The Notion of a Random Variable

- A random variable, x, is a variable whose variations are due to chance/randomness. A random variable can be considered as a function, which assigns a value to the outcome of an experiment.
- For example, in a coin tossing experiment, the corresponding random variable, x, can assume the values $x_1 = 0$ if the result of the experiment is "heads" and $x_2 = 1$ if the result is "tails."
- denote a random variable with a lower case roman, such as x, and the values it takes once an experiment has been performed, with mathmode italics, such as x.

The Notion of a Random Variable

- A random variable is described in terms of a set of **probabilities** if its values are of a discrete nature, or in terms of a **probability density function** (pdf) if its values lie anywhere within an interval of the real axis (non-countably infinite set).

Definitions of Probability

- Relative Frequency Definition: The probability, $P(A)$, of an event, A, is the limit

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n},$$

where n is the total number of trials and n_A the number of times event A occurred.

- In practice, one can use

$$P(A) \approx \frac{n_A}{n},$$

for large enough values of n. However, care must be taken on how large n must be, especially when $P(A)$ is very small.

- From a physical reasoning point of view, probability can also be understood as a measure of our uncertainty concerning the corresponding event.

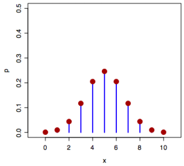
Definitions of Probability

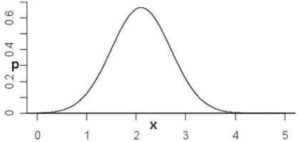
- Axiomatic Definition

1. The probability of an event A, $P(A)$ is a nonnegative number
$$P(A) \geq 0$$
2. The probability of an event C, which is certain to occur, is equal to one,
$$P(C) = 1$$
3. If two events, A and B, are mutually exclusive (they cannot occur simultaneously), then the probability of occurrence of either A or B (denoted as $A \cup B$) is given by
$$P(A \cup B) = P(A) + P(B)$$

Random Variables

- Informally, a random variable (r.v.) X denotes possible outcomes of an event
- Can be **discrete** (i.e., finite many possible outcomes) or **continuous**



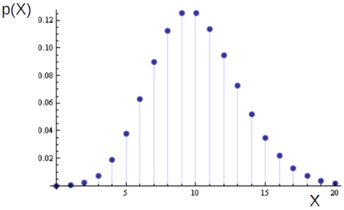


- Some examples of **discrete r.v.**
  - A random variable $X \in \{0, 1\}$ denoting outcomes of a coin-toss
  - A random variable $X \in \{1, 2, \ldots, 6\}$ denoteing outcome of a dice roll
- Some examples of **continuous r.v.**
  - A random variable $X \in (0, 1)$ denoting the bias of a coin
  - A random variable X denoting heights of students in CS
  - A random variable X denoting time to get to your hall from the department

Discrete Random Variables

- For a discrete r.v. X, $p(x)$ denotes the probability that $p(X = x)$
- $p(x)$ is called the probability mass function (PMF)

$$\begin{aligned} p(x) &\geq 0 \\ p(x) &\leq 1 \\ \sum_x p(x) &= 1 \end{aligned}$$



Discrete Random Variables

- A discrete random variable, x, can take any value from a finite or a countably infinite set, $\mathcal{X}$. The probability of an event "$\mathrm{x} = x$" is denoted as
$$P(\mathrm{x} = x) \text{ or simply } P(x).$$
- Assuming that no two values in $\mathcal{X}$ can occur simultaneously and that an experiment always returns a value, we have that
$$\sum_{x \in \mathcal{X}} P(x) = 1,$$
and $\mathcal{X}$ is known as the sample or state space.
- Joint probability: The joint probability of two events A and B to occur simultaneously is denoted as $P(A, B)$.
- Given two random variables $\mathrm{x} \in \mathcal{X}$ and $\mathrm{y} \in \mathcal{Y}$, the following sum rule is obtained
$$P(x) = \sum_{y \in \mathcal{Y}} P(x, y).$$

Discrete Random Variables

- Conditional probability: The conditional probability of an event A given another event B, is denoted as $P(A|B)$ and it is defined as

$$P(A|B) := \frac{P(A, B)}{P(B)}$$

- The above definition gives rise to the following product rule

$$P(A, B) = P(A|B)P(B)$$

- Expressed in terms of two random variables, x and y, we have

$$P(x, y) = P(x|y)P(y)$$

- $P(x)$ and $P(y)$ are also known as the marginal probabilities to be distinguished from the joint and the conditional ones.
- Statistical independence: Two random variables, x and y, are said to be statistically independent if and only if

$$P(x, y) = P(x)P(y),\ \forall x \in \mathcal{X},\ y \in \mathcal{Y}$$

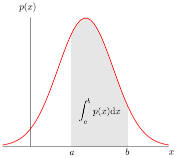
- Bayes Theorem: This important and elegant theorem is a direct consequence of the product rule and the symmetry property of the joint probability, i.e., $P(x, y) = P(y, x)$, and it is given by the following two equations,

$$\begin{aligned} P(x|y) &= \frac{P(y|x)P(x)}{P(y)} \\ P(y|x) &= \frac{P(x|y)P(y)}{P(x)} \end{aligned}$$

- What this theorem says is that, our **uncertainty** as expressed by the conditional probability $P(y|x)$ of an output variable, say y, given the value of an input, x, can be expressed the other way round; that is, in terms of the (uncertainty) conditional, $P(x|y)$ and the two marginal probabilities, $P(x)$ and $P(y)$.

Continuous Random Variables

- For a continuous r.v. X, a probability $p(X = x)$ is meaningless
- Instead we use $p(X = x)$ or $p(x)$ to denote the probability density at $X = x$
- For a continuous r.v. X, we can only talk about probability within an interval $X \in (x, x + \delta x)$
  - $p(x)\delta x$ is the probability that $X \in (x, x + \delta x)$ as $\delta x \to 0$



- The probability density $p(x)$ satisfies the following

$$p(x) \geq 0 \quad \text{and} \quad \int_x p(x)dx = 1 \qquad \text{(note: for continuous r.v., } p(x) \text{ can be } > 1\text{)}$$

- $p(.)$ can mean different things depending on the context
  - $p(X)$ denotes the distribution (PMF/PDF) of an r.v. X
  - $p(X = x)$ or $p(x)$ denotes the **probability** or **probability density** at point x
- Actual meaning should be clear from the context (but be careful)
- Exercise the same care when $p(.)$ is a specific distribution (Bernoulli, Beta, Gaussian, etc.)
- The following means **drawing a random sample** from the distribution $p(X)$

$$x \sim p(X)$$

Continuous Random Variables

- A continuous random variable, x, can take values anywhere in an in interval in the real axis $\mathbb{R}$.
- The starting point to develop tools for describing such variables is to build bridges with what we know from the discrete random variables case.
- The cumulative distribution function (cdf) is defined as

$$F(x) := P(\mathrm{x} \leq x).$$

  That is, cdf is the probability of the discrete event: "x takes any value less or equal to x".
- Thus, we can write

$$P(x_1 < \mathrm{x} \leq x_2) = F(x_2) - F(x_1).$$

- Assuming $F(x)$ to be differentiable, the probability density function (pdf), denoted with lower case p, is defined as

$$p(x) := \frac{dF(x)}{dx}.$$

- Then, it is readily seen that

$$P(x_1 < \mathrm{x} \leq x_2) = \int_{x_1}^{x_2} p(x)dx,$$

and

$$F(x) = \int_{-\infty}^{x} p(z)dz.$$

- Since an event is certain to occur in $-\infty < \mathrm{x} < +\infty$, we have that

$$\int_{-\infty}^{+\infty} p(x)dx = 1.$$

- The previously stated rules, for the discrete random variables case, are also valid for the continuous ones, i.e.,

$$p(x|y) = \frac{p(x,y)}{p(y)}, \quad p(x) = \int_{-\infty}^{+\infty} p(x,y)dy.$$

Mean, Variance and Covariance

- Two of the most useful quantities associated with a random variable, x, are:
  - The mean value, which is defined as:

$$\mathbb{E}[\mathrm{x}] := \int_{-\infty}^{+\infty} x p(x) dx.$$

  - The variance, which is defined as:

$$\sigma_x^2 := \int_{-\infty}^{+\infty} \left(x - \mathbb{E}[\mathrm{x}]\right)^2 p(x) dx,$$

  with integrations substituted by summations for the case of discrete variables, e.g.,

$$\mathbb{E}[\mathrm{x}] := \sum_{x \in \mathcal{X}} x P(x).$$

- More general, when a function f is involved, we have,

$$\mathbb{E}[f(\mathrm{x})] := \int_{-\infty}^{+\infty} f(x) p(x) dx.$$

Mean, Variance and Covariance

- It can readily be deduced from the respective definitions that, the mean value with respect to two random variables can be written as:

$$\mathbb{E}[\mathrm{x}, \mathrm{y}] := \mathbb{E}_{\mathrm{x}}\left[\mathbb{E}_{\mathrm{y}|\mathrm{x}}[f(\mathrm{x}, \mathrm{y})]\right].$$

- Given two random variables, x, y, their covariance is defined as

$$\mathsf{cov}(\mathrm{x}, \mathrm{y}) := \mathbb{E}\left[(\mathrm{x} - \mathbb{E}[\mathrm{x}])(\mathrm{y} - \mathbb{E}[\mathrm{y}])\right]$$

- Their correlation is defined as

$$r_{x,y} := \mathbb{E}[\mathrm{xy}] = \mathsf{cov}(\mathrm{x}, \mathrm{y}) + \mathbb{E}[\mathrm{x}]\mathbb{E}[\mathrm{y}]$$

- A random vector is a collection of random variables, $\mathbf{x} := [\mathrm{x}_1, \ldots, \mathrm{x}_l]^T$ and their joint pdf is denoted as

$$p(\boldsymbol{x}) = p(x_1, \ldots, x_l),\ \boldsymbol{x} = [x_1, \ldots, x_l]^T$$

Mean, Variance and Covariance

- The covariance matrix of a random vector, $\mathbf{x} \in \mathbb{R}^l$, is defined as

$$\mathsf{Cov}(\mathbf{x}) := \mathbb{E}\big[(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^T\big],$$

or

$$\mathsf{Cov}(\mathbf{x}) = \begin{bmatrix} \mathsf{cov}(\mathrm{x}_1, \mathrm{x}_1) & \dots & \mathsf{cov}(\mathrm{x}_1, \mathrm{x}_l) \\ \vdots & \ddots & \vdots \\ \mathsf{cov}(\mathrm{x}_l, \mathrm{x}_1) & \dots & \mathsf{cov}(\mathrm{x}_l, \mathrm{x}_l) \end{bmatrix}$$

- Similarly, the correlation matrix of $\mathbf{x}$ is defined as

$$R_x := \mathbb{E}\left[\mathbf{x}\mathbf{x}^T\right],$$

or

$${}_x \quad = \quad \begin{bmatrix} \mathbb{E}[\mathrm{x}_1, \mathrm{x}_1] & \dots & \mathbb{E}[\mathrm{x}_1, \mathrm{x}_l] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\mathrm{x}_l, \mathrm{x}_1] & \dots & \mathbb{E}[\mathrm{x}_l, \mathrm{x}_l] \end{bmatrix} = \mathsf{Cov}(\mathbf{x}) + \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{x}^T]$$

- Important Property: The covariance as well as the correlation matrices are positive semidefinite.
- A matrix A is called positive semidefinite, if

$$\boldsymbol{y}^T A \boldsymbol{y} \geq 0, \ \forall \boldsymbol{y} \in \mathbb{R}^l,$$

  and it is called positive definite if the inequality is a strict one.
- **Proof:** For the covariance matrix, we have

$$\boldsymbol{y}^T \mathbb{E}\left[\left(\mathbf{x} - \mathbb{E}[\mathbf{x}]\right)\left(\mathbf{x} - \mathbb{E}[\mathbf{x}]\right)^T\right] \boldsymbol{y} = \mathbb{E}\left[\left(\boldsymbol{y}^T\left(\mathbf{x} - \mathbb{E}[\mathbf{x}]\right)\right)^2\right] \geq 0.$$

- Let $\mathbf{x}$, $\mathbf{y}$ be two random vectors, which are related via a transform,

$$\mathbf{y} = \boldsymbol{f}(\mathbf{x}).$$

- The vector function $\boldsymbol{f}$ is assumed to be invertible. That is, there is a uniquely defined vector function, denoted as $\boldsymbol{f}^{-1}$, so that,

$$\mathbf{x} = \boldsymbol{f}^{-1}(\mathbf{y}).$$

- Given the pdf, $p_{\mathbf{x}}(\boldsymbol{x})$, of $\mathbf{x}$, it can be shown that,

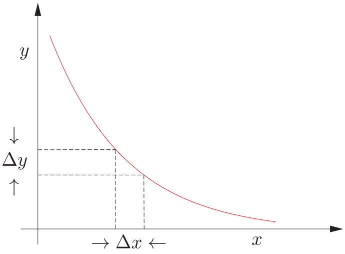
$$p_{\mathbf{y}}(\boldsymbol{y}) = \left. \frac{p_{\mathbf{x}}(\boldsymbol{x})}{\left|\det(J(\mathbf{y};\mathbf{x}))\right|} \right|_{\boldsymbol{x}=\boldsymbol{f}^{-1}(\boldsymbol{y})},$$

where the Jacobian matrix of the transformation is defined as

$$J(\mathbf{y};\mathbf{x}) := \frac{\partial(y_1, y_2, \ldots, y_l)}{\partial(x_1, x_2, \ldots, x_l)} := \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_l} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_l}{\partial x_1} & \cdots & \frac{\partial y_l}{\partial x_l} \end{bmatrix}.$$

Transformation of Random Variables

- We have denoted as $\det(\cdot)$ the determinant of a matrix and $|\cdot|$ the absolute value.
- For the case of two random variables, the previous formula becomes
$$p_{\mathrm{y}}(y) = \left.\frac{p_{\mathrm{x}}(x)}{\left|\frac{dy}{dx}\right|}\right|_{x=f^{-1}(y)}.$$
- The proof of the previous formula can be justified by carefully looking at the following figure and noting that $p(x)|\Delta x| = p(y)|\Delta y|$.



Example

- Let the two random vectors $\mathbf{x}$ and $\mathbf{y}$ be related by a linear transform, via an invertible matrix A,

$$\mathbf{y} = A\mathbf{x}.$$

- Then, it is easily checked out that the Jacobian matrix is equal to the matrix A,

$$J(\mathbf{y}; \mathbf{x}) = A.$$

- Thus, we readily obtain that,

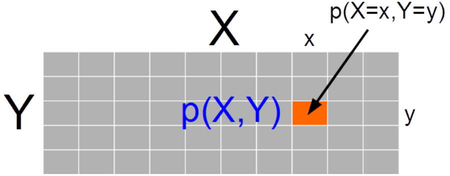
$$p_{\mathbf{y}}(\boldsymbol{y}) = \frac{p_{\mathbf{x}}(A^{-1}\boldsymbol{x})}{|\text{det}A|}.$$

Joint Probability Distribution

Joint probability distribution $p(X, Y)$ models probability of co-occurrence of two r.v. X, Y

For discrete r.v., the joint PMF $p(X, Y)$ is like a table (that sums to 1)

$$\sum_x \sum_y p(X = x, Y = y) = 1$$

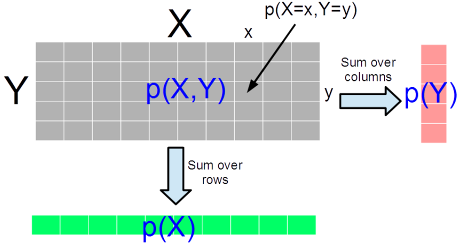


For continuous r.v., we have joint PDF $p(X, Y)$

$$\int_x \int_y p(X = x, Y = y) dx dy = 1$$

Marginal Probability Distribution

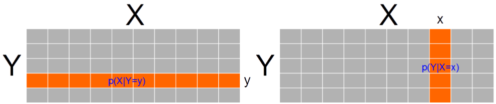
- Intuitively, the probability distribution of one r.v. regardless of the value the other r.v. takes
- For discrete r.v.'s: $p(X) = \sum_y p(X, Y = y)$, $\quad p(Y) = \sum_x p(X = x, Y)$
- For discrete r.v. it is the sum of the PMF table along the rows/columns



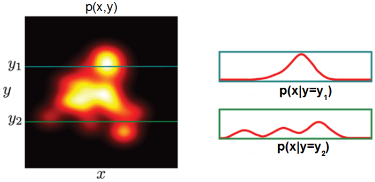
- For continuous r.v.: $p(X) = \int_y p(X, Y = y)dy$, $\quad p(Y) = \int_x p(X = x, Y)dx$
- Note: Marginalization is also called "integrating out" (especially in Bayesian learning)

Conditional Probability Distribution

- Probability distribution of one r.v. given the value of the other r.v.
- Conditional probability $p(X|Y=y)$ or $p(Y|X=x)$: like taking a slice of $p(X,Y)$
- For a discrete distribution:



- For a continuous distribution[1]:



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Some Basic Rules

- **Sum rule:** Gives the marginal probability distribution from joint probability distribution
  - For discrete r.v.: $p(X) = \sum_Y p(X, Y)$
  - For continuous r.v.: $p(X) = \int_Y p(X, Y)dY$
- **Product rule:** $p(X, Y) = p(Y|X)p(X) = p(X|Y)p(Y)$
- **Bayes rule**: Gives conditional probability

$$\boxed{p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}}$$

  - For discrete r.v.: $p(Y|X) = \frac{p(X|Y)p(Y)}{\sum_Y p(X|Y)p(Y)}$
  - For continuous r.v.: $p(Y|X) = \frac{p(X|Y)p(Y)}{\int_Y p(X|Y)p(Y)dY}$
- Also remember the **chain rule**

$$p(X_1, X_2, \ldots, X_N) = p(X_1)p(X_2|X_1)\ldots p(X_N|X_1, \ldots, X_{N-1})$$

CDF and Quantiles

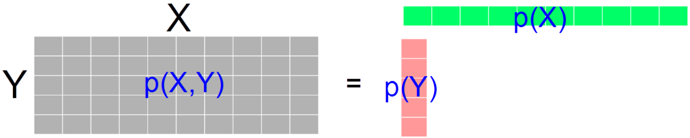
- Cumulative distribution function (CDF): $F(x) = p(X \leq x)$
- $\alpha \leq 1$ quantile is defined as the x_α s.t.

$$p(X \leq x_\alpha) = \alpha$$

Independence

- X and Y are independent ($X \perp\!\!\!\perp Y$) when knowing one tells nothing about the other

$$
\begin{aligned}
p(X|Y=y) &= p(X)\\
p(Y|X=x) &= p(Y)\\
p(X,Y) &= p(X)p(Y)
\end{aligned}
$$



- $X \perp\!\!\!\perp Y$ is also called **marginal independence**
- **Conditional independence** ($X \perp\!\!\!\perp Y|Z$): independence given the value of another r.v. Z

$$p(X,Y|Z=z) = p(X|Z=z)p(Y|Z=z)$$

Expectation

- **Expectation** or **mean** μ of an r.v. with PMF/PDF $p(X)$

$$\mathbb{E}[X] = \sum_x xp(x) \qquad \text{(for discrete distributions)}$$
$$\mathbb{E}[X] = \int_x xp(x)dx \qquad \text{(for continuous distributions)}$$

- **Note:** The definition applies to **functions of r.v.** too (e.g., $\mathbb{E}[f(X)]$)
- **Note:** Expectations are always w.r.t. the underlying probability distribution of the random variable involved, so sometimes we'll write this explicitly as $\mathbb{E}_{p()}[.]$, unless it is clear from the context
- **Linearity of expectation**

$$\mathbb{E}[\alpha f(X) + \beta g(Y)] = \alpha\mathbb{E}[f(X)] + \beta\mathbb{E}[g(Y)]$$

  (a very useful property, true even if X and Y are not independent)
- **Rule of iterated/total expectation**

$$\mathbb{E}_{p(X)}[X] = \mathbb{E}_{p(Y)}[\mathbb{E}_{p(X|Y)}[X|Y]]$$

Variance and Covariance

- **Variance** σ^2 (or "spread" around mean μ) of an r.v. with PMF/PDF $p(X)$

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mu^2$$

- **Standard deviation:** $\text{std}[X] = \sqrt{\text{var}[X]} = \sigma$
- For two scalar r.v.'s x and y, the **covariance** is defined by

$$\text{cov}[x, y] = \mathbb{E}\left[\{x - \mathbb{E}[x]\}\{y - \mathbb{E}[y]\}\right] = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y]$$

- For **vector** r.v. $\boldsymbol{x}$ and $\boldsymbol{y}$, the **covariance matrix** is defined as

$$\text{cov}[\boldsymbol{x}, \boldsymbol{y}] = \mathbb{E}\left[\{\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}]\}\{\boldsymbol{y}^T - \mathbb{E}[\boldsymbol{y}^T]\}\right] = \mathbb{E}[\boldsymbol{x}\boldsymbol{y}^T] - \mathbb{E}[\boldsymbol{x}]\mathbb{E}[\boldsymbol{y}^\top]$$

- Cov. of components of a vector r.v. $\boldsymbol{x}$: $\text{cov}[\boldsymbol{x}] = \text{cov}[\boldsymbol{x}, \boldsymbol{x}]$
- **Note:** The definitions apply to functions of r.v. too (e.g., $\text{var}[f(X)]$)
- **Note:** Variance of sum of independent r.v.'s: $\text{var}[X + Y] = \text{var}[X] + \text{var}[Y]$

KL Divergence

- KullbackLeibler divergence between two probability distributions $p(X)$ and $q(X)$

$$
\begin{aligned}
KL(p||q) &= \int p(X)\log\frac{p(X)}{q(X)}dX = -\int p(X)\log\frac{q(X)}{p(X)}dX \qquad \text{(for continuous distributions)} \\
KL(p||q) &= \sum_{k=1}^{K} p(X=k)\log\frac{p(X=k)}{q(X=k)} \qquad \text{(for discrete distributions)}
\end{aligned}
$$

- It is non-negative, i.e., $KL(p||q) \geq 0$, and zero if and only if $p(X)$ and $q(X)$ are the same
- For some distributions, e.g., Gaussians, KL divergence has a closed form expression
- KL divergence is not symmetric, i.e., $KL(p||q) \neq KL(q||p)$

Entropy

- Entropy of a continuous/discrete distribution $p(X)$

$$
\begin{aligned}
H(p) &= -\int p(X)\log p(X)dX \\
H(p) &= -\sum_{k=1}^{K} p(X=k)\log p(X=k)
\end{aligned}
$$

- In general, a peaky distribution would have a smaller entropy than a flat distribution
- Note that the KL divergence can be written in terms of expetation and entropy terms

$$KL(p||q) = \mathbb{E}_{p(X)}[-\log q(X)] - H(p)$$

- Some other definition to keep in mind: conditional entropy, joint entropy, mutual information, etc.

Transformation of Random Variables

Suppose $\boldsymbol{y} = f(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x} + \mathbf{b}$ be a linear function of an r.v. $\boldsymbol{x}$

Suppose $\mathbb{E}[\boldsymbol{x}] = \boldsymbol{\mu}$ and $\text{cov}[\boldsymbol{x}] = \boldsymbol{\Sigma}$

- Expectation of $\boldsymbol{y}$
$$\mathbb{E}[\boldsymbol{y}] = \mathbb{E}[\mathbf{A}\boldsymbol{x} + \mathbf{b}] = \mathbf{A}\boldsymbol{\mu} + \mathbf{b}$$
- Covariance of $\boldsymbol{y}$
$$\text{cov}[\boldsymbol{y}] = \text{cov}[\mathbf{A}\boldsymbol{x} + \mathbf{b}] = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T$$

Likewise if $y = f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x} + b$ is a scalar-valued linear function of an r.v. $\boldsymbol{x}$:

- $\mathbb{E}[y] = \mathbb{E}[\boldsymbol{a}^T\boldsymbol{x} + b] = \boldsymbol{a}^T\boldsymbol{\mu} + b$
- $\text{var}[y] = \text{var}[\boldsymbol{a}^T\boldsymbol{x} + b] = \boldsymbol{a}^T\boldsymbol{\Sigma}\boldsymbol{a}$

Common Probability Distributions

Important: We will use these extensively to model **data** as well as **parameters**

Some **discrete distributions** and what they can model:

- **Bernoulli:** Binary numbers, e.g., outcome (head/tail, 0/1) of a coin toss
- **Binomial:** Bounded non-negative integers, e.g., # of heads in n coin tosses
- **Multinomial:** One of K (>2) possibilities, e.g., outcome of a dice roll
- **Poisson:** Non-negative integers, e.g., # of words in a document
- .. and many others

Some **continuous distributions** and what they can model:

- **Uniform:** numbers defined over a fixed range
- **Beta:** numbers between 0 and 1, e.g., probability of head for a biased coin
- **Gamma:** Positive unbounded real numbers
- **Dirichlet:** vectors that sum of 1 (fraction of data points in different clusters)
- **Gaussian:** real-valued numbers or real-valued vectors
- .. and many others

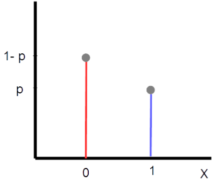
Discrete Distributions

Bernoulli Distribution

- Distribution over a binary r.v. $x \in \{0, 1\}$, like a coin-toss outcome
- Defined by a probability parameter $p \in (0, 1)$

$$P(x = 1) = p$$

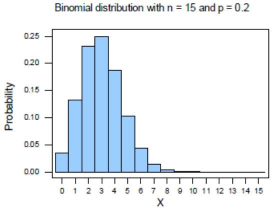
- Distribution defined as: $\text{Bernoulli}(x; p) = p^x(1 - p)^{1-x}$



- Mean: $\mathbb{E}[x] = p$
- Variance: $\text{var}[x] = p(1 - p)$

Binomial Distribution

- Distribution over number of successes m (an r.v.) in a number of trials
- Defined by two parameters: total number of trials (N) and probability of each success $p \in (0, 1)$
- Can think of Binomial as multiple independent Bernoulli trials
- Distribution defined as
$$\text{Binomial}(m; N, p) = \binom{N}{m} p^m (1-p)^{N-m}$$



- Mean: $\mathbb{E}[m] = Np$
- Variance: $\text{var}[m] = Np(1-p)$

Multinoulli Distribution

- Also known as the **categorical distribution** (models categorical variables)
- Think of a random assignment of an item to one of K bins - a K dim. binary r.v. $\boldsymbol{x}$ with single 1 (i.e., $\sum_{k=1}^{K} x_k = 1$): **Modeled by a multinoulli**

$$\underbrace{[0 \quad 0 \quad 0 \quad \ldots 0 \quad 1 \quad 0 \quad 0]}_{\text{length} = K}$$

- Let vector $\boldsymbol{p} = [p_1, p_2, \ldots, p_K]$ define the probability of going to each bin
  - $p_k \in (0, 1)$ is the probability that $x_k = 1$ (assigned to bin k)
  - $\sum_{k=1}^{K} p_k = 1$
- The multinoulli is defined as: $\text{Multinoulli}(\boldsymbol{x}; \boldsymbol{p}) = \prod_{k=1}^{K} p_k^{x_k}$
- Mean: $\mathbb{E}[x_k] = p_k$
- Variance: $\text{var}[x_k] = p_k(1 - p_k)$

Multinomial Distribution

- Think of repeating the Multinoulli N times
- Like distributing N items to K bins. Suppose x_k is count in bin k

$$0 \le x_k \le N \quad \forall\ k = 1, \dots, K, \qquad \sum_{k=1}^{K} x_k = N$$

- Assume probability of going to each bin: $\boldsymbol{p} = [p_1, p_2, \dots, p_K]$
- Multonomial models the bin allocations via a discrete vector $\boldsymbol{x}$ of size K

$$\begin{bmatrix} x_1 & x_2 & \dots x_{k-1} & x_k & x_{k-1} \dots & x_K \end{bmatrix}$$

- Distribution defined as

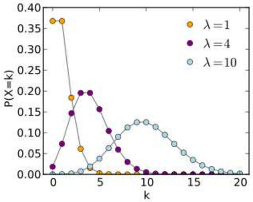
$$\text{Multinomial}(\boldsymbol{x}; N, \boldsymbol{p}) = \binom{N}{x_1 x_2 \dots x_K} \prod_{k=1}^{K} p_k^{x_k}$$

- Mean: $\mathbb{E}[x_k] = Np_k$
- Variance: $\text{var}[x_k] = Np_k(1 - p_k)$
- Note: For $N = 1$, multinomial is the same as multinoulli

Poisson Distribution

- Used to model a non-negative integer (count) r.v. k
- Examples: number of words in a document, number of events in a fixed interval of time, etc.
- Defined by a positive rate parameter λ
- Distribution defined as

$$\text{Poisson}(k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad k = 0, 1, 2, \ldots$$



- Mean: $\mathbb{E}[k] = \lambda$
- Variance: $\text{var}[k] = \lambda$

The Empirical Distribution

- Given a set of points $\phi_1, \ldots, \phi_K$, the empirical distribution is a discrete distribution defined as

$$p_{emp}(A) = \frac{1}{K} \sum_{k=1}^{K} \delta_{\phi_k}(A)$$

where $\delta_\phi(.)$ is the **dirac function** located at ϕ, s.t.

$$\delta_\phi(A) = \begin{cases} 1 & \text{if } \phi \in A \\ 0 & \text{if } \phi \in A \end{cases}$$

- The "weighted" version of the empirical distribution is

$$p_{emp}(A) = \sum_{k=1}^{K} w_k \delta_{\phi_k}(A) \qquad \qquad (\text{where } \sum_{k=1}^{K} w_k = 1)$$

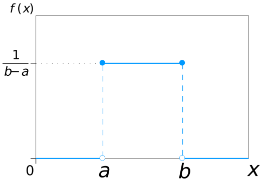
and the weights and points $(w_k, \phi_k)_{k=1}^{K}$ together define this discrete distribution

Continuous Distributions

Uniform Distribution

- Models a continuous r.v. x distributed uniformly over a finite interval $[a, b]$

$$\text{Uniform}(x; a, b) = \frac{1}{b-a}$$

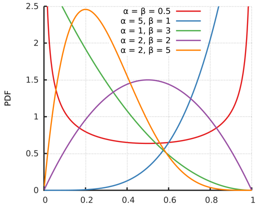


- Mean: $\mathbb{E}[x] = \frac{(b+a)}{2}$
- Variance: $\text{var}[x] = \frac{(b-a)^2}{12}$

Beta Distribution

- Used to model an r.v. p between 0 and 1 (e.g., a probability)
- Defined by two **shape parameters** α and β

$$\text{Beta}(p; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

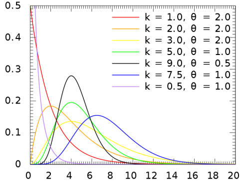


- Mean: $\mathbb{E}[p] = \frac{\alpha}{\alpha+\beta}$
- Variance: $\text{var}[p] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- Often used to model the probability parameter of a Bernoulli or Binomial (also **conjugate** to these distributions)

Gamma Distribution

- Used to model positive real-valued r.v. x
- Defined by a **shape parameters** k and a **scale parameter** θ

$$\text{Gamma}(x; k, \theta) = \frac{x^{k-1} e^{-\frac{x}{\theta}}}{\theta^k \Gamma(k)}$$



- Mean: $\mathbb{E}[x] = k\theta$
- Variance: $\text{var}[x] = k\theta^2$
- Often used to model the rate parameter of Poisson or exponential distribution (conjugate to both), or to model the inverse variance (precision) of a Gaussian (conjuate to Gaussian if mean known)

Dirichlet Distribution

- Used to model non-negative r.v. vectors $\boldsymbol{p} = [p_1, \ldots, p_K]$ that sum to 1

$$0 \leq p_k \leq 1, \quad \forall k = 1, \ldots, K \quad \text{and} \quad \sum_{k=1}^{K} p_k = 1$$

- Equivalent to a distribution over the $K - 1$ dimensional simplex
- Defined by a K size vector $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_K]$ of positive reals
- Distribution defined as

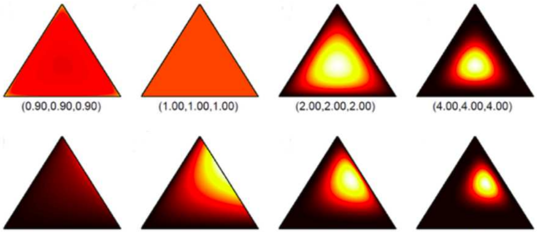
$$\text{Dirichlet}(\boldsymbol{p}; \boldsymbol{\alpha}) = \frac{\Gamma(\sum_{k=1}^{K} \alpha_k)}{\prod_{k=1}^{K} \Gamma(\alpha_k)} \prod_{k=1}^{K} p_k^{\alpha_k - 1}$$

- Often used to model the probability vector parameters of Multinoulli/Multinomial distribution
- Dirichlet is conjugate to Multinoulli/Multinomial
- **Note:** Dirichlet can be seen as a generalization of the Beta distribution. Normalizing a bunch of Gamma r.v.'s gives an r.v. that is Dirichlet distributed.

Dirichlet Distribution

- For $\boldsymbol{p} = [p_1, p_2, \ldots, p_K]$ drawn from Dirichlet$(\alpha_1, \alpha_2, \ldots, \alpha_K)$
  - Mean: $\mathbb{E}[p_k] = \frac{\alpha_k}{\sum_{k=1}^K \alpha_k}$
  - Variance: $\text{var}[p_k] = \frac{\alpha_k(\alpha_0 - \alpha_k)}{\alpha_0^2(\alpha_0+1)}$ where $\alpha_0 = \sum_{k=1}^K \alpha_k$
- Note: $\boldsymbol{p}$ is a point on $(K-1)$-simplex
- Note: $\alpha_0 = \sum_{k=1}^K \alpha_k$ controls how peaked the distribution is
- Note: α_k's control where the peak(s) occur

Plot of a 3 dim. Dirichlet (2 dim. simplex) for various values of $\boldsymbol{\alpha}$:



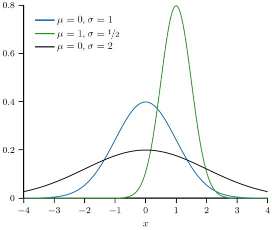
(0.90,0.90,0.90) (1.00,1.00,1.00) (2.00,2.00,2.00) (4.00,4.00,4.00)

(0.85,1.50,2.00) (1.00,1.76,2.35) (1.70,3.00,4.00) (3.40,6.00,8.00)

Univariate Gaussian Distribution

- Distribution over real-valued scalar r.v. x
- Defined by a scalar **mean** μ and a scalar **variance** σ^2
- Distribution defined as

$$\mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

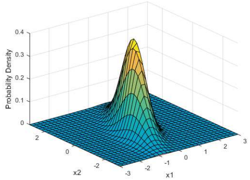


- Mean: $\mathbb{E}[x] = \mu$
- Variance: $\text{var}[x] = \sigma^2$
- Precision (inverse variance) $\beta = 1/\sigma^2$

Multivariate Gaussian Distribution

- Distribution over a multivariate r.v. vector $\boldsymbol{x} \in \mathbb{R}^D$ of real numbers
- Defined by a **mean vector** $\boldsymbol{\mu} \in \mathbb{R}^D$ and a $D \times D$ **covariance matrix** $\boldsymbol{\Sigma}$

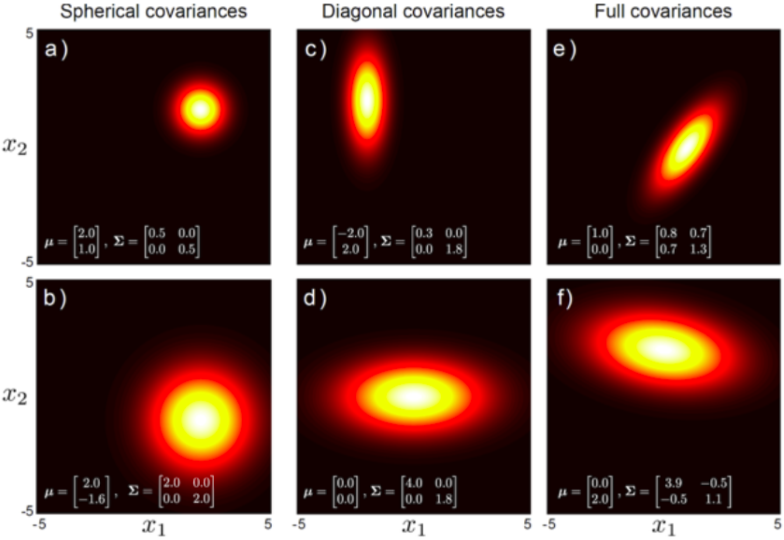
$$\mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^D |\boldsymbol{\Sigma}|}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$$



- The covariance matrix $\boldsymbol{\Sigma}$ must be symmetric and positive definite
  - All eigenvalues are positive
  - $\boldsymbol{z}^\top \boldsymbol{\Sigma} \boldsymbol{z} > 0$ for any real vector $\boldsymbol{z}$
- Often we parameterize a multivariate Gaussian using the inverse of the covariance matrix, i.e., the **precision matrix** $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$

Multivariate Gaussian: The Covariance Matrix

The covariance matrix can be spherical, diagonal, or full



Multivariate Gaussian: Marginals and Conditionals

- Given $\boldsymbol{x}$ having multivariate Gaussian distribution $\mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$. Suppose

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}, \quad \boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}$$

- The marginal distribution is simply

$$p(\boldsymbol{x}_a) = \mathcal{N}(\boldsymbol{x}_a|\boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa})$$

- The conditional distribution is given by

$$\begin{aligned} p(\mathbf{x}_a|\mathbf{x}_b) &= \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_{a|b}, \boldsymbol{\Lambda}_{aa}^{-1}) \\ \boldsymbol{\mu}_{a|b} &= \boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{aa}^{-1}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b) \end{aligned}$$

Thus marginals and conditionals of Gaussians are Gaussians

Multivariate Gaussian: Marginals and Conditionals

- Given the conditional of an r.v. $\boldsymbol{y}$ and marginal of r.v. $\boldsymbol{x}$, $\boldsymbol{y}$ is conditioned on

$$
\begin{aligned}
p(\mathbf{y}|\mathbf{x}) &= \mathcal{N}\left(\mathbf{y}|\mathbf{A}\mathbf{x}+\mathbf{b}, \mathbf{L}^{-1}\right) \\
p(\mathbf{x}) &= \mathcal{N}\left(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}\right)
\end{aligned}
$$

- Marginal of $\boldsymbol{y}$ and "reverse" conditional are given by

$$
\begin{aligned}
p(\mathbf{x}|\mathbf{y}) &= \mathcal{N}(\mathbf{x}|\boldsymbol{\Sigma}\{\mathbf{A}^{\mathrm{T}}\mathbf{L}(\mathbf{y}-\mathbf{b})+\boldsymbol{\Lambda}\boldsymbol{\mu}\}, \boldsymbol{\Sigma}) \\
p(\mathbf{y}) &= \mathcal{N}(\mathbf{y}|\mathbf{A}\boldsymbol{\mu}+\mathbf{b}, \mathbf{L}^{-1}+\mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}})
\end{aligned}
$$

  where $\boldsymbol{\Sigma} = (\boldsymbol{\Lambda} + \mathbf{A}^\top \mathbf{L}\mathbf{A})^{-1}$

- Note that the "reverse conditional" $p(\boldsymbol{x}|\boldsymbol{y})$ is basically the posterior of $\boldsymbol{x}$ is the prior is $p(\boldsymbol{x})$
- Also note that the marginal $p(\boldsymbol{y})$ is the predictive distribution of $\boldsymbol{y}$ after integrating out $\boldsymbol{x}$
- Very useful property for probabilistic models with Gaussian likelihoods and/or priors. Also very handly for computing **marginal likelihoods**.

Gaussians: Product of Gaussians

- Pointwise multiplication of two Gaussians is another (unnormalized) Gaussian

$$\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})\,\mathcal{N}(\mathbf{x}; \boldsymbol{\nu}, \mathbf{P}) = \frac{1}{Z}\mathcal{N}(\mathbf{x}; \boldsymbol{\omega}, \mathbf{T}),$$

where

$$\begin{aligned}
\mathbf{T} &= (\boldsymbol{\Sigma}^{-1} + \mathbf{P}^{-1})^{-1} \\
\boldsymbol{\omega} &= \mathbf{T}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \mathbf{P}^{-1}\boldsymbol{\nu}) \\
Z^{-1} &= \mathcal{N}(\boldsymbol{\mu}; \boldsymbol{\nu}, \boldsymbol{\Sigma} + \mathbf{P}) = \mathcal{N}(\boldsymbol{\nu}; \boldsymbol{\mu}, \boldsymbol{\Sigma} + \mathbf{P})
\end{aligned}$$

Multivariate Gaussian: Linear Transformations

- Given a $\boldsymbol{x} \in \mathbb{R}^d$ with a multivariate Gaussian distribution

$$\mathcal{N}(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Consider a linear transform of $\boldsymbol{x}$ into $\boldsymbol{y} \in \mathbb{R}^D$

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x} + \mathbf{b}$$

  where $\mathbf{A}$ is $D \times d$ and $\mathbf{b} \in \mathbb{R}^D$

- $\boldsymbol{y} \in \mathbb{R}^D$ will have a multivariate Gaussian distribution

$$\mathcal{N}(\boldsymbol{y}; \mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$$

Some Other Important Distributions

- Wishart Distribution and Inverse Wishart (IW) Distribution: Used to model $D \times D$ p.s.d. matrices
  - Wishart often used as a conjugate prior for modeling precision matrices, IW for covariance matrices
  - For $D = 1$, Wishart is the same as gamma dist., IW is the same as inverse gamma (IG) dist.
- Normal-Wishart Distribution: Used to model mean and precision matrix of a multivar. Gaussian
  - Normal-Inverse Wishart (NIW): : Used to model mean and cov. matrix of a multivar. Gaussian
  - For $D = 1$, the corresponding distr. are Normal-Gamma and Normal-Inverse Gamma (NIG)
- Student-t Distribution (a more robust version of Normal distribution)
  - Can be thought of as a mixture of infinite many Gaussians with different precisions (or a single Gaussian with its precision/precision matrix given a gamma/Wishart prior and integrated out)

Typical Distributions for Continuous Variables: The Gaussian

- The Central Limit Theorem: Consider N mutually independent random variables, each following its own distribution with mean values μ_i and variances $\sigma_i^2,\ i = 1, 2, \ldots, N$. Define a new random variable as their sum, i.e.,

$$\mathrm{x} = \sum_{i=1}^{N} \mathrm{x}_i.$$

Then, the mean and variance of the new variable are given by,

$$\mu = \sum_{i=1}^{N} \mu_i, \quad \text{and} \quad \sigma_x^2 = \sum_{i=1}^{N} \sigma_i^2.$$

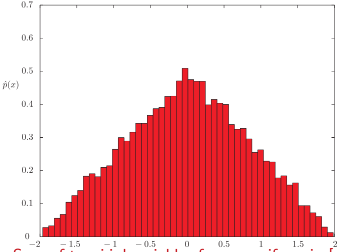
- It can be shown that, as $N \longrightarrow \infty$ the distribution of the normalized variable

$$\mathrm{z} = \frac{\mathrm{x} - \mu}{\sigma},$$

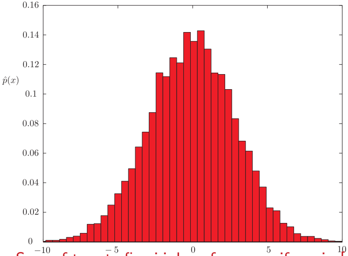
tends to the standard normal distribution, $\mathcal{N}(z|0, 1)$.

Typical Distributions for Continuous Variables: The Gaussian

- The Central Limit Theorem is one of the most important theorems in probability and statistics and it partly explains the popularity of the Gaussian distribution.
- In practice, even summing up a relatively small number of random variables, one can obtain a good approximation to a Gaussian. For example, if the individual pdfs are smooth enough and the random variables are identically and independently distributed (iid), a number between 5 to 10 may be sufficient.



Sum of two i.i.d variables from a uniform in [-1,1]



Sum of twenty five i.i.d r.v from a uniform in [-1,1]