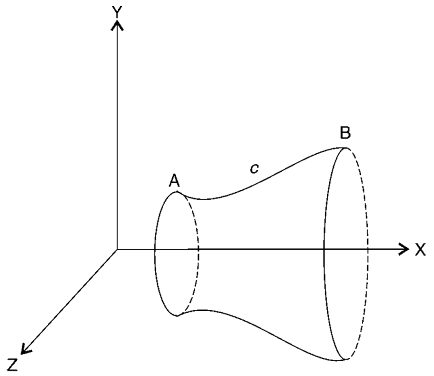
Variational Calculus Examples

Minimal surface area

Example 1: Find the curve c passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ such that the rotation of the curve c about x-axis generates a surface of revolution having minimum surface area.



Solution: The surface area S generated by revolving the curve c defined by $y(x)$ about the x-axis is

$$S[y(x)] = \int_A^B 2\pi y\, ds = \int_{x=x_1}^{x_2} 2\pi y \sqrt{1 + y'^2}\, dx \tag{1}$$

To find the extremal $y(x)$ which minimizes (1). Here $f = y\sqrt{1 + y'^2}$ which is independent of x. The Euler's equation is

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0 \quad \text{or} \quad f - y'\frac{\partial f}{\partial y'} = \text{constant} = c_1 \tag{2}$$

Substituting f and $\frac{\partial f}{\partial y'}$, we have

$$y\sqrt{1 + y'^2} - y'\frac{yy'}{\sqrt{1 + y'^2}} = c_1 \tag{3}$$

$$\frac{y(1 + y'^2) - yy'^2}{\sqrt{1 + y'^2}} = \frac{y}{\sqrt{1 + y'^2}} = c_1 \tag{4}$$

Put $y' = \sinh t$, then from (4)

$$\frac{y}{\sqrt{1 + \sinh^2 t}} = \frac{y}{\cosh t} = c_1 \quad \text{or} \quad y = c_1 \cosh t \tag{5}$$

So

$$dx = \frac{dy}{y'} = \frac{c_1 \sinh t\, dt}{\sinh t} = c_1\, dt \tag{6}$$

Integrating

$$x = c_1 t + c_2 \tag{7}$$

where c_2 is the constant of integration. Eliminating 't' between (5) and (7)

$$t = \frac{x - c_2}{c_1} \tag{8}$$

therefore

$$y = c_1 \cosh t = c_1 \cosh\left(\frac{x - c_2}{c_1}\right) \tag{9}$$

Equation (9) represents a two-parameter family of catenaries. The two constants c_1 and c_2 are determined using the end (boundary) conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Geodesics

Example 2: Find the geodesics on a sphere of radius 'a'.

Solution: In spherical coordinates r, θ, ϕ, the differential of arc length on a sphere is given by

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (r \sin\theta d\phi)^2 \tag{10}$$

Since $r = a =$ constant, $dr = 0$. So

$$\left(\frac{ds}{d\theta}\right)^2 = a^2 + a^2 \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2 \tag{11}$$

Integrating with respect to θ between θ_1 and θ_2,

$$s = \int_{\theta_1}^{\theta_2} a\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta \tag{12}$$

Here

$$f = a\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2} \tag{13}$$

is independent of ϕ, but is a function of θ and $\frac{d\phi}{d\theta}$. Denoting $\frac{d\phi}{d\theta} = \phi'$, the Euler's equation reduces to

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial \phi'} = \text{constant} \tag{14}$$

$$\text{i.e.,} \quad a \cdot \frac{1}{2}\frac{2\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta\phi'^2}} = k = \text{constant} \tag{15}$$

Squaring

$$a^2 \sin^4\theta\phi'^2 = k^2(1 + \sin^2\theta\phi'^2) \tag{16}$$

$$\text{or} \quad \frac{d\phi}{d\theta} = \phi' = \frac{k}{\sin\theta\sqrt{\sin^2\theta - k^2}} = \frac{k\csc^2\theta}{\sqrt{1 - k^2\csc^2\theta}} \tag{17}$$

Integrating, we get

$$\phi(\theta) = \int \frac{k\csc^2\theta\, d\theta}{\sqrt{(1 - k^2) - (k\cot\theta)^2}} + c_2 \tag{18}$$

$$\phi(\theta) = \cos^{-1}\left(\frac{k\cot\theta}{\sqrt{1 - k^2}}\right) + c_2 \tag{19}$$

where c_2 is constant of integration. Rewriting

$$\frac{k\cot\theta}{\sqrt{1 - k^2}} = \cos(\phi - c_2) = \cos\phi\cos c_2 + \sin\phi\sin c_2 \tag{20}$$

$$\text{or} \quad \cot\theta = A\cos\phi + B\sin\phi \tag{21}$$

where

$$A = \frac{(\cos c_2)\sqrt{1 - k^2}}{k}, \qquad B = \frac{(\sin c_2)\sqrt{1 - k^2}}{k} \tag{22}$$

Multiplying by $a\sin\theta$, we have

$$a\cos\theta = Aa\sin\theta\cos\phi + Ba\sin\theta\sin\phi \tag{23}$$

Since $r = a$, the spherical coordinates are $x = a\sin\theta\cos\phi, y = a\sin\theta\sin\phi, z = a\cos\theta$, so

$$z = Ax + By \tag{24}$$

which is the equation of a plane, passing through the origin (0, 0, 0) (since no constant term), the center of the sphere. This plane cuts the sphere along a great circle. Hence the great circle is the geodesic on the sphere.

Variational problems. f is dependent on x, y, y'

Example 3: Find a complete solution of the Euler-Lagrange equation for

$$\int_{x_1}^{x_2} [y^2 - (y')^2 - 2y \cosh x] \, dx \tag{25}$$

Solution: Here $f(x, y, y') = y^2 - (y')^2 - 2y \cosh x$, which is a function of x, y, y'. The Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{26}$$

Differentiating (25) partially with respect to y and y', we get

$$\frac{\partial f}{\partial y} = 2y - 2\cosh x \tag{27}$$

$$\frac{\partial f}{\partial y'} = -2y' \tag{28}$$

Substituting (27) and (28) in (26), we have

$$2y - 2\cosh x - \frac{d}{dx}(-2y') = 0 \tag{29}$$

$$y'' + y = \cosh x \tag{30}$$

The complementary function of (30) is

$$y_c = c_1 \cos x + c_2 \sin x \tag{31}$$

and a particular integral of (30) is

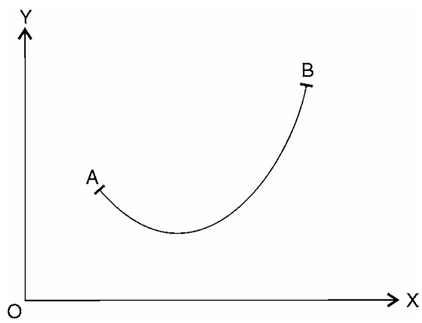
$$y = \frac{1}{2} \cosh x \tag{32}$$

Thus the complete solution of the Euler-Lagrange Equation (30) is

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{1}{2} \cosh x \tag{33}$$

Catenary

Example 4: Determine the shape an absolutely flexible, inextensible homogeneous and heavy rope of given length L suspended at the points A and B.



Solution: The rope in equilibrium takes a shape such that its center of gravity occupies the lowest position. Thus, to find the minimum of the y-coordinate of the center of gravity of the string given by

$$I[y(x)] = \frac{\int_{x_1}^{x_2} y\sqrt{1 + y'^2} \, dx}{\int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx} \tag{34}$$

subject to the constraint

$$J[y(x)] = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = L = \text{constant} \tag{35}$$

Thus, to minimize the numerator in R.H.S. of (34) subject to (35). Form

$$H = y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2} = (y+\lambda)\sqrt{1+y'^2} \tag{36}$$

where λ is a Lagrangian multiplier. Here H is independent of x. So the Euler equation is

$$H - y'\frac{\partial H}{\partial y'} = \text{constant} = k_1 \tag{37}$$

$$\text{i.e.,}\quad (y+\lambda)\sqrt{1+y'^2} - y'(y+\lambda)\frac{y'}{\sqrt{1+y'^2}} = k_1 \tag{38}$$

$$\frac{(y+\lambda)(1+y'^2) - y'(y+\lambda)y'}{\sqrt{1+y'^2}} = k_1 \tag{39}$$

$$\frac{(y+\lambda)(1+y'^2-y'^2)}{\sqrt{1+y'^2}} = \frac{y+\lambda}{\sqrt{1+y'^2}} = k_1 \tag{40}$$

$$y+\lambda = k_1\sqrt{1+y'^2} \tag{41}$$

Put $y' = \sinh t$, where t is a parameter, in (41) Then

$$y+\lambda = k_1\sqrt{1+\sinh^2 t} = k_1\cosh t \tag{42}$$

Now

$$dx = \frac{dy}{y'} = \frac{k_1 \sinh t\,dt}{\sinh t} = k_1\,dt \tag{43}$$

Integrating

$$x = k_1 t + k_2 \tag{44}$$

Eliminating 't' between (42) and (44), we have

$$y+\lambda = k_1\cosh t = k_1\cosh\left(\frac{x-k_2}{k_1}\right) \tag{45}$$

Equation (45) is the desired curve, which is a catenary.

Note: The three unknowns λ, k_1, k_2 will be determined from the two boundary conditions (curve passing through A and B) and the constraint (35).

Catenary

Example 4*: One isoperimetric problem is the catenary which is the shape a uniform rope or chain of fixed length l that minimizes the gravitational potential energy. Let the rope have a uniform mass per unit length of σ kg/m.

Solution: The gravitational potential energy is

$$U = \sigma g\int y ds = \sigma g\int \sqrt{dx^2+dy^2} = \sigma g\int y\sqrt{1+y'^2}\,dx \tag{46}$$

The constraint is that the length be a constant l

$$l = \int ds = \int \sqrt{1+y'^2}\,dx \tag{47}$$

Thus the function is $f(y,y';x) = y\sqrt{1+y'^2}$ while the integral constraint sets $g = \sqrt{1+y'^2}$. These need to be inserted into the Euler equation by defining

$$F = f + \lambda g = (y+\lambda)\sqrt{1+y'^2} \tag{48}$$

Note that this case is one where $\frac{\partial F}{\partial x} = 0$ and λ is a constant; also defining $z = y + \lambda$ then $z' = y'$. Therefore, the Euler's equations can be written in the integral form

$$F - z'\frac{\partial F}{\partial z'} = c = \text{constant} \tag{49}$$

Inserting the relation $F = z\sqrt{1+z'^2}$ gives

$$z\sqrt{1+z'^2} - z'\frac{zz'}{\sqrt{1+z'^2}} = c \tag{50}$$

where c is an arbitrary constant. This simplifies to

$$\frac{z}{\sqrt{1+z'^2}} = c \tag{51}$$

$$z^2 = c^2(1+z'^2) \tag{52}$$

$$z'^2 = \left(\frac{z}{c}\right)^2 - 1 \tag{53}$$

The integral of this is

$$z = c\cosh\left(\frac{x+b}{c}\right) \tag{54}$$

where b and c are arbitrary constants fixed by the locations of the two fixed ends of the rope.

Minimal travel cost

Example 5: Assume that the cost of flying an aircraft at height z is $e^{-\kappa z}$ per unit distance of flight-path, where κ is a positive constant. Consider that the aircraft flies in the (x, z)-plane from the point $(-a, 0)$ to the point $(a, 0)$ where $z = 0$ corresponds to ground level, and where the z-axis points vertically upwards. Find the extremal for the problem of minimizing the total cost of the journey.

Solution: The differential arc-length element of the flight path, ds, can be written as

$$ds = \sqrt{dx^2 + dz^2} = \sqrt{1+z'^2}\,dx \tag{55}$$

where $z' = \frac{dz}{dx}$. Thus, the cost integral to be minimized is

$$C = \int_{-a}^{+a} e^{-\kappa z} ds = \int_{-a}^{+a} e^{-\kappa z}\sqrt{1+z'^2}\,dx \tag{56}$$

The function of this integral is

$$f = e^{-\kappa z}\sqrt{1+z'^2} \tag{57}$$

The partial differentials required for the Euler equations are

$$\frac{\partial f}{\partial z} = -\kappa e^{-\kappa z}\sqrt{1+z'^2} \tag{58}$$

$$\frac{\partial f}{\partial z'} = \frac{z' e^{-\kappa z}}{\sqrt{1+z'^2}} \tag{59}$$

Therefore, the Euler's equation equals

$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) = \frac{\partial f}{\partial z} \tag{60}$$

$$\frac{d}{dx}\left(\frac{z' e^{-\kappa z}}{\sqrt{1+z'^2}}\right) = -\kappa e^{-\kappa z}\sqrt{1+z'^2} \tag{61}$$

$$\frac{-\kappa z'^2 e^{-\kappa z}}{\sqrt{1+z'^2}} + \frac{z'' e^{-\kappa z}}{\sqrt{1+z'^2}} - \frac{z'^2 e^{-\kappa z} z''}{\sqrt{1+z'^2}^3} = -\kappa e^{-\kappa z}\sqrt{1+z'^2} \tag{62}$$

This can be simplified by multiplying the radical to give

$$-\kappa z'^2 - \kappa^2 z'^4 + z'' - z'' z'^2 = -\kappa - \kappa z'^2 \tag{63}$$

Canceling terms gives

$$z'' + \kappa(1 + z'^2) = 0 \tag{64}$$

Separating the variables leads to

$$\arctan z' = \int \frac{dz'}{1 + z'^2} = \int -\kappa dx = -\kappa x + c_1 \tag{65}$$

Integration gives

$$z(x) = \int^x \tan(c_1 - \kappa x)\, dx = \frac{\ln(\cos(c_1 - \kappa x)) - \ln(\cos(c_1 + \kappa a))}{\kappa} + c_2 \tag{66}$$

$$z(x) = \frac{1}{\kappa} \ln\left(\frac{\cos(c_1 - \kappa x)}{\cos(c_1 + \kappa a)}\right) + c_2 \tag{67}$$

Using the initial condition that $z(-a) = 0$ gives $c_2 = 0$. Similarly, the final condition $z(a) = 0$ implies that $c_1 = 0$. Thus, Euler's equation has determined that the optimal trajectory that minimizes the cost integral C is

$$z(x) = \frac{1}{\kappa} \ln\left(\frac{\cos(\kappa x)}{\cos(\kappa a)}\right) \tag{68}$$

This example is typical of problems encountered in economics.