Zero-sum games

November 28, 2018

1 Conflict and cooperation

Definition 1. *Game theory is a decision theory in conflict situations.*

Definition 2. *The game is a mathematical model in conflict situations.*

A theory of games, introduced in 1921 by Emile Borel, was established in 1928 by John von Neumann and Oskar Morgenstern, to develop it as a means of decision making in complex economic systems. In their book "The Theory of Games and Economic Behaviour", published in 1944, they asserted that the classical mathematics developed for applications in mechanics and physics fail to describe the real processes in economics and social life.

2 Definition of a two-person zero-sum game in normal form

Definition 3. *The system*

$$\Gamma = (X, Y, K), \tag{2.1}$$

where X and Y are nonempty sets, and the function $K : X \times Y \to R^1$, is called a two-person zero-sum game in normal form.

The elements $x \in X$ and $y \in Y$ are called the *strategies* of players 1 and 2, respectively, in the game Γ, the elements of the Cartesian product $X \times Y$ (i.e. the pairs of strategies (x, y), where $x \in X$ and $y \in Y$) are called *situations*, and the function K is the payoff of Player 1. Player 2's payoff in situation (x, y) is equal to $[-K(x, y)]$; therefore the function K is also called the *payoff function* of the game Γ and the game Γ is called a *zero-sum game*. Thus, in order to specify the game Γ, it is necessary to define the sets of strategies X, Y for players 1 and 2, and the payoff function K given on the set of all situations $X \times Y$.

The game Γ is interpreted as follows. Players simultaneously and independently choose strategies $x \in X, y \in Y$. Thereafter Player 1 receives the payoff equal to $K(x, y)$ and Player 2 receives the payoff equal to $(-K(x, y))$.

Definition 4. *Definition. The game* $\Gamma' = (X', Y', K')$ *is called a subgame of the game* $\Gamma = (X, Y, K)$ *if* $X' \subset X, Y' \subset Y$*, and the function* $K' : X' \times Y' \to R^1$ *is a restriction of function* K *on* $X' \times Y'$*.*

This section focuses on two-person zero-sum games in which the strategy sets of the players' are finite.

Definition 5. *Definition. Two-person zero-sum games in which both players have finite sets of strategies are called matrix games.*

Suppose that Player 1 in matrix game (2.1) has a total of m strategies. Let us order the strategy set X of the first player, i.e. set up a one-to-one correspondence between the sets $M = \{1, 2, \ldots, m\}$ and X. Similarly, if Player 2 has n strategies, it is possible to set up a one-to-one correspondence between the sets $N = \{1, 2, \ldots, n\}$ and Y. The game Γ is then fully defined by specifying the matrix $A = \{a_{ij}\}$, where $a_{ij} = K(x_i, y_i)$, $(i, j) \in M \times N$, $(x_i, y_j) \in X \times Y, i \in M, j \in N$ (whence comes the name of the game — the matrix game). In this case the game Γ is realized as follows. Player 1 chooses row $i \in M$ and Player 2 (simultaneously and independently from Player 1) chooses column $j \in N$. Thereafter Player 1 receives the payoff (a_{ij}) and Player 2 receives the payoff $(-a_{ij})$. If the payoff is equal to a negative number, then we are dealing with the actual loss of Player 1.

Denote the game Γ with the payoff matrix A by Γ_A and call it the $(m \times n)$ game according to the dimension of matrix A.

Example 1. Defense of the city. This example is known in literature as Colonel Blotto game. Colonel Blotto has m regiments and his enemy has n regiments. The enemy is defending two posts. The post will be taken by Colonel Blotto if when attacking the post he is more powerful in strength on this post. The opposing parties are two separate regiments between the two posts.

Define the payoff to the Colonel Blotto (Player 1) at each post. If Blotto has more regiments than the enemy at the post (Player 2), then his payoff at this post is equal to the number of the enemy's regiments plus one (the occupation of the post is equivalent to capturing of one regiment). If Player 2 has more regiments than Player 1 at the post, Player 1 loses his regiments at the post plus one (for the lost of the post). If each side has the same number of regiments at the post, it is a draw and each side gets zero. The total payoff to Player 1 is the sum of the payoffs at the two posts.

The game is zero-sum. We shall describe strategies of the players. Suppose that $m > n$. Player 1 has the following strategies: $x_0 = (m, 0)$ – to place all of the regiments at the first post; $x_1 = (m - 1, 1)$ – to place $(m - 1)$ regiments at the first post and one at the second; $x_2 = (m - 2, 2), \ldots, x_{m-1} = (1, m - 1), x_m = (0, m)$. The enemy (Player 2) has the following strategies: $y_0 = (n, 0)$, $y_1 = (n - 1, 1), \ldots, y_n = (0, n)$.

Suppose that the Player 1 chooses strategy x_0 and Player 2 chooses strategy y_0. Compute the payoff a_{00} of Player 1 in this situation. Since $m > n$, Player 1 wins at the first post. His payoff is $n + 1$ (one for holding the post). At the second post it is draw. Therefore $a_{00} = n + 1$. Compute a_{01}. Since $m > n - 1$, then in the first post Player 1's payoff is $n-1+1=n$. Player 2 wins at the second post. Therefore the loss of Player 1 at this post is one. Thus, $a_{01}=n-1$. Similarly, we obtain $a_{0j}=n-j+1-1=n-j,\ 1\le j\le n$. Further, if $m-1>n$ then $a_{10}=n+1+1=n+2,\ a_{11}=n-1+1=n,\ a_{1j}=n-j+1-1-1=n-j-1,\ \underline{2\le j\le n}$. In a general case (for any m and n) the elements $a_{ij},\ i=\overline{0,m},\ j=\overline{0,n}$, of the payoff matrix are computed as follows:

$$a_{ij}=K(x_i,y_j)=\begin{cases} n+2 & \text{if } m-i>n-j,\quad i>j,\\ n-j+1 & \text{if } m-i>n-j,\quad i=j,\\ n-j-i & \text{if } m-i>n-j,\quad i<j,\\ -m+i+j & \text{if } m-i<n-j,\quad i>j,\\ j+1 & \text{if } m-i=n-j,\quad i>j,\\ -m-2 & \text{if } m-i<n-j,\quad i<j,\\ -i-1 & \text{if } m-i=n-j,\quad i<j,\\ -m+i-1 & \text{if } m-i<n-j,\quad i=j,\\ 0 & \text{if } m-i=n-j,\quad i=j. \end{cases}$$

Thus, with $m=4, n=3$, considering all possible situations, we obtain the payoff matrix A of this game:

$$A=\begin{array}{c} \\ x_0\\ x_1\\ x_2\\ x_3\\ x_4 \end{array}\begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array}\\ \begin{bmatrix} 4 & 2 & 1 & 0\\ 1 & 3 & 0 & -1\\ -2 & 2 & 2 & -2\\ -1 & 0 & 3 & 1\\ 0 & 1 & 2 & 4 \end{bmatrix} \end{array}.$$

Example 2. Game of Evasion. Players 1 and 2 choose integers i and j from the set $\{1,\dots,n\}$. Player 1 wins the amount $|i-j|$. The game is zero-sum. The payoff matrix is square $(n\times n)$ matrix, where $a_{ij}=|i-j|$. For $n=4$, the payoff matrix A has the form

$$A=\begin{array}{c} \\ 1\\ 2\\ 3\\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\\ \begin{bmatrix} 0 & 1 & 2 & 3\\ 1 & 0 & 1 & 2\\ 2 & 1 & 0 & 1\\ 3 & 2 & 1 & 0 \end{bmatrix} \end{array}.$$

Example 3. Discrete Type Game.

Players approach one another by taking n steps. After each step a player may or may not fire a bullet, but during the game he may fire only once. The probability that the player will hit his opponent (if he shoots) on the k-th step is assumed to be k/n $(k\le n)$.

A strategy for Player 1 (2) consists in taking a decision on shooting at the i-th (j-th) step. Suppose that $i<j$ and Player 1 makes a decision to shoot at the i-th step and Player 2 makes a decision to shoot at the j-th step. The payoff a_{ij} to Player 1 is then determined by

$$a_{ij}=\frac{i}{n}-\left(1-\frac{i}{n}\right)\frac{j}{n}=\frac{n(i-j)+ij}{n^2}.$$

Thus the payoff a_{ij} is the difference in the probabilities of hitting the opponent and failing to survive. In the case $i > j$, Player 2 is the first to fire and $a_{ij} = -a_{ji}$. If however, $i = j$, then we set $a_{ij} = 0$. Accordingly, if we set $n = 5$, the game matrix multiplied by 25 has the form

$$A = \begin{bmatrix} 0 & -3 & -7 & -11 & -15 \\ 3 & 0 & 1 & -2 & -5 \\ 7 & -1 & 0 & 7 & 5 \\ 11 & 2 & -7 & 0 & 15 \\ 15 & 5 & -5 & -15 & 0 \end{bmatrix}.$$

Example 4. Attack-Defense Game. Suppose that Player 1 wants to attack one of the targets $c_1, \ldots, c_n$ having positive values $\tau_1 > 0, \ldots, \tau_n > 0$. Player 2 defends one of these targets. We assume that if the undefended target c_i is attacked, it is necessarily destroyed (Player 1 wins τ_i) and the defended target is hit with probability $1 > \beta_i > 0$ (the target c_i withstands the attack with probability $1 - \beta_i > 0$), i.e. Player 1 wins (on the average) $\beta_i \tau_i,\ i = 1, 2, \ldots, n$.

The problem of choosing the target for attack (for Player 1) and the target for defense (for Player 2) reduces to the game with the payoff matrix

$$A = \begin{bmatrix} \beta_1\tau_1 & \tau_1 & \ldots & \tau_1 \\ \tau_2 & \beta_2\tau_2 & \ldots & \tau_2 \\ \ldots & \ldots & \ldots & \ldots \\ \tau_n & \tau_n & \ldots & \beta_n\tau_n \end{bmatrix}.$$

Example 5. Discrete Search Game.

There are n cells. Player 2 hide an object in one of n cells and Player 1 wishes to find it. In examining the i-th cell, Player 1 exerts $\tau_i > 0$ efforts, and the probability of finding the object in the i-th cell (if it is concealed there) is $0 < \beta_i \leq 1,\ i = 1, 2, \ldots, n$. If the object is found, Player 1 receives the amount α. The players' strategies are the numbers of cells wherein the players respectively hide and search for the object. Player 1's payoff is equal to the difference in the expected receipts and the efforts made in searching for the object. Thus, the problem of hiding and searching for the object reduces to the game with the payoff matrix

$$A = \begin{bmatrix} \alpha\beta_1 - \tau_1 & -\tau_1 & -\tau_1 & \ldots & -\tau_1 \\ -\tau_2 & \alpha\beta_2 - \tau_2 & -\tau_2 & \ldots & -\tau_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\tau_n & -\tau_n & -\tau_n & \ldots & \alpha\beta_n - \tau_n \end{bmatrix}.$$

Example 6. Noisy Search Suppose that Player 1 is searching for a mobile object (Player 2) for the purpose of detecting it. Player 2's objective is the opposite one (i.e. he seeks to avoid being detected). Player 1 can move with velocities $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3$ and Player 2 with velocities $\beta_1 = 1,\ \beta_2 = 2,\ \beta_3 = 3$, respectively. The range of the detecting device used by Player 1, depending on the velocities of the players is determined by the matrix

$$D = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 4 & 5 & 6 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} \end{array}.$$

Strategies of the players are the velocities, and Player 1's payoff in the situation (α_i, β_j) is assumed to be the search efficiency $a_{ij} = \alpha_i \delta_{ij}$, $i = \overline{1,3}, j = \overline{1,3}$, where δ_{ij} is an element of the matrix D. Then the problem of selecting velocities in a noisy search can be represented by the game with matrix

$$A = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 4 & 5 & 6 \\ 6 & 8 & 10 \\ 3 & 6 & 9 \end{bmatrix} \end{array}.$$

3 Solution of matrix games

Consider a two-person zero-sum game $\Gamma = (X, Y, K)$. In this game each of the players seeks to maximize his payoff by choosing a proper strategy. But for Player 1 the payoff is determined by the function $K(x, y)$, and for Player 2 it is determined by $(-K(x, y))$, i.e. the players' objectives are directly opposite. Note that the payoff of Player 1 (2) (the payoff function) is determined on the set of situations $(x, y) \in X \times Y$. Each situation, and hence the player's payoff do not depend only on his own choice, but also on what strategy will be chosen by his opponent whose objective is directly opposite. Therefore, seeking to obtain the maximum possible payoff, each player must take into account the opponent's behavior.

In the theory of games it is supposed that the behavior of both players is rational, i.e. they wish to obtain the maximum payoff, assuming that the opponent is acting in the best (for himself) possible way. What maximal payoff can Player 1 guarantee himself? Suppose player 1 chooses strategy x. Then, at worst case he will win $\min_y K(x, y)$. Therefore, Player 1 can always guarantee himself the payoff $\max_x \min_y K(x, y)$. If the max and min are not reached, Player 1 can guarantee himself obtaining the payoff arbitrarily close to the quantity

$$\underline{v} = \sup_{x \in X} \inf_{y \in Y} K(x, y), \tag{3.1}$$

which is called the *lower value* of the game.

The principle of constructing strategy x based on the maximization of the minimal payoff is called the *maximin principle*, and the strategy x selected by this principle is called the *maximin strategy* of Player 1.

For Player 2 it is possible to provide similar reasonings. Suppose he chooses strategy y. Then, at worst, he will lose $\max_x K(x, y)$. Therefore, the second player can always guarantee himself the payoff $-\min_y \max_x K(x,y)$. The number

$$\overline{v} = \inf_{y\in Y} \sup_{x\in X} K(x,y) \tag{3.2}$$

is called the *upper value of the game* Γ.

The principle of constructing a strategy y, based on the minimization of maximum losses, is called the *minimax principle*, and the strategy y selected for this principle is called the *minimax strategy* of Player 2. It should be stressed that the existence of the minimax (maximin) strategy is determined by the reachability of the extremum in (3.1), (3.2).

Consider the $(m \times n)$ matrix game Γ_A. Then the extrema in (3.1) and (3.2) are reached and the lower and upper values of the game are, respectively equal to

$$\underline{v} = \max_{1\le i\le m} \min_{1\le j\le n} a_{ij}, \tag{3.3}$$

$$\overline{v} = \min_{1\le j\le n} \max_{1\le i\le m} a_{ij}. \tag{3.4}$$

The minimax and maximin for the game Γ_A can be found by the following scheme

$$\begin{array}{c}
\left[\begin{array}{cccc}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2n} \\
\dots & \dots & \dots & \dots \\
a_{m1} & a_{m2} & \dots & a_{mn}
\end{array}\right]
\left.\begin{array}{c}
\min_j a_{1j} \\
\min_j a_{2j} \\
\dots \\
\min_j a_{mj}
\end{array}\right\} \max_i \min_j a_{ij} \\
\underbrace{\begin{array}{cccc}
\max_i a_{i1} & \max_i a_{i2} & \dots & \max_i a_{in}
\end{array}}_{\min\limits_j \max\limits_i a_{ij}}
\end{array}$$

Thus, in the game Γ_A with the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 5 & 3 & 8 \\ 6 & 0 & 1 \end{bmatrix}$$

the lower value (maximin) $\underline{v}$ and the maximin strategy i_0 of the first player are $\underline{v} = 3$, $i_0 = 2$, respectively, and the upper value (minimax) $\overline{v}$ and the minimax strategy j_0 of the second player are $\overline{v} = 3$, $j_0 = 2$, respectively.

Definition 6. *In the two-person zero-sum game* $\Gamma = (X, Y, K)$ *the point* (x^*, y^*) *is called an equilibrium point, or a saddle point, if*

$$K(x, y^*) \le K(x^*, y^*), \tag{3.5}$$

$$K(x^*, y) \ge K(x^*, y^*) \tag{3.6}$$

for all $x \in X$ *and* $y \in Y$.

The set of all equilibrium points in the game Γ will be denoted as

$$Z(\Gamma),\ Z(\Gamma) \subset X \times Y.$$

Definition 7. *The value of the payoff function of the I-player in an equilibrium point is called the game value and is denoted by v.*

Equilibrium situations have the following properties.

Theorem 1. *Let (x_1^*, y_1^*), (x_2^*, y_2^*) be two arbitrary saddle points in the two-person zero-sum game Γ. Then:*

1. $K(x_1^*, y_1^*) = K(x_2^*, y_2^*)$;
2. $(x_1^*, y_2^*) \in Z(\Gamma)$, $(x_2^*, y_1^*) \in Z(\Gamma)$.

Proof. From the definition of a saddle point for all $x \in X$ and $y \in Y$ we have

$$K(x, y_1^*) \le K(x_1^*, y_1^*) \le K(x_1^*, y); \tag{3.7}$$

$$K(x, y_2^*) \le K(x_2^*, y_2^*) \le K(x_2^*, y). \tag{3.8}$$

We substitute x_2^* into the left-hand side of the inequality (3.7), y_2^* into the right-hand side, x_1^* into the left-hand side of the inequality (3.8) and y_1^* into the right-hand side. Then we get

$$K(x_2^*, y_1^*) \le K(x_1^*, y_1^*) \le K(x_1^*, y_2^*) \le K(x_2^*, y_2^*) \le K(x_2^*, y_1^*).$$

From this it follows that:

$$K(x_1^*, y_1^*) = K(x_2^*, y_2^*) = K(x_2^*, y_1^*) = K(x_1^*, y_2^*). \tag{3.9}$$

Show the validity of the second statement. Consider the point (x_2^*, y_1^*). From (3.7) -(3.9), we then have

$$K(x, y_1^*) \le K(x_1^*, y_1^*) = K(x_2^*, y_1^*) = K(x_2^*, y_2^*) \le K(x_2^*, y) \tag{3.10}$$

for all $x \in X, y \in Y$. The inclusion $(x_1^*, y_2^*) \in Z(\Gamma)$ can be proved in much the same way.

From the theorem it follows that the payoff function takes the same values at all saddle points. Therefore, it is meaningful to introduce the following definition.

For any game $\Gamma = (X, Y, K)$ the following proposition takes place.

Lemma 1. *In matrix game Γ*

$$\underline{v} \ \le \ \overline{v} \tag{3.11}$$

or

$$\sup_{x \in X} \inf_{y \in Y} K(x, y) \ \le \ \inf_{y \in Y} \sup_{x \in X} K(x, y). \tag{3.12}$$

Proof. Let $x \in X$ random player strategy 1. Then we have

$$K(x, y) \le \sup_{x \in X} K(x, y).$$

Hence we get

$$\inf_{y\in Y} K(x,y) \leq \inf_{y\in Y} \sup_{x\in X} K(x,y).$$

Now we note that on the right-hand side of the last inequality constant, and the value $x \in X$ was chosen arbitrarily. Therefore, inequality

$$\sup_{x\in X} \inf_{y\in Y} K(x,y) \leq \inf_{y\in Y} \sup_{x\in X} K(x,y).$$

Theorem 2 (Necessary and sufficient conditions for the existence of the saddle point in the game $\Gamma = (X, Y, K)$)**.** *For the existence of the saddle point in the game $\Gamma = (X, Y, K)$, it is necessary and sufficient that the quantities*

$$\min_y \sup_x K(x,y), \;\; \max_x \inf_y K(x,y) \tag{3.13}$$

exist and the following equality holds:

$$\underline{v} = \max_x \inf_y K(x,y) = \min_y \sup_x K(x,y) = \overline{v}. \tag{3.13}$$

Proof.

Necessity. Let $(x^*, y^*) \in Z(\Gamma)$. Then for all $x \in X$ and $y \in Y$ the following inequality holds:

$$K(x,y^*) \leq K(x^*,y^*) \leq K(x^*,y) \tag{3.14}$$

and hence

$$\sup_x K(x,y^*) \leq K(x^*,y^*). \tag{3.15}$$

At the same time, we have

$$\inf_y \sup_x K(x,y) \leq \sup_x K(x,y^*). \tag{3.16}$$

Comparing (3.15) and (3.16) we get

$$\inf_y \sup_x K(x,y) \leq \sup_x K(x,y^*) \leq K(x^*,y^*). \tag{3.17}$$

In the similar way we get the inequality

$$K(x^*,y^*) \leq \inf_y K(x^*,y) \leq \sup_x \inf_y K(x,y). \tag{3.18}$$

On the other hand, the inverse inequality (3.12) holds. Thus, we get

$$\sup_x \inf_y K(x,y) = \inf_y \sup_x K(x,y), \tag{3.19}$$

and finally we get

$$\min_y \sup_x K(x,y) = \sup_x K(x,y^*) = K(x^*,y^*),$$

$$\max_x \inf_y K(x,y) = \inf_y K(x^*,y) = K(x^*,y^*),$$

i.e. the exterior extrema of the min sup and max inf are reached at the points y^* and x^* respectively.

Sufficiency. Suppose there exist the min sup and max inf

$$\max_x \inf_y K(x,y) = \inf_y K(x^*,y); \tag{3.20}$$

$$\min_y \sup_x K(x,y) = \sup_x K(x,y^*) \tag{3.21}$$

and the equality (3.13) holds. We shall show that (x^*,y^*) is a saddle point. Indeed,

$$K(x^*,y^*) \geq \inf_y K(x^*,y) = \max_x \inf_y K(x,y); \tag{3.22}$$

$$K(x^*,y^*) \leq \sup_x K(x,y^*) = \min_y \sup_x K(x,y). \tag{3.23}$$

By (3.13) the min sup is equal to the max inf, and from (3.22), (3.23) it follows that the min sup is also equal to the $K(x^*,y^*)$, i.e. the inequalities in (3.22), (3.23) are satisfied as equalities. Now we have

$$K(x^*,y^*) = \inf_y K(x^*,y) \leq K(x^*,y),$$

$$K(x^*,y^*) = \sup_x K(x,y^*) \geq K(x,y^*)$$

for all $x \in X$ and $y \in Y$, i.e. $(x^*,y^*) \in Z(\Gamma)$.

The proof shows that the common value of the min sup and max inf is equal to $K(x^*,y^*) = v$, the value of the game, and any min sup (max inf) strategy $y^*(x^*)$ is optimal in terms of the theorem, i.e. the point (x^*,y^*) is a saddle point.

The proof of the theorem yields the following statement.

Corollary 1. *If the* min sup *and* max inf *in (3.12) exist and are reached on* $\overline{y}$ *and* $\overline{x}$*, respectively, then*

$$\max_x \inf_y K(x,y) = K(\overline{x},\overline{y}) = \min_y \sup_x K(x,y). \tag{3.24}$$

The games, in which saddle points exist, are called *strictly determined*. Therefore, this theorem establishes the criterion for strict determination of the game and can be restated as follows. For the game to be strictly determined it is necessary and sufficient that the min sup and max inf in (3.12) exist and the equality (3.13) is satisfied.

Note that, in the game Γ_A, the extrema in (3.12) are always reached and the theorem may be reformulated in the following form.

Corollary 2. *For the* $(m \times n)$ *matrix game to be strictly determined it is necessary and sufficient that the following equalities hold*

$$\min_{j=1,2,\ldots,n} \max_{i=1,2,\ldots,m} \alpha_{ij} = \max_{i=1,2,\ldots,m} \min_{j=1,2,\ldots,n} \alpha_{ij}. \tag{3.25}$$

For example, in the game with the matrix $\begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 4 \\ 0 & -2 & 7 \end{bmatrix}$ the situation (2,1) is a saddle point. In this case

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = 2.$$

On the other hand, the game with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not have a saddle point, since

$$\min_j \max_i a_{ij} = 1 > \max_i \min_j a_{ij} = 0.$$

4 Mixed extension of a game

Definition 8. *The random variable whose values are strategies of a player is called a mixed strategy of the player.*

Thus, for the matrix game Γ_A, a mixed strategy of Player 1 is a random variable whose values are the row numbers $i \in M$, $M = \{1, 2, \ldots, m\}$. A similar definition applies to Player 2's mixed strategy whose values are the column numbers $j \in N$ of the matrix A.

Considering the above definition of *mixed strategies*, the former strategies will be referred to as *pure strategies*. Since the random variable is characterized by its distribution, the mixed strategy will be identified in what follows with the probability distribution over the set of pure strategies. Thus, Player 1's mixed strategy x in the game is the m-dimensional vector

$$x = (\xi_1, \ldots, \xi_m), \ \sum_{i=1}^{m} \xi_i = 1, \xi_i \geq 0, i = 1, \ldots, m. \tag{4.1}$$

Similarly, Player 2's mixed strategy y is the n-dimensional vector

$$y = (\eta_1, \ldots, \eta_n), \ \sum_{j=1}^{n} \eta_j = 1, \eta_j \geq 0, j = 1, \ldots, n. \tag{4.2}$$

In this case, $\xi_i \geq 0$ and $\eta_j \geq 0$ are the probabilities of choosing the pure strategies $i \in M$ and $j \in N$, respectively, when the players use mixed strategies x and y.

For example, the net strategy 1 for the first player – $x = (1, 0, \ldots, 0)$.

Definition 9. *The pair (x, y) of mixed strategies in the matrix game Γ_A is called the situation in mixed strategies.*

We shall define the payoff of Player 1 at the point (x, y) in mixed strategies for the $(m \times n)$ matrix game Γ_A as the mathematical expectation of his payoff provided that the players use mixed strategies x and y, respectively. The players choose their strategies independently; therefore the mathematical expectation of payoff $K(x, y)$ in mixed strategies $x = (\xi_1, \dots, \xi_m)$, $y = (\eta_1, \dots, \eta_n)$ is equal to

$$K(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \xi_i \eta_j = (xA)y = x(Ay) \tag{4.3}$$

The function $K(x, y)$ is continuous in $x \in X$ and $y \in Y$. Notice that when one player uses a pure strategy (i or j, respectively) and the other uses a mixed strategy (y or x), the payoffs $K(i, y), K(x, j)$ are computed by formulas

$$K(i, y) = K(u_i, y) = \sum_{j=1}^{n} a_{ij} \eta_j = a_i y, i = 1, \dots, m,$$

$$K(x, j) = K(x, w_j) = \sum_{i=1}^{m} a_{ij} \xi_i = x a^j, j = 1, \dots, n,$$

where a_i, a^j are respectively the ith row and the jth column of the $(m \times n)$ matrix A.

Thus, from the matrix game $\Gamma_A = (M, N, A)$ we have arrived at a new game $\overline{\Gamma}_A = (X, Y, K)$, where X and Y are the sets of mixed strategies in the game Γ_A and K is the payoff function in mixed strategies (mathematical expectation of the payoff). The game $\overline{\Gamma}_A$ will be called a *mixed extension* of the game Γ_A. The game Γ_A is a subgame for $\overline{\Gamma}_A$, i.e. $\Gamma_A \subset \overline{\Gamma}_A$.

Definition 10. *The point (x^*, y^*) in the game $\overline{\Gamma}_A$ forms a saddle point and the number $v = K(x^*, y^*)$ is the value of the game $\overline{\Gamma}_A$ if for all $x \in X$ and $y \in Y$*

$$K(x, y^*) \le K(x^*, y^*) \le K(x^*, y). \tag{4.4}$$

The strategies (x^*, y^*) appearing in the saddle point are called optimal. Moreover, by Theorem 2, the strategies x^* and y^* are respectively the maximin and minimax strategies, since the exterior extrema in (3.12) are reachable (the function $K(x, y)$ is continuous on the compact sets X and Y).

Lemma 2. *Let Γ_A and $\Gamma_{A'}$ be two $(m \times n)$ matrix games, where*

$$A' = \alpha A + B, \ \alpha > 0, \alpha = const,$$

and B is the matrix with the same elements β, i.e. $\beta_{ij} = \beta$ for all i and j. Then $Z(\overline{\Gamma}_{A'}) = Z(\overline{\Gamma}_A)$, $\overline{v}_{A'} = \alpha \overline{v}_A + \beta$, where $\overline{\Gamma}_{A'}$ and $\overline{\Gamma}_A$ are the mixed extensions of the games $\Gamma_{A'}$ and Γ_A, respectively, and $\overline{v}_{A'}, \overline{v}_A$ are the values of the games $\overline{\Gamma}_{A'}$ and $\overline{\Gamma}_A$.

Proof. Both matrices A and A' are of dimension $m \times n$; therefore the sets of mixed strategies in the games $\Gamma_{A'}$ and Γ_A coincide. We shall show that for any situation in mixed strategies (x, y) the following equality holds

$$K'(x, y) = \alpha K(x, y) + \beta, \tag{4.5}$$

where K' and K are Player 1's payoffs in the games $\overline{\Gamma}_{A'}$ and $\overline{\Gamma}_A$, respectively.

Indeed, for all $x \in X$ and $y \in Y$ we have

$$K'(x,y) = xA'y = \alpha(xAy) + xBy = \alpha K(x,y) + \beta.$$

From Scale Lemma it then follows that $Z(\overline{\Gamma}_{A'}) = Z(\overline{\Gamma}_A)$, $\overline{v}_{A'} = \alpha \overline{v}_A + \beta$.

Example 7. Verify that the strategies $y^* = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $x^* = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ are optimal and $v = 0$ is the value of the game $\overline{\Gamma}_A$ with matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 3 & -1 \end{bmatrix}.$$

We shall simplify the matrix A (to obtain the maximum number of zeros). Adding a unity to all elements of the matrix A, we get the matrix

$$A' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

Each element of the matrix A' can be divided by 2. The new matrix is of the form

$$A'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}.$$

By the lemma we have $v_{A''} = \frac{1}{2}v_{A'} = \frac{1}{2}(v_A + 1)$. Verify that the value of the game Γ_A is equal to 1/2. Indeed, $K(x^*, y^*) = Ay^* = 1/2$. On the other hand, for each strategy $y \in Y, y = (\eta_1, \eta_2, \eta_3)$ we have $K(x^*, y) = \frac{1}{2}\eta_1 + \frac{1}{2}\eta_2 + \frac{1}{2}\eta_3 = \frac{1}{2} \cdot 1 = \frac{1}{2}$, and for all $x = (\xi_1, \xi_2, \xi_3), x \in X$, $K(x, y^*) = \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 + \frac{1}{2}\xi_3 = \frac{1}{2}$. Consequently, the above-mentioned strategies x^*, y^* are optimal and $v_A = 0$.

In what follows, whenever the matrix game Γ_A is mentioned, we shall mean its mixed extension $\overline{\Gamma}_A$.

Theorem 3. *The main theorem of matrix games. Any matrix game has a saddle point in mixed strategies. [von Neumann, 1928].*

Proof. Let Γ_A be an arbitrary $(m \times n)$ game with a strictly positive matrix $A = \{a_{ij}\}$, i.e. $a_{ij} > 0$ for all $i = \overline{1,m}$ and $j = \overline{1,n}$. Show that in this case the theorem is true. To do this, we shall consider an auxiliary linear programming problem

$$\min xu, \ \ xA \geq w, \ \ x \geq 0 \tag{4.6}$$

and its dual problem

$$\max yw, \ \ Ay \leq u, \ \ y \geq 0, \tag{4.7}$$

where $u = (1, \ldots, 1) \in R^m$, $w = (1, \ldots, 1) \in R^n$. From the strict positivity of the matrix A it follows that there exists a vector $x > 0$ for which $xA > w$, i.e. problem (4.6) has a feasible solution. On the other hand, the vector $y = 0$ is a feasible solution of problem (4.7). And it can by easily seen that there exist a feasible solution of (4.7) y' with $|y'| > 0$. Therefore, by the duality theorem of linear programming, both problems (4.6) and (4.7) have optimal solutions $\overline{x}, \overline{y}$, respectively, and

$$\overline{x}u = \overline{y}w = \Theta > 0. \tag{4.8}$$

Consider vectors $x^* = \overline{x}/\Theta$ and $y^* = \overline{y}/\Theta$ and show that they are optimal strategies for the players 1 and 2 in the game $\overline{\Gamma}_A$, respectively and the value of the game is equal to $1/\Theta$.

Indeed, from (4.8) we have

$$x^*u = (\overline{x}u)/\Theta = (\overline{y}w)/\Theta = y^*w = 1,$$

and from feasibility of $\overline{x}$ and $\overline{y}$ for problems (4.6), (4.7), it follows that $x^* = \overline{x}/\Theta \geq 0$ and $y^* = \overline{y}/\Theta \geq 0$, i.e. x^* and y^* are the mixed strategies of players 1 and 2 in the game Γ_A.

Let us compute a payoff to Player 1 at (x^*, y^*):

$$K(x^*, y^*) = x^*Ay^* = (\overline{x}A\overline{y})/\Theta^2. \tag{4.9}$$

On the other hand, from the feasibility of vectors $\overline{x}$ and $\overline{y}$ for problems (4.6),(4.7) and equality (4.8), we have

$$\Theta = w\overline{y} \leq (\overline{x}A)\overline{y} = \overline{x}(A\overline{y}) \leq \overline{x}u = \Theta. \tag{4.10}$$

Thus, $\overline{x}A\overline{y} = \Theta$ and (4.9) implies that

$$K(x^*, y^*) = 1/\Theta. \tag{4.11}$$

Let $x \in X$ and $y \in Y$ be arbitrary mixed strategies for players 1 and 2. The following inequalities hold:

$$K(x^*, y) = (x^*A)y = (\overline{x}A)y/\Theta \geq (wy)/\Theta = 1/\Theta, \tag{4.12}$$

$$K(x, y^*) = x(Ay^*) = x(A\overline{y})/\Theta \leq \ (xu)/\Theta = 1/\Theta. \tag{4.13}$$

Comparing (4.11)–(4.13), we have that (x^*, y^*) is a saddle point and $1/\Theta$ is the value of the game Γ_A with a strictly positive matrix A.

Now consider the $(m \times n)$ game $\Gamma_{A'}$ with an arbitrary matrix $A' = \{a'_{ij}\}$. Then there exists such constant $\beta > 0$ that the matrix $A = A' + B$ is strictly positive, where $B = \{\beta_{ij}\}$ is an $(m \times n)$ matrix, $\beta_{ij} = \beta$, $i = \overline{1, m}$, $j = \overline{1, n}$. In the game Γ_A there exists a saddle point (x^*, y^*) in mixed strategies, and the value of the game equals $v_A = 1/\Theta$, where Θ is determined as in (4.8).

From Lemma 2, it follows that $(x^*, y^*) \in Z(\overline{\Gamma}_{A'})$ is a saddle point in the game $\Gamma_{A'}$ in mixed strategies and the value of the game is equal to $v_{A'} = v_A - \beta = 1/\Theta - \beta$. This completes the proof of Theorem.

Informally, the existence of a solution in the class of mixed strategies implies that, by randomizing the set of pure strategies, the players can always eliminate uncertainty in choosing their strategies they have encountered before the game starts. Note that the mixed strategy solution does not necessarily exist in zero-sum games. Examples of such games with an infinite number of strategies are given in Secs. 2.3, 2.4.

Notice that the proof of theorem is constructive, since the solution of the matrix game is reduced to a linear programming problem, and the solution algorithm for the game $\Gamma_{A'}$ is as follows.

1. By employing the matrix A', construct a strictly positive matrix $A = A' + B$, where $B = \{\beta_{ij}\}, \beta_{ij} = \beta > 0$.

2. Solve the linear programming problems (4.6), (4.7). Find vectors $\overline{x}, \overline{y}$ and a number Θ (see (4.8)).

3. Construct optimal strategies for the players 1 and 2, respectively,
$$x^* = \overline{x}/\Theta, \; y^* = \overline{y}/\Theta.$$

4. Compute the value of the game $\Gamma_{A'}$
$$v_{A'} = 1/\Theta - \beta.$$

Example 9. Consider the matrix game Γ_A determined by the matrix
$$A = \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}.$$

Associated problems of linear programming are of the form
$$\begin{array}{ll} \min \xi_1 + \xi_2, & \max \eta_1 + \eta_2, \\ 4\xi_1 + 2\xi_2 \geq 1, & 4\eta_1 \leq 1 \\ 3\xi_2 \geq 1, & 2\eta_1 + 3\eta_2 \leq 1, \\ \xi_1 \geq 0, \xi_2 \geq 0, & \eta_1 \geq 0, \eta_2 \geq 0. \end{array}$$

Note that, these problems may be written in the equivalent form with constraints in the form of equalities
$$\begin{array}{ll} \min \xi_1 + \xi_2, & \max \eta_1 + \eta_2, \\ 4\xi_1 + 2\xi_2 - \xi_3 = 1, & 4\eta_1 + \eta_3 = 1 \\ 3\xi_2 - \xi_4 = 1, & 2\eta_1 + 3\eta_2 + \eta_4 = 1, \\ \xi_1 \geq 0, \xi_2 \geq 0, \xi_3 \geq 0, \xi_4 \geq 0, & \eta_1 \geq 0, \eta_2 \geq 0, \eta_3 \geq 0, \eta_4 \geq 0. \end{array}$$

Thus, any method of solving the linear programming problems can be used to solve the matrix games. The simplex method is most commonly used to solve such problems. Its systematic discussion may be found in [Ashmanov (1981)], [Gale (1960)], [Hu (1970)].

In a sense, the linear programming problem is equivalent to the matrix game Γ_A. Indeed, consider the following direct and dual problems of linear programming
$$\begin{array}{l} \min xu \\ xA \geq w, \\ x \geq 0, \end{array} \tag{4.14}$$

$$
\begin{aligned}
&\max yw \\
&Ay \le u, \\
&y \ge 0.
\end{aligned} \tag{4.15}
$$

Let $\overline{X}$ and $\overline{Y}$ be the sets of optimal solutions of the problems (4.14) and (4.15), respectively. Denote $(1/\Theta)\overline{X} = \{\overline{x}/\Theta \,|\overline{x} \in X\}$, $(1/\Theta)\overline{Y} = \{\overline{y}/\Theta \,|\overline{y} \in Y\}$, $\Theta > 0$.

Theorem 4. *Let Γ_A be the $(m \times n)$ game with the positive matrix A (all elements are positive) and let there be given two dual problems of linear programming (4.14) and (4.15). Then the following statements hold.*

1. *Both linear programming problems have a solution ($\overline{X} \neq \oslash$ and $\overline{Y} \neq \oslash$), in which case*
$$\Theta = \min_x xu = \max_y yw.$$

2. *The value v_A of the game Γ_A is*
$$v_A = 1/\Theta,$$
and the strategies
$$x^* = \overline{x}/\Theta,\ \ y^* = \overline{y}/\Theta$$
are optimal, where $\overline{x} \in \overline{X}$ is an optimal solution of the direct problem (4.14) and $\overline{y} \in \overline{Y}$ is the solution of the dual problem (4.15).

3. *Any optimal strategies $x^* \in X^*$ and $y^* \in Y^*$ of the players can be constructed as shown above, i.e.*

$$X^* = (1/\Theta)\overline{X},\ \ Y^* = (1/\Theta)\overline{Y}.$$

Proof. Statements 1, 2 and inclusions $(1/\Theta)\overline{X} \subset X^*$, $(1/\Theta)\overline{Y} \subset Y^*$, immediately follow from the proof of Theorem 3.

Show the inverse inclusion. To do this, consider the vectors $x^* = (\xi_1^*, \dots, \xi_m^*) \in X^*$ and $\overline{x} = (\overline{\xi}_1, \dots, \overline{\xi}_m)$, where $\overline{x} = \Theta x^*$. Then for all $j \in N$ we have

$$\overline{x}a^j = \Theta x^* a^j \ge \Theta(1/\Theta) = 1,$$

in which case $\overline{x} \ge 0$, since $\Theta > 0$ and $x^* \ge 0$. Therefore $\overline{x}$ is a feasible solution to problem (4.14).

Let us compute the value of the objective function

$$\overline{x}u = \Theta x^* u = \Theta = \min_x xu,$$

i.e. $\overline{x} \in \overline{X}$ is an optimal solution to problem (4.14).

The inclusion $Y^* \subset (1/\Theta)\overline{Y}$ can be proved in a similar manner. This completes the proof of the theorem.

5 Dominance of strategies

The complexity of solving a matrix game increases as the dimensions of the matrix A increase. In some cases, however, the analysis of payoff matrices permits a conclusion that some pure strategies do not appear in the spectrum of optimal strategy. This can result in replacement of the original matrix by the payoff matrix of a smaller dimension.

Definition 11. *Strategy x' of Player 1 is said to dominate strategy x'' in the $(m \times n)$ game Γ_A if the following inequalities hold for all pure strategies $j \in \{1, \dots, n\}$ of Player 2*

$$x'a^j \geq x''a^j. \tag{5.1}$$

Similarly, strategy y' of Player 2 dominates his strategy y'' if for all pure strategies $i \in \{1, \dots, m\}$ of Player 1

$$a_i y' \leq a_i y''. \tag{5.2}$$

If inequalities (5.1), (5.2) are satisfied as strict inequalities, then we are dealing with a *strict dominance.* A special case of the dominance of strategies is their equivalence.

Definition 12. *Strategies x' and x'' of Player 1 are equivalent in the game Γ_A if for all $j \in \{1, \dots, n\}$*

$$x'a^j = x''a^j.$$

We shall denote this fact by $x' \sim x''$.

For two equivalent strategies x' and x'' the following equality holds (for every $y \in Y$)

$$K(x', y) = K(x'', y).$$

Similarly, strategies y' and y'' of Player 2 are equivalent $(y' \sim y'')$ in the game Γ_A if for all $i \in \{1, \dots, m\}$

$$y'a_i = y''a_i.$$

Hence we have that for any mixed strategy $x \in X$ of Player 1 the following equality holds

$$K(x, y') = K(x, y'').$$

For pure strategies the above definitions are transformed as follows. If Player 1's pure strategy i' dominates strategy i'' and Player 2's pure strategy j' dominates strategy j'' of the same player, then for all $i = 1, \dots, m$; $j = 1, \dots, n$ the following inequalities hold

$$a_{i'j} \geq a_{i''j}, \;\; a_{ij'} \leq a_{ij''}.$$

Definition 13. *The strategy* $x''(y'')$ *of Player 1(2) is dominated if there exists a strategy* $x' \neq x''$ $(y' \neq y'')$ *of this player which dominates* $x''(y'')$*; otherwise strategy* $x''(y'')$ *is an undominated strategy.*

Similarly, strategy $x''(y'')$ of Player 1(2) is *strictly dominated* if there exists a strategy $x'(y')$ of this player which strictly dominates $x''(y'')$, i.e. for all $j = \overline{1,n}(i = \overline{1,m})$ the following inequalities hold

$$x'a^j > x''a^j, \ a_iy' < a_iy'';$$

otherwise strategy $x''(y'')$ of Player 1(2) is not strictly dominated.

Show that players playing optimally do not use dominated strategies. This establishes the following assertion.

Theorem 5. *If, in the game* $\overline{\Gamma}_A$*, strategy* x' *of one of the players dominates an optimal strategy* x^**, then strategy* x' *is also optimal.*

Proof. Let x' and x^* be strategies of Player 1. Then, by dominance,

$$x'a^j \geq x^*a^j$$

for all $j = \overline{1,n}$. Hence, using the optimality of strategy x^*, we get

$$v_A = \min_j x^*a^j \geq \min_j x'a^j \geq \ \min_j x^*a^j = v_A$$

for all $j = \overline{1,n}$. Therefore, by Theorem 11, strategy x' is also optimal.

Thus, an optimal strategy can be dominated only by another optimal strategy. On the other hand, no optimal strategy is strictly dominated; hence the players when playing optimally must not use strictly dominated strategies.

Theorem 6. *If, in the game* $\overline{\Gamma}_A$*, strategy* x^* *of one of the players is optimal, then strategy* x^* *is not strictly dominated.*

Proof. For definiteness, let x^* be an optimal strategy of Player 1. Assume that x^* is strictly dominated, i.e. there exist such strategy $x' \in X$ that

$$x'a^j > x^*a^j, \ j = 1, 2, \ldots, n.$$

Hence

$$\min_j x'a^j > \min_j x^*a^j.$$

However, by the optimality of $x^* \in X$, the equality $\min\limits_j x^*a^j = v_A$ is satisfied. Therefore, the strict inequality

$$\max_x \min_j xa^j > v_A$$

holds and this contradicts to the fact that v_A is the value of the game. The contradiction proves the theorem.

It is clear that the reverse assertion is generally not true. Thus, in the game with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ the first and second strategies of Player 1 are not strictly dominated, although they are not optimal.

On the other hand, it is intuitively clear that if the ith row of the matrix A (the jth column) is dominated, then there is no need to assign positive probability to it. Thus, in order to find optimal strategies instead of the game Γ_A, it suffices to solve a subgame $\Gamma_{A'}$, where A' is the matrix obtained from the matrix A by deleting the dominated rows and columns.

Before proceeding to a precise formulation and proof of this result, we will introduce the notion of an *extension of mixed strategy x at the ith place.* If $x = (\xi_1, \ldots, \xi_m) \in X$ and $1 \le i \le m+1$, then the extension of strategy x at the ith place is called the vector $\overline{x}_i = (\xi_1, \ldots, \xi_{i-1}, 0, \xi_i, \ldots, \xi_m) \in R^{m+1}$. Thus the extension of vector $(1/3, 2/3, 1/3)$ at the 2nd place is the vector $(1/3, 0, 2/3, 1/3)$; the extension at the 4th place is the vector $(1/3, 2/3, 1/3, 0)$; the extension at the 1st place is the vector $(0, 1/3, 2/3, 1/3)$.

Theorem 7. *Let Γ_A be an $(m \times n)$ game. We assume that the ith row of matrix A is dominated (i.e. Player 1's pure strategy i is dominated) and let $\Gamma_{A'}$ be the game with the matrix A' obtained from A by deleting the ith row. Then the following assertions hold.*

1. $v_A = v_{A'}$.
2. *Any optimal strategy y^* of Player 2 in the game $\Gamma_{A'}$ is also optimal in the game Γ_A.*
3. *If x^* is an arbitrary optimal strategy of Player 1 in the game $\Gamma_{A'}$ and $\overline{x}^*_i$ is the extension of strategy x^* at the ith place, then $\overline{x}^*_i$ is an optimal strategy of that player in the game Γ_A.*
4. *If the ith row of the matrix A is strictly dominated, then an arbitrary optimal strategy $\overline{x}^*$ of Player 1 in the game Γ_A can be obtained from an optimal strategy x^* in the game $\Gamma_{A'}$ by the extension at the ith place.*

Proof. Without loss of generality, we may assume, that the last mth row is dominated. Let $x = (\xi_1, \ldots, \xi_m)$ be a mixed strategy which dominates the row m. If $\xi_m = 0$, then from the dominance condition for all $j = 1, 2, \ldots, n$ we get

$$\sum_{i=1}^{m} \xi_i \alpha_{ij} = \sum_{i=1}^{m-1} \xi_i \alpha_{ij} \ge \alpha_{mj},$$

$$\sum_{i=1}^{m-1} \xi_i = 1,\ \xi_i \ge 0,\ i = 1, \ldots, m-1. \tag{5.3}$$

Otherwise ($\xi_m > 0$), consider the vector $x' = (\xi'_1, \ldots, \xi'_m)$, where

$$\xi'_i = \begin{cases} \xi_i/(1-\xi_m), & i \ne m, \\ 0, & i = m. \end{cases} \tag{5.4}$$

Components of the vector x are non-negative, $(\xi_i' \geq 0, i = 1, \dots, m)$ and $\sum_{i=1}^{m} \xi_i' = 1$. On the other hand, for all $i = 1, \dots, n$ we have

$$\frac{1}{1-\xi_m} \sum_{i=1}^{m} \xi_i \alpha_{ij} \geq \alpha_{mj} \frac{1}{1-\xi_m} \sum_{i=1}^{m} \xi_i$$

or

$$\frac{1}{1-\xi_m} \sum_{i=1}^{m-1} \xi_i \alpha_{ij} \geq \alpha_{mj} \frac{1}{1-\xi_m} \sum_{i=1}^{m-1} \xi_i.$$

Considering (5.4) we get

$$\sum_{i=1}^{m-1} \xi_i' \alpha_{ij} \geq \alpha_{mj} \sum_{i=1}^{m-1} \xi_i' = \alpha_{mj},\ j = 1, \dots, n,$$

$$\sum_{i=1}^{m-1} \xi_i' = 1,\ \xi_i' \geq 0,\ i = 1, \dots, m-1. \tag{5.5}$$

Thus, from the dominance of the mth row it always follows that it does not exceed a convex linear combination of the remaining $m-1$ rows (see 5.5).

Let $(x^*, y^*) \in Z(\Gamma_{A'})$ be a saddle point in the game $\Gamma_{A'}$, $x^* = (\xi_1^*, \dots, \xi_{m-1}^*)$, $y^* = (\eta_1^*, \dots, \eta_n^*)$. To prove assertions 1, 2 and 3 of the theorem, it suffices to show that $K(x_m^*, y^*) = v_{A'}$ and

$$\sum_{j=1}^{n} \alpha_{ij} \eta_j^* \leq v_{A'} \leq \sum_{i=1}^{m-1} \alpha_{ij} \xi_i^* + 0 \cdot \alpha_{mj} \tag{5.6}$$

for all $i = 1, \dots, m,\ j = 1, \dots, n.$

The following theorem is presented without the proof.

Theorem 8. *Let Γ_A be an $(m \times n)$ game. Assume that the jth column of the matrix A is dominated and $\Gamma_{A'}$ is the game having the matrix A' obtained from A by deleting the jth column. Then the following assertions are true.*

1. $v_A = v_{A'}$.

2. *Any optimal strategy x^* of Player 1 in the game $\Gamma_{A'}$ is also optimal in the game Γ_A.*

3. *If y^* is an arbitrary optimal strategy of Player 2 in the game $\Gamma_{A''}$ and $\overline{y}_j^*$ is the extension of strategy y at the jth place, then $\overline{y}_j^*$ is an optimal strategy of Player 2 in the game Γ_A.*

4. *Further, if the jth column of the matrix A is strictly dominated, then an arbitrary optimal strategy $\overline{y}^*$ of Player 2 in the game Γ_A can be obtained from an optimal strategy y^* in the game $\Gamma_{A'}$ by extension at the jth place.*

To summarize: The theorems yield an algorithm for reducing the dimension of a matrix game. Thus, if the matrix row (column) is not greater (not smaller) than a convex linear combination of the remaining rows (columns) of the matrix, then to find a solution of the game, this row (column) can be deleted. In this case, an extension of optimal strategy in the truncated matrix game yields an optimal solution of the original game. If the inequalities are satisfied as strict inequalities, the set of optimal strategies in the original game can be obtained by extending the set of optimal strategies in the truncated game; otherwise this procedure may cause a loss of optimal strategies. An application of these theorems is illustrated by the following example.

Example 10. Consider the game with the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 5 & 3 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 7 & 3 & 0 & 6 \end{bmatrix}.$$

Since the 3rd row a_3 dominates the 1st row ($a_3 \geq a_1$), then, by deleting the 1st row, we obtain

$$A_1 = \begin{bmatrix} 5 & 3 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 7 & 3 & 0 & 6 \end{bmatrix}.$$

In this matrix the 1st column a^3 dominates the 3rd column a^1. Hence we get

$$A_2 = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 6 \end{bmatrix}.$$

In the latter matrix no row (column) is dominated by the other row (column). At the same time, the 1st column a^1 is dominated by the convex linear combination of columns a^2 and a^3, i.e. $a^1 \geq 1/2a^2 + 1/2a^3$, since $3 > 1/2 + 1/2 \cdot 3$, $1 = 1/2 \cdot 2 + 1/2 \cdot 0$, $3 = 0 \cdot 1/2 + 1/2 \cdot 6$. By eliminating the 1st column, we obtain

$$A_3 = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 6 \end{bmatrix}.$$

In this matrix the 1st row is equal to the linear convex combination of the second and third rows with a mixed strategy $x = (0, 1/2, 1/2)$, since $1 = 1/2 \cdot 2 + 0 \cdot 1/2$, $3 = 0 \cdot 1/2 + 6 \cdot 1/2$. Thus, by eliminating the 1st row, we obtain the matrix

$$A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}.$$

The players' optimal strategies x^* and y^* in the game with this matrix are $x^* = y^* = (3/4, 1/4)$, in which case the game value v is $3/2$.

The latter matrix is obtained by deleting the first two rows and columns; hence the players' optimal strategies in the original game are extensions of these strategies at the 1st and 2nd places, i.e. $\overline{x}^*_{12} = \overline{y}^*_{12} = (0, 0, 3/4, 1/4)$.

6 Properties of optimal strategies and value of the game

Consider the properties of optimal strategies which, in some cases, assist in finding the value of the game and a saddle point.

Let $(x^*, y^*) \in X \times Y$ be a saddle point in mixed strategies for the game Γ_A. It turns out that, to test the point (x^*, y^*) for a saddle, it will suffice to test the inequalities (4.4) only for $i \in M$ and $j \in N$, not for all $x \in X$ and $y \in Y$, since the following statement is true.

Theorem 9. *For the situation (x^*, y^*) to be an equilibrium (saddle point) in the game Γ_A, and the number $v = K(x^*, y^*)$ be the value, it is necessary and sufficient that the following inequalities hold for all $i \in M$ and $j \in N$:*

$$K(i, y^*) \leq K(x^*, y^*) \leq K(x^*, j). \tag{6.1}$$

Proof. Necessity. Let (x^*, y^*) be a saddle point in the game Γ_A. Then

$$K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y)$$

for all $x \in X$, $y \in Y$. Hence, in particular, for $u_i \in X$ and $w_j \in Y$ we have

$$K(i, y^*) = K(u_i, y^*) \leq K(x^*, y^*) \leq K(x^*, w_j) = K(x^*, j)$$

for all $i \in M$ and $j \in N$.

Sufficiency. Let (x^*, y^*) be a pair of mixed strategies for which the inequalities (6.1) hold. Also, let $x = (\xi_1, \ldots, \xi_m) \in X$ and $y = (\eta_1, \ldots, \eta_n) \in Y$ be arbitrary mixed strategies for the players 1 and 2, respectively. Multiplying the first and second inequalities (6.1) by ξ_i and η_j, respectively, and summing, we get

$$\sum_{i=1}^{m} \xi_i K(i, y^*) \leq K(x^*, y^*) \sum_{i=1}^{m} \xi_i = K(x^*, y^*), \tag{6.2}$$

$$\sum_{j=1}^{n} \eta_j K(x^*, j) \geq K(x^*, y^*) \sum_{j=1}^{n} \eta_j = K(x^*, y^*). \tag{6.3}$$

In this case we have

$$\sum_{i=1}^{m} \xi_i K(i, y^*) = K(x, y^*), \tag{6.4}$$

$$\sum_{j=1}^{n} \eta_j K(x^*, j) = K(x^*, y). \tag{6.5}$$

Substituting (6.4), (6.5) into (6.2) and (6.3), respectively, and taking into account the arbitrariness of strategies $x \in X$ and $y \in Y$, we obtain saddle point conditions for the pair of mixed strategies (x^*, y^*).

Corollary 3. *Let* (i^*, j^*) *be a saddle point in the game* Γ_A*. Then the situation* (i^*, j^*) *is also a saddle point in the game* $\overline{\Gamma}_A$*.*

Example 11. Solution of the Evasion-type Game. Suppose the players select integers i and j between 1 and n, and Player 1 wins the amount $a_{ij} = |i - j|$, i.e. the distance between the numbers i and j.

Suppose the first player uses strategy $x^* = (1/2, 0, \ldots, 0, 1/2)$. Then

$$K(x^*, j) = 1/2|1 - j| + 1/2|n - j| = 1/2(j - 1) + 1/2(n - j) = (n - 1)/2$$

for all $1 \leq j \leq n$.

a) Let $n = 2k+1$ be odd. Then Player 2 has a pure strategy $j^* = (n+1)/2 = k + 1$ such that

$$a_{ij^*} = |i - (n + 1)/2| = |i - k - 1| \leq k = (n - 1)/2$$

for all $i = 1, 2, \ldots, n$.

b) Let $n = 2k$ be even. Then Player 2 has a strategy
$y^* = (0, 0, \ldots, 1/2, 1/2, 0, \ldots, 0)$, where $\eta^*_k = 1/2$, $\eta^*_{k+1} = 1/2$, $\eta^*_j = 0$, $j \neq k+1$, $j \neq k$, and

$$K(j, y^*) = 1/2|i - k| + 1/2|i - k - 1| \leq 1/2k + 1/2(k - 1) = (n - 1)/2$$

for all $1 \leq i \leq n$.

It can be easily seen that the value of the game is $v = (n - 1)/2$, Player 1 has optimal strategy x^*, and Player 2's optimal strategy is j^* if $n = 2k + 1$, and y^* if $n = 2k$.

Theorem 10. *Let* Γ_A *be an* $(m \times n)$ *game. For the situation in mixed strategies, let* (x^*, y^*) *be an equilibrium (saddle point) in the game* $\overline{\Gamma}_A$*, it is necessary and sufficient that the following equality holds*

$$\max_{1 \leq i \leq m} K(i, y^*) = \min_{1 \leq j \leq n} K(x^*, j). \tag{6.6}$$

Proof.

Necessity. If (x^*, y^*) is a saddle point, then, by Theorem 9, we have

$$K(i, y^*) \leq K(x^*, y^*) \leq K(x^*, j)$$

for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. Therefore

$$K(i, y^*) \leq K(x^*, j)$$

for each i and j. Suppose the opposite is true, i.e. (6.6) is not satisfied. Then

$$\max_{1 \leq i \leq m} K(i, y^*) < \min_{1 \leq j \leq n} K(x^*, j).$$

Consequently, the following inequalities hold

$$K(x^*,y^*)=\sum_{i=1}^{m}\xi_i^* K(i,y^*)\le \max_{1\le i\le m} K(i,y^*)<\min_{1\le j\le n} K(x^*,j)$$

$$\le \sum_{j=1}^{n}\eta_j^* K(x^*,j)=K(x^*,y^*).$$

The obtained contradiction proves the necessity of the Theorem assertion.

Sufficiency. Let a pair of mixed strategies $(\tilde{x},\tilde{y})$ be such that $\max\limits_i K(i,\tilde{y})=\min\limits_j K(\tilde{x},j)$. Show that in this case $(\tilde{x},\tilde{y})$ is a saddle point in the game $\overline{\Gamma}_A$.

The following relations hold

$$\min_{1\le j\le n} K(\tilde{x},j)\le \sum_{j=1}^{n}\tilde{\eta}_j K(\tilde{x},j)=K(\tilde{x},\tilde{y})$$

$$=\sum_{i=1}^{m}\tilde{\xi}_i K(i,\tilde{y})\le \max_{1\le i\le m} K(i,\tilde{y}).$$

Hence we have

$$K(i,\tilde{y})\le \max_{1\le i\le m} K(i,\tilde{y})=K(\tilde{x},\tilde{y})=\min_{1\le j\le n} K(\tilde{x},j)\le K(\tilde{x},j)$$

for all $1\le i\le m$ and $1\le j\le n$, then, by the Theorem 9, $(\tilde{x},\tilde{y})$ is the saddle point in the game $\overline{\Gamma}_A$.

From the proof it follows that any one of the numbers in (6.6) is the value of the game.

Theorem 11. *The following relation holds for the matrix game* Γ_A

$$\max_x \min_j K(x,j)=v_A=\min_y \max_i K(i,y), \tag{6.7}$$

in which case the extrema are achieved on the players' optimal strategies.

This theorem follows from the Theorems 2 and 10, and its proof is left to the reader.

Theorem 12. *In the matrix game* Γ_A *the players' sets of optimal mixed strategies* X^* *and* Y^* *are convex polyhedra.*

Proof. By Theorem 9, the set X^* is the set of all solutions of the system of inequalities

$$xa^j\ge v_A,\ j\in N,$$

$$xu=1,$$

$$x\ge 0,$$

where $u = (1, \ldots, 1) \in R^m$, v_A is the value of the game. Thus, X^* is a convex polyhedral set. On the other hand, $X^* \subset X$, where X is a convex polyhedron. Therefore X^* is bounded and a convex polyhedron.

In a similar manner, it may be proved that Y^* is a convex polyhedron.

As an application of Theorem 11, we shall provide a geometric solution to the games with two strategies for one of the players $(2 \times n)$ and $(m \times 2)$ games. This method is based on the property that the optimal strategies x^* and y^* deliver exterior extrema in the equality

$$v_A = \max_x \min_j K(x, j) = \min_y \max_i K(i, y).$$

Example 12. *$(2 \times n)$ game.* We shall examine the game in which Player 1 has two strategies and Player 2 has n strategies. The matrix is of the form

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \end{bmatrix}.$$

Suppose Player 1 chooses mixed strategy $x = (\xi, 1 - \xi)$ and Player 2 chooses pure strategy $j \in N$. Then a payoff to Player 1 at (x, j) is

$$K(x, j) = \xi\alpha_{1j} + (1 - \xi)\alpha_{2j}. \tag{6.8}$$

Geometrically, the payoff is a straight line segment with coordinates (ξ, K). Accordingly, to each pure strategy j corresponds a straight line. The graph of the function

$$H(\xi) = \min_j K(x, j)$$

is the lower envelope of the family of straight lines (6.8). This function is concave as the lower envelope of the family of concave (linear in the case) function. The point ξ^*, at which the maximum of the function $H(\xi)$ is achieved with respect to $\xi \in [0, 1]$, yields the required optimal solution $x^* = (\xi^*, 1 - \xi^*)$ and the value of the game $v_A = H(\xi^*)$.

For definiteness, we shall consider the game with the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 1 & 4 & 0 \end{bmatrix}.$$

For each $j = 1, 2, 3, 4$ we have: $K(x, 1) = -\xi + 2$, $K(x, 2) = 2\xi + 1$, $K(x, 3) = -3\xi + 4$, $K(x, 4) = 4\xi$. The lower envelope $N(\xi)$ of the family of straight lines $\{K(x, j)\}$ and the lines themselves, $K(x, j)$, $j = 1, 2, 3, 4$ are shown in Fig. 1.1. The maximum $H(\xi^*)$ of the function $H(\xi)$ is found as the intersection of the first and the fourth lines. Thus, ξ^* is a solution of the equation.

$$4\xi^* = -\xi^* + 2 = v_A.$$

Hence we get the optimal strategy $x^* = (2/5, 3/5)$ of Player 1 and the value of the game is $v_A = 8/5$. Player 2's optimal strategy is found from the following reasonings. Note that in the case studied $K(x^*, 1) = K(x^*, 4) = v_A = 8/5$.

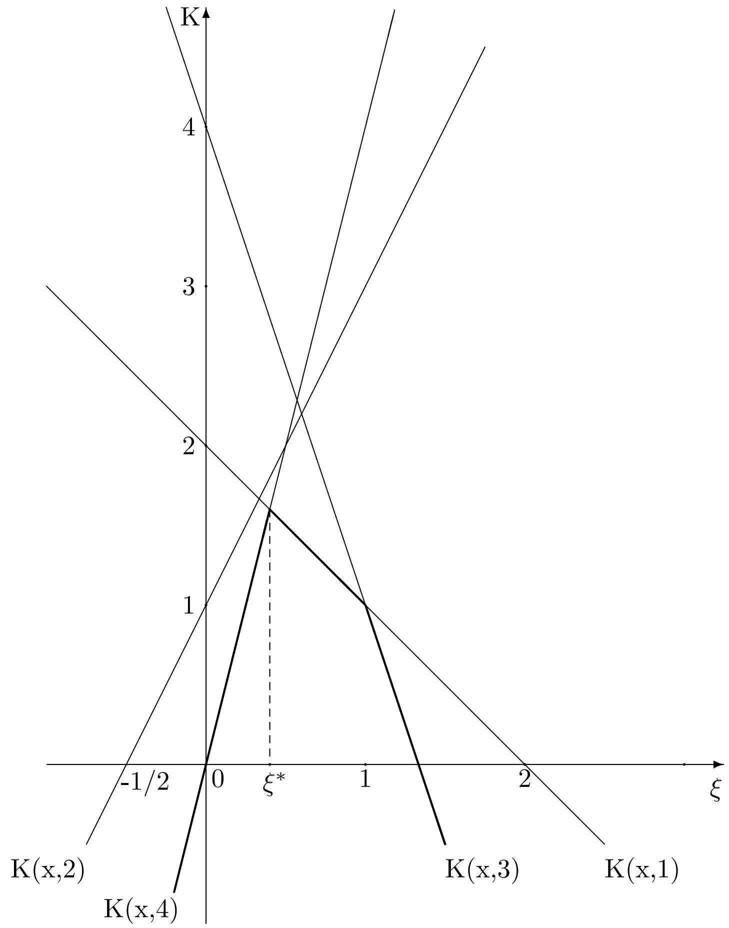

Figure 1:

For the optimal strategy $y^* = (\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*)$ the following equality must hold

$$v_A = K(x^*, y^*) = \eta_1^* K(x^*, 1) + \eta_2^* K(x^*, 2) + \eta_3^* K(x^*, 3) + \eta_4^* K(x^*, 4).$$

In this case $K(x^*, 2) > 8/5$, $K(x^*, 3) > 8/5$; therefore $\eta_2^* = \eta_3^* = 0$, and η_1^*, η_4^* can be found from the conditions

$$\eta_1^* + 4\eta_4^* = 8/5,$$

$$2\eta_1^* = 8/5.$$

Thus, $\eta_1^* = 4/5$, $\eta_4^* = 1/5$ and the optimal strategy of Player 2 is $y^* = (4/5, 0, 0, 1/5)$.

Example 13. $(m \times 2)$ *game.* In this example, Player 2 has two strategies and Player 1 has m strategies. The matrix A is of the form

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \dots & \dots \\ \alpha_{m1} & \alpha_{m2} \end{bmatrix}.$$

This game can be analyzed in a similar manner. Indeed, let $y = (\eta, 1-\eta)$ be an arbitrary mixed strategy of Player 2. Then Player 1's payoff in situation (i, y) is

$$K(i, y) = \alpha_{i1}\eta + \alpha_{i2}(1-\eta) = (\alpha_{i1} - \alpha_{i2})\eta + \alpha_{i2}.$$

The graph of the function $K(i, y)$ is a straight line. Consider the upper envelope of these straight lines, i.e. the function

$$H(\eta) = \max_i [(\alpha_{i1} - \alpha_{i2})\eta + \alpha_{i2}].$$

The function $H(\eta)$ is convex (as the upper envelope of the family of convex functions).

The point of minimum η^* of the function $H(\eta)$ yields the optimal strategy $y^* = (\eta^*, 1-\eta^*)$ and the value of the game is $v_A = H(\eta^*) = \min\limits_{\eta \in [0,1]} H(\eta)$.

We shall provide a theorem that is useful in finding a solution of the game.

Theorem 13. *Let $x^* = (\xi_1^*, \dots, \xi_m^*)$ and $y^* = (\eta_1^*, \dots, \eta_n^*)$ be optimal strategies in the game $\overline{\Gamma}_A$ and v_A be the value of the game. Then for any i, for which $K(i, y^*) < v_A$, there must be $\xi_i^* = 0$, and for any j such that $v_A < K(x^*, j)$ there must be $\eta_j^* = 0$.*

Conversely, if $\xi_i^* > 0$, then $K(i, y^*) = v_A$, and if $\eta_j^* > 0$, then $K(x^*, j) = v_A$.

Proof. Suppose that for some $i_0 \in M$, $K(i_0, y^*) < v_A$ and $\xi_{i_0}^* \neq 0$. Then we have

$$K(i_0, y^*)\xi_{i_0}^* < v_A \xi_{i_0}^*.$$

For all $i \in M$, $K(i, y^*) \leq v_A$, therefore

$$K(i, y^*)\xi_i^* \leq v_A \xi_i^*.$$

Consequently, $K(x^*, y^*) < v_A$, which contradicts to the fact that v_A is the value of the game. The second part of the Theorem can be proved in a similar manner.

This result is a counterpart of the complementary stackness theorem [Hu (1970)] or, as it is sometimes called the canonical equilibrium theorem for the linear programming problem [Gale (1960)].

Definition 14. *Player 1's (2's) pure strategy $i \in M (j \in N)$ is called an essential or active strategy if there exists the player's optimal strategy $x^* = (\xi_1^*, \dots, \xi_m^*)$ $(y^* = (\eta_1^*, \dots, \eta_n^*))$ for which $\xi_i^* > 0$ $(\eta_j^* > 0)$.*

From the definition, and from the latter theorem, it follows that for each essential strategy i of Player 1 and any optimal strategy $y^* \in Y^*$ of Player 2 in the game Γ_A the following equality holds:

$$K(i, y^*) = a_i y^* = v_A.$$

A similar equality holds for any essential strategy $j \in N$ of Player 2 and for the optimal strategy $x^* \in X^*$ of Player 1

$$K(x^*, j) = a^j x^* = v_A.$$

If the equality $a_i y = v_A$ holds for the pure strategy $i \in M$ and mixed strategy $y \in Y$, then the strategy i is the best reply to the mixed strategy y in the game Γ_A.

Thus, using this terminology, the theorem can be restated as follows. If the pure strategy of the player is essential, then it is the best reply to any optimal strategy of the opponent.

A knowledge of the optimal strategy spectrum simplifies to finding a solution of the game. Indeed, let M_{X^*} be the spectrum of Player 1's optimal strategy x^*. Then each optimal strategy $y^* = (\eta_1^*, \dots, \eta_n^*)$ of Player 2 and the value of the game v satisfy the system of inequalities

$$a_i y^* = v,\ i \in M_{x^*},$$

$$a_i y^* \le v,\ i \in M \setminus M_{x^*},$$

$$\sum_{j=1}^{n} \eta_j^* = 1,\ \eta_j^* \ge 0,\ j \in N.$$

Thus, only essential strategies may appear in the spectrum M_{x^*} of any optimal strategy x^*.

7 Methods for solving matrix games

Analytical solution method

Example 14. Consider the analytical solution of Attack and Defence game Let us consider the game with the $(n \times n)$ matrix A.

$$A = \begin{bmatrix} \beta_1\tau_1 & \tau_1 & \dots & \tau_1 \\ \tau_2 & \beta_2\tau_2 & \dots & \tau_2 \\ \dots & \dots & \dots & \dots \\ \tau_n & \tau_n & \dots & \beta_n\tau_n \end{bmatrix}.$$

Here $\tau_i > 0$ is the value and $0 < \beta_i < 1$ is the probability of hitting the target $C_i, i = 1, 2, \dots, n$ provided that it is defended.

Let $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n$. We shall define the function φ, of integers $1, 2, \ldots, n$ as follows:

$$\varphi(k) = \Big\{\sum_{i=k}^{n}(1-\beta_i)^{-1} - 1\Big\}/\sum_{i=k}^{n}(\tau_i(1-\beta_i))^{-1} \tag{7.1}$$

and let $l \in \{1, 2, \ldots, n\}$ be an integer which maximize the function $\varphi(k)$, i.e.

$$\varphi(l) = \max_{k=1,2,\ldots,n} \varphi(k). \tag{7.2}$$

We shall establish properties of the function $\varphi(k)$. Denote by R one of the signs of the order relation $\{>, =, <\}$. In this case

$$\varphi(k) R \varphi(k+1) \tag{7.3}$$

if and only if

$$\tau_k R \varphi(k),\ k = 1, 2, \ldots, n-1,\ \tau_0 \equiv 0. \tag{7.4}$$

Indeed, from (7.1) we obtain

$$\frac{\varphi(k)}{\tau_k}\frac{(1-\beta_k)^{-1}}{\sum_{i=k+1}^{n}(\tau_i(1-\beta_i))^{-1}} + \varphi(k) = \varphi(k+1)$$

$$+\frac{(1-\beta_k)^{-1}}{\sum_{i=k+1}^{n}(\tau_i(1-\beta_i))^{-1}}.$$

Then we have

$$\left(\frac{\varphi(k)}{\tau_k} - 1\right)\frac{(1-\beta_k)^{-1}}{\sum_{i=k+1}^{n}(\tau_i(1-\beta_i))^{-1}} + \varphi(k) = \varphi(k+1). \tag{7.5}$$

Note that the coefficient in (7.5) placed after brackets, is positive. Therefore, from (7.5) we obtained equivalence of relations (7.3) and (7.4).

Now, since $\varphi(l) \geq \varphi(l-1)$ or $\varphi(l) \geq \varphi(l+1)$, (in this case $\tau_{l-1} \leq \varphi(l-1)$ or $\tau_l \geq \varphi(l)$), then from relations (7.2), (7.3) we have

$$\tau_{l-1} \leq \varphi(l) \leq \tau_l. \tag{7.6}$$

Find optimal strategies in the game Γ_A. Recall that we have inequalities $\tau_1 \leq \tau_2 \leq \ldots \tau_n$. Then the optimal strategies $x^* = (\xi_1^*, \ldots, \xi_n^*)$ and $y^* = (\eta_1^*, \ldots, \eta_n^*)$ for players 1 and 2 respectively, are as follows:

$$\xi_i^* = \begin{cases} 0,\ i = 1, \ldots, l-1, \\ (\tau_i(1-\beta_i))^{-1}/\sum_{j=l}^{n}(\tau_j(1-\beta_j))^{-1},\ i = l, \ldots, n, \end{cases} \tag{7.7}$$

$$\eta_j^* = \begin{cases} 0,\ j = 1, \ldots, l-1, \\ (\tau_j - \varphi(l))/(\tau_j(1-\beta_j)),\ j = l, \ldots, n, \end{cases} \tag{7.8}$$

and the value of the game is

$$v_A = \varphi(l).$$

We have that $\xi_i^* \geq 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^n \xi_i^* = 1$. From the definition of $\varphi(l)$ and (7.6) we have that $\eta_j^* \geq 0$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n \eta_j^* = 1$.

Let $K(x^*, j)$ be a payoff of Player 1 at (x^*, j). Similarly, let $K(i, y^*)$ be a payoff at (i, y^*). Substituting (7.7), (7.8) into the payoff function and using the assumption that the values of targets do not decrease, and using (7.6), we obtain

$$K(x^*, j) = \begin{cases} \displaystyle\sum_{i=l}^{n} \tau_i \xi_i^* = \varphi(l) + \sum_{j=l}^{n} (\tau_j(1-\beta_j))^{-1} > \varphi(l), \ j = \overline{1, l-1}, \\ \displaystyle\sum_{i=l}^{n} \tau_i \xi_i^* - (1-\beta_j)\tau_j \xi_j^* = \varphi(l), \ j = \overline{l, n}, \end{cases}$$

$$K(i, y^*) = \begin{cases} \tau_i \leq \varphi(l), \ i = \overline{1, l-1}, \\ \tau_i - \tau_i(1-\beta_i)\eta_i^* = \varphi(l), \ i = \overline{l, n}. \end{cases}$$

Thus, for all $i, j = 1, \ldots, n$ the following inequalities hold

$$K(i, y^*) \leq \varphi(l) \leq K(x^*, j).$$

Then, by Theorem 9, x^* and y^* are optimal and $v_A = \varphi(l)$. This completes the solution of the game.

Iterative method

Consider Brown-Robinson iterative method (fictitious play method). This method employs a repeated fictitious play of game having a given payoff matrix. One repetition of the game is called a play. Suppose the game is played with an $(m \times n)$ matrix $A = \{a_{ij}\}$. In the 1st play both players choose arbitrary pure strategies. In the kth play each player chooses the pure strategy which maximizes his expected payoff against the observed empirical probability distribution of the opponents pure strategies for $(k-1)$ plays.

Thus, we assume that in the first k plays Player 1 uses the ith strategy ξ_i^k times $(i = 1, \ldots, m)$ and Player 2 uses the jth strategy η_j^k times $(j = 1, \ldots, n)$. In the $(k+1)$ play, Player 1 will then use i_{k+1} strategy and Player 2 will use his j_{k+1} strategy, where

$$\overline{v}^k = \max_i \sum_j a_{ij} \eta_j^k = \sum_j a_{i_{k+1}j} \eta_j^k$$

and

$$\underline{v}^k = \min_j \sum_i a_{ij} \xi_i^k = \sum_i a_{ij_{k+1}} \xi_i^k.$$

Let v be the value of the matrix game Γ_A. Consider the expressions

$$\overline{v}^k / k = \max_i \sum_j \alpha_{ij} \eta_j^k / k = \sum_j \alpha_{i_{k+1}j} \eta_j^k / k,$$

$$\underline{v}^k/k = \min_j \sum_i \alpha_{ij}\xi_i^k/k = \sum_i \alpha_{ij_{k+1}}\xi_i^k/k.$$

The vectors $x^k = (\xi_1^k/k, \dots, \xi_m^k/k)$ and $y^k = (\eta_1^k/k, \dots, \eta_n^k/k)$ are mixed strategies for the players 1 and 2, respectively; hence, by the definition of the value of the game we have

$$\max_k \underline{v}^k/k \le v \le \min_k \overline{v}^k/k.$$

We have thus obtained an iterative process which enables us to find an approximate solution of the matrix game, the degree of approximation to the true value of the game being determined by the length of the interval $[\max_k \underline{v}^k/k, \min_k \overline{v}^k/k]$. Convergence of the algorithm is guaranteed by the Theorem [Robinson (1950)].

Theorem 14.

$$\lim_{k\to\infty}(\min_k \overline{v}^k/k) = \lim_{k\to\infty}(\max_k \underline{v}^k/k) = v.$$

Example 15. Find an approximate solution to the game having the matrix

$$A = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}.$$

Denote Player 1's strategies by α, β, γ, and Player 2's strategies by a, b, c. Suppose the players first choose strategies α and a, respectively. If Player 1 chooses strategy α, then Player 1 can receive one of the payoffs (2,1,3). If Player 2 chooses strategy a, then Player 1 can receive one of the payoffs (2,3,1). In the 2nd and 3rd plays, Player 1 chooses strategy β and Player 2 chooses strategy b, since these strategies ensure the best result, etc.

Table 1 shows the results of plays, the players' strategies, the accumulated payoff, and the average payoff.

Thus, for 12 plays, we obtain an approximate solution

$$x^{12} = (1/4, 1/6, 7/12), \quad y^{12} = (1/12, 7/12, 1/3)$$

and the accuracy can be estimated by the number 5/12. The principal disadvantage of this method is its low speed of convergence which decreases as the matrix dimension increases. This also results from the nonmonotonicity of sequences $\overline{v}^k/k$ and $\underline{v}^k/k$.

Consider another iteration algorithm which is free of the above-mentioned disadvantages.

Monotonic iterative method of solving matrix games. [Sadovsky (1978)].

We consider a mixed extension $\Gamma_A = (X, Y, K)$ of the matrix game with the $(m \times n)$ matrix A.

Denote by $x^N = (\xi_1^N, \dots, \xi_m^N) \in X$ the approximation of Player 1's optimal strategy at the Nth iteration, and by $c^N \in R^N$, $c^N = (\gamma_1^N, \dots, \gamma_n^N)$ an auxiliary vector. Algorithm makes it possible to find (exactly and approximately) an optimal strategy for Player 1 and a value of the game v.

Play No	Player 1's choice	Player 2's choice	Player 1's payoff			Player 2's payoff			$\frac{\overline{v}^k}{k}$	$\frac{\underline{v}^k}{k}$
			α	β	γ	a	b	c		
1	α	a	2	3	1	2	1	3	3	1
2	β	b	3	3	3	5	1	4	3/2	1/2
3	β	b	4	3	5	8	1	5	5/3	1/3
4	γ	b	5	3	7	9	3	6	7/4	3/4
5	γ	b	6	3	9	10	5	7	9/5	5/5
6	γ	b	7	3	11	11	7	8	11/6	7/6
7	γ	b	8	3	13	12	9	9	13/7	9/7
8	γ	c	14	4	14	13	12	10	14/8	10/8
9	γ	c	14	5	15	14	12	11	15/9	11/9
10	γ	c	17	6	16	15	14	12	17/10	12/10
11	α	c	20	7	17	17	15	15	20/11	15/11
12	α	b	21	7	19	19	16	18	21/12	16/12

Table 1:

At the start of the process, Player 1 chooses an arbitrary vector of the form $c^0 = a_{i_0}$, where a_{i_0} is the row of the matrix A having the number i_0.

Iterative process is constructed as follows. Suppose the $N-1$ iteration is performed and vectors x^{N-1}, c^{N-1} are obtained. Then x^N and c^N are computed from the following iterative formulas

$$x^N = (1-\alpha_N)x^{N-1} + \alpha_N \tilde{x}^N, \tag{7.9}$$

$$c^N = (1-\alpha_N)c^{N-1} + \alpha_N \tilde{c}^N, \tag{7.10}$$

where $0 \le \alpha_N \le 1$. Vectors $\tilde{x}^N$ and $\tilde{c}^N$ will be obtained below.

We consider the vector $c^{N-1} = (\gamma_1^{N-1}, \ldots, \gamma_n^{N-1})$ and select indices j_k on which the minimum is achieved

$$\min_{j=1,\ldots,n} \gamma_j^{N-1} = \gamma_{j_1}^{N-1} = \gamma_{j_2}^{N-1} = \ldots = \gamma_{j_k}^{N-1}.$$

Denote

$$\underline{v}^{N-1} = \min_{j=1,\ldots,n} \gamma_j^{N-1} \tag{7.11}$$

and $J^{N-1} = \{j_1, \ldots, j_k\}$ be the set of indices on which (7.11) is achieved.

Let $\Gamma^N \subset \Gamma_A$ be a subgame of the game Γ_A with the matrix $A^N = \{a_{ij}^{N-1}\}$, $i = 1, \ldots, m$, and the index $j^{N-1} \in J^{N-1}$. Solve the subgame and find an optimal strategy $\tilde{x}^N \in X$ for Player 1. Let $\tilde{x}^N = (\tilde{\xi}_1^N, \ldots, \tilde{\xi}_m^N)$.

Compute the vector $\tilde{c}^N = \sum_{i=1}^m \tilde{\xi}_i^N a_i$. Suppose the vector $\tilde{c}^N$ has components $\tilde{c}^N = (\tilde{\gamma}_1^N, \ldots, \tilde{\gamma}_n^N)$. Consider the $(2 \times n)$ game with matrix

$$\begin{bmatrix} \gamma_1^{N-1} & \ldots & \gamma_n^{N-1} \\ \tilde{\gamma}_1^N & \ldots & \tilde{\gamma}_n^N \end{bmatrix}.$$

Find Player 1's optimal strategy $(\alpha_N, 1-\alpha_N),\ \ 0 \le \alpha_N \le 1$ in this subgame.

Substituting the obtained values $\tilde{x}^N, \tilde{c}^N, \alpha_N$ into (7.9), (7.10), we find x^N and c^N. We continue the process until the equality $\alpha_N = 0$ is satisfied or the required accuracy of computations is achieved. Convergence of the algorithm is guaranteed by the following theorem [Sadovsky (1978)].

Theorem 15. *Let $\{\underline{v}^N\}, \{x^N\}$ be the iterative sequences determined by (7.9), (7.11). Then the following assertions are true.*

1. $\underline{v}^N > \underline{v}^{N-1}$, i.e. the sequence $\{\underline{v}^{N-1}\}$ strictly and monotonically increases

2.

$$\lim_{N\to\infty} \underline{v}^N = \underline{v} = v \tag{7.12}$$

3. $\lim\limits_{N\to\infty} x^N = x^*$, where $x^* \in X^*$ is an optimal strategy of Player 1.

Example 16. By employing a monotonic algorithm, solve the game with the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Iteration 0. Suppose Player 1 chooses the 1st row of the matrix A, i.e. $x^* = (1,0,0)$ and $c^0 = a_1 = (2,1,3)$. Compute $\underline{v}^0 = \min\limits_j \gamma_j^0 = \gamma_2^0 = 1,\ J^0 = 2$.

Iteration 1. Consider the subgame $\Gamma^1 \subset \Gamma$ having the matrix

$$A^1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

An optimal strategy $\tilde{x}^1$ of Player 1 is the vector $\tilde{x}^1 = (0,0,1)$. Then $\tilde{c}^1 = a_3 = (1,2,1)$. Solve the (2×3) game with the matrix $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Note that the 3rd column of the matrix is dominated and so we consider the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Because of the symmetry, Player 1's optimal strategy in this game is the vector $(\alpha_N, 1-\alpha_N) = (1/2, 1/2)$.

We compute x^1 and c^1 by formulas (7.9), (7.10). We have

$$\begin{aligned} &x^1 = 1/2x^0 + 1/2\tilde{x}^1 = (1/2, 0, 1/2), \\ &c^1 = 1/2c^0 + 1/2\tilde{c}^1 = (3/2, 3/2, 2), \\ &\underline{v}^1 = \min\nolimits_j \gamma_j^1 = \gamma_1^1 = \gamma_2^1 = 3/2 > \underline{v}^0 = 1. \end{aligned}$$

The set of indices is of the form $J^1 = \{1,2\}$.

Iteration 2. Consider the subgame $\Gamma^2 \subset \Gamma$ having the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

The first row in this matrix is dominated; hence it suffices to examine the submatrix

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

Player 1's optimal strategy in this game is the vector $(1/4, 3/4)$; hence $\tilde{x}^2 = (0, 1/4, 3/4)$.

Compute $\tilde{c}^2 = 1/4a_2 + 3/4a_3 = (3/2, 3/2, 1)$ and consider the (2×3) game with the matrix

$$\begin{bmatrix} 3/2 & 3/2 & 1 \\ 3/2 & 3/2 & 2 \end{bmatrix}.$$

The second strategy of Player 1 dominates the first strategy and hence $\alpha_2 = 0$. This completes the computations $x^* = x^1 = (1/2, 0, 1/2)$; the value of the game is $v = \underline{v}^1 = 3/2$, and Player 2's optimal strategy is of the form $y^* = (1/2, 1/2, 0)$.

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