Fourier Analysis is a very important analysis tool whose ideas are used in all areas of science and engineering. Underlying idea, simply put, is to decompose any given function into oscillatory components; such a decomposition is technically called as a Fourier Transform. As an example, a sound signal is decomposed into component sinusoidal waves whose frequencies are then determined i.e. the Fourier transform of the sound signal will have large amplitudes for these component frequencies. (Side note: Inverse Fourier Transform will convert the amplitudes and frequencies into a time dependent signal).

Consider a periodic signal with period T, i.e. $f(t+T) = f(t)$. The basic idea is to use cosine and sine functions which have this period to construct approximations to function $f(t)$, i.e.

$$f(t) = \sum_{n\in\mathbf{Z}^+} a_n \cos\left(2\pi n \frac{t}{T}\right) + b_n \sin\left(2\pi n \frac{t}{T}\right) \tag{1}$$

where $\mathbf{Z}^+ = \{0, 1, 2, 3, \cdots\}$ is the set of non-negative integers. Clearly each term in summation has a period of T/n, and hence also a period of T. For a given $f(t)$, the coefficients a_n and b_n can be found as follows:

$$a_m = \frac{1}{T}\int_0^T \, dt\ f(t) \cos\left(2\pi m \frac{t}{T}\right), \quad b_m = \frac{1}{T}\int_0^T \, dt\ f(t) \sin\left(2\pi m \frac{t}{T}\right) \tag{2}$$

[**SIDE-NOTE:** For signals with out a period (i.e. with infinite period), with $f(t)$ being the signal, then we aim for the following decomposition:

$$f(t) = \int_{-\infty}^{\infty} d\nu\ \mathcal{F}(\nu)\ e^{2\pi i\nu t} \tag{3}$$

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Let us take an examples:

- if $f(t) = A\cos(2\pi\nu_0 t)$, then clearly $\mathcal{F}(\nu) = \frac{1}{2}\left(\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right)$, where δ is the Dirac Delta function. That is, the cos function is essentially two components of equal intensity
- if $f(t) = A\sin(2\pi\nu_0 t)$, then clearly $\mathcal{F}(\nu) = \frac{1}{2i}\left(\delta(\nu-\nu_0) - \delta(\nu+\nu_0)\right)$.
- If $f(t)$ is a real function, then $\mathcal{F}(\nu)$ has even symmetry. i.e. $\mathcal{F}(-\nu) = +\mathcal{F}(\nu)$
- If $f(t)$ is a purely imaginary function (no real component) then $\mathcal{F}$ has odd symmetry, i.e. $\mathcal{F}(-\nu) = -\mathcal{F}(\nu)$

Inverse transform defined as $\mathcal{F}(\nu) = \int_{-\infty}^{\infty} dt\ f(t)e^{-2\pi i\nu t}$ provides a method to calculate the amplitude $\mathcal{F}(\nu)$.

J. Fourier [1822 AD ?] suggested that any $f(t)$ can be decomposed as given above; such decomposition has several nice properties in interpretation of the signal and stability of numerical analysis that have made Fourier Analysis so pervasive. When, instead of time dependent signal, we have position dependent signal $f(x)$, Fourier Transform $\mathcal{F}$ will have wavelength (or wave vector $k = 2\pi/\lambda$) components i.e. $\mathcal{F}(k)$. Extension to 2- or 3- dimensions is straightforward; $f(\vec{r}) = \int d\vec{k}\ \mathcal{F}(\vec{k}) e^{i\vec{k}\cdot\vec{r}}$.

Curves in 2-dimensional Euclidean space can be thought of as parametric functions i.e. $x(t)$ and $y(t)$ with t being the parameter. Closed curves i.e. the curves that loop back onto themselves after length T, can be thought as satisfying the condition $x(t+T) = x(t)$ and simultaneously $y(t+T) = y(t)$. Fourier Analysis of $x(t)$ and $y(t)$ will then involve trigonometric functions

$$x(t) = \int d\nu \hat{X}(\nu) e^{2\pi i\nu t} \text{ and } y(t) = \int d\nu \hat{Y}(\nu) e^{2\pi i\nu t} \tag{4}$$

can be simplified by use of complex numbers $z(t) = x(t) + iy(t)$, with $z(t+T) = z(t)$, as

$$z(t) = \int d\nu\ C(\nu) e^{2\pi i\nu t} \tag{5}$$

where the complex Fourier Coefficients $C(\nu)$ have to be determined; clearly requirement of $z(t+T) = z(t)$ would mean that $\nu T = n$ where $n \in \mathcal{Z}$, the set of all integers. It can be easily shown that

$$\int_0^T dt\ z(t)\ e^{-2\pi i\mu t} = T\ C(\mu) \implies C(\mu) = \frac{1}{T}\int_0^T dt\ z(t)\ e^{-2\pi i\mu t} \tag{6}$$

providing a scheme to determine the values of Fourier Coefficients $C(\nu)$ by numerical integration (of the second equality in above equation using provided ‘equally spaced’ values of $z(t)$); this is also known as Inverse Fourier Transform.

References:

1. **Fourier Series and Fourier Transform:** Chapter-11 of “Advanced Engineering Mathematics” by Kreyzig and Westwig (10th Ed) discusses setting up, convergence of Fourier Series for 1-dimensional functions; also in the same chapter is the discussion on Fourier Transforms (and their inverses: Equations (4,5,6) above)