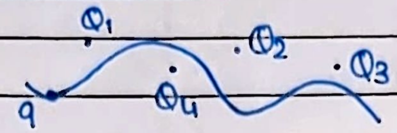
Khud injure ho gaya

ELECTRODYNAMICS

~ Lectures by Proffessor Diganta Das, compiled by Aaryan Shah

*] Electrostatics:



$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4}$

•] Principle of Superposition:

- Force on the test charge due to a given charge is independant of all other charges.

Q] Why is principle of superposition valid?

→ Principle of superposⁿ is valid for systems that exhibit linear dependance on the material response & where geometry of system doesn't change significantly due to the forces.

∴ We have a linear dependance on every other charge

(∵ Coulomb's law : $F \propto q(Q_1 + Q_2 + \dots)$)

*] Coulomb's Law:

- $\vec{F} = \dfrac{K q_1 q_2}{r^2} \cdot \hat{r}$, $K = \dfrac{1}{4\pi\varepsilon_0}$ → permitivity of free space

$q_1 \longleftarrow r \longrightarrow q_2$

$\varepsilon_0 = 8.85 \times 10^{-2} \dfrac{C^2}{Nm^2}$

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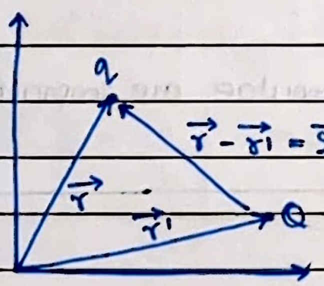


$\Rightarrow \vec{F} = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{qQ}{|\vec{r} - \vec{r'}|^2} \hat{\mathscr{r}}$

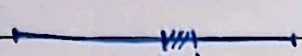
$\therefore \vec{F} = \dfrac{1}{4\pi\varepsilon_0} \left[\dfrac{qQ_1}{|\vec{r} - \vec{r_1'}|^2} \hat{\mathscr{r}}_1 + \dfrac{qQ_2}{|\vec{r} - \vec{r_2'}|^2} \hat{\mathscr{r}}_2 + \dots \dots \right]$

$= q\vec{E}$

$\therefore \boxed{\vec{F} = q\vec{E}}$ where $\vec{E} = \dfrac{1}{4\pi\varepsilon_0} \left[\dfrac{Q_1}{|\vec{r} - \vec{r_1'}|^2} \hat{\mathscr{r}}_1 + \dfrac{Q_2}{|\vec{r} - \vec{r_2'}|^2} \hat{\mathscr{r}}_2 + \dots \right]$

→ has some physical meaning, not just a calculation tool

Systems with charge densities:

Ex:  $Q, L, \lambda \; ; \; \lambda = \frac{Q}{L}$

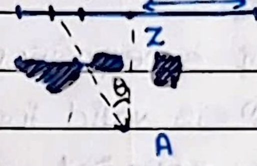
$dq = \lambda dl$

Ex:  $Q, A, \sigma \; ; \; \sigma = \frac{Q}{A}$

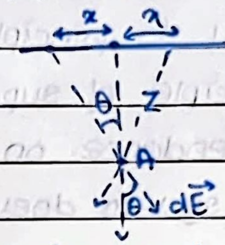
$dq = \sigma dA$

Q] Find Field @ A.

$dq = \lambda dx$ $\quad L \quad$ $Q,$ [redacted] $2L, \lambda$



$\Longrightarrow$



→ [redacted]

$$d\vec{E} = 2 \times K \frac{\lambda dx}{z^2 + x^2} \cos\theta$$

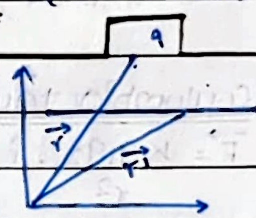
↳ ∵ sin cancel

$$\Rightarrow E = 2K\lambda \int_0^L \frac{dx}{z^2+x^2} \cdot \frac{z}{\sqrt{z^2+x^2}}$$

$$\therefore \vec{E} = \frac{2K\lambda}{z} \times \frac{L^2}{\sqrt{z^2+L^2}} \hat{z} \quad \because (x = z\tan\theta)$$

q

NOTE: ① $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}\,')\, dl}{|\vec{r} - \vec{r}\,'|^2} \cdot \hat{\imath}$ (Line)

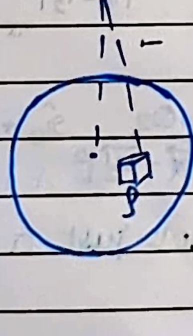


② $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \iint \frac{\sigma(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} \hat{\imath}\, ds$ (Surface)

③ $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} \hat{\imath}\, dV$ (Volume)

→ For most practical cases these formulae are impractical, that's why use trix.

Ex: consider sphere:



$dV = dx \cdot dy \cdot dz$

$= r^2 \sin\theta\, d\theta\, d\phi\, dr$

TOO COMPLICATED TO INTEGRATE, NEED NEW CONCEPT

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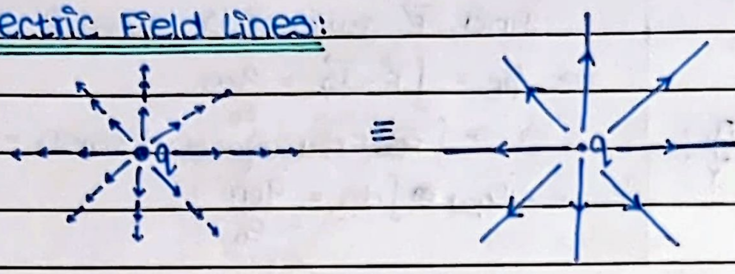
*] Electric Field Lines:

→ 

→ decreasing magnitude is understood by density of field lines in a region

o] Flux:

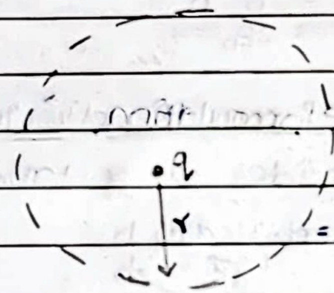
- Amount of any physical quantity flowing through a [redacted] unit surface @ a given time.

$\Rightarrow \boxed{\phi_F = \int \vec{F} \cdot \vec{dA}}$

$\Rightarrow \phi_E = \iint \vec{E} \cdot \vec{dA}$

→ Now, closed surface:

$\boxed{\phi_E = \oint \vec{E} \cdot \vec{dA}}$



Find flux through sphere?

$= \oint \frac{q}{4\pi\varepsilon_0 r^2} d(4\pi r^2)$

$= \frac{q}{\varepsilon_0}$

*] Gauss Law: $\boxed{\phi_E = \oint \vec{E} \cdot \vec{dA} = \frac{q_{en}}{\varepsilon_0}}$

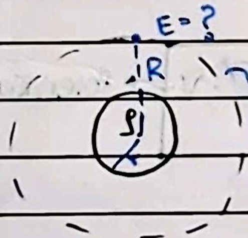
Divergence Theorem: You can convert a closed surface integral to a volume integral i.e. $\oint \vec{E} \cdot \vec{dA} = \int (\vec{\nabla} \cdot \vec{E}) dV$

$\Rightarrow \int (\vec{\nabla} \cdot \vec{E}) dV = \frac{1}{\varepsilon_0} \int \rho dV$

$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$ → Really Really Nice Formula

⊛ Gauss's Divergence Formula of Electrostatics

Ex:



$E = ?$, R, ρ ↝ Gaussian surface

$\Rightarrow \phi_E = \int \vec{E} \cdot \vec{ds} = \frac{q}{\varepsilon_0}$, $q = \rho \times \frac{4}{3}\pi r^3$

$\Rightarrow \int E ds \cos\theta = \frac{q}{\varepsilon_0}$, $\theta = 0$

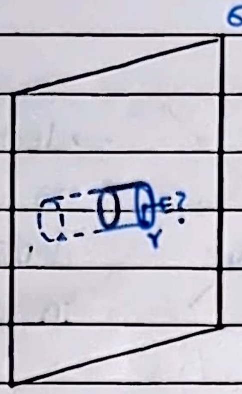
$\Rightarrow \int E ds = \frac{q}{\varepsilon_0} \longrightarrow E \int ds = \frac{q}{\varepsilon_0}$

$\longrightarrow E \times 4\pi R^2 = \frac{q}{\varepsilon_0}$

$\longrightarrow E = \frac{q}{4\pi\varepsilon_0 R^2}$

$\therefore \underline{\underline{E = \frac{kq}{R^2}}}$

Q]



Find $\vec{E}$ just near sheet?

$\Rightarrow \phi_E = \int \vec{E}\cdot\vec{ds} = \frac{q_{en}}{\varepsilon_0}$

$= \int$ [redacted] $E_{flat\,surfaces}\cdot ds\cos\theta = \frac{q_{en}}{\varepsilon_0}$

$\rightarrow E_{flat}$ [redacted] $\int ds = \frac{q_{en}}{\varepsilon_0}$

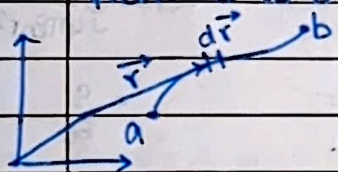
$\Rightarrow E\cdot \underbrace{2\pi r^2}_{\text{2 flat (exiting flux)}} = \frac{\sigma\pi r^2}{\varepsilon_0}$

$\Rightarrow E = \frac{\sigma\cancel{\pi r^2}}{2\cancel{\pi r^2}\varepsilon_0}$

$\therefore \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

*] Reformulating $\vec{E}$ in terms of scalar potential:

- Consider a [redacted] stationary charge distribution being moved from a to b.



$\Rightarrow V = \int_a^b \vec{E}\cdot\vec{dl} = \int_a^b \frac{q}{4\pi\varepsilon_0 r^2}\cdot dr$

$= \frac{-q}{4\pi\varepsilon_0 r}\Big|_a^b$

$\therefore V_{AB} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$

$\Rightarrow V_{BA} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$

$\rightarrow \oint \vec{E}\cdot\vec{dl} = 0 \rightarrow \int(\vec{\nabla}\times\vec{E})\cdot\vec{ds} = 0$ ∵ (Stokes theorem)

$\oint_c \vec{A}\cdot\vec{dl} = \iint_s (\vec{\nabla}\times\vec{A})\cdot\vec{ds}$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$

? ask

$\Rightarrow -(V_B - V_A) = \int_a^b \vec{E}\cdot\vec{dl}$

$= -\int_a^b (\nabla\cdot V)\cdot\vec{dl} = \int_a^b \vec{E}\cdot\vec{dl} \longrightarrow \boxed{\vec{E} = -\vec{\nabla}\cdot V}$ → Potential

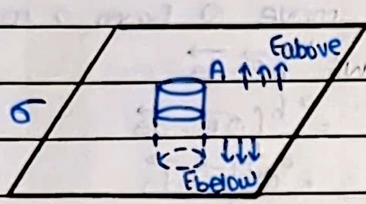
*] How to find potential:

[redacted]

$\rightarrow$ Consider $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\cdot V) \Rightarrow \boxed{\nabla^2\cdot V = \frac{-\rho}{\varepsilon_0}}$

↳ Poisson's Equation

*] Boundary condition on $\vec{E}$ & $\vec{V}$:



$$\Rightarrow \oint \vec{E}\cdot\vec{dA} = \frac{\sigma A}{\varepsilon_0}$$
$$= \int \vec{E}\cdot|dA|\cdot\hat{n} = \frac{\sigma A}{\varepsilon_0}$$
flat surface

$\rightarrow \vec{E}\cdot\hat{n}|_{above} = E^{\perp}_{above}$, $\vec{E}\cdot\hat{n}|_{below} = E^{\perp}_{below}$

$\Rightarrow A(E^{\perp}_{above} - E^{\perp}_{below}) = \frac{\sigma A}{\varepsilon_0}$

$\Rightarrow E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$, where $E^{\perp}_{above} = -E^{\perp}_{below}$

$\therefore$ $E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$; $E^{\perp}_{above} = -E^{\perp}_{below}$

lly, $E^{\parallel}_{above} = E^{\parallel}_{below}$

$\Rightarrow$ Now, $\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ & $V_{above} - V_{below} = -\int \vec{E}\cdot\vec{dl}$

$\rightarrow$ Now as the path length shrinks so does $-\int \vec{E}\cdot\vec{dl} \rightarrow 0$

$\therefore V_{above} = V_{below}$

However, the gradient of V inherits the discontinuity in E as $E = -\nabla V$ $\therefore$ $\nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$

$$\Rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

where $\frac{\partial V}{\partial n} = \nabla V\cdot\hat{n} \longrightarrow$ Normal Derivative of V (rate of change ⊥)

NOTE: ① $E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$:

② $E^{\perp}_{above} = -E^{\perp}_{below} \Rightarrow |\vec{E}^{\perp}| = \frac{\sigma}{2\varepsilon_0}$; $E^{\parallel}_{above} = E^{\parallel}_{below}$

③ $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

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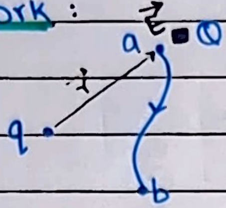
*] Work :



- Find WD to move Q from a to b?

$\longrightarrow \vec{F} = Q\vec{E}$; $W = \int_a^b \vec{F}\cdot\vec{ds}$

$$= \int_a^b Q\vec{E}\cdot\vec{ds}$$

$$= Q\int_a^b \vec{E}\cdot\vec{ds}$$

Now, we also know that $V = -\int E\cdot dr$

$$= Q\int E\cdot ds\cos\theta$$

$$\Rightarrow -Q\int E\,ds \quad \because \text{(Force opposing)}$$

$$\Rightarrow \boxed{WD = Q(V(b) - V(a))}$$

- WD depends only on boundary conditions & is independant of the path $\longrightarrow$ Electrostatic is CONSERVATIVE in nature

$\longrightarrow$ To get n particles from ∞ together:

$$WD = \frac{1}{4\pi\varepsilon_0}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n} \frac{q_i q_j}{r_{ij}}\right] \equiv \text{PE of the system}$$

$$= \frac{K}{2}\left[\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} \frac{q_i q_j}{r_{ij}}\right]$$ cause now q_1q_2 & q_2q_1 both considered

$$\Rightarrow W.D. = \frac{1}{2}\left[\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ i\neq j}}^{n} K\frac{q_j}{r_{ij}}\right)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} q_i V(r)\right] \longrightarrow$$ Energy of Point Charge Distribution i.e. Discrete Distribution

$\Longrightarrow$ For continuous distribution: $WD = \frac{1}{2}\int \rho V d\tau$

$\hookrightarrow$ charge density

Now, from Gauss law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow WD = \frac{1}{2}\int \varepsilon_0 \vec{\nabla}\cdot\vec{E}\, V d\tau$

$\Rightarrow WD = \frac{\varepsilon_0}{2}\int \vec{\nabla}\cdot\vec{E}\, V d\tau \longrightarrow$ Integrating by parts

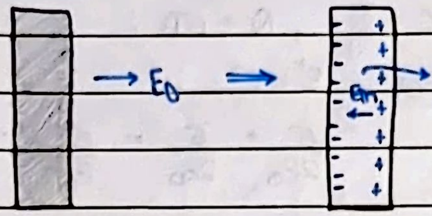
$$\Rightarrow WD = \frac{\varepsilon_0}{2}\left[-\int \vec{E}\cdot(\vec{\nabla}V)\, d\tau + \oint V\vec{E}\, d\vec{a}\right]$$

$$= \frac{\varepsilon_0}{2}\left[\int E^2 d\tau + \underbrace{\oint V\vec{E}\cdot d\vec{a}}_{\propto 1/r}\right] \quad \because \vec{E} = -\vec{\nabla}V$$

$\propto 1/r$ so when $V\uparrow$ $\oint\downarrow$

$$\longrightarrow \boxed{W.D. = \frac{\varepsilon_0}{2}\int E^2 d\tau}$$ for big volume

*] **Conductors:**



E_{in}: Charge separation produces opposing $\vec{E}$ until it cancels out $\vec{E_0}$.

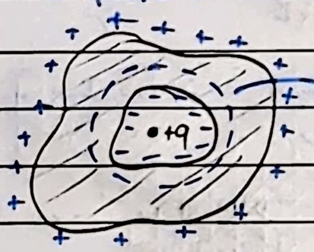
★ $\Rightarrow$ $\vec{E}_{net}$ inside conductor $= 0$.

$\to$ $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow 0 = \frac{\rho}{\varepsilon_0}$ $\therefore \rho = 0$ ★ $\therefore$ charge only lies on surface

$\to$ $V = \int \vec{E}\cdot d\vec{l} \to V = 0$ $\therefore$ ★ Potential diff @ any two pts inside conductor is 0 i.e. V same throughout conductor.

★ $\to$ $\vec{E}$ is perpendicular to surface of conductor

Ex:



$\to$ Applying Gauss law in this Gaussian surface

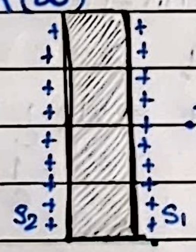
$\phi_E = \oint E\cdot dA = \frac{q_{en}}{\varepsilon_0} \Rightarrow q + Q_{in} = 0$

$\therefore$ $Q_{in} = -q$.

lly, bahar $+q$.

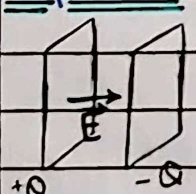
Ex: $\vec{E}$ for infinite sheet $= \sigma/2\varepsilon_0$ (derived before)

$\Rightarrow$ for conductor (∞)



$\vec{E} = \frac{\sigma}{2\varepsilon_0}\Big|_{S_1} + \frac{\sigma}{2\varepsilon_0}\Big|_{S_2}$ $\therefore$ $\vec{E} = \frac{\sigma}{\varepsilon_0}$

*] **Capacitor:**



$V = V_f - V_i = -\int_{r_-}^{r_+} \vec{E}\cdot d\vec{r}$

$r_+ \to$ location of +ve plate

$r_- \to$ location of -ve plate

- $\vec{E} \propto Q$
- $V \propto Q$

$\Rightarrow$ $\boxed{C = \frac{Q}{V}}$

$C \to$ capacitance

P.T.O.

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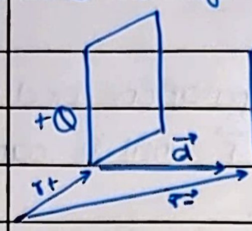
[#] Find C of the system

→ 

$\Rightarrow C = \frac{Q}{V}$, $Q = \sigma A$

$E_{net} = E^{+} + E^{-}$

$= \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$ $\therefore E_{net} = \frac{\sigma}{\varepsilon_0}$

$\rightarrow \vec{d} = \vec{r_-} - \vec{r_+}$

Now, $V = -\int_{r_-}^{r_+} E \cdot dr$; E inside constant

$\Rightarrow V = \vec{E} \int_{-r}^{r_+} \cdot d\vec{r} \rightarrow \boxed{V = Ed}$

$\therefore V = \frac{\sigma}{\varepsilon_0} d$ $\therefore C = \frac{\cancel{\sigma} A \varepsilon_0}{\cancel{\sigma} d}$ $\therefore \boxed{C_{sys} = \frac{\varepsilon_0 A}{d}}$

•] Work Done to charge capacitors:

- consider charge q @ any moment

$\Rightarrow V = \frac{q}{C}$ $\therefore dW = \frac{q}{C} dq$

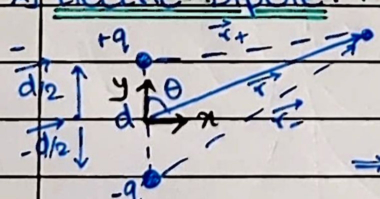
$\rightarrow \int dW = \int \frac{q\,dq}{C} \Rightarrow W = \frac{1}{C} \int q\,dq$

$= \frac{1}{C} \frac{q^2}{2}$

$\Rightarrow W = \frac{(CV)^2}{2C} = \frac{1}{2} CV^2$

$\therefore \boxed{W = \frac{1}{2} CV^2}$

*] Electric Dipole:



$\rightarrow \vec{r} = \vec{r_+} + \vec{d/2}$

$\vec{r} = \vec{r_-} - \vec{d/2}$

$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{|\vec{r_+}|} - \frac{q}{|\vec{r_-}|} \right)$

Now, $|\vec{r_+}| = \sqrt{r^2 + d^2/4 - rd\cos\theta}$

$|\vec{r_-}| = \sqrt{r^2 + d^2/4 + rd\cos\theta}$

$$\rightarrow \frac{1}{r_+} = \frac{1}{r}\left(1-\frac{d}{2r}\cos\theta\right) \quad \& \quad \frac{1}{r_-} = \frac{1}{r}\left(1+\frac{d}{2r}\cos\theta\right) \quad \because \text{(Binomial)}$$

$$\Rightarrow V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\left(1-\frac{d}{2r}\cos\theta\right) - \frac{1}{r}\left(1+\frac{d}{2r}\cos\theta\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\cancel{1} - \frac{d}{2r}\cos\theta \cancel{-1} - \frac{d}{2r}\cos\theta\right]$$

$$= \frac{-q\,d\cos\theta}{4\pi\varepsilon_0 r^2} \quad \therefore \quad \boxed{V(\vec{r}) = \frac{kq}{r^2}\cdot d\cos\theta}$$

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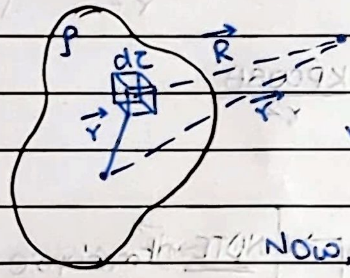

$$\vec{r'} + \vec{R} = \vec{r} \quad \therefore \vec{R} = \vec{r} - \vec{r'}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r'})\,d\tau}{R}$$

Now, $R^2 = r^2 + (r')^2 - 2rr'\cos\theta$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right]$$

$$\Rightarrow \frac{1}{R} = \frac{1}{r} \times \frac{1}{\sqrt{1+(r'/r)(r'/r - 2\cos\theta)}} \quad \text{Let } \varepsilon = (r'/r)(r'/r - 2\cos\theta)$$

$$= \frac{1}{r} \times \frac{1}{\sqrt{1+\varepsilon}}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \frac{1}{2}\left(\frac{r'}{r}\right)^2(3\cos^2\theta - 1) + \frac{1}{2}\left(\frac{r'}{r}\right)^3(5\cos^3\theta - 3\cos\theta) + \dots\dots\right)$$

$$\Rightarrow \underbrace{V(\vec{r})}_{\text{Multipole Expansion}} = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\underbrace{\int \rho(r')\,d\tau}_{\text{Monopole Expansion}} + \frac{1}{r^2}\underbrace{\int r'\cos\theta\,\rho(r')\,d\tau}_{\text{Dipole Expansion}} + \dots\dots\right]$$

$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\int \vec{r'}\cos\theta\,\rho(r')\,d\tau = \frac{1}{4\pi\varepsilon_0 r^2}\int \hat{r}\cdot\vec{r'}\,\rho(\vec{r'})\,d\tau$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\,\hat{r}\cdot\int \vec{r'}\,\rho(\vec{r'})\,d\tau = \boxed{\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = V_{dipole}}$$

where $\boxed{\underbrace{\vec{p}}_{\text{dipole moment}} = \int \vec{r'}\,\rho(\vec{r'})\,d\tau}$

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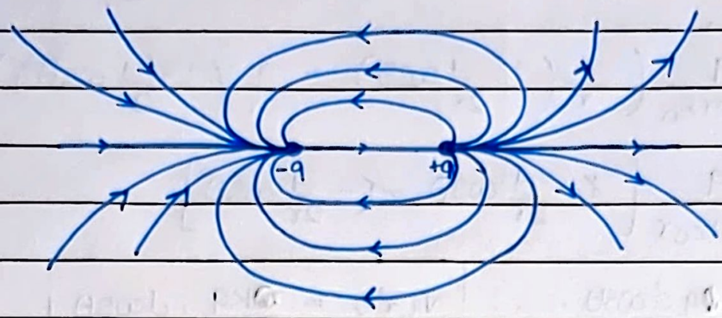
• Electric Field of Dipole:



HW: Calculate $\vec{E}_{dipole}$ & crosscheck drawing.

→ Now, as calculated before $V_{dipole} = \frac{kqd\cos\theta}{r^2}$ $\left(\frac{\hat{r}\cdot p}{4\pi\varepsilon_0 r^2}\right)$

(*) Dipole Moment : $p = q \cdot d$

$$\Rightarrow \vec{E}_{dipole} = \frac{kp\cos\theta}{r^2}$$

(*) Now, $E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$

$$E_\theta = \frac{1}{r}\frac{-\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

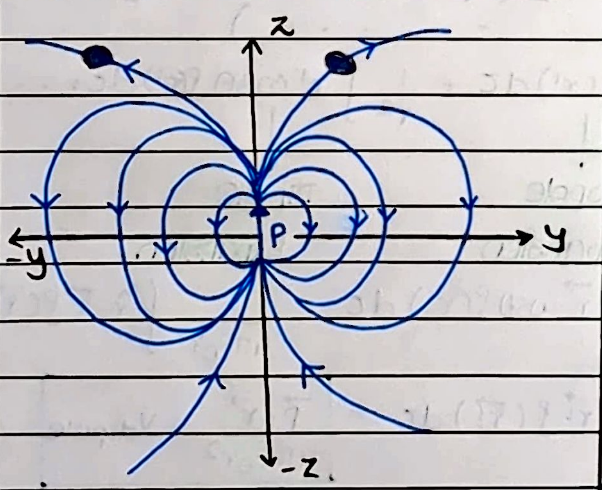
$$E_\phi = \frac{-1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

NOTE:
1] $\vec{E}_{monopole} \propto 1/r^2$
2] $\vec{E}_{dipole} \propto 1/r^3$
IIly, 3] $\vec{E}_{quadropole} \propto 1/r^4$
4] $\vec{E}_{octople} \propto 1/r^5$
& so on

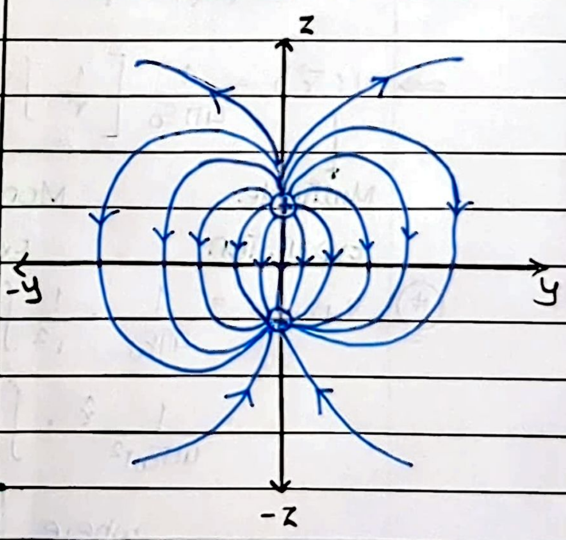
$$\therefore \boxed{\vec{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\cdot\hat{\theta}\right)}$$

equivalent only @ r >>> d

a] Field of a pure dipole



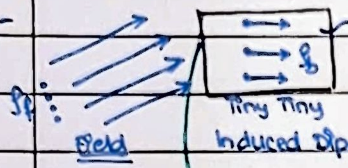
b] Field of a physical dipole



*] Insulators / Dielectrics:

- Charge no free
- $\exists$ induced dipole $\therefore \vec{p} = c\vec{E}_{external}$ $\because (\vec{p} \propto \vec{E}_{external})$

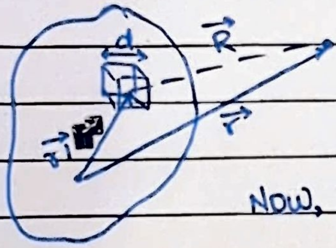
•] [redacted] Field / Potential:

- 

Polarization: $\vec{P}$ = dipole moment / volume

Polarized Object

#



$\vec{r}' + \vec{R} = \vec{r}$ $\Rightarrow V(r) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{\vec{p}(r')\cdot\hat{R}}{R^2}$

$\vec{R} = \vec{r} - \vec{r}'$

Now, $\vec{p}(r') = \vec{P}\underbrace{d^3}_{\text{volume}}$ [redacted] $\vec{r}' \equiv \vec{P}d\tau'$

$$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{R}\cdot\vec{P}}{R^2}d\tau' \quad , \text{Now, } \vec{\nabla}\left(\frac{1}{R}\right) = \frac{\hat{R}}{R^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\int \vec{P}\cdot\vec{\nabla}\left(\frac{1}{R}\right)d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\int_V \vec{\nabla}\cdot\left(\frac{\vec{P}}{R}\right)d\tau' - \int \frac{1}{R}\vec{\nabla}\cdot\vec{P}\,d\tau'\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint \frac{\vec{P}\cdot\hat{n}}{R}da' + \int \frac{-\vec{\nabla}\cdot\vec{P}}{R}d\tau'\right] \quad \because (\text{Divergence Theorem})$$

On comparing to [redacted] std. formula:

$$\rightarrow \boxed{V(r) = \int \frac{\sigma_b \cdot da'}{R} + \int \frac{\rho_b d\tau'}{R} \quad \text{where } \sigma_b = \vec{P}\cdot\hat{n} \ \& \ \rho_b = -\vec{\nabla}\cdot\vec{P}}$$

Potential due to a Polarized Dielectric

$\rightarrow$ Ily, $\boxed{\vec{E} = \dfrac{-\vec{P}}{3\varepsilon_0}}$ $\rightarrow$ Field due to polarized Dielectric

$\Rightarrow \rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P}$ $\Rightarrow \varepsilon_0\vec{\nabla}\cdot\vec{E} = \rho_f - \vec{\nabla}\cdot\vec{P}$ $\because (\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0)$

$\Rightarrow \nabla\cdot(\varepsilon_0\vec{E} + \vec{P}) = \rho_f$

$\vec{D}$ = Electric Displacement

$\therefore \boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$ where $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

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NOTE: When charge distribution is not symmetric, the tough part about formulae is $\iiint$ thus, we tend to use

Poisson's Eqn: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

OR

Laplace Eqn: $\nabla^2 V = 0$ (region where $\rho = 0$)

*] Magnetostatics:

- Was discovered when it was seen that two current carrying wires kept near each other attract/repel.
- Dir^n given by right hand thumb rule.

◦] Lorentz:

→ $\vec{F}_{mag} = q(\vec{v} \times \vec{B})$ (Axiom) like Coulomb

∴ Total Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

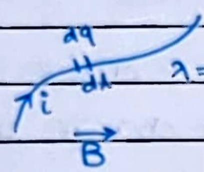
→ F_{mag} don't do any work:

↳ $\vec{F} = q\, vB\sin\theta\, \hat{v}_\perp$: $(v\perp)$

↳ $\int dW = \int F \cdot ds \longrightarrow \int q\frac{vB\sin\theta}{dt}\, d\hat{x}_\perp \cdot dx = 0$ (essentially force ⊥ to displacement)

*] Force on current carrying wire placed in B:

→



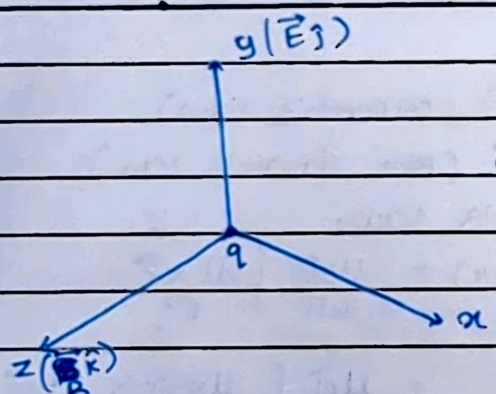
λ = line charge density

$$\longrightarrow d\vec{F}_{mag} = dq\,(\vec{v} \times \vec{B}) = \lambda dl\,(\vec{v} \times \vec{B}) = (\lambda\vec{v} \times \vec{B})\,dl = (\vec{I} \times \vec{B})\,dl$$

∴ $\boxed{F_{mag} = \int (\vec{I} \times \vec{B})\,dl}$

∴ for uniform current: $\boxed{F_{mag} = i\int d\vec{l} \times \vec{B}}$

(*)



what will happen?

→ cycloid along y-axis

Applications: cyclotron

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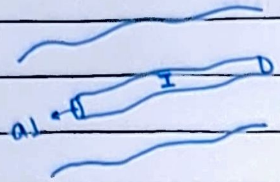
*] **How to calculate $\vec{B}$:**

- Magnetic field due to a "steady current"

•] **Current Density ($\vec{J}$):**



$\rightarrow \boxed{\vec{J} = \frac{\vec{I}}{a_\perp}}$

∴ What will be total current?

$\rightarrow \oint \vec{J}\cdot d\vec{s} = \int \vec{\nabla}\cdot\vec{J}\,dV$ ∵ (Divergence Theorem)

$\Rightarrow \int \vec{\nabla}\cdot\vec{J}\,dV = -\frac{d}{dt}\int \rho\,dV$ ∵ essentially total charge out = rate of change of charge

∴ $\boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}}$ → Continuity Equation

↳ local charge conservation

→ Now, if $\frac{\partial \rho}{\partial t} = 0$, $\rightarrow \vec{\nabla}\cdot\vec{J} = 0$ (charge in = charge out)

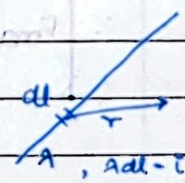
↳ no charge accumulation

For steady current: $\boxed{\vec{\nabla}\cdot\vec{J} = 0}$

•] **Biot - Savart Law:**

- $\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\vec{r}}{r^2}}$ ⇒ for steady current: $\boxed{\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2}}$

↳ Unit: Tesla (Te) T

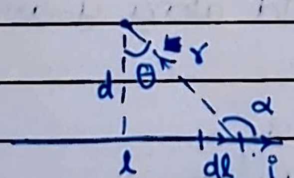


↳ **Analogy to Electrostatics:**

1] Stationary charge → $\vec{E}$ (Coulomb's law)

2] Steady current → $\vec{B}$ (Biot Savart's law)

•] **Application of Biot Savart's Law:**



$\Rightarrow B(r) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2}$

$= \frac{\mu_0 I}{4\pi}\int \frac{dl\sin\alpha}{r^2} = \frac{\mu_0 I}{4\pi}\int \frac{dl\cos\theta}{r^2}$

$\frac{1}{r^2} = \frac{\cos^2\theta}{d^2} \Rightarrow \frac{\mu_0 I}{4\pi}\int dl\cos\theta\,\frac{\cos^2\theta}{d^2}$

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$\Rightarrow B(r) = \frac{\mu_0 i}{4\pi}\int \frac{dl \cos^3\theta}{d^2}$, Now, $\tan\theta = \frac{l}{d} \rightarrow dl = \frac{d}{\cos^2\theta}d\theta$

$= \frac{\mu_0 i}{4\pi}\int \frac{d}{\cos^2\theta}\frac{\cos^3\theta}{d^2}\,d\theta = \frac{\mu_0 i}{4\pi d}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta$

[redacted] $\Rightarrow \boxed{B = \frac{\mu_0 i}{4\pi d}(\sin\theta_2 - \sin\theta_1)}$

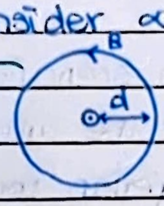
→ For Infinitely long wire $\theta_2 = \pi/2,\ \theta_1 = -\pi/2$:

$\Rightarrow \boxed{B = \frac{\mu_0 i}{2\pi d}}$

$\vec{\nabla}\times\vec{B} = ?$ Curl of B & $\vec{\nabla}\cdot\vec{B} = ?$

→ consider ∞ long wire with outward current:

Amperian Loop



$\rightarrow B = \frac{\mu_0 i}{2\pi d}$

Now, we will calculate its line integral:

$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \oint B\,dl = \oint \frac{\mu_0 i}{2\pi d}\,dl = \frac{\mu_0 i}{\cancel{2\pi d}}\cdot\cancel{2\pi d} = \mu_0 i$

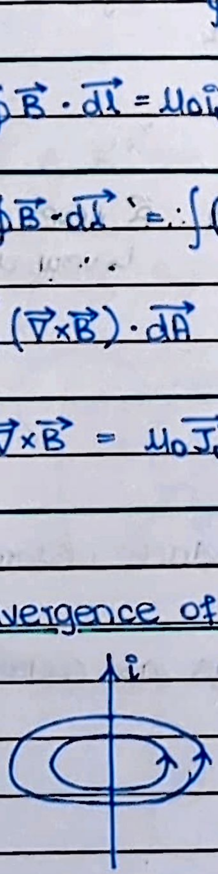
$\therefore \boxed{\oint \vec{B}\cdot\vec{dl} = \mu_0 i_{net}}$ → Ampere's Law

$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \int(\vec{\nabla}\times\vec{B})\cdot\vec{dA}$ $\because$ (Stokes Law)

$\therefore \int(\vec{\nabla}\times\vec{B})\cdot\vec{dA} = \mu_0 I_{net} \Rightarrow \int(\vec{\nabla}\times\vec{B})\cdot\vec{dA} = \mu_0\int \vec{J}\cdot\vec{dA}$

$\therefore \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{net}}$ → Ampere's Law Calculus version

Divergence of $\vec{B}$:

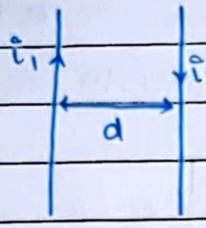


$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ $\because$ each loop closed

↳ no origin like charge q for lines to come from

⇒ NO MAGNETIC MONOPOLES EXIST

Q] Calculate force on the other wire (per unit length)

→ 

$\therefore B_{12} = \frac{\mu_0 i_1}{2\pi d}$

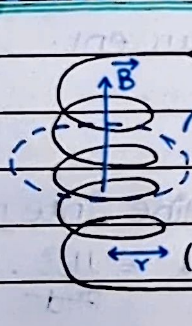
$\vec{F_2} = i_2 \int \vec{dl} \times \vec{B}$

$= i_2 \int dl B \sin 90°$

$\Rightarrow \frac{\vec{F_2}}{dl} = i_2 \cdot \frac{\mu_0 i_1}{2\pi d}$ $\therefore \frac{\vec{F_1}}{dl} = \frac{i_1 \mu_0 i_2}{2\pi d}$ $\because$ (NLM-III)

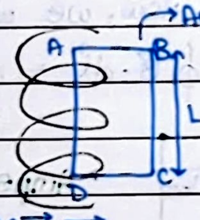
$\therefore \boxed{\frac{F}{dl} = \frac{\mu_0 i_1 i_2}{2\pi d}}$

*] $\vec{B}$ due to Solenoid:

→ 

We have to consider two scenario inside the cylinder & outside the cylinder

(n windings) per unit length

$\therefore$ 

$\Rightarrow \oint \vec{B}\cdot\vec{dl} = \mu_0 i n L$

$\Rightarrow \int_A^B \cancel{\vec{B}\cdot\vec{dl}}^{\,0} + \int_B^C \cancel{\vec{B}\cdot\vec{dl}}^{\,0} + \int_C^D \cancel{\vec{B}\cdot\vec{dl}}^{\,0} + \int_D^A \vec{B}\cdot\vec{dl}$

assume very long cylinder $\therefore \vec{B}$ never comes out

↳ very close approx.

$\Rightarrow \int_D^A \vec{B}\cdot\vec{dl} = \mu_0 i n L \Rightarrow B\cdot\cancel{L} = \mu_0 i n \cancel{L}$

$\therefore \boxed{\vec{B}_{inside} = \mu_0 n i \,(\hat{z})}$

NOTE: ① Just like capacitors are used to produce uniform $\vec{E}$, solenoids are used to produce $\vec{B}$.

② Ampere's law has similar shortcomings as Gauss law vis-a-vis symmetry

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*] Magnetic ~~Potential~~ Vector Potential ($\vec{A}$):

→ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ & $\vec{\nabla}\cdot\vec{B} = 0$

⇒ $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ → magnetic vector potential

| $\vec{\nabla}\cdot\vec{E} = 0$ & $\vec{F} = -\vec{\nabla}V$
 $V \sim q/r$ |

→ $\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

⇒ $\mu_0\vec{J} = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{A})} - \nabla^2\vec{A}$ | Poisson's Eqn : $\frac{\rho}{\varepsilon_0} = -\nabla^2 V$

→ is there a way to make this disappear?

(*) # $\vec{E} = -\vec{\nabla}V$

if $V' = V + \lambda$ & $\vec{\nabla}\lambda = 0$ then V & V' give same $\vec{E}$

Gauge Transf. } lly, $\vec{B} = \vec{\nabla}\times\vec{A}$

if $\vec{A}' = \vec{A} + \vec{f}$ & $\vec{\nabla}\times\vec{f} = 0$ then $\vec{A}$ & $\vec{A}'$ give same $\vec{B}$

→ Say $f = \vec{\nabla}\chi$:

$\exists$ f s.t. $\vec{\nabla}\times\vec{f} = 0$ & for $\vec{A}' = \vec{A} + \vec{f} \Rightarrow \vec{\nabla}\cdot\vec{A}' = 0$ → (*)

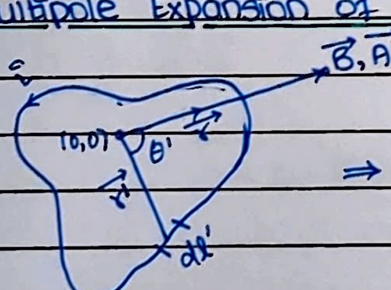
⇒ $\mu_0\vec{J} = \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{A}'}) - \nabla^2(\vec{A}')$ ∵ ($\vec{A}'$ gives same B)

0 ∵ (From (*))

∴ $\boxed{\mu_0\vec{J} = -\nabla^2(\vec{A}')}$ → Vector Equation, can be used for simple cases, tough to use for harder situations

NOTE: $\vec{E}$ & $\vec{B}$ are invariant under Gauge Transformation

°] Multipole Expansion of $\vec{A}$:



$$\Rightarrow \vec{A}(r) = \frac{\mu_0 i}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{\text{Monopole Term}} + \underbrace{\frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}'}_{\text{Dipole Term}} + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos\theta' - \frac{1}{2}\right)d\vec{l}' + \dots\right]$$

⇒ $\vec{A}(r)_{monopole} = \frac{\mu_0 i}{4\pi r}\oint d\vec{l}' = \frac{\mu_0 i}{4\pi r} \times 0 = 0$

∴ $\boxed{\vec{A}(r)_{monopole} = 0}$ → No monoples exist

P.T.O.

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$\Rightarrow \vec{A}(r)_{dipole} = \frac{\mu_0 i}{4\pi r^2} \oint r' \cos\theta' \, d\vec{l}'$

$= \frac{\mu_0 i}{4\pi r^2} \int \vec{r}' \cdot \hat{r} \, d\vec{l}'$

\# Identity [redacted] des calculus:

- $\vec{c}$ = any constant vector

$\Rightarrow \oint \vec{c} \cdot \vec{r} \cdot d\vec{l} = \frac{1}{2} \oint (\vec{r}' \times d\vec{l}) \times \vec{c}$

$\Rightarrow \vec{A}(r)_{dipole} = \frac{\mu_0 i}{4\pi r^2} \left(\frac{1}{2} \oint (\vec{r}' \times d\vec{l}') \times \hat{r} \right)$

$= \frac{\mu_0 i}{4\pi r^2} (\vec{a} \times \hat{r}) \quad \because \left(\vec{a} = \frac{1}{2} \oint \vec{r}' \times d\vec{l}' \right)$

$= \frac{\mu_0}{4\pi r^2} (i\vec{a}) \times \hat{r}$

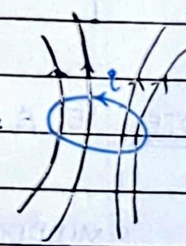
$\Rightarrow \boxed{\vec{A}(r)_{dipole} = \frac{\mu_0 (i\vec{a}) \times \text{[redacted]} \hat{r}}{4\pi r^2}} \Rightarrow$ • Magnetic Dipole:

$\rightarrow \vec{m} = i \cdot \vec{a}$

$= i \oint d\vec{a}$

for small radius:

$\longrightarrow$ loops act like magnets.



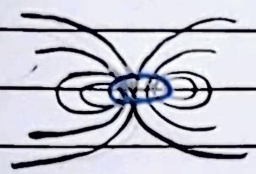
HW: $\Rightarrow$ Calculate $\vec{B}_{dipole}$:

$\vec{B} = \vec{\nabla} \times \vec{A}$

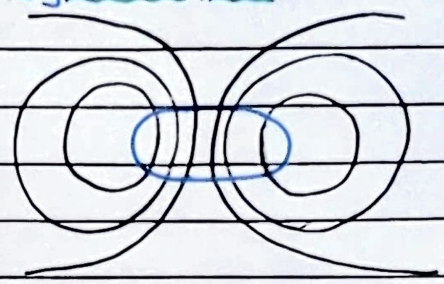
$= \vec{\nabla} \times \left(\frac{\mu_0 \vec{m} \times \hat{r}}{4\pi r^2} \right)$

°] Field lines of dipoles:

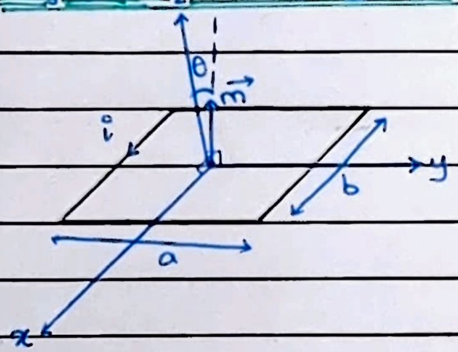
1] Pure Dipole:



2] Physical Dipole:

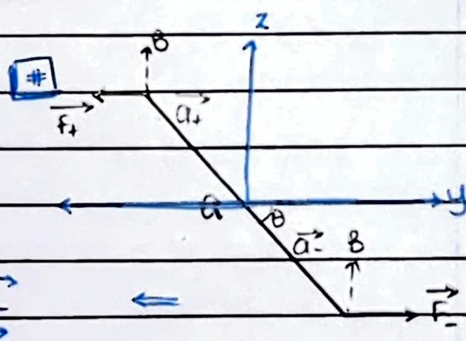


*] Magnetic Moment & Torque:



→ Consider $\vec{B}$ along z:

$\Rightarrow \vec{m} = iab$



$\vec{\tau} = \vec{a_+} \times \vec{F_+} + \vec{a_-} \times \vec{F_-}$ ⇐

$= \vec{a_+} \times \vec{F_+} + \vec{a_+} \times \vec{F_+}$

$= 2a_+F_+ \sin\theta$ [redacted]

$\Rightarrow |\vec{\tau}| = \frac{2aF\sin\theta}{2}$ $\therefore \tau = aF\sin\theta$

$\vec{F} = dq\, \vec{v} \times \vec{B}$

$= \frac{dq}{dt} \vec{dl} \times \vec{B} = i\vec{dl} \times \vec{B} \rightarrow \vec{F} = ibB$

$\Rightarrow \tau = aibB\sin\theta$

$= mB\sin\theta$ $\therefore$ $\boxed{\tau = \vec{m} \times \vec{B}}$

→ This is the Idea behind Paramagnetism

*] Magnets??:

- $\frac{e^2}{4\pi\varepsilon_0 R^2} = \frac{mv^2}{R}$, Now, introduce $\vec{B}$ $\Rightarrow \frac{e^2}{4\pi\varepsilon_0 R^2} + ev'B = \frac{mv'}{R^2}$

Atom

$\Rightarrow ev'B = \frac{m}{R}(v'+v)(v'-v)$

$\therefore \boxed{\Delta v = \frac{eBR}{2m}}$

$\Rightarrow ev'B = \frac{m}{R}(2v')(\Delta v) \because (v' \approx v)$

*] **Magnetization:**

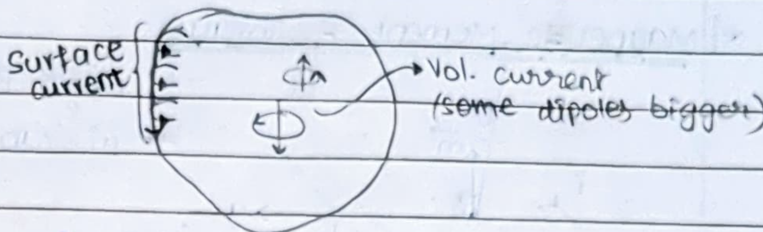
- $\vec{M}$ = dipole moment / volume

$\Rightarrow \vec{A}_{dipole} = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\text{[redacted]}\,\hat{r}}{r^2}$

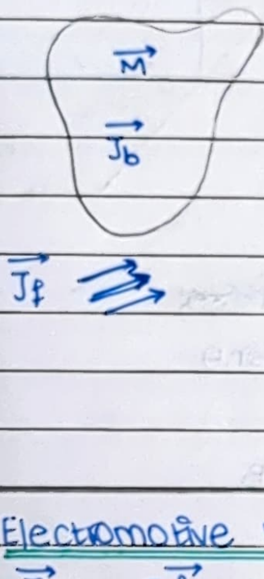
$\rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(r')\,\hat{r}}{r^2}\,d\tau'$ (For unit volume $d\tau'$)

$$= \frac{\mu_0}{4\pi}\left[\int \frac{1}{r}\underbrace{\left(\vec{\nabla}\times\vec{M}(r')\right)}_{\text{volume current}} d\tau' + \oint \frac{\overbrace{\vec{M}(r')\times d\vec{a}'}^{\text{surface current}}}{r}\right]$$

$\Rightarrow$ $\boxed{\begin{aligned}\vec{J}_b &= \vec{\nabla}\times\vec{M}\\ \vec{K}_b &= \vec{M}\times\hat{n}\end{aligned}}$



°] **$\vec{B}$ inside magnetized object:**



$\rightarrow \vec{J} = \vec{J}_b + \vec{J}_f$

$\frac{\vec{\nabla}\times\vec{B}}{\mu_0} = \vec{J}_b + \vec{J}_f$

$= \vec{\nabla}\times\vec{M} + \vec{J}_f$

$\Rightarrow \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f}$ where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

*] **Electromotive force:**

$\rightarrow \vec{J} = \sigma\vec{f} \rightarrow$ Driving force of current

$\hookrightarrow$ conductivity $\left(\sigma = \frac{1}{\rho}\right) \rightarrow$ resistivity

$= \sigma\left(\vec{E} + \vec{v}\times\vec{B}\right)$

$= \sigma\vec{E}$ $\because$ (v_{avg} for $e^{\ominus} = 0$ because Brownian)

Now, for relative practicality $\sigma \rightarrow \infty$ for conductors

$\Rightarrow \frac{I}{\sigma} = \vec{E} = \vec{f} \rightarrow 0 \quad \because \sigma \rightarrow \infty$

$\hookrightarrow$ Req: Driving force for current is not too high

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Now, what extent does the battery field affect conduction. $\exists$ some secondary external force

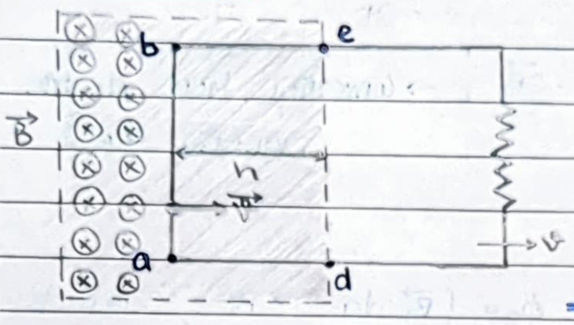
$$\Rightarrow \vec{f} = \vec{f_s} + \vec{E}$$

$$\Rightarrow \oint \vec{f}\cdot\vec{dl} = \oint(\vec{f_s}+\vec{E})\cdot\vec{dl}$$
$$= \oint \vec{f_s}\cdot\vec{dl} + \cancel{\oint \vec{E}\cdot\vec{dl}}$$

$$\Rightarrow \boxed{\oint \vec{f}\cdot\vec{dl} = \oint \vec{f_s}\cdot\vec{dl} = \mathcal{E}}$$ → EMF (line integral of the work done)

$$\rightarrow \mathcal{E} = \oint \vec{f_s}\cdot\vec{dl} = -\oint \vec{E}\cdot\vec{dl} = -\int_a^b \vec{E}\cdot\vec{dl}$$

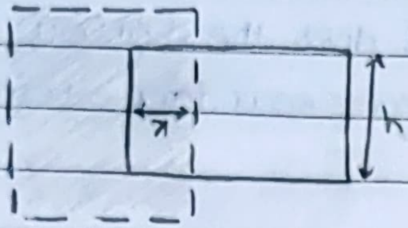
Motional EMF:



$$\rightarrow \mathcal{E} = \oint \vec{f}_{mag}\cdot\vec{dl}$$
$$= \oint vBdl$$
$$= vBh$$

→ Is the $\vec{f}_{mag}$ doing work?

→ But we know f_{mag} doesn't do work



$\Rightarrow \Phi_B = \int \vec{B} \cdot \vec{da}$

$= Bhx$

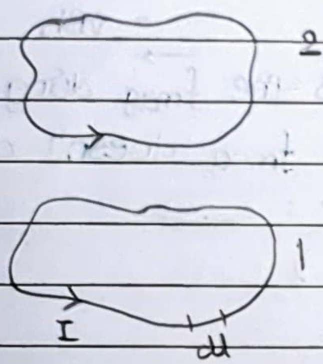
$\Rightarrow \frac{d\Phi_B}{dt} = Bh\frac{dx}{dt} = -Bhv = -\mathcal{E}$

$\therefore \boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}$ → Faraday ⓧ Lens Law
↳ time varying magnetic field creates electric field.

$\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \vec{da}$

$\int (\vec{\nabla} \times \vec{E}) \cdot \vec{da} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \vec{da}$

$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$ → conversion from electric to magnetic field



$\Phi_2 = \int \vec{B_1} \cdot da$ → Biot Savart's
↓
$B \propto I$

$= M_{21} I$ → Only geometry dependant

$\Rightarrow M_{21} = \frac{\mu_0}{4\pi} \int \frac{\vec{dl_1} \cdot \vec{dl_2}}{r^2}$

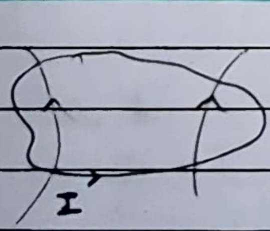
⊛ $M = M_{21} = M_{12}$ → Mutual Inductance

*] **Mutual Inductance:**

$\boxed{\mathcal{E}_2 = -M\frac{dI_1}{dt} = -\frac{d\Phi_2}{dt}}$

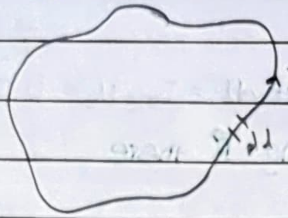
*] **Self Inductance:**

⇒ Self linking $\quad \mathcal{E} \propto \frac{dI}{dt}$ → Back EMF ⇒ energy required to setup current



$\Rightarrow \boxed{\mathcal{E} = -L\frac{dI}{dt} = -\frac{d\Phi}{dt}}$

Energy required to set up current:



$0 \to I$?

$\Rightarrow dW = V\,dq$

$= -\mathcal{E}\,dq$

$\Rightarrow \dfrac{dW}{dt} = -\mathcal{E}\dfrac{dq}{dt}$

$=$ [redacted] $-\mathcal{E}i = Li\dfrac{dI}{dt}$

$\Rightarrow \int dW = \int LI\,dI \Rightarrow \boxed{W = \frac{1}{2}LI^2}$

$\rightarrow$ Now, $W = \frac{1}{2}LI^2$

$= \frac{1}{2}I(LI) = \frac{1}{2}I(\phi_{self})$

$= \frac{1}{2}I\int \vec{B}\cdot d\vec{a}$

$= \frac{1}{2}I\oint \vec{A}\cdot d\vec{l}$ $\because$ (Stokes Theorem) (last)

$= \frac{1}{2}\oint (\vec{A}\cdot\vec{I})\,d\vec{l}$

$= \frac{1}{2}\int (\vec{A}\cdot\vec{\nabla}\times\vec{B})\,d\tau$ $\because$ (Divergence Thrm)

$\therefore \boxed{E = \frac{1}{2\mu_0}\int B^2\,d\tau}$ $\rightarrow \frac{1}{2\mu_0}\int B^2 d\tau = \frac{1}{2}LI^2$

*] Maxwell's Equations:

1] $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

2] $\vec{\nabla}\cdot\vec{B} = 0$

3] $\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

4] $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

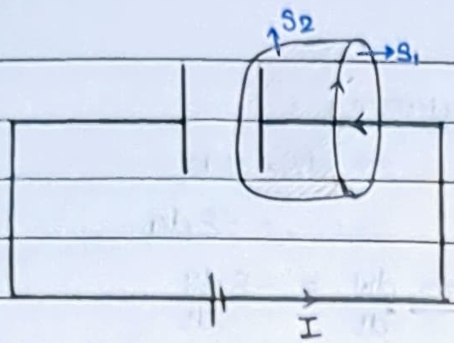
$\rightarrow$ He noticed some inconsistencies:

NOTE: divergence(curl) = 0

$\rightarrow 0 = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\cdot\dfrac{\partial\vec{B}}{\partial t} = -\dfrac{\partial}{\partial t}\vec{\nabla}\cdot\vec{B} = 0$ ✓

$\rightarrow 0 = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot\vec{J}$ $\rightarrow$ only zero for steady current

$\therefore$ Something is wrong with Ampere's Law



$\rightarrow \oint B \cdot dl = I_{enc}\mu_0 = \mu_0 I \quad \because$ (for S_1)

$\Rightarrow \oint B \cdot dl = I_{enc}\mu_0 = 0 \cdot I = 0 \quad \because$ (for S_2)

wrong : $\vec{B}$ there

$\rightarrow$ Now, $0 = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J} = -\varepsilon_0 \mu_0 \frac{\partial \rho}{\partial t}$

$= -\varepsilon_0 \mu_0 \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{E}$

Now, $\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$ $\rightarrow$ Mathematical term added to negate $\mu_0 \vec{\nabla} \cdot \vec{J}$

$\rightarrow$ verified experimentally by Hertz

NOTE: Final Equations of Maxwell:

1] $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

2] $\vec{\nabla} \cdot \vec{B} = 0$

3] $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$

4] $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

*] Energy Conservation & Poynting Theorem:

d^3x

E, B

$\rightarrow U = \int d^3x \left(\frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{1}{2\mu_0} \vec{B} \cdot \vec{B} \right)$

$\Rightarrow \frac{dU}{dt} = \int d^3x \left(\frac{\varepsilon_0}{2} \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{2\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} \right)$

Now, $-\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{E}(\vec{\nabla} \times \vec{B}) - \vec{B}(\vec{\nabla} \times \vec{E})$

$\rightarrow \frac{dU}{dt} = \int d^3x \left(\frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \vec{E} \cdot \vec{J} - \frac{1}{\mu_0} \vec{B} \cdot (\vec{\nabla} \times \vec{E}) \right)$

$= \underbrace{-\int d^3x (\vec{J} \cdot \vec{E})} - \underbrace{\frac{1}{\mu_0} \oint (\vec{E} \times \vec{B}) \cdot d\vec{S}}$

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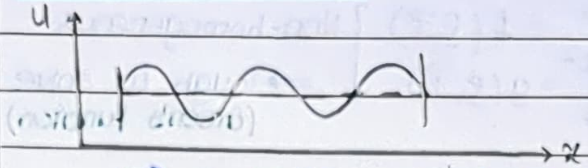
*] Electro-Magnetic Waves:

→ 

$$\Rightarrow \boxed{\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}}$$ → wave eqn.

→ $c \sim \text{P.}\sqrt{\frac{T}{\mu}}$

→ Now, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2\cdot\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$$ → eqn. of motion of $\vec{E}$

→ $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8$ m/s

IIIy, $\Rightarrow \boxed{\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}}$ → eqn. of motion of $\vec{B}$.

→ $\left.\begin{aligned}\vec{E}(z,t) &= \vec{E_0}(z,t)e^{i(kz-\omega t)}\\ \vec{B}(z,t) &= \vec{B_0}(z,t)e^{i(kz-\omega t)}\end{aligned}\right\}$ where $k = \frac{2\pi}{\lambda}$, $k = \frac{\omega}{v}$

→ Now, $\vec{\nabla}\cdot\vec{E} = 0$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}\right) = 0$$

$$\Rightarrow \cancel{\frac{\partial E_{0x}}{\partial x}}^{\,0} + \cancel{\frac{\partial E_{0y}}{\partial y}}^{\,0} + \frac{\partial E_{0z}}{\partial z} = 0 \rightarrow \boxed{E_{0z} = 0,\ B_{0z} = 0}$$

Essentially have zero component along dirn of propogatn

→ Now, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \rightarrow \boxed{-kE_{0y} = \omega B_{0x}}$

→ Now, $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ ∵ $(\vec{J} = 0)$ $\rightarrow \boxed{kE_{0x} = \omega B_{0y}} \Rightarrow \boxed{\frac{E_0}{B_0} = v}$

⇒ 

∴ E⊥B & $\vec{E}$ & $\vec{B}$ along z ($\hat{k}$)

$$\rightarrow \boxed{\vec{B} = \frac{1}{c}\hat{z}\times\vec{E}}$$

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Consider a situation with $\rho \neq 0, \vec{J} \neq 0$:

→ Now, $\nabla^2 E - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = f(\rho, \vec{J})$

||ly, $\nabla^2 B - \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} = g(\rho, \vec{J})$

} Non-homogeneous ⟹ Tough to solve (Green's function)

Consider a conductor:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \boxed{\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \sigma \frac{\partial E}{\partial t}} \rightarrow \boxed{\nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} + \mu_0 \sigma \frac{\partial B}{\partial t}}$$