Problem 1
Consider the function $f(x, y) = x^2 + y^2 + \beta xy + x + 2y$. For what values of β, does this function have a unique global minimum ?

Solution
A function have to be strictly convex for it to have unique global minima.

$$\nabla f = \begin{bmatrix} \frac{\delta f}{\delta x} \\ \frac{\delta f}{\delta y} \end{bmatrix}$$

$$= \begin{bmatrix} 2x + \beta y + 1 \\ 2y + \beta x + 2 \end{bmatrix}$$

Hessian, H $= \begin{bmatrix} \frac{\delta^2 f}{\delta x^2} & \frac{\delta^2 f}{\delta y \delta x} \\ \frac{\delta^2 f}{\delta x \delta y} & \frac{\delta^2 f}{\delta y^2} \end{bmatrix} = \begin{bmatrix} 2 & \beta \\ \beta & 2 \end{bmatrix} = 4 - \beta^2$

Hessian must be greater than zero for function to be strictly convex.

So, $4 - \beta^2 > 0 \implies \beta^2 < 4$

Problem 2
Suppose the one dimensional function $f(x_k + \alpha d_k)$ is unimodal and differentiable. Let α^* be the minimum of the function. If any $\alpha \geq \alpha^*$ is selected, show that $\nabla f(x_{k+1})^T d_k > 0$

Solution let $t(\alpha) = f(x_k + \alpha d_k)$.

Since, $t(\alpha)$ is unimodal $t(\alpha)$ has one global minima and one local minima.

So as α^* is the minimum of the function then for any $\alpha > \alpha^*$.

Hence, $\frac{dt}{d\alpha} > 0$

$\frac{dt}{d\alpha} = \frac{d}{d\alpha}(f(x_k + \alpha d_k)) = \nabla f_{k+1}^T d_k$

As $\frac{dt}{d\alpha} > 0$, So, $\nabla f(x_{k+1}^T d_k) > 0$

Problem 3
Consider the following problem. $f(x, y) = exp(-\frac{1}{3}x^3 + x - y^2)$. Suppose you want to do it using pure Newton's method. Is $x_0 = (-1, 1)$ good starting point ?

Solution Newton method can be applied only in decent direction. So, we formulate the problem as minimize $f(x, y) = -exp(-\frac{1}{3}x^3 + x - y^2)$

$\frac{\delta f}{\delta x} = (-x^2 + 1)exp(-\frac{1}{3}x^3 + x - y^2)$

$\frac{\delta f}{\delta y} = -2yexp(-\frac{1}{3}x^3 + x - y^2)$

$\frac{\delta^2 f}{\delta x^2} = (-x^2 + 1)^2 exp(-\frac{1}{3}x^3 + x - y^2) - 2xexp(-\frac{1}{3}x^3 + x - y^2)$

$\frac{\delta^2 f}{\delta y^2} = -2exp(-\frac{1}{3}x^3 + x - y^2) + 4y^2 exp(-\frac{1}{3}x^3 + x - y^2)$

$\frac{\delta^2 f}{\delta y \delta x} = -2y(-x^2 + 1)exp(-\frac{1}{3}x^3 + x - y^2)$

$\frac{\delta^2 f}{\delta x \delta y} = -2y(-x^2+1)exp(-\frac{1}{3}x^3 + x - y^2)$

Hessian, H $= \begin{bmatrix} \frac{\delta^2 f}{\delta x^2} & \frac{\delta^2 f}{\delta y \delta x} \\ \frac{\delta^2 f}{\delta x \delta y} & \frac{\delta^2 f}{\delta y^2} \end{bmatrix}$

$$= \begin{bmatrix} (-x^2+1)^2 exp(-\frac{1}{3}x^3 + x - y^2) - 2x exp(-\frac{1}{3}x^3 + x - y^2) & -2y(-x^2+1)exp(-\frac{1}{3}x^3 + x - y^2) \\ -2y(-x^2+1)exp(-\frac{1}{3}x^3 + x - y^2) & -2exp(-\frac{1}{3}x^3 + x - y^2) + 4y^2 exp(-\frac{1}{3}x^3 + x - y^2) \end{bmatrix}$$

$H_{x_0} = \begin{bmatrix} -2exp(-5/2) & 0 \\ 0 & -2exp(-5/2) \end{bmatrix}$

Since this is a negative definite matrix, hence X_0 is not a good starting point.

Problem 4
Find the rectangle of given perimeter that has greatest area by using Lagrange multiplier theorem. Verify it using second order conditions.

Solution Let x and y be the length and width of the rectangle. The given perimeter is $x + y = c$ where c is given. So, the problem becomes

maximize $z = xy$
subject to $x + y = c$

We can rewrite the problem as

minimize $z = -xy$
subject to $x + y = c$

So, the lagrangian problem becomes
$L(\lambda) = -xy + \lambda(x + y - c)$

$\frac{\delta L}{\delta x} = -y + \lambda$

$\frac{\delta L}{\delta y} = -x + \lambda$

$\frac{\delta L}{\delta \lambda} = x + y - c$

$\frac{\delta L}{\delta x} = 0 \implies y = \lambda$

$\frac{\delta L}{\delta y} = 0 \implies x = \lambda$

$\frac{\delta L}{\delta \lambda} = 0 \implies x + y = c \implies x = y = c/2$

So, $x^* = y^* = c/2$

Border Hessian Matrix $H^B = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}$

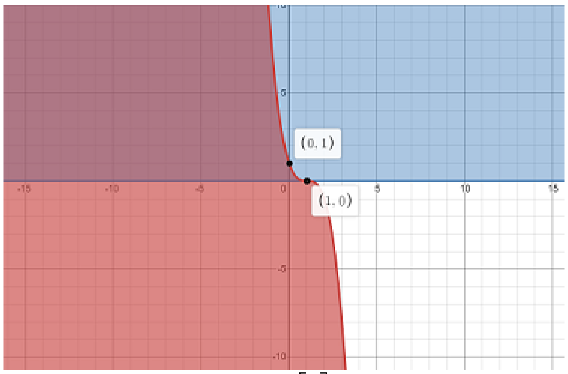
$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

$= -2$

Since $|H^B| < 0$, so the objective function is minimized.

Problem 5
Consider
Minimize $f(x,y) = -x$
Subject to $y - (1 - x^3)\ -y \leq 0$
a. Find the optimal solution by solving it graphically.
b. Do Lagrange multiplier exist ? How could you say without actually solving the problem ?

Solution a) Ploted the graph using an online tool which looks like below



So, the solution is $X^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) Here, $\Delta s_1, \Delta s_2$ are linearly dependent. So lagrange multiplier does not exists.