GRAPH THEORY

GRAPH → A mathematical model

Definition → A Graph $G=(V,E)$ consists of a non-empty set V of vertices or nodes and a set E of edges. Each edge has either one or two vertices associated with it called endpoints. An edge is said to connect its endpoints

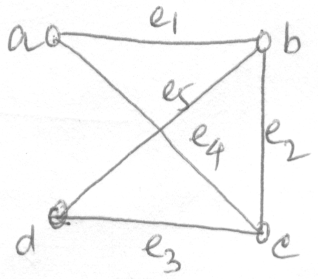


Fig 1.

$V=\{a,b,c,d\}$ $|V|=4$

$E=\{e_1,e_2,e_3,e_4,e_5\}$ $|E|=5$

* The connection made by edges matters, not the geometry (length of the edges, whether edges cross, how vertices are depicted and so on do not matter)

Infinite Graph: A Graph with infinite vertex set

Finite Graph: A Graph with finite vertex set

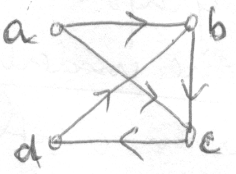
Notations:

* $V(G)$ – Set of Vertices (Domain) || Defining G in terms of logic notation

$E(G)$ – Represents Relation : $E(G) \subseteq V(G) \times V(G)$

Type of Graph:

① Directed Graph (Di-Graph) $G=(V,E)$ with a set of vertices (V) and set E of directed edges. Each edge is an ordered pair of vertices (u,v) where $u,v \in V$

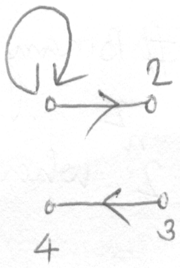


$V=\{a,b,c,d\}$ $E=\{(a,b),(b,c),(a,c),(d,b),(c,d)\}$

$E(G)=V(G)\times V(G)$ represents relation

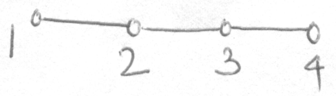
Given, $V(G_1)=\{1,2,3,4\}$

$E(G_1)=\{(1,1),(1,2),(3,4)\}$

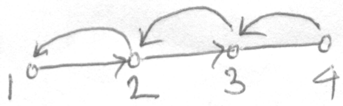


G_1

if $E(G)=\{(1,2),(2,3),(3,4)\}$



(2) Undirected Graph → Does not have directed edges

 $\equiv$ 

A special case of directed graph

$E(G) = \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}$

$\equiv \{\{1,2\}, \{2,3\}, \{3,4\}\}$ — Set notation

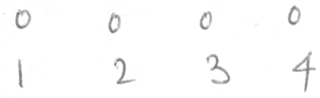
Note:

For a Directed Graph

$E(G) = V(G) \times V(G)$

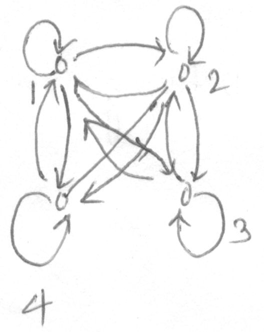
A binary relation = Directed Graph

Case 1: If $V(G) = \{1,2,3,4\}$
$E(G) = \phi$ ‖ empty/null graph



Case 2: If $V(G) = \{1,2,3,4\}$

$E(G) = V(G) \times V(G) = \{(2,1), (1,2), (1,1), (3,1), (1,3), (4,1), (1,4), (3,2), (2,3), (2,2), (4,2), (2,4), (4,3), (3,4), (3,3), (4,4)\}$



If $V(G) = \{1, 2, \ldots n\}$

Q: How many different directed graphs are possible?

Ans: Any binary rel^n corresponds to a directed graph.
Similarly, given a DAG, one can derive the underlying relation

Directed graph = # Binary relations

$= 2^{n^2}$ where n is no. of vertices

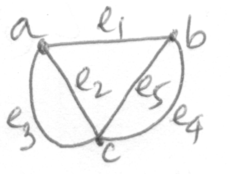
Given, a set of n items, ~~no. of~~ Size of cross product
i.e. if $|V| = n$ then $|V \times V| = n^2$

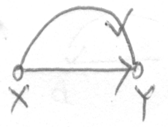
$\therefore$ # Binary relations $= 2^{n^2}$

Terminologies

(i) Simple graphs → each edge connects two different vertices and no two edges connect the same pair of vertices

(ii) Multigraphs → graphs that may have multiple edges connecting the same two vertices



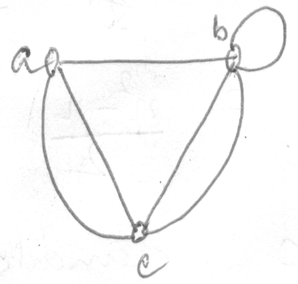


When 'm' different edges connect vertices u & v then we say $\{u, v\}$ is an edge of multiplicity m

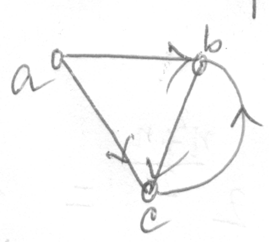
For ex: in the two graphs above, $\{a, c\}$ & $\{b, c\}$ have & $\{x, y\}$ multiplicity 2

(iii) ~~A~~ Self Loop → An edge that connects a vertex to itself is called Self loop

(iv) Pseudograph → may include loops, as well as multiple edges connecting the same pair of vertices.



* Simple directed graph has no loops & no multiple edges.



Consider the (product set – Diagonal elements) ⇓ self loops

⇒

Directed Simple graphs = no. of irreflexive Binary Relations of 'n' vertices

$= 2^{n^2 - n}$

Undirected Simple graphs = no. of irreflexive Binary Relations of 'n' vertices

$= 2^{\frac{n^2 - n}{2}}$

Undirected Graphs

$V(G) = \{1, 2, \ldots n\}$

$E(G) \subseteq V(G) \times V(G)$

$1 \rightleftarrows 2 \equiv 1 - 2$

$\{(1,2), (2,1)\}$ $\quad \{\{1,2\}\}$

$a - b \equiv a \rightleftarrows b$

The underlying relation is always symmetric

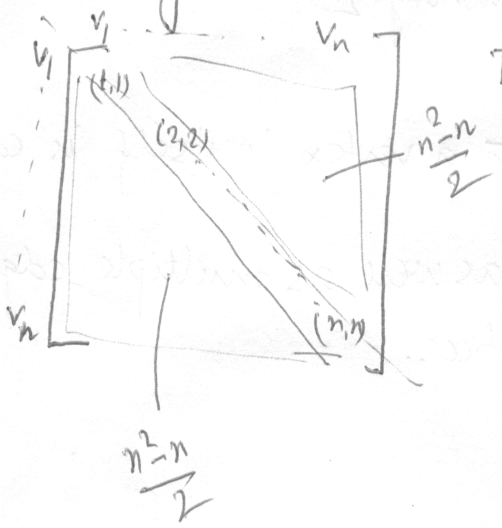
Q: How many possible undirected graphs are possible on 'n' nodes/vertices

Ans: Any undirected graph corresponds to symmetric binary relation

$1 - 2 - 3$ $\quad \left\{\begin{matrix}(1,2), (2,1) \\ (2,3), (3,2)\end{matrix}\right\} \equiv \{\{1,2\}, \{2,3\}\}$

\# undirected graphs = \# symmetric binary rel^{n}s on 'n' nodes

To Count \# Symmetric rel^{n}s ~~on~~ on 'n' nodes using matrix representation



The diagonal elements set $= \{(1,1), (2,2), \ldots (n,n)\}$

$|D| = n$

\# elements in the lower triangle

$= \cancel{\frac{n^2 - n}{2}} \quad \frac{\text{Total} - \text{Diagonal}}{2}$

$= \frac{n^2 - n}{2}$

$\because$ Lower Δ is symmetric to upper Δ $\therefore$ elements (v_i, v_j) & (v_j, v_i) are included

$\therefore$ \# Binary rel^n $= \cancel{\#\text{elements}}\; 2^{\#\text{elements in any one } \Delta} \times 2^{\#\text{element in diag}}$

$= 2^{\frac{n^2-n}{2}} \times 2^n = 2^{\frac{n^2+n}{2}} =$

all possible subsets of ~~(v_i,v_j)~~ (v_i, v_j) & its counterpart

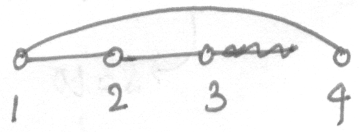
Undirected Graphs

Neighbourhood of a vertex $v \in V(G)$ ⟹ vertices adjacent to v and belong to E

$$N_G(v) = \{ u \mid \{v, u\} \in E(G) \}$$

$V(G) = \{1, 2, 3, 4\}$

$E(G) = \{ \{1,2\}, \{2,3\}, \{1,4\} \}$



$N_G(1) = \{2, 4\}$

$N_G(3) = \{2\}$

$N_G(2) = \{1,3\}$ $N_G(4) = \{1\}$

Degree of a Vertex $v \in V(G)$

$$d_G(v) = |N_G(v)|$$

$d_G(1) = 2$

$d_G(3) = 1$ $d_G(2) = 2$ $d_G(4) = 1$

Degree sequence of a graph G

$$D(G) = (d_{v_1}(G), d_{v_2}(G), \ldots\ldots d_{v_n}(G))$$

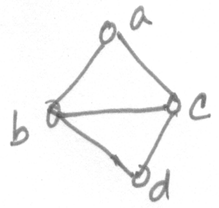
$$D(G) = (2, 1, 2, 1)$$

If a graph G is given, ⟹ D(G) exists

Q: If a sequence is given, are these sequences graphic?
↳ given an arbitary vector, does there exist a graph whose degree sequence correspond to the vector?

Ex: Is $(1,1,3,4,2)$ a graphic sequence ⟹ trying to construct a graph on 5 vertices s.t. $D(G) = \{1,1,3,4,2\}$ $d_G(v_1) = 1$

Observation:



$\forall v \in V(G)$ $0 \le d_G(v) \le n-1$ (degree of any vertex) ‖ ex: b, c are adj. to all

$n = |V(G)|$

Is $(4,3,2,1)$ graphic? ↳ graph on 4 vertices

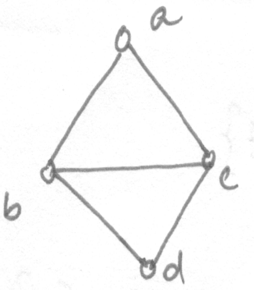
$D(G) = \{4,3,2,1\}$

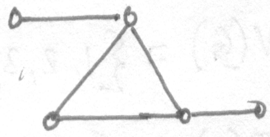
$D_G(v) = 4$ but any node in a the graph having 4 vertices can atmax have degree = 3

no graph exists corresponding to it

$(4,3,2,1)$ is not graphic

Q: Is $(3,3,3,3,3)$ is graphic?
↳ not graphic – why ??





$D(G) = (1,3,2,3,1)$
↳ $S = 10$
↓
$\underline{2 \times |E(G)|}$

$(2,3,3,2)$, Degree sum $(S) = \sum_{v_i \in V(G)} d_G(v)$

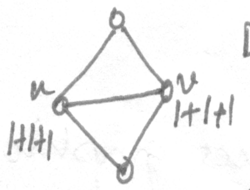
$S = 2+3+3+2 = 10$

Observation: For any G, $S = \sum d_G(v_i)$ is even

every edge contributes 1 to each of the vertices it connects

∴ in a graph, degree of a vertex is contributed by each edges it connects

each edge is counted once for left hand pt. & once for rt hand point

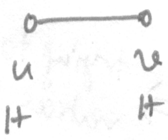


$D(G) = (2,3,3,2)$

$S = \sum d|v_i| = 10$

$\sum d_G(v_i) = 10 = 2 \times 5 = 2|E(G)|$

⇒ ~~Sum of~~



is counted once w.r.t $d_G(u)$

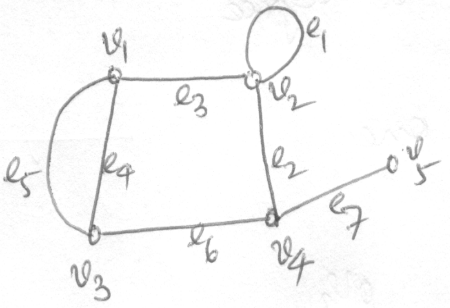
" " again " $d_G(v)$

∴ $\sum d_G(v_i) = 2 \times |E(G)|$

Sum of degrees is always even.

...ence & Degree

When a vertex v_i is an end vertex of some edge e_j, v_i & e_j are said to be incident (on or to) each other



e_2, e_6 & e_7 are incident with v_4

Two non parallel edges are adjacent if they are incident on a common Vertex. For ex: e_2 & e_7 are adjacent

Two vertices are adjacent if they are the end vertices of same edge
v_4 & v_5 are adjacent while v_1 & v_4 are not

The no. of edges incident on a vertex v_i, with self-loop counted twice is called degree, $d(v_i)$ of vertex v_i

$$d(v_1) = d(v_3) = d(v_4) = 3 \quad d(v_2) = 4 \quad d(v_3) = 1$$

$\boxed{\text{Degree is also called Valency}}$ **degree sequence: (3, 3, 3, 4, 1)**

Let us Consider a Graph G with e edges and n vertices

$v_1, v_2 \ldots v_n$.

∵ each edge contributes two degrees, the sum of degrees of all vertices in G is twice the no. of edges

i.e. $\boxed{\sum_{i=1}^{n} d(v_i) = 2e}$ ⇒ **the sum of degree sequence of ~~vertices~~ vertices in a graph are even and equal to twice no. of edges** *

degree sequence sum

Ex: $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = 3+4+3+3+1 = 14 = 2 \times \#\text{edges}$

Theorem: The no. of vertices of odd degree in a graph is even

Proof: Consider vertices with odd & even degree separately.

$$\underbrace{\sum_{i=1}^{n} d(v_i)}_{2\times\#\text{edges} \Downarrow \text{even}} = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k)$$

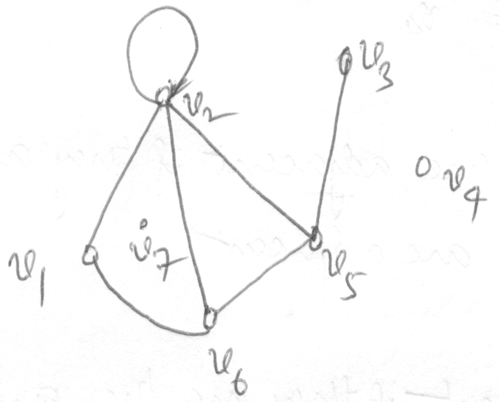
∵ LHS is even & 1st term of RHS is even
∴ $\sum_{\text{odd}} d(v_k)$ is also even

***Given a Sequence (3,3,3,3,5) Is it graphic?**
Soln: No., ∵ the sum of sequence 3+3+3+3+5 = 17 ⇒ not even

Isolated Vertex, Pendant Vertex & Null Graph

Isolated Vertex → Vertex with no incident edge
i.e. Vertices with zero degree

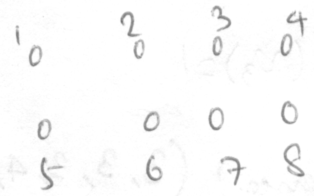
Pendent Vertex — Vertex with degree one



v_4, v_7 → isolated vertices
v_3 — pendant vertex

Two adjacent edges are said to be in series if their common vertex is of degree 2. Ex: Two edges incident on v_1 are in series

Null Graph → Graph without edges



$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ $E = \phi$

$|V| = 8$

If $V = \phi$ → then there is no graph