19 Canonical Transformations

Given a Hamiltonian $H = H(q_j, p_j, t)$, the motion of the system is found by integration of the Hamilton equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{19.1}$$

For the case of a cyclic coordinate, we have, as we know,

$$\frac{\partial H}{\partial q_i} = 0, \quad \text{i.e.,} \quad \dot{p}_i = 0.$$

Hence, the corresponding momentum is constant: $p_i = \beta_i =$ constant.

Whether or not H contains cyclic coordinates depends in general on the coordinates adopted for describing a problem. This is immediately seen from the following example: If a circular motion in a central field is described in Cartesian coordinates, there is no cyclic coordinate. If we use polar coordinates (ϱ, φ), the angular coordinate is cyclic (angular momentum conservation).

A mechanical problem would therefore be greatly simplified if one could find a coordinate transformation from the set p_i, q_i to a new set of coordinates P_i, Q_i with

$$Q_i = Q_i(p_j, q_j, t), \qquad P_i = P_i(p_j, q_j, t), \tag{19.2}$$

where all coordinates Q_i for the problem were cyclic. Then all momenta are constant, $P_i = \beta_i$, and the new Hamiltonian H' is then only a function of the constant momenta P_i; hence, $H' = H'(P_j)$. Then

$$\dot{Q}_i = \frac{\partial H'(P_j)}{\partial P_i} = \omega_i = \text{constant}, \qquad \dot{P}_i = -\frac{\partial H'(P_j)}{\partial Q_i} = 0.$$

Then integration with respect to time leads to

$$Q_i = \omega_i t + \omega_0, \qquad P_i = \beta_i = \text{constant}.$$

Here, we presupposed that the new coordinates (P_i, Q_i) again satisfy the (canonical) Hamilton equations, with a new Hamiltonian $H'(P_j, Q_j, t)$. This is an essential requirement for a coordinate transformation of the form (19.2) to make it *canonical*.

Just as p_i is the canonical momentum corresponding to q_i $(p_i = \partial L/\partial \dot{q}_i)$, P_i shall be the canonical momentum to Q_i. A pair (q_i, p_i) is called *canonically conjugate* if the Hamilton equations hold for q_i and p_i. The transformation from one pair of

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canonically conjugate coordinates to another pair is called a *canonical transformation*. Then

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H'}{\partial Q_i}. \tag{19.3}$$

At the moment, we do not yet require that all Q_i be cyclic. This case will be considered later (Chap. 20).

In the new coordinates, we require Hamilton's principle to be maintained. Thus, for fixed instants of time, t_1 and t_2, we have both

$$\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt = 0$$

and

$$\delta \int_{t_1}^{t_2} L'(Q_j, \dot{Q}_j, t)\, dt = 0.$$

Thus, the difference

$$\delta \int (L - L')\, dt = 0 \tag{19.4}$$

also vanishes.

We observe that (19.4) will then be fulfilled even if the old and new Lagrangians differ by a total time derivative of a function F:

$$L - L' = \frac{dF}{dt}, \quad \text{because} \quad \delta \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = \delta\big(F|_{t_2} - F|_{t_1}\big) = 0,$$

since the variation of a constant equals zero. As we shall see, the function F mediates the transformation (p_i, q_i) to (P_i, Q_i). F is therefore also called a *generating function*. In the general case, F will be a function of the old and the new coordinates; together with the time t it involves $4n + 1$ coordinates:

$$F = F(p_j, q_j, P_j, Q_j, t).$$

But since simultaneously there are $2n$ transformation equations

$$Q_i = Q_i(p_j, q_j, t), \qquad P_i = P_i(p_j, q_j, t), \tag{19.5}$$

F involves only $2n + 1$ independent variables. F must contain both a coordinate from the old coordinate set p_i (or q_i) and one of the new P_i (or Q_i) to enable us to establish a relation between the systems. Hence, there are *four possibilities for a generating function*:

$$\begin{aligned} F_1 &= F(q_j, Q_j, t), \qquad F_2 = F(q_j, P_j, t), \\ F_3 &= F(p_j, Q_j, t), \qquad F_4 = F(p_j, P_j, t). \end{aligned} \tag{19.6}$$

Each of these functions has $2n + 1$ independent variables. The dependency must be selected in a suitable way, according to the actual problem. We now derive the transformation rules of the form (19.2) from a generating function of type F_1.

Because

$$L = L' + \frac{dF}{dt} \quad \text{and} \quad L = \sum p_i \dot{q}_i - H, \tag{19.7}$$

we have

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - H' + \frac{dF}{dt}. \tag{19.8}$$

For the total time derivative of F_1 we then have

$$\frac{dF_1}{dt} = \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}. \tag{19.9}$$

We insert this expression into (19.8), which yields

$$\sum p_i \dot{q}_i - \sum P_i \dot{Q}_i - H + H' = \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}.$$

By comparing the coefficients, we obtain

$$\begin{aligned} p_i &= \frac{\partial F_1(q_j, Q_j, t)}{\partial q_i}, \\ P_i &= -\frac{\partial F_1(q_j, Q_j, t)}{\partial Q_i}, \\ H' &= H + \frac{\partial F_1(q_j, Q_j, t)}{\partial t}. \end{aligned} \tag{19.10}$$

We are now prepared to derive the transformation equations for a generating function of the type F_2, which is also denoted by S:

$$F_2 \equiv S = S(q_j, P_j, t).$$

For the derivation, we will use a comparison of coefficients as for F_1; therefore, we require that F_2 be composed as follows:

$$F_2(q_j, P_j, t) = \sum_i P_i Q_i + F_1(q_j, Q_j, t), \tag{19.11}$$

since then we can consider the problem analogously to F_1. We imagine the Q_i as being expressed through the second equation of (19.10), i.e., through

$$P_i = -\frac{\partial F_1(q_j, Q_j, t)}{\partial Q_i}.$$

According to (19.8), we have

$$\begin{aligned}\sum_i p_i\dot{q}_i - H &= \sum_i P_i\dot{Q}_i - H' + \frac{d}{dt}F_1\\ &= \sum_i P_i\dot{Q}_i - H' + \frac{d}{dt}\Big(F_2(q_j,P_j,t) - \sum_i P_iQ_i\Big).\end{aligned}$$

This leads to

$$\begin{aligned}\sum_i p_i\dot{q}_i - \sum_i P_i\dot{Q}_i - H + H' &= \frac{d}{dt}\Big(F_2(q_j,P_j,t) - \sum_i P_iQ_i\Big)\\ &= \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t}\\ &\quad - \sum_i \dot{P}_iQ_i - \sum_i P_i\dot{Q}_i\\ \sum_i p_i\dot{q}_i + \sum_i \dot{P}_iQ_i - H + H' &= \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t}.\end{aligned}$$

Comparing again the coefficients now yields the equations

$$\begin{aligned}p_i &= \frac{\partial F_2(q_j,P_j,t)}{\partial q_i}, \qquad Q_i = \frac{\partial F_2(q_j,P_j,t)}{\partial P_i},\\ H'(P_j,Q_j,t) &= H(p_j,q_j,t) + \frac{\partial F_2(q_j,P_j,t)}{\partial t}.\end{aligned} \tag{19.12}$$

The first two relations allow us to determine the transformation equations $q_i = q_i(Q_j,P_j,t)$ and $p_i = p_i(Q_j,P_j,t)$, which by insertion into the third equation of (19.12) yield the new Hamiltonian $H'(P_j,Q_j,t)$.

The transformation equations for the other types of generating functions are obtained analogously, by choosing an appropriate sum which enables us to use the methods of the first two problems.

From (19.10) and (19.12), we obtain the dependence of the new coordinates (P_i,Q_i) on the old (p_i,q_i) and vice versa. For the case F_1, from

$$p_i = \frac{\partial F_1(q_j,Q_j,t)}{\partial q_i}$$

follow the equations $p_i = p_i(q_j,Q_j,t)$, which can be solved for the Q_i:

$$Q_i = Q_i(p_j,q_j,t).$$

Insertion into the equations

$$P_i = -\frac{\partial F_1(q_j,Q_j,t)}{\partial Q_i}$$

then enables us to calculate

$$P_i = P_i(p_j,q_j,t).$$

We now understand the name *generating function* for F: The function F determines the canonical transformation

$$Q_i = Q_i(p_j, q_j, t), \qquad P_i = P_i(p_j, q_j, t)$$

through equations of the type (19.10) or (19.12).

By means of Legendre transformations, we may furthermore define generating functions $F_3(p_j, Q_j, t)$ and $F_4(p_j, P_j, t)$. Based on the Legendre transformation

$$F_3(p_j, Q_j, t) = F_1(q_j, Q_j, t) - \sum_i q_i p_i \tag{19.13}$$

we obtain with a similar derivation the canonical transformation rules

$$q_i = -\frac{\partial F_3(p_j, Q_j, t)}{\partial p_i}, \qquad P_i = -\frac{\partial F_3(p_j, Q_j, t)}{\partial Q_i},$$
$$H' = H + \frac{\partial F_3(p_j, Q_j, t)}{\partial t}.$$

Starting finally from

$$F_4(p_j, P_j, t) = F_3(p_j, Q_j, t) + \sum_i Q_i P_i \tag{19.14}$$

the following transformation rules emerge

$$q_i = -\frac{\partial F_4(p_j, P_j, t)}{\partial p_i}, \qquad Q_i = \frac{\partial F_4(p_j, P_j, t)}{\partial P_i}, \qquad H' = H + \frac{\partial F_4(p_j, P_j, t)}{\partial t}.$$

Calculating the second derivatives of the generating functions $F_{1,2,3,4}$, we find the following relations to apply between old and new coordinates under a *canonical* transformation

$$\begin{aligned}
\frac{\partial Q_i}{\partial q_k} &= \frac{\partial^2 F_2}{\partial q_k \partial P_i} = \frac{\partial p_k}{\partial P_i}, \qquad & \frac{\partial Q_i}{\partial p_k} &= \frac{\partial^2 F_4}{\partial p_k \partial P_i} = -\frac{\partial q_k}{\partial P_i},\\
\frac{\partial P_i}{\partial q_k} &= -\frac{\partial^2 F_1}{\partial q_k \partial Q_i} = -\frac{\partial p_k}{\partial Q_i}, \qquad & \frac{\partial P_i}{\partial p_k} &= -\frac{\partial^2 F_3}{\partial p_k \partial Q_i} = \frac{\partial q_k}{\partial Q_i}.
\end{aligned} \tag{19.15}$$

Exactly the existence of these mutual relations between old and new coordinates distinguishes a *canonical* transformation from a *general* transformation (19.2) of the system's coordinates. For the latter, (19.15) do not hold.

In the preceding derivation, the Hamiltonians $H(q_j, p_j, t)$ and $H'(Q_j, P_j, t)$ were conceived as alternative descriptions of the *same* dynamical system. On the other hand, we may as well conceive H and H' as describing *different* dynamical system. A canonical transformation of H into H' then establishes a correlation of both dynamical systems. This way, it is sometimes possible to find the solution of a given dynamical system by canonically transforming it into a second system that is easier to solve. The solution of the original system is then obtained by canonically back transforming the solution of the second system. With examples 19.4 and 21.16, we shall work out the solutions of the *damped* and the *time-dependent* harmonic oscillators, respectively, by canonically transforming these systems into the *ordinary* harmonic oscillator.

EXAMPLE

19.1 Example of a Canonical Transformation

Let the generating function be given by

$$F_1(q_j, Q_j) = \sum_k q_k Q_k.$$

According to (19.10) the particular transformation rules follow as

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i, \qquad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i, \qquad H'(P_j, Q_j) = H(q_j, p_j).$$

The example shows the in the Hamiltonian formalism, the momentum and position coordinates play equivalent parts.

EXAMPLE

19.2 Point Transformations

We consider the canonical transformation that is defined by the particular generating function

$$F_2(q_j, P_j, t) = \sum_k P_k f_k(q_j, t),$$

with arbitrary differentiable functions $f_k(q_j, t)$. The transformation rules (19.12) for this F_2 follow as

$$Q_i = f_i(q_j, t), \qquad p_i = \sum_k P_k \frac{\partial f_k}{\partial q_i},$$

$$H'(Q_j, P_j, t) = H(q_j, p_j, t) + \sum_k P_k \frac{\partial f_k}{\partial t}.$$

The new position coordinates Q_i thus emerge as functions of the original position coordinates q_i, without any dependence on the momentum coordinates. Transformations of this type are referred to as *point transformations*. This class of transformations is generally canonical as we can always construct the corresponding generating function.

The particular case $f_k(q_j, t) = q_k$ then defines the *identical* transformation

$$Q_i = q_i, \qquad p_i = \sum_k P_k \delta_{ki} = P_i, \qquad H'(Q_j, P_j, t) = H(q_j, p_j, t).$$

EXAMPLE

19.3 Harmonic Oscillator

The kinetic energy $T(p)$ and the potential energy $V(q)$ of a particle be given by

$$T(p) = \frac{p^2}{2m}, \qquad V(q) = \frac{1}{2}kq^2 = \frac{1}{2}m\omega^2 q^2, \quad \omega^2 = \frac{k}{m}, \; m, k, \omega = \text{const.},$$

Example 19.3

with m denoting the particle's mass, k a characteristic constant of the oscillator, and ω its characteristic frequency. The Hamiltonian of this system is then

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{19.16}$$

The canonical equations and the equation of motion follow as

$$\dot{q}(t) = \frac{\partial H}{\partial p} = \frac{p(t)}{m}, \qquad -\dot{p}(t) = \frac{\partial H}{\partial q} = q(t)\, m\omega^2, \qquad \ddot{q} + \omega^2 q = 0. \tag{19.17}$$

The direct way to evaluate the dynamics of this system is to integrate the equation of motion. Here, we choose the "detour" over the canonical transformation formalism, namely to map our system into another system with Hamiltonian H' whose canonical equations are even easier to solve. As the "target Hamiltonian" H', we choose

$$H'(P) = \omega P. \tag{19.18}$$

A simple transformation $(H; q, p) \mapsto (H'; P, Q)$, that provides this mapping is obviously

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q, \qquad p = \sqrt{2m\omega P} \cos Q, \qquad H' = H. \tag{19.19}$$

We observe that the new momentum P has acquired the dimension of an *action*, whereas the new position coordinate Q is now dimensionless, i.e. an *angle*. In order to ensure that the *form* of the canonical equations is maintained in the new coordinates, we must test whether this transformation is actually *canonical*. To this end, we must find a generating function that yields the transformation rules (19.19).

We try a generating function of the form $F_1(q, Q, t)$. The transformation rules (19.19) are first cast into the particular functional form that corresponds to the generating function $F_1(q, Q, t)$, hence into the form $p = p(q, Q, t)$ and $P = P(q, Q, t)$

$$p = m\omega q \cot Q, \qquad P = \frac{1}{2}m\omega q^2 \frac{1}{\sin^2 Q}, \qquad H' = H. \tag{19.20}$$

We must now find a function $F_1(q, Q, t)$ that yields these particular transformation rules according to the general prescriptions (19.10). Thus, $F_1(q, Q, t)$ must satisfy

$$\frac{\partial F_1}{\partial q} = m\omega q \cot Q, \qquad \frac{\partial F_1}{\partial Q} = -\frac{1}{2}m\omega q^2 \frac{1}{\sin^2 Q}, \qquad \frac{\partial F_1}{\partial t} = 0.$$

Obviously, such a function exists and is given by

$$F_1(q, Q) = \frac{1}{2}m\omega q^2 \cot Q.$$

The transformation (19.19) thus establishes indeed a *canonical* transformation. This may be observed also from the fact that the rules (19.20) satisfy the symmetry conditions (19.15)

$$\frac{\partial P}{\partial q} = \frac{m\omega q}{\sin^2 Q} = -\frac{\partial p}{\partial Q}.$$

Example 19.3

With the evidence of the transformation (19.19) being canonical, it is ensured that the transformed system (19.18) constitutes on its part a Hamiltonian system — and hence the maintains the canonical form of the canonical equations. Explicitly, the canonical equations of the transformed system are

$$\dot{Q}(t) = \frac{\partial H'}{\partial P} = \omega, \qquad \dot{P}(t) = -\frac{\partial H'}{\partial Q} = 0.$$

These equations are thus equivalent to the original canonical equations (19.17) that emerged from the original Hamiltonian (19.16). As H' does not depend on Q, we observe that the new canonical position coordinate Q is *cyclic*, hence that its conjugate canonical momentum P represents a *conserved quantity*. The canonical equations for $\dot{Q}(t)$ and $\dot{P}(t)$ can be immediately integrated, yielding

$$Q(t) = \omega t + Q(0), \qquad P(t) = P(0).$$

The system's dynamics are thus completely solved in the simplest possible manner. Inserting the solution functions $Q(t)$ and $P(t)$ into the transformation rules (19.19), we obtain the solutions in the original coordinates $q(t)$ and $p(t)$

$$q(t) = \sqrt{\frac{2P(0)}{m\omega}}\sin\big(\omega t + Q(0)\big), \qquad p(t) = \sqrt{2m\omega P(0)}\cos\big(\omega t + Q(0)\big).$$

The trigonometric functions can finally be split by means of the addition theorems. According to (19.19), the values $\sin Q(0)$ and $\cos Q(0)$ can then be expressed in terms of the initial conditions $q(0)$ and $p(0)$ of the original system

$$q(t) = q(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t, \qquad p(t) = -q(0)\,m\omega\sin\omega t + p(0)\cos\omega t.$$

As expected, we find the solution of the harmonic oscillator exactly in the form as we would have obtained by a direct integration of the canonical equations (19.17).

At this point, one could well argue that overall effort needed to solve the canonical equations (19.17) along the "detour" over the canonical transformation method is even *larger* than that for the direct solution. But this is only due to the simplicity of the original system. The example here was just chosen to demonstrate the *method* consisting of three steps: (i) forth transformation of the initial conditions into a second system, (ii) solving on that basis the dynamics of the second system, and (iii) transforming back the obtained solution into the original system coordinates. We may depict both alternatives by means of the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(H; q(0), p(0)) \arrow[r, "\substack{\text{Solution of the canonical equations}\\ \text{of Hamiltonian } H}"] \arrow[d, "\mathcal{M}(0)", "\substack{\text{Canonical forth}\\ \text{transformation}\\ \text{of Hamiltonian}\\ \text{and initial}\\ \text{conditions}}"'] & (q(t), p(t)) \\
(H'; Q(0), P(0)) \arrow[r, "\substack{\text{Solution of the canonical equations}\\ \text{of Hamiltonian } H'}"] & (Q(t), P(t)) \arrow[u, "\mathcal{M}^{-1}(t)"', "\substack{\text{Canonical back}\\ \text{transformation}\\ \text{of the solution}}"]
\end{tikzcd}$$

Example 19.3

In the next Example 19.4, we will show that the method to determine the dynamics of a given system by transforming it into a second system that is easier to solve can indeed *reduce* the overall effort, as compared to a direct solution of the original system. This will become obvious with Example 21.16, where we treat the *time-dependent* damped harmonic oscillator. This case has long been thought of as possessing no analytic solution. Yet, the solution of this problem by means of a generalized canonical transformation is fairly straightforward. The price to pay is that we must find the appropriate generating function.

EXAMPLE

19.4 Damped Harmonic Oscillator

The Hamiltonian of the damped harmonic oscillator is explicitly time dependent

$$H(q,p,t) = \frac{p^2}{2m}\,e^{-2\gamma t} + \frac{1}{2}m\omega^2 e^{2\gamma t} q^2, \tag{19.21}$$

with the abbreviations $2\gamma = \beta/m$ and $\omega^2 = k/m$. As before, m stands for the mass of the moving point particle, β for the friction coefficient, and k for the oscillator's constant. The canonical equations follow as

$$\dot{q}(t) = \frac{\partial H}{\partial p} = \frac{p(t)}{m}\,e^{-2\gamma t}, \qquad -\dot{p}(t) = \frac{\partial H}{\partial q} = q(t)\,m\omega^2\,e^{2\gamma t}.$$

In the left-hand side equation, we see that the *canonical* momentum $p(t)$ no longer coincides with the *kinetic* momentum $p_{\mathrm{kin}}(t) = m\dot{q}(t)$, provided that $\gamma \neq 0$,

$$p(t) = m\dot{q}(t)\,e^{2\gamma t} = p_{\mathrm{kin}}(t)\,e^{2\gamma t}.$$

We may combine the two first-order equations into one second-order equation for $q(t)$ to obtain the equation of motion of the damped harmonic oscillator in its common form

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0. \tag{19.22}$$

Instead of solving this equation directly by means of an appropriate Ansatz function, we will first map the Hamiltonian (19.21) by means of a canonical transformation into the Hamiltonian of an *undamped* harmonic oscillators. In present case, the canonical transformation will be based on a generating function of type F_2, namely

$$F_2(q,P,t) = e^{\gamma t}\,qP - \frac{1}{2}m\gamma\,e^{2\gamma t} q^2.$$

According to (19.12), the subsequent transformation rules follow as

$$\begin{aligned}
p &= \frac{\partial F_2}{\partial q} = e^{\gamma t}\,P - m\gamma\,e^{2\gamma t} q \\
Q &= \frac{\partial F_2}{\partial P} = e^{\gamma t} q \\
H' - H &= \frac{\partial F_2}{\partial t} = \gamma e^{\gamma t} qP - m\gamma^2 e^{2\gamma t} q^2 = \gamma QP - m\gamma^2 Q^2.
\end{aligned}$$

Example 19.4

As the new position coordinate Q solely depends on the old position coordinate q, we are dealing here with a particular case of the general class of *point transformations*. Furthermore, the relation between old and new coordinates is obviously *linear*. We may thus express the transformation rules in matrix form. Solving for the old coordinates this yields

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} e^{-\gamma t} & 0 \\ -m\gamma e^{\gamma t} & e^{\gamma t} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{19.23}$$

According to the rule (19.12) for the mapping of the Hamiltonians, we get the *new* Hamiltonian $H'(Q, P, t)$ by expressing the original Hamiltonian $H(q, p, t)$ via (19.23) in terms of the new coordinates Q, P and, moreover, by adding $\partial F_2/\partial t$

$$\begin{aligned} H' &= \frac{1}{2} m^{-1} e^{-2\gamma t} \big(-m\gamma e^{\gamma t} Q + e^{\gamma t} P\big)^2 + \frac{1}{2} m\omega^2 e^{2\gamma t} e^{-2\gamma t} Q^2 + \gamma Q P - m\gamma^2 Q^2 \\ &= \frac{1}{2} m^{-1} (P - m\gamma Q)^2 + \frac{1}{2} m\omega^2 Q^2 + \gamma Q P - m\gamma^2 Q^2. \end{aligned}$$

In the present example, we thus find a transformed Hamiltonian H' that no longer depends on time explicitly

$$H'(Q, P) = \frac{P^2}{2m} + \frac{1}{2} m\tilde{\omega}^2 Q^2, \qquad \tilde{\omega}^2 = \omega^2 - \gamma^2.$$

We now observe that H' emerges as exactly the Hamiltonian of an *undamped* harmonic oscillator with angular frequency $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$. Its solution is already known from Example 19.3

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} \cos\tilde{\omega}t & m^{-1}\tilde{\omega}^{-1}\sin\tilde{\omega}t \\ -m\tilde{\omega}\sin\tilde{\omega}t & \cos\tilde{\omega}t \end{pmatrix} \begin{pmatrix} Q(0) \\ P(0) \end{pmatrix}. \tag{19.24}$$

The solution functions $q(t)$ and $p(t)$ of the *damped* harmonic oscillator now follows as the *product* the solution (19.24) and the canonical forth and back transformations, given by (19.23) and its inverse

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} e^{-\gamma t} & 0 \\ -m\gamma e^{\gamma t} & e^{\gamma t} \end{pmatrix} \begin{pmatrix} \cos\tilde{\omega}t & m^{-1}\tilde{\omega}^{-1}\sin\tilde{\omega}t \\ -m\tilde{\omega}\sin\tilde{\omega}t & \cos\tilde{\omega}t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m\gamma & 1 \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}.$$

On the right-hand side, the initial conditions $Q(0)$, $P(0)$ of the transformed system were expressed through those of the original system, $q(0)$, $p(0)$. Explicitly, according to the inverse transformation of (19.23) at $t = 0$, we have $Q(0) = q(0)$ and $P(0) = m\gamma q(0) + p(0)$. The determinants of all matrices are unity and hence the determinant of the combined linear mapping $(q(0), p(0)) \mapsto (q(t), p(t))$. This is in agreement with the requirement of Liouville's theorem.

In the form of the product of three matrices, it becomes obvious that the solution method via canonical transformation consists of the *three* steps, as sketched at the end of Example 19.3. We may finally express the solution of the damped harmonic oscillator (19.22) concisely by multiplying the matrices

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = R(t) \begin{pmatrix} q(0) \\ p(0) \end{pmatrix},$$

with

Example 19.4

$$R(t) = \begin{pmatrix} e^{-\gamma t}[\cos\tilde{\omega}t + \gamma\tilde{\omega}^{-1}\sin\tilde{\omega}t] & e^{-\gamma t} m^{-1}\tilde{\omega}^{-1}\sin\tilde{\omega}t \\ -e^{\gamma t} m\,\omega^2\tilde{\omega}^{-1}\sin\tilde{\omega}t & e^{\gamma t}[\cos\tilde{\omega}t - \gamma\tilde{\omega}^{-1}\sin\tilde{\omega}t] \end{pmatrix},$$

$$\tilde{\omega}^2 = \omega^2 - \gamma^2.$$

The present example shows that the task of solving the equation of motion of a given dynamical system can be facilitated if we succeed to represent it as the transformed solution of a another system that is easier to solve. But this works only if we can find an appropriate generating function.

EXAMPLE

19.5 Infinitesimal Time Step

We consider the particular canonical transformation that is generated by the function

$$F_2(q_j, P_j, t) = \sum_i q_i P_i + H(q_j, p_j, t)\,\delta t. \tag{19.25}$$

Herein, H stand for the Hamiltonian of the given dynamical system, and δt for an infinitesimal interval on the time axis. From the general form of transformation rules for generating functions of type F_2 we obtain the particular rules for (19.25) as

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} = P_i + \frac{\partial H}{\partial q_i}\delta t = P_i - \frac{dp_i}{dt}\delta t, \\ Q_i &= \frac{\partial F_2}{\partial P_i} = q_i + \frac{\partial H}{\partial P_i}\delta t \overset{\text{1st order in } \delta t}{=} q_i + \frac{\partial H}{\partial p_i}\delta t = q_i + \frac{dq_i}{dt}\delta t, \\ H' &= H + \frac{\partial F_2}{\partial t} = H + \frac{\partial H}{\partial t}\delta t = H + \frac{dH}{dt}\delta t. \end{aligned}$$

In last rightmost terms of these equations, the canonical equations were inserted, respectively. Solving for the transformed quantities, this means

$$\begin{aligned} P_i &= p_i + \dot{p}_i\,\delta t, \\ Q_i &= q_i + \dot{q}_i\,\delta t, \\ H' &= H + \dot{H}\,\delta t. \end{aligned}$$

We now observe that the particular generating function (19.25) defines precisely the canonical transformation that pushes the system ahead by an *infinitesimal* time step δt. As any canonical transformation can be applied an arbitrary number of times in sequence, we can conclude that the transformation along *finite* time steps is also canonical. This is an important result: the *time evolution* of a Hamiltonian system constitutes a particular canonical transformation. As already stated, the class of canonical transformations are characterized by their property to map Hamiltonian systems into

Example 19.5

Hamiltonian systems. It is thus ensured that a Hamiltonian system remains a Hamiltonian system in the course of its time evolution.

EXAMPLE

19.6 General Form of Liouville's Theorem

With the theory of canonical transformations at hand, we may cast Liouville's theorem into the following general form: the volume element $dV = dq_1 \dots dq_n \, dp_1 \dots dp_n$ of a Hamiltonian system with n degrees of freedom is *invariant* with respect to canonical transformations,

$$dQ_1 \dots dQ_n \, dP_1 \dots dP_n \overset{\text{can. transf.}}{=} dq_1 \dots dq_n \, dp_1 \dots dp_n .$$

For *general* transformations of the system's coordinates, the transformation of the volume element dV is determined by the determinant D of its Jacobi matrix

$$dQ_1 \dots dQ_n \, dP_1 \dots dP_n = D \, dq_1 \dots dq_n \, dp_1 \dots dp_n ,$$
$$D = \frac{\partial(Q_1, \dots, Q_n, P_1, \dots, P_n)}{\partial(q_1, \dots, q_n, p_1, \dots, p_n)} .$$

Liouville's theorem thus states that the determinant D of the transformation's Jacobi matrix is *unity* in case that the transformation is *canonical*.

For the sake of transparency, we first prove Liouville's theorem for the case of a system with *one* degree of freedom, i.e., for $n = 1$. For such a system, the determinant D of the Jacobi matrix emerging from a transition from old coordinates q, p to new coordinates $Q = Q(q, p)$, $P = P(q, p)$ is given by

$$D = \frac{\partial(Q, P)}{\partial(q, p)} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} .$$

According to the general rule for the partial derivatives of the *inverse functions* $q = q(Q, P)$, $p = p(Q, P)$ we have

$$\frac{\partial p}{\partial P} = \frac{1}{D}\frac{\partial Q}{\partial q} ,$$

which means that the determinant D of the Jacobi matrix can be expressed as

$$D = \frac{\partial Q}{\partial q}\left[\frac{\partial p}{\partial P}\right]^{-1} . \tag{19.26}$$

In the particular case of a *canonical* transformation, the transformation rules can be derived from a generating function $F_2(q, P, t)$, as stated in (19.12),

$$Q = \frac{\partial F_2}{\partial P} , \qquad p = \frac{\partial F_2}{\partial q} .$$

Inserting the expressions for Q and p, the determinant D is equivalently expressed as

$$D = \frac{\partial^2 F_2}{\partial q \partial P}\left[\frac{\partial^2 F_2}{\partial P \partial q}\right]^{-1} = 1 .$$

But this equals unity as the partial derivatives may be interchanged.

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The proof for the general case of systems with n degrees of freedom is worked out analogously. In the case of a Hamiltonian system, the determinant D of the Jacobi matrix that is associated with a general transformation of the system's coordinates has an *even* number of rows (columns). We assume the transformation to be invertible. Then, we may express the new position coordinates $Q_i = Q_i(q_j, p_j)$ as functions of the old coordinates, and the old momenta as functions of the new coordinates, $p_i = p_i(Q_j, P_j)$. The determinant of the associated Jacobi matrix is the represented by

$$D = \frac{\partial(Q_1, \ldots, Q_n)}{\partial(q_1, \ldots, q_n)} \left[\frac{\partial(p_1, \ldots, p_n)}{\partial(P_1, \ldots, P_n)} \right]^{-1}, \tag{19.27}$$

which generalizes the relation (19.26). Provided that a generating function $F_2(q_j, P_j, t)$ exists, then the transformation is referred to as *canonical*, and the transformation rules are given by (19.12). Inserting Q_i and p_i yields

$$D = \left| \left(\frac{\partial^2 F_2}{\partial q_j \partial P_i} \right) \right| \left| \left(\frac{\partial^2 F_2}{\partial P_j \partial q_i} \right) \right|^{-1} = 1,$$

which is again unity as we may interchange the sequence of partial derivatives and due to the fact that determinants of transposed matrices coincide.

We finally remark that the generating function F_2 used here in this proof is completely equivalent to the other types of generating function. For, the determinant (19.27) of the transformation's Jacobi matrix has the equivalent representations

$$\begin{aligned} D &= (-1)^n \frac{\partial(P_1, \ldots, P_n)}{\partial(q_1, \ldots, q_n)} \left[\frac{\partial(p_1, \ldots, p_n)}{\partial(Q_1, \ldots, Q_n)} \right]^{-1} \\ &= \frac{\partial(P_1, \ldots, P_n)}{\partial(p_1, \ldots, p_n)} \left[\frac{\partial(q_1, \ldots, q_n)}{\partial(Q_1, \ldots, Q_n)} \right]^{-1} \\ &= (-1)^n \frac{\partial(Q_1, \ldots, Q_n)}{\partial(p_1, \ldots, p_n)} \left[\frac{\partial(q_1, \ldots, q_n)}{\partial(P_1, \ldots, P_n)} \right]^{-1}. \end{aligned}$$

The result $D = 1$ for a canonical transformation then follows in the same way as above by inserting the rules into the appropriate generating function F_1, F_3, or F_4.

With the result of Example 19.5, we know that the time evolution of a Hamiltonian system can be conceived as a particular canonical transformation whose generating function is based on the Hamiltonian H. This yields the more special version of Liouville's theorem from Chap. 18, where it was stated that the volume element dV of a Hamiltonian system is *invariant* in the course of the system's time evolution.

EXAMPLE

19.7 Canonical Invariance of the Poisson Brackets

For a Hamiltonian system $H(q_j, p_j, t)$ of n degrees of freedom, and for two differentiable functions $F(q_j, p_j, t)$, $G(q_j, p_j, t)$ of the canonical variables and time t, the

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Poisson bracket[1] of F and G is defined by

$$[F,G]=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right). \tag{19.28}$$

A special case is established if we set up the Poisson brackets of the canonical variables q_i and p_i. As these variables are required to not depend on each other, we immediately get

$$[q_i,q_j]=0, \qquad [p_i,p_j]=0, \qquad [q_i,p_j]=\delta_{ij}. \tag{19.29}$$

We first convince ourselves that the same relations hold for canonically transformed coordinates Q_i and P_i, hence that the *fundamental* Poisson brackets (19.29) are *invariant* under canonical transformations. Making use of the relations (19.15), we find

$$\begin{aligned}
[Q_i,Q_j]&=\sum_{k=1}^{n}\left[\frac{\partial Q_i}{\partial q_k}\frac{\partial Q_j}{\partial p_k}-\frac{\partial Q_i}{\partial p_k}\frac{\partial Q_j}{\partial q_k}\right]\\
&=\sum_{k=1}^{n}\left[\frac{\partial p_k}{\partial P_i}\frac{\partial Q_j}{\partial p_k}+\frac{\partial q_k}{\partial P_i}\frac{\partial Q_j}{\partial q_k}\right]=\frac{\partial Q_j}{\partial P_i}=0\\
[P_i,P_j]&=\sum_{k=1}^{n}\left[\frac{\partial P_i}{\partial q_k}\frac{\partial P_j}{\partial p_k}-\frac{\partial P_i}{\partial p_k}\frac{\partial P_j}{\partial q_k}\right]\\
&=\sum_{k=1}^{n}\left[-\frac{\partial p_k}{\partial Q_i}\frac{\partial P_j}{\partial p_k}-\frac{\partial q_k}{\partial Q_i}\frac{\partial P_j}{\partial q_k}\right]=-\frac{\partial P_j}{\partial Q_i}=0\\
[Q_i,P_j]&=\sum_{k=1}^{n}\left[\frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k}-\frac{\partial Q_i}{\partial p_k}\frac{\partial P_j}{\partial q_k}\right]\\
&=\sum_{k=1}^{n}\left[\frac{\partial p_k}{\partial P_i}\frac{\partial P_j}{\partial p_k}+\frac{\partial q_k}{\partial P_i}\frac{\partial P_j}{\partial q_k}\right]=\frac{\partial P_j}{\partial P_i}=\delta_{ij}.
\end{aligned} \tag{19.30}$$

We are now prepared to show that the Poisson bracket of two arbitrary functions $F(q_j,p_j,t)$ and $G(q_j,p_j,t)$ establishes likewise a canonical invariant. The time t

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[1] *Siméon Denis Poisson*, French mathematician and physicist, b. June 21, 1781, Pithiviers, France–d. April 25, 1840, Paris, France. Descending from a simple social background—his father was a soldier—Poisson had good teachers who recognized his extraordinary gifts and made it possible for him to begin studies at the École Polytechnique in Paris in 1798. There, his mathematical talents were recognized by Laplace and Lagrange. Poisson became an assistant professor, and, in 1806, a full professor at the École Polytechnique, where he energetically worked to improve teaching and the formation of students.

His research initially was focused on the theory of ordinary and partial differential equations, which he applied to many different physical problems. Thus, Poisson developed further the mechanics of Laplace and Lagrange, and studied problems related to the propagation of sound, elasticity, and static electricity. He later turned his interests towards the theory of probabilities, and recognized the seminal nature of the Law of Large Numbers.

Many ideas and concepts are named after Poisson, such as the Poisson equation in potential theory, the Poisson bracket of mechanics, the Poisson ratio in elasticity, and the Poisson distribution in statistics.

Example 19.7

as the common independent variable of both the original and the transformed system is not transformed. We may thus restrict ourselves to the nested mapping

$$F(q_j, p_j) = F\big(Q_k(q_j, p_j), P_k(q_j, p_j)\big),$$
$$G(q_j, p_j) = G\big(Q_k(q_j, p_j), P_k(q_j, p_j)\big).$$

Applying the chain rule, one finds

$$[F, G]_{q,p} = \sum_k \sum_i \sum_j \Bigg\{ \left(\frac{\partial F}{\partial Q_i} \frac{\partial Q_i}{\partial q_k} + \frac{\partial F}{\partial P_i} \frac{\partial P_i}{\partial q_k} \right) \left(\frac{\partial G}{\partial Q_j} \frac{\partial Q_j}{\partial p_k} + \frac{\partial G}{\partial P_j} \frac{\partial P_j}{\partial p_k} \right) - \left(\frac{\partial F}{\partial Q_i} \frac{\partial Q_i}{\partial p_k} + \frac{\partial F}{\partial P_i} \frac{\partial P_i}{\partial p_k} \right) \left(\frac{\partial G}{\partial Q_j} \frac{\partial Q_j}{\partial q_k} + \frac{\partial G}{\partial P_j} \frac{\partial P_j}{\partial q_k} \right) \Bigg\}.$$

Multiplying and recollecting the terms for Poisson brackets with respect to the coordinates Q_i, P_j yields the equivalent expression

$$[F, G]_{q,p} = \sum_i \sum_j \Bigg\{ \frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial Q_j} [Q_i, Q_j] + \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial P_j} [P_i, P_j] + \frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial P_j} [Q_i, P_j] - \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial Q_j} [Q_j, P_i] \Bigg\}. \tag{19.31}$$

Equation (19.31) holds for any invertible coordinate transformation. In the particular case that the transformation is *canonical*, then in addition the relations (19.30) for the fundamental Poisson brackets apply. In that case, (19.31) simplifies to

$$[F, G]_{q,p} = \sum_i \sum_j \left(\frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial Q_j} \right) \delta_{ij} = [F, G]_{Q,P}.$$

The Poisson bracket $[F, G]$ is thus uniquely determined by functions F and G and independent from the underlying coordinate system, provided that a transformation of the coordinate system is canonical.

EXAMPLE

19.8 Poisson's Theorem

Poisson's theorem embodies an important benefit of the Poisson bracket formalism: if two invariants I_1 and I_2 of a given dynamical system are known, then it is possible to directly construct a third invariant I_3. In order to demonstrate this, we first derive the general rule for the total time derivative of a Poisson bracket

$$\frac{d}{dt}[F, G] = \left[\frac{dF}{dt}, G \right] + \left[F, \frac{dG}{dt} \right].$$

The proof is easily worked out by directly calculating the total time derivative of the Poisson bracket's definition from (19.28)

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$$
\begin{aligned}
\frac{d}{dt}[F,G] &= \sum_{i=1}^{n}\bigg[\frac{\partial F}{\partial q_i}\frac{d}{dt}\left(\frac{\partial G}{\partial p_i}\right)+\frac{\partial G}{\partial p_i}\frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right)-\frac{\partial F}{\partial p_i}\frac{d}{dt}\left(\frac{\partial G}{\partial q_i}\right)\\
&\qquad -\frac{\partial G}{\partial q_i}\frac{d}{dt}\left(\frac{\partial F}{\partial p_i}\right)\bigg]\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\bigg[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial G}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial G}{\partial p_j}\frac{dp_j}{dt}+\frac{\partial G}{\partial t}\right)\\
&\qquad -\frac{\partial G}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial F}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial F}{\partial p_j}\frac{dp_j}{dt}+\frac{\partial F}{\partial t}\right)\\
&\qquad +\frac{\partial G}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial F}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial F}{\partial p_j}\frac{dp_j}{dt}+\frac{\partial F}{\partial t}\right)\\
&\qquad -\frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial G}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial G}{\partial p_j}\frac{dp_j}{dt}+\frac{\partial G}{\partial t}\right)\bigg]\\
&= \sum_{i=1}^{n}\bigg[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{dG}{dt}\right)-\frac{\partial G}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{dF}{dt}\right)+\frac{\partial G}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)\\
&\qquad -\frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{dG}{dt}\right)\bigg]\\
&= \left[F,\frac{dG}{dt}\right]+\left[\frac{dF}{dt},G\right].
\end{aligned}
$$

If both $I_1 \equiv F$ as well as $I_2 \equiv G$ are invariants of motion, i.e., if $dF/dt \equiv 0$ and $dG/dt \equiv 0$, we conclude

$$
\frac{d}{dt}[F,G] \equiv 0. \tag{19.32}
$$

With $I_3 \equiv [F,G]$ we have then found another, possibly trivial, invariant of the system.

We remark that Poisson's theorem in the form of (19.32) only applies for invariants F and G, whose total time derivatives vanish *identically*.

In case that $dF/dt = 0$ and $dG/dt = 0$ represent only *implicit functions*, we *cannot* infer that the Poisson brackets $[dF/dt, G]$ and $[F, dG/dt]$ vanish. The reason is that the construction of a Poisson bracket does not constitute an *algebraic* but an *analytic* operation. In the latter case, we must impose the stronger condition that the partial derivatives of dF/dt and dG/dt with respect to the q_i and the p_i all vanish

$$
\frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)=0,\qquad \frac{\partial}{\partial p_i}\left(\frac{dF}{dt}\right)=0 \quad\Rightarrow\quad \left[\frac{dF}{dt},G\right]=0.
$$

EXAMPLE

19.9 Invariants of the Plane Kepler System

The Hamiltonian for a plane and time-independent Kepler system is given by

$$
H(q_j,p_j)=\frac{1}{2}\left(p_1^2+p_1^2\right)-\frac{\mu}{r},\qquad r=\sqrt{q_1^2+q_1^2},\qquad \mu=G(m_1+m_2).
$$

Herein, G denotes the gravitational constant, and m_1, m_2 the masses of the respective bodies. The canonical equations are obtained as

Example 19.8

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = p_i, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = -\mu\frac{q_i}{r^3}.$$

The angular momentum $D = q_1 p_2 - q_2 p_1$ constitutes an invariant of all systems with central force fields. We verify this by directly calculating the time derivative of D and subsequently inserting the canonical equations

$$\begin{aligned}\frac{dD}{dt} &= q_1\dot{p}_2 + \dot{q}_1 p_2 - q_2\dot{p}_1 - \dot{q}_2 p_1\\ &= -\frac{\mu}{r^3}q_1q_2 + p_1p_2 + \frac{\mu}{r^3}q_2q_1 - p_2p_1\\ &\equiv 0.\end{aligned}$$

Another invariant of this system is given by

$$R_1 = q_1 p_2^2 - q_2 p_1 p_2 - \mu\frac{q_1}{r}.$$

We convince ourselves of this fact again by direct calculation of the time derivative of R_1

$$\begin{aligned}\frac{dR_1}{dt} &= \dot{q}_1 p_2^2 + 2q_1 p_2\dot{p}_2 - \dot{q}_2 p_1 p_2 - q_2\dot{p}_1 p_2 - q_2 p_1\dot{p}_2 - \mu\frac{\dot{q}_1}{r}\\ &\quad + \mu\frac{q_1}{r^3}(q_1\dot{q}_1 + q_2\dot{q}_2)\\ &= p_1p_2^2 - 2\frac{\mu}{r^3}q_1q_2p_2 - p_1p_2^2 + \frac{\mu}{r^3}q_1q_2p_2 + \frac{\mu}{r^3}q_2^2p_1\\ &\quad - \frac{\mu}{r^3}p_1\left(q_1^2+q_2^2\right) + \frac{\mu}{r^3}q_1(q_1p_1+q_2p_2)\\ &\equiv 0.\end{aligned}$$

According to Poisson's theorem, the function $R_2 = [D, R_1]$ then represents another invariant of the system

$$\begin{aligned}R_2 \overset{\text{Def}}{=} [D, R_1] &= \frac{\partial D}{\partial q_1}\frac{\partial R_1}{\partial p_1} - \frac{\partial D}{\partial p_1}\frac{\partial R_1}{\partial q_1} + \frac{\partial D}{\partial q_2}\frac{\partial R_1}{\partial p_2} - \frac{\partial D}{\partial p_2}\frac{\partial R_1}{\partial q_2}\\ &= -q_2p_2^2 + q_2\left(p_2^2 - \frac{\mu}{r} + \mu\frac{q_1^2}{r^3}\right)\\ &\quad - p_1(2q_1p_2 - q_2p_1) + q_1\left(p_1p_2 - \frac{\mu}{r^3}q_1q_2\right)\\ &= q_2p_1^2 - q_1p_1p_2 - \mu\frac{q_2}{r}.\end{aligned}$$

Example 19.8

We can prove this easily by directly calculating dR_2/dt. The invariants R_1 and R_2 constitute the components of the Runge–Lenz[2] vector. We will get back to the Runge–Lenz vector in Example 21.21.

[2] *Carl David Tolmé Runge*, German mathematician and physicist, b. August 30, 1856, Bremen, Germany–d. January 3, 1927, Göttingen, Germany. Runge came from a family of merchants and grew up in Havana and Bremen. He took up studies of literature at Munich, but soon switched to mathematics and physics. As a student in Munich, he met Max Planck, which was the beginning of a lifelong friendship. Runge finished his studies with a thesis on differential geometry, supervised by Weierstrass, and became a professor of mathematics in Hanover in 1886. In 1906, he took up a professorship in Göttingen. Runge worked on the numerical solution of equations—the Runge–Kutta method for the solution of differential equations is named after him—and on spectroscopy. He did spectroscopical measurements himself and contributed eminently to the understanding of the spectral series of various atoms. Runge applied his results to the new field of the analysis of stellar spectra. In a textbook on vector analysis, Runge described the derivation, originally found by Gibbs, of conserved quantity of the Kepler problem. This discussion was then referred to by Wilhelm Lenz in his early quantum mechanical treatment of the hydrogen atom. The corresponding conserved quantity has become known as the Runge–Lenz vector.

Wilhelm Lenz, German physicist, b. February 8, 1888, Frankfurt am Main, Germany–d. April 30, 1957, Hamburg, Germany. Lenz attended the same school in Frankfurt as Otto Hahn, and took up studies of mathematics and physics in Göttingen in 1906. He obtained his Ph.D. in 1911 with Arnold Sommerfeld in Munich and became Sommerfelds assistant. In 1921, Lenz became professor of theoretical physics in Hamburg. Among his students and assistants in Hamburg were Pascual Jordan, Wolfgang Pauli, and Hans Jensen, who was awarded the Nobel Prize in physics in 1963 for the development of the shell model of the atomic nucleus. Lenz' contributions to the early quantum mechanics of hydrogen-like atoms renewed interest in the Runge–Lenz vector, which, actually, had been known long before. A simple model for the description of ferromagnets developed by Lenz and proposed as a thesis topic to one of his students is well known today by the name of the student: the Ising model.