MIOT Graph Theory, Autumn 2022 - 2023
Quiz 1

Maximum marks: 30 **Time: 13-Sep-2023** **Duration: 45 minutes**

Q1. Prove or disprove: The complement of a simple disconnected graph must be connected. **[5]**

Solution : The statement is true.

Let G be a simple disconnected graph and u, v ∈ V (G). G' is the complement of G.
If u and v belong to different components of G, then the edge uv ∈ E(G'). If u and v ∈ to the same component of G, choose a vertex w in another component of G. (G has at least two components, since it is disconnected.) But then the edges uw and wv belong to E(G'). That is, in all cases there is a uv path in G'.

Q2. (a) Is the following sequence a graphic sequence (5; 5; 5; 4; 2; 1; 1; 1) ?
If it is graphic, produce a realization of the sequence, else prove why it is not graphic. **[4]**

Solution :

(5; 5; 5; 4; 2; 1; 1; 1) is graphic () (4; 4; 3; 1; 0; 1; 1), i.e., (4; 4; 3; 1; 1; 1; 0) is graphic ()
(3; 2; 0; 0; 1; 0), i.e., (3; 2; 1; 0; 0; 0) is graphic.

The last sequence is not graphic, since a vertex of degree 3 in a simple graph must have three neighbours each of positive degree.

(b) Show that there exists a simple graph with 12 vertices and 28 edges such that degree of each vertex is either 3 or 5. **[5]**

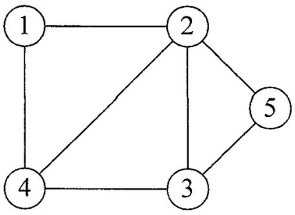
Solution :

Suppose there are p vertices with degree 3. Therefore, 3p+5(12-p) = 2*28. Thus, p=2. Thus, if there exists a graph, it has 2 vertices with degree 3 and 10 vertices with degree 5. thus the degree sequence is
v= [5 5 5 5 5 5 5 5 5 5 3 3].

To show that this is a graphic sequence, we demonstrate that v is a graphic vector and after 9 iterations we get a zero vector.

$$
\begin{aligned}
v_1 &= [5 \quad 5 \quad 5 \quad 5 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3] \\
v_2 &= [4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3 \quad 3 \quad 3] \\
v_3 &= [4 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3] \\
v_4 &= [3 \quad 3 \quad 3 \quad 3 \quad 2 \quad 2 \quad 2 \quad 2] \\
v_5 &= [2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2] \\
v_6 &= [2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1] \\
v_7 &= [2 \quad 1 \quad 1 \quad 1 \quad 1] \\
v_8 &= [1 \quad 1 \quad 0 \quad 0] \\
v_9 &= [0 \quad 0 \quad 0]
\end{aligned}
$$

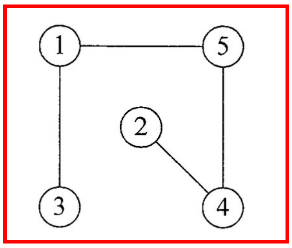
Q3. Identify whether the following graph is Bipartite :



The graph is non-bipartite.
Consider the situation, where the nodes are divided in to two sets A and B obeying the bipartite property that nodes having edges between them should not be placed in the same set. Following such a constraint nodes 1,2,3,4 could be grouped as follows : Set A = {1,3} Set B ={2,4}
However, node 5 cannot be placed in either of the sets since it has edge with both 2 which belongs to Set A and also with 3 which belongs to Set B.

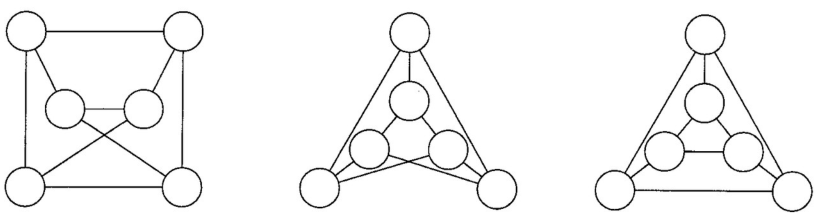
Obtain the complement of the Graph? Is the complement bipartite? **[6]**

Solution :



Bipartite graph since the nodes can be divided into two sets X and Y such that nodes having edges between them should not be placed in the same set. Following such a constraint nodes 1,2,3,4,5 could be grouped as follows : Set X = {1,2} Set Y ={3,4,5}

Q4. Determine if the following three graphs are isomorphic **[3]**



(a) (b) (c)

Solution : Figure (a) and (b) are isomorphic to K_{33}, and thus are isomorphic to each other.
However, (c) is not isomorphic to K_{33}

Q5. Show that the sum of the diagonal elements of the second power of adjacency matrix is twice the number of edges of the graph. **[4]**

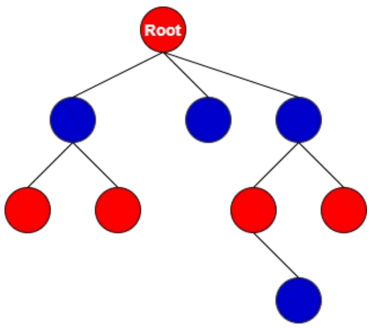
Using the definition of an adjacency matrix A, it follows that $[A^2]_{ii}$ is equal to the number of edges connected to i, i.e. the degree of i: d_i. Now recall that $\sum_i d_i = 2e$. Finally use the fact that the eigenvalues of A^2 are λ^2_i along with the fact that their sum is equal to the trace of A^2.

Q6. Show that tree is a bipartite graph **[3]**

Solution :

The proof is based on the fact that every bipartite graph is 2-chromatic.

Suppose a tree G(V, E). Let R be the root of the tree (any vertex can be taken as root). As we know there is only one path from R to any other vertex of the tree. Assign colour red to vertices at an even distance from R and the colour blue to vertices at an odd distance from R.



Vertices adjacent to each other cannot have the same colour. If 2 vertices with the same colour are adjacent then there exists more than one path from the root to these vertices.

For example, suppose two vertices v1 and v2 that are red and are adjacent. Both v1, v2 must be at an even distance from the root. But if we visit v2 through v1, it implies v2 is also at an odd distance from the root. This means there are two paths to reach v2, one of even length and the other of odd length. But there is exactly one path between any two vertices in a tree. Therefore, **the same colour vertices are not adjacent to each other**.

We can also say that 2 paths from root to any vertex implies there is a cycle in the tree which is not possible.We can bipartition the vertices by placing red vertices in one set and blue vertices in another set.

Hence, we can say that **every tree is bipartite**.
