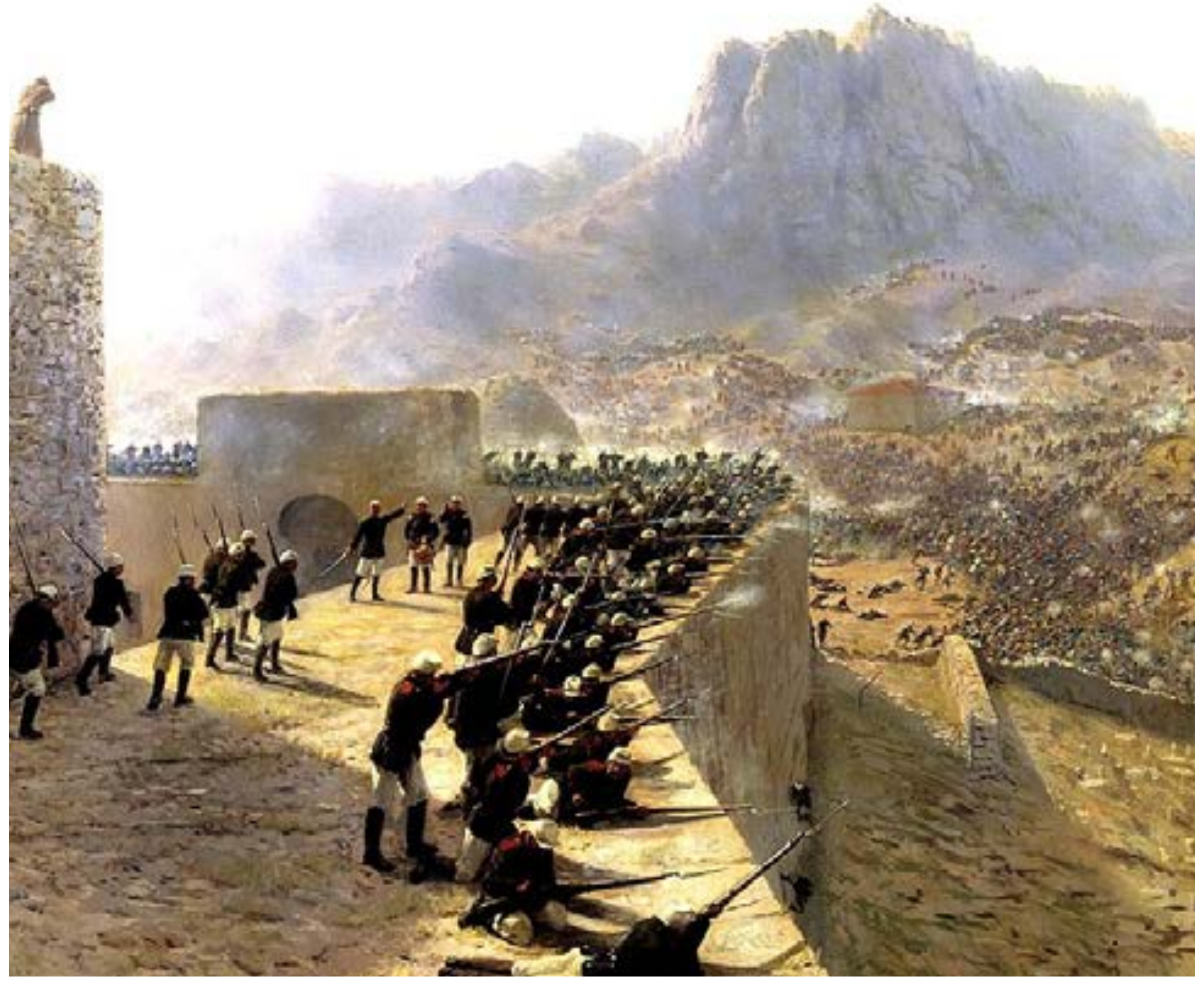
Saint Petersburg
State
University

Zero-sum Games in Normal Form and Matrix Games

O. Petrosian

PhD

Colonel Blotto Game

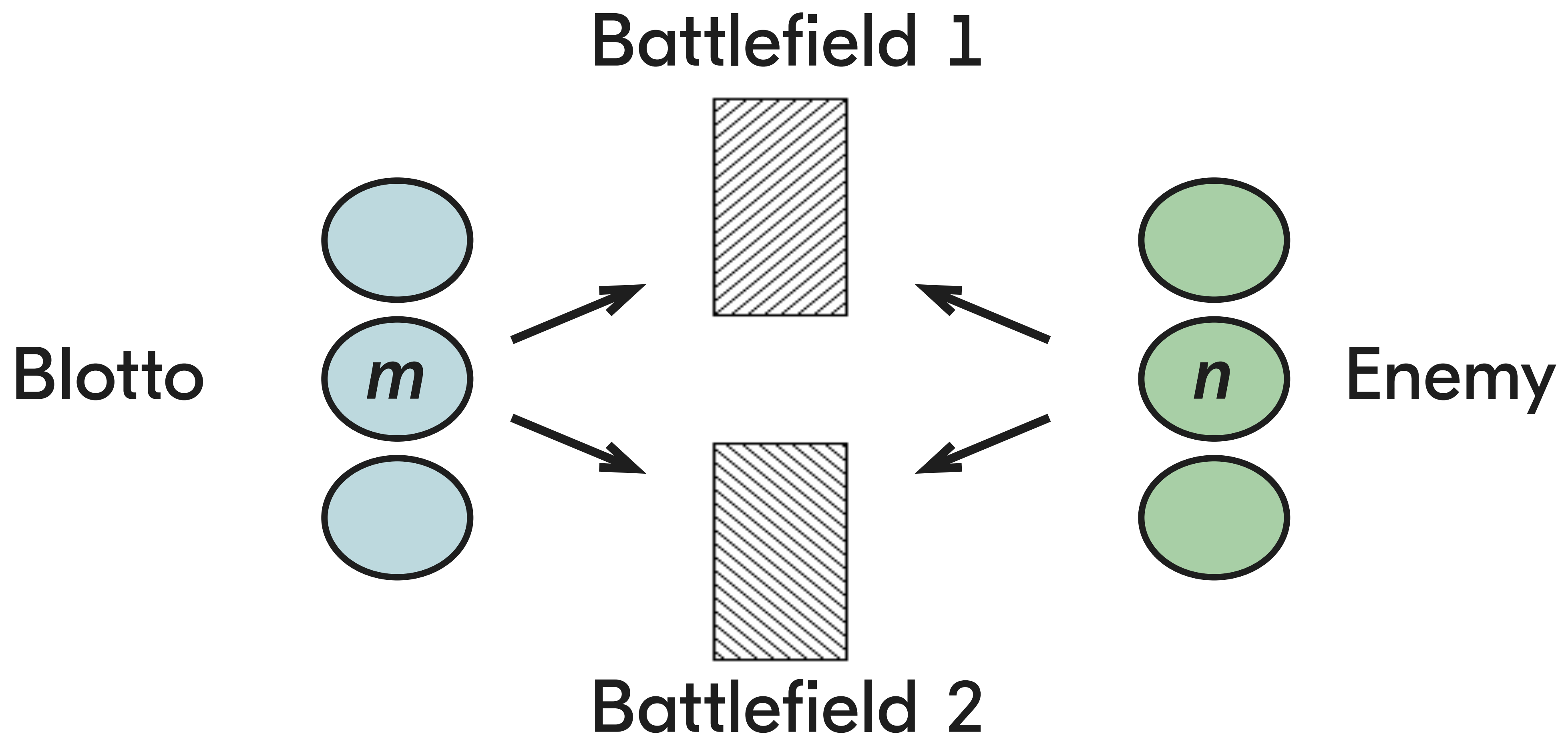


“Defence of Bayezet during the Russo-Turkish War”,
L. Lagorio, 1891

- Colonel Blotto has *m* regiments.
- His enemy has *n* regiments.

Colonel Blotto has to find optimal allocation of regiments for 2 battlefields.

Colonel Blotto Game



- On each battlefield the side that allocates more regiments wins.
- Neither side knows how many regiments the opposing side allocates on each battlefield.

Zero-sum Games in Normal Form

Definition.
The system

$$\Gamma = (X, Y, K),$$

where X and Y are strategy sets of players 1 and 2 correspondingly and the function $K: X \times Y \longrightarrow R^1$, is called two-person zero-sum game in normal form.

- $x \in X$ is the strategy of player 1,
  $y \in Y$ is the strategy of player 2.
- $(x, y) \in X \times Y$ is the strategy profile the game Γ.
- $K(x, y)$ is the payoff function of player 1,
  $[-K(x, y)]$ is the payoff function of player 2.

SPbSU

Zero-sum Games in Normal Form

Definition.
The system

$$\Gamma = (X, Y, K),$$

where X and Y are strategy sets of players 1 and 2 correspondingly, and the function $K: X \times Y \longrightarrow R^1$, is called a two-person zero-sum game in normal form.

Colonel Blotto game.

- $x = (x_1, x_2) \in X$, where $x_1 + x_2 = m$, $x_i \geq 0$, $i = 1, 2$.
  $y = (y_1, y_2) \in Y$, where $y_1 + y_2 = n$, $y_i \geq 0$, $i = 1, 2$.
- $K(x, y) = h_1(x, y) + h_2(x, y)$, $h_i(x, y) = \begin{cases} y_i + 1, \text{ if } x_i > y_i \text{ (Blotto's victory)}, \\ 0, \text{ if } x_i = y_i \text{ (drow)}, \\ -(x_i + 1), \text{ if } x_i < y_i \text{ (Blotto's defeat)}, \end{cases}$

  $[-K(x, y)]$ — payoff function of player 2.

Matrix Games

Definition.
Two-person zero-sum games in which both players have finite sets of strategies are called matrix games.

Notations.

- $\Gamma_A = (X, Y, K)$ is the matrix game.
- $x_i \in X$, where $i \in \{0, 1, \dots, m\}$ is the strategy of player 1,
  $y_j \in Y$, where $j \in \{0, 1, \dots, n\}$ is the strategy of player 2.
- $(x_i, y_j) \in X \times Y$ is the strategy profile in game $\Gamma_A = (X, Y, K)$.
- $K(x_i, y_j) = a_{i,j}$ is the payoff function of player 1,
  $[-K(x_i, y_j)] = -a_{i,j}$ is the payoff function of player 2.

Matrix Games

Definition.
Two-person zero-sum games in which both players have finite sets of strategies are called matrix games.

Colonel Blotto game ($m = 4$, $n = 3$):

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Strategies and payoffs.
$x_i = (m - i, i)$,
$y_j = (n - j, j)$.

$$a_{ij} = \begin{cases} n + 2, & \text{if } m - i > n - j,\ i > j, \\ n - j + 1, & \text{if } m - i > n - j,\ i = j, \\ n - j - i, & \text{if } m - i > n - j,\ i < j, \\ -m + i + j, & \text{if } m - i < n - j,\ i > j, \\ j + 1, & \text{if } m - i = n - j,\ i > j, \\ -m - 2, & \text{if } m - i < n - j,\ i < j, \\ -i - 1, & \text{if } m - i = n - j,\ i < j, \\ -m + i - 1, & \text{if } m - i < n - j,\ i = j, \\ 0, & \text{if } m - i = n - j,\ i = j. \end{cases}$$

References

1. Owen, G. (1982). Game Theory. London: Academic Press.

2. Peters, H. (2008). Game Theory. A Multi-Leveled Approach.
   Berlin: Springer-Verlag

3. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.

Saint Petersburg
State
University
www.spbu.ru

Saint Petersburg
State
University

Saddle Point

O. Petrosian

PhD

Maximin and Minimax Strategies

Definition.
Maximin strategy of player 1 is the strategy x_{i_0} which satisfies:

$$\max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) = \min_{y_i \in Y} K(x_{i_0}, y_j) = \underline{v}$$

here $\underline{v}$ is called the lower value of the game.

$$\begin{array}{c} \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m0} & a_{m1} & \dots & a_{mn} \end{bmatrix} \end{array} \left.\begin{array}{l} \min_j a_{0j} \\ \min_j a_{1j} \\ \dots \\ \min_j a_{mj} \end{array}\right\} \max_i \min_j a_{ij}$$

Maximin and Minimax Strategies

Definition.
Minimax strategy of player 2 is the strategy y_{j_0} which satisfies:

$$\min_{y_i \in Y} \max_{x_j \in X} K(x_i, y_j) = \max_{x_i \in X} K(x_i, y_{j_0}) = \overline{v}$$

here $\overline{v}$ is called the upper value of the game.

$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m0} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\underbrace{\max_i a_{i0} \quad \max_i a_{i1} \quad \cdots \quad \max_i a_{in}}_{\min_j \max_i a_{ij}}$$

Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

How Colonel Blotto should behave, what strategy should he choose?

Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Suppose, the enemy chooses strategy y_1,
then Colonel Blotto has to choose strategy x_1:

$$\max_{x_i \in X} K(x_i, y_1) = K(x_1, y_1).$$

Colonel Blotto Game

$$A = \begin{array}{c c} & \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Colonel Blotto does not know in advance what strategy the enemy will choose!

Colonel Blotto Game

$$A = \begin{array}{c c} & \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Colonel Blotto can ensure himself the payoff:

$$\underline{v} = \max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) = a_{0,3} = a_{4,0} = 0.$$

Whatever the behavior of the enemy, Colonel Blotto will receive not less than 0 choosing the maxmin strategies x_0 or x_4!

Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Similarly for the enemy,
he can be sure that he will loose not more than:

$$\overline{v} = \min_{y_j \in Y} \max_{x_i \in X} K(x_i, y_j) = a_{1,1} = a_{3,2} = 3.$$

Whatever the behavior of Colonel Blotto, the enemy will not lose more than 3 choosing the minmax strategies $\boldsymbol{y_1}$ or $\boldsymbol{y_2}$!

Colonel Blotto Game

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

In this game

$$\underline{v} = \max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) \neq \min_{y_j \in Y} \max_{x_i \in X} K(x_i, y_j) = \overline{v},$$

$$0 \neq 3.$$

Another Example

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_0 & y_1 & y_2 \end{array} \\ \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & 4 \\ 0 & -2 & 7 \end{pmatrix} \end{array}$$

In this game

$$\max_{x_i \in X} \min_{y_j \in Y} K(x_i, y_j) = \min_{y_j \in Y} \max_{x_i \in X} K(x_i, y_j) = K(x_1, y_0) = 2.$$

Saddle Point

Definition.
In the two-person zero-sum game $\mathbf{\Gamma} = (X, Y, K)$ strategy profile (x^*, y^*) is called saddle point, if

$$K(x, y^*) \leq K(x^*, y^*),\ \forall x \in X,$$

$$K(x^*, y) \geq K(x^*, y^*),\ \forall y \in Y.$$

Existence of Saddle Point

Theorem (necessary and sufficient conditions for the existence of saddle point).
Saddle point in the game **Γ** exists, if and only if

$$\underline{v} = \max_{x \in X} \min_{y \in Y} K(x, y) = \min_{y \in Y} \max_{x \in X} K(x, y) = \overline{v}.$$

References

1. Fudenberg, D. & Tirole. (2000). J. Game Theory. Cambridge: MIT-press.

2. Kolokoltsov, V. N. & Malafeyev, O. A. (2010). Understanding Game Theory: Introduction to the Analysis of Many Agent Systems with Competition and Cooperation. Singapore: World Scientific.

3. Owen, G. (1982). Game Theory. London: Academic Press.

4. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.

5. Mazalov, V. V. (2014). Mathematical game theory and applications. New York: Wiley.

6. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.

Saint Petersburg
State
University
www.spbu.ru

Saint Petersburg
State
University

Mixed Extension

O. Petrosian

PhD

Mixed Strategies

What Colonel Blotto needs to do to avoid disclosing information about his strategy?

$$
\begin{array}{cc}
 & \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\begin{pmatrix}
4 & 2 & 1 & 0 \\
1 & 3 & 0 & -1 \\
-2 & 2 & 2 & -2 \\
-1 & 0 & 3 & 1 \\
0 & 1 & 2 & 4
\end{pmatrix}
\end{array}
$$

Mixed Strategies

Definition.
Mixed strategy of the player is a probability distribution defined over the set of pure strategies.

Mixed strategies of players 1 and 2 have the following form:

$$x = (\xi_0, \dots, \xi_m), \sum_{i=0}^{m} \xi_i = 1, \xi_i \geq 0, i = 0, \dots, m.$$

$$y = (\eta_0, \dots, \eta_n), \sum_{j=0}^{n} \eta_j = 1, \eta_j \geq 0, j = 0, \dots, n.$$

where ξ_i и $\eta_j \geq 0$ are the probability of choosing pure strategies i and j by the first and the second player respectively.
In what follows, by X, Y we will denote sets of mixed strategies.

Mixed Strategies

Definition.
Mixed strategy of the player is a probability distribution defined over the set of pure strategies.

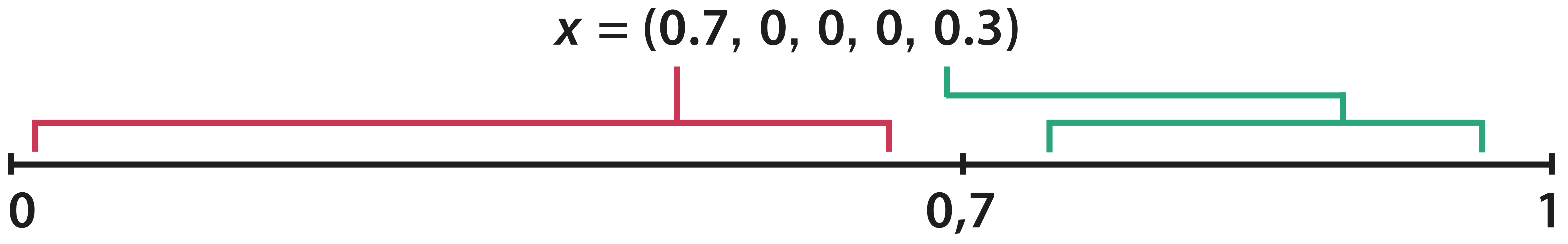
Colonel Blotto Game.
Suppose $x = (0.2, 0.2, 0.2, 0.2, 0.2)$, $y = (0.25, 0.25, 0.25, 0.25)$.

$$\begin{array}{cc} & \begin{array}{cccc} 0.25 & 0.25 & 0.25 & 0.25 \end{array} \\ \begin{array}{c} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{array} & \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Mixed Strategies

Suppose $x = (0.7, 0, 0, 0, 0.3)$.

How does Colonel Blotto realize this strategy?



Random number table

0.83	0.48	0.88	0.81	0.37	0.09	0.56	0.88	0.29	0.37
0.26	0.43	0.65	0.08	0.97	0.26	0.28	0.53	0.61	0.42
0.41	0.63	0.84	0.04	0.42	0.61	0.05	0.50	0.67	0.75
0.45	0.53	0.33	0.19	0.10	0.39	0.53	0.04	0.24	0.79
0.22	0.98	0.54	0.77	0.04	0.55	0.76	0.13	0.32	0.46
0.15	0.42	0.86	0.35	0.24	0.83	0.85	0.36	0.49	0.11

References

1. Fudenberg, D. & Tirole. (2000). J. Game Theory. Cambridge: MIT-press.
2. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.
3. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.
4. Owen, G. (1982). Game Theory. Academic Press.
5. Mazalov, V. V. (2014). Mathematical game theory and applications. New York: Wiley.
6. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.

Saint Petersburg
State
University
www.spbu.ru

Saint Petersburg
State
University

Saddle Point in Mixed Strategies

O. Petrosian

PhD

Payoff in Mixed Strategies

Definition.
Pair (x, y) of mixed strategies in the matrix game Γ_A is called the strategy profile in mixed strategies.

Definition.
Payoff in mixed strategies $x = (\xi_0, \dots, \xi_m)$, $y = (\eta_0, \dots, \eta_n)$ is defined as mathematical expectation of payoff in pure strategies:

$$K(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \xi_i a_{ij} \eta_j = (xA)y = x(Ay).$$

Payoff in Mixed Strategies

Suppose $x = (0.7, 0, 0, 0, 0.3)$, $y = (0.1, 0.9, 0, 0)$:

$$\begin{array}{cc} & \begin{array}{cccc} 0.1 & 0.9 & 0 & 0 \end{array} \\ \begin{array}{c} 0.7 \\ 0 \\ 0 \\ 0 \\ 0.3 \end{array} & \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

Payoff in strategy profile (x, y):

$$K(x, y) = \sum_{i=0}^{4} \sum_{j=0}^{3} \xi_i a_{ij} \eta_j = 0.7(0.1 \cdot 4 + 0.9 \cdot 2 + 0 \cdot 1 + 0 \cdot 0) +$$

$$+ 0(\ldots) + 0(\ldots) + 0(\ldots) + 0.3(0.1 \cdot 0 + 0.9 \cdot 1 + 0 \cdot 2 + 0 \cdot 4) = 1.81.$$

Saddle Point in Mixed Strategies

Theorem (main theorem of matrix games).
Any matrix game has a saddle point in mixed strategies.

There always exists strategy profile in mixed strategies (x^*, y^*), such that:

$$K(x, y^*) \leq K(x^*, y^*), \forall x \in X,$$
$$K(x^*, y) \geq K(x^*, y^*), \forall y \in Y.$$

Also the following equality is satisfied:

$$K(x^*, y^*) = \max_{x \in X} \min_{y \in Y} K(x, y) = \min_{y \in Y} \max_{x \in X} K(x, y).$$

Payoff in saddle point is called value of the game and denoted by v.

Saddle Point in Mixed Strategies

Suppose $x^* = \left(\frac{4}{9}, 0, \frac{1}{9}, 0, \frac{4}{9}\right)$, $y^* = \left(\frac{7}{90}, \frac{32}{90}, \frac{48}{90}, \frac{3}{90}\right)$:

$$\begin{array}{c} \\ \frac{4}{9} \\ 0 \\ \frac{1}{9} \\ 0 \\ \frac{4}{9} \end{array} \begin{array}{c} \begin{array}{cccc} \frac{7}{90} & \frac{32}{90} & \frac{48}{90} & \frac{3}{90} \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array}$$

For strategy profile (x^*, y^*) the following holds:

$$K(x^*, y^*) = \max_{x \in X} \min_{y \in Y} K(x, y) = \min_{y \in Y} \max_{x \in X} K(x, y) = \frac{14}{9}.$$

Therefore, (x^*, y^*) is a saddle point in the game Γ_A.

Properties of Optimal Strategies and Game Value

Theorem.
Strategy profile (x^*, y^*) in mixed strategies is a saddle point in the game Γ_A, if and only if the following equality holds:

$$\min_{y_j \in Y} K(x^*, y_j) = \max_{x_i \in X} K(x_i, y^*).$$

Example

Suppose $x = (\xi, 1 - \xi)$, $y = (\eta, 1 - \eta)$:

$$\begin{array}{cc} & \begin{array}{cc} \eta & 1-\eta \end{array} \\ \begin{array}{c} \xi \\ 1-\xi \end{array} & \begin{pmatrix} 6 & 5 \\ 3 & 7 \end{pmatrix} \end{array}$$

$K(x_1, y^*) = 6\eta^* + 5(1 - \eta^*)$ $\qquad$ $\eta^* + 5 = 7 - 4\eta^*$

$K(x_2, y^*) = 3\eta^* + 7(1 - \eta^*)$ $\qquad$ $\eta^* = 0.4$

$K(x_1, y^*) = K(x_2, y^*)$ $\qquad$ $y^* = (\eta^*, 1 - \eta^*) = (0.4, 0.6)$

$K(x^*, y_1) = 6\xi^* + 3(1 - \xi^*)$ $\qquad$ $3\xi^* + 3 = 7 - 2\xi^*$

$K(x^*, y_2) = 5\xi + 7(1 - \xi^*)$ $\qquad$ $\xi^* = 0.8$

$K(x^*, y_1) = K(x^*, y_2)$ $\qquad$ $x^* = (\xi^*, 1 - \xi^*) = (0.8, 0.2)$

$$K(x^*, y^*) = 5.4$$

SPbSU

References

1. Fudenberg, D. & Tirole, J. (2000). Game Theory. Cambridge: MIT-press.
2. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.
3. Mazalov, V. V. (2014). Mathematical game theory and applications. New York: Wiley.
4. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.
5. von Neumann, J., Morgenstern, O. (1970). Theory of games and economic behavior. Princeton: University Press.

Saint Petersburg
State
University

www.spbu.ru

Saint Petersburg
State
University

Dominance of Strategies

O. Petrosian

PhD

Dominance of Strategies

$$A = \begin{pmatrix} 4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\ 0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\ 3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\ 2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\ 4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\ 0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\ 2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0 \end{pmatrix}$$

How to find a saddle point in this game?

Dominance of Strategies

Definition.
Strategy x' (y') of player 1 (2) dominates the strategy x'' (y''), if the following inequalities hold:

$$x'a^j \geq x''a^j,\ j \in \{1, \dots, n\}$$
$$(y'\ a_i \leq y''a_i,\ i \in \{1, \dots, m\}).$$

Definition.
Strategy x'' (y'') of player 1 (2) is dominated if there exists a strategy $x' \neq x''$ ($y' \neq y''$) of player 1 (2), which dominates x'' (y'').

Strategy x'' (y'') of player 1 (2) is strictly dominated if there exists a strategy x' (y') for which the inequalities above are strict.

Dominance of Strategies

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{array} &
\begin{pmatrix}
4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\
0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\
3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\
0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\
2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

- Strategies x_6, x_7, x_5, x_7, x_4, x_7, x_5, x_4, y_4, y_6 strictly dominate strategies x_0, x_1, x_2, x_3, x_8, x_9, x_{10}, x_{11}, y_0, y_8 correspondingly.
- Therefore, strategies x_0, x_1, x_2, x_3, x_8, x_9, x_{10}, x_{11}, y_0, y_8 are strictly dominated.

Dominance of Strategies

Theorem.
If strategy x' dominates an optimal strategy x^*, then strategy x' is also optimal*.

Theorem.
If strategy x^* is optimal, then it is not strictly dominated.

* — optimal strategies are the strategies from the saddle point.

Dominance of Strategies

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \end{array} &
\begin{pmatrix}
4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\
0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\
3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\
0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\
2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

Strategies x_0, x_1, x_2, x_3, x_8, x_9, x_{10}, x_{11}, y_0, y_8 are not included with positive probabilities in optimal strategies.

Dominance of Strategies

Denote by A' matrix obtained from A by deleting the i-th row.

By $\overline{x_i^*}$ denote the extension of strategy x^* at the i-th place $\overline{x_i^*} = (x_1^*, \dots, x_{i-1}^*, 0, x_i^*, x_{i+1}^*, \dots, x_n^*)$.

Theorem.
Suppose that the i-th row of matrix A is dominated, then:

- $v_A = v_{A'}$.
- Any optimal strategy y^* of player 2 in the game $\Gamma_{A'}$ is optimal in the game Γ_A.
- If x^* is optimal strategy of player 1 in the game $\Gamma_{A'}$, then $\overline{x_i^*}$ is optimal in the game Γ_A.

Dominance of Strategies

$$
\begin{array}{c}
\\ x_0 \\ x_1 \\ x_3 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11}
\end{array} \\
\begin{pmatrix}
4 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 5 \\
0 & 2 & 1 & 5 & 0 & 1 & 0 & 6 & 2 & 3 & 0 & 1 \\
3 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 2 & 2 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 \\
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\
0 & 3 & 0 & 4 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 5 \\
2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

Optimal strategies in this game are

$x^* = y^* = (0, 0, 0, 0, 0, 0, \frac{3}{4}, \frac{1}{4}, 0, 0, 0, 0),$

game value is $v = \frac{3}{2}$.

Dominance of Strategies

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \end{array} \\
\begin{array}{c} x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\ 3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\ 4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\ 0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6 \end{pmatrix}
\end{array}
$$

Optimal strategies in this game:

$x^*= (0, 0, \frac{3}{4}, \frac{1}{4})$,

$y^*= (0, 0, 0, 0, 0, 0, \frac{3}{4}, \frac{1}{4}, 0, 0, 0, 0)$.

Dominance of Strategies

Denote by A' matrix obtained from A by deleting the j-th column.
By $\overline{y_j^*}$ denote the extension of strategy y^* at the j-th place.

Theorem.
Suppose that the j-th column of matrix A of the game Γ_A is dominated, then:

- $v_A = v_{A'}$.
- Any optimal strategy x^* of player 1 in the game $\Gamma_{A'}$ is optimal in the game Γ_A.
- If y^* is optimal strategy of player 2 in the game $\Gamma_{A'}$, then $\overline{y_j^*}$ is optimal in the game Γ_A.

References

1. Vorob'ov, N. N. (1994). Foundations of Game Theory: Noncooperative Games. Basel: Birkhäuser.
2. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.
3. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.
4. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.
5. Mazalov, V. V. (2014). Mathematical game theory and applications. New York: Wiley.

Saint Petersburg
State
University
www.spbu.ru

Saint Petersburg
State
University

Dominance of Strategies

O. Petrosian

PhD

SPbSU

Dominance of Strategies

$$\begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} \end{array} \\
\begin{pmatrix}
2 & 4 & 1 & 1 & 2 & 1 & 1 & 0 & 4 & 2 & 1 & 0 \\
3 & 2 & 3 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 3 \\
4 & 3 & 3 & 0 & 3 & 1 & 2 & 0 & 4 & 1 & 2 & 6 \\
0 & 4 & 3 & 6 & 0 & 3 & 0 & 6 & 2 & 3 & 2 & 6
\end{pmatrix}
\end{array}$$

- Strategies y_4, y_4, y_5, y_7, y_6, y_5, y_6, y_7 strictly dominate strategies y_0, y_1, y_2, y_3, y_8, y_9, y_{10}, y_{11} correspondingly.
- Therefore, strategies y_0, y_1, y_2, y_3, y_8, y_9, y_{10}, y_{11} are strictly dominated.

Dominance of Strategies

Strategy x_5 dominates the strategy x_4:

$$
\begin{array}{c c}
 & \begin{array}{cccc} y_4 & y_5 & y_6 & y_7 \end{array} \\
\begin{array}{c} x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix}
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{c c}
 & \begin{array}{cccc} y_4 & y_5 & y_6 & y_7 \end{array} \\
\begin{array}{c} x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix}
\end{array}
$$

Dominance of Strategies

Strategy y_6 dominates the strategy y_4:

$$
\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}
\begin{array}{c}
\begin{array}{cccc} y_4 & y_5 & y_6 & y_7 \end{array} \\
\begin{pmatrix} 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix}
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array}
\begin{array}{c}
\begin{array}{ccc} y_5 & y_6 & y_7 \end{array} \\
\begin{pmatrix} 3 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 6 \end{pmatrix}
\end{array}
$$

Dominance of Strategies

Strategy $y = (0, 1/2, 1/2)$ dominates the strategy y_5:

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccc} y_5 & y_6 & y_7 \end{array} \\ \begin{pmatrix} 3 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 6 \end{pmatrix} \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{cc} y_6 & y_7 \end{array} \\ \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

Dominance of Strategies

Strategy $x = (0, 1/2, 1/2)$ dominates the strategy x_5:

$$\begin{array}{cc} & \begin{array}{cc} y_6 & y_7 \end{array} \\ \begin{array}{c} x_5 \\ x_6 \\ x_7 \end{array} & \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

$$\downarrow$$

$$\begin{array}{cc} & \begin{array}{cc} y_6 & y_7 \end{array} \\ \begin{array}{c} x_6 \\ x_7 \end{array} & \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

Optimal strategies are $x^* = y^* = \left(\frac{3}{4}, \frac{1}{4}\right)$, game value is $v = \frac{3}{2}$.

Colonel Blotto Game

Suppose $m = 3$, $n = 1$. Pure strategies of players $x_i = (m - i, i)$, $y_j = (n - j, j)$.

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{pmatrix} 2 & 0 \\ 3 & 1 \\ 1 & 3 \\ 0 & 2 \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \end{array}$$

Optimal strategies are $x^* = y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, game value is $v = 2$.

References

1. Straffin, Ph. D. (1993). Game Theory and Strategy. Washington: MAA notes.
2. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.
3. Mazalov, V. V. (2014). Mathematical game theory and applications. New York: Wiley.
4. Vorob'ov, N. N. (1994). Foundations of Game Theory: Noncooperative Games. Basel: Birkhäuser.
5. Peters, H. (2008). Game Theory. A Multi-Leveled Approach. Berlin: Springer-Verlag.

Saint Petersburg
State
University
www.spbu.ru

Saint Petersburg
State
University

Iterative Solution Method for Matrix Games

O. Petrosian

PhD

Iterative Method

Iterative Brown-Robinson method is an iterative procedure for constructing a sequence of $(\underline{v}^k, \overline{v}^k)$ converging to the game value.

- On each iteration, players 1 and 2 use pure strategies.
- Choice of pure strategy on the current iteration is based on the accumulated payoff.

Iterative Method

Algorithm:

- **Iteration** 0:
  x_{i_0}, y_{i_0} are the arbitrary initial pure strategies.
- **Iteration** 1:
  x_{i_1}: $\max\limits_i a_{i,j_0} = \overline{v}^1$; y_{i_1}: $\min\limits_j a_{i_0,j} = \underline{v}^1$.
  …
- **Iteration** $k + 1$:
  $x_{i_{k+1}}$: $\max\limits_i \sum_j a_{i,j}\,\eta_j^k / k = \overline{v}^k$; $y_{j_{k+1}}$: $\min\limits_j \sum_i a_{ij}\,\xi^k / k = \underline{v}^k$,
  where ξ_i^k and η_j^k is the number of chosing pure strategies x_i, y_j correspondingly in k iterations.
  …

Accuracy of the algorithm is defined by $\varepsilon = \max\limits_k \overline{v}^k - \min\limits_k \underline{v}^k$.

Iterative method

$x^k = (\xi_1^k / k, \ldots, \xi_m^k / k)$ and $y^k = (\eta_1^k / k, \ldots, \eta_n^k / k)$ are the frequencies of pure strategies.

Interval for the game value:

$$v \in \left[\max_k \overline{v}^k / k, \min_k \underline{v}^k / k\right].$$

Theorem (convergence of algorithm).

$$\lim_{k \to +\infty} \left(\min_k \underline{v}^k / k \right) = \lim_{k \to +\infty} \left(\max_k \overline{v}^k / k \right) = v.$$

Colonel Blotto Game

Solve Colonel Blotto game using iterative method:

$$
\begin{array}{c c}
 & \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\begin{pmatrix}
4 & 2 & 1 & 0 \\
1 & 3 & 0 & -1 \\
-2 & 2 & 2 & -2 \\
-1 & 0 & 3 & 1 \\
0 & 1 & 2 & 4
\end{pmatrix}
\end{array}
$$

Colonel Blotto Game

№	Choice of 1 player	Choice of 1 player	Payoff of player 1					Payoff of player 2				vk dash below	vk dash above
			x0	x1	x2	x3	x4	y0	y1	y2	y3		
1	x0	y2	4	1	-2	-1	0	4	2	1	0	0,00	4,00
2	x0	y3	4	0	-4	0	4	8	4	2	0	0,00	2,00
3	x4	y3	4	-1	-2	1	8	8	5	4	4	1,33	2,67
4	x4	y3	4	-2	-4	2	12	8	6	6	8	1,50	3,00
5	x4	y2	5	-2	-2	5	16	8	7	8	12	1,40	3,20
6	x4	y1	7	1	0	5	17	8	8	10	16	1,33	2,83
7	x4	y1	9	4	2	5	17	8	9	12	20	1,14	2,43
8	x4	y0	13	5	0	4	17	8	10	14	24	1,00	2,13
9	x4	y0	17	6	-2	3	17	8	11	16	28	0,89	1,89
10	x4	y0	21	7	-2	2	17	8	12	18	32	0,80	2,10
11	x0	y0	25	8	-4	1	17	12	14	19	32	1,09	2,27
12	x0	y0	29	9	-6	0	17	16	16	20	32	1,33	2,42
789	x0	y0	1234	1033	1208	1052	1228	1184	1202	1243	1348	1,50	1,56
790	x0	y0	1238	1034	1206	1051	1228	1188	1204	1244	1348	1,50	1,57
791	x0	y0	1242	1035	1204	1050	1228	1192	1206	1245	1348	1,51	1,57
792	x0	y0	1246	1036	1202	1049	1228	1196	1208	1246	1348	1,51	1,57
793	x0	y0	1250	1037	1200	1048	1228	1200	1210	1247	1348	1,51	1,58
794	x0	y0	1254	1038	1198	1047	1228	1204	1212	1248	1348	1,52	1,58
795	x0	y0	1258	1039	1196	1046	1228	1208	1214	1249	1348	1,52	1,58
796	x0	y0	1262	1040	1194	1045	1228	1212	1216	1250	1348	1,52	1,59
797	x0	y0	1266	1041	1192	1044	1228	1216	1218	1251	1348	1,53	1,59
798	x0	y0	1270	1042	1190	1043	1228	1220	1220	1252	1348	1,53	1,59
799	x0	y1	1272	1045	1192	1043	1229	1224	1222	1253	1348	1,53	1,59
800	x0	y1	1274	1048	1194	1043	1230	1228	1224	1254	1348	1,53	1,59

Colonel Blotto Game

$$
\begin{array}{cc}
 & \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\
\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} &
\begin{pmatrix}
4 & 2 & 1 & 0 \\
1 & 3 & 0 & -1 \\
-2 & 2 & 2 & -2 \\
-1 & 0 & 3 & 1 \\
0 & 1 & 2 & 4
\end{pmatrix}
\end{array}
$$

Solution on the iteration № 800:

- $x^{800} = (0.433, 0, 0.098, 0, 0.470)$, $y^{800} = (0.073, 0.429, 0.443, 0.056)$.
- $v^{800} = 1.555675$.
- [1.53, 1.59] is the interval for game value on the interation 800.

Other Methods for Solving Matrix Games

Other methods for solving matrix games:

- Graphic-analytical method (for $[2 \times n]$, $[m \times 2]$ games).
- Linear programming method.

References

1. Petrosyan, L. A., Zenkevich, N. A., (2016). Game theory. Singapore: World Scientific.
2. Vorob'ov, N. N. (1994). Foundations of Game Theory: Noncooperative Games. Basel: Birkhäuser.

SPbSU

Saint Petersburg
State
University

www.spbu.ru