Exercises on solving $A\mathbf{x} = \mathbf{b}$ and row reduced form R

Problem 8.1: (3.4 #13.(a,b,d) *Introduction to Linear Algebra:* Strang) Explain why these are all false:

a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

b) The system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.

c) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Solution:

a) The coefficient of $\mathbf{x}_p$ must be one.

b) If $\mathbf{x}_n \in \mathbf{N}(A)$ is in the nullspace of A and $\mathbf{x}_p$ is one particular solution, then $\mathbf{x}_p + \mathbf{x}_n$ is also a particular solution.

c) There's always $\mathbf{x}_n = 0$.

Problem 8.2: (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices $[U \ \ 0]$ and $[U \ \ \mathbf{c}]$ to $[R \ \ 0]$ and $[R \ \ \mathbf{d}]$. Solve $R\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{d}$.

Check your work by plugging your values into the equations $U\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{c}$.

Solution: First we transform $[U \ \ 0]$ into $[R \ \ 0]$:

$$[U \ \ 0] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [R \ \ 0].$$

We now solve $R\mathbf{x} = \mathbf{0}$ via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},$$

where we used the free variable $x_2 = -1$. ($c\mathbf{x}$ is a solution for all c.)

We check that this is a correct solution by plugging it into $U\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

Next, we transform $[U\ \ \mathbf{c}]$ into $[R\ \ \mathbf{d}]$:

$$[U\ \ \mathbf{c}] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [R\ \ \mathbf{d}].$$

We now solve $R\mathbf{x} = \mathbf{d}$ via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix},$$

where we used the free variable $x_2 = 1$.

Finally, we check that this is the correct solution by plugging it into the equation $U\mathbf{x} = \mathbf{c}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark$$

Problem 8.3: (3.4 #36.) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

Solution: **Yes.** In order to check that $A = C$ as matrices, it is enough to check that $A\mathbf{y} = C\mathbf{y}$ for all vectors $\mathbf{y}$ of the correct size (or just for the standard basis vectors, since multiplication by them "picks out the columns"). So let $\mathbf{y}$ be any vector of the correct size, and set $\mathbf{b} = A\mathbf{y}$. Then $\mathbf{y}$ is certainly a solution to $A\mathbf{x} = \mathbf{b}$, and so by our hypothesis must also be a solution to $C\mathbf{x} = \mathbf{b}$; in other words, $C\mathbf{y} = \mathbf{b} = A\mathbf{y}$.