Problem 1
Is the intersection of two convex sets convex ? Is the union of two convex sets convex ?

Solution
1. Intersection of two convex sets

Let A and B be two convex sets.

Let $x, y \in A \cap B$
$\implies x, y \in A$ and $x, y \in B$
$\implies x\lambda + (1-\lambda)y \in A, \forall\, \lambda \in [0,1]$ and $x\lambda + (1-\lambda)y \in B, \forall\, \lambda \in [0,1]$ ...(Since A, B are convex sets)
$\implies x\lambda + (1-\lambda)y \in A \cap B, \forall\, \lambda \in [0,1]$

Hence, $A \cap B$ is a convex set. Therefore, intersection of any two convex sets is also a convex set.

2. Union of two convex sets

We can prove that union of two convex sets is not convex using counter example.

We already know that $[4,5]$ and $[6,7]$ are convex set.
But, $5 * t + (1-t) * 6$ does not belong to $[4,5] \cup [6,7]$ for any $t \in [0,1]$.

Hence, union of two convex sets may not be convex.

Problem 2
Prove that the following sets are convex.
(a) **Polyhedra**: Sets of the form $K = \{\ x \in \mathbb{R}_n : \langle a_i, x\rangle \le b_i\ for\ i = 1,2,...,m\ \}$ where $a_i \in \mathbb{R}_n$ and $b_i \in \mathbb{R}$ for $i = 1,2,...,m$
(b) **Ellipsoids**: Sets of the form $K = \{\ x \in \mathbb{R}^n : x^T A x \le 1\ \}$ where $A \in \mathbb{R}^{nxn}$ is a PD matrix.
(c) **Unit balls in l_p norms for $p \ge 1$**: $B_p(a,1) := \{\ x \in \mathbb{R}^n : \|\ x - a\ \|_p \le 1\ \}$ where $a \in \mathbb{R}^n$ is a vector.

Solution
(a) Let $x, y \in K$.

Hence, $\langle a_i, x\rangle \le b_i \implies \lambda\langle a_i, x\rangle \le \lambda b_i\ \forall\, i = 1,2,...,m$ and $\lambda \in [0,1]$
Similarly, $\langle a_i, y\rangle \le b_i \implies (1-\lambda)\langle a_i, y\rangle \le (1-\lambda)b_i\ \forall\, i = 1,2,...,m$ and $\lambda \in [0,1]$

Adding above two inequalities, we get,

$\lambda\langle a_i, x\rangle + (1-\lambda)\langle a_i, y\rangle \le \lambda b_i + (1-\lambda)b_i\ \forall\, i = 1,2,...,m$ and $\lambda \in [0,1]$
$\implies \lambda * a_i.x + (1-\lambda)a_i.x \le b_i\ \forall\, i = 1,2,...,m$ and $\lambda \in [0,1]$
$\implies a_i(\lambda x + (1-\lambda)y) \le b_i\ \forall\, i = 1,2,...,m$ and $\lambda \in [0,1]$
$\implies \langle a_i, \lambda x + (1-\lambda y)\rangle \le b_i \forall\, i = 1,2,...,m$ and $\lambda \in [0,1]$

Hence, $x, y \in K \implies \lambda x + (1-\lambda y) \in K$. Hence, the polyhedra set is a convex set.

(b) Since, $A \in \mathbb{R}^{nxn}$ hence, A will be a positive definite matrix. Hence, $A = (A^{1/2})^2$ for a uniquely defined symmetrix positive definite matrix $A^{1/2}$.

Setting, $\|\ x\ \|_A = \|\ A_x^{1/2}\ \|_2$. Hence, we have a norm on R^n. Then, we get,

$x^T A x = [(x^T)A^{1/2}][A^{1/2}x] = \|\ A^{1/2}x\ \|_2^2 = \|\ x\ \|_Q^2$.

We can redfine K as $K = \{x \in \mathbb{R}^n : \|\ x\ \|_A^2 \le 1\}$ where $A \in \mathbb{R}^{nxn}$ is a postive definite matrix.

Let $m, n \in K$.

$\| \lambda m + (1-\lambda) n \|_A^2 \leq \| \lambda m \|_A^2 + \| (1-\lambda) n \|_A^2$

$\leq \lambda \| m \|_A^2 + (1-\lambda) \| n \|_A^2$

$\leq \lambda.1 + (1-\lambda).1$

≤ 1

Hence, $\lambda m + (1-\lambda) n \in K$, so K is a convex set.

(c) For the l_p norm we know that,

$\| f + g \|_p \leq \| f \|_p + \| g \|_p$
$\lambda \| f \|_p = \| \lambda f \|_p$

Let $x, y \in B_p(a, 1)$, So
$\| x - a \|_p \leq 1$
$\| y - a \|_p \leq 1$

We can combine above 2 equations with $\lambda \in [0, 1]$
$\lambda \| x - a \|_p + (1-\lambda) \| y - a \|_p \leq \lambda + (1-\lambda)$
$\implies \| \lambda(x-a) \|_p + \| (1-\lambda)(y-a) \| \leq 1$ (Property of l_p norm)
$\implies \| \lambda(x-a) + (1-\lambda)(y-a) \| \leq 1$ (Property of l_p norm)
$\implies \| \lambda x + (1-\lambda) y - a \| \leq 1$

Hence, $\lambda x + (1-\lambda) y \in B_p(a, 1)$
Hence, $B_p(a, 1)$ is a convex set.