5 Coupled Oscillation and Normal Modes

5.1 Simple Harmonic Oscillators

Consider a system under the influence of a restoring force proportional to the displacement from the origin. The force is given by:

$$F(x) = -kx$$

where k is a positive constant, and x represents the displacement from the equilibrium position.

Equation of Motion

By applying Newton's second law, $F = ma$, where $a = \frac{d^2x}{dt^2}$ is the acceleration, we get the equation of motion for the system:

$$m\frac{d^2x}{dt^2} = -kx$$

Rearranging this, we obtain:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

where we define the natural angular frequency ω_0 as:

$\omega_0 \sim \sqrt{\frac{k}{m}}$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

This is the standard form of the equation for simple harmonic motion (SHM), where ω_0 is the angular frequency of oscillation.

$x = A\sin(\omega t -$

Solution to the Equation of Motion

The general solution to the equation of motion for a simple harmonic oscillator is:

$$x(t) = A\sin(\omega_0 t - \delta)\,,$$

Here, A is the amplitude of oscillation, and δ is a phase constant that depends on the initial conditions of the system. This solution represents periodic oscillations with angular frequency ω_0.

5.2 Damped Oscillators (tutorial and assignment)

In the ideal case of a simple harmonic oscillator, once it is set in motion, it would continue oscillating indefinitely. However, in reality, the oscillations eventually stop due to energy dissipation, typically in the form of damping forces. A common model for such a system includes a damping force proportional to the velocity, given by:

$F_d \propto x$

$$F_d = -b\dot{x}$$

where b is the damping coefficient, and $\dot{x}$ is the velocity. The equation of motion for the damped harmonic oscillator becomes:

$$m\ddot{x} + b\dot{x} + kx = 0$$

which can be rewritten as:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

where $\beta = \frac{b}{2m}$ is the damping parameter, and $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the undamped system. The general solution to this equation, which will be derived in more detail in an upcoming tutorial or assignment, is:

$$x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2-\omega_0^2}\,t\right) + A_2 \exp\left(-\sqrt{\beta^2-\omega_0^2}\,t\right)\right]$$

This solution describes the motion of the system, including both the exponential decay of amplitude due to the damping factor $e^{-\beta t}$ and the oscillatory motion governed by the term involving $\sqrt{\beta^2-\omega_0^2}$, assuming underdamped conditions where $\beta < \omega_0$. In figure (8) the solutions for damped oscillation is shown for different relative values between damping term and the frequency.

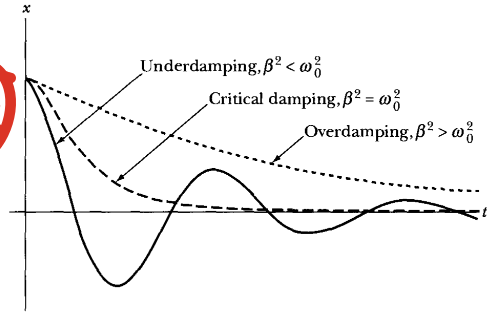

Figure 8: Damped oscillator solutions.

5.3 Two coupled Oscillators

Consider a system consisting of two equal masses, $m_1 = m_2 = m$, moving in a horizontal plane. These masses are coupled by a spring with spring constant k_{12}. Each mass is also attached to a spring with spring constant k, where the other ends of the springs are fixed to rigid walls. Let x_1 and x_2 represent the displacements of the masses from their equilibrium positions. With reference to the figure 9, the Lagrangian of the system is given by:

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k_{12}(x_2-x_1)^2 - \frac{1}{2}x_2^2\,. \tag{54}$$

The equation of motions of the two masses are

$$m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 = 0\,, \quad m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = 0\,. \tag{55}$$

Before we try to solve the equations, notice that that they can be written in a compact form as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = 0\,, \tag{56}$$

where square matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix},$$

and the column vector $\mathbf{X}$ is

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Mathematically, the equation (56) is a generalization of the SHO equation of motion $m\ddot{x} + kx = 0$ such that the system have a dof $\mathbf{X}$ with mass $\mathbf{M}$ and spring constant $\mathbf{K}$.

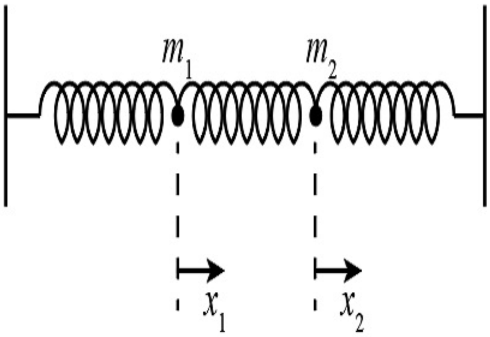

Figure 9: Two mass points connected by a spring of spring constant k_{12}(middle spring). The leftmost and rightmost strings have identical spring constant k and are attached to a rigid wall.

$x_1 = Ae^{i\omega t}$ $x_2 = Be^{i\omega t}$

Normal Frequencies

In trying to solve (56) it is reasonable to assume that there could be solutions where both the masses move in oscillatory motions with same frequency but different amplitudes. We assume the solutions of the type

$$x_1(t) = Ae^{i\omega t} \qquad x_1(t) = Be^{i\omega t} \tag{57}$$

The ω is the frequency to be determined and A, B are complex amplitudes that can be determined from the initial conditions. Note that the assumed solutions have complex parts which are unphysical. The actual motion must be given by real solutions only, for example $x_1(t) = A\cos(\omega t - \delta_1)$ $x_2(t) = A\cos(\omega t - \delta_2)$, where $\delta_{1,2}$ are phases. The reason for choosing exponential solution is that it lets us combine the two solutions into a column due to same time dependance of the type $e^{i\omega t}$

$$\mathbf{X} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\omega t}. \tag{58}$$

$x = ve^{rt}$

Substituting the solutions (58) in to the EoM (56) leads to

$$-\omega^2 \mathbf{M} \mathbf{a} e^{i\omega t} + \mathbf{K} \mathbf{a} e^{i\omega t} = 0\,, \qquad \text{or} \quad (-\omega^2 \mathbf{M} + \mathbf{K})\mathbf{a} = 0\,. \tag{59}$$

where $\mathbf{a} = \begin{pmatrix} A \\ B \end{pmatrix}$. The equation is known as the generalized eigenvalue equation. A non-trivial solution exist if the determinant of the matrix $-\omega^2\mathbf{M} + \mathbf{K}$ vanishes, *i.e.*,

$$\begin{vmatrix} k + k_{12} - m\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - m\omega^2 \end{vmatrix} = 0\,,$$

which yields

$$(k + k_{12} - m\omega^2)^2 - k_{12}^2 = 0\,,$$

Solving for ω we get

$$\omega = \sqrt{\frac{k + k_{12} \pm k_{12}}{m}}\,.$$

So we have two *eigenfrequencies*

$$\boxed{\omega_1 = \sqrt{\frac{k + 2k_{12}}{m}}\,, \quad \omega_2 = \sqrt{\frac{k}{m}}}\,. \tag{60}$$

The eigenfrequencies are also known as the *normal frequencies* of the system.

Normal Modes

Corresponding to each of the frequencies, we investigate the amplitude of solutions (58).

First Normal Mode: Let

$$\omega = \omega_1 = \sqrt{\frac{k + 2k_{12}}{m}}\,.$$

Substituting in equation (5.3) yields

$$\begin{pmatrix} k + k_{12} - k - 2k_{12} & -k_{12} \\ -k_{12} & k + k_{12} - k - 2k_{12} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0\,,$$

which yields that $A = -B$, *i.e.,* the two masses oscillate with the same amplitude but their motions are out of phase. This is the first normal mode that can be written as

$$\mathbf{X} = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_1 t}\,. \tag{61}$$

1st → $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Second Normal Mode: Let

$$\omega = \omega_2 = \sqrt{\frac{k}{m}}\,.$$

Substituting in equation (5.3) yields

$$\begin{pmatrix} k + k_{12} - k & -k_{12} \\ -k_{12} & k + k_{12} - k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0\,,$$

which yields that $A = B$, *i.e.,* the two masses oscillate with the same amplitude and their motions are in phase. This is the second normal mode

$$\mathbf{X} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_2 t}\,. \tag{62}$$

2nd $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ → in

out phase

Normal Coordinates

One can write the general solution of the problem as

$$\mathbf{X} = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_1 t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_2 t}\,. \tag{63}$$

To be more general we can write

$$\mathbf{X} = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_1 t} + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_1 t} + B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_2 t} + B_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_2 t}\,. \tag{64}$$

Looking at the individual solutions

$$\begin{aligned} x_1(t) &= \quad A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t} + B_1 e^{i\omega_2 t} + B_2 e^{-i\omega_2 t}\,, \\ x_2(t) &= -A_1 e^{i\omega_1 t} - A_2 e^{-i\omega_1 t} + B_1 e^{i\omega_2 t} + B_2 e^{-i\omega_2 t}\,, \end{aligned} \tag{65}$$

we see that each of the coordinates x_1 and x_2 have complex time-dependance – this is not surprising since the EoMs (55) are coupled.

To simplify the complex frequency dependance we define a linear superposition of $x_{1,2}$

$$\eta_1 = \frac{1}{2}(x_1 - x_2)\,, \quad \eta_2 = \frac{1}{2}(x_1 + x_2)\,.$$

Taking a similar linear superposition of the two equations EoMs (55) we get

$$m\ddot{\eta}_1 + (k + 2k_{12})\eta_1 = 0\,, \quad m\ddot{\eta}_2 + k\eta_2 = 0\,, \tag{66}$$

η_1

These equations show that the solutions of η_1 oscillates frequency $\omega_1 = \sqrt{(k+2k_{12})/m}$ and η_2 oscillates with frequency $\omega_2 = \sqrt{k/m}$. It is easy to see that for the first normal mode when $\omega = \omega_1$

$$\eta_1 = A_1 e^{i\omega_1 t}\,, \quad \eta_2 = 0\,,$$

and for the second normal mode

$$\eta_1 = 0\,, \quad \eta_2 = B_1 e^{-\omega_2 t}\,.$$

The η_1 and η_2 are known as the normal modes and the most general solutions can be written as

$$\eta_1(t) = E_1 e^{i\omega_1 t} + E_2 e^{i\omega_1 t}\,, \quad \eta_2(t) = F_1 e^{i\omega_2 t} + F_2 e^{i\omega_2 t} \tag{67}$$

The four constants $E_{1,2}$ and $F_{1,2}$ can be determined from the initial conditions of the system. Let us chose as initial conditions $x_1(0) = -x_2(0)$ and $\dot{x}_1 = -\dot{x}_2$. In this case we see that $\eta_2(0) = 0$ and $\dot{\eta}_2(0) = 0$ which gives $F_1 = F_2 = 0$, *i.e.*, $\eta_2(t) = 0$ for all times. This is the antisymmetrical mode where the two particles oscillate in the opposite directions (*out of phase*) with frequency ω_1. On the other hans for the initial conditions $x_1(0) = x_2(0)$ and $\dot{x}_1 = \dot{x}_2$ we find that $\eta_1(t) = 0$ for all the times t – two masses oscillate *in phase* with frequency ω_2. This is the symmetrical solution. The fact that the antisymmetrical mode has a higher frequency is a general result.

5.4 General problem of coupled oscillator

The previous system consists of two degrees of freedom. To generalize for n degrees of freedom, we consider a system of n generalized coordinates $q_k = \{q_1, ...q_n\}$. At the stable equilibrium of the system

$$q_k = q_{k0}\,, \quad \dot{q}_k = 0\,, \quad \ddot{q}_k = 0 \quad \text{for all } k.$$

We assume that the generalized coordinates are not explicit functions of time and the system is conservative. So kinetic energy will be quadratic function of generalized velocity, and the potential is function of coordinates only. We are interested in the motion of the system within the immediate neighborhood of the stable configuration. Let us define the deviations from the generalized coordinates by η_k

$$q_k = q_{k0} + \eta_k\,.$$

About the equilibrium of the system the potential can be Taylor expanded as

$$U(q_k) = U(q_{k0}) + \sum_k \frac{\partial U}{\partial q_k}\bigg|_{q_{k0}} \eta_k + \frac{1}{2}\sum_{j,k} \frac{\partial^2 U}{\partial q_j \partial q_k}\bigg|_{q_{j0},q_{k0}} \eta_j \eta_k + \cdots\,.$$

(In many books the displacement from the equilibrium is denoted by η_k, *i.e.*, $q_k \to q_{k0} + \eta_k$). The second term vanishes at the equilibrium, and the constant $U(q_{k0})$ can be set to zero by a proper choice of coordinates. Retaining terms to order $\mathcal{O}(\eta_k^2)$ in the expansion the potential becomes

$$U(q_k) = \frac{1}{2}\sum_{j,k} A_{jk}\eta_j\eta_k\,, \qquad A_{jk} = \frac{\partial^2 U}{\partial q_j \partial q_k}\bigg|_{q_{j0},q_{k0}}\,. \tag{68}$$

Here A_{jk}, where j,k both runs from 1 to n is a $n \times n$ symmetric matrix. There is also a series expansion of the kinetic term

$$T = \frac{1}{2} m_{jk}\dot{\eta}_j\dot{\eta}_k\,. \tag{69}$$

where m_{jk} are components of a $n \times n$ matrix and may be functions of generalized coordinates q_k also. In fact m_{jk} can be expanded in terms of the η_k but as we are neglecting terms of the order $\mathcal{O}(\eta_k^2)$ only the first term in that series expansion is relevant. In most cases there will be no cross terms in the kinetic energy. The Lagrangian of the system therefore is

$$L(\eta_k, \dot{\eta}_k, t) = \frac{1}{2} m_{jk}\dot{\eta}_j^2 - \frac{1}{2} A_{jk}\eta_j\eta_k\,.$$

The Lagrange's equation of motions

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_j}\right) - \frac{\partial L}{\partial \eta_j} = 0\,,$$

yield

$$\left(m_{jk}\ddot{\eta}_j + A_{jk}\eta_k\right) = 0\,. \tag{70}$$

This is a set of n second order linear homogeneous differential equations with constant coefficients. We assume solutions of the type

$$\eta_j(t) = a_j e^{i\omega t}\,,$$

where a_j are real amplitudes and ω is frequency. Substituting the solutions in the EoMs we get

$$\sum_j (A_{jk} - \omega^2 m_{jk})a_j = 0\,. \tag{71}$$

This is also a set of n homogeneous linear equations that a_j must satisfy(these a_j were A, B in (**??**)). Non-trivial solutions exist if the following determinant vanishes

$$\left|A_{jk} - \omega^2 m_{jk}\right| = 0\,.$$

This is an equation of degree n is ω^2 – hence there will be n roots which may be labeled as ω_r^2 where r runs from 1 to n. These are the *eigenfrequencies* or *charataristic frequencies*. Substituting an eigenfrequency in (72) determine the ratio of the a_j coefficients $a_1 : a_2 : a_3 : ...... : a_n$ for each of the ω_r. Since there are n values of ω_r there will be n set of ratios of a_j – these are called the *eigenvectors*. To be more specific we introduce the notation a_{jr} that denotes a set $\{a_j\}$ for ω_r. An important property of the eigenvectors is that they are are orthogonal – in other words the set $\{a_{j1}\}$, $\{a_{j2}\}$, $\{a_{j3}\}$....$\{a_{jn}\}$ forms a complete basis.

Once the eigenfrequencies are obtained the general solutions can be written as a linear superposition

$$q_j(t) = \sum_r \alpha_r a_{jr} e^{i\omega_r t}\,. \tag{72}$$

where we have introduced the constants α_r since the vectors $\{a_{j1}\}$, $\{a_{j2}\}$, $\{a_{j3}\}$....$\{a_{jn}\}$ are orthogonal to each other. Though solutions have complex part only the real parts are physical. We now define

$$\eta_r = \alpha_r e^{i\omega_r t}\,,$$

so that the solutions read

$$q_j = \sum_r a_{jr}\eta_r\,. \tag{73}$$

Note that the η_r satisfy

$$\ddot{\eta}_r + \omega_r^2 \eta_r = 0\,. \tag{74}$$

So, these are the normal coordinates of the problem. The constants α_r are determined from the initial conditions of the problem.

So, here is the step-by-step process to solve coupled oscillation problems

- From the Lagrangian obtain the set of equations of motion (70) for the coordinates q_j.

$$\sum_j \left(A_{jk}q_j + m_{jk}\ddot{q}_j\right) = 0\,.$$

These are n equations.

- Assume oscillatory solutions of the type $q_j = a_j e^{i\omega t}$ where a_j are arbitrary coefficients and omega are frequencies to be determined. Substituting these solutions in the above equation gives the characteristic equation

$$\sum_j (A_{jk} - \omega^2 m_{jk}) a_j = 0\,.$$

  This is a set of n equations for the coefficients a_j. For non-trivial solutions to exist (*i.e.,* not all the a_j to be zero) the following must satisfy

$$\left| A_{jk} - \omega^2 m_{jk} \right| = 0\,. \tag{75}$$

  This is called the characteristic determinant. This is a n degree equation in ω^2 – hence there will be n roots called the eigenfrequencies.

- Once the eigenfrequencies are determined substitute them one by one in the characteristic equation to find the ratios of the a_j coefficients. Since there are n eigenfrequencies there are n such ratios. To distinguish between them denote them by a_{jr} for a frequency ω_r. These are useful later to write down the general solution. The sets a_{j1}, a_{j2} etc so obtained are orthogonal to each other

- Define normal coordinates

$$\eta_r = \alpha_r e^{i\omega_r t}\,,$$

  where α_r are arbitrary constants to be determined from the initial conditions of the problem.

Tutorial: Three coupled pendulums from John Taylor's book