1 Books

- Classical Dynamics: Thornton & Marion
- An Introduction to Classical Mechanics: Kleppner & Kolenkow
- Classical Mechanics: Doglus Gregory
- Classical Mechanics: Goldstein
- Classical Mechanics: John Taylor

2 Elementary Principles

2.1 Single Particle Mechanics

inertial frame → 2nd law valid

Position vector of a single particle in a Cartesian coordinate system is

$$\vec{r}(t) = \hat{i}x(t) + \hat{j}y(t) + \hat{k}z(t)\,.$$

Since I am lazy to type these notes, from now onwards I will not show the time dependence of the position vector. Velocity of the particle is

$$\vec{v} = \frac{d\vec{r}}{dt}\,, \tag{1}$$

and linear momentum is

$$\vec{p} = m\vec{v}\,, \tag{2}$$

where m the mass of the particle. If the force acting on the particle is $\vec{F}$ then from Newtons's second law it follows that

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} = \frac{d}{dt}(m\vec{v})\,. \tag{3}$$

What the Newton's 2nd law tells us is that, unless acted on by a force, an object is either is at rest or in uniform motion relative to the coordinate system. The reference frame in which this law is valid is called the *inertial frame* or *Galilean frame.* Unless otherwise mentioned, we will take a coordinate system attached firmly to the earth as an inertial frame. Acceleration of the particle is

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}\,. \tag{4}$$

If $\vec{F} = 0$ then $\vec{p}$ =constant:- *in the absence of a force, linear momentum is conserved.* This is conservation of linear momentum.

The work done by $\vec{F}$ to move the particle from point 1 to point 2 is

$$W_{12} = \int_1^2 \vec{F}.d\vec{r} = m\int_1^2 \frac{d\vec{v}}{dt}.\vec{v}dt = \frac{m}{2}(v_2^2 - v_1^2) = T_2 - T_1 \tag{5}$$

where $v_{1,2}$ are velocities and $T_{1,2}$ are the kinetic energies at the two end points

$$T_{1,2} = \frac{1}{2}mv^2\,.$$

Conservative force: There are certain forces for which the work done in moving a particle from point 1 to point 2 depends only on the initial and final positions, and is independent of the path taken. These forces are known as conservative forces. In the case of a conservative force, if you move a particle from point 1 to point 2 and then return it from point 2 to point 1, the total work done over the entire journey will be zero, *i.e.,*

$$\int_1^2 \vec{F}.d\vec{r} + \int_2^1 \vec{F}.d\vec{r} = \boxed{\oint \vec{F}.d\vec{r} = 0}\,. \tag{6}$$

Vector analysis have taught us that if any vector $\vec{F}$ satisfies an equation like (6), then the vector can be written as (Exercise)

$$\vec{F} = -\vec{\nabla} V\,, \tag{7}$$

where $V \equiv V(\vec{r}) = V(x, y, z)$ is a scalar function of position coordinates and $\vec{\nabla}$ is known as the gradiant operator *gradiant* of a scalar function (the concept of gradiant will be discussed in a tutorial session)

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial x} + \vec{k}\frac{\partial}{\partial x}\,.$$

In classical mechanics the function V is known as the *potential energy* or simply *potential*. Force is unique if a constant is added to the potential, *i.e.*, $\vec{F} = -\vec{\nabla}(V + \text{const.})$, therefore absolute value of a potential is immaterial – what is physical is the potential difference.

For a conservative force, the work done in moving a particle from point 1 to point 2 is

$$W_{12} = \int_1^2 \vec{F}.d\vec{r} = -\int_1^2 \vec{\nabla} V.d\vec{r} = V_1 - V_2\,. \tag{8}$$

We have already found in (5) that $W_{12} = T_2 - T_1$. Hence we get

$$T_2 - T_1 = V_1 - V_2 \Rightarrow T_1 + V_1 = T_2 + V_2 = E. \tag{9}$$

In other words, the total energy $E = T + V$ is conserved in case of conservative field. Example: motion of a particle under gravity (when air resistance neglected).

In equation (8) we have

$$\int \vec{F}.d\vec{r} = -\int \vec{\nabla} V.d\vec{r}\,,$$

which implies that the force can be written as the gradiant of the potential

$$\boxed{\vec{F} = -\vec{\nabla} V\,.} \tag{10}$$

2.2 System of many Particles

2.2.1 Constraints

In most problems, the Newton's equation alone is insufficient to fully describe a system because there are *constraints* that limit the motion. To understand what is meant by constraints, consider a single particle moving along a straight line on the x-axis. In this case, the motion is constrained by the conditions:

$$y = 0, \quad z = 0.$$

If the particle moves on the surface of a sphere with radius R, the constraint is:

$$x^2 + y^2 + z^2 = R^2.$$

Similarly, in a multi-particle system—such as a gas inside a balloon—the molecules are constrained to move within the surface of the balloon. For example, in a two-particle system where the distance between the particles is always fixed at ℓ, the constraint equation is:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = \ell^2.$$

In most of the problems we will discuss, the constraints can be expressed as equations. These are referred to as *holonomic constraints*. However, there are systems where the constraints are such that they can not be expressed by equations involving coordinates. Hence, dependent coordinates can not be fully eliminated. They are called *non-holonomic constraints*. A classic example is that of a disk of radius R that rolls on a surface without slipping. If $\vec{v}$ is the linear velocity of the disk and $\vec{\omega}$ then the following constraints apply

$$v = \omega R\,.$$

Some more examples of constrained system are

- Cars on road
- Simple pendulum
- Double pendulum (2nd pendulum hanging from the mass of the first)
- Particle on a surface

For a constrained system, a straightforward approach to apply Newton's law run into the following difficulties

- One has to simultaneously solve the Newton's law and the equations of constraints
- Equation of constraints can be enforced by force of constraints. Usually, the forces of constraints are unknown. In the example of many particle system (11), the interparticle forces f_{ji} are constraint forces that are unknown. Take an example of a system of many particles, where each particle experiences two types of forces: an external force $\vec{F}_i^e$ and an interaction force $\vec{F}_{ji}$ due to the j^{th} particle. The dynamics of the system are governed by Newton's second law, which for each particle can be written as:

  ext. interaction

  $$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i^e + \sum_{j\neq i} \vec{F}_{ji} \,. \tag{11}$$

  In this problem the forces of constraints F_{ij} are unknown making the Newtonian setup unsuitable to solve the problem. There are many such examples where the forces of constraints are unknown.

The first problems can be overcome by introducing the concept of *generalized coordinates* and the second one by using *Lagrange's equation of motion* instead of Newtons law.

2.2.2 Generalized coordinates

$3 \times N$

The number of independent coordinates required to specify the configuration of a system of N free particles (no constraints) in a three-dimensional Cartesian coordinate system is $3N$. In a Cartesian coordinate system the coordinates are (x_1, y_1, z_1) for the first particle, (x_2, y_2, z_2) for the second particle, and so on. The coordinate set if referred to as as *degrees of freedom (dof)* of the system $f = 3N$.

If k holonomic constraints are imposed, then there exists k independent equations, each relating the coordinates of the system. The equations allow us to eliminate k coordinates, reducing the number of independent coordinates variables to $f = 3N - k$. Consequently, the system will have

$$f = 3N - k \,, \tag{12}$$

degrees of freedom.

So far we have been thinking in terms of Cartesian coordinates. If a system has f degrees of freedom then one can introduce a set of f independent coordinates called the *generalized coordinates*. What is meant by "independent" is that there is no functional relation between the generalized coordinates. A set of f generalized coordianates is usually denoted by q_i where $i = 1, 2, 3...f$. The generalized coordinates are not always Cartesian. It can be polar, cylindrical, or mix of many coordinate systems.

Lets take an example of a single particle moving on the surface of a sphere of radius R. The equation of constraint is $x^2 + y^2 + z^2 = R^2$. In this case if two coordinates x, y are determined then z is given as $z = \sqrt{R^2 - x^2 - y^2}$. Hence the degrees of freedom of motion of a particle on surface of a sphere is 3.1 - 1 = 2. Alternatively the motion can be described in terms of spherical polar coordinates (R, θ, ϕ)

$f = 3N - k$

$$x = R\sin\theta\cos\phi \,, \quad y = R\sin\theta\sin\phi \,, \quad z = R\cos\theta \tag{13}$$

Since R is a constant, there are only two independent coordinates $q = \{\theta, \phi\}$, consistent with the previous discussion. These are the generalized coordinates of the system. The equations (13) express the functional relationship between old (Cartesian) and the generalized coordinates.

So in a system of N particles with k constraints, there exist $f = 3N - k$ generalized coordinates $q_1, q_2, ...q_{3N-k}$, and one can always express the old coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2)....(x_N, y_N, z_N)$ in terms of the generalized coordinates

$$
\begin{aligned}
x_1 &= f_1(q_1, q_2...q_{3N-k}, t)\,, \\
y_1 &= f_2(q_1, q_2...q_{3N-k}, t)\,, \\
z_1 &= f_3(q_1, q_2...q_{3N-k}, t)\,, \\
... &= ...\,, \\
... &= ...\,, \\
... &= ...\,, \\
z_N &= f_{3N}(q_1, q_2...q_{3N-k}, t)\,,
\end{aligned}
$$

If t is absent in the above functions then constraints are independent of time – scleronomic constraints. We will mostly deal with these types. These equations can be inverted to solve for the generalized coordinates.

Once the set of generalized coordinates of a system is known the set is said to completely determine the configuration of the system. It means that the position vectors of each of the particle system can be written in terms of the generalized coordinates.

Example: Consider a system of two particles shown in figure. The genaralized coordinates are x – distance of the first particle from the origin, and θ – the angle that the second particle makes with respect to the varticle. The position coordinates are

$$\vec{r}_1 = x\vec{i}\,, \qquad \vec{r}_2 = (x + a\cos\theta)\vec{i} - (a\sin\theta)\vec{k}\,.$$

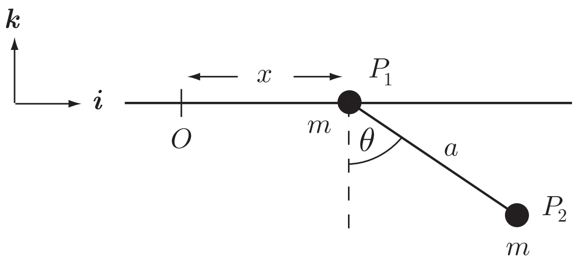

Figure 1: The generalized coordinates are x and θ

Exercise: Three particles P_1, P_2, P_3, where P_1, P_2 are connected by a light rod of length a and P_2, P_3 connected by a light rod of length b. The system *i)* slides on a horizontal surface, *ii)* moves in a three-dimensional space.

Consider a system of N particles described by f degrees of freedom. So the position vector of all the particles are given by a set of f generalized coordinates $q \equiv q_1, q_2....q_f$, *i.e.*, $\vec{r}_i = \vec{r}_i(q)$ for all the $i = 1...N$. The velocity of the i^{th} particle is given by

$$\vec{v}_i = \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \cdots = \sum_j^f \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\,. \tag{14}$$

The time derivative of generalized coordinates are called *generalized velocity*

$$\dot{q}_i = \frac{dq_i}{dt}\,. \tag{15}$$

The coordinates q_i and the generalized velocity $\dot{q}_i$ are independent of each other since at time $t = 0$, we can chose $q_i(0)$ and $\dot{q}_i(0)$ independently. From (14) one can show that the generalized kinetic energy is given as

$$T = \sum_j^f \sum_k^f a_{jk}(q)\dot{q}_j\dot{q}_k \,, \tag{16}$$

where (exercise!)

$$a_{jk} = \frac{1}{2}\sum_{i=1}^N m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}.\frac{\partial \vec{r}_i}{\partial q_k}\right) .$$

It shows that the kinetic energy is homogeneous and quadratic in the generalized coordinates.

One can also define a generalized force as

$$Q_j = -\frac{\partial V}{\partial q_j} \,. \tag{17}$$