Cosological Correlators from Scattering Amplitudes

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Abstract: These notes review some of the basic computations for cosmological correlators. These have been reviewed in several places and we discuss the same, sometimes from a slightly different point of view. Towards the end we discuss some recent progress on the relation between these correlators and scattering amplitudes.

[1]Send typos/correction/suggestions to the email address above!

Contents

References:

1. Very useful set of notes can be found on https://github.com/ddbaumann/cosmo-correlators.

2. Video Lectures by Nima are still the best lectures on the subject for section 2. These can be found in https://youtu.be/CU7m8m-9Hw4. Some parts of these lectures can be complemented by the notes [1].

3. QFT in the Schrodinger Picture can be found in [2].

4. Discussion of the Poles for the wave function with recursion relations can be found in [3] and the correlator in [4, 5]. The importance of one particular pole, the relation to flat space limit, can be found in [6].

5. Simple examples of performing loop integrals are given in detail with attached mathematica notebooks in [7].

6. Discussion of dressing Rules can be found in [8].

1 Motivation

The canonical motivation for studying cosmological correlators is usually to better understand inflation. However I personally think these are more ubiquitous objects and therefore more widely applicable even in areas that do not have anything to do with cosmology.

1. **Inflationary Correlators**: Most of the correlation functions for scalars and gravitons can be related to imprints left on the CMB or gravitational waves, which therefore, can be used to infer properties of inflation. However beyond the 2-pt function we have not seen anything and are unlikely to see anything beyond the 3-pt functions in the next 100 years. Although, even knowing properties about the 3-pt function imposes some consistency conditions on higher point functions.

2. **QFT in Curved Spacetime**: Apart from flat spacetime, there is no good notion of S-matrix (gauge invariant, field redefinition invariant) in other spacetimes. Hence the only "calculables" are correlation functions. In some cases and with some approximations, we can promote these calculables to observables. But generically even for answering concrete thought experimental questions, we can only compute correlation functions. Hence one can open their favorite QFT book and ask "how does page xyz generalize to dS?", almost 80% of the time that will not be known. Even basic questions involving unitarity, causality, cluster decomposition, etc. is not well understood in curved space.

3. **Expectation Values**: The name cosmological correlators is misleading as such objects can be computed in places even away from cosmology. For example, expectation values like $\langle\psi|h_{ij}|\psi\rangle$, $\langle\psi|\vec{E}|\psi\rangle$, enables one to compute expectation values of gravitiational and electric field in any general setting.

4. **Schwinger-Keldysh Path Integrals**: Computation of such expectation values naturally leads to path integrals which are different from the standard Feynman path integrals. For example, standard Feynman path integrals allow one to compute $\langle \text{out}|\text{in}\rangle$ correlators, however expectation values are examples of $\langle \text{in}|\text{in}\rangle$ correlators and require Schwinger-Keldysh path integrals.

5. **Special Functions**: In flat space one starts encountering functions (beyond Logarithms) while computing loops at higher points, or while stuyding String amplitudes at tree level. However in dS, one already encounters special functions while doing tree level computations. Hence the analytical structure of dS correlators are far less clearer than in flat space, as the same questions that are "easy" in flat space, are already "hard" in dS. Example: Single exchange in ϕ^3 theory.

6. **AdS/CFT**: Most of the literature has focused on the computation of the wave function. That is partly inspired by AdS/CFT. The ground state wave function is given by the

path integral

$$\psi[\phi(\vec{x})] = \int_{\varphi(-\infty,\vec{x})(0)=0}^{\varphi(0,\vec{x})=\phi(\vec{x})} D\varphi e^{-S[\varphi]} \tag{1.1}$$ {WF}

which is the same path integral one uses to compute the AdS partition function. Hence upto some analytical continutation these are similar objects for certain fields in the theory (example: light fields in dS). Thus a lot of attention has been given to compute this object including many recent discussions in dS quantum gravity (connection between the norm of the wave function and the sphere partition function).

7. **(Hidden) Simplicities in Path Integrals**: The whole process of computing the correlator from the wave function requires two kind of path integrals. Firstly the one you use to define the wave function, second, the one you use to get the correlator from the wave function,

$$\langle\psi|\phi(\vec{x}_1)\cdots\phi(\vec{x}_n)|\psi\rangle = \int D\phi|\psi[\phi]|^2\phi(\vec{x}_1)\cdots\phi(\vec{x}_n) \tag{1.2}$$ {corrfromWF}

That is why it was always thought that the correlation function is much harder than the wave function. However in the last few years we have found other ways of evaluating the correlator, which demonstrate a hidden simplicity in the entire structure.

8. **Polytopes**: A general theme in the study of amplitudes has been a way to repackage the amplitude such that its physical properties are manifest from a completely different point of view, usually mathematical. This has been very successful in flat space for the S-matrix and several examples have helped compute things beyond the standard approaches like Feynman diagrams and BCFW, eg: hidden connection between $\mathrm{tr}(\phi^3)$ theory and YM. There is a similar ongoing program in cosmology but hasn't been much application to gauge theories unlike in flat space.

2 Wave Functions in Flat Space

{sec:wf}

The most well-known path integral: the one with periodic boundary conditions. Typically for all these computations we use Euclidean time (but not very essential) $t \to i\tau$. This computes the partition function

$$Z = \int_{\phi(0)}^{\phi(\beta)=\phi(0)} D\phi e^{-S_E[\phi]} \tag{2.1}$$

Similarly, Dirichlet boundary conditions computes the **ground state** wave function

$$\Psi_0[\phi] = \int_{\varphi(-\infty)=0}^{\varphi(0)=\phi} D\varphi e^{-S} \tag{2.2}$$

{dirichlet}

This can also be worked out in the Lorentzian setting with some $i\epsilon$ prescriptions,

$$\Psi_0[\phi] = \int_{\varphi(-\infty(1-i\epsilon))=0}^{\varphi(0)=\phi} D\varphi e^{iS} \tag{2.3}$$

There is an interesting discussion on boundary conditions here Expand. To see why the path integral computes the ground state wave function we go back to the definition of the path integral,

$$\int_{\varphi(T)=\Phi'}^{\varphi(0)=\Phi} D\varphi e^{iS} = \langle\Phi'|e^{-iHT}|\Phi\rangle \tag{2.4}$$

One can extract the vacuum wave function using $T \to -\infty(1-i\epsilon)$, to get,

$$\langle\Phi'|e^{-iHT}|\Phi\rangle = \sum_{n=0}^{\infty} \langle\Phi'|e^{-iHT}|n\rangle \langle n|\Phi\rangle = \sum_{n=0}^{\infty} \langle\Phi'|n\rangle\, e^{-iTE_n} \langle n|\Phi\rangle \propto \psi_0^*(\Phi')\psi_0(\Phi) \tag{2.5}$$

In the final step we have used the fact that dominant term from the sum in the limit $T \to \infty$ comes from $E_n \to 0$, which essentially picks up the ground state. Thus,

$$\int_{\varphi(-\infty+i\epsilon)=0}^{\varphi(0)=\Phi} D\phi e^{iS} \propto \psi_{vac}(\Phi) \tag{2.6}$$

A more familiar equation is the Schrodinger equation and one can derive that from here. This roughly becomes equivalent to saying that

$$\psi(\vec{x},t) = e^{-i\int_0^t Hdt}\psi(\vec{x},0) \equiv \int Dx e^{iS}\psi(\vec{x},0) \tag{2.7}$$

Thus when we ask for the ground state of the wave function we are really asking "what happens to a state at a given time which is the Gaussian at early times?".
Of course, without any interactions or any time dependence in the mass, etc. the state would continue to remain a Gaussian.

2.1 Free Theory

We see how the Gaussian comes about from a path integral computation. Consider the free action of a HO,

$$S = \int_{-\infty}^{0} dt \left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\omega^2\varphi^2\right). \tag{2.8}$$

This is a quadratic PI, and hence can be evaluated exactly using the saddle point method,

$$\int_{\varphi\big(-\infty(1-i\varepsilon)\big)\to 0}^{\varphi(0)=\phi} \mathcal{D}\varphi\ e^{iS} = e^{iS[\varphi_{\text{cl}}]} \equiv e^{iS[\phi]}, \tag{2.9}$$

where φ_{cl} satisfies

$$\ddot{\varphi}_{\text{cl}} + \omega^2\varphi_{\text{cl}} = 0, \qquad \varphi_{\text{cl}} = Ae^{i\omega t} + Be^{-i\omega t}. \tag{2.10}$$

The boundary conditions are

$$\varphi\big(-\infty(1-i\varepsilon)\big) \to 0, \qquad \varphi(0) = \phi. \tag{2.11}$$

From the former we are forced to have $B = 0$. The latter then fixes the classical solution to be

$$\varphi_{\text{cl}}(t) = \phi\, e^{i\omega t}. \tag{2.12}$$

Plugging this solution in, we get the classical action

$$\begin{aligned} S[\varphi_{\text{cl}}] &= \int_{-\infty}^{0} dt \left(\frac{1}{2}\dot{\varphi}_{\text{cl}}^2 - \frac{1}{2}\omega^2\varphi_{\text{cl}}^2\right) \\ &= \int_{-\infty}^{0} dt\ \frac{1}{2}\partial_t(\dot{\varphi}_{\text{cl}}\varphi_{\text{cl}}) - \frac{1}{2}\int_{-\infty}^{0} dt\ \varphi_{\text{cl}}(\ddot{\varphi}_{\text{cl}} + \omega^2\varphi_{\text{cl}}) \\ &= \frac{1}{2}\dot{\varphi}_{\text{cl}}(0)\,\varphi_{\text{cl}}(0) = \frac{i\omega\phi^2}{2}. \end{aligned} \tag{2.13}$$

Hence,

$$\psi[\varphi_{\text{cl}}] = e^{iS[\varphi_{\text{cl}}]} = e^{-\frac{1}{2}\omega\phi^2}. \tag{2.14}$$

2.2 Interacting Theory

To get more interesting dynamics we consider something that has interactions. For example, consider the ϕ^4 theory with the action

$$S_{\phi^4} = -\int_0^{\infty} dtd^3x\frac{1}{2}\big((\partial_t\phi)^2 - (\partial_i\phi)^2 - m^2\phi^2\big) - \frac{\lambda}{4!}\int_0^{\infty} dtd^3 \tag{2.15}$$ {phi4lag}

The path integral we would like to do is[1]

$$\psi(\phi) = \int_{\varphi(0,\vec{x})=0}^{\varphi(0,\vec{x})=\phi(\vec{x})} D\varphi e^{-\int_0^{\infty} dtd^3x\frac{1}{2}\big((\partial_t\phi)^2-(\partial_i\phi)^2-m^2\phi^2\big)-\frac{\lambda}{4!}\int_0^{\infty} dtd^3x\varphi^4} \tag{2.16}$$ {phi4ex}

[1]For this discussion, Lorentzian vs Euclidean will not be important, though these are very important for other things like divergences, gravity, etc.

There are multiple ways of attacking this path integral. For example, by construction of a generating functional and then differentiating that, this is conventionally done in the AdS/CFT literature, for example, in section 5 of the book [9] or in [10]. For tree level examples this is also reviewed using the saddle point approximation in https://github.com/ddbaumann/cosmo-correlators. We will follow a direct brute force approach to evaluating this. This is roughly similar to the treatment found in [1]. We demonstrate how the path integral in (2.16) is done to $O(\lambda)$ and the appearance of Witten diagrams. We expand the path integral in λ and obtain

$$\psi(\phi) = \int_{\varphi(0,\vec{x})=0}^{\varphi(\infty,\vec{x})=\phi(\vec{x})} D\varphi e^{-\int d^4x \frac{1}{2}\left((\partial_t\phi)^2-(\partial_i\phi)^2-m^2\phi^2\right)}\left(1-\frac{\lambda}{4!}\int d^4x\varphi^4(t,\vec{x})\right) \tag{2.17}$$

where we use the shorthand $\int d^4x = \int_0^\infty dt \int d^3x$. The $O(\lambda^0)$ term is something that we saw for the free theory. To see it explicitly we can pull out the contribution from the free factor which we evaluated in the previous section and write the wave function as

$$\psi(\phi) = e^{-\frac{1}{2}\int d^3k\phi(\vec{k})\omega_k\phi(-\vec{k})} \int_{\varphi(0,\vec{x})=0}^{\varphi(\infty,\vec{x})=\phi(\vec{x})} D\varphi e^{-S_0[\varphi]}\left(1-\frac{\lambda}{4!}\int d^4x\varphi^4(t,\vec{x})\right) \tag{2.18}$$

where now

$$S_0 = \int d^3k dt \varphi(t,-\vec{k})(\partial^2-\omega_k^2)\varphi(t,\vec{k}) \tag{2.19}$$

with $\omega_k^2 = k^2+m^2$. The $O(\lambda)$ term can be done in a way we do path integrals with boundary conditions. To ease up notations we denote

$$\psi_1(\phi) = \int d^4x \int_{\varphi(0,\vec{x})=0}^{\varphi(0,\vec{x})=\phi(\vec{x})} D\varphi\varphi^4(t,\vec{x})e^{-S_0[\phi]} \tag{2.20}$$

such that the wave function $\psi(\phi)$ becomes

$$\psi(\phi) = e^{-\frac{1}{2}\int d^3k\phi(\vec{k})\omega_k\phi(-\vec{k})}\left(1-\frac{\lambda}{4!}\psi_1(\phi)\right)$$

To find $\psi_1(\phi)$ we work in **momentum space**, where this is defined by taking a Fourier transform in 3-dimensions. This is also the generic situation in dS space[2] and therefore is a good idea. Denoting the Fourier transform of $\varphi(t,\vec{x})$ as

$$\varphi(t,\vec{x}) = \int d^3k\varphi(t,\vec{k})e^{i\vec{k}\cdot\vec{x}} \tag{2.21}$$

we get the following integral for $\psi_1(\phi)$,

$$\psi_1(\phi) = \int_0^\infty dt \int d^3k_1\cdots d^3k_4\delta(\vec{k}_1+\cdots\vec{k}_4)\int_{\varphi(\infty,\vec{x})=0}^{\varphi(0,\vec{k})=\phi(\vec{k})} D\varphi\varphi(t,\vec{k}_1)\cdots\varphi(t,\vec{k}_4)e^{-S_0} \tag{2.22}$$ {psi1}

[2]Homogeneity and Isotropy would imply that a function $f(\vec{x}_1,\vec{x}_2) \equiv f(|\vec{x}_1-\vec{x}_2|)$.

where the $\delta(\vec{k}_1 + \cdots + \vec{k}_4)$ arises from the d^3x integral and reflects spatial translational invariance. To perform the path integral in $\varphi(t, \vec{k})$, it is convenient to split the field into a *classical* and a *quantum* part which have the following properties (this trick is done in many places, including [11]).

$$\varphi(t, \vec{k}) = \varphi_c(t, \vec{k}) + \varphi_q(t, \vec{k}) \tag{2.23}$$

with the split such that the field $\varphi_c(t, \vec{k})$ satisfies the classical equations of motion,

$$(\partial_t^2 - k^2)\varphi_c(t, \vec{k}) = 0 \tag{2.24}$$ {eomphic}

and the boundary conditions,

$$\begin{aligned} &\varphi_c(0, \vec{k}) = \phi(\vec{k}), \qquad \varphi_c(\infty, \vec{k}) = 0, \\ &\varphi_q(0, \vec{k}) = 0, \qquad \varphi_q(\infty, \vec{k}) = 0. \end{aligned} \tag{2.25}$$ {bndycond}

Since the field $\varphi_c(t, \vec{k})$ is chosen to satisfy the boundary conditions of the field $\varphi(t, \vec{k})$, it automatically provides the boundary conditions satisfied by the field $\varphi_q(t, \vec{k})$. With this split we note that the action itself has a very simple form

$$\begin{aligned} S_0 &= \int d^3k dt \varphi(t, -\vec{k})(\partial^2 - k^2)\varphi(t, \vec{k}) \\ &= \int d^3k dt \varphi_c(t, -\vec{k})(\partial^2 - k^2)\varphi_c(t, \vec{k}) + \int d^3k dt \varphi_q(t, -\vec{k})(\partial^2 - k^2)\varphi_c(t, \vec{k}) \\ &\quad + \int d^3k dt \varphi_c(t, -\vec{k})(\partial^2 - k^2)\varphi_q(t, \vec{k}) + \int d^3k dt \varphi_q(t, -\vec{k})(\partial^2 - k^2)\varphi_q(t, \vec{k}) \end{aligned} \tag{2.26}$$

The first two terms on the RHS are zero since φ_c satisfies the EOM. The third term is also zero because of an integration by parts and using the boundary conditions of the fields. Therefore the only remaining term is the fourth term,

$$S_0 = \int d^3k dt \varphi_q(t, -\vec{k})(\partial_t^2 - k^2)\varphi_q(t, \vec{k}) \tag{2.27}$$

Using this and the definition of the field φ_c we can find the explicit solutions of the fields by the Green functions method. The field φ_c is particularly simple, as we can easily solve (2.24) with the boundary conditions (2.25) to give,

$$\varphi_c(t, \vec{k}) = \phi(\vec{k}) e^{-\omega_k t} \tag{2.28}$$

The mode e^{-kt} is selected from the boundary condition at $t \to \infty$ (In Lorentzian spacetime it would be supplemented with an $i\epsilon$ prescription). These are referred to as the *Bulk-Boundary propagators* and are normally obtained as a Bessel function. The field φ_q can be represented by the Green function method,

$$\varphi_q(t, \vec{k}) = \int_0^\infty dt' G(t, t', \vec{k}) \varphi_q(t', \vec{k}) \tag{2.29}$$

where the function $G(t,t',\vec{k})$ is obtained from,

$$(\partial_t^2+\omega_k^2)G(t,t',\vec{k})=\delta(t-t') \tag{2.30}$$

and satisfies

$$G(t,t',\vec{k})=G(t',t,\vec{k}),\qquad G(0,t',\vec{k})=G(\infty,t',\vec{k})=0. \tag{2.31}$$

Without the boundary conditions this would be the same as the Feynman Green function we learn in amplitudes. However due to the additional boundary conditions we get

$$G(t,t',\vec{k})=\frac{1}{2\omega_k}\Big[\Theta(t-t')e^{-\omega_k(t-t')}+\Theta(t'-t)e^{-\omega_k(t'-t)}-e^{-\omega_k(t+t')}\Big] \tag{2.32}$$ {bulkbulk}

These are known as the *Bulk-Bulk propagators* and are conventionally expressed in terms of Bessel K, I functions. Where the two Bessel functions appear as a consequence of the boundary conditions on the fields. Note that due to the last term in the Green function above it is not invariant time translations.

We can now evaluate (2.22). We first evaluate the path integrals over the fields $\varphi(t,\vec{k})$ and then evaluate the time integrals.

$$\begin{aligned}&\int_{\varphi(\infty,\vec{x})=0}^{\varphi(0,\vec{k})=\phi(\vec{k})}D\varphi\varphi(t,\vec{k}_1)\cdots\varphi(t,\vec{k}_4)e^{-S_0}\\&=\int_{\varphi_q(\infty,\vec{x})=0}^{\varphi_q(0,\vec{k})=0}D\varphi_q\big(\varphi_c(t,\vec{k}_1)+\varphi_q(t,\vec{k}_1)\big)\cdots\big(\varphi_c(t,\vec{k}_4)+\varphi_q(t,\vec{k}_4)\big)e^{-\int d^3kdt\varphi_q(t,\vec{-}k)(\partial_t^2+k^2)\varphi(t,\vec{k})}\end{aligned} \tag{2.33}$$

Since the action is even in φ_q the odd-point terms involving them are zero. Thus the remaining terms include

$$\begin{aligned}&\int_{\varphi(0,\vec{x})=0}^{\varphi(0,\vec{k})=\phi(\vec{k})}D\varphi\varphi(t,\vec{k}_1)\cdots\varphi(t,\vec{k}_4)e^{-S_0}\\&=\varphi_c(t,\vec{k}_1)\cdots\varphi_c(t,\vec{k}_4)\int D\varphi_q e^{-S_0}+\sum_{\text{perms}}\varphi_c(t,\vec{k}_1)\varphi_c(t,\vec{k}_2)\int D\varphi_q\varphi_q(t,\vec{k}_3)\varphi_q(t,\vec{k}_4)e^{-S_0}\\&\quad+\int D\varphi_q\varphi_q(t,\vec{k}_1)\cdots\varphi_q(t,\vec{k}_4)e^{-S_0}\end{aligned} \tag{2.34}$$ {phi4exp1}

We thus have the contribution to the 4-pt wavefunction coefficient, the 2-pt wave function coefficient and the normalization term. The integral $\int D\varphi_q e^{-S_0}$ can be absorbed in the normalization of the wave function.

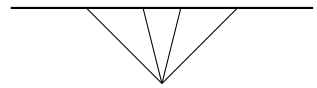
Thus we are left with the following contributions to the wave function.

The first term in (2.34) becomes

$$\begin{aligned}&\varphi_c(t,\vec{k}_1)\cdots\varphi_c(t,\vec{k}_4)\int D\varphi_q e^{-S_0}\\&\longrightarrow\int d^3k_1\cdots d^3k_4\phi(\vec{k}_1)\cdots\phi(\vec{k}_4)\delta(\vec{k}_1+\cdots k_4)\int_0^\infty dte^{-t(\omega_{k_1}+\cdots\omega_{k_4})}\end{aligned} \tag{2.35}$$

which can be diagrammatically represented as,

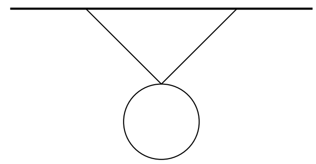
$$\int d^3k_1 \cdots d^3k_4 \phi(\vec{k}_1) \cdots \phi(\vec{k}_4) \delta(\vec{k}_1 + \cdots k_4) \int_0^\infty dt e^{-t(\omega_{k_1} + \cdots \omega_{k_4})} =$$



(2.36)

This is a contact diagram.
Similarly, the second term in (2.34) becomes a tadpole,

$$\sum_{\text{perms}} \varphi_c(t, \vec{k}_1) \varphi_c(t, \vec{k}_2) \int D\varphi_q \varphi_q(t, \vec{k}_3) \varphi_q(t, \vec{k}_4) e^{-S_0} \longrightarrow$$

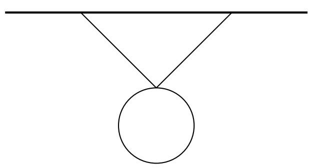


(2.37)

where the integral over the φ_q fields gives the bulk-bulk propagator,

$$\int D\varphi_q \varphi_q(t, \vec{k}_3) \varphi_q(t, \vec{k}_4) e^{-S_0} = \delta(\vec{k}_3 + \vec{k}_4) G(t, t, \vec{k}_3) \tag{2.38}$$

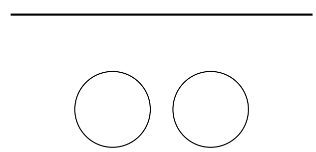
Through this we can compute the integral over $\vec{k}_3$ to give the following integral representation



$$= \int d^3k_1 d^3k_2 \phi(\vec{k}_1) \phi(\vec{k}_2) \delta(\vec{k}_1 + \vec{k}_2) \int d^3k' \int_0^\infty dt \; G(t, t, \vec{k}) \tag{2.39}$$

The final term in (2.34) is a disconnected diagram and is given as

$$\int D\varphi_q \varphi_q(t, \vec{k}_1) \cdots \varphi_q(t, \vec{k}_4) e^{-S_0} =$$



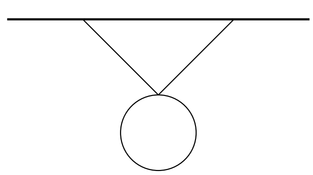
(2.40)

These are the analogues of disconnected graphs and can be ommitted by working with $\log[\psi]$. Such diagrams are called **Witten diagrams** and they compute the **wave function coefficients**,

$$\begin{aligned} \psi(\phi) = e^{\int d^3k \phi(\vec{k}) k^2 \phi(-\vec{k})} \Big[1 + \lambda \int d^3k_1 d^3k_2 \delta(\vec{k}_1 + \vec{k}_2) \phi(\vec{k}_1) \phi(\vec{k}_2) \psi_2(\vec{k}_1, \vec{k}_2) \\ + \lambda \int d^3k_1 \cdots d^3k_4 \delta(\vec{k}_1 + \cdots + \vec{k}_4) \phi(\vec{k}_1) \cdots \phi(\vec{k}_4) \psi_4(\vec{k}_1, \cdots, \vec{k}_4) \Big] \end{aligned} \tag{2.41}$$

where for the example above the coefficients ψ_2 and ψ_4 are given as,

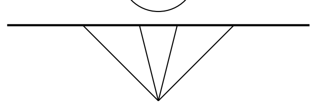
$$\psi_2(\vec{k}_1, \vec{k}_2) =$$



$$= \int d^3k' \int_0^\infty dt \; e^{-k_1 t} G(t, t, \vec{k}'),$$

(2.42)

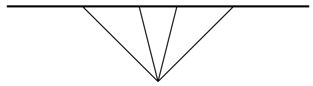
$$\psi_4(\vec{k}_1, \cdots, \vec{k}_4) =$$



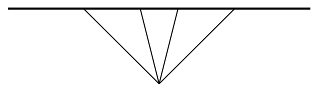
$$= \int_0^\infty dt e^{-(\omega_{k_1} + \omega_{k_2} + \omega_{k_3} + \omega_{k_4})t}$$

2.3 Time Integrals

For the contact graphs in ϕ^4 theory we get

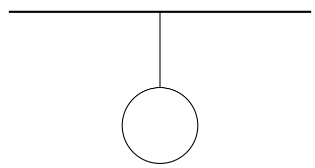
$\psi_4(\vec{k}_1, \cdots, \vec{k}_4) =$  $= \int_0^\infty dt e^{-(\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4})t}$ (2.43)

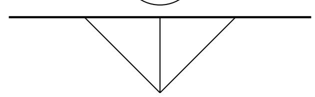
We shall often suppress the spatial momentum conserving delta function, $\delta(\vec{k}_1 + \cdots + \vec{k}_4)$. In this case the time integrals are easy to perform and gives simple poles in the energies,

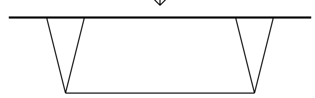
 $= \dfrac{1}{\omega_{k_1} + \cdots + \omega_{k_4}}$ (2.44)

This is the crucial difference between a scattering amplitude and a wave function. For an amplitude we would get a delta function in the energies however for the wave function we get poles. If we work with mode functions in dS then we can get more complicated functions of this variable. We return back to the interpretation of these singularities in section 2.5.

We can now directly evaluate Witten diagrams to obtain the corrections to the wave function. For example, we can repeat the same procedure for ϕ^3 theory. In this case, the first two corrections are given as

$\psi_1(\vec{k}) =$ 

$\psi_3(\vec{k}_1, \vec{k}_2, \vec{k}_3) =$  (2.45)

$\psi_4(\vec{k}_1, \cdots, \vec{k}_4) =$ 

Consider the 3rd diagram. That is a single exchange and has the following integral representation

$$\psi_4(\vec{k}_1, \cdots, \vec{k}_4) = \int_0^\infty dt_1 dt_2 e^{-(\omega_{k_1}+\omega_{k_2})t_1} e^{-(\omega_{k_3}+\omega_{k_4})t_2} G(t_1, t_2, \vec{k}) \tag{2.46}$$ {phi3int}

where $\vec{k} = \vec{k}_1 + \vec{k}_2$. By using the Green function given in (2.32) we can evaluate the time integrals very easily,

$$\begin{aligned}\psi_4(\vec{k}_1, \cdots, \vec{k}_4) &= \frac{1}{2(\omega_{12}+\omega_k)(\omega_{12}+\omega_{34})\omega_k} + \frac{1}{2(\omega_{34}+\omega_k)(\omega_{12}+\omega_{34})\omega_k} - \frac{1}{2(\omega_{12}+\omega_k)(\omega_{34}+\omega_k)\omega_k} \\ &= \frac{1}{(\omega_{12}+\omega_k)(\omega_{34}+\omega_k)(\omega_{12}+\omega_{34})}\end{aligned} \tag{2.47}$$ {phi3int1}

where we have used the shorthand $\omega_{12} = \omega_{k_1} + \omega_{k_2}$. Where we again obtain an answer which has poles in the sum of energies. Each of these poles have a physical meaning which is connected to the structure of the time integrals and we shall discuss them in section 2.5. However there is another physical way to see how this arises as shown below.

2.4 Connection with Lippman-Schwinger

Solving for the perturbative correction of the wavefunction can be done by either the path integral as discussed above or by solving the Schrodinger equation. The latter is not usually discussed in textbooks but there are some which discuss them [2, 12]. In terms of differential equations the question we are really solving is

$$H\psi = E\psi \tag{2.48}$$

where E is the ground state energy. The free Hamiltonian is given as,

$$H_0 = \int d^3k \frac{\partial^2}{\partial\phi(\vec{k})\partial\phi(-\vec{k})} + \omega_k^2 \phi(\vec{k})\phi(-\vec{k}) \tag{2.49}$$

The first term is just the square of the conjugate momentum. By applying this to the Gaussian ground state

$$\psi_0^{free}(\phi) = e^{-\int d^3k \omega_k \phi_{\vec{k}} \phi_{-\vec{k}}} \tag{2.50}$$

we get the ground state energy,

$$E_0^{free} = V \int d^3k \omega_k \tag{2.51}$$

with V denoting the volume of space, often represented by $\delta^3(0)$. The excited states of this can be obtained by acting with the ladder operators,

$$a_{\vec{k}}^\dagger = \omega_k \phi(\vec{k}) - \frac{\partial}{\partial\phi(\vec{k})}, \qquad a_{\vec{k}} = \omega_k \phi(\vec{k}) + \frac{\partial}{\partial\phi(\vec{k})}. \tag{2.52}$$

For example,

$$\begin{aligned} \psi_{k_1}^{free}(\phi) &= \frac{1}{\sqrt{\omega_{k_1}}} a_{\vec{k}_1}^\dagger \psi_0^{free}(\phi), \\ \psi_{k_1,k_2}^{free}(\phi) &= \frac{1}{\sqrt{\omega_{k_1}\omega_{k_2}}} a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger \psi_0^{free}(\phi). \end{aligned} \tag{2.53}$$

The energy of these states are given as

$$E_{k_1}^{free} - E_0 = \omega_{k_1}, \qquad E_{k_1,k_2}^{free} - E_0 = \omega_{k_1} + \omega_{k_2} \tag{2.54}$$

For the interacting theory corresponding the action (2.15) the Hamiltonian is given as

$$H = H_0 + \frac{\lambda}{4!} \int d^3x \phi^4(\vec{x}) \tag{2.55}$$

Just like we work out the corrections to the energy and states in quantum mechanics we can repeat the same here. For example, the first order correction to n-particle state is given by the **Lippmann-Schwinger equation**

$$\psi_n^{(1)}(\phi) = \sum_{m \neq n} \frac{\langle \psi_m^{free} | H_{int} | \psi_n^{free} \rangle}{E_n^{free} - E_m^{free}} \psi_m^{free}(\phi) \tag{2.56}$$

When applied to the interaction $\lambda\phi^4$, we see how the poles in the energy arises. They are simply a consequence of the denominators in the equation above! For example the contribution to the ground state to first order, i.e., $n = 0$, receives a correction from $m = 0, 2, 4$. The term with $m = 4$ is given as,

$$\begin{aligned} \psi_0^{(1)}(\phi)\Big|_{m=4} &= \lambda \int d^3x \int d^3k_1 \cdots d^3k_4 \frac{\langle \psi^{free}_{k_1,\cdots,k_4} | \phi^4(x) | \psi^{free}_0 \rangle}{E^{free}_{k_1,\cdots,k_4} - E^{free}_0} \psi^{free}_{k_1,\cdots,k_4}(\phi) \\ &= \lambda \int d^3x \int d^3k_1 \cdots d^3k_4 \frac{\langle \psi^{free}_{k_1,\cdots,k_4} | \phi^4(x) | \psi^{free}_0 \rangle}{\omega_{k_1} + \omega_{k_2} + \omega_{k_3} + \omega_{k_4}} \psi^{free}_{k_1,\cdots,k_4}(\phi) \end{aligned} \tag{2.57}$$

The rest of the computation is exactly similar to how we manipulate with ladder operators while doing S-matrix computations. This shows how we obtain these corrections for the wave functions from a more familiar QM langauge and explains the origins of the mysterious poles.

2.5 Poles and Residues

{sec:poles}

2.5.1 Poles

We now understand the origin of the poles in different physical ways. However there is another very simple integral argument to understand the poles. Consider the following integral

$$I(k) = \int_0^\infty dt e^{-kt} \tag{2.58}$$

For any such integral the poles can arise in two ways. One, from the edge of the limit (known as pinch singularity) and the one where the integrand itself is singular. For the case of $k \neq 0$ there is no value in which the integral is singular. However for the particular case of $k \to 0$ the integral is singular as $t \to \infty$. For the physical applications, in place of k we have sums of energies. Hence the only place the singularities can arise for such integrals is when the sums of energies go to zero, example, $\omega_{k_1} + \cdots + \omega_{k_4}$. This can not occur for any real value of the momenta but only appears in the complex plane after analytical continutation. For the case of ϕ^3 theory (2.46) we have two such time integrals

$$\begin{aligned} I_1 &= \int_0^\infty dt_1 \int_0^{t_1} dt_2 e^{-(k_{12}+k)t_1} e^{(k_{34}-k)t_2}, \\ I_2 &= \int_0^\infty dt_1 dt_2 e^{-(k_{12}+k)t_1} e^{(k_{34}+k)t_2} \end{aligned} \tag{2.59}$$

For the second integral it is very easy to see that the singularties arise from $k_{12} + k \to 0$ and $k_{34} + k \to 0$. For the first integral, the singularities arise in the limit $t_2 \to t_1, t_1 \to \infty$ which gives

$$I_1 = \int^\infty dt_1 e^{-(k_{12}+k_{34})t_1} \tag{2.60}$$

which is singular when $k_{12} + k_{34} \to 0$. The second place a singularity can aris is when $t_2 \to 0, t_1 \to \infty$ which leads to

$$I_1 = \int^{\infty} dt_1 e^{-(k_{12}+k)t_1} \tag{2.61}$$

which gives $k_{12}+k \to 0$ as confirmed by the explicit evaluation of the integral (2.47). Thus we recover the singularities of the final answer. While this method of checking for the singularities work for these simple examples it is in general inconvenient to apply them. Plus, in this process we also encounter spurious singularities like $\frac{1}{k}$ which do not appear in the final answer as seen in (2.46). However the general takeaway from this is that the singularities will only appear when the sum of energies entering a vertex is zero. For the scattering amplitude these reflect the energy conserving delta functions at each vertex.

2.5.2 Resiudes

Since we have seen that we get poles in the unphysical planes we can compute their resiudes. For the contact diagram it is a very simple structure. By computing the residue at $\omega_{k_1} \cdots \omega_{k_4} \to 0$ we get λ. In this case it is equivalent to computing the time integrals from $(-\infty, \infty)$ as that effectively enforces the energy conservation and thus is equivalent to evaluating the resiude $\omega_{k_1} = -\omega_{k_1} - \omega_{k_2} - \omega_{k_3}$.
For the exchange diagram it is more interesting as there are more residues to consider. Let us denote the wave function via

$$\psi_4 = \int_{-\infty}^{0} dt_1 dt_2 e^{ix_1 t_1} e^{ix_2 t_2} G(t_1, t_2, \vec{k}) = \frac{1}{(x_1+x_2)(x_1+\omega_k)(x_2+\omega_k)} \tag{2.62}$$ {psi4phi3}

where $x_1 = \omega_{k_1} + \omega_{k_2}$ and $x_2 = \omega_{k_3} + \omega_{k_4}$. For this integral we can compute the residue at the total energy pole which gives

$$\operatorname*{Res}_{x_2=-x_1} \psi_4 = \frac{1}{\omega_k^2 - x_1^2} = \frac{1}{\vec{k}^2 + m^2 - (\omega_{k_1} + \omega_{k_2})^2} \tag{2.63}$$ {resEpole}

which is the familiar $\frac{1}{s+m^2}$ where s is the Mandelstam for the s-channel. It is easy to see that we obtain the same result by changing the limits of the time integrals in (2.62) to $(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} dt_1 dt_2 e^{ix_1 t_1} e^{ix_2 t_2} G(t_1, t_2, \vec{k}) = \frac{\delta(x_1+x_2)}{\vec{k}^2 + m^2 - (\omega_{k_1} + \omega_{k_2})^2} + \delta(x_1+k)\delta(x_2+k)\# \tag{2.64}$$

where the second term on the RHS vanishes for generic scattering process and comes from the boundary term in the Green function (2.32). Hence this integral picks up the contribution from the total energy conserving delta function

$$\int_{-\infty}^{\infty} dt_1 dt_2 e^{ix_1 t_1} e^{ix_2 t_2} G(t_1, t_2, \vec{k}) = \frac{\delta(x_1+x_2)}{\vec{k}^2 + m^2 - (\omega_{k_1} + \omega_{k_2})^2} \tag{2.65}$$

which then matches with the contribution from the residue evaluated at the pole (2.63). This is known as the **flat space limit**. In dS this ends up computing the high energy limit of the scattering process as the Bessel functions as $t \to \infty$ reduce to massless plane waves in flat space. An intuitive way to understand that is that the frequency of the mode functions get blue-shifted as $t \to \infty$.

By a similar set of arguments one can also evaluate the **partial energy singularities**. These correspond to enforcing energy conservation only in a subset of the time integrals. For instance by changing the limits of the time integral $t_2 \to (-\infty, \infty)$ we get

$$\int_{-\infty}^{0} dt_1 \int_{-\infty}^{\infty} dt_2 e^{ix_1 t_1} e^{ix_2 t_2} G(t_1, t_2, \vec{k}) = \frac{\delta(x_2 + \omega_k)}{\omega_k^2 - x_1^2} \longrightarrow \frac{1}{x_1 + \omega_k} \frac{1}{x_1 - \omega_k} \delta(x_2 + \omega_k) \quad (2.66)$$

where this reduces to an amplitude (in this case it is trivial as it's a 3-pt) times a lower point wave function coefficient (the leftmost factor on the RHS). This can be extended to higher point functions and spins. The factor in the middle is general sensitive to the spin and the scaling dimension of the particle exchanged [13, 14].

2.6 Recursions

We can exploit properties of the Green function in (2.32) to derive useful recursion relations. These follow by noting that the Green function satisfies

$$(\partial_{t_1} + \partial_{t_2}) G(t_1, t_2, k) = e^{-\omega_k (t_1 + t_2)} \quad (2.67)$$

Therefore consider the single exchange graph

$$\psi_4 = \int_0^\infty dt_1 dt_2 e^{-x_1 t_1} e^{-x_2 t_2} G(t_1, t_2, k) \quad (2.68)$$

and insert the differential operator $(\partial_{t_1} + \partial_{t_2})$ inside this

$$\int_0^\infty dt_1 dt_2 (\partial_{t_1} + \partial_{t_2}) e^{-x_1 t_1} e^{-x_2 t_2} G(t_1, t_2, k) = 0 \quad (2.69)$$

The RHS follows from the boundary conditions satisfied by the Green function. By expanding the derivatives on the LHS we obtain a term proportional to ψ_4 and a product of contact terms,

$$(x_1 + x_2)\psi_4 = \int_0^\infty dt_1 e^{-(x_1 + \omega_k) t_1} \int_0^\infty dt_2 e^{-(x_1 + \omega_k) t_2} \implies \psi_4 = \frac{1}{(x_1 + x_2)(x_1 + \omega_k)(x_2 + \omega_k)} \quad (2.70)$$

This is a nice alternative form of using the Lippmann Schwinger equation which directly makes the physical poles manifest avoids running into the spurious poles. This can be easily extended to spin 1 [15] and $\frac{1}{2}$ [14].

2.7 Reconstruction

2.8 Loops

3 In-In Correlators

3.1 Wave function to Correlators

Until now we have studied the wave function in some detail. However to compute the correlator we need to average over the field configurations at $t = 0$. This requires us to do the path integral (1.2),

$$\langle\psi|\phi_1\cdots\phi_n|\psi\rangle = \int D\phi|\psi(\phi)|^2\phi_1\cdots\phi_n \tag{3.1}$$

where $\phi_i \equiv \phi(\vec{k}_i)$. Let us consider the case of $n = 4$ and demonstrate the evaluation of this path integral. The probability distribution $|\psi(\phi)|^2$ is given as

$$\begin{aligned}|\psi(\phi)|^2 = e^{-2\int \mathrm{Re}\psi_2(\vec{k})\phi_k\phi_{-k}d^3k}\Big[1 + \int 2\mathrm{Re}\psi_3\phi_1\phi_2\phi_3 d^3k_1d^3k_2d^3k_3 + \int 2\mathrm{Re}\psi_4\phi_1\cdots\phi_4 d^3k_1\cdots d^3k_4 \\ + \frac{1}{2}\int 2\mathrm{Re}\psi_3\phi_1\phi_2\phi_3d^3k_1d^3k_2d^3k_3\int 2\mathrm{Re}\psi_3\phi_1'\phi_2'\phi_3'd^3k_1'd^3k_2'd^3k_3' + \cdots\Big]\end{aligned} \tag{3.2}$$

where the term in the 2nd line comes by expanding an expontial, $e^{-x^2-\lambda x^3} = e^{-x^2}(1 - \lambda x^3 - \frac{1}{2}\lambda^2(x^3)^2 + \cdots)$. Thus we can use this probability distribution $|\psi(\phi)|^2$ with the quadratic part of this effective action given by $2\mathrm{Re}\psi_2$ to compute the 4-pt correlator and obtain the following contribution

$$\langle\psi|\phi_1\cdots\phi_4|\psi\rangle = \frac{1}{2\mathrm{Re}\psi_2(k_1)\cdots 2\mathrm{Re}\psi_2(k_4)}\Big[\mathrm{Re}\psi_4(\vec{k}_1,\cdots,\vec{k}_4) + \frac{\mathrm{Re}\psi_3(\vec{k}_1,\vec{k}_2,\vec{k})\mathrm{Re}\psi_3(\vec{k}_3,\vec{k}_4,\vec{k})}{\mathrm{Re}\psi_2(\vec{k})}\Big] \tag{3.3}$$

This computation is independent of the background of spacetime. In dS, the factors in the front represent the conversion from dimension Δ to the shadow dimension $d - \Delta$. We will usually strip these factors off and represent the correlator as

$$\langle\psi|\phi_1\cdots\phi_4|\psi\rangle = \mathrm{Re}\psi_4(\vec{k}_1,\cdots,\vec{k}_4) + \frac{\mathrm{Re}\psi_3(\vec{k}_1,\vec{k}_2,\vec{k})\mathrm{Re}\psi_3(\vec{k}_3,\vec{k}_4,\vec{k})}{\mathrm{Re}\psi_2(\vec{k})} \tag{3.4}$$

This shows that knowing the wave function coefficients is in general sufficient to determine the correlation functions. There is an additional subtlety while considering loops as we can get loop integrals while multiplying product of two wave functions. These are known as classical loops and will be discussed later. Note that only the first term in this expansion will have the total energy pole and thus the flat space limit of both these objects are the same.
By applying this formula for the ϕ^3 case we get

$$\langle\psi|\phi_1\cdots\phi_4|\psi\rangle = \frac{1}{(\omega_{12}+\omega_{34})(\omega_{12}+\omega_k)(\omega_{34}+\omega_k)} + \frac{1}{\omega_k(\omega_{12}+\omega_k)(\omega_{34}+\omega_k)} \tag{3.5}$$ {psi34ptcorr}

Thus we see the appearance of an extra pole $\frac{1}{\omega_k}$. The example above hints that the extra contribution to the correlator can be obtained by replacing the total energy pole with $\frac{1}{\omega_k}$. This indeed ends up being true for general diagrams [5] where the idea is to draw the Witten diagram which gives the wave function, and to that, replace certain poles in a similar manner. Example,

$$\langle\phi_1\cdots\phi_4\rangle^{(2)} = \psi_4^{(2)} + \frac{\psi_6^{(1)}}{\psi_2} + \frac{\psi_4\psi_4}{\psi_2\psi_2} \tag{3.6}$$

where the terms on the RHS denote the bubble contribution to the wavefunction; the diagram obtained by cutting one leg of the bubble, i.e, a double exchange (a 6-pt graph in ϕ^4 theory) and finally the diagrams obtained by cutting two intermediate legs of the bubble (product of 4-pt contact). This process illustrates some features of the correlator which was not apparent before, however it also hides another structure that can be made apparent by going to Fourier space.

3.2 Fourier Space

Before explaining the structure of the dressing rule let us consider an analog of Fourier space [16] for these time integrals in presence of the boundary conditions. The standard Fourier transform which gives the Feynman (time ordered) Green function is given by

$$\frac{1}{2k}\left[\Theta(t_1-t_2)e^{-k(t_1-t_2)}+\Theta(t_2-t_1)e^{-k(t_2-t_1)}\right]=\int_{-\infty}^{\infty}\frac{e^{ip(t_1-t_2)}}{p^2+k^2} \tag{3.7}$$

However the Green functions for the wave function have an additional term on the LHS which indicate that they satisfy Dirichlet boundary condition. This can be made apparent on the RHS by replacing the exponentials with sines,

$$\frac{1}{2k}\left[\Theta(t_1-t_2)e^{-k(t_1-t_2)}+\Theta(t_2-t_1)e^{-k(t_2-t_1)}-e^{-k(t_1+t_2)}\right]=\int_{-\infty}^{\infty}\frac{\sin(pt_1)\sin(pt_2)}{p^2+k^2} \tag{3.8}$$

Using this representation we shall be able to conveniently derive a new integrand and study its properties. For example, applying this formula to the single exchange diagram we get,

$$\begin{aligned}\psi_4 &= \int_0^{\infty} dt_1 dt_2 e^{-x_1t_1}e^{-x_2t_2}G(t_1,t_2,\vec{k})\\ &= \int_0^{\infty} dt_1 dt_2 e^{-x_1t_1}e^{-x_2t_2}\int_{-\infty}^{\infty}\frac{\sin(pt_1)\sin(pt_2)}{p^2+\omega_k^2}\\ &= \int_{-\infty}^{\infty} dp\frac{p^2}{(p^2+x_1^2)(p^2+x_2^2)}\frac{1}{p^2+\omega_k^2}\end{aligned} \tag{3.9}$$ {WFsingleex}

While at this stage the integral looks fairly uninteresting and a simple rewriting of the integral above it is has some interesting properties. For instace, it can be used to combine several graphs of spin-1 under a single integrand give. However it has the disadvantage that there is nothing nice about this representation while applied to loops. For example, when applied to the buble graph in ϕ^4 theory we get (see appendix E of [8])

$$\psi_4^{(2)}=\int d^3l\int_{-\infty}^{\infty}dp_1dp_2\frac{1}{(p_1^2+l^2)(p_2^2+(\vec{l}+\vec{k})^2)}\mathcal{P}_{1/2}(k_{12};p_1,p_2)\mathcal{P}_{1/2}(k_{34};p_1,p_2) \tag{3.10}$$ {psi4bub}

where the function $\mathcal{P}_{1/2}(k;p_1,p_2)$ is,

$$\mathcal{P}_{1/2}(k;p_1,p_2)=\frac{k}{2}\left[\frac{1}{k^2+(p_1-p_2)^2}+\frac{1}{k^2+(p_1+p_2)^2}\right] \tag{3.11}$$

However as shown below it has an interesting use when applied to the correlator.

3.3 Dressing Rules

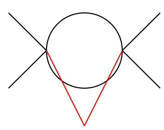
By using the Fourier space representation of the bulk-bulk propagators we obtain a similar representation for the in-in correlator (3.5),

$$\langle \phi_1 \cdots \phi_4 \rangle = \int_{-\infty}^{\infty} \frac{dp x_1 x_2}{(p^2+x_1^2)(p^2+x_2^2)} \frac{1}{p^2+\omega_k^2} \tag{3.12}$$

While this looks very similar to the the wave function in (3.9) the Kernel also applies to loop diagrams! In flat space the same kernel works for any theory without derivative interactions. For example, the 1-loop contribution to the correlator (which includes a contribution from the bubble graph in (3.10)) is given as,

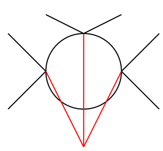
$$\langle \phi_1 \cdots \phi_4 \rangle^{(2)} = \int_{-\infty}^{\infty} \frac{dp x_1 x_2}{(p^2+x_1^2)(p^2+x_2^2)} \int \frac{d^4L}{L^2(L+K)^2} \tag{3.13}$$

where $L^\mu = (l_0, \vec{l})$, $d^4L = dl_0 d^3l$ and $K^\mu = (p, \vec{k})$. This shows that after computing the contributions from all the wave function coefficients at $O(\lambda^2)$ to the correlator we find a way to bring the extra Kernel integrals outside the 4D-loop integral for the amplitude. This can be expressed in a more diagrammatic way via,

$\langle \phi_1 \cdots \phi_4 \rangle^{(2)} =$ 

$$= \int_{-\infty}^{\infty} \frac{dp\, x_1 x_2}{(p^2+x_1^2)(p^2+x_2^2)} \int \frac{d^4L}{L^2(L+P)^2} \tag{3.14}$$

where the red lines denote the auxiliary propagators which impose a loop integral over the energy variable of the flat space Feynman diagram.
One can now apply this rule for higher point graphs, example,

$\langle \phi_1 \cdots \phi_6 \rangle^{(s)} =$ 

$$= \int_{-\infty}^{\infty} \frac{dp_1 dp_2 k_{12} k_{34} k_{56}}{(p_1^2+k_{12}^2)(p_2^2+k_{34}^2)\left((p_1+p_2)^2+k_{56}^2\right)} \int \frac{d^4L}{L^2(L^2+P_1)^2(L+P_2)^2}$$

3.3.1 Proof via In-out

There is an elegant proof that can be given for the rule above using the in-out formalism.

4 Correlators of Gauge Theories and Gravity

4.1 Wave function for Spinning Theories

The wave function for spinning theories follows a very similar story as the scalar ones. Upon obtaining the bulk-boundary and bulk-bulk propagators, computation is exactly similar.

However for spinning theories one must note that the wave functions are also gauge invariant. This states that

$$\psi(A_i) = \psi(A_i + \partial_i \lambda) \tag{4.1}$$ {gaugeinv}

This condition is implied by the fact that the wave function satisfies the Gauss law,

$$\partial_i \frac{\partial \psi(\vec{A}_i)}{\partial A_i} = 0 \tag{4.2}$$ {gauss}

Clasically this is simply the statement $\vec{\nabla} \cdot \vec{E} = 0$. To see this explicitly expand the RHS of (4.1) to obtain

$$\begin{aligned} \psi(A_i + \partial_i \lambda) &= \psi(A_i) + \int d^3x \partial_i \lambda \frac{\partial \psi(\vec{A})}{\partial A_i} \\ &= \psi(A_i) + \int d^3x \partial_i \left(\lambda \frac{\partial \psi(\vec{A})}{\partial A_i} \right) - \int d^3x \lambda \partial_i \frac{\partial \psi(\vec{A})}{\partial A_i} \end{aligned} \tag{4.3}$$

The term in the middle vanishes as $\lambda(|\vec{x}| \to \infty) \to 0$ and the last term vanishes as a consequence of (4.2) thereby establishing (4.1). This statement is also true with matter fields,

$$\psi(A_i + \partial_i \lambda, \phi + i\lambda\phi, \phi^\dagger - i\lambda\phi^\dagger) = \psi(A_i, \phi, \phi^\dagger) \tag{4.4}$$

where the Gauss constraint is now given as

$$\partial_i \frac{\partial \psi(A_i)}{\partial A_i} = i\Big(\phi \frac{\partial}{\partial \phi} - \phi^\dagger \frac{\partial}{\partial \phi^\dagger}\Big)\psi \tag{4.5}$$

The invariance of the wave function under this set of gauge transformations has an interesting application in SK path integrals.

4.2 Spinning Correlators via Schwinger-Keldysh

5 Application of Dressing Rules

5.1 Loops

5.2 Cutkosky Rules

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