}{k} \quad (1)$$

for some  $k \in (0, |W|]$

**Corollary 2.** If  $|U| = |W|$ , then  $U$   $k$ -interferes with  $W$  if and only if  $W$   $k$ -interferes with  $U$ .

We restrict the upper range of  $k$  to  $|W|$  for convenience, as beyond that all values of  $\frac{|W|}{k}$  will be less than 1.

**Definition 3. ((r,T,k)-Subgraph capacity)** Given a vertex set  $V = \{v_1, \dots, v_n\}$ , the  $(r, T, k)$ -subgraph capacity of  $V$  is the expected maximum number of subgraphs of expected size  $r$  that can be collected subject to the constraint that for any randomly pick subgraph  $U$ ,

$$\mathbb{E}[X] \leq T \quad (2)$$

where  $X$  is the number of interferences caused due to  $U$ .

First, we consider the case where all subgraphs are of the same size.

**Lemma 4.** Given a vertex set  $V$  with  $n$  vertices and two subsets  $U, W$  of size  $r$ , the probability that  $U$   $k$ -interferes with  $W$  is

$$\sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}}$$

*Proof.* if  $V = \{v_1, \dots, v_n\}$ , we can represent a subset  $U$  as a vector of length  $n$ ,  $u$  defined by

$$u_i = \begin{cases} 1 & \text{if } v_i \in U \\ 0 & \text{if } v_i \notin U \end{cases}$$

With this representation,  $U, W$  intersect at the indices where both vectors  $u, w$  have a 1. Let  $Y$  be a random variable denoting the number of indices where both  $u, w$  have a 1. Then

$$\mathbb{P}(Y = y) = \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}} \quad (3)$$

This follows from the fact that given the first array  $U$ , we already know where the 1's are located. We can pick the  $y$  intersecting 1's for the second array in  $\binom{r}{y}$  ways implicitly placing 0's in the remaining spots. We then fill the remaining  $n - r$  indices corresponding to the 0's in the first array with  $r - y$  1's in  $\binom{n-r}{r-y}$  ways. Finally we divide by the total number of possible subgraphs  $\binom{n}{r}$ . Then the probability that  $U$   $k$ -interferes with  $W$  is

$$\sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}} \quad (4)$$

□

**Theorem 5.** Given a vertex set  $V$  with  $n$  vertices, the  $(r, T, k)$ -subgraph capacity of  $V$  is

$$\left\lceil \frac{T}{\sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}}} + 1 \right\rceil$$

*Proof.* Suppose we have  $M$  subgraphs in the collection. Pick two subgraphs  $U, W$ . From lemma 4., we know that the probability of  $U$   $k$ -interfering with  $W$  is  $\sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}}$ . Since all subgraphs have the same size, by corollary 2. this is the probability that  $U, W$  pair will cause 2  $k$ -interferences. So the expected number of interferences caused by one pair is

$$2 \sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}}$$

We know that there are  $\binom{M}{2} = M(M-1)/2$  such pairings so the expected number of total interferences is

$$2 \frac{M(M-1)}{2} \sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}} = M(M-1) \sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}}$$

Since there are  $M$  subgraphs, the expected number of interferences by picking one subgraph is

$$\frac{M(M-1)}{M} \sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}} = (M-1) \sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}}$$

From equation (2), we have

$$(M-1) \sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}} \leq T \implies M \leq \frac{T}{\sum_{y=\lceil \frac{r}{k} \rceil}^r \frac{\binom{r}{y} \binom{n-r}{r-y}}{\binom{n}{r}}} + 1$$

The  $(r, T, k)$ -subgraph capacity of  $V$  is the largest integer  $M$  that satisfies equation (6). □

Now we consider the case where the subgraphs might have different sizes but have expected size  $r$ .

**Lemma 6.** *Given a vertex set  $V$  with  $n$  vertices and two subsets  $U, W$  of size  $r_u, r_w$ , the probability that  $U$   $k$ -interferes with  $W$  is*

$$\sum_{y=\lceil \frac{r_w}{k} \rceil}^{r_w} \frac{\binom{r_u}{y} \binom{n-r_u}{r_w-y}}{\binom{n}{r_w}}$$

*Proof.* We use the same representation as in lemma 4. Note that in this case,

$$\mathbb{P}(Y = y) = \frac{\binom{r_u}{y} \binom{n-r_u}{r_w-y}}{\binom{n}{r_w}} \quad (5)$$

This follows from the fact that given the first array  $U$ , we already know where the 1's are located. We can pick the  $y$  intersecting 1's for the second array in  $\binom{r_u}{y}$  ways implicitly placing 0's in the remaining spots. We then fill the remaining  $n - r_u$  indices corresponding to the 0's in the first array with  $r_w - y$  1's in  $\binom{n-r_u}{r_w-y}$  ways. Finally we divide by the total number of possible subgraphs  $\binom{n}{r_w}$ . Then the probability that  $U$   $k$ -interferes with  $W$  is

$$\sum_{y=\lceil \frac{r_w}{k} \rceil}^{r_w} \frac{\binom{r_u}{y} \binom{n-r_u}{r_w-y}}{\binom{n}{r_w}} \quad (6)$$

□

**Theorem 7.** *Given a vertex set  $V$  with  $n$  vertices, the  $(r, T, k)$ -subgraph capacity of  $V$  is*

*placeholder*

*Proof.* Suppose we have  $M$  subgraphs  $U_1, \dots, U_m$  with sizes  $r_1, \dots, r_m$ . Pick two subgraphs  $U_i, U_j$ . From lemma 6., we know that the expected number of interferences caused by this pair is

$$\sum_{y=\lceil \frac{r_j}{k} \rceil}^{r_j} \frac{\binom{r_i}{y} \binom{n-r_i}{r_j-y}}{\binom{n}{r_j}} + \sum_{y=\lceil \frac{r_i}{k} \rceil}^{r_i} \frac{\binom{r_j}{y} \binom{n-r_j}{r_i-y}}{\binom{n}{r_i}}$$

We then sum over all possible pairings to get the expected number of total interferences:

$$\sum_{(i,j) \in (1,m) \times (1,m), i \neq j} \left( \sum_{y=\lceil \frac{r_j}{k} \rceil}^{r_j} \frac{\binom{r_i}{y} \binom{n-r_i}{r_j-y}}{\binom{n}{r_j}} + \sum_{y=\lceil \frac{r_i}{k} \rceil}^{r_i} \frac{\binom{r_j}{y} \binom{n-r_j}{r_i-y}}{\binom{n}{r_i}} \right)$$

Since there are  $M$  subgraphs, the expected number of interferences by picking one subgraph is

$$\frac{1}{M} \sum_{(i,j) \in (1,m) \times (1,m), i \neq j} \left( \sum_{y=\lceil \frac{r_j}{k} \rceil}^{r_j} \frac{\binom{r_i}{y} \binom{n-r_i}{r_j-y}}{\binom{n}{r_j}} + \sum_{y=\lceil \frac{r_i}{k} \rceil}^{r_i} \frac{\binom{r_j}{y} \binom{n-r_j}{r_i-y}}{\binom{n}{r_i}} \right)$$

TODO: we need to use the fact that the expected value is  $r$  to simplify this then plug it into equation (2) to solve for  $M$ . Maybe helpful:

$$\mathbb{E} \left[ \binom{X}{k} \right] = \sum_0^{\infty} \binom{i}{k} \mathbb{P}(X = i)$$

We hope the final result will be similar to the result in theorem 5.

□