Indian Institute of Technology Patna
Department of Electrical Engineering
EE322 - Mathematical Methods in EE
Spring - 2015
Quiz - 2
10 April 2015

There are 5 problems. They carry equal marks.

$(5 \times 2 = 10)$

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1. Consider the function

$$f(x,y) = x^2 + y^2 + \beta xy + x + 2y$$

For what values of β, does this function have a unique global minimum?

In order this function to have a unique global minimum, It must be strict convex.

$$\nabla G = \begin{bmatrix} 2x + \beta y + 1 \\ 2y + \beta x + 2 \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & \beta \\ \beta & 2 \end{bmatrix} \text{ — 1 pt}$$

Hessian must be +ve definite if the function is strict convex.

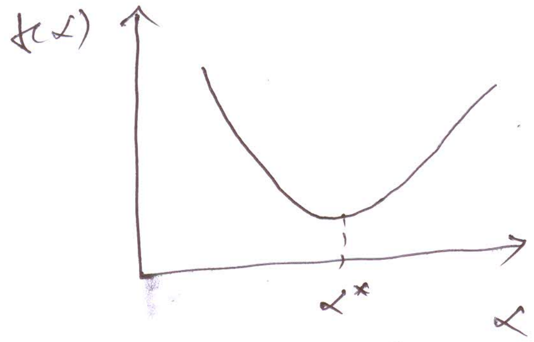
H will be +ve definite for $\beta^2 < 4$. — 1 pt

2. Suppose the one dimensional function $f(\mathbf{x}_k + \alpha \mathbf{d}_k)$ is unimodal and differentiable. Let α^* be the minimum of the function. If any $\alpha > \alpha^*$ is selected, show that $\nabla f(\mathbf{x}_{k+1})^T \mathbf{d}_k > 0$.

Let

$$f(\alpha) = f(x_k + \alpha d_k)$$

Since $f(\alpha)$ is unimodal function,



$\rightarrow$ 1 pt

If you choose $\alpha > \alpha^*$, $\frac{df}{d\alpha} > 0$.

$$\frac{df}{d\alpha} = \frac{d}{d\alpha}\left(f(x_k + \alpha d_k)\right)$$

$$= \nabla f_{k+1}^T d_k$$

As $\frac{df}{d\alpha} > 0$, $\rightarrow$ 1 pt

$$\nabla f_{k+1}^T d_k > 0$$

3. Consider the following problem.

$$\text{maximize} \quad f(x, y) = \exp(-\frac{1}{3}x^3 + x - y^2)$$

Suppose you want to do it using pure Newton's method. Is $\mathbf{x}_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ a good starting point?

Minimize $-f(x)$

If the Newton's Method to go in a descent direction, H must be +ve definite → (1pt)

$$\nabla f = -\begin{bmatrix} (-x^2+1)\exp(-\frac{1}{3}x^3 + x - y^2) \\ -2y\exp(-\frac{1}{3}x^3 + x - y^2) \end{bmatrix}$$

$$H = -\begin{bmatrix} (-x^2+1)^2\exp(-\frac{1}{3}x^3 + x - y^2) + (-2x)\exp(-\frac{1}{3}x^3 + x - y^2) & (-2y)(-x^2+1)\exp(-\frac{1}{3}x^3 + x - y^2) \\ (-2y)(-x^2+1)\exp(-\frac{1}{3}x^2 + x - y^2) & 4y^2\exp(-\frac{1}{3}x^3 + x - y^2) - 2\exp(-\frac{1}{3}x^3 + x - y^2) \end{bmatrix}$$

$$H\Big|_{x_0} = \begin{bmatrix} -2\exp(-5/3) & 0 \\ 0 & -2\exp(-5/3) \end{bmatrix} \rightarrow \text{(1pt)}$$

As H is a −ve definite, x_0 is not a good starting point

4. Find the rectangle of given perimeter that has greatest area by using Lagrange multiplier theorem. Verify it using second order conditions.

Let x and y be the length and breadth of a rectangle.

$x + y = c$ (given) $\longrightarrow$ (some constant)

area $= xy$

$$\left.\begin{aligned} \max \quad & f(x,y) = xy \\ \text{s.t} \quad & x + y = c \end{aligned}\right\} \rightarrow \begin{aligned} & \min(-xy) \\ & x + y = c \end{aligned}$$

$L = -xy + \lambda(x + y - c)$

$\nabla L_x = 0 \Rightarrow$

$-y + \lambda = 0$

$-x + \lambda = 0$

$x + y = c$

1 Pt

By solving this

$\boxed{x = y = c/2}$; $\lambda = c/2$

It is the optimal solution.

Square is the answer

To verify second order condition,

$\nabla^2 L_{xx} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$d^T \nabla^2_{Lxx} d > 0$

$\forall \nabla h^T d = 0$

$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0$

$d_1 + d_2 = 0$

$\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -2d_1d_2$

$= 2d_1^2$

It is always > 0

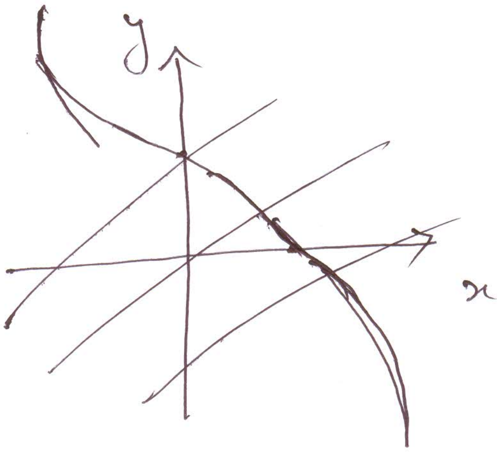
$\rightarrow$ 1 Pt

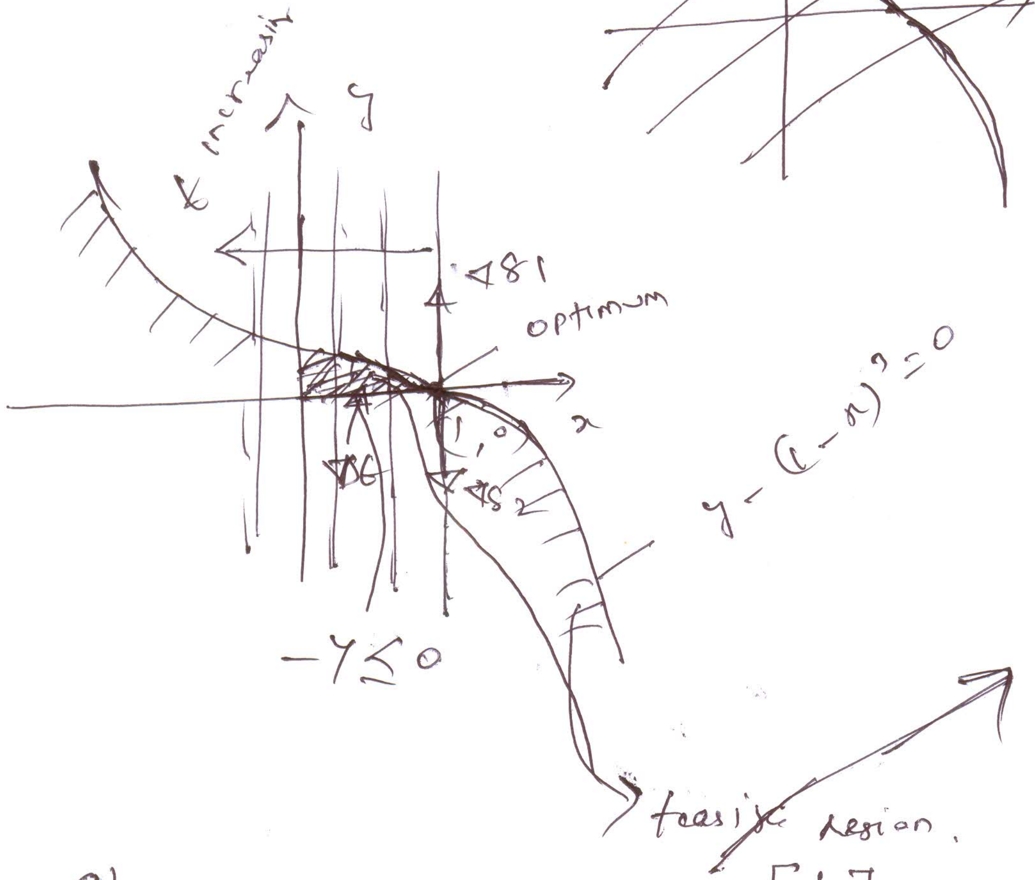
5. Consider

$$\begin{aligned} \text{minimize} \quad & f(x,y) = -x \\ \text{subject to} \quad & y - (1-x)^3 \le 0 \\ & -y \le 0 \end{aligned}$$

(a) Find the optimal solution by solving it graphically.

(b) Do Lagrange multipliers exist? How could you say without actually solving the problem?



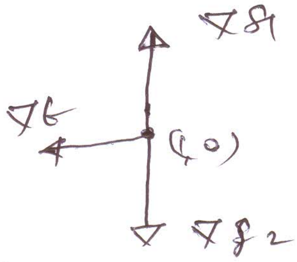


1 pt

a) optimum is $x^* = \begin{bmatrix}1\\0\end{bmatrix}$

b) at optimum, ∇g_1 and ∇g_2 are linearly dependent,

"Lagrange multipliers do not exist"



1 pt