1 Results

Definition 1. (**Interference**) Given two sets U, W, we say U *$k-$interferes* with W if

$$|U \cap W| \geq \frac{|W|}{k} \tag{1}$$

for some $k \in (0, |W|]$

Corollary 2. *If $|U| = |W|$, then U k-interferes with W if and only if W k-interferes with U.*

We restrict the upper range of k to $|W|$ for convenience, as beyond that all values of $\frac{|W|}{k}$ will be less than 1.

Definition 3. (**(r,T,k)-Subgraph capacity**) Given a vertex set $V = \{v_1, ..., v_n\}$, the *(r, T, k)-subgraph capacity* of V is the expected maximum number of subgraphs of expected size r that can be collected subject to the constraint that for any randomly picked subgraph U,

$$\mathbb{E}[X] \leq T \tag{2}$$

where X is the number of interferences caused due to picking U.

First, we consider the case where all subgraphs are of the same size.

Lemma 4. *Given a vertex set V with n vertices and two subsets U, W of size r, the probability that U k-interferes with W is*

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$$

Proof. if $V = \{v_1, ..., v_n\}$, we can represent a subset U as a vector of length n, u defined by

$$u_i = \begin{cases} 1 & \text{if } v_i \in U \\ 0 & \text{if } v_i \notin U \end{cases}$$

With this representation, U, W intersect at the indices where both vectors u, w have a 1. Let Y be a random variable denoting the number of indices where both u, w have a 1. Then

$$\mathbb{P}(Y = y) = \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}} \tag{3}$$

This follows from the fact that given the first array U, we already know where the 1's are located. We can pick the y intersecting 1's for the second array in $\binom{r}{y}$ ways implicitly placing 0's in the remaining spots. We then fill the remaining $n - r$ indices corresponding to the 0's in the first array with $r - y$ 1's in $\binom{n-r}{r-y}$ ways. Finally we divide by the total number of possible subgraphs $\binom{n}{r}$. Then the probability that U k-interferes with W is

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}} \tag{4}$$

$\square$

Theorem 5. *Given a vertex set V with n vertices, the (r, T, k)-subgraph capacity of V is*

$$\left\lfloor \frac{T}{\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}} + 1 \right\rfloor$$

Proof. Suppose we have M subgraphs in the collection. Pick an arbitrary subgraph U. From lemma 4., we know that the probability of U k-interfering with another subgraph is $\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$. This can be interpreted as the expected number of k-interferences caused by picking U with one subgraph. Since there are $M - 1$ other subgraphs, the expected number of k-interferences caused by picking U is

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}(M - 1) \tag{5}$$

From equation (2), we have

$$\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}(M-1) \leq T \implies M \leq \frac{T}{\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}} + 1 \tag{6}$$

The (r, T, k)-subgraph capacity of V is the largest integer M that satisfies equation (6). $\square$

Alternate proof. Suppose we have M subgraphs in the collection. Pick two subgraphs U, W. From lemma 4., we know that the probability of U k-interfering with W is $\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$. Since all subgraphs have the same size, by corollary 2. this is the probability that U, W pair will cause 2 k-interferences. So the expected number of interferences caused by one pair is

$$2 \sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$$

We know that there are $\binom{M}{2} = M(M-1)/2$ such pairings so the expected number of total interferences is

$$2\frac{M(M-1)}{2} \sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}} = M(M-1) \sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$$

Since there are M subgraphs, the expected number of interferences by picking one subgraph is

$$\frac{M(M-1)}{M} \sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}} = (M-1) \sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}$$

From equation (2), we have

$$(M-1) \sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}} \leq T \implies M \leq \frac{T}{\sum_{y=\lceil \frac{r}{k} \rceil}^{r} \frac{\binom{r}{y}\binom{n-r}{r-y}}{\binom{n}{r}}} + 1$$

The (r, T, k)-subgraph capacity of V is the largest integer M that satisfies equation (6).

$\square$

Now we consider the case where the subgraphs might have different sizes but have expected size r.

Lemma 6. *Given a vertex set V with n vertices and two subsets U, W of size r_u, r_w, the probability that U k-interferes with W is*

$$\sum_{y=\lceil \frac{r_w}{k} \rceil}^{r_w} \frac{\binom{r_u}{y}\binom{n-r_u}{r_w-y}}{\binom{n}{r_w}}$$

Proof. We use the same representation as in lemma 4. Note that in this case,

$$\mathbb{P}(Y = y) = \frac{\binom{r_u}{y}\binom{n-r_u}{r_w-y}}{\binom{n}{r_w}} \tag{7}$$

This follows from the fact that given the first array U, we already know where the 1's are located. We can pick the y intersecting 1's for the second array in $\binom{r_u}{y}$ ways implicitly placing 0's in the remaining spots. We then fill the remaining $n - r_u$ indices corresponding to the 0's in the first array with $r_w - y$ 1's in $\binom{n-r_u}{r_w-y}$ ways. Finally we divide by the total number of possible subgraphs $\binom{n}{r_w}$. Then the probability that U k-interferes with W is

$$\sum_{y=\lceil \frac{r_w}{k} \rceil}^{r_w} \frac{\binom{r_u}{y}\binom{n-r_u}{r_w-y}}{\binom{n}{r_w}} \tag{8}$$

$\square$

Theorem 7. *Given a vertex set V with n vertices, the (r, T, k)-subgraph capacity of V is*

placeholder

Remark. I am not sure if we can use the first proof approach used in theorem 5.

Proof. Suppose we have M subgraphs $U_1, ..., U_M$ with sizes $r_1, ..., r_M$. Pick two subgraphs U_i, U_j. From lemma 6., we know that the expected number of interferences caused by this pair is

$$\sum_{y=\lceil \frac{r_j}{k} \rceil}^{r_j} \frac{\binom{r_i}{y}\binom{n-r_i}{r_j-y}}{\binom{n}{r_j}} + \sum_{y=\lceil \frac{r_i}{k} \rceil}^{r_i} \frac{\binom{r_j}{y}\binom{n-r_j}{r_i-y}}{\binom{n}{r_i}}$$

We then sum over all possible pairings to get the expected number of total interferences:

$$\sum_{(i,j)\in(1,M)\times(1,M), i\neq j} \left(\sum_{y=\lceil \frac{r_j}{k} \rceil}^{r_j} \frac{\binom{r_i}{y}\binom{n-r_i}{r_j-y}}{\binom{n}{r_j}} + \sum_{y=\lceil \frac{r_i}{k} \rceil}^{r_i} \frac{\binom{r_j}{y}\binom{n-r_j}{r_i-y}}{\binom{n}{r_i}} \right)$$

Since there are M subgraphs, the expected number of interferences by picking one subgraph is

$$\frac{1}{M} \sum_{(i,j)\in(1,M)\times(1,M), i\neq j} \left(\sum_{y=\lceil \frac{r_j}{k} \rceil}^{r_j} \frac{\binom{r_i}{y}\binom{n-r_i}{r_j-y}}{\binom{n}{r_j}} + \sum_{y=\lceil \frac{r_i}{k} \rceil}^{r_i} \frac{\binom{r_j}{y}\binom{n-r_j}{r_i-y}}{\binom{n}{r_i}} \right)$$

TODO: we need to use the fact that the expected value is r to simplify this then plug it into equation (2) to solve for M. Maybe helpful:

$$\mathbb{E}\left[\binom{X}{k}\right] = \sum_{0}^{\infty} \binom{i}{k} \mathbb{P}(X = i)$$

We hope the final result will be similar to the result in theorem 5.

$\square$