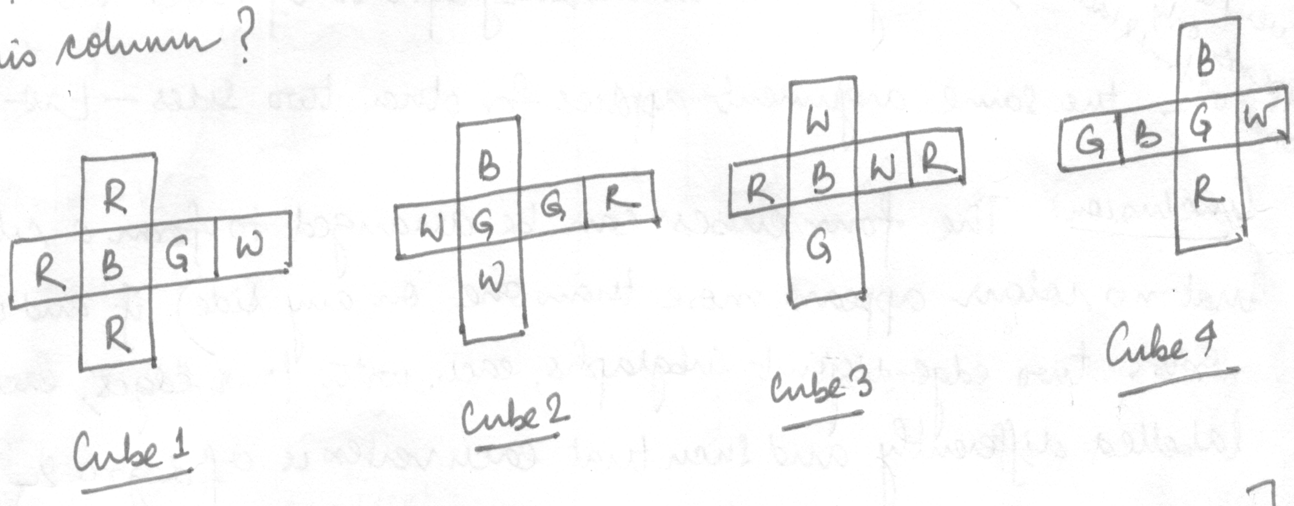
Solving a problem using Graph Theory

1. Convert the phycical problem to a graph-theory problem
2. Solve the graph theory problem

Example:

Given, four cubes. Six faces of the cube are coloured Blue, Green, Red or White. Is it possible to stack the cubes one on top of another to form a column such that no colour appear twice on any of the four sides of this column?

Solution:



Trial-error method: Try $3 \times 24 \times 24 \times 24 = 41,472$ possibility [IMPRACTICAL]

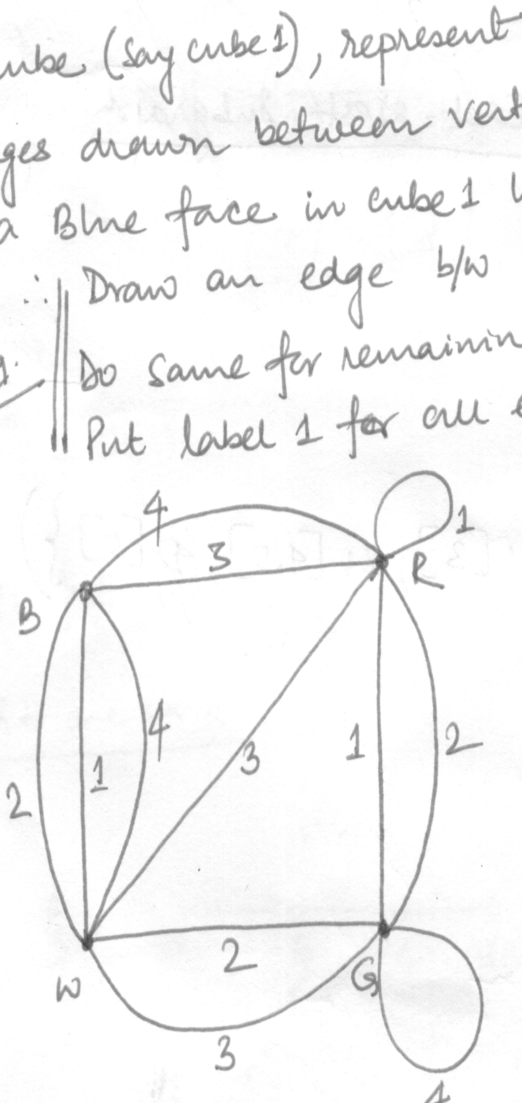
Graphical Solution: Graph with four edges B, G, R, W (One for each colour)

Pick a cube (say cube 1), represent its three pairs of opposite faces by three edges drawn between vertices of ~~op~~ appropriate colours.

ex: a Blue face in cube 1 has white face opposite to it

∴ Draw an edge b/w vertices B & W in the graph

step 1:
- Draw an edge b/w vertices B & W in the graph
- Do same for remaining two pairs of faces in Cube 1
- Put label 1 for all edges resulting from Cube 1; Repeat for other cubes



Step 2: For the graph obtained:

degree of each vertex = total no. of faces with corresponding colour

⇒ # Blue faces = 5

Green " = 6

Red " = 7

White " = 6

Cubes (say facing north & South).

A Subgraph (with four edges) will represent these eight faces — four facing north & four facing South

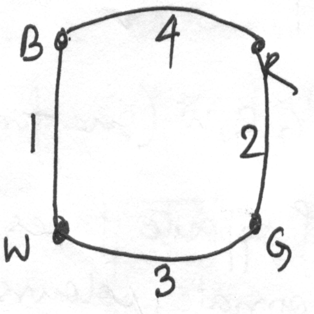
* - each of the four edges in this graph will have different labels — 1, 2, 3, 4 $\Rightarrow$ one edge from each cube

* - no colour occurs twice either on north (front) Side or South (back) Side

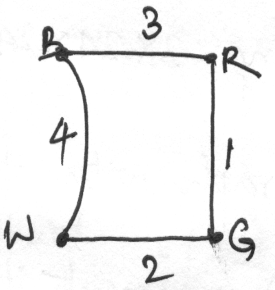
every colour appears only once on front & back $\Rightarrow$ every vertex in the subgraph is of vertex two

- the same argument applies for other two sides — East & West

<u>Conclusion</u>! The four cubes can be arranged to form a column such that no colour appears more than one on any side) if and only if there exists two edge-disjoint subgraphs, each with four edges, each edges labelled differently and such that each vertex is of degree 2



North-South Subgraph



East-West Subgraph

* https://sagecell.sagemath.org

```
G = Graph({0:[1,2,3], 1:[3], 2:[4,5], 4:[5]})
G.show()
G.Complement()
G.adjacency-matrix()
G.degree_sequence()
```

...plementary Graphs

G and G' are same
↓
Structurally

G & G^c are isomorphic to each other

Isomorphism

G, H are isomorphic if

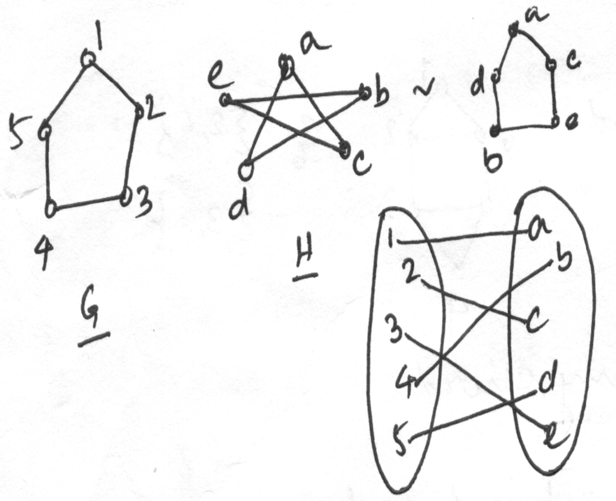
∃ Bijection (one-one map)

$V(H) \leftrightarrow V(G)$

$f: G \rightarrow H$

$\forall \ \{u,v\} \in E(G)$ iff $\{f(u), f(v)\} \in E(H)$

if u,v are adjacent then w.r.t map. f(u) & f(v) must be adjacent in H



G H

$\Rightarrow \{u,v\} \notin E(G)$

iff

$\{f(u), f(v)\} \notin E(H)$

G & H are isomorphic

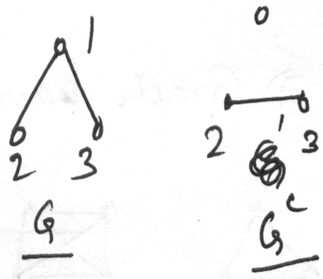
Complement of a Graph

Given, $G:(V(G), E(G))$

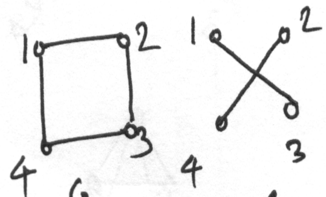
$V(G^c) = V(G)$

$E(G^c) = \{(u,v) \mid \{u,v\} \notin E(G)\}$ → edges not present in G

include missing edges of G

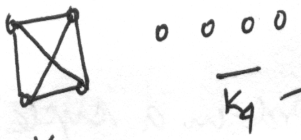


G G^c

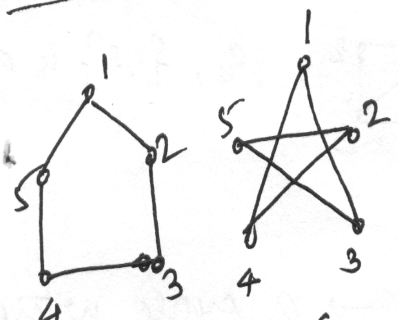


G G^c

If $|E(G)| = l$ $|E(G^c)| = {}^nC_2 - l$



K_4 $\overline{K_4}$ → edgeless graph



G G^c

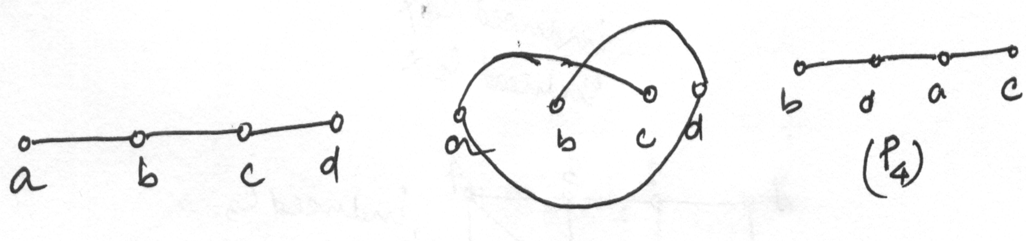
5 edges $\quad {}^5C_2 - 5 = 5$ edges

∴ G and G^c are isomorphic

∴ G is self-complementary ⇒

Comp of G_5 is G_5 itself

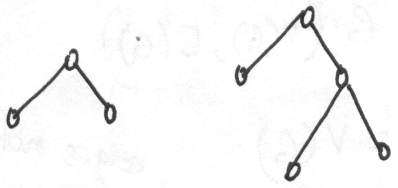
P_4 :



(P_4)

∴ P_4 is Self-Complementary

TREES → Connected Acyclic Graphs [any vertex can be reached from any other vertex]

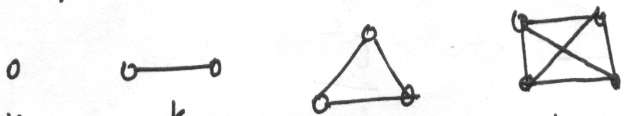


T1 T2

$|V(T)| = n$

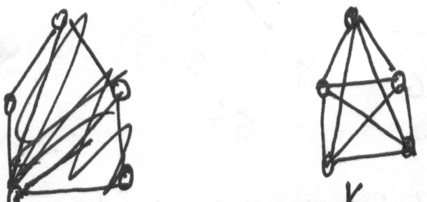
$|E(T)| = n-1$

Complete Graph ⇒ Graph having ~~a~~ max. no. of edges



K_1 K_2 K_3 K_4

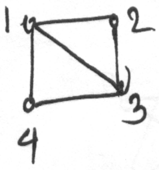
$|E(G)| = {}^nC_2 = \frac{n(n-1)}{2}$ $0 \le E(G) \le \frac{n(n-1)}{2}$



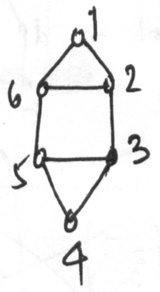
K_5

~~Induced Cycle~~

Chord → a chord in a cycle C is an edge joining pair of non-consecutive vertices in C

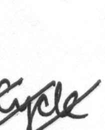


C_4, $\{1,3\}$ is a chord



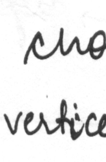
C_6 $\{2,6\}$ $\{3,5\}$

Induced cycle → a cycle without any chord

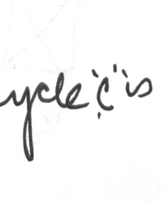


C_4

C_4 but not induced

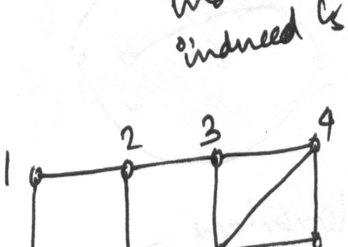


induced



not induced

induced C_3
induced C_4 ✗
induced C_5 ✗



induced C_3
induced C_4
induced C_6 ✗