Problem 1
If $A = U\Sigma V^T$ is the SVD of a square invertible matrix A, what is the SVD of A^{-1} ?

Solution The SVD of a mxn matrix A is given by $A = UWV^T$ where,
U: m x n orthonormal eigenvectors of AA^T
V^T: n x n orthonormal eigenvectors of A^TA
W: n x n diagonal matrix having diagonal elements as the square roots of the eigenvalues of A^TA.

Since, U & V are orthogonal, $U^T = U^{-1}$ and $V^T = V^{-1}$

A diagonal matrix D with non-zero entries will have an inverse as

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 & \dots & 0 \\ 0 & 1/d_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1/d_n \end{bmatrix}$$

where $d_1, d_1, \dots, d_n$ are elements of D. Hence, the diagonal matrix W can also be inverted in the same way.

Hence, $A^{-1} = (UWV^T)^{-1} = (V^T)^{-1}W^{-1}U^{-1} = VW^{-1}U^T$.

Problem 2
If $A = U\Sigma V^T$ is the SVD of a square invertible matrix A, what is the SVD of A^{-1} ?

Solution The Eigendecomposition of a matrix is as follows.

The matrix U is formed by placing the eigenvector of A as its column. The Λ is a diagonal matrix where diagonal elements are eigenvalues.

So, $AU = U\Lambda$. Hence, $A = U\Lambda U^{-1}$

In positive semi-definite matrix, the eigenvalues are greater than or equal to zero. Also, any two eigenvector are orthogonal.

Hence, U is a orthogonal matrix and if we normalize its eigenvetor then it becomes orthonormal.

So, we can write $A = U\Lambda U^{-1}$.

The eigendecomposition of A satisfies all the requirement of having SVD.

Problem 3
If A is not positive semidefinite, how can we modify signs in A = $U\Lambda U^T$

Solution If A is not positive semi-definite then there there exists eigenvalues which are not positive. In such case, we can take absolute value of eigen vectors to form the diagonal matrix Λ.

Problem 4
Show that for a square A = USU^T one has $AA^T = US^2U^T$ and A^TA =V^TS^2V , and that these are eigen decompositions.

Solution Given, $\mathrm{A} = \mathrm{USU^T}$

$$\begin{aligned} \text{So,} \quad & \mathrm{A^T} = (\mathrm{USU^T})^\mathrm{T} \\ \implies & \mathrm{A^T} = (\mathrm{U^T})^\mathrm{T}\mathrm{S^T}\mathrm{U^T} \\ \implies & \mathrm{A^T} = \mathrm{US^TU^T} \end{aligned}$$

$$
\begin{aligned}
\text{So,} \quad & \mathrm{A^TA = USU^TUS^TU^T} \\
\implies & \mathrm{A^TA = USS^TU^T} \\
\implies & \mathrm{A^TA = US^2U^T}
\end{aligned}
$$

$$
\begin{aligned}
\text{So,} \quad & \mathrm{AA^T = USU^TUS^TU^T} \\
\implies & \mathrm{AA^T = USS^TU^T} \\
\implies & \mathrm{AA^T = US^2U^T}
\end{aligned}
$$