Electronic Journal of Applied Statistical Analysis
Vol. 00, Issue 00, 00, 0-0
DOI: DOI–xxxx

Survival Analysis of Random Censoring with Inverse Maxwell Distribution: An Application to Guinea Pigs Data

C. P. Yadav*[a], Jitendra Kumar[b], and M. S. Panwar[c]

[a]*Saw Swee Hock School of Public Health, National University of Singapore, Singapore-117549*
[b]*Directorate of Economics and Statistics, Planning Department, Delhi-110054, India.*
[c]*Department of Statistics, Banaras Hindu University,, Varanasi-221005, India*

January 13, 2023

In real-life situations, performing an experiment up to a certain period of time or getting the desired number of failures is time-consuming and costly. Many of the available observations remain censored and only give the survival information of testing units up to a noted time and not about the exact failure times. In this study, we consider inverse Maxwell distribution having an upside-down hazard rate as a survival lifetime model. The censoring time is also assumed to follow the inverse Maxwell distribution with a different parameter. The probability of failure of an item before censoring and expected and observed time on the test is derived from a random censoring scheme. The maximum likelihood estimators with their confidence intervals for the parameters are obtained for a randomly censored setup. We obtain the Bayes estimators by taking the inverted gamma distribution as a prior under squared error loss function. In Bayesian analysis, the two techniques, i.e. Markov Chain Monte Carlo and Tierney-Kadane approximation methods are used for estimation purposes. For checking the performances of proposed estimators, we perform an extensive simulation study. A real data, guinea pigs, is analyzed to support the proposed study.

keywords: Random censoring, Inverse Maxwell distribution, Fisher information matrix, M-H algorithm, T-K approximation.

*Corresponding authors: chandraprakashy29@gmail.com

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

1 Introduction

In life-testing experiments, researchers conduct tests on human beings, electrical appliances, nature and many more aspects. In such studies, the primary objective is to understand the basic nature of the observed lifetimes. Generally, conducting life-testing experiments is time taking and expensive which demands a large amount of money, labor and time. For reducing the cost and time of the experiments, various types of censoring schemes are developed in the literature. These censoring schemes are helpful to perform an experiment in limited sources and time as well. Popular censoring schemes are Type-I and Type-II in which the experiment time and the maximum number of failures are being fixed in advance respectively. But, in these censoring schemes either we have to spend more time or do not get the desired number of failures. Several times these censoring schemes are not pertinent to the nature of life testing experiments and hence the purpose is not solved. There are some other censoring schemes to allow the experimental units to be removed while running the experiment. These types of censoring schemes are known as progressive and hybrid censoring schemes. These schemes have also been studied in detail in the literature under consideration of different lifetime distributions. A special type of censoring scheme known as random censoring in literature occurs when the item is lost or removed randomly from the experiment before its failure under the study. A sample is randomly censored when both the experimental unit and censoring time points are random and independent of each other's outcomes. A subject who moves away from the experimental environment before the event of interest occurs is considered a randomly censored value. In real life such situations occur often, especially in clinical trials, the patients do not complete the course of treatment and they leave the study due to several factors before the termination point of the experiment.

In a random censoring scheme, the entry time and the exit time of units in the experiment are random. So each unit has a censoring time that is statistically independent with corresponding failure times. In this case, the observed data is obtained by taking the minimum of the censoring time and failure time. For random censoring, various distributions such as exponential, Rayleigh, gamma and Weibull, etc. for the failure time and censoring time have been considered. Nandi and Dewan (2010) analyzes Marshall-Olkin Bivariate Weibull distribution in the presence of random censoring and obtains the parameter estimate by the Expectation-Maximization (E-M) algorithm. Kumar and Garg (2014) discusses the parameter estimation of generalized inverted Rayleigh distribution under the random censoring scheme. Krishna et al. (2015) presentes the maximum likelihood (ML) and Bayes estimator assuming Maxwell distribution under the random censoring scheme. In the continuation, Krishna and Goel (2017) discusses the parameter estimation of geometric distribution under random censoring data setup. In recent study, Kumar and Kumar (2019) analyzes the estimation procedure of inverse Weibull distribution under a random censoring setup.

Many well-known lifetime models are used in life testing experiments. But for particular real data, the search for a more suitable model is always in demand. The Maxwell model is one of the known distributions for life testing experiments. Many authors have discussed several studies in different scenarios for Maxwell distribution. Tyagi and

Bhattacharya (1989) present the Bayesian analysis of Maxwell distribution for velocity. Whereas, Bekker and Roux (2005) discusses the statistical estimation for Maxwell distribution under ML, Bayes, and E-Bayesian approaches and compares the efficiency for all methods. Krishna and Malik (2009) analyze the setup of Maxwell distribution under the Type-II censoring scheme and obtain the ML and Bayes estimate of the parameter of Maxwell distribution. Tomer and Panwar (2015) obtain the ML and Bayes estimates of Maxwell distribution under Type-I progressively hybrid censoring scheme and also calculate the expected number of failures in treatment. Modi and Gill (2015) discuss the length-based weighted Maxwell distribution and derive its statistical properties. Sharma et al. (2017) derive the statistical properties of extended Maxwell distribution. Sharma et al. (2018) analyze area-biased Maxwell distribution under classical and Bayesian approaches. They also discuss the experimental study with simulation and real data. Chaturvedi and Vyas (2019) propose a Gamma-Maxwell distribution and obtain the parameter and reliability characteristics of the given distribution.

Sometimes inverse distributions are more suitable in lifetime experiments according to important characteristics of given data. In the last decade, many authors have discussed the behavior of hazard function for several inverse distributions. The inverse Maxwell distribution ($InvMWD$) is obtained by inverting the Maxwell random variate. The behavior of the hazard function for $InvMWD$ is inverse of the bathtub. Singh and Srivastava (2012) discuss the Bayesian estimation procedure of $InvMWD$ with size-biased sampling. Singh and Srivastava (2014) shows the survival behavior of $InvMWD$ and other statistical properties. They also obtained the parameter estimation in different circumstances. Tomer and Panwar (2020) review the important statistical properties of $InvMWD$ in detail and obtained the estimate of the parameter under classical and Bayesian paradigms with applications in different scenarios. Yadav et al. (2021) study the $InvMWD$ under the Marshall-Olkin family and obtained point and interval estimates under ML approach. The authors give applications of the proposed distributions to deal with different lifetime data problems. Kishan et al. (2021) discusses the three component competing risks model for $InvMWD$ under ML and Bayesian approaches. Authors report the simulation and real data results for the proposed estimators.

Let us define the underlying lifetime distribution $InvMWD$. If Y is the Maxwell random variable then a new random variable $X = \frac{1}{Y}$ follows $InvMWD$. The probability density function (pdf) of $InvMWD$ is given as

$$f(x;\theta) = \frac{4}{\sqrt{\pi}} \frac{1}{x^4 \theta^{\frac{3}{2}}} \exp\left(-\frac{1}{\theta x^2}\right); \quad x > 0, \theta > 0 \tag{1}$$

The survival function at a given time t of $InvMWD$ is given by

$$S(t;\theta) = \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{1}{\theta t^2}\right), \tag{2}$$

where quantity $\gamma(a,z) = \int_0^z u^{a-1} e^{-u}\, du$ is a lower incomplete gamma function. The pdf and survival function of $InvMWD$ are presented in Figure 1 and Figure 2 for different values of θ.

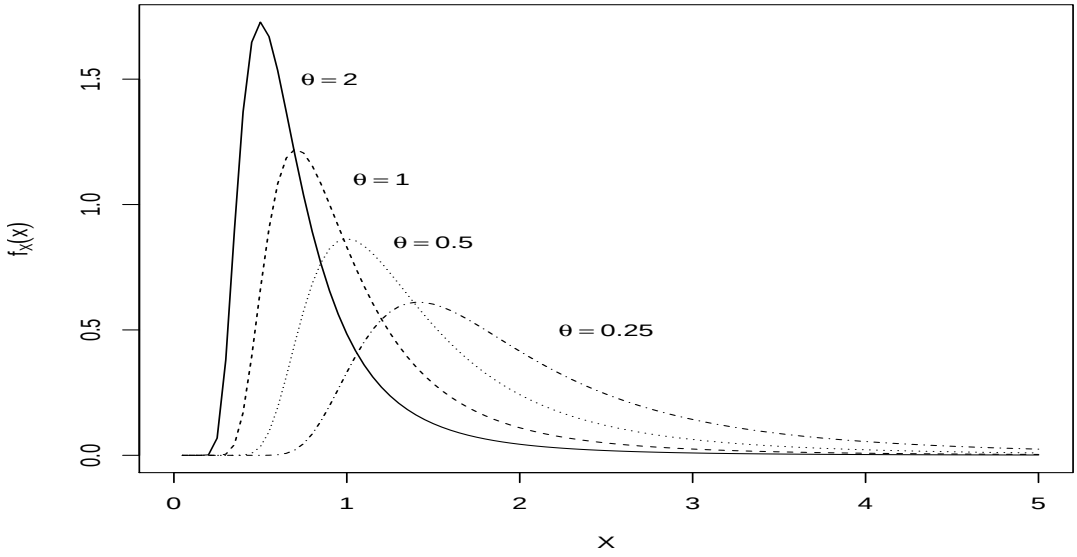

Figure 1: Density plot of $InvMWD$ for different values of θ.

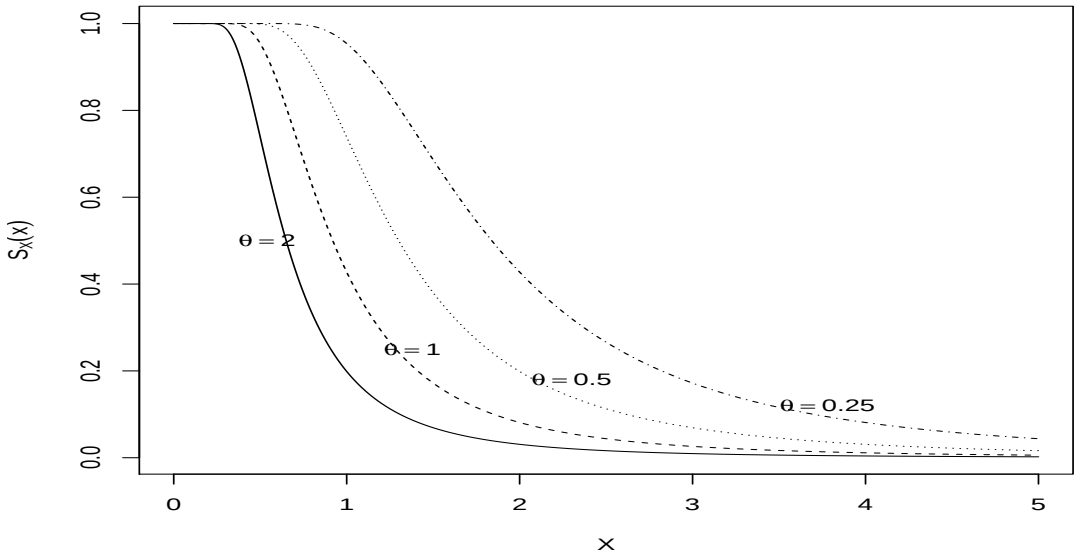

Figure 2: Survival plot of $InvMWD$ for different values of θ.

The hazard function of $InvMWD$ is given by

$$h(x;\theta) = \frac{2}{x^4\theta^{\frac{3}{2}}} \exp\left(-\frac{1}{\theta x^2}\right)\left[\gamma\left(\frac{3}{2}, \frac{1}{\theta x^2}\right)\right]^{-1}. \tag{3}$$

The hazard function of $InvMWD$ is an upside-down bathtub in nature, i.e. it increases sharply in the initial phase and then after reaching a peak point it deepens gradually.

This means $InvMWD$ represents the lifetime of such individuals who have an increased chance of failing in the early age of life span and after survival up to a specific age, the rate of failure starts decreasing as age increases. Tomer and Panwar (2020) have discussed the nature of the hazard function of $InvMWD$ in detail. The hazard rate function for the $InvMWD$ is given for different values of parameter θ in Figure 3. In

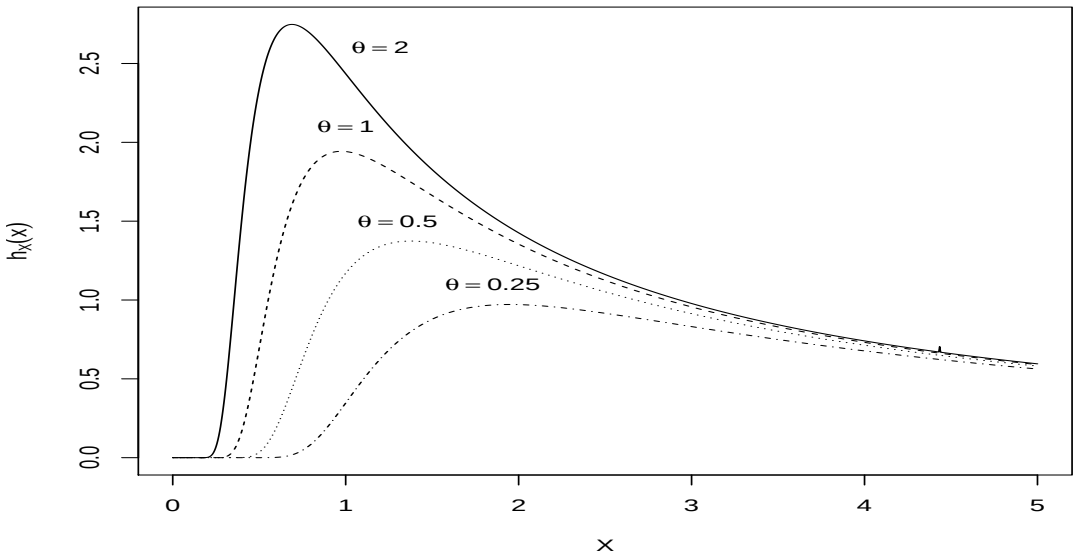

Figure 3: The hazard rate function of $InvMWD$ for different values of θ.

this study, we consider $InvMWD$ as a lifetime model under the random censoring setup and obtain the estimates of the parameters under classical and Bayesian approaches. In introduction, we review the literature on censoring schemes and formulation and applications of $InvMWD$. In Section 2, we derive the mathematical formulation for the considered model under random censoring. In this case, both failure time and censoring time are considered to follow the $InvMWD$ with different parameters. The expression of the probability of failure before censoring time and observed time to test are also obtained. In Section 3, the ML estimators are obtained with their asymptotic confidence intervals ($ACIs$) for the unknown parameters The Bayesian estimation procedure for parameter estimation under squared error loss functions by using the inverted gamma prior is discussed in Section 4. The Bayes estimates are obtained by using the Markov Chain Monte Carlo ($MCMC$) algorithm and the Tierney-Kadane ($T-K$) approximation method. The simulation study is presented in Section 5 and finally, Section 6 dedicated to real data analysis to study the applications of $InvMWD$ under random censoring.

2 Setup of Problem

2.1 $InvMWD$ Sample with Randomly Censoring Scheme

Suppose there are n individuals in the study. Let us denote the time to event associated to these individuals as X_1,X_2,...,X_n. These lifetimes are assumed to be independent and identically distributed (*i.i.d.*) with *pdf* $f_X(x;\theta)$ and cumulative distribution function (*cdf*) $F_X(x;\theta)$, respectively. Also, let T_1, T_2,...,T_n be the random censoring times for these individuals with *pdf* and *cdf* $f_T(t;\lambda)$ and $F_T(t;\lambda)$, respectively. Moreover, let us assume that the random variables $X_i's$ and $T_i's$, $i = 1, 2, ..., n$ are mutually independent. Note that, among $X_i's$ and $T_i's$, only one will actually be observed at any particular time.

Under this setup, we observe the actual time as $Y_i = min(X_i, T_i); i = 1, 2, \ldots, n$. Define indicator variable D_i, as

$$D_i = \begin{cases} 1, & \text{if } X_i \leq T_i \\ 0, & \text{if } X_i > T_i, \end{cases}$$

i.e., the indicator variable D_i takes value 1 when the i^{th} individual experiences the event at time X_i, and 0 if the he/she leaves the study before experiencing event of interest. Thus, the indicator variable D_i is a Bernoulli random variable with parameter p and the probability mass function of D_i is given by

$$P[D_i = j] = p^j(1-p)^{(1-j)}; \quad j = 0, 1 \tag{4}$$

Since, we assume $X_i's$ and $T_i's$ to be independent, therefore $Y_i's$ and $D_i's$ will also be independent.

The joint density function of Y and D can be defined as

$$f_{Y,D}(y, d; \theta, \lambda) = \{f_X(y;\theta)\,(1 - F_T(y;\lambda))\}^d \{f_T(y;\lambda)\,(1 - F_X(y;\theta))\}^{1-d}. \tag{5}$$

$y, \lambda, \theta \geq 0, d = 0, 1$.

We assume X and T follow $InvMWD$ with parameter θ and λ, respectively. Using the *pdf* and *cdf* of $InvMWD$ from (1) and (2), the joint density can be rewritten as

$$f_{Y,D}(y, d; \theta, \lambda) = \left[\frac{4}{\sqrt{\pi}} \frac{1}{y^4 \theta^{\frac{3}{2}}} \exp\left(-\frac{1}{\theta y^2}\right) \left\{\frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{1}{\lambda y^2}\right)\right\}\right]^d \left[\frac{4}{\sqrt{\pi}} \frac{1}{y^4 \lambda^{\frac{3}{2}}} \exp\left(-\frac{1}{\lambda y^2}\right) \left\{\frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{1}{\theta y^2}\right)\right\}\right]^{1-d}.$$

In such studies, researchers may be interested to know the probability of failure for individuals. We wish to obtain the mathematical form for probability of failure. Under

Table 1: Probability of failure (p) before censoring time for different combination values of θ and λ.

$\theta \backslash \lambda$	0.5	1.0	2.0	3.0
0.5	0.50	0.71	0.86	0.96
1.0	0.29	0.50	0.71	0.89
2.0	0.14	0.29	0.50	0.76
3.0	0.05	0.11	0.24	0.50

$InvMWD$ model, the mathematical expression is given as

$$\begin{aligned} P\left[\text{an item fails (d=1)}\right] = P\left[X \leq T\right] &= \int_0^\infty P\left[X \leq t | T = t\right] f_T(t) dt \\ &= \int_0^\infty \left[\int_0^t f_X(x,\theta) dx\right] f_T(t,\lambda) dt \\ &= \frac{4}{\sqrt{\pi}\lambda^{\frac{3}{2}}} \int_0^\infty t^{-4} \exp\left(-\frac{1}{\lambda t^2}\right) \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, \frac{1}{\theta t^2}\right) dt \\ &= \frac{8}{\pi\lambda^{\frac{3}{2}}} \int_0^\infty t^{-4} \exp\left(-\frac{1}{\lambda t^2}\right) \Gamma\left(\frac{3}{2}, \frac{1}{\theta t^2}\right) dt, \end{aligned}$$

where, $\Gamma(a,z) = \int_z^\infty u^{a-1} e^{-u}\, du$ is a upper incomplete gamma function. We can solve probability value by numerically for different values of θ and λ. Table 1 shows the probability of failure (p) before the censoring time for different values of θ and λ. We observe that an increase in the values of p with increasing values of λ, for a fixed value of θ while a decrease in the values of p with increasing values of θ for a fixed value of λ.

2.2 Expected Time on Test

In lifetime experiments, the cost of the experiment depends on the time on test, hence the researchers are interested to estimate the total time on test in such experiments. We derive the mathematical expression of expected time on test (ETT) and obtain its values varying θ, λ and n in the random censoring scenario. Let us define the variable $Z = \max(Y_1, Y_2, ..., Y_n)$, then the cdf of Z is given by

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P\left[max(Y_1, Y_2, ..., Y_n) \leq z\right] = \left[P(Y_1 \leq z)\right]^n ; z > 0. \end{aligned}$$

Since $Y_i,\ i = 1, 2, ..., n$ are $i.i.d.$ random variables, thus we have

$$\begin{aligned} P\left[Y_i \leq z\right] &= P\left[min\left(X_i, T_i\right) \leq z\right] \\ &= 1 - P\left[min(X_i, T_i) > z\right] \\ &= 1 - P\left[X_i > z\right] P\left[T_i > z\right] \qquad (X_i,\ T_i \text{ are independent}) \\ &= 1 - \frac{4}{\pi} \gamma\left(\frac{3}{2}, \frac{1}{\theta z^2}\right) \gamma\left(\frac{3}{2}, \frac{1}{\lambda z^2}\right). \end{aligned}$$

Using above expression, we get the *cdf* of Z as follows

$$F_Z(z) = \left[1 - \frac{4}{\pi}\gamma\left(\frac{3}{2}, \frac{1}{\theta z^2}\right)\gamma\left(\frac{3}{2}, \frac{1}{\lambda z^2}\right)\right]^n. \tag{6}$$

Now, the desired ETT can be written as

$$\begin{aligned} ETT = E(Z) &= \int_0^\infty (1 - F_Z(z))\, dz \\ &= \int_0^\infty \left(1 - \left[1 - \frac{4}{\pi}\gamma\left(\frac{3}{2}, \frac{1}{\theta z^2}\right)\gamma\left(\frac{3}{2}, \frac{1}{\lambda z^2}\right)\right]^n\right) dz. \end{aligned} \tag{7}$$

In addition to ETT, one more quantity, observed time on test ($OBTT$) is of great interest in such studies. In case of random censored sample, $OBTT$ can be given by quantity $Z = max(Y_1, Y_2, \ldots, Y_n)$.

In case of uncensored (complete) sample, we derive ETT and $OBTT$. Let us define $V = max(X_1, X_2, ..., X_n)$, where $X_1, X_2, ..., X_n$ is the observed sample values. Now, the distribution function of V is given by

$$F_V(v) = P(V \leq v) = P\left[max(X_1, X_2, ..., X_n) \leq v\right] = \left[P(X_1 \leq v)\right]^n.$$

Since $X_i's$ are i.i.d., therefore, the expected value of V is given by

$$E(V) = \int_0^\infty \left[1 - F_V(v)\right] dv = \int_0^\infty \left[1 - \left[P(X_1 \leq v)\right]^n\right] dv.$$

Thus, for our underlying distribution $InvMWD(\theta)$, expression for ETT can be defined as

$$ETT = \int_0^\infty \left[1 - \left\{\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}, \frac{1}{\lambda v^2}\right)\right\}^n\right] dv. \tag{8}$$

The $OBTT$ for complete sample case is given by $V = max(X_1, X_2, \ldots, X_n)$.

We obtain the simulated results for ETT and OBTT using underlying distribution. We generate 5000 randomly censored samples using the defined setup in subsection 2.1. By using (7) and (8), the value of ETT and $OBTT$ under randomly censored data for different values of θ, λ and n are obtained. Average estimated values (EV) of $OBTT$ along with mean square error (MSE) and ETT are reported in Table 2.

3 Maximum Likelihood Estimation

3.1 Point Estimation

Consider n individuals are in an experiment under random censoring setup. Then the observed sample will be $Y_1, Y_2, ..., Y_n$. Under $InvMWD$ and observed data, the likelihood

Table 2: ETT and OBTT for different combination of λ and θ with varying sample size.

$\theta\backslash\lambda$		0.5			1.0			2.0		
			$OBTT$			$OBTT$			$OBTT$	
	n	ETT	EV	MSE	ETT	EV	MSE	ETT	EV	MSE
	50	2.6269	2.6239	0.0491	2.2001	2.2011	0.0379	1.8266	1.8158	0.0261
0.5	60	2.7181	2.7251	0.0531	2.2772	2.2837	0.0415	1.8923	1.8974	0.0280
	80	2.8670	2.8617	0.0634	2.4029	2.4094	0.0489	1.9994	2.0014	0.0347
	50	2.2001	2.1928	0.0356	1.8575	1.8536	0.0260	1.5557	1.5529	0.0176
1.0	60	2.2772	2.2917	0.0419	1.9220	1.9328	0.0293	1.6102	1.6082	0.0204
	80	2.4029	2.4072	0.0460	2.0273	2.0239	0.0303	1.6991	1.6953	0.0215
	50	1.8266	1.8244	0.0271	1.5557	1.5683	0.0201	1.3134	1.3110	0.0137
2.0	60	1.8923	1.8866	0.0264	1.6102	1.6154	0.0210	1.3590	1.3655	0.0147
	80	1.9994	1.9941	0.0292	1.6991	1.7018	0.0233	1.4335	1.4365	0.0164

function of parameters is given by

$$\begin{aligned} L(y,d;\theta,\lambda) &= \prod_{i=1}^{n} f_{Y,D}(y,d;\theta,\lambda) \\ &\propto \theta^{-\frac{3m}{2}}\lambda^{-\frac{3(n-m)}{2}}\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n}\frac{d_i}{y_i^2}\right)\exp\left(-\frac{1}{\lambda}\sum_{i=1}^{n}\frac{(1-d_i)}{y_i^2}\right)\prod_{i=1}^{n}\gamma\left(\frac{3}{2},\frac{1}{\lambda y_i^2}\right)^{d_i} \\ &\quad \prod_{i=1}^{n}\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)^{(1-d_i)}, \end{aligned} \tag{9}$$

where, $m=\sum_{i=1}^{n} d_i$. Now, on taking the logarithm of above equation, the log-likelihood function can be written in the following form

$$\begin{aligned} l(y,d;\theta,\lambda) \propto &-\frac{3m}{2}\ln(\theta)-\frac{3(n-m)}{2}\ln(\lambda)-\frac{1}{\theta}\sum_{i=1}^{n}\frac{d_i}{y_i^2}-\frac{1}{\lambda}\sum_{i=1}^{n}\frac{(1-d_i)}{y_i^2} \\ &+\sum_{i=1}^{n} d_i \ln\left\{\gamma\left(\frac{3}{2},\frac{1}{\lambda y_i^2}\right)\right\}+\sum_{i=1}^{n}(1-d_i)\ln\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta y_i^2}\right)\right\}. \end{aligned} \tag{10}$$

Taking the partially differentiation of log-likelihood function with respect to θ and λ and then equating to zero, we get the ML estimator $\hat{\theta}$ and $\hat{\lambda}$ of θ and λ, respectively, as

follows

$$\hat{\theta} = \frac{2}{3m}\left[\sum_{i=1}^{n}\frac{d_i}{y_i^2} - \frac{1}{\hat{\theta}^{\frac{1}{2}}}\sum_{i=1}^{n}\frac{(1-d_i)\exp\left(-\frac{1}{\hat{\theta}y_i^2}\right)}{y_i^3\gamma\left(\frac{3}{2},\frac{1}{\hat{\theta}y_i^2}\right)}\right], \tag{11}$$

$$\hat{\lambda} = \frac{2}{3(n-m)}\left[\sum_{i=1}^{n}\frac{(1-d_i)}{y_i^2} - \frac{1}{\hat{\lambda}^{\frac{1}{2}}}\sum_{i=1}^{n}\frac{d_i\exp\left(-\frac{1}{\hat{\lambda}y_i^2}\right)}{y_i^3\gamma\left(\frac{3}{2},\frac{1}{\hat{\lambda}y_i^2}\right)}\right]. \tag{12}$$

The above equations of θ and λ are not in closed form, so we use the numerical iteration methods for obtaining estimates.

3.2 Interval Estimation

The confidence intervals are measures of uncertainty in the sampling methods. Since the distribution of parameters θ and λ are not in closed-form so we find the observed Fisher information matrix as

$$I(\underline{\hat{\xi}}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\Bigg|_{(\theta=\hat{\theta},\lambda=\hat{\lambda})},$$

where, $\hat{\xi}=(\hat{\theta},\hat{\lambda})$ is vector of ML estimates. Also the elements of matrix are given as

$$a_{11} = -\frac{\partial^2 l(y,d;\theta,\lambda)}{\partial\theta^2},\ \ a_{12} = -\frac{\partial^2 l(y,d;\theta,\lambda)}{\partial\theta\partial\lambda} = a_{21}\ \text{ and }\ a_{22} = -\frac{\partial^2 l(y,d;\theta,\lambda)}{\partial\lambda^2}$$

The observed variance-covariance matrix is obtained by inverting Fisher matrix $I(\underline{\hat{\xi}})$ as

$$I^{(-1)}(\underline{\hat{\xi}}) = \begin{bmatrix} Var(\hat{\theta}) & Cov(\hat{\theta},\hat{\lambda}) \\ Cov(\hat{\lambda},\hat{\theta}) & Var(\hat{\lambda}) \end{bmatrix}$$

Thus, using the asymptotic normality of estimators, we get the $100(1-\alpha)\%$ confidence limits for $\hat{\theta}$ and $\hat{\lambda}$ by $\hat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\theta})}$ and $\hat{\lambda} \pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\lambda})}$, respectively, where $z_{(\frac{\alpha}{2})}$ is upper $100(\frac{\alpha}{2})^{th}$ percentile of standard normal variate.

The detailed expressions used in constructing the asymptotic confidence interval (ACI) are given as

$$\frac{\partial^2 l(y,d;\theta,\lambda)}{\partial\theta^2}\bigg|_{\theta=\hat{\theta}} = \frac{3m}{2\hat{\theta}^2} - \frac{2}{\hat{\theta}^3}\sum_{i=1}^{n}\frac{d_i}{y_i^2} - \sum_{i=1}^{n}(1-d_i)\psi(y,\hat{\theta}),$$

$$\frac{\partial^2 l(y,d;\theta,\lambda)}{\partial\theta\partial\lambda}\bigg|_{\theta=\hat{\theta},\lambda=\hat{\lambda}} = \frac{\partial^2 l(y,d;\theta,\lambda)}{\partial\theta\partial\lambda}\bigg|_{\lambda=\hat{\lambda},\theta=\hat{\theta}} = 0$$

and

$$\frac{\partial^2 l(y,d;\theta,\lambda)}{\partial\lambda^2}\bigg|_{\lambda=\hat{\lambda}} = \frac{3(n-m)}{2\hat{\lambda}^2} - \frac{2}{\hat{\lambda}^3}\sum_{i=1}^{n}\frac{(1-d_i)}{y_i^2} - \sum_{i=1}^{n}d_i\psi(y,\hat{\lambda}),$$

where,

$$\psi(z,\eta)=\xi(z,\eta)\left[\frac{1}{\eta^2z^2}-\frac{5}{2\eta}+\xi(z,\eta)\right]\ \ and\ \ \xi(z,\eta)=\frac{1}{z^3}e^{\frac{-1}{\eta z^2}}\left[\eta^{\frac{5}{2}}\gamma\left(\frac{3}{2},\frac{1}{\eta z^2}\right)\right]^{-1}.$$

Sometimes using this method, the lower bound of ACIs may come negative. In order to overcome this issue, one approach is to replace the lower bound by zero and another is to apply logarithmic transformation to obtain the asymptotic normality of ln $(\hat{\theta})$ and ln $(\hat{\lambda})$ as

$$\frac{\ln\hat{\theta}-\ln\theta}{Var(\ln\hat{\theta})}\sim N(0,1)\qquad\&\qquad\frac{\ln\hat{\lambda}-\ln\lambda}{Var(\ln\hat{\lambda})}\sim N(0,1).$$

Therefore, using the above property $100(1-\alpha)\%$ ACIs of θ and λ in this manner can be obtained by

$$\left[\hat{\theta}\,\exp\left(-z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\ln\hat{\theta})}\right),\ \ \hat{\theta}\,\exp\left(z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\ln\hat{\theta})}\right)\right]$$

and

$$\left[\hat{\lambda}\,\exp\left(-z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\ln\hat{\lambda})}\right),\ \ \hat{\lambda}\,\exp\left(z_{\frac{\alpha}{2}}\sqrt{\widehat{V}ar(\ln\hat{\lambda})}\right)\right].$$

4 Bayesian Estimation

Bayesian inferential techniques provide a regular method for the integration of prior information drawn from other imaging methods. In the Bayesian framework, we use the Bayes theorem to update the probability of a related event with some prior information. So, we consider the parameter as a random variable that follows a distribution known as the prior distribution.

4.1 Choice of Prior Distribution

In some experiments, strong prior information is available which is based on the joint information of the nature of failure and past empirical experience. So, we choose informative prior to the experiment as like any probabilistic model for the experiment. But when the past information is neither easy nor existent, it is a very cumbersome situation to select an appropriate prior for the population parameter. This is a very extreme situation where earlier experimentation was not available, so we prefer non-informative priors. Here, we consider that the prior distributions of θ and λ are inverted gamma distribution, denoted by $IG(a_1,b_1)$ and $IG(a_2,b_2)$, respectively, and given as

$$\pi_1^*(\theta)=\frac{{b_1}^{a_1}}{\Gamma a_1\theta^{a_1+1}}\exp\left(-\frac{b_1}{\theta}\right);\quad\theta,a_1,b_1>0.$$

and

$$\pi_2^*(\lambda)=\frac{{b_2}^{a_2}}{\Gamma a_2\lambda^{a_2+1}}\exp\left(-\frac{b_2}{\lambda}\right);\quad\lambda,a_2,b_2>0.$$

where a_1, a_2, b_1 and b_2 are the hyper-parameter of prior distribution. Since θ and λ are independent, so the joint prior density is obtain by multiplying both priors density and written up to proportionality constants as follows

$$\pi^*(\theta,\lambda) \propto \frac{1}{\theta^{a_1+1}\lambda^{a_2+1}} \exp\left\{-\left(\frac{b_1}{\theta}+\frac{b_2}{\lambda}\right)\right\}. \tag{13}$$

4.2 Posterior Distribution

For obtaining the posterior distribution, we have to merge the likelihood function in (9) and the joint prior density in (13). The required posterior distribution of (θ,λ) for given observation, comes out to be as follows

$$\begin{aligned}\pi(\theta,\lambda;y,d) &= \frac{L(\theta,\lambda;y,d)\pi^*(\theta,\lambda)}{\int_0^\infty\int_0^\infty L(\theta,\lambda;y,d)\pi^*(\theta,\lambda)d\theta d\lambda}\\ &= C^{-1}L(\theta,\lambda;y,d)\pi^*(\theta,\lambda),\end{aligned} \tag{14}$$

where,

$$\begin{aligned}C &= \int_0^\infty\int_0^\infty L(\theta,\lambda;y,d)\pi^*(\theta,\lambda)d\theta d\lambda\\ &= \int_0^\infty \theta^{-\left(\frac{3m}{2}+a_1+1\right)}\prod_{i=1}^n \exp\left(-\frac{d_i}{\theta y_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta {y_i}^2}\right)\right\}^{(1-d_i)}\exp(-\frac{b_1}{\theta})\,d\theta\\ &\quad \int_0^\infty \lambda^{-\left(\frac{3(n-m)}{2}+a_2+1\right)}\prod_{i=1}^n \exp\left(-\frac{(1-d_i)}{\lambda y_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\lambda {y_i}^2}\right)\right\}^{d_i}\exp(-\frac{b_2}{\lambda})\,d\lambda\\ &= C_1C_2 = \int_0^\infty g_1(\theta)d\theta\int_0^\infty g_2(\lambda)d\lambda.\end{aligned}$$

Here, quantities $g_1(\theta)$ and $g_2(\lambda)$ can be given below by

$$g_1(\theta) = \int_0^\infty \theta^{-\left(\frac{3m}{2}+a_1+1\right)}\prod_{i=1}^n \exp\left(-\frac{d_i}{\theta y_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta {y_i}^2}\right)\right\}^{(1-d_i)}\exp(-\frac{b_1}{\theta})\,d\theta$$

and

$$g_2(\lambda) = \int_0^\infty \lambda^{-\left(\frac{3(n-m)}{2}+a_2+1\right)}\prod_{i=1}^n \exp\left(-\frac{(1-d_i)}{\lambda y_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\lambda {y_i}^2}\right)\right\}^{d_i}\exp(-\frac{b_2}{\lambda})\,d\lambda.$$

Now, the posterior distribution, up to proportionality constant is written as below

$$\begin{aligned}\pi(\theta,\lambda;y,d) \propto\ & \theta^{-\left(\frac{3m}{2}+a_1+1\right)}\lambda^{-\left(\frac{3(n-m)}{2}+a_2+1\right)}\prod_{i=1}^n \exp\left(-\frac{d_i}{\theta y_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\theta {y_i}^2}\right)\right\}^{(1-d_i)}\\ & \prod_{i=1}^n \exp\left(-\frac{(1-d_i)}{\lambda y_i^2}\right)\left\{\gamma\left(\frac{3}{2},\frac{1}{\lambda {y_i}^2}\right)\right\}^{d_i}\exp\left\{-\left(\frac{b_1}{\theta}+\frac{b_2}{\lambda}\right)\right\}.\end{aligned} \tag{15}$$

The Bayes estimate of $k(\xi) = k(\theta, \lambda)$, say, is the function of θ and λ under squared error loss function (SELF), can be obtained by as follows

$$E\left\{k(\xi)\right\} = \frac{\int_0^\infty \int_0^\infty k(\xi)\pi(\theta, \lambda; y, d)d\theta d\lambda}{\int_0^\infty \int_0^\infty \pi(\theta, \lambda; y, d)d\theta d\lambda}. \tag{16}$$

We observe that the direct solution of the ratio of integral in (16) is not possible. In this regards, we have to discuss some Bayesian approximation techniques which are available in literature. Here we used the two technique for drive the Bayes estimate of parameter as (i) Markov Chain Monte Carlo (MCMC) method and (ii) Tierney-Kadane (T-K) approximation method.

From posterior density (15), we obtain the full conditional distributions for parameters θ and λ as

$$\pi_1(\theta|y, d) \propto \theta^{-\left(\frac{3m}{2}+a_1+1\right)} \exp\left\{-\frac{1}{\theta}\left(b_1 + \sum_{i=1}^{n} \frac{d_i}{y_i^2}\right)\right\} \prod_{i=1}^{n} \left\{\gamma\left(\frac{3}{2}, \frac{1}{\theta {y_i}^2}\right)\right\}^{(1-d_i)}, \tag{17}$$

$$\pi_2(\lambda|y, d) \propto \lambda^{-\left(\frac{3(n-m)}{2}+a_2+1\right)} \exp\left\{-\frac{1}{\lambda}\left(b_2 + \sum_{i=1}^{n} \frac{(1-d_i)}{y_i^2}\right)\right\} \prod_{i=1}^{n} \left\{\gamma\left(\frac{3}{2}, \frac{1}{\lambda {y_i}^2}\right)\right\}^{d_i}. \tag{18}$$

We observe that the marginal posterior distributions of θ and λ cannot be obtained in the closed form, which is essential to obtain the Bayes estimates of parameters. But for observed data, both are independent of each other.

4.3 MCMC Method

In MCMC method, the Metropolis-Hastings (M-H) algorithm [Chen et al. (2012)] is well know method to generate the random sample from the posterior density. The algorithm was first proposed by Metropolis et al. (1953) and later generalized by Hastings (1970). Roberts and Smith (1994) discuss the conditions require for the convergence of the algorithm. Hitchcock (2003) discuss the history and origin of the M-H algorithm in detail. Using the M-H algorithm, we geneerate a Markov chain whose stationary distribution is approximately same as the posterior distribution of interest.

Here, the full conditional posterior densities of θ and λ defined in (17) and (18) are independent of each other, so we can draw the Bayesian samples from them by the M-H algorithm independently. The necessary steps to generate samples by the M-H algorithm from conditional densities are given as follows:

1. Set t=1 and take the initial value of parameters $\theta^{(0)} = \hat{\theta}$ and $\lambda^{(0)} = \hat{\lambda}$.
2. Generate candidate points θ^* from proposal density $q_1 \sim N\left(\hat{\theta}, Var(\hat{\theta})\right)$ and λ^* from proposal density $q_2 \sim N\left(\hat{\lambda}, Var(\hat{\lambda})\right)$, now produce the points u_1 and u_2 from a uniform distribution U(0,1). Then compute an acceptance ratio

$$r_1 = \frac{\pi_1(\theta^*; y, d)q_1(\theta^{(t-1)})}{\pi_1(\theta^{(t-1)}; y, d)q_1(\theta^*)}, \quad r_2 = \frac{\pi_2(\lambda^*; y, d)q_2(\lambda^{(t-1)})}{\pi_2(\lambda^{(t-1)}; y, d)q_2(\lambda^*)}.$$

3. Let $P_1(\theta^{(t-1)}, \theta^*) = min(r_1, 1)$, then set $\theta^{(t)} = \theta^*$ if $u_1 \leq P_1(\theta^{(t-1)}, \theta^*)$, otherwise $\theta^{(t)} = \theta^{(t-1)}$. Similarly let $P_2(\lambda^{(t-1)}, \lambda^*) = min(r_2, 1)$, set $\lambda^{(t)} = \lambda^*$ if $u_2 \leq P_2(\lambda^{(t-1)}, \lambda^*)$, otherwise $\lambda^{(t)} = \lambda^{(t-1)}$.

4. set t=t+1.

5. Repeat steps (2)-(4) N' times to get the sequence $\theta^1, \theta^2, \ldots, \theta^{N'}$ and $\lambda^1, \lambda^2, \ldots, \lambda^{N'}$ where N' is a large number.

From the generated samples, we discard the first few values to remove the dependency of initial value effects. Also, by using *cumsum* and ACF plots, we diagnose the stationary of the chains. In the end, we get a sample of size N, based on which we draw the required inferences. Additionally, we calculated the Bayesian credible intervals ($BCIs$) and highest posterior density (HPD) intervals of the parameter using the method proposed by Chen and Shao (1999).

4.4 Tierney-Kadane Approximation

In continuation, we use the T-K approximation [Tierney and Kadane (1986)] to find out the Bayes estimates as an alternative to the MCMC algorithm. This method is used to reduce the ratio of two integrals into finite expression. The posterior expectation of any parametric function $k(\xi)$ can be expressed in the following form

$$I(y) = \frac{\int_\xi k(\xi) e^{l(\xi)+\eta(\xi)} d\xi}{\int_\xi e^{l(\xi)+\eta(\xi)} d\xi}, \tag{19}$$

where $l(\xi)$ and $\eta(\xi)$ are the logarithm of likelihood function and logarithm of joint prior distribution, respectively. Now, let us define the functions $G(\xi)$ and $G^*(\xi)$, respectively, as given by:

$$G(\xi) = \frac{l(\xi) + \eta(\xi)}{n} \tag{20}$$

and

$$G^*(\xi) = G(\xi) + \frac{\ln k(\xi)}{n} \tag{21}$$

If $\hat{\xi}$ and $\hat{\xi}^*$ are the estimate value of vectors which maximizes by (20) and (21), respectively, then the function $I(y)$ is approximated by

$$I(y) = \sqrt{\frac{|S^*|}{|S|}} e^{n\{G^*(\hat{\xi}^*) - G(\hat{\xi})\}} \tag{22}$$

where $|S^*|$ and $|S|$ are the negative of inverse of Hessian of $G(\hat{\xi})$ and $G^*(\hat{\xi})$ respectively computed at $\hat{\xi}$ and $\hat{\xi}^*$.

For obtaining the Bayes estimate of θ, consider $k(\xi) = \theta$ and by using the log-likelihood function in (10) and joint prior density in (13). In this case, we define the function $G(\xi)$

given in (20) as below

$$G(\xi) = -\frac{3m}{2n}\ln(\theta) - \frac{3(n-m)}{2n}\ln(\lambda) - \frac{1}{n\theta}\sum_{i=1}^{n}\frac{d_i}{y_i^2} - \frac{1}{n\lambda}\sum_{i=1}^{n}\frac{(1-d_i)}{y_i^2} + \frac{1}{n}\sum_{i=1}^{n} d_i \ln\left\{\gamma\left(\frac{3}{2}, \frac{1}{\lambda y_i^2}\right)\right\} + \frac{1}{n}\sum_{i=1}^{n}(1-d_i)\ln\left\{\gamma\left(\frac{3}{2}, \frac{1}{\theta y_i^2}\right)\right\} - \frac{(a_1+1)}{n}\ln(\theta) - \frac{(a_2+1)}{n}\ln(\lambda) - \frac{b_1}{n\theta} - \frac{b_2}{n\lambda} \tag{23}$$

Let $\hat{\xi} = (\hat{\theta}, \hat{\lambda})$ be the function of ML estimate of θ and λ and obtained by maximizing (23). Hence, the function S is define as

$$|S| = \begin{bmatrix} -\frac{\partial^2 G(\xi)}{\partial\theta^2} & -\frac{\partial^2 G(\xi)}{\partial\theta\partial\lambda} \\ -\frac{\partial^2 G(\xi)}{\partial\lambda\partial\theta} & -\frac{\partial^2 G(\xi)}{\partial\lambda^2} \end{bmatrix}\Bigg|^{-1}_{(\hat{\theta},\hat{\lambda})},$$

Also, by using (21), let us define the second term such as

$$G^*(\xi) = G(\xi) + \frac{\ln\theta}{n} \tag{24}$$

The ML estimates of $\hat{\xi}^* = (\hat{\theta}^*, \hat{\lambda}^*)$ is obtained by maximizing (24) with respect to θ and λ. We can obtain the negative of inverse hessian of $G^*(\theta, \lambda)$ in similar manner done above, say $|S^*|$. Thus using the above quantities, we get the required Bayes estimates of θ by using (22). For obtaining the Bayes estimate of λ, we consider $k(\xi) = \lambda$. The rest of the calculations can be done similarly as in the case of Bayes' estimate of θ by using the T-K approximation method.

5 Simulation Study

In this section, we give illustrations based on the simulation study. We generated samples for the random censoring setup by assuming failure and censoring time distribution both as $InvMWD$ with corresponding parameters θ and λ. A simulation study is done by setting different initial values of parameters θ and λ and the failure of the system is obtained by taking the minimum censoring time and lifetimes. For simulation study, we take two sets for parameters as $(\theta, \lambda) = \{(1.65, 0.85), (0.85, 1.15)\}$. We compute ML estimates for θ and λ along with MSE, AB for different sample sizes.

In Bayesian analysis, we use the inverted gamma prior for both parameters. We take arbitrary values for hyper-parameters of prior distribution as $a_1 = a_2 = 2$ and $b_1 = b_2 = 3$. In this regard, the Bayes estimate of parameters is obtained under SELF by using the (i) MCMC method and (ii) T-K approximation method. In the MCMC method, the M-H algorithm is used for generating the sample form full conditional of θ and λ. We use the normal distribution as proposal density along with ML estimate as location and *variance* as the scale parameter. From a long chain of length 500000

obtained from the MH algorithm, first 5000 values are removed to discard the effect of the initial values of the parameter. A proper thinning interval is chosen to avoid the auto-correlation. Thus, we finally left with reduced stationary chain based on which Bayes estimates are obtained. We report Bayesian credible and HPD intervals along with coverage probability and shapes. In the T-K approximation method, we obtain the point estimates along with MSE and AB. We have calculated results based on $N = 5000$ simulated data. The ML and Bayes estimates of θ and λ are with their MSE and AB discussed in Table 3 for different values of n. Since, the Bayes estimate of θ and λ obtained by using MCMC method and T-K approximation method denoted by θ_M, λ_M and θ_{TK}, λ_{TK}, respectively. We present the BCI and HPD intervals of θ and λ with respectively coverage probability and shape for various sample sizes and presented in Table 4. We use statistical software **R** for computation purposes throughout the study. On the behalf of the simulation study, we conclude that

1. The MSE and AB decrease with the increase in sample sizes.
2. The MSE and AB are less in the case of Bayesian than that of the classical approach.
3. The MCMC and T-K approximation methods perform equally well under different sample sizes.
4. We see an improvement in MSE and AB in the case of T-K approximation only for a large sample size. That indicates the better performance of the T-K approximation method over the MCMC method.
5. For interval estimation, it can be concluded that BCI and HPD intervals provide narrower limits compared to asymptotic confidence intervals.
6. We observed that the length of ACI is greater than Bayesian intervals. And, the length of the BCI interval is greater than the HPD intervals.

$$HPD \leq BCI \leq ACI$$

7. The shape values based on Bayesian confidence intervals are greater than one, which indicates that the distribution of parameters is positively skewed.

5.1 Single Sample Based Study

Here we consider a simulated sample for analysis to show that how one can use the results obtained in the previous sections, to solve a real life problem. In this scenario, we consider that individual and censoring time follow $InvMWD$ with parameters θ and λ. For simulated data from the considered $InvMWD$ population, we consider $\theta = 2$ and $\lambda = 1.5$ and generate an observed sample under random censoring of size $n = 40$. The observed sample is 0.3182, 0.3694+, 0.378, 0.3922+ ,0.4066+, 0.4332, 0.4351+, 0.4399, 0.443, 0.4457+, 0.4931+, 0.4449, 0.4552, 0.4607, 0.4701, 0.4749, 0.4868, 0.5095, 0.5254, 0.5373,

Table 3: The ML and Bayes estimates of θ and λ along with their MSE and AB for different sample sizes taking $\theta = 1.65$ and $\lambda = 0.85$.

n		$\hat{\theta}$	$\hat{\lambda}$	$\tilde{\theta}_M$	$\tilde{\lambda}_M$	$\tilde{\theta}_{TK}$	$\tilde{\lambda}_{TK}$
	Estimate	1.6558	0.8417	1.6582	0.8891	1.6675	0.8734
50	MSE	0.0363	0.0153	0.0360	0.0137	0.0394	0.0142
	AB	0.1587	0.0998	0.1470	0.0939	0.1581	0.0941
	Estimate	1.6490	0.8461	1.6724	0.8795	1.6533	0.8816
60	MSE	0.0311	0.0114	0.0281	0.0124	0.0280	0.0129
	AB	0.1387	0.0851	0.1293	0.0892	0.1322	0.0885
	Estimate	1.6525	0.8465	1.6832	0.8727	1.6686	0.8698
80	MSE	0.0267	0.0089	0.0258	0.0097	0.0211	0.0088
	AB	0.1310	0.0766	0.1238	0.0811	0.1165	0.0741

Table 4: The ML and Bayes estimates of θ and λ along with their MSE and AB for different sample sizes taking $\theta = 0.85$ and $\lambda = 1.15$.

n		$\hat{\theta}$	$\hat{\lambda}$	$\tilde{\theta}_M$	$\tilde{\lambda}_M$	$\tilde{\theta}_{TK}$	$\tilde{\lambda}_{TK}$
	Estimate	0.8545	1.1597	0.8909	1.1930	0.8780	1.1704
50	MSE	0.0114	0.0217	0.0108	0.0190	0.0115	0.0182
	AB	0.0843	0.1162	0.0830	0.1121	0.0858	0.1082
	Estimate	0.8511	1.1516	0.8840	1.1879	0.8766	1.1739
60	MSE	0.0097	0.0155	0.0119	0.0189	0.0090	0.0151
	AB	0.0780	0.0990	0.0851	0.1070	0.0763	0.0946
	Estimate	0.8418	1.1517	0.8709	1.1709	0.8715	1.1670
80	MSE	0.0078	0.0118	0.0077	0.0124	0.0075	0.0108
	AB	0.0699	0.0863	0.0667	0.0890	0.0686	0.0819

0.5392+, 0.5405, 0.5520+, 0.5557+, 0.5569, 0.5813, 0.6106, 0.6107+, 0.6300+, 0.6308, 0.6362, 0.6369, 0.6375, 0.6963, 0.7118, 0.7524, 0.7578, 0.8257+, 0.8416+, 0.9386+, where $y+$ denoting the observed censored time. On the basis of the observed sample, the estimated values of parameters and other functions are obtained. We consider the hyper-parameter values as the same in the simulation study. Figure 4 shows the *cumsum* and *acf* plot of simulated data and similarly Figure 5 present the data *iteration* and marginal *posterior* density plot of given data. Figures are given in Appendix (8). The ML and Bayes estimates of parameters and related functions are presented in Table 5.

Table 5: Estimated values for ACI, BCI and HPD intervals along with average lengths (ALs), shape and coverage probabilities (CPs) of parameters for $n = 50$, 60 and 80.

$\theta = 1.65$	$\lambda = 0.85$	ACI		BCI		HPD	
n		$\hat{\theta}$	$\hat{\lambda}$	$\tilde{\theta}$	$\tilde{\lambda}$	$\tilde{\theta}^*$	$\tilde{\lambda}^*$
	(LL,UL)	(1.2757, 2.0359)	(0.6105, 1.0729)	(1.3237, 2.0672)	(0.6802, 1.1487)	(1.3053, 2.0350)	(0.6672, 1.1251)
50	Shape	1.0000	1.0000	1.2237	1.2439	1.0699	1.0683
	CP	0.94	0.94	0.94	0.96	0.94	0.96
	(LL,UL)	(1.3034, 1.9946)	(0.6345, 1.0576)	(1.3620, 2.0469)	(0.6883, 1.1135)	(1.3460, 2.0186)	(0.6778, 1.0946)
60	Shape	1.0000	1.0000	1.2080	1.2247	1.0661	1.0696
	CP	0.94	0.94	0.95	0.94	0.94	0.95
	(LL,UL)	(1.3526, 1.9525)	(0.6632, 1.0297)	(1.4090, 2.0058)	(0.7056, 1.0714)	(1.3988, 1.9863)	(0.6968, 1.0573)
80	Shape	1.0000	1.0000	1.1775	1.1899	1.0696	1.0599
	CP	0.93	0.93	0.92	0.94	0.92	0.95
$\theta = 0.85$	$\lambda = 1.15$						
	(LL,UL)	(0.6389, 1.0701)	(0.8883, 1.4312)	(0.6957, 1.1313)	(0.9489, 1.4937)	(0.6851, 1.1119)	(0.9364, 1.4705)
50	Shape	1.0000	1.0000	1.2326	1.2317	1.0788	1.0861
	CP	0.95	0.94	0.96	0.97	0.97	0.97
	(LL,UL)	(0.6552, 1.0470)	(0.9055, 1.3977)	(0.7052, 1.1002)	(0.9636, 1.4588)	(0.6960, 1.0838)	(0.9522, 1.4385)
60	Shape	1.0000	1.0000	1.2105	1.2090	1.0684	1.0675
	CP	95	0.94	0.92	0.94	0.92	0.93
	(LL,UL)	(0.6737, 1.0099)	(0.9388, 1.3647)	(0.7161, 1.0536)	(0.9763, 1.4000)	(0.7088, 1.0409)	(0.9674,1.3838)
80	Shape	1.0000	1.0000	1.1811	1.1779	1.0534	1.0509
	CP	0.93	0.95	0.94	0.93	0.95	0.94

Table 6: The ML and Bayes estimated values, ACI, BCI, HPD for θ and λ, ETT and $OBTT$ for simulated data set.

Estimate Value	$\hat{\theta}$	$\hat{\lambda}$
ML	2.0733	1.5611
ACI	(1.8421, 2.3044)	(1.3748, 1.7474)
BE_{TK}	2.0864	1.5842
BE_{MC}	2.0834	1.5872
BCI	(1.6233, 2.6649)	(1.2155, 2.0766)
HPD	(1.5956, 2.6246)	(1.2033, 2.0281)
ETT	1.3521	
$OBTT$	0.9386	

6 Real Data Study

In this section, we illustrate estimation procedures as discussed in the previous sections with the help of real data. The survival times (in days) of 72 Guinea Pigs were reported by Bjerkedal et al. (1960). The authors discussed a detailed description of the study of Guinuea Pigs experiment which is conducted in the Tuberculosis Research Laboratory, Chamblee, Ga. In this study, two independent studies on the survival of experimental animals infected with a variable number of virulent human tubers are studied. In this study, groups of animals infected with different numbers of virulent tubercles are compared. Material is provided by two independent studies comprising a total of 1414 guinea pigs randomly allocated into two experimental groups to unchallenged control groups and challenged with different numbers of virulent human tubercle bacilli. The number of animals included in the present material was 467 for study M with an average weight at the challenge was 475 grams and 947 for study P with an average weight at the challenge was 467 grams. In the two experiments, equal numbers of males and females were classified according to weight and randomized among the six experimental groups (regimens) of each study. In the case of the infection by the same dose in a group of livestock or animals, at the first death of an animal in a group, it shows the least resistance and most resistance for the last to die in the same group and defines the survival time of pigs. All the animals have provided the same number of tubercle bacilli doses. The author considered the proposed method in the studies (i) causes of death other than tuberculosis, (ii) failure of the challenge to be ineffective, and (iii) variation in the number of bacilli in challenge doses from the same bacterial suspension. Finally, they show a relationship between survival time and the number of tubercular bacilli responsible for infection and also compare the studies carried out by M and P groups. They obtain the mean survival time for survivals groups.

The nature of hazard of data is found to be uni-modal and shown by Kundu and Howlader (2010). The same data set is used for analysis purpose by Tomer and Panwar

(2020). In the Figure 6 and 7, ECDF and QQ plots are given and it can be seen that the survival time of Guinea Pigs time data fits $InvMWD$ well. Further, the K-S test statistics (D) is found to be 0.2223 along with p value greater than 0.05, indicates that there is no evidence to reject the hypothesis that the considered model is suitable for the data. Also, for ease of analysis we divide the observations by 100. Further, we artificially introduce the random censoring in the data by generating 72 censoring time points uniformly in the range of observations by fixing seed value 100. The random censored samples are obtained as: 0.12, 0.15, 0.22, 0.24, 0.24, 0.32, 0.32, 0.33, 0.34, 0.38, 0.38, 0.43, 0.44, 0.48, 0.52 0.53, 0.54, 0.54, 0.55, 0.56, 0.57, 0.57+, 0.58, 0.58, 0.59, 0.59+, 0.60, 0.60, 0.60 0.60, 0.61, 0.62, 0.63, 0.65, 0.65, 0.67, 0.68, 0.70, 0.70, 0.72, 0.73, 0.76, 0.76, 0.81 0.83, 0.84, 0.84+, 0.85, 0.87, 0.88+, 0.89+, 0.91, 0.96, 0.96+, 0.98, 0.98+, 1.01+, 1.04+, 1.10, 1.21, 1.31+, 1.42+, 1.43, 1.46, 1.62+, 1.74+, 1.74+, 1.75, 1.75, 1.78+, 2.11, 2.65+, where $y+$ denoting the censored time. There are 56 observations which are observed as exact failure time and 16 as randomly censored. For this sample ML and Bayes estimates are given in Table 7. The estimates under Bayesian approach are calculated under non-informative priors. Figure 8 shows the *cumsum* and acf plot of simulated data and similarly Figure 9 shows the data *iteration* and marginal *posterior* density plot of given data. All figures are given in Appendix (8).

Table 7: The ML and Bayes estimates for Guinea Pigs data set

Estimate Value	$\hat{\theta}$	$\hat{\lambda}$
ML	3.0588	0.4632
ACI	(2.4813, 3.6363)	(0.3350, 0.5914)
BE_{TK}	3.0861	0.4701
BE_{MC}	3.0943	0.4724
BCI	(2.5575, 3.6951)	(0.3626, 0.6046)
HPD	(2.5565, 3.6868)	(0.3618, 0.5987)
ETT	2.3049	
$OBTT$	2.6500	

7 Concluding Remarks

In this study, we discuss the estimation procedure of $InvMWD$ under the random censoring setup. We obtain the probability of failure of an item before censoring time for different combinations of parameter values. Also, we calculate the observed and expected time on the test under complete and censored scenarios. Parameter estimators along with their confidence intervals under classical and Bayesian approaches are derived. In the simulation study, it is observed that the parameters are consistent with increase in sample size as the mean square error and absolute bias reduce. Finally, simulated and survival data are analyzed in support of the proposed model. The parameter estimates along with interval estimates are reported for the real data. All the results are found

to be satisfactory and support the proposed study. In some real life situations the dependence assumption between lifetime and censored time observations may be more feasible, we recommend this for future work. Further the presence of covariates in the model may be challenging to deal with.

Acknowledgements

The authors are very thankful to the Editor and Reviewers for their fruitful comments which improved the quality of the current study.

Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

Bekker, A. and Roux, J. (2005). Reliability characteristics of the Maxwell distribution: A Bayes estimation study. *Communications in Statistics-Theory and Methods*, 34(11):2169–2178.

Bjerkedal, T. et al. (1960). Acquisition of resistance in Guinea Pies infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene*, 72(1):130–48.

Chaturvedi, A. and Vyas, S. (2019). Generalized Gamma-Maxwell distribution: Properties and estimation of reliability functions. *Journal of Statistics and Management Systems*, 22(8):1425–1444.

Chen, M.-H. and Shao, Q.-M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computational and Graphical Statistics*, 8(1):69–92.

Chen, M.-H., Shao, Q.-M., and Ibrahim, J. G. (2012). *Monte Carlo methods in Bayesian computation*. Springer Science & Business Media.

Hastings, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. 57(1):97–109.

Hitchcock, D. B. (2003). A history of the Metropolis–Hastings algorithm. *The American Statistician*, 57(4):254–257.

Kishan, R., Yadav, C., Kumar, J., and Bisht, S. S. (2021). Competing risk analysis of lifetime data using inverse Maxwell distribution. *International Journal of Agricultural and Statistical Sciences*, 17(1):285–293.

Krishna, H. and Goel, N. (2017). Maximum likelihood and Bayes estimation in randomly censored geometric distribution. *Journal of Probability and Statistics*, 2017:1–12.

Krishna, H. and Malik, M. (2009). Reliability estimation in Maxwell distribution with Type-II censored data. *International Journal of Quality & Reliability Management*, (2):184–195.

Krishna, H., Vivekanand, and Kumar, K. (2015). Estimation in Maxwell distribution with randomly censored data. *Journal of Statistical Computation and Simulation*, 85(17):3560–3578.

Kumar, K. and Garg, R. (2014). Estimation of the parameters of randomly censored generalized inverted Rayleigh distribution. *International Journal of Agricultural and Statistical Sciences*, 10(1):147–155.

Kumar, K. and Kumar, I. (2019). Estimation in inverse Weibull distribution based on randomly censored data. *Statistica*, 79(1):47–74.

Kundu, D. and Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. *Computational Statistics & Data Analysis*, 54(6):1547–1558.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E. (1953). Equation of state calculations by fast computing machines. *The journal of chemical physics*, 21(6):1087–1092.

Modi, K. and Gill, V. (2015). Length-biased weighted Maxwell distribution. *Pakistan Journal of Statistics and Operation Research*, 11(4):465–472.

Nandi, S. and Dewan, I. (2010). An EM algorithm for estimating the parameters of bivariate Weibull distribution under random censoring. *Computational Statistics & Data Analysis*, 54(6):1559–1569.

Roberts, G. O. and Smith, A. F. (1994). Simple conditions for the convergence of the Gibbs sampler and Metropolis-Hastings algorithms. *Stochastic processes and their applications*, 49(2):207–216.

Sharma, V. K., Bakouch, H. S., and Suthar, K. (2017). An extended Maxwell distribution: Properties and applications. *Communications in Statistics-Simulation and Computation*, 46(9):6982–7007.

Sharma, V. K., Dey, S., Singh, S. K., and Manzoor, U. (2018). On length and area-biased Maxwell distributions. *Communications in Statistics-Simulation and Computation*, 47(5):1506–1528.

Singh, K. and Srivastava, R. (2012). Bayesian estimation of parameter of inverse Maxwell distribution via size-biased sampling. *International Journal of Science and Research (IJSR)*, 3(9):2319–7064.

Singh, K. L. and Srivastava, R. (2014). Inverse Maxwell distribution as a survival model, genesis and parameter estimation. *Research Journal of Mathematical and Statistical Sciences*, 2(7):23–28.

Tierney, L. and Kadane, J. B. (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association*, 81(393):82–86.

Tomer, S. K. and Panwar, M. (2015). Estimation procedures for Maxwell distribution under Type-I progressive hybrid censoring scheme. *Journal of Statistical Computation and Simulation*, 85(2):339–356.

Tomer, S. K. and Panwar, M. (2020). A review on inverse Maxwell distribution with

its statistical properties and applications. *Journal of Statistical Theory and Practice*, 14:1–25.

Tyagi, R. and Bhattacharya, S. (1989). Bayes estimation of the Maxwell's velocity distribution function. *Statistica*, 29(4):563–567.

Yadav, C., Kumar, J., and Panwar, M. (2021). Statistical analysis of Marshall-Olkin inverse Maxwell distribution: Estimation and application to real data. *Reliability: Theory & Applications*, 16(3 (63)):249–272.

8 Appendix

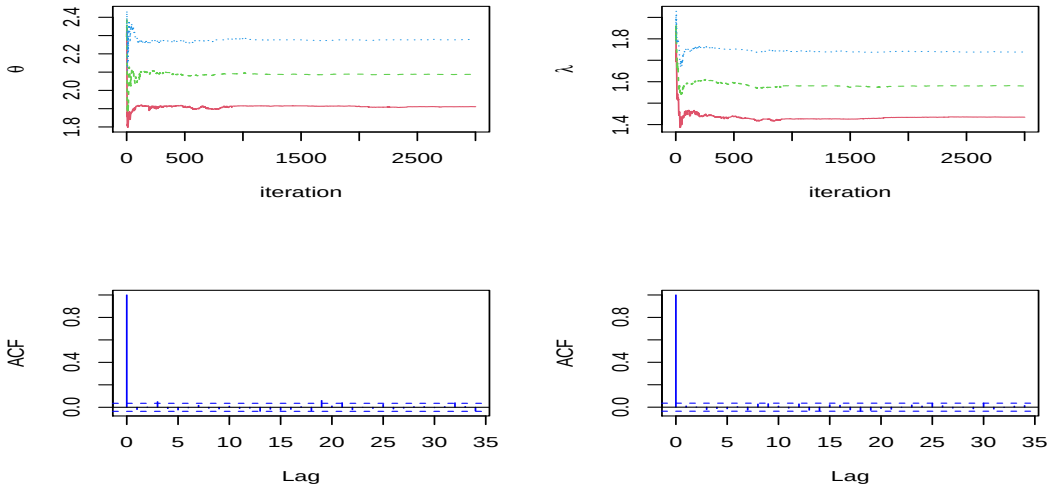

Figure 4: Cumsum and ACF plots based on posterior distribution for simulated data.

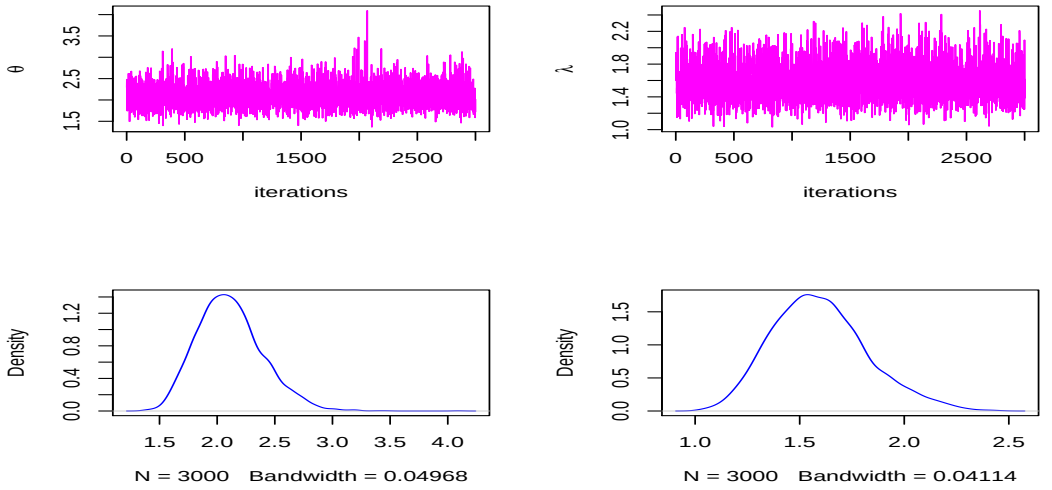

Figure 5: Trace and density plots based on posterior distribution for simulated data.

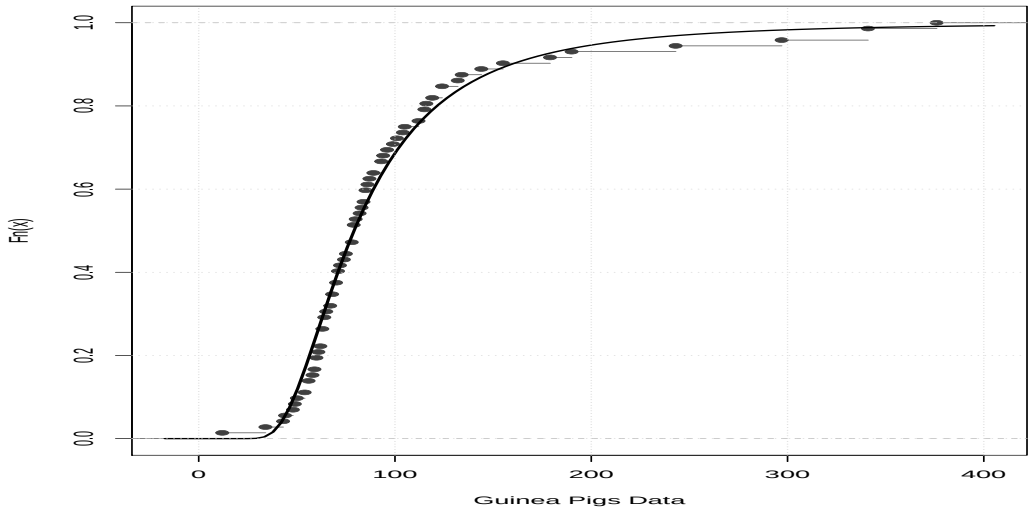

Figure 6: ECDF plot for Guinea Pigs Data.

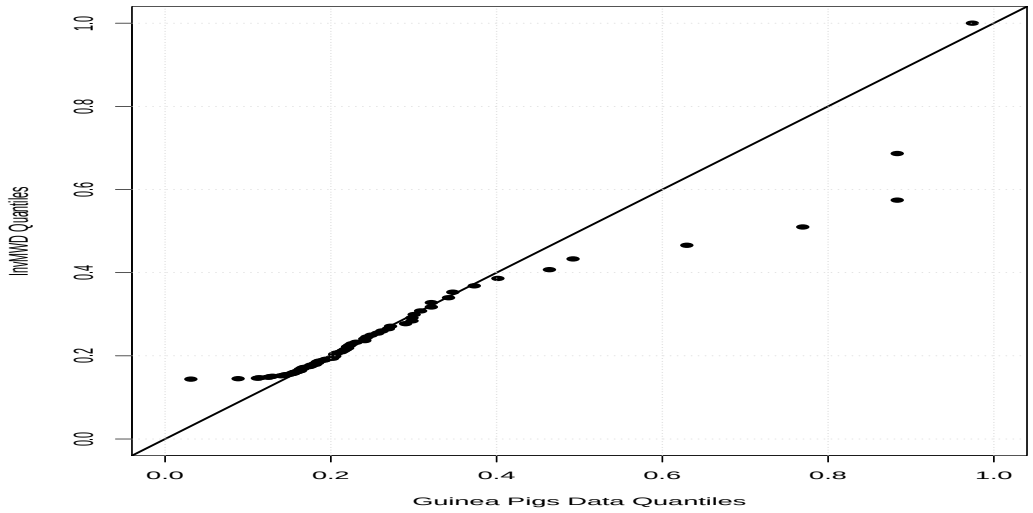

Figure 7: QQ plot for Guinea Pigs data.

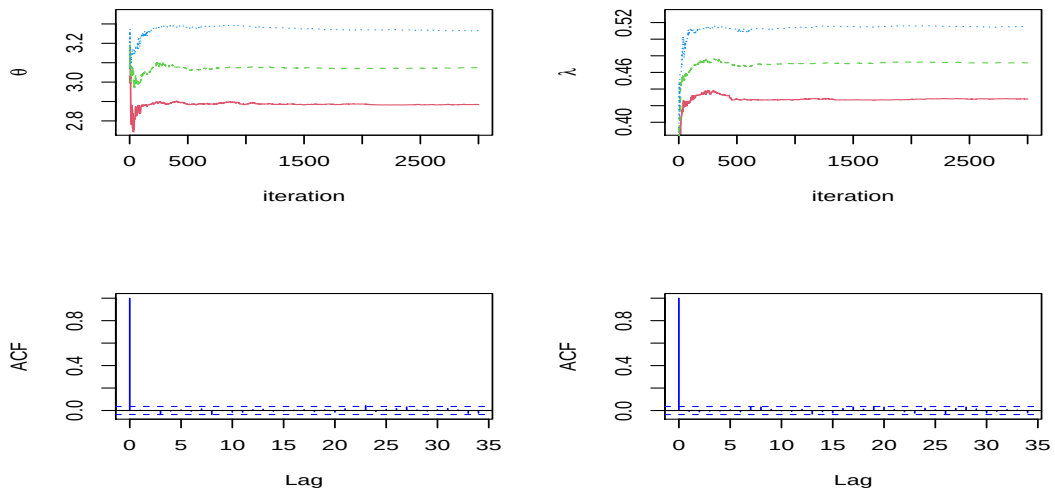

Figure 8: Cumsum and ACF plots based on posterior distribution for Guinea Pigs data.

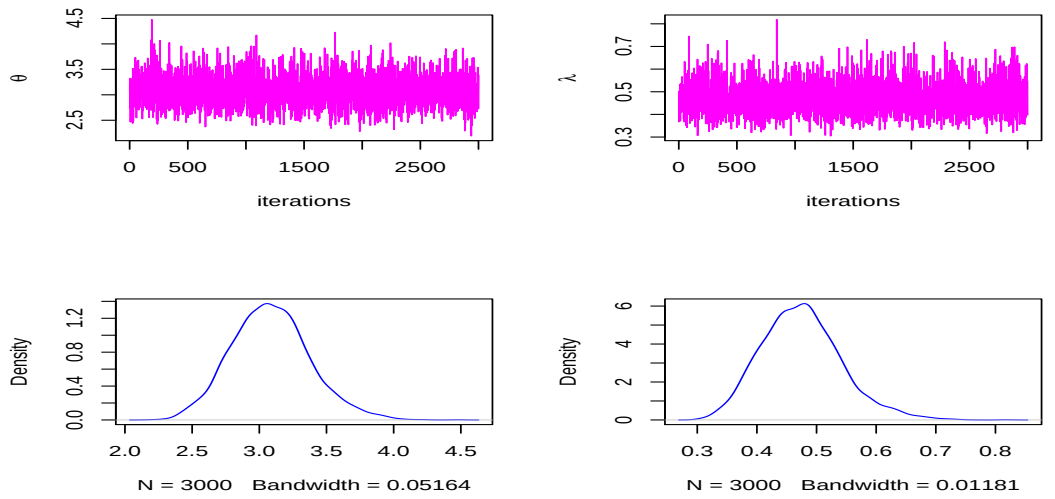

Figure 9: Trace and density plots based on posterior distribution for Guinea Pigs data.