3 Lagrangian formulation

In this section, we will explore a new formulation of classical mechanics that addresses the limitations of Newtonian mechanics. To reiterate, the need for this new formulation arises from the following challenges associated with Newton's laws:

- In a system with constraints, we must simultaneously solve both Newton's second law and the equations that describe those constraints.
- The forces of constraint are often unknown, making it difficult to apply Newton's laws directly in many situations.
- Newton's second law is not invariant under coordinate transformations. For example, the equation of motion for a free particle in a two-dimensional plane, expressed in Cartesian coordinates, is given by:

$$m\hat{x}\frac{d^2x}{dt^2} + m\hat{y}\frac{d^2y}{dt^2} = \vec{F}\,.$$

However, in spherical coordinates, the same equation takes a more complex form:

$$m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + m\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\theta} = \vec{F}\,.$$

This shows that the form of the equation depends on the choice of coordinates, which complicates analysis.

To overcome these limitations, we will derive Lagrange's equation of motion in this section. This formulation provides a more general and coordinate-independent way to describe the motion of a system between times t_1 and t_2.

3.1 Hamilton's Principle

Configuration space: Consider a system with f degrees of freedom. We can represent the system in an f-dimensional space, where the coordinate system consists of f orthogonal axes, labeled $Q_1, Q_2, \ldots, Q_f$. This f-dimensional space is referred to as a *hyperspace*. In this hyperspace, a single point corresponds to a specific set of f coordinates, $\{q_1, q_2, q_3, \ldots, q_f\}$, which collectively describe the instantaneous configuration of the system. This set of coordinates is called the system's *configuration*. The f-dimensional space in which these coordinates reside is known as the *configuration space*. For example, the configuration space of a particle moving in three-dimensional space is a 3-dimensional Cartesian coordinate system.

Motion of a system in configuration space: Since the configuration of the system at any instant t is represented by a single point in the configuration space, it follows that the motion of the system corresponds to a curve in this space. This curve is a function of the generalized coordinates, with time acting as the parameter. However, since the generalized coordinates do not necessarily correspond to any physical coordinate system, the trajectory in the configuration space does not necessarily resemble the actual path traced by the system in real space.

Lagrangian formulation is based on the *Hamilton's* principle[1]. The principle is valid for systems where all forces can be derived from a generalized scalar potential that may be a function of coordinates and velocities. If the potential is explicit function of coordinates only then the system is also conservative. Consider such a system that evolves from time t_1 at point $A \equiv \{q_1(t_1), q_2(t_1).....q_f(t_1)\}$ to another time t_2 point $B \equiv \{q_1(t_2), q_2(t_2).....q_f(t_2)\}$. According to the Hamilton's principle, the motion will be such that the line integral

$$\boxed{S = \int_{t_1}^{t_2}\Big(T(q_i,\dot{q}_i,t) - V(q_i,t)\Big)dt = \int_{t_1}^{t_2} L(q_i,\dot{q}_i,t)dt}\,, \tag{18}$$

[1]Please read *Feynman Lectures on Physics, Vol-2, Chapter 19* for an excellent introduction to this topic.

has a stationary value for the actual path of the system. Here, the value of the integral S is called the *action* and the *functional* (function of a function is called a functional)

$$\boxed{L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, t)} . \tag{19}$$

is called the Lagrangian of the system. The meaning of "stationary" is that for any path that is infinitesimally close to the original path, the change in the action is

$$\boxed{\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0} . \tag{20}$$

For a holonomic system, the Lagrange equation of motion can be derived from the Hamilton's principle.

3.2 Variational Calculas

The Hamilton's principle (20) is equivalent to the mathematical problem in *calculus of variation*. In one dimension, the problem of calculus of variation is as follows: suppose there is a path $y = y(x)$ between x_1 and x_2. There is a function $f(y, \dot{y}, x)$ that is defined on the path where $y' = dy/dx$. In the calculus of variation, one finds the path $y = y(x)$ given the stationary integral

$$J = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx . \tag{21}$$

Here x acts as the parameter of the path. The meaning of stationary is that $\delta J = 0$ for any path that is infinitesimally close to the original path. We can denote a neighbourhood path as $y(x) + \alpha\eta(x)$ such that α is a parameter and $\eta(x_1) = 0, \eta(x_2) = 0$. To be specific all paths can be written as

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x) , \tag{22}$$

such that $\alpha = 0$ correspond to the original path. The required properties of the function $\eta(x)$ is that it is continuous and nonsingular and has continuous first and second derivative between x_1 and x_2.

Our aim is to find the actual path for which the value of the integral (21) is extremum – *i.e.,* for an infinitesimal deviation of the path from the original path, the deviation δJ. Since an infinitesimally close path to the original path is parametrized by a non-zero value of α one can write Hamilton's principle as

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0 .$$

Differentiating (18) w.r.to α we get (from now onwards we omit $|_{\alpha=0}$ sign for simplicity)

$$0 = \left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha} dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} \right) dx .$$

We have used Leibniz formula of differentiating an integration. Though we have used partial derivative inside the integration, we remember that since α is independent of x we could have used $d/d\alpha$ as well. Since x and α are independent, the last term within the bracket vanishes. The second term can be written as

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right) dx ,$$

where we have interchanged d/dx and $\partial/\partial\alpha$ since α and x are independent. Further integrating by parts we obtain

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right) dx = \left.\frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha} dx .$$

Since all the paths have the common end-points $(x_1, y(x_1))$ and $(x_2, y(x_2))$, and the end points are always fixed, the first terms is zero in the above line check yourself. Hence we get

$$0 = \frac{dJ}{d\alpha}\bigg|_{\alpha=0} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'}\right)\frac{dy}{d\alpha}dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x)dx\,.$$

Since $y \equiv y(x, \alpha)$ are all independent paths, their variations $\eta(x) = dy/d\alpha$ are independent of each other. From the "fundamental lemma" of calculus it implies that the terms in the right-hand-side of the parenthesis vanishes

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{df}\left(\frac{\partial f}{\partial y'}\right) = 0}\,. \tag{23}$$

This is known the Euler-Lagrange equation. Solution of the equation yields the path $y = y(x)$ for which (21) is stationary. In other words, the solution of the EL equation (23) gives the path $y = y(x)$.

3.2.1 The brachistochrone problem

The statement of the problem is as follows: there is a particle that falls under the influence of gravity only from a higher point (shown in figure as 1) from rest to a lower point (point 2). The problem is the find the path for which the time taken is minimum.

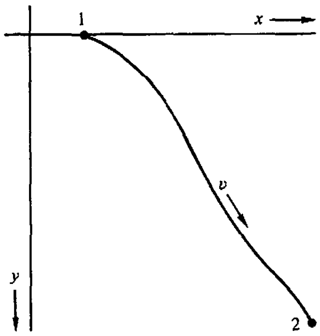

Figure 2: The brachistochrone problem.

If $v(t)$ is the instantaneous velocity of the particle and it travels a path ds in the interval dt then the total time taken is

$$t_{12} = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{v}\,. \tag{24}$$

The origin of the coordinate is at the point 1 and the y-axis vertically downwards. According to the conservation theorem

$$\frac{1}{2}mv^2 = mgy\,,$$

from which the velocity can be obtained as $v = \sqrt{2gy}$. Hence, the integral t_{12} can be written as

$$t_{12} = \int_1^2 \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}dx\,.$$

The integral t_{12} can be identified with J and the function f is

$$f = \sqrt{\frac{1 + y'^2}{2gy}}\,,$$

so that it satisfies the equation (26)

$$-\frac{\sqrt{1+y'^2}}{2y\sqrt{y}} - \frac{d}{dx}\left(\frac{1}{\sqrt{y}} \cdot y' \cdot \frac{1}{\sqrt{1+y'^2}}\right) = 0\,,$$
$$\implies -2yy'' = 1 + y'^2\,,$$

where y'' double derivative of y with respect to x. After multiplying both sides by y' and then rearranging, it can be integrated as

$$\int \frac{2y'y''}{1+y'^2} = -\int \frac{y'}{y}\,, \qquad \implies \ln(1+y'^2) = -\ln y + A\,,$$
$$\implies 1 + y'^2 = \frac{B}{y}\,,$$

where $B = e^A$. We need to integrate it one more time. To this end we write the last line above as

$$\frac{\sqrt{y}dy}{\sqrt{B-y}} = \pm dx\,.$$

To integrate it we make a change in variable as $y = B\sin^2\phi$ so that $dy = 2B\sin\phi\cos\phi d\phi$. Integrating we get

$$B(2\phi - \sin 2\phi) = \pm 2x - C\,,$$

where C is a constant. Redefining $\theta = 2\phi$ we get the parametric solutions as

$$x = \pm a(\theta - \sin\theta) \pm d\,, \quad y = a(1-\cos\theta)\,,$$

where $a = B/2$ and $d = C/2$. Using the initial condition $(x, y) = (0, 0)$ yields $d = 0$. On the other hand we assume that x always increase during the motion, so that only positive sign can be chosen. Hence the final solution is

$$x = a(\theta - \sin\theta)\,, \quad y = a(1-\cos\theta)\,. \tag{25}$$

This is the parametric equation of a curve that is called a cycloid.
Practice problems: shortest distance between two points in a plane is a straight line, shape of a wire hanging freely under its own weight,

Some other problems that will be discussed in class or in tutorials

- Suppose there is a curve in the $x-y$ lane between two fixed points (x_1, y_1) and (x_2, y_2). The plane is revolved around the y-axis. The problem is to find the curve for which the surface area is minimum.
- The catenary curve
- Geodesic on a given geometry

3.3 Euler-Lagrange Equation of motion

We now discuss what the Hamilton's principle (20) yields for the generalized coordinates. The derivation presented in section 3.2 can be generalized (exercise/assignment) to show that each of the generalized coordinates satisfy the following equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,. \tag{26}$$

The equation (26) is known as the Euler-Lagrange equation of motion (ELEoM). Solution of this equation gives trajectory $q_i = q_i(t)$.

Following examples are discussed in class – practice problem

- Free particle
- Freely falling particle under gravity
- Simple pendulum
- Double pendulum
- Pendulum where the pivot point moves in horizontal axis
- Atwood's machine

3.4 Conservation theorems

3.4.1 Conjugate/Canonical/generalized momentum

Consider a system in motion under a force that can be derived from a potential, which is a function of position alone. To describe the system, we use Cartesian coordinates $x_i = (x, y, z)$. For example, this could represent the motion of a particle under the influence of gravity. In this context, the first term in equation (26) corresponds to

$$\frac{\partial L}{\partial x_i} = \frac{\partial T}{\partial x_i} - \frac{\partial V(x,y,z)}{\partial x_i} = F_{x_i}\,,$$

and from the second term we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}} - \frac{\partial V(x,y,z)}{\partial \dot{x}}\right),\\ &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\sum \frac{m^2}{2}(\dot{x}_i^2)\right) = \frac{d}{dt}(m_i\dot{x}_i) = \frac{d}{dt}(p_i)\,.\end{aligned}$$

This leads to Newton's second law, and the x_i^{th} component of the linear momentum can be obtained from the Lagrangian as

$$m\dot{x}_i = \frac{\partial L}{\partial \dot{x}_i}\,.$$

The result can be further generalized to define *generalized momentum* or the *canonical momentum* or *conjugate momentum* for the coordinate q_i as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\,. \tag{27}$$

If the coordinates q_i are not Cartesian, the corresponding momenta p_i may not have the dimensions of linear momentum. Furthermore, for velocity-dependent potentials, even in the case of Cartesian coordinates, the generalized coordinates q_i are not necessarily identical to the mechanical momentum. An example of this is the motion of a charged particle in an uniform electromagnetic field, where the Lagrangian is

$$L = \frac{m^2}{2}\dot{r}^2 - Q\phi + Q\vec{A}.\dot{\vec{r}}\,,$$

where $\vec{r}$ is the position vector of the particle, Q is the charge, and $\vec{A}$ is the magnetic vector potential. In this example the x-component of the mechanical momentum of the particle is $m\dot{x}$, but the generalized momentum is

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + QA_x\,.$$

If the Lagrangian does not depend on a given coordinate q_i, but depends on its corresponding velocity $\dot{q}_i$, then the coordinate is said to be *cyclic* (or *ignorable*). In this case, the Euler-Lagrange equation implies that $\frac{\partial L}{\partial q_i} = 0$, and the equation of motion becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \Longrightarrow p_i = \text{constant}\,,$$

so, *generalized momentum corresponding to generalized coordinate is conserved.* In the previous example, of both ϕ and $\vec{A}$ are independent of r then

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + QA_x = \text{constant}\,.$$

Here, what is conserved is not the mechanical linear momentum $m\dot{x}$ but sum of $m\dot{x}$ and QA_x.

3.4.2 Energy conservation

Another conservation theorem is obtained for Lagrangian that has no explicit time dependence, *i.e.,*

$$\frac{\partial L}{\partial t} = 0\,.$$

so the total time deribvative of the Lagrangian is

$$\begin{aligned}
\frac{dL}{dt} &= \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\,, \\
&= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\,, \\
&= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) = \sum_i \frac{d}{dt}\Big(p_i\dot{q}_i\Big)\,.
\end{aligned}$$

which gives

$$\frac{d}{dt}\left(\sum_i p_i\dot{q}_i - L\right) = 0\,, \tag{28}$$

i.e., the function inside the parenthesis is a constant of motion. It is called the energy function

$$h(q_i, \dot{q}_i; t) = \sum_i p_i\dot{q}_i - L\,. \tag{29}$$

It can be shown that if L has no explicit time dependence and the potential is dependent of position coordinate only then the energy function is equal to the total energy of the system.

Examples: Simple pendulum, double pendulum, coupled pendulum, etc