Problem 1
What is dual norm. Derive the dual norm of the L_1, L_2 and L_∞.

Solution Let $\|w\|$ be a generic norm of vector w.

The dual norm is defined as follows.

$\|x\| \star = max\langle$ w, x $\rangle$ such that $\|$ w $\| \leq 1$.

Hence, we get the following result.

$\langle w, z\rangle \leq \|w\|\|z\| \star$

Dual norm of L_1 and L_∞

Let $||z|| = \Sigma|z_i| = ||Z||_1(l_1 norm)$.

maximize $\Sigma z_i y_i$ for $\Sigma|z_i| \leq 1$.

$= max|y_i| = ||y||_\infty$.

Hence, the dual norm of L_1 is L_∞. Since we know that the dual norm of the dual norm of is the original norm. Hence, the dual norm of L_∞ is L_1.

Dual norm of L_2
We can see from the below equation that dual norm of L_2 is the L_2 itself.

$$\max_{||z||_2 \leq 1} \quad z^T y \leq ||z||_2||y||_2 \leq ||y||_2 \tag{1}$$

The equality is obtained when,

$$z = \begin{cases} ||y||_2^{-1}.y, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

Problem 2
Gram-Schmidt Let $(1,-1,1,1),(1,0,1,0),(0,1,0,1)$ be a linearly independent set in R^4. Find an orthonormal set $v1, v2, v3$ st L((1, 1,1,1), (1,0,1,0), (0,1,0,1)) = L(v1, v2, v3)

Solution step-1

$$\tilde{\mathbf{u_1}} = \tilde{\mathbf{v_1}} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \tilde{\mathbf{e_1}} = \frac{\tilde{\mathbf{u_1}}}{|\tilde{\mathbf{u_1}}|} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

step-2

$$\tilde{\mathbf{u_2}} = \tilde{\mathbf{v_2}} - \mathrm{proj}_{\tilde{\mathbf{u_1}}}(\tilde{\mathbf{v_2}}) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \tilde{\mathbf{e_2}} = \frac{\tilde{\mathbf{u_2}}}{|\tilde{\mathbf{u_2}}|} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

step-3

$$\tilde{\mathbf{u_3}} = \tilde{\mathbf{v_3}} - \mathrm{proj}_{\tilde{\mathbf{u_1}}}(\tilde{\mathbf{v_3}}) - \mathrm{proj}_{\tilde{\mathbf{u_2}}}(\tilde{\mathbf{v_3}}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tilde{\mathbf{e_3}} = \frac{\tilde{\mathbf{u_3}}}{|\tilde{\mathbf{u_3}}|} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

answer:

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\} \approx \left\{ \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.707106781186548 \\ 0 \\ 0.707106781186548 \end{bmatrix} \right\}.$$

Problem 3
Prove that every real, symmetric matrix X has the decomposition $X = Q\Lambda Q^T$. Q is an orthogonal matrix and Λ is a diagonal matrix with eigenvalues as elements.

Solution
Let n be the order of real symmetric matrix A.
And $x_1, x_2, \ldots, x_n$ be the eigenvector of A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Let $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$ be the eigenvalue matrix of A. And $Q = [x_1, x_2, ..., x_n]$ be the eigenvector matrix of A.

Then, we have,

$$AQ = A[x_1, x_2, ..., x_n]$$

$$= [Ax_1, Ax_2, ..., Ax_n]$$

$$= [\lambda_1 x_1, \lambda_2 x_2, ..., \lambda_n x_n]$$

$$= Q\Lambda$$

Hence, $AQ = Q\Lambda$.

Or, $A = Q\Lambda Q^{-1}$
If we can prove that Q is orthogonal then we would get our desired result of $A = Q\Lambda Q^T$ as for any orthogonal matrix, $Q^{-1} = Q^T$

We know that for any real matrix A and any vectors x and y, we have,

$$\langle Ax, y \rangle = \langle x, A^T y \rangle$$

Let x, y be the eigenvector of A corresponding to distinct eigenvalue λ_1 and λ_2. We already know that A is a real, symmetric matrix. Then,

$$\lambda_1 \langle x, y \rangle = \langle \lambda_1 x, y \rangle = \langle Ax, y \rangle = \langle x, A^T y \rangle = \langle x, Ay \rangle = \langle x, \lambda_2 y \rangle = \lambda_2 \langle x, y \rangle$$

Therefore, $(\lambda_1 - \lambda_2)\langle x, y \rangle = 0$. Since $(\lambda_1 - \lambda_2) \neq 0$ as they are distinct eigenvalues. Therefore, x and y are orthogonal to each other.

Similarly, we can prove that any two eigenvector of A are orthogonal to each other. Hence, the matrix Q is orthogonal.

Hence, $A = Q\Lambda Q^T$

Problem 4
Can an orthogonal matrix have an entry $U_{ij} > 1$? Why?

Solution Yes. Since, all diagonal matrix are orthogonal and a diagonal matrix can have a diagonal element greater than 1. Hence, it is possible for an orthogonal matrix to have an element greater than 1.

Problem 5
When is a diagonal matrix orthogonal?

Solution Every diagonal matrix has the property that it is orthogonal. Hence, every diagonal matrix is already orthogonal.

Problem 6
When is an upper triangular matrix orthogonal ?

Solution Let A be an upper triangular matrix which is also orthogonal. Then, as per orthogonality, $A^{-1} = A^T$.

Also, note that the inverse of an upper triangular matrix is also an upper triangular matrix. Hence, A^T is both upper triangular and lower triangular matrix, i.e., a diagonal matrix. It implies that A is also a diagonal matrix.

Hence, an upper triangular matrix will be orthogonal when it is diagonal.

Problem 7
Is the inverse of an orthogonal matrix orthogonal ?

Solution We know that, $A^T = A^{-1}$

Taking inverse, $(A^{-1})^{-1} = A$

Taking transpose, $(A^T)^T = A$

Hence, $(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$

Problem 8
What are eigenvalues and eigenvectors of a diagonal matrix ?

Solution The eigenvalues of a diagonal matrix are present as the elements of that diagonal matrix. And its eigenvector form a canonical basis for space K^n

Problem 9
Can you find vectors that attain each of the equality cases in $\lambda_{min} \leq \frac{x^TAx}{||x||^2} \leq \lambda_{max}$?

Solution The given inequality is known as Rayleigh Inequality. Here, A is any symmetric matrix of order n and x is any vector of $\mathbb{R}^n$.

One vector for which equality can be attain in both the cases is when $x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and A is an identity matrix of order 3.

Then λ_{min} and λ_{max} both are 1 and $||x||^2 = 1.1 + 0.0 + 0.0 = 1$ and $x^TAx = 1$.

There are many other vectors as well. Any column vector of an identity matrix will also satisfy this property.