Supplementary Materials: A Multilevel Approach for Estimating Menopause Duration using Recall Data

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Abstract: This document includes calculations for a two-cause setup in recall-based competing risks data, posterior plots for menopause data, and simulation tables.

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1 Calculation for Two Causes Setup

1.1 Likelihood Construction

Under two causes setup, we can write the likelihood function under observed data vector d as

$$
\begin{aligned}
L(\theta|d) = &\prod_{i=1}^{n_{r_1}} \Big[f(t_i,\theta_1)\bar{F}(t_i,\theta_2)\{1-\psi_1(s_i,t_i)\}\Big] \prod_{i=n_{r_1}+1}^{n_{r_2}} \Big[f(t_i,\theta_2)\bar{F}(t_i,\theta_1)\{1-\psi_2(s_i,t_i)\}\Big] \\
&\prod_{i=n_r+1}^{n_{nr}} \Big[\int_0^{s_i} f(u,\theta_1)\bar{F}(u,\theta_2)\psi_1(s_i,u)du + \int_0^{s_i} f(u,\theta_2)\bar{F}(u,\theta_1)\psi_2(s_i,u)du\Big] \\
&\prod_{i=n_{nr}+1}^{n} \bar{F}(s_i;\ \theta_1,\theta_2). \qquad (1.1)
\end{aligned}
$$

The symbols n_{r_1} and n_{r_2} denotes the number of individuals who had experienced the event of interest before monitoring time and recalled the exact causes of occurrence of the event as 1 and 2 respectively.

Assume $T \sim \mathcal{W}(\alpha_k,\beta_k)$; $k=1,2$ for causes 1 and 2 and denote the parameter vector by $\Theta = (\alpha_1,\beta_1,\lambda_1,\alpha_2,\beta_2,\lambda_2)$. Putting all the functional forms of the densities in (1.1), the likelihood function can be written as below

$$
\begin{aligned}
L(\Theta|d) \;=\; &\alpha_1^{n_{r_1}}\beta_1^{n_{r_1}}\alpha_2^{n_{r_2}-n_{r_1}}\beta_2^{n_{r_2}-n_{r_1}} \prod_{i=1}^{n_{r_1}} t_i^{\alpha_1-1}\exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_1(s_i-t_i)\} \\
&\prod_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2-1}\exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_2(s_i-t_i)\} \prod_{i=n_{nr}+1}^{n} \exp\{-\beta_1 s_i^{\alpha_1} - \beta_2 s_i^{\alpha_2}\} \\
&\prod_{i=n_r+1}^{n_{nr}} \Big[\int_0^{s_i} \alpha_1\beta_1 u^{\alpha_1-1}\exp\{-\beta_1 u^{\alpha_1} - \beta_2 u^{\alpha_2}\}\Big(1-\exp\{-\lambda_1(s_i-u)\}\Big)du \\
&+ \int_0^{s_i} \alpha_2\beta_2 u^{\alpha_2-1}\exp\{-\beta_1 u^{\alpha_1} - \beta_2 u^{\alpha_2}\}\Big(1-\exp\{-\lambda_2(s_i-u)\}\Big)du\Big]. \qquad (1.2)
\end{aligned}
$$

1.2 Classical Inference

The likelihood function (1.2) is incompatible to deal further. Since for the non-recall observations, partial information is available on time to event and causes of the event is also unknown. We treat it as a missing data problem and apply the E-M algorithm for point estimation.

1.2.1 Expectation Maximization Algorithm

Since any observation under non-recall is exposed to two cause, hence a latent variable Z_i following Bernoulli distribution with probability of success P_i is introduced. The probability of success $(Z_i = 1)$ is defined as

$$P_i = \frac{I_k(s_i)}{\sum\limits_k I_k(s_i)}; \qquad i = n_r + 1, n_r + 2, \dots, n_{nr}, \;\; k = 1, 2, \tag{1.3}$$

where, $I_1(.)$ and $I_2(.)$ is given by

$$I_1(s_i) = \int\limits_0^{s_i} \alpha_1 \beta_1 u^{\alpha_1 - 1} \exp\{-\beta_1 u^{\alpha_1} - \beta_2 u^{\alpha_2}\} \Big(1 - \exp\{-\lambda_1(s_i - u)\}\Big) du$$

and

$$I_2(s_i) = \int\limits_0^{s_i} \alpha_2 \beta_2 u^{\alpha_2 - 1} \exp\{-\beta_1 u^{\alpha_1} - \beta_2 u^{\alpha_2}\} \Big(1 - \exp\{-\lambda_2(s_i - u)\}\Big) du.$$

With the help of these probabilities and introduced latent variable, we can assign exact causes to each observation under non-recall. Incorporating these, the likelihood can be written as

$$L(\Theta|d) = \alpha_1^{n_{r_1}} \beta_1^{n_{r_1}} \alpha_2^{n_{r_2}-n_{r_1}} \beta_2^{n_{r_2}-n_{r_1}} \prod_{i=1}^{n_{r_1}} t_i^{\alpha_1-1} \exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_1(s_i - t_i)\}$$
$$\prod_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2-1} \exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_2(s_i - t_i)\} \prod_{i=n_{nr}+1}^{n} \exp\{-\beta_1 s_i^{\alpha_1} - \beta_2 s_i^{\alpha_2}\}$$
$$\prod_{i=n_r+1}^{n_{nr}} \left[\int_0^{s_i} \alpha_1\beta_1 u^{\alpha_1-1} \exp\{-\beta_1 u^{\alpha_1} - \beta_2 u^{\alpha_2}\}\Big(1 - \exp\{-\lambda_1(s_i - u)\}\Big) du\right]^{z_i}$$
$$\prod_{i=n_r+1}^{n_{nr}} \left[\int_0^{s_i} \alpha_2\beta_2 u^{\alpha_2-1} \exp\{-\beta_1 u^{\alpha_1} - \beta_2 u^{\alpha_2}\}\Big(1 - \exp\{-\lambda_2(s_i - u)\}\Big) du\right]^{1-z_i}. \tag{1.4}$$

Following equivalent quantity approach, in light of given causes, quantity T^* is introduced, which lies in the interval $(0, S)$. Now the likelihood function (1.4) in the light of conditional density of T^* can be re-written as

$$L(\Theta|d) = \alpha_1^{n_{r_1}} \beta_1^{n_{r_1}} \alpha_2^{n_{r_2}-n_{r_1}} \beta_2^{n_{r_2}-n_{r_1}} \prod_{i=1}^{n_{r_1}} t_i^{\alpha_1-1} \exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_1(s_i - t_i)\}$$
$$\prod_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2-1} \exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_2(s_i - t_i)\} \prod_{i=n_{nr}+1}^{n} \exp\{-\beta_1 s_i^{\alpha_1} - \beta_2 s_i^{\alpha_2}\}$$
$$\prod_{i=n_r+1}^{n_{nr}} \left[\alpha_1\beta_1 t_i^{*\alpha_1-1} \exp\{-\beta_1 t_i^{*\alpha_1} - \beta_2 t_i^{*\alpha_2}\}\Big(1 - \exp\{-\lambda_1(s_i - t_i^*)\}\Big)\right]^{z_i}$$
$$\prod_{i=n_r+1}^{n_{nr}} \left[\alpha_2\beta_2 t_i^{*\alpha_2-1} \exp\{-\beta_1 t_i^{*\alpha_1} - \beta_2 t_i^{*\alpha_2}\}\Big(1 - \exp\{-\lambda_2(s_i - t_i^*)\}\Big)\right]^{1-z_i}. \tag{1.5}$$

Also, for making the functional form of non-recall probabilities compatible, we introduce two latent variables U^* and V^* corresponding to causes 1 and 2 following exponential distribution with mean $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ truncated at $(S - T^*)$. For non-recall observations, let us denote the missing data by $\underline{D}^* = (Z, T^*, U^*, V^*)$. For i^{th} observation belonging to non-recall category, denote the missing data by $d^* = (z_i, t_i^*, u_i^*, v_i^*)$. The complete data can be written as (d, d^*). Now under the complete data the likelihood function (1.5) can be written as

$$L_c(\Theta|d,d^*) = \alpha_1^{n_{r_1}+Z^*} \beta_1^{n_{r_1}+Z^*} \lambda_1^{Z^*} \alpha_2^{n_{r_2}+n_{nr}-n_r-Z^*} \beta_2^{n_{r_2}+n_{nr}-n_r-Z^*} \lambda_2^{n_{nr}-n_r-Z^*}$$
$$\prod_{i=1}^{n_{r_1}} t_i^{\alpha_1-1} \exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_1(s_i - t_i)\}$$
$$\prod_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2-1} \exp\{-\beta_1 t_i^{\alpha_1} - \beta_2 t_i^{\alpha_2} - \lambda_2(s_i - t_i)\}$$
$$\prod_{i=n_r+1}^{n_{nr}} \left[t_i^{*\alpha_1-1} \exp\{-\beta_1 t_i^{*\alpha_1} - \beta_2 t_i^{*\alpha_2} - \lambda_1 u_i^*\}\right]^{z_i} \prod_{i=n_{nr}+1}^{n} \exp\{-\beta_1 s_i^{\alpha_1} - \beta_2 s_i^{\alpha_2}\}$$
$$\prod_{i=n_r+1}^{n_{nr}} \left[t_i^{*\alpha_2-1} \exp\{-\beta_1 t_i^{*\alpha_1} - \beta_2 t_i^{*\alpha_2} - \lambda_2 v_i^*\}\right]^{1-z_i}, \qquad (1.6)$$

where, $Z^* = \sum_{i=n_r+1}^{n_{nr}} z_i$. Taking natural logarithm of (1.6), the log-likelihood function can be written as

$$\begin{aligned} l_c(\Theta|d,d^*) &= (n_{r_1}+Z^*)\ln(\alpha_1) + (n_{r_1}+Z^*)\ln(\beta_1) + Z^*\ln(\lambda_1) + (n_{r_2}+n_{nr}-n_r-Z^*)\ln(\alpha_2) \\ &+ (n_{r_2}+n_{nr}-n_r-Z^*)\ln(\beta_2) + (n_{nr}-n_r-Z^*)\ln(\lambda_2) + (\alpha_1-1)\sum_{i=1}^{n_{r_1}}\ln(t_i) - \beta_1\sum_{i=1}^{n_{r_1}} t_i^{\alpha_1} \\ &- \beta_2\sum_{i=1}^{n_{r_1}} t_i^{\alpha_2} - \lambda_1\sum_{i=1}^{n_{r_1}}(s_i-t_i) + (\alpha_2-1)\sum_{i=n_{r_1}+1}^{n_{r_2}}\ln(t_i) - \beta_1\sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_1} - \beta_2\sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2} \\ &- \lambda_2\sum_{i=n_{r_1}+1}^{n_{r_2}}(s_i-t_i) + (\alpha_1-1)\sum_{i=1}^{n_{nr}} z_i\ln(t_i^*) - \beta_1\sum_{i=n_r+1}^{n_{nr}} z_i t_i^{*\alpha_1} - \beta_2\sum_{i=n_r+1}^{n_{nr}} z_i t_i^{*\alpha_2} \\ &- \lambda_1\sum_{i=n_r+1}^{n_{nr}} z_i u_i^* + (\alpha_2-1)\sum_{i=n_r+1}^{n_{nr}}(1-z_i)\ln(t_i^*) - \beta_1\sum_{i=n_r+1}^{n_{nr}}(1-z_i)t_i^{*\alpha_1} \\ &- \beta_2\sum_{i=n_r+1}^{n_{nr}}(1-z_i)t_i^{*\alpha_2} - \lambda_2\sum_{i=n_r+1}^{n_{nr}}(1-z_i)v_i^* - \beta_1\sum_{i=n_{nr}+1}^{n} s_i^{\alpha_1} - \beta_2\sum_{i=n_{nr}+1}^{n} s_i^{\alpha_2}. \end{aligned} \qquad (1.7)$$

We define the quantity Q as

$$
\begin{aligned}
Q\Big(\Theta|\hat{\Theta}^{(m)}\Big) &= (n_{r_1}+P)\ln(\alpha_1) + (n_{r_1}+P)\ln(\beta_1) + P\ln(\lambda_1) + (n_{r_2}+n_{nr}-n_r-P)\ln(\alpha_2) \\
&+ (n_{r_2}+n_{nr}-n_r-P)\ln(\beta_2) + (n_{nr}-n_r-P)\ln(\lambda_2) + (\alpha_1-1)\sum_{i=1}^{n_{r_1}}\ln(t_i) - \beta_1\sum_{i=1}^{n_{r_1}} t_i^{\alpha_1} \\
&- \beta_2\sum_{i=1}^{n_{r_1}} t_i^{\alpha_2} - \lambda_1\sum_{i=1}^{n_{r_1}}(s_i-t_i) + (\alpha_2-1)\sum_{i=n_{r_1}+1}^{n_{r_2}}\ln(t_i) - \beta_1\sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_1} - \beta_2\sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2} \\
&- \lambda_2\sum_{i=n_{r_1}+1}^{n_{r_2}}(s_i-t_i) + (\alpha_1-1)\sum_{i=n_r+1}^{n_{nr}} P_i E\Big[\ln(t_i^*)\mid t_i^*<s_i\Big] - \beta_1\sum_{i=n_r+1}^{n_{nr}} P_i E\Big[t_i^{*\alpha_1}\mid t_i^*<s_i\Big] \\
&- \beta_2\sum_{i=n_r+1}^{n_{nr}} P_i E\Big[t_i^{*\alpha_2}\mid t_i^*<s_i\Big] - \lambda_1\sum_{i=n_r+1}^{n_{nr}} P_i E\Big[u_i^*\mid u_i^*<(s_i-t_i^*)\Big] \\
&+ (\alpha_2-1)\sum_{i=n_r+1}^{n_{nr}}(1-P_i)E\Big[\ln(t_i^*)\mid t_i^*<s_i\Big] - \beta_1\sum_{i=n_r+1}^{n_{nr}}(1-P_i)E\Big[t_i^{*\alpha_1}\mid t_i^*<s_i\Big] \\
&- \beta_2\sum_{i=n_r+1}^{n_{nr}}(1-P_i)E\Big[t_i^{*\alpha_2}\mid t_i^*<s_i\Big] - \lambda_2\sum_{i=n_r+1}^{n_{nr}}(1-P_i)E\Big[v_i^*\mid v_i^*<(s_i-t_i^*)\Big] \\
&- \beta_1\sum_{i=n_{nr}+1}^{n} s_i^{\alpha_1} - \beta_2\sum_{i=n_{nr}+1}^{n} s_i^{\alpha_2}. \qquad (1.8)
\end{aligned}
$$

Since $z_i \sim \mathcal{B}(1, P_i)$, hence $E(z_i) = P_i$ and $E(Z^*) = E(\sum_{i=n_r+1}^{n_{nr}} z_i) = P$. The conditional densities of U^* and V^* can be calculated by using density below

$$
h_Y\Big(y_i|y_i<(s_i-t_i^*), z_i, \lambda_k\Big) = \frac{h(y_i;\ \lambda_k)}{1-\bar{H}\Big((s_i-t_i^*);\ \lambda_k\Big)} = \frac{\lambda_k\exp\{-\lambda_k y_i\}}{1-\exp\{-\lambda_k(s_i-t_i^*)\}} \qquad (1.9)
$$

where $h(.)$ and $\bar{H}(.)$ denotes the truncated density and survival functions of Y at point $(S-T^*)$.

The expectation terms used in E-step of the E-M algorithm can be calculated from conditional densities of latent variables given below

$$\xi_1(t_i^*;\ \alpha_1,\beta_1) = E\Big[t_i^{*\alpha_1}|t_i^*<s_i,k=1,\alpha_1,\beta_1\Big] = \frac{\int_0^{s_i} u^{2\alpha_1-1}\alpha_1\beta_1\exp\{-\beta_1 u^{\alpha_1}\}du}{1-\exp\{-\beta_1 s_i^{\alpha_1}\}}$$
$$= \frac{1-(1+\beta_1 s_i^{\alpha_1})\exp\{-\beta_1 s_i^{\alpha_1}\}}{\beta_1\Big(1-\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)},$$

$$\xi_2(t_i^*;\ \alpha_1,\beta_1,\alpha_2) = E\Big[t_i^{*\alpha_2}|t_i^*<s_i,k=1,\alpha_1,\beta_1\Big] = \frac{\int_0^{s_i} u^{\alpha_1+\alpha_2-1}\alpha_1\beta_1\exp\{-\beta_1 u^{\alpha_1}\}du}{1-\exp\{-\beta_1 s_i^{\alpha_1}\}}$$
$$= \frac{\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_1}\right)-\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_1},\beta_1 s_i^{\alpha_1}\right)}{(\beta_1)^{\alpha_2/\alpha_1}\Big(1-\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)},$$

$$\xi_3(t_i^*;\ \alpha_1,\beta_1) = E\Big[\ln(t_i^*)|t_i^*<s_i,k=1,\alpha_1,\beta_1\Big] = \frac{\int_0^{s_i} u^{\alpha_1-1}\ln(u)\alpha_1\beta_1\exp\{-\beta_1 u^{\alpha_1}\}du}{1-\exp\{-\beta_1 s_i^{\alpha_1}\}},$$

$$\xi_4(t_i^*;\ \alpha_1,\beta_1) = E\Big[t_i^{*\alpha_1}\ln(t_i^*)|t_i^*<s_i,k=1,\alpha_1,\beta_1\Big] = \frac{\int_0^{s_i} u^{2\alpha_1-1}\ln(u)\alpha_1\beta_1\exp\{-\beta_1 u^{\alpha_1}\}du}{1-\exp\{-\beta_1 s_i^{\alpha_1}\}}$$
$$= \frac{K_2(s_i)-\ln(\beta_1)\Big(1-(1+\beta_1 s_i^{\alpha_1})\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)}{\alpha_1\beta_1\Big(1-\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)},$$

$$\xi_5(t_i^*;\ \alpha_1,\beta_1,\alpha_2) = E\Big[t_i^{*\alpha_2}\ln(t_i^*)|t_i^*<s_i,k=1,\alpha_1,\beta_1\Big]$$
$$= \frac{\int_0^{s_i} u^{\alpha_1+\alpha_2-1}\ln(u)\alpha_1\beta_1\exp\{-\beta_1 u^{\alpha_1}\}du}{1-\exp\{-\beta_1 s_i^{\alpha_1}\}}$$
$$= \frac{K_3(s_i)-\ln(\beta_1)\left\{\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_1}\right)-\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_1},\beta_1 s_i^{\alpha_1}\right)\right\}}{\alpha_1(\beta_1)^{\alpha_2/\alpha_1}\Big(1-\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)},$$

$$\xi_6(u_i^*;\ \lambda_1) = E\Big[u_i^*|u_i^*<(s_i-t_i^*),k=1,\lambda_1\Big] = \frac{\int_0^{s_i-t_i^*} u\lambda_1\exp\{-\lambda_1 u\}}{1-\exp\{-\lambda_1(s_i-t_i^*)\}}$$
$$= \frac{1}{\lambda_1}\left[\frac{1-\{1+\lambda_1(s_i-t_i^*)\}\exp\{-\lambda_1(s_i-t_i^*)\}}{1-\exp\{-\lambda_1(s_i-t_i^*)\}}\right],$$

$$\xi_7(t_i^*;\ \alpha_1,\beta_1) = E\Big[t_i^{*\alpha_1}\Big(\ln(t_i^*)\Big)^2|t_i^*<s_i,k=1,\alpha_1,\beta_1\Big]$$
$$= \frac{\int_0^{s_i} u^{2\alpha_1-1}\Big(\ln(u)\Big)^2\alpha_1\beta_1\exp\{-\beta_1 u^{\alpha_1}\}du}{1-\exp\{-\beta_1 s_i^{\alpha_1}\}}$$
$$= \frac{K_4(s_i)+\Big(\ln(\beta_1)\Big)^2\Big(1-(1+\beta_1 s_i^{\alpha_1})\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)-2\ln(\beta_1)K_2(s_i)}{\alpha_1^2\beta_1\Big(1-\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)},$$

$$\xi_8(t_i^*;\ \alpha_1,\beta_1,\alpha_2) = E\Big[t_i^{*\alpha_2}\Big(\ln(t_i^*)\Big)^2|t_i^*<s_i,k=1,\alpha_1,\beta_1\Big]$$
$$= \frac{\int_0^{s_i} u^{\alpha_1+\alpha_2-1}\Big(\ln(u)\Big)^2\alpha_1\beta_1\exp\{-\beta_1 u^{\alpha_1}\}du}{1-\exp\{-\beta_1 s_i^{\alpha_1}\}}$$
$$= \frac{K_5(s_i)+\Big(\ln(\beta_1)\Big)^2\left\{\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_1}\right)-\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_1},\beta_1 s_i^{\alpha_1}\right)\right\}-2\ln(\beta_1)K_3(s_i)}{\alpha_1^2(\beta_1)^{\alpha_2/\alpha_1}\Big(1-\exp\{-\beta_1 s_i^{\alpha_1}\}\Big)},$$

$$\xi_9(t_i^*;\ \alpha_1,\alpha_2,\beta_2) = E\Big[t_i^{*\alpha_1}|t_i^* < s_i, k=2,\alpha_2,\beta_2\Big] = \frac{\int_0^{s_i} u^{\alpha_1+\alpha_2-1}\alpha_2\beta_2\exp\{-\beta_2 u^{\alpha_2}\}du}{1-\exp\{-\beta_2 s_i^{\alpha_2}\}}$$
$$= \frac{\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right) - \Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_2}, \beta_2 s_i^{\alpha_2}\right)}{(\beta_2)^{\alpha_1/\alpha_2}\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)},$$

$$\xi_{10}(t_i^*;\ \alpha_2,\beta_2) = E\Big[t_i^{*\alpha_2}|t_i^* < s_i, k=2,\alpha_2,\beta_2\Big] = \frac{\int_0^{s_i} u^{2\alpha_2-1}\alpha_2\beta_2\exp\{-\beta_2 u^{\alpha_2}\}du}{1-\exp\{-\beta_2 s_i^{\alpha_2}\}}$$
$$= \frac{1-(1+\beta_2 s_i^{\alpha_2})\exp\{-\beta_2 s_i^{\alpha_2}\}}{\beta_2\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)},$$

$$\xi_{11}(t_i^*;\ \alpha_2,\beta_2) = E\Big[\ln(t_i^*)|t_i^* < s_i, k=2,\alpha_2,\beta_2\Big] = \frac{\int_0^{s_i} u^{\alpha_2-1}\ln(u)\alpha_2\beta_2\exp\{-\beta_2 u^{\alpha_2}\}du}{1-\exp\{-\beta_2 s_i^{\alpha_2}\}}$$
$$= \frac{K_6(s_i) - \ln(\beta_2)\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)}{\alpha_2\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)},$$

$$\xi_{12}(t_i^*;\ \alpha_1,\alpha_2,\beta_2) = E\Big[t_i^{*\alpha_1}\ln(t_i^*)|t_i^* < s_i, k=2,\alpha_2,\beta_2\Big] = \frac{\int_0^{s_i} u^{\alpha_1+\alpha_2-1}\ln(u)\alpha_2\beta_2\exp\{-\beta_2 u^{\alpha_2}\}du}{1-\exp\{-\beta_2 s_i^{\alpha_2}\}}$$
$$= \frac{K_7(s_i) - \ln(\beta_2)\left\{\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right) - \Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_2}, \beta_2 s_i^{\alpha_2}\right)\right\}}{\alpha_2(\beta_2)^{\alpha_1/\alpha_2}\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)},$$

$$\xi_{13}(t_i^*;\ \alpha_2,\beta_2) = E\Big[t_i^{*\alpha_2}\ln(t_i^*)|t_i^* < s_i, k=2,\alpha_2,\beta_2\Big] = \frac{\int_0^{s_i} u^{2\alpha_2-1}\ln(u)\alpha_2\beta_2\exp\{-\beta_2 u^{\alpha_2}\}du}{1-\exp\{-\beta_2 s_i^{\alpha_2}\}}$$
$$= \frac{K_8(s_i) - \ln(\beta_2)\Big(1-(1+\beta_2 s_i^{\alpha_2})\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)}{\alpha_2\beta_2\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)},$$

$$\xi_{14}(v_i^*;\ \lambda_2) = E\Big[v_i^*|v_i^* < (s_i - t_i^*), k=2,\lambda_2\Big] = \frac{\int_0^{s_i-t_i^*} u\lambda_2\exp\{-\lambda_2 u\}}{1-\exp\{-\lambda_2(s_i-t_i^*)\}}$$
$$= \frac{1}{\lambda_2}\left[\frac{1-\Big(1+\lambda_2(s_i-t_i^*)\Big)\exp\{-\lambda_2(s_i-t_i^*)\}}{1-\exp\{-\lambda_2(s_i-t_i^*)\}}\right],$$

$$\xi_{15}(t_i^*;\ \alpha_1,\alpha_2,\beta_2) = E\Big[t_i^{*\alpha_1}\Big(\ln(t_i^*)\Big)^2|t_i^* < s_i, k=2,\alpha_2,\beta_2\Big]$$
$$= \frac{\int_0^{s_i} u^{\alpha_1+\alpha_2-1}\Big(\ln(u)\Big)^2\alpha_2\beta_2\exp\{-\beta_2 u^{\alpha_2}\}du}{1-\exp\{-\beta_2 s_i^{\alpha_2}\}}$$
$$= \frac{K_9(s_i) + \Big(\ln(\beta_2)\Big)^2\left\{\Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right) - \Gamma\left(\frac{\alpha_1+\alpha_2}{\alpha_2}, \beta_2 s_i^{\alpha_2}\right)\right\} - 2\ln(\beta_2)K_7(s_i)}{\alpha_2^2(\beta_2)^{\alpha_1/\alpha_2}\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)},$$

$$\xi_{16}(t_i^*;\ \alpha_2,\beta_2) = E\Big[t_i^{*\alpha_2}\Big(\ln(t_i^*)\Big)^2|t_i^* < s_i, k=2,\alpha_2,\beta_2\Big]$$
$$= \frac{\int_0^{s_i} u^{2\alpha_2-1}\Big(\ln(u)\Big)^2\alpha_2\beta_2\exp\{-\beta_2 u^{\alpha_2}\}du}{1-\exp\{-\beta_2 s_i^{\alpha_2}\}}$$
$$= \frac{K_{10}(s_i) + \Big(\ln(\beta_2)\Big)^2\Big(1-(1+\beta_2 s_i^{\alpha_2})\exp\{-\beta_2 s_i^{\alpha_2}\}\Big) - 2\ln(\beta_2)K_8(s_i)}{\alpha_2^2\beta_2\Big(1-\exp\{-\beta_2 s_i^{\alpha_2}\}\Big)},$$

where,

$$
\begin{aligned}
K_1(y) &= \int_0^{\beta_1 y^{\alpha_1}} \ln(y)\exp\{-y\}dy, \quad K_2(y) = \int_0^{\beta_1 y^{\alpha_1}} y\ln(y)\exp\{-y\}dy,\\
K_3(y) &= \int_0^{\beta_1 y^{\alpha_1}} y^{\alpha_2/\alpha_1}\ln(y)\exp\{-y\}dz, \quad K_4(y) = \int_0^{\beta_1 y^{\alpha_1}} y\Big(\ln(y)\Big)^2\exp\{-y\}dy\\
K_5(z) &= \int_0^{\beta_1 y^{\alpha_1}} y^{\alpha_2/\alpha_1}\Big(\ln(y)\Big)^2\exp\{-y\}dy, \quad K_6(y) = \int_0^{\beta_2 y^{\alpha_2}} \ln(y)\exp\{-y\}dy,\\
K_7(y) &= \int_0^{\beta_2 y^{\alpha_2}} y^{\alpha_1/\alpha_2}\ln(y)\exp\{-y\}dy, \quad K_8(y) = \int_0^{\beta_2 y^{\alpha_2}} y\ln(y)\exp\{-y\}dy\\
K_9(y) &= \int_0^{\beta_2 y^{\alpha_2}} y^{\alpha_1/\alpha_2}\Big(\ln(y)\Big)^2\exp\{-y\}dy, \quad K_{10}(y) = \int_0^{\beta_2 y^{\alpha_2}} y\Big(\ln(y)\Big)^2\exp\{-y\}dy.
\end{aligned}
$$

After updating the missing data with help of expectation terms calculated in the E-step, the log-likelihood is maximized in the M-step and let at the m^{th} iteration $\hat{\Theta}^{(m)}$ be the vector of unknown parameter's estimate of Θ. Then the updated value of parameters at $(m+1)^{th}$ iteration can be computed by using the following expressions

$$\hat{\alpha}_1^{(m+1)} = \frac{\Big(n_{r_1} + P^{(m)}\Big)}{A} \tag{1.10}$$

$$\hat{\alpha}_2^{(m+1)} = \frac{\Big(n_{r_2} + n_{nr} - n_r - P^{(m)}\Big)}{B} \tag{1.11}$$

$$\hat{\beta}_1^{(m+1)} = \frac{\Big(n_{r_1} + P^{(m)}\Big)}{\left[\begin{array}{l} \sum_{i=1}^{n_{r_1}} t_i^{\hat{\alpha}_1^{(m+1)}} + \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\hat{\alpha}_1^{(m+1)}} + \sum_{i=n_r+1}^{n_{nr}} P_i^{(m)}\xi_1\Big(t_i^*;\ \alpha_1{}^{(m)}, \beta_1{}^{(m)}\Big) \\ \quad + \sum_{i=n_r+1}^{n_{nr}}\Big(1-P_i^{(m)}\Big)\xi_9\Big(t_i^*;\ \alpha_1{}^{(m)}, \alpha_2{}^{(m)}, \beta_2{}^{(m)}\Big) + \sum_{i=n_{nr}+1}^{n} s_i^{\hat{\alpha}_1^{(m+1)}} \end{array}\right]} \tag{1.12}$$

$$\hat{\lambda}_1^{(m+1)} = \frac{P^{(m)}}{\sum_{i=1}^{n_{r_1}}(s_i - t_i) + \sum_{i=n_r+1}^{n_{nr}} P_i^{(m)}\xi_6\Big(u_i^*;\ \lambda_1{}^{(m)}\Big)} \tag{1.13}$$

$$\hat{\beta}_2^{(m+1)} = \frac{\Big(n_{r_2} + n_{nr} - n_r - P^{(m)}\Big)}{\left[\begin{array}{r} \sum_{i=1}^{n_{r_1}} t_i^{\hat{\alpha}_2^{(m+1)}} + \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\hat{\alpha}_2^{(m+1)}} + \sum_{i=n_r+1}^{n_{nr}} P_i^{(m)}\xi_2\Big(t_i^*;\ \alpha_1{}^{(m)}, \beta_1{}^{(m)}, \alpha_2{}^{(m)}\Big) \\ + \sum_{i=n_r+1}^{n_{nr}}\Big(1-P_i^{(m)}\Big)\xi_{10}\Big(t_i^*;\ \alpha_2{}^{(m)}, \beta_2{}^{(m)}\Big) + \sum_{i=n_{nr}+1}^{n} s_i^{\hat{\alpha}_2^{(m+1)}} \end{array}\right]} \tag{1.14}$$

$$\hat{\lambda}_2^{(m+1)} = \frac{\Big(n_{nr} - n_r - P^{(m)}\Big)}{\sum_{i=n_{r_1}+1}^{n_{r_2}}(s_i - t_i) + \sum_{i=n_r+1}^{n_{nr}}\Big(1-P_i^{(m)}\Big)\xi_{14}\Big(v_i^*;\ \lambda_2{}^{(m)}\Big)} \tag{1.15}$$

where,

$$A = -\sum_{i=1}^{n_{r_1}} \ln(t_i) + \hat{\beta}_1^{(m)} \sum_{i=1}^{n_{r_1}} t_i^{\hat{\alpha}_1^{(m)}} \ln(t_i) + \hat{\beta}_1^{(m)} \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\hat{\alpha}_1^{(m)}} \ln(t_i) - \sum_{i=n_r+1}^{n_{nr}} P_i^{(m)} \xi_3\Big(t_i^*;\ \alpha_1{}^{(m)}, \beta_1{}^{(m)}\Big)$$
$$+ \hat{\beta}_1^{(m)} \sum_{i=n_r+1}^{n_{nr}} P_i^{(m)} \xi_4\Big(t_i^*;\ \alpha_1{}^{(m)}, \beta_1{}^{(m)}\Big) + \hat{\beta}_1^{(m)} \sum_{i=n_r+1}^{n_{nr}} \Big(1 - P_i^{(m)}\Big) \xi_{12}\Big(t_i^*;\ \alpha_1{}^{(m)}, \alpha_2{}^{(m)}, \beta_2{}^{(m)}\Big)$$
$$+ \hat{\beta}_1^{(m)} \sum_{i=n_{nr}+1}^{n} s_i^{\hat{\alpha}_1^{(m)}} \ln(s_i);$$
$$B = -\sum_{i=n_{r_1}+1}^{n_{r_2}} \ln(t_i) + \hat{\beta}_2^{(m)} \sum_{i=1}^{n_{r_1}} t_i^{\hat{\alpha}_2^{(m)}} \ln(t_i) + \hat{\beta}_2^{(m)} \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\hat{\alpha}_2^{(m)}} \ln(t_i) + \hat{\beta}_2^{(m)} \sum_{i=n_{nr}+1}^{n} s_i^{\hat{\alpha}_2^{(m)}} \ln(s_i)$$
$$+ \hat{\beta}_2^{(m)} \sum_{i=n_r+1}^{n_{nr}} P_i^{(m)} \xi_5\Big(t_i^*;\ \alpha_1{}^{(m)}, \alpha_2{}^{(m)}, \beta_2{}^{(m)}\Big) + \hat{\beta}_2^{(m)} \sum_{i=n_r+1}^{n_{nr}} (1 - P_i^{(m)}) \xi_{13}\Big(t_i^*;\ \alpha_2{}^{(m)}, \beta_2{}^{(m)}\Big)$$
$$- \sum_{i=n_r+1}^{n_{nr}} \Big(1 - P_i^{(m)}\Big) \xi_{11}\Big(t_i^*;\ \alpha_2{}^{(m)}, \beta_2{}^{(m)}\Big)$$

1.2.2 Observed Fisher Information Matrix

The terms used for the construction of observed Fisher information matrix are given below

$$\frac{\partial^2 Q}{\partial \alpha_1^2} = -\frac{(n_{r_1} + P)}{\alpha_1^2} - \beta_1 \sum_{i=1}^{n_{r_1}} t_i^{\alpha_1} \Big(\ln(t_i)\Big)^2 - \beta_1 \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_1} \Big(\ln(t_i)\Big)^2 - \beta_1 \sum_{i=n_r+1}^{n_{nr}} P_i \xi_7\Big(t_i^*;\ \alpha_1, \beta_1\Big)$$
$$- \beta_1 \sum_{i=n_r+1}^{n_{nr}} (1 - P_i) \xi_{15}\Big(t_i^*;\ \alpha_1, \alpha_2, \beta_2\Big) - \beta_1 \sum_{i=n_{nr}+1}^{n} s_i^{\alpha_1} \Big(\ln(s_i)\Big)^2,$$
$$\frac{\partial^2 Q}{\partial \beta_1^2} = -\frac{\Big(n_{r_1} + P\Big)}{\beta_1^2}; \quad \frac{\partial^2 Q}{\partial \beta_2^2} = -\frac{(n_{r_2} + n_{nr} - n_r - P)}{\beta_2^2},$$
$$\frac{\partial^2 Q}{\partial \alpha_2^2} = -\frac{\Big(n_{r_2} + n_{nr} - n_r - P\Big)}{\alpha_2^2} - \beta_2 \sum_{i=1}^{n_{r_1}} t_i^{\alpha_2} \Big(\ln(t_i)\Big)^2 - \beta_2 \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2} \Big(\ln(t_i)\Big)^2$$
$$- \beta_2 \sum_{i=n_r+1}^{n_{nr}} P_i \xi_8\Big(t_i^*;\ \alpha_1, \beta_1, \alpha_2\Big) - \beta_2 \sum_{i=n_r+1}^{n_{nr}} (1 - P_i) \xi_{16}\Big(t_i^*;\ \alpha_2, \beta_2\Big) - \beta_2 \sum_{i=n_{nr}+1}^{n} s_i^{\alpha_2} \Big(\ln(s_i)\Big)^2,$$
$$\frac{\partial^2 Q}{\partial \lambda_1^2} = -\frac{P}{\lambda_1^2}; \quad \frac{\partial^2 Q}{\partial \lambda_2^2} = -\frac{\Big(n_{nr} - n_r - P\Big)}{\lambda_2^2},$$

$$\frac{\partial^2 Q}{\partial \beta_1 \partial \alpha_1} = -\sum_{i=1}^{n_{r_1}} t_i^{\alpha_1} \ln(t_i) - \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_1} \ln(t_i) - \sum_{i=n_r+1}^{n_{nr}} P_i \xi_4\Big(t_i^*;\ \alpha_1, \beta_1\Big) - \sum_{i=n_{nr}+1}^{n} s_i^{\alpha_1} \ln(s_i)$$
$$- \sum_{i=n_r+1}^{n_{nr}} \Big(1 - P_i\Big) \xi_{12}\Big(t_i^*;\ \alpha_1, \alpha_2, \beta_2\Big) = \frac{\partial^2 Q}{\partial \alpha_1 \partial \beta_1},$$
$$\frac{\partial^2 Q}{\partial \beta_2 \partial \alpha_2} = -\sum_{i=1}^{n_{r_1}} t_i^{\alpha_2} \ln(t_i) - \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2} \ln(t_i) - \sum_{i=n_r+1}^{n_{nr}} P_i \xi_5\Big(t_i^*;\ \alpha_1, \beta_1, \alpha_2\Big) - \sum_{i=n_{nr}+1}^{n} s_i^{\alpha_2} \ln(s_i)$$
$$- \sum_{i=n_r+1}^{n_{nr}} \Big(1 - P_i\Big) \xi_{13}\Big(t_i^*;\ \alpha_2, \beta_2\Big) = \frac{\partial^2 Q}{\partial \alpha_2 \partial \beta_2}.$$

1.3 Bayesian Inference

All the inferences in the Bayesian paradigm are drawn from the posterior distribution. The likelihood in (1.2) indicates that it is difficult to study the properties of posterior analytically. So, we use likelihood given in (1.6) in this section for further study.

1.3.1 Prior Distribution

Consider $\alpha_k,\ \beta_k,\ \lambda_k;\ \ (k = 1, 2)$ as Gamma distributed with parameters $(a_l, b_l);\ \ l = 1, 2, \ldots, 6$. Assuming the independence of priors the joint prior density is written up to proportionality constants as below

$$\pi(\Theta) \propto {\alpha_1}^{a_1-1} {\beta_1}^{a_2-1} {\lambda_1}^{a_3-1} {\alpha_2}^{a_4-1} {\beta_2}^{a_5-1} {\lambda_2}^{a_6-1} \exp\left\{-(\alpha_1 b_1 + \beta_1 b_2 + \lambda_1 b_3 + \alpha_2 b_4 + \beta_2 b_5 + \lambda_2 b_6)\right\} \tag{1.16}$$

1.3.2 Gibbs Sampling

For applying the Gibbs Sampling the posterior distribution is required which can be obtained by merging complete likelihood (1.6) and the joint prior (1.16). The posterior distribution up-to proportionality constant is written below

$$\begin{aligned}
\Pi(\Theta|d,d^*) &\propto L_c(\Theta|d,d^*)\pi(\Theta)\\
&\propto \alpha_1^{(n_{r_1}+Z^*+a_1-1)}\beta_1^{(n_{r_1}+Z^*+a_2-1)}\lambda_1^{(Z^*+a_3-1)}\alpha_2^{(n_{r_2}+n_{nr}-n_r-Z^*+a_4-1)}\beta_2^{(n_{r_2}+n_{nr}-n_r-Z^*+a_5-1)}\\
&\lambda_2^{(n_{nr}-n_r-Z^*+a_6-1)}\prod_{i=1}^{n_{r_1}} t_i^{\alpha_1-1}\exp\{-\beta_1 t_i^{\alpha_1}-\beta_2 t_i^{\alpha_2}-\lambda_1(s_i-t_i)\}\prod_{i=n_{nr}+1}^{n}\exp\{-\beta_1 s_i^{\alpha_1}-\beta_2 s_i^{\alpha_2}\}\\
&\prod_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2-1}\exp\{-\beta_1 t_i^{\alpha_1}-\beta_2 t_i^{\alpha_2}-\lambda_2(s_i-t_i)\}\prod_{i=n_r+1}^{n_{nr}}\left[t_i^{*\alpha_1-1}\exp\{-\beta_1 t_i^{*\alpha_1}-\beta_2 t_i^{*\alpha_2}-\lambda_1 u_i^*\}\right]^{z_i}\\
&\prod_{i=n_r+1}^{n_{nr}}\left[t_i^{*\alpha_2-1}\exp\{-\beta_1 t_i^{*\alpha_1}-\beta_2 t_i^{*\alpha_2}-\lambda_2 v_i^*\}\right]^{1-z_i}\\
&\exp\{-\alpha_1 b_1-\beta_1 b_2-\lambda_1 b_3-\alpha_2 b_4-\beta_2 b_5-\lambda_2 b_6\} \qquad (1.17)
\end{aligned}$$

In order to draw the samples from posterior, we write the full conditionals for parameters based on the posterior distribution (1.17) which are given below

$$\begin{aligned}
\alpha_1|\Theta_{(-\alpha_1)} &\propto \alpha_1^{(n_{r_1}+Z^*+a_1-1)}\left(\prod_{i=1}^{n_{r_1}} t_i^{\alpha_1-1}\right)\left(\prod_{i=n_r+1}^{n_{nr}} t_i^{*z_i(\alpha_1-1)}\right)\\
&\exp\left(-\beta_1\sum_{i=1}^{n_{r_1}} t_i^{\alpha_1}-\beta_1\sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_1}-\beta_1\sum_{i=n_r+1}^{n_{nr}} z_i t_i^{*\alpha_1}\right)\\
&\exp\left(-\beta_1\sum_{i=n_r+1}^{n_{nr}}(1-z_i)t_i^{*\alpha_1}-\beta_1\sum_{i=n_{nr}+1}^{n} s_i^{\alpha_1}-\alpha_1 b_1\right), \qquad (1.18)\\
\alpha_2|\Theta_{(-\alpha_2)} &\propto \alpha_2^{(n_{r_2}+n_{nr}-n_r-Z^*+a_4-1)}\left(\prod_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2-1}\right)\left(\prod_{i=n_r+1}^{n_{nr}} t_i^{*(1-z_i)(\alpha_2-1)}\right)\\
&exp\left(-\beta_2\sum_{i=1}^{n_{r_1}} t_i^{\alpha_2}-\beta_2\sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2}\right)\\
&\exp\left(-\beta_2\sum_{i=n_r+1}^{n_{nr}} z_i t_i^{*\alpha_2}-\beta_2\sum_{i=n_r+1}^{n_{nr}}(1-z_i)t_i^{*\alpha_2}-\beta_2\sum_{i=n_{nr}+1}^{n} s_i^{\alpha_2}-\alpha_2 b_4\right),\\
&\qquad (1.19)
\end{aligned}$$

$$\beta_1|\Theta_{(-\beta_1)} \sim \mathcal{G}\left(n_{r_1} + Z^* + a_2, \sum_{i=1}^{n_{r_1}} t_i^{\alpha_1} + \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_1} + \sum_{i=n_r+1}^{n_{nr}} z_i t_i^{*\alpha_1} + \sum_{i=n_r+1}^{n_{nr}} (1-z_i) t_i^{*\alpha_1} + \sum_{i=n_{nr}+1}^{n} s_i^{\alpha_1} + b_2\right) \tag{1.20}$$

$$\lambda_1|\Theta_{(-\lambda_1)} \sim \mathcal{G}\left(Z^* + a_3, \sum_{i=1}^{n_{r_1}} (s_i - t_i) + \sum_{i=n_r+1}^{n_{nr}} z_i u_i^* + b_3\right), \tag{1.21}$$

$$\beta_2|\Theta_{(-\beta_2)} \sim \mathcal{G}\left(\tilde{Z}^*, \sum_{i=1}^{n_{r_1}} t_i^{\alpha_2} + \sum_{i=n_{r_1}+1}^{n_{r_2}} t_i^{\alpha_2} + \sum_{i=n_r+1}^{n_{nr}} z_i t_i^{*\alpha_2} + \sum_{i=n_r+1}^{n_{nr}} (1-z_i) t_i^{*\alpha_2} + \sum_{i=n_{nr}+1}^{n} s_i^{\alpha_2} + b_5\right), \tag{1.22}$$

$$\lambda_2|\Theta_{(-\lambda_2)} \sim \mathcal{G}\left(n_{nr} - n_r - Z^* + a_6, \sum_{i=n_{r_1}+1}^{n_{r_2}} (s_i - t_i) + \sum_{i=n_r+1}^{n_{nr}} (1-z_i) v_i^* + b_6\right). \tag{1.23}$$

Here, $\tilde{Z}^* = (n_{r_2} + n_{nr} - n_r - Z^* + a_5)$. From equations (1.18) to (1.23), we can see that the full conditionals of α_1 and α_2 are not in standard density form. So, samples from these full conditionals are generated by using Metropolis-Hasting (M-H) algorithm taking Normal distribution as proposal density. Further full conditionals of β_1, λ_1, β_2 and λ_2 are following Gamma distribution with varying shapes and scales and observations from these can be drawn directly for given values of latent variables z, t^*, u^* and v^* and hyper-parameters.

1.3.3 Data Augmentation Algorithm

The algorithm used for generating samples from full conditionals using nested Gibbs sampling is given below:

- **Step 1**: Set initial values of parameters, say $\Theta^{(0)} = \left(\alpha_1^{(0)}, \beta_1^{(0)}, \lambda_1^{(0)}, \alpha_2^{(0)}, \beta_2^{(0)}, \lambda_2^{(0)}\right)$ and generate $z_i \sim \mathcal{B}(1, P_i)$; $\quad i = n_r + 1, n_r + 2, \cdots, n_{nr}$.

- **Step 2**: For given values of z_i and Θ, generate observations on t_i^* using expression

$$t_i^* = \left[-\frac{1}{\beta_k} \ln \left\{ 1 - w_i \left(1 - \exp\{-\beta_k s_i^{\alpha_k}\} \right) \right\} \right]^{1/\alpha_k}; \quad i = n_r + 1, n_r + 2, \ldots, n_{nr}, \ k = 1, 2, \tag{1.24}$$

where $w \sim \mathcal{U}(0,1)$.

- **Step 3**: Based on generated values of z_i, t_i^* and Θ from previous steps, observations on u_i^* and v_i^* can be generated by using expression

$$y_i^* = -\frac{1}{\lambda_k} \ln \left[1 - w_i \Big(1 - \exp\{-\lambda_k (s_i - t_i^*)\} \Big) \right]; \quad i = n_r + 1, n_r + 2, \ldots, n_{nr}, \ k = 1, 2. \tag{1.25}$$

- **Step 4**: For given values of z_i, t_i^*, u_i^* and v_i^* generate observations on $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ using M-H algorithm taking normal as proposal density.

- **Step 5**: In the next step observations on $\beta_1^{(1)}$, $\lambda_1^{(1)}$, $\beta_2^{(1)}$ and $\lambda_2^{(1)}$ can be generated for given $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ using (1.20) to (1.23) respectively.

Now, the current state is $\Theta^{(1)} = \left(\alpha_1^{(1)}, \beta_1^{(1)}, \lambda_1^{(1)}, \alpha_2^{(1)}, \beta_2^{(1)}, \lambda_2^{(1)} \right)$. Steps 1 to 5 are replicated M times to obtain a sequence of random variables $\left(\Theta^{(1)}, \Theta^{(2)}, \cdots, \Theta^{(M)} \right)$. After discarding the burn-in from generated chains, the stationarity of the chains are checked by trace plot and Gelman and Rubin's test statistics, we left out with a reduced chain of length M'. All the Bayesian inferences are drawn based on these samples.

2 Posterior Plots for Menopause data

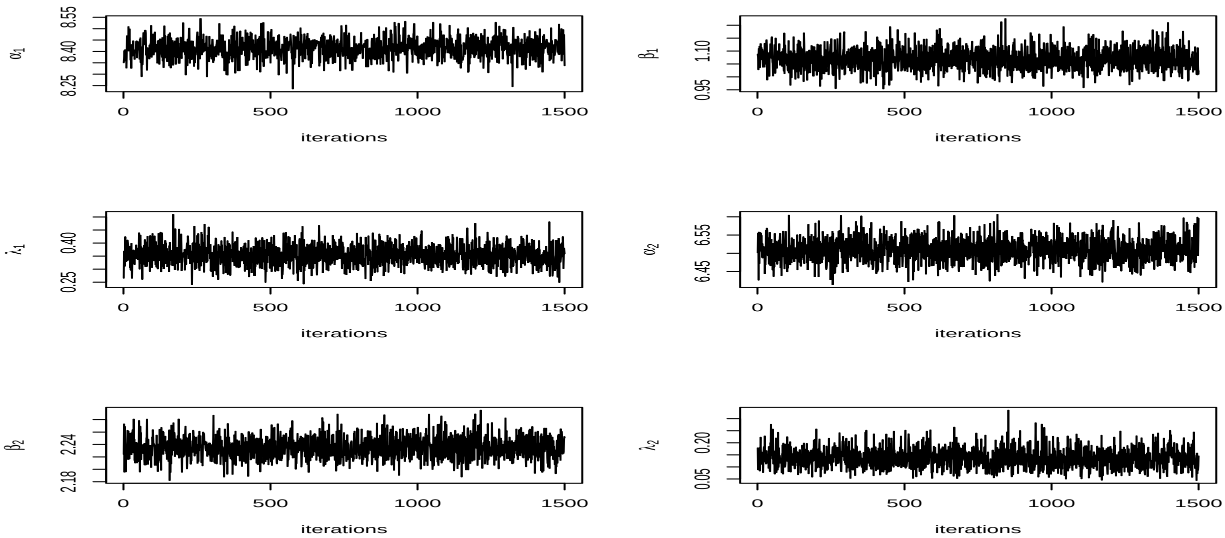

Figure 1: MCMC trace plots for posterior samples when causes are known for non-recall.

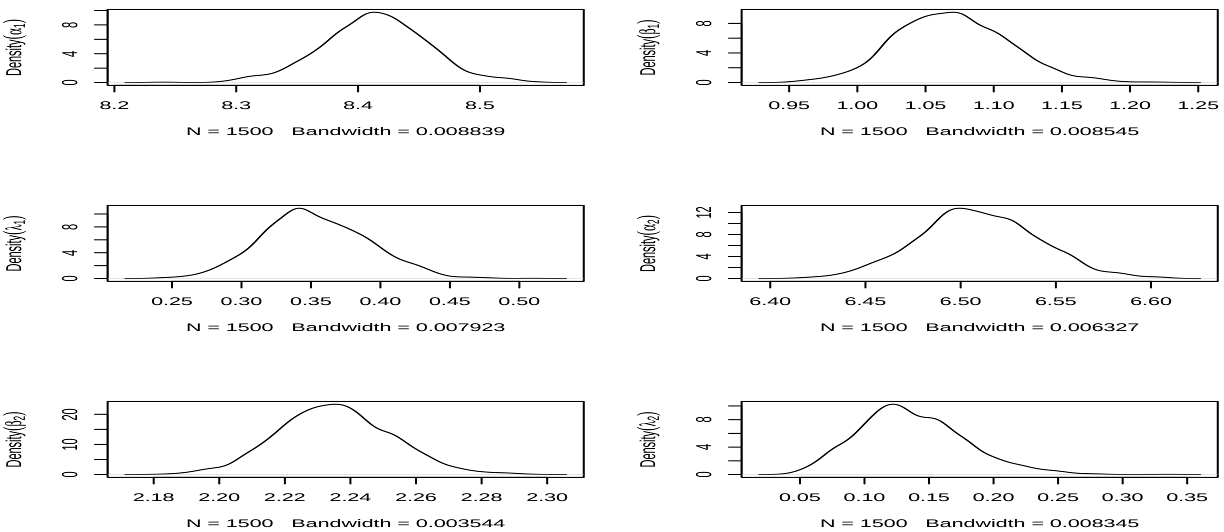

Figure 2: MCMC density plots for posterior samples when causes are known for non-recall.

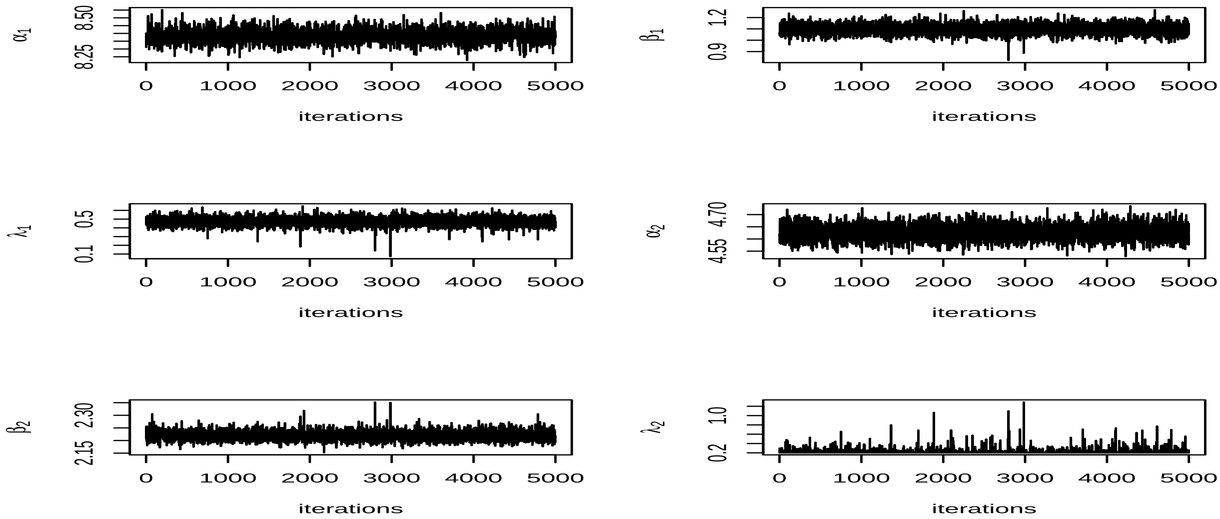

Figure 3: MCMC trace plots for posterior samples when causes are unknown for non-recall.

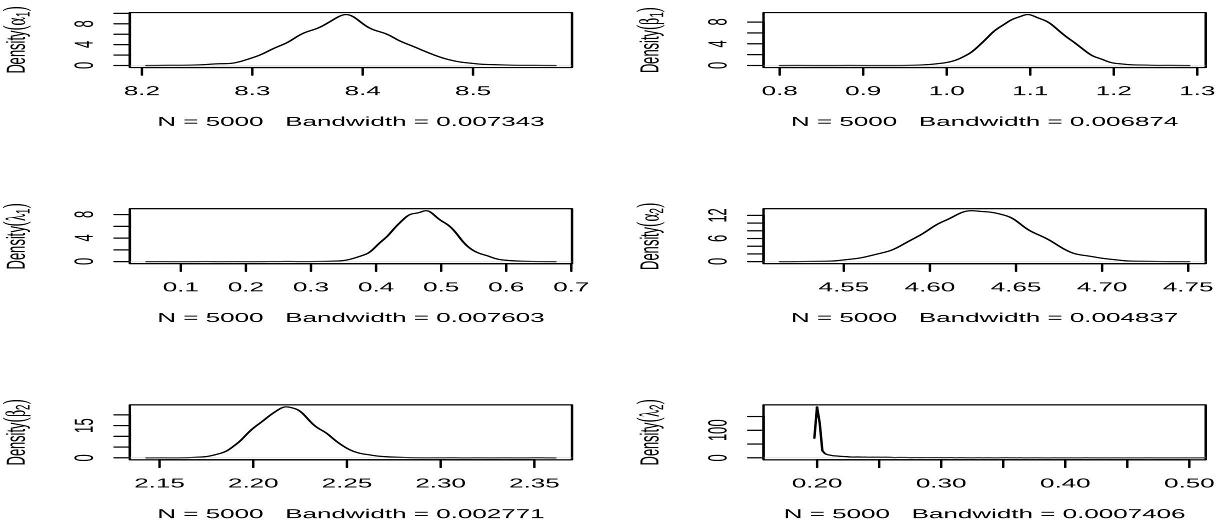

Figure 4: MCMC density plots for posterior samples when causes are unknown for non-recall.

3 Simulation Tables

Table 1: Mean square error and absolute bias for point estimates under uniform monitoring points for varying sample sizes.

			ML			*Bayes*		
$(\lambda_1,\ \lambda_2)$		$n \longrightarrow$	80	150	250	80	150	250
(0.10, 0.12)	$\hat{\alpha}_1$	MSE	0.0177	0.0121	0.0073	0.0160	0.0111	0.0052
		AB	0.1048	0.0881	0.0690	0.0916	0.0707	0.0503
	$\hat{\beta}_1$	MSE	0.0717	0.0441	0.0390	0.0414	0.0229	0.0214
		AB	0.2258	0.1651	0.1616	0.1745	0.1250	0.1238
	$\hat{\lambda}_1$	MSE	0.0204	0.0117	0.0084	0.0191	0.0101	0.0075
		AB	0.1128	0.0928	0.0760	0.0955	0.0761	0.0558
	$\hat{\alpha}_2$	MSE	0.0174	0.0083	0.0073	0.0109	0.0061	0.0054
		AB	0.1008	0.0710	0.0648	0.0917	0.0607	0.0404
	$\hat{\beta}_2$	MSE	0.0909	0.0488	0.0465	0.0756	0.0261	0.0242
		AB	0.2513	0.1830	0.1789	0.1893	0.1324	0.1305
	$\hat{\lambda}_2$	MSE	0.0235	0.0112	0.0070	0.0204	0.0101	0.0050
		AB	0.1289	0.0913	0.0716	0.0915	0.0882	0.0573
(0.17, 0.18)	$\hat{\alpha}_1$	MSE	0.0204	0.0126	0.0076	0.0187	0.0121	0.0053
		AB	0.1145	0.0889	0.0696	0.0919	0.0809	0.0604
	$\hat{\beta}_1$	MSE	0.0770	0.0502	0.0451	0.0535	0.0250	0.0242
		AB	0.2319	0.1856	0.1740	0.1953	0.1372	0.1303
	$\hat{\lambda}_1$	MSE	0.0322	0.0179	0.0123	0.0202	0.0125	0.0102
		AB	0.1510	0.1095	0.0918	0.1090	0.0870	0.0672
	$\hat{\alpha}_2$	MSE	0.0205	0.0103	0.0086	0.0190	0.0101	0.0068
		AB	0.1058	0.0824	0.0654	0.0921	0.0608	0.0505
	$\hat{\beta}_2$	MSE	0.0965	0.0590	0.0484	0.0574	0.0340	0.0260
		AB	0.2610	0.1962	0.1869	0.1967	0.1479	0.1352
	$\hat{\lambda}_2$	MSE	0.0289	0.0183	0.0111	0.0153	0.0120	0.0101
		AB	0.1427	0.1186	0.0874	0.0939	0.1034	0.0684

Table 2: Mean square error and absolute bias for point estimates under exponential monitoring points for varying sample sizes.

			ML			$Bayes$		
$(\lambda_1,\ \lambda_2)$		$n \longrightarrow$	80	150	250	80	150	250
(0.10, 0.12)	$\hat{\alpha}_1$	MSE	0.0216	0.0118	0.0094	0.0208	0.0102	0.0054
		AB	0.1191	0.0923	0.0816	0.1011	0.0805	0.0702
	$\hat{\beta}_1$	MSE	0.1292	0.0800	0.0710	0.0743	0.0570	0.0573
		AB	0.3249	0.2540	0.2422	0.2341	0.1857	0.1685
	$\hat{\lambda}_1$	MSE	0.0047	0.0032	0.0017	0.0041	0.0030	0.0010
		AB	0.0577	0.0454	0.0322	0.0466	0.0264	0.0222
	$\hat{\alpha}_2$	MSE	0.0212	0.0106	0.0096	0.0182	0.0087	0.0078
		AB	0.1058	0.0845	0.0839	0.0911	0.0705	0.0702
	$\hat{\beta}_2$	MSE	0.1001	0.0673	0.0671	0.0793	0.0369	0.0346
		AB	0.2797	0.2288	0.2039	0.2136	0.1668	0.1590
	$\hat{\lambda}_2$	MSE	0.0063	0.0039	0.0027	0.0030	0.0028	0.0019
		AB	0.0643	0.0472	0.0395	0.0348	0.0313	0.0300
(0.17, 0.18)	$\hat{\alpha}_1$	MSE	0.0218	0.0131	0.0099	0.0215	0.0120	0.0073
		AB	0.1207	0.0944	0.0827	0.0102	0.0813	0.0704
	$\hat{\beta}_1$	MSE	0.1298	0.1288	0.0963	0.0751	0.0678	0.0591
		AB	0.3320	0.3033	0.2879	0.2351	0.2311	0.1932
	$\hat{\lambda}_1$	MSE	0.0094	0.0052	0.0029	0.0059	0.0037	0.0028
		AB	0.0725	0.0567	0.0428	0.0471	0.0460	0.0409
	$\hat{\alpha}_2$	MSE	0.0216	0.0126	0.0107	0.0210	0.0091	0.0086
		AB	0.1066	0.0954	0.0891	0.0914	0.0906	0.0803
	$\hat{\beta}_2$	MSE	0.1385	0.1251	0.1237	0.0905	0.0636	0.0525
		AB	0.3352	0.3287	0.3140	0.2300	0.2197	0.2034
	$\hat{\lambda}_2$	MSE	0.0111	0.0066	0.0030	0.0072	0.0048	0.0026
		AB	0.0815	0.0635	0.0451	0.0542	0.0526	0.0428

Table 3: Average length, shape and coverage probability for interval estimates under uniform monitoring points for varying sample sizes.

			ACI			*HPD*		
$(\lambda_1,\ \lambda_2)$		$n \longrightarrow$	80	150	250	80	150	250
(0.10, 0.12)		*AL*	0.5069	0.3749	0.2818	0.4863	0.3350	0.2603
	$\hat{\alpha}_1$	*Shape*	1.0000	1.0000	1.0000	1.0092	1.0176	1.0068
		CP	0.9533	0.9067	0.9200	1.0000	1.0000	1.0000
		AL	0.6731	0.5357	0.3920	0.6589	0.5204	0.3871
	$\hat{\beta}_1$	*Shape*	1.0000	1.0000	1.0000	1.0741	1.0531	1.0242
		CP	0.8733	0.8780	0.8640	0.9600	0.9733	0.9500
		AL	0.2066	0.1424	0.1106	0.2021	0.1298	0.1051
	$\hat{\lambda}_1$	*Shape*	1.0000	1.0000	1.0000	1.1080	1.0692	1.0623
		CP	0.8460	0.8360	0.8390	1.0000	1.0000	1.0000
		AL	0.5029	0.3805	0.2881	0.4850	0.3769	0.2612
	$\hat{\alpha}_2$	*Shape*	1.0000	1.0000	1.0000	1.0899	1.0549	1.0132
		CP	0.9333	0.9467	0.9400	1.0000	1.0000	1.0000
		AL	0.6902	0.5318	0.3931	0.6732	0.5150	0.3852
	$\hat{\beta}_2$	*Shape*	1.0000	1.0000	1.0000	1.0926	1.0479	1.0433
		CP	0.8707	0.8747	0.8560	0.9000	0.9533	0.9900
		AL	0.2507	0.1434	0.1083	0.2448	0.1225	0.1055
	$\hat{\lambda}_2$	*Shape*	1.0000	1.0000	1.0000	1.0462	1.0401	1.0168
		CP	0.8500	0.8467	0.8420	0.9933	1.0000	0.9900
(0.17, 0.18)		*AL*	0.5328	0.3771	0.2944	0.5029	0.3564	0.2720
	$\hat{\alpha}_1$	*Shape*	1.0000	1.0000	1.0000	1.0996	1.0699	1.0363
		CP	0.9500	0.9000	0.9300	1.0000	1.0000	1.0000
		AL	0.7362	0.5427	0.3959	0.7173	0.5361	0.3915
	$\hat{\beta}_1$	*Shape*	1.0000	1.0000	1.0000	1.0908	1.0804	1.0574
		CP	0.8000	0.8710	0.8610	0.9600	0.9500	0.9500
		AL	0.2806	0.1888	0.1483	0.2749	0.1793	0.1280
	$\hat{\lambda}_1$	*Shape*	1.0000	1.0000	1.0000	1.0902	1.0581	1.0376
		CP	0.8505	0.8520	0.8440	1.0000	1.0000	1.0000
		AL	0.5395	0.3849	0.2947	0.5147	0.3832	0.2822
	$\hat{\alpha}_2$	*Shape*	1.0000	1.0000	1.0000	1.0153	1.0009	1.0001
		CP	0.9750	0.9650	0.9400	1.0000	1.0000	1.0000
		AL	0.7508	0.5320	0.4007	0.7499	0.5189	0.3934
	$\hat{\beta}_2$	*Shape*	1.0000	1.0000	1.0000	1.0876	1.0771	1.0574
		CP	0.8150	0.8690	0.8630	0.9500	0.9350	1.0000
		AL	0.2904	0.1983	0.1479	0.2719	0.1865	0.1289
	$\hat{\lambda}_2$	*Shape*	1.0000	1.0000	1.0000	1.0849	1.0497	1.0301
		CP	0.8560	0.8485	0.8440	1.0000	0.9950	1.0000

Table 4: Average length, shape and coverage probability for interval estimates under exponential monitoring points for varying sample sizes.

			ACI			HPD		
$(\lambda_1,\ \lambda_2)$		$n \longrightarrow$	80	150	250	80	150	250
(0.10, 0.12)		AL	0.4719	0.3428	0.2550	0.4515	0.3301	0.2466
	$\hat{\alpha}_1$	$Shape$	1.0000	1.0000	1.0000	1.0100	1.0073	1.0014
		CP	0.8600	0.8900	0.8760	1.0000	1.0000	1.0000
		AL	0.5699	0.4394	0.3411	0.5441	0.4250	0.3319
	$\hat{\beta}_1$	$Shape$	1.0000	1.0000	1.0000	1.0941	1.0370	1.0235
		CP	0.8900	0.8810	0.8320	0.9200	0.9300	0.9800
		AL	0.1558	0.1026	0.0842	0.1321	0.1012	0.0798
	$\hat{\lambda}_1$	$Shape$	1.0000	1.0000	1.0000	1.1336	1.0895	1.0869
		CP	0.9100	0.8769	0.8680	1.0000	0.9700	0.9800
		AL	0.4666	0.3400	0.2590	0.4563	0.3296	0.2517
	$\hat{\alpha}_2$	$Shape$	1.0000	1.0000	1.0000	1.0198	1.0025	1.0023
		CP	0.9000	0.8900	0.8200	1.0000	1.0000	1.0000
		AL	0.6050	0.4514	0.3422	0.5862	0.4328	0.3350
	$\hat{\beta}_2$	$Shape$	1.0000	1.0000	1.0000	1.0770	1.0426	1.0090
		CP	0.8600	0.8947	0.8526	0.9800	0.9800	1.0000
		AL	0.1873	0.1250	0.0925	0.1748	0.1096	0.0887
	$\hat{\lambda}_2$	$Shape$	1.0000	1.0000	1.0000	1.1124	1.0923	1.0645
		CP	0.8200	0.8700	0.8700	1.0000	0.9800	0.9400
(0.17, 0.18)		AL	0.4861	0.3592	0.2649	0.4784	0.3412	0.2579
	$\hat{\alpha}_1$	$Shape$	1.0000	1.0000	1.0000	1.0036	1.0024	1.0010
		CP	0.8800	0.8500	0.8760	1.0000	1.0000	1.0000
		AL	0.5847	0.4939	0.3608	0.5780	0.4799	0.3544
	$\hat{\beta}_1$	$Shape$	1.0000	1.0000	1.0000	1.1434	1.1118	1.0916
		CP	0.8373	0.8200	0.8780	0.9800	0.9500	0.9800
		AL	0.2219	0.1638	0.1238	0.2089	0.1438	0.1187
	$\hat{\lambda}_1$	$Shape$	1.0000	1.0000	1.0000	1.1455	1.1322	1.1019
		CP	0.8067	0.8760	0.9150	0.9867	0.9900	1.0000
		AL	0.4761	0.3728	0.2599	0.4575	0.3532	0.2689
	$\hat{\alpha}_2$	$Shape$	1.0000	1.0000	1.0000	1.0498	1.0298	1.0019
		CP	0.9200	0.8000	0.8740	1.0000	1.0000	1.0000
		AL	0.6539	0.5139	0.3601	0.6492	0.5085	0.3499
	$\hat{\beta}_2$	$Shape$	1.0000	1.0000	1.0000	1.1408	1.1145	1.0940
		CP	0.8367	0.8205	0.9070	0.9667	0.9400	0.9500
		AL	0.2252	0.1674	0.1251	0.2117	0.1559	0.1137
	$\hat{\lambda}_2$	$Shape$	1.0000	1.0000	1.0000	1.1403	1.1273	1.1014
		CP	0.8740	0.8685	0.8720	0.9867	0.9900	0.9700