# 4 Motion under Central Force

This chapter is and application of the Lagrangian formulation developed in the previous chapters. The topic of discussion is motion of a two-body system under the influence of conservative central force describe by some potential $U(r)$ which is a function of their radial distance $\vec{r}$. We will start by reviewing some basic concepts of rotational motion

## 4.1 Rotational Motion: Review

A particle rotates about an axis in a circle. If the radius vector $\mathbf{r}$ makes an angle $d\theta$ in time $dt$ then the instantaneous angular velocity is

$$\omega = \frac{d\theta}{dt} = \dot{\theta}\,. \tag{31}$$

The linear velocity is

$$v = \frac{d}{dt}(rd\theta) = r\omega\,. \tag{32}$$

The direction of $\omega$ is along the axis and determined by the right-hand rule, and direction of $v$ is perpendicular to the $\mathbf{r}$. So we can write the above relation in vectorial form as

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}\,. \tag{33}$$

This relates the angular velocity with the linear velocity. As directions are determined by right-hand rule, you can also write

$$\boldsymbol{\omega} = \frac{|\mathbf{v}|}{|\mathbf{r}|}\hat{\mathbf{r}} \times \hat{\mathbf{v}}\,. \tag{34}$$

The angular momentum of the particle about $O$ is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}\,, \tag{35}$$

and the *torque* or moment of force about the same origin is

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \dot{\mathbf{p}}\,, \tag{36}$$

In linear motion rate of change of momentum can be equated to force. In rotational motion, the rate of change of angular momentum is given as

$$\begin{aligned}\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) &= \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}\,,\\ &= \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{N}\end{aligned} \tag{37}$$

So the rate of change of angular momentum is equal to the applied torque – *in the absence of any torque, the angular momentum is conserved*. This is the conservation of angular momentum.

## 4.2 Equivalent one-body problem

Consider the motion of two masses points $m_1$ and $m_2$ where the only force is their interaction potential $U$. We assume that $U$ is a function of the position vector between the two particles. With reference to the figure 3, the potential is a function of $\mathbf{r}_2 - \mathbf{r}_1$, $U(|\mathbf{r}_2 - \mathbf{r}_1|)$. The distance between the two masses are not fixed, so the system has six dof represented by $\mathbf{r}_1$ and $\mathbf{r}_2$. As the potential is a function of the difference between $\mathbf{r}_1$ and $\mathbf{r}_2$ only, the system can also be described by a different set of six coordinates. They are the vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and the position vector of the CoM of the system $\mathbf{R}$. So the Lagrangian of the system is

$$L = T(\mathbf{R}, \mathbf{r}) - U(|\mathbf{r}|)\,.$$

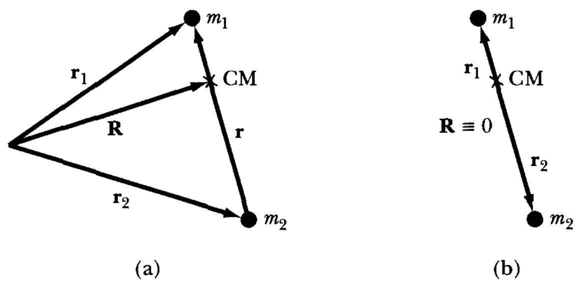


Figure 3: In the figures to the left, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{R}$ are the position vectors of the two point masses and the CoM of the system. In the figure to the left the position vectors of the two point masses are shown by $\mathbf{r}_1'$ and $\mathbf{r}_2'$.

The equation for CoM is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\,,$$

and relative to the CoM individual position vectors are (exercise)

$$\mathbf{r}_1' = -\frac{m_2}{m_1 + m_2}\mathbf{r}\,, \qquad \mathbf{r}_2' = \frac{m_1}{m_1 + m_2}\mathbf{r}\,.$$

The kinetic energy can be written as the kinetic energy of the CoM of the system, and that of the individual masses with respect to the CoM

$$T(\mathbf{R}, \mathbf{r}) = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}m_1\dot{\mathbf{r}}_1'^2 + \frac{1}{2}m_1\dot{\mathbf{r}}_1'^2\,.$$

After some simplifications (verify) the total Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(|\mathbf{r}|),. \tag{38}$$

where we have defined the reduced mass of the system as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}\,.$$

The first thing to notice about the Lagrangian (38) is that there are three cyclic coordinates $\mathbf{R}$. So the CoM is either at rest or moving uniformly. Since none of the EoM of $\mathbf{r}$ contains $\mathbf{R}$ from here onwards we ignore $\mathbf{R}$ and work with the Lagrangian

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(|\mathbf{r}|),. \tag{39}$$

## 4.3 Integrals of motion

The potential $U(|\mathbf{r}|)$ being a function of $\mathbf{r}$ the force $\mathbf{F} = -\vec{\nabla}U(|\mathbf{r}|)\hat{\mathbf{r}}$ acts along $\hat{\mathbf{r}}$ meaning it is a conservative force. From (37) it follows

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}\,.$$

is conserved (exerice – easy to show following (37)). There are two possibilities: either $\mathbf{L} = 0$ or it is a constant. The first case implies $\mathbf{r} = \dot{\mathbf{r}}$, *i.e.,* the motion is in a straight line, and we are not interested in this problem. The second scenario is possible if $\mathbf{r}$ lies in a plane that is constant. As the motion is on a

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(|r|)$$

plane there are two degrees of freedom. Introducing polar coordinates $(r = |\mathbf{r}|, \theta)$ the Lagrangian (39) can be re-written as

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)\,. \tag{40}$$

Since the angular momentum is conserved, it is expected that the angle $\theta$ is cyclic which is clearly seen in the Lagrangian (40). The corresponding conjugate momentum $p_\theta$ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} \tag{41}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad m r^2 \dot{\theta} = \ell$$

From the EL EoM of $\theta$ it follows that

$$\dot{p}_\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(\mu r^2 \dot{\theta}) = \frac{\partial L}{\partial \theta} = 0\,,$$

which when integrated yields

$$\boxed{m r^2 \dot{\theta} = \ell = \text{constant}}\,. \tag{42}$$

It is the magnitude of the constant angular momentum $\mathbf{L}$.

The constant angular momentum is nothing but Kepler's second law in disguise. To see why it is so, consider an infinitesimal time interval $dt$ within which the radius vector of the system sweeps an infinitesimal area

$$dA = \frac{1}{2} r \cdot r d\theta = \frac{1}{2} r^2 d\theta\,,$$

$$dA: \frac{1}{2} r \cdot r\, d\theta = \frac{1}{2} r^2 d\theta$$

Dividing by the time interval, we obtain the *areal velocity*

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2\mu}\,. \tag{43}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2\mu}$$

which is constant in time. This is known as the *Kepler's second law* of planetary motion which states that *the radius vectors of planets sweeps equal area in unit time at any point in their orbits.* This law was empirically discovered by Kepler.

As the motion corresponding to the $\theta$ coordinate is constant, using (42) the Lagrangian of the system can be re-rewritten as

$$\boxed{L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu^2 r^2} - U(r)}\,. \tag{44}$$

$$\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2}$$

This Lagrangian does not explicitly depend on time so the total energy of the system

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu^2 r^2} + U(r) = \text{const.} \tag{45}$$

This is another integral of motion. We can re-write

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{\ell^2}{2mr^2}\right)} \Longrightarrow t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r) - \frac{\ell^2}{2mr^2}\right)}}$$

From this one can in principle solve $r$ as a function of time. But the integral on the RHS is difficult to perform. Instead of solving $r$ and $\theta$ as a function of time, we solve the equation of orbits *i.e.*, $r \equiv r(\theta)$.

## 4.4 Classifications of orbits

Before solving the orbits for a given potential $U(r)$ lets classify the orbits. If the radial velocity $\dot{r} = 0$ then from (45) we get

$$E = \frac{1}{2}\frac{\ell^2}{\mu^2 r^2} + U(r)\,.$$

This is quadratic in $r$ and has two roots, say $r = r_{\max}$ and $r = r_{\min}$. Radial velocity vanishes at the turning point of the motion – so the motion is between the annular region $r = r_{\max}$ ~~and $r = r_{\min}$. For example the motion of the earth~~ around the sun has a max and min radius vector. For some specific combinations of $E, \ell, \mu$ and $U(r)$ it is possible $r_{\min} = r_{\max}$ – this is an example of circular motion where the radial velocity is zero all the time.

For motions that have both $r_{\max} \neq r_{\min}$ are called bounded. There are two types of orbits in bounded motions – open and closed. Closed orbits are those that are repeated after the radius vector has made a finite number of excursion between $r_{\max}$ and $r_{\min}$. In other words the orbits close on itself. But orbits that do not close on itself, are called open orbits. An example of open orbit is shown in figure 4.

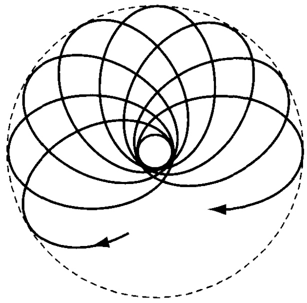

Figure 4: An open orbit

Orbits can also be examined in the following way. By defining an effective potential

$$V' = \frac{1}{2}\frac{\ell^2}{\mu^2 r^2} + U(r)\,.$$

we rewrite (45) as

$$\boxed{\mu \dot{r}^2 = 2(E - V'(r))}\,, \tag{46}$$

The second term $\ell^2/2\mu^2 r^2$ is called the *centrifugal force*, though it is not a force in the usual sense. In figure 5 we plot $V'(r)$ against $r$ for inverse square potential

$$U(r) = -\frac{k}{r}\,,$$

which correspond to the inverse square force

$$F = -\frac{\partial}{\partial r}U(r) = -\frac{k}{r^2}\,,$$

like the gravitational force. In the In figure 5 we also have four values of the total energy $E_1$, $E_3$, $E_3$, $E_4$. For $E = E_1$ the radius vector can not be smaller than $r_1$ otherwise $V'$ is greater than the total energy and hence the velocity is imaginary following (46). A similar situation is obtained for $E = E_2$. In these cases motion is unbounded as shown in the left figure . For $E = E_3$, the motion is bounded between $r_1$ and $r_2$ as shown in the right figure . For $E = E_4$ the motion is circular.

## 4.5 Equations of motion

As mentioned in the previous section, our interest is to obtain the equation for the orbits *i,e,.* a functional dependance bewtween $\theta$ and $r$. We note that

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr = \frac{\ell}{\mu r^2 \dot{r}}dr\,.$$

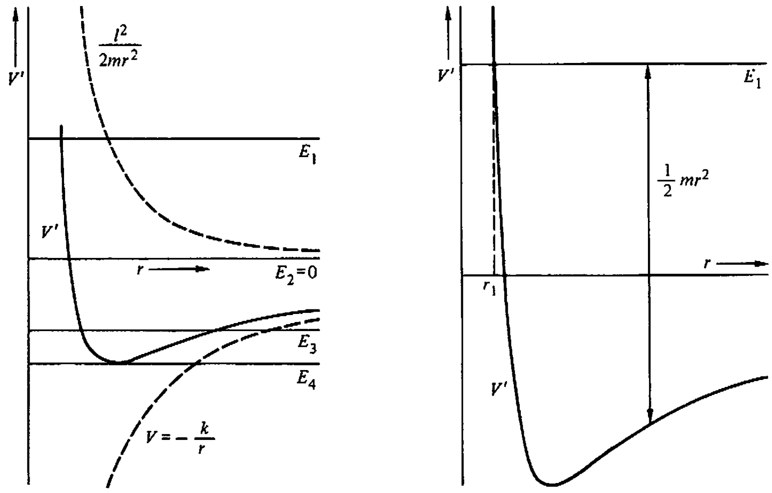


Figure 5: Equivalent one dimensional potential for inverse square law.

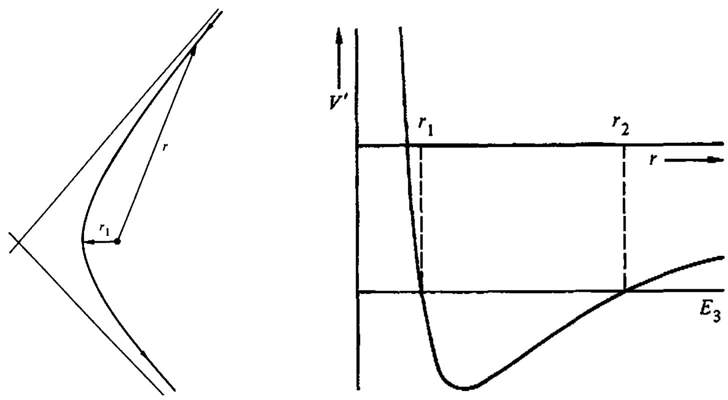


Figure 6: Bound and unbound motion.

So from (45) we get

$$\theta(r) = \int \frac{\pm(\ell/r^2)dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu^2 r^2}\right)}}\,. \tag{47}$$

This is the solution for $\theta$ in terms of $r$. To obtain the solution for $r$ in terms of time we use the El EoM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial r}\right) - \frac{\partial L}{\partial r} = 0\,,$$

which gives

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r)\,. \tag{48}$$

In the above equation, if we replace $\dot{\theta}$ in terms of $\ell$ we get the EoM for $r$ in terms of time. However, in central force problems, the equation of orbit *i.e.,* $r \equiv r(\theta)$ is of interests. Therefore we convert the above equation as a differential equation for $r$ in terms of the $\theta$. To this end, we cast this in a suitable form by change of variable

$$u = \frac{1}{r}\,.$$

We compute

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot r}{\dot\theta} = -\frac{\mu}{\ell}\dot r\,.$$

In the last line angular $\ell = \mu r^2\dot\theta$ has been used. The second derivative of the above is

$$\frac{du^2}{d\theta^2} = \frac{d}{d\theta}\Big(-\frac{\mu}{\ell}\dot r\Big) = \frac{dt}{d\theta}\frac{d}{dt}\Big(-\frac{\mu}{\ell}\dot r\Big) = -\frac{\mu}{\ell\dot\theta}\ddot r = -\frac{\mu^2}{\ell^2}r^2\ddot r\,.$$

which can be written as

$$\ddot r = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}\,.$$

Substituting this in (48) and replacing $\dot\theta$ by $\ell$ we get the EoM for $u$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F(u)}\,, \tag{49}$$

or in terms of $r$

$$\boxed{\frac{d^2}{d\theta^2}\Big(\frac{1}{r}\Big) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)}\,. \tag{50}$$

Solution of this equation gives the orbit.

Few examples of equation of orbit

## 4.6 Kepler's Problem

Kepler's problem concerns motion of planets around the sun under inverse squared force

$$F(r) = -\frac{GMm}{r^2} = -\frac{k}{r^2}\,,$$

which correspond to a potential of the type

$$U(r) = -\frac{k}{r}\,.$$

$r = \dfrac{\alpha}{1+\epsilon\cos\theta}$

The equation of orbit for this potential is

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}(-ku^2) = \frac{\mu k}{\ell^2}\,.$$

The solution of this equation is

$$u = \frac{\mu k}{\ell^2} + A\cos\theta\,, \Longrightarrow \boxed{r = \frac{\alpha}{1+\epsilon\cos\theta}}\,, \tag{51}$$

where $A$ is a constant and

$\alpha = \dfrac{\ell^2}{\mu k}$

$$\alpha = \frac{\ell^2}{\mu k}\,, \quad \epsilon = A\frac{\ell^2}{\mu k}\,.$$

The equation (51) represents the orbital equation for the inverse-square law. Before we compute the positive constant $\epsilon$, which determines the behavior of the orbits, we first examine some general characteristics of the orbits. The behavior of the orbits differs for $\epsilon < 1$ and $\epsilon \geq 1$. For $\epsilon < 1$, the denominator in (51) never vanishes, ensuring that the radius vector $r$ remains bounded for all values of the angle $\theta$. On the other hand, for $\epsilon \geq 1$, the denominator can vanish for certain values of $\theta$, causing the radius vector

$u = \dfrac{\mu k}{\ell^2} + A\cos\theta \rightarrow r = \dfrac{\alpha}{1+\epsilon\cos\theta}$ $\quad \epsilon = A\dfrac{\ell^2}{\mu k}$

to become unbounded and approach infinity. Clearly, the value $\epsilon = 1$ serves as the boundary between the bounded and unbounded solutions. We will soon see that this boundary is related to the total energy $E$ of the system.

For $\epsilon < 1$, the denominator of (51) oscillates between $1 \pm \epsilon$. Hence the radius vector oscillates between a minimum and a maximum

$$r_{\min} = \frac{\alpha}{1+\epsilon}\,, \qquad r_{\max} = \frac{\alpha}{1-\epsilon}\,.$$

The $r = r_{\min}$ which occurs for $\theta = 0$ is called the *perihelion*, and $r = r_{\max}$ which occurs at $\theta = \pi$ is the *aphelion* of the orbit. The orbit also has a period of $2\pi$ so that $r(2\pi) = r(0)$ and the orbit closes on itself after one revolution. We can show (exercise) that the orbit is nothing but an ellipse. To do so we consider the plane of the orbit as the $X - Y$ plane. We introduce

$$a = \frac{\alpha}{1-\epsilon^2}\,, \qquad b = \frac{\alpha}{\sqrt{1-\epsilon^2}}\,, \qquad d = a\epsilon\,,$$

and write (51) as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

which is the standard equation for an ellipse where the center $C$ and the origin $\mathcal{O}$ are separated by a distance $d$ as shown in figure 7. The distances $a, b$ are the semi-minor and the semi-major axes and it is related to the $\epsilon$ as

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}}\,,$$

which means that $\epsilon$ is the eccentricity of the ellipse. So *the orbits of planets are elliptical with the sun at one of the two focuses* – this is *Kepler's first law* of planetary motion.

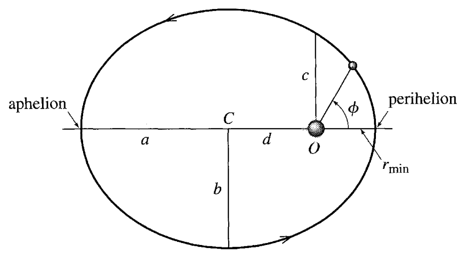


Figure 7: Elliptical orbit of a planet given by equation (51). The sun is at the origin $\mathcal{O}$ which is also one of the focus of the ellipse. The center is at $C$. The distances $a, b$ are called the semi-major and the semi-minor axes. The parameter $\alpha = \mu k/\ell^2$ the value of the radius vector when $\theta = 90^o$. The closest and the farthest from the sun is called perihelion and the aphelion.

To determine the eccentricity $\epsilon$ in terms of total energy $E$, we note that at $r = r_{\min}$ the radial velocity $\dot r = 0$ and the (46) gives

$$E = V'(r_{\min}) = \frac{\ell^2}{2\mu r_{\min}^2} - \frac{k}{r_{\min}} = \frac{1}{2r_{\min}}\left(\frac{\ell^2}{\mu r_{\min}} - 2k\right).$$

We have just found that $r_{\min} = \alpha/(1+\epsilon)$. Substituting this in the above equation and solving for $\epsilon$ we get

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}\,. \tag{52}$$

The orbits can be classified according to the total energy as

- Hyperbola: $\epsilon > 1 \implies E > 0$: Orbits of spacecraft escaping the earth's gravity during interplanetary missions or flyby encounters with other planets.
- Parabola: $\epsilon = 1 \implies E = 0$: Comets with non-periodic orbits
- Circle: $\epsilon = 0 \implies E = -\frac{\mu k^2}{2\ell^2}$
- Ellipse: $0 < \epsilon < 1 \implies -\frac{\mu k^2}{2\ell^2} < E < 0$

$T = \int dA$

We finally discuss the time period of elliptical orbits. From (43)

$T = \frac{2\mu}{\ell} A$

$$\tau = \int dt = \int \frac{2\mu}{\ell} dA = \frac{2\mu}{\ell} A \, .$$

For an ellipse $A = \pi ab$. From previous expressions of $a, b$ in terms of eccentricity we have $b = \sqrt{\alpha a}$. Substituting in the expressions of $\tau$ and squaring we get

$$\tau^2 = \frac{4\pi^2 \mu}{k} a^3 \, . \tag{53}$$

This is Kepler's third law which states that *squared of the time period is proportional to the cube of the semimajor axis of the ellipse*.

$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$