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Abstract

We consider estimation of multicomponent stress-strength reliability under classical as well as Bayesian approaches when stress-strength components follow lognormal distributions and data are observed under progressive censoring. Various estimates are obtained by sequentially considering one parameter common and unknown. Different expressions for the reliability are evaluated under these cases. For both the cases, we obtain different estimates of considered parametric function using likelihood, Lindley and repeated sampling methods. Asymptotic and credible intervals are also obtained. Extensive simulations are conducted to examine the behavior of all estimates under various censoring schemes. We finally present analysis of a real life example from application purposes.

Keywords: Bayes estimate, interval estimation, lognormal, maximum likelihood estimate, stress-strength components.

Mathematics Subject Classification: 62F10, 62F15, 62N02

1 Introduction

In many situations, we often require to obtain inferences for multicomponent systems. Now a days systems/products are designed and developed using several components so that required commercial purposes may be achieved through the multi-functionality feature of the product. Reliability inference is an important measure for improving the product quality. Usually life tests are conducted to assess reliability of products/systems. Through proper experimentation appropriate inferences for a system can be derived. We note that components of a multicomponent system can be sequentially arranged in different manner. Some examples of such systems are hanging bridges, racer cars with V-8 engine, powerful multi-engines airplanes, multi-censor smartphones, and many other electronic equipment. Inference upon reliability is widely studied

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in literature. We refer to Al-Hemyari and Al-Dolami [1], Jia et al. [2] and Al-Hemyari [3] for some useful results on this aspect. Initial discussion on a stress-strength model (SM) is given in Birnbaum [4], see also, Birnbaum and MaCarty [5]. Inference upon such a model has found wide applications in many areas. In this regard we refer to Kundu and Gupta [6], Hussain [7], Nadar and Kizilaslan [8] etc. for a detailed discussion. In reliability applications, the failure of a SM is observed when stress Y exceeds strength X of the system under consideration. Accordingly the reliability under this set up is given by $R = P(Y < X)$. Several important applications of SMs are well discussed by Kotz et al. [9]. Also see Ghitany et al. [10], Yadav et al. [11], Nadarajah et al. [12], Kumari et al. [13] for various illustrative discussion on SMs.

Initial study on reliability estimation of the SM for a multicomponent system is discussed in Bhattacharyya and Johnson [14]. A system having k strength components and a stress force works good provided s $(1 \leq s \leq k)$ or more strength components work good. We call it s-out-of-k: G system. The reliability under such a set up is

$$\begin{aligned} R_{s,k} &= P[\text{at least s of } (X_1, X_2, \ldots, X_k) \text{ exceed } Y] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} (1 - F_X(y))^i (F_X(y))^{k-i} dF_Y(y), \end{aligned} \tag{1}$$

where $(X_1, X_2, \ldots, X_k)$ denote iid strength variables with distribution $F_X(\cdot)$. Also stress Y has distribution $F_Y(\cdot)$. Many researchers have investigated such inference problems. Hanagal [15] obtained estimation results on SM reliability when components follow multivariate exponential distributions. We further refer to Rao and Kantam [16] for some important results for logarithm of logistic probability models. Inferences for (generalized) exponential models are discussed in the article by Rao [17]. In similar note we refer to Rao [18, 19, 20] for useful inferences on SMs. Recently Kizilaslan and Nadar [21] also obtained various estimation results for the SM when components follow two dimensional Kumaraswamy distributions. Numerical evaluation of all estimators of reliability is presented and comments are presented. Several new models are proposed and studied in recent past. The family of Marshall-Olkin bivariate Weibull models have found wide applications in practice due to its ability to model various phenomena. Nadar and Kizilaslan [22] also discuss this distribution under the SM framework. Inverted family of distributions are widely used in analysis and several articles have discussed inferences for this family. In this regard Kizilaslan [23] investigated the SM using such families of distribution. Most of these results are obtained under complete sample cases. Similar results abound under censored data cases also. Kumaraswamy distribution is widely used in survival analysis. Recently, Kohansal [24] obtained inferences for this model under the SM framework when samples are recorded using progressive censoring. Various estimates of reliability are compared numerically and author provided useful remarks based on these comparisons. Censoring plays vital role in reliability evaluation. Progressive censoring has been widely applied in such investigation. Failure times from this scheme are recorded as follow. If N systems are tested then failure times of n $(n \leq N)$ systems are recorded by implementing different stages. When first system fails, R_1 systems are removed from the test. When second system fails, R_2 systems are removed from surviving $(N - R_1 - 2)$ systems. Finally when failure of nth system is observed, the test concluded and remaining $(R_n = N - n - R_1 - \cdots - R_{n-1})$ systems are removed. Various properties of this method are discussed in depth in treatise of Balakrishnan and Aggrawala [25] and Balakrishnan and Cramer [26]. See also Wu et al. [27], Rastogi and Tripathi [28] Maurya et al. [29], Maurya et al. [30], Mahto et al. [31], Chandra et al. [32], Hua and Gui [33] for important literature on this method of inference.

Estimation of the SM reliability is considered when strength and stress variables follow lognormal distributions. Also data are obtained from progressive type II censoring. The distribution function of a lognormal distribution is

$$F_X(x,\eta,\xi) = \Phi\Big(\frac{\ln x - \eta}{\sqrt{\xi}}\Big),\ 0 < x < \infty,\ \ -\infty < \eta < \infty, 0 < \xi < \infty, \tag{2}$$

and its density is given by

$$f(x,\eta,\xi) = \frac{1}{\sqrt{\xi}\,x}\,\phi\Big(\frac{\ln x - \eta}{\sqrt{\xi}}\Big),\ 0 < x < \infty,\ \ -\infty < \eta < \infty, 0 < \xi < \infty, \tag{3}$$

where $\Phi(.)$ and $\phi(.)$ are distribution and density functions of $N(0,1)$ variable. The scalars η and ξ are parameters of this distribution. Here we denote this model as $LN(\eta,\xi)$. This model is very popular in failure time analysis. This distribution is highly flexible as it can be used to analyze different physical phenomena. We further note that hazard rate is the ratio of density to the survival function. We see that hazard rate of lognormal distribution depends upon both the model parameters. In fact hazard rate is zero in the beginning and then increases to its maximum value. Finally it again goes down to zero as x approaches infinity. Due to this fact lognormal distribution can adequately be used to analyze many physical phenomena. Applications of the lognormal distribution in field of practical studies are diverse and include actuarial science, business, economics, and in lifetime analysis of electronic components, see Serfling, [34]. This probability model has its applications in analyzing homogeneous loss data (Punzo et al., [35]) and is also useful in the study of heterogeneous data. Further lognormal distribution is used in modeling data which may be normally distributed except for the fact that it may be skewed in nature, see Limpert, Stahel, and Abbt [36]. The wide range of applicability of this distribution makes it a suitable model for analyzing various SMs as well. Thus we consider making useful inferences for the reliability under multicomponent system. Proposed results are evaluated numerically and remarks are presented. We derive these inferences under progressively censored systems by assuming different parametric restrictions on this lifetime model.

In Section 2, we present different estimators of the SM reliability for a multicomponent system when the parameter ξ is common to stress and strength components. Likelihood estimators of reliability are evaluated under given scheme. The Lindley method and importance sampling are applied to compute corresponding Bayes estimates. In Section 3, estimation problem is considered when η is common parameter and various estimates of the reliability are obtained. Expression for the SM parameter is obtained. Particularly MH algorithm is applied for Bayesian computations. In Section 4, we numerically compare all proposed estimators of the SM reliability from simulations. A practical example is analyzed as well. Finally some concluding remarks are given in Section 5.

2 Estimation when ξ is common parameter

2.1 MLE

We now compute the MLE of SM reliability. Let $X_1, X_2, \ldots, X_k$ follow $LN(\eta_1,\xi)$ distribution, Y is $LN(\eta_2,\xi)$ distributed and both are independent where η_1,η_2 and ξ are unknown parameters. Since distribution function of this particular model does not exist in closed forms and so associated expression of reliability will appear in integral form only. In this regard, we obtained $R_{s,k}$ as follows:

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} \int_{0}^{\infty} \left[1 - \Phi\Big(\frac{\ln y - \eta_1}{\sqrt{\xi}}\Big)\right]^{i} \left[\Phi\Big(\frac{\ln y - \eta_1}{\sqrt{\xi}}\Big)\right]^{k-i} \frac{1}{\sqrt{\xi}\, y}\, \phi\Big(\frac{\ln y - \eta_2}{\sqrt{\xi}}\Big)\, dy$$

$$= \sum_{i=s}^{k} \sum_{j=0}^{i} \binom{k}{i}\binom{i}{j}(-1)^{j} \int_{0}^{\infty} \left[\Phi\Big(\frac{\ln y - \eta_1}{\sqrt{\xi}}\Big)\right]^{j+k-i} \frac{1}{\sqrt{\xi}\, y}\, \phi\Big(\frac{\ln y - \eta_2}{\sqrt{\xi}}\Big)\, dy,$$

Considering the transformation $\frac{\ln y - \eta_2}{\sqrt{\xi}} = y$ we get the following expression

$$R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{i} \binom{k}{i}\binom{i}{j} (-1)^{j} \int_{-\infty}^{\infty} \left[\Phi\Big(y + \frac{\eta_2 - \eta_1}{\sqrt{\xi}}\Big)\right]^{j+k-i} \phi(y)\, dy. \tag{4}$$

In order to compute the desired MLE of this parametric function, estimates of parameters η_1, η_2 and ξ are evaluated. Let N identical systems be put on a test where each system consists of K components. Then a total of n systems each with k components are realized under considered censoring scheme. The data of form $\{X_{i1}, \ldots, X_{ik}\},\ i = 1, 2, \ldots, n$ are observed from the $LN(\eta_1, \xi)$ distribution when the censoring scheme is $\{K, k, R_1, \ldots R_k\}$. Also samples $\{Y_1, Y_2, \ldots, Y_n\}$ are observed from $LN(\eta_2, \xi)$ distribution against scheme $\{N, n, S_1, \ldots, S_n\}$. Now, likelihood of η_1, η_2 and ξ is given by

$$L(\eta_1, \eta_2, \xi \mid x, y) = c_1 \prod_{i=1}^{n} \left[c_2 \prod_{j=1}^{k} f(x_{ij})\, [1 - F(x_{ij})]^{R_j}\right] f(y_i)\, [1 - F(y_i)]^{S_i},$$

where

$$c_1 = N(N - S_1 - 1)\cdots(N - S_1 - \cdots - S_{n-1} - n + 1),$$
$$c_2 = K(K - R_1 - 1)\cdots(K - R_1 - \cdots - R_{k-1} - k + 1).$$

This likelihood function under observed data turns out to be,

$$\begin{aligned} L(\eta_1, \eta_2, \xi \mid x, y) \;\; &= \;\; c_1\, c_2^{n} \frac{1}{\xi^{n(k+1)/2}} \prod_{i=1}^{n} \left[1 - \Phi\Big(\frac{\ln y_i - \eta_2}{\sqrt{\xi}}\Big)\right]^{S_i} \frac{1}{y_i}\, \phi\Big(\frac{\ln y_i - \eta_2}{\sqrt{\xi}}\Big) \\ & \prod_{i=1}^{n} \prod_{j=1}^{k} \left[1 - \Phi\Big(\frac{\ln x_{ij} - \eta_1}{\sqrt{\xi}}\Big)\right]^{R_j} \frac{1}{x_{ij}}\, \phi\Big(\frac{\ln x_{ij} - \eta_1}{\sqrt{\xi}}\Big). \end{aligned}$$

Now the log-likelihood is

$$\begin{aligned} l(\eta_1, \eta_2, \xi \mid x, y) \;\; &\propto \;\; -(n(k+1)/2)\, \ln\xi - \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n}\sum_{j=1}^{k} \ln x_{ij} + \sum_{i=1}^{n} \Big\{ \ln\phi\Big(\frac{\ln y_i - \eta_2}{\sqrt{\xi}}\Big)\Big\} \\ & + \sum_{i=1}^{n} S_i \Big\{ \ln\Big[1 - \Phi\Big(\frac{\ln y_i - \eta_2}{\sqrt{\xi}}\Big)\Big] \Big\} + \sum_{i=1}^{n}\sum_{j=1}^{k} \Big\{ \ln\phi\Big(\frac{\ln x_{ij} - \eta_1}{\sqrt{\xi}}\Big)\Big\} \\ & + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \Big\{ \ln\Big[1 - \Phi\Big(\frac{\ln x_{ij} - \eta_1}{\sqrt{\xi}}\Big)\Big] \Big\}. \end{aligned} \tag{5}$$

We rewrite this function as

$$l(\eta_1,\eta_2,\xi\mid x,y) \;\propto\; -(n(k+1)/2)\,\ln\xi-\sum_{i=1}^{n}\ln y_i-\sum_{i=1}^{n}\sum_{j=1}^{k}\ln x_{ij}+\sum_{i=1}^{n}\{\ln\phi(w_i)\}+\sum_{i=1}^{n}\sum_{j=1}^{k}\{\ln\phi(z_{ij})\}$$
$$+\sum_{i=1}^{n}S_i\{\ln[1-\Phi(w_i)]\}+\sum_{i=1}^{n}\sum_{j=1}^{k}R_j\{\ln[1-\Phi(z_{ij})]\}, \tag{6}$$

where $w_i=\frac{\ln y_i-\eta_2}{\sqrt{\xi}}$ and $z_{ij}=\frac{\ln x_{ij}-\eta_1}{\sqrt{\xi}}$. The MLEs of η_1, η_2 and ξ are computed from the associated likelihood equations obtained below

$$\frac{\partial l}{\partial\eta_1}=\frac{1}{\sqrt{\xi}}\Big[\sum_{i=1}^{n}\sum_{j=1}^{k}z_{ij}+\sum_{i=1}^{n}\sum_{j=1}^{k}R_j\frac{\phi(z_{ij})}{1-\Phi(z_{ij})}\Big]=0, \tag{7}$$

$$\frac{\partial l}{\partial\eta_2}=\frac{1}{\sqrt{\xi}}\Big[\sum_{i=1}^{n}w_i+\sum_{i=1}^{n}S_i\frac{\phi(w_i)}{1-\Phi(w_i)}\Big]=0, \tag{8}$$

and

$$\frac{\partial l}{\partial\xi}=\frac{1}{2\xi}\Big[-n(k+1)+\sum_{i=1}^{n}w_i^2+\sum_{i=1}^{n}S_i\frac{w_i\phi(w_i)}{1-\Phi(w_i)}+\sum_{i=1}^{n}\sum_{j=1}^{k}z_{ij}^2+\sum_{i=1}^{n}\sum_{j=1}^{k}R_j\frac{z_{ij}\phi(z_{ij})}{1-\Phi(z_{ij})}\Big]=0. \tag{9}$$

Respective maximum likelihood estimators $\hat{\eta}_1,\hat{\eta}_2$ and $\hat{\xi}$ of η_1,η_2 and ξ are computed numerically by solving the above nonlinear likelihood equations. Finally the estimator $\hat{R}_{s,k}$ of $R_{s,k}$ is obtained from these estimates. Next MLEs based interval of the reliability is obtained using the given censored samples.

2.2 Asymptotic intervals

We observed that MLEs of unknown parameters $\theta=(\eta_1,\eta_2,\xi)$ cannot be simplified much and hence associated probability distribution is difficult to obtain. So exact confidence interval is difficult to derive. We try to construct approximate confidence interval (ACI) based on large sample approximation. In this connection we write observed Fisher information matrix of θ as

$$I(\theta)=[I_{ij}]_{3\times3}=\begin{pmatrix}-\frac{\partial^2 l}{\partial\eta_1^2} & -\frac{\partial^2 l}{\partial\eta_1\partial\eta_2} & -\frac{\partial^2 l}{\partial\eta_1\partial\xi}\\ -\frac{\partial^2 l}{\partial\eta_2\partial\eta_1} & -\frac{\partial^2 l}{\partial\eta_2^2} & -\frac{\partial^2 l}{\partial\eta_2\partial\xi}\\ -\frac{\partial^2 l}{\partial\xi\partial\eta_1} & -\frac{\partial^2 l}{\partial\xi\partial\eta_2} & -\frac{\partial^2 l}{\partial\xi^2}\end{pmatrix}.$$

We find that $\hat{R}_{s,k}$ is like a normal variable having mean $R_{s,k}$ and variance

$$\hat{V}(\hat{R}_{s,k})=\Delta\hat{R}_{s,k}^T I^{-1}(\hat{\theta})\Delta\hat{R}_{s,k},$$

where $\Delta\hat{R}_{s,k}=\left(\frac{\partial R_{s,k}}{\partial\eta_1},\frac{\partial R_{s,k}}{\partial\eta_2},\frac{\partial R_{s,k}}{\partial\xi}\right)\Big|_{\theta=\hat{\theta}}$. Thus, $100(1-\alpha)\%$ confidence interval of $R_{s,k}$ is

$$\left(\hat{R}_{s,k}-z_{\alpha/2}\sqrt{\hat{V}(\hat{R}_{s,k})},\ \hat{R}_{s,k}+z_{\alpha/2}\sqrt{\hat{V}(\hat{R}_{s,k})}\right)$$

where $z_{\alpha/2}$ denotes the upper $z_{\alpha/2}$th quantile of $N(0,1)$ variable. Note that, the coefficient of fisher information matrix $I_{ij},\ i,j=1,2,3$ and the coefficient of $\Delta R_{s,k}$ is calculated in subsection 2.3.1. We are not presenting these expressions here for the sake of brevity.

2.3 Bayesian estimation

Bayes estimators of the SM parameter are investigated against considered censoring. It is assumed parameters η_1 and η_2 are distributed as normal $N(p_1, \xi/q_1)$ and $N(p_2, \xi/q_2)$ distributions and ξ has inverse gamma $IG(p_3, q_3/2)$ distribution with density $IG(p_0, q_0) \propto \xi^{-(p_0+1)}\, e^{-q_0/\xi}$, $\xi > 0$, $p_0 > 0$, $q_0 > 0$. We mention that we have considered normal priors for η_1 and η_2. Similarly we take into account an inverse gamma prior distribution for the parameter ξ. Note that parameters η_1 and η_2 assume their values on the whole real line. We thus considered normal distributions for these parameters as their priors. Similarly parameter ξ takes has its values on the positive part of the real line. Accordingly we take an inverse gamma prior for ξ. We also mention that these prior considerations lead to reasonably simplified posterior distribution which is useful for further computations.

After some algebraic computations, the joint posterior of η_1, η_2 and ξ given the censored data is expressed as

$$\begin{aligned}
\pi(\eta_1, \eta_2, \xi \mid x, y) \;\propto\;& N_{\eta_1|\xi}\Big(\frac{\sum_{i=1}^{n}\sum_{j=1}^{k}\ln x_{ij} + p_1 q_1}{nk + q_1}, \frac{\xi}{nk+q_1}\Big) N_{\eta_2|\xi}\Big(\frac{\sum_{i=1}^{n}\ln y_i + p_2 q_2}{n+q_2}, \frac{\xi}{n+q_2}\Big) \\
& IG_{\xi}\Big(\frac{n(k+1)}{2} + p_3, \frac{1}{2}\Big\{q_3 + q_1 p_1^2 + \sum_{i=1}^{n}\sum_{j=1}^{k}(\ln x_{ij})^2 + \sum_{i=1}^{n}(\ln y_i)^2 + q_2 p_2^2 \\
& - \frac{(\sum_{i=1}^{n}\sum_{j=1}^{k}\ln x_{ij} + p_1 q_1)^2}{nk+q_1} - \frac{(\sum_{i=1}^{n}\ln y_i + p_2 q_2)^2}{n+q_2}\Big\}\Big) \\
& \prod_{i=1}^{n}\prod_{j=1}^{k}\Big[1 - \Phi\Big(\frac{\ln x_{ij} - \eta_1}{\sqrt{\xi}}\Big)\Big]^{R_j} \prod_{i=1}^{n}\Big[1 - \Phi\Big(\frac{\ln y_i - \eta_2}{\sqrt{\xi}}\Big)\Big]^{S_i}. \qquad (10)
\end{aligned}$$

The Bayes estimator δ_B of multicomponent reliability against squared error loss is obtained as the posterior mean of $R_{s,k}$, that is $\delta_B = E(R_{s,k} \mid x, y)$ where

$$E(R_{s,k} \mid x, y) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} R_{s,k}\, \pi(\eta_1, \eta_2, \xi \mid x, y)\, d\eta_1 d\eta_2 d\xi}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \pi(\eta_1, \eta_2, \xi \mid x, y)\, d\eta_1 d\eta_2 d\xi}.$$

It is seen that the above estimators appears in the form of integral expressions only. This may not be simplified in closed form expressions as posterior distribution is not specified. The Lindley method (Lindley, [37]) and importance sampling algorithm are discussed to obtain the required estimate of the reliability.

2.3.1 Lindley method

This method is very useful in Bayesian computations. We note that posterior mean of $h(\theta)$ is

$$E(h(\theta) \mid x, y) = \frac{\int h(\theta) e^{l(\theta)+\rho(\theta)} d\theta}{\int e^{l(\theta)+\rho(\theta)} d\theta}. \qquad (11)$$

where $l(\theta)$, $\rho(\theta)$ are log-likelihood and logarithm of prior of $\theta = (\theta_1, \theta_2, \theta_3)$, respectively. The required estimated is now written as

$$E[h(\theta) \mid x, y] = \Big[h(\theta) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(h_{ij} + 2h_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} L_{ijk}\sigma_{ij}\sigma_{kl}h_l\Big]_{\theta=\hat{\theta}}, \qquad (12)$$

where $i,j,k,p=1,2,3,\ h_i=\partial h/\partial\theta_i,\ h_{ij}=\partial^2 h/\partial\theta_i\partial\theta_j,\ \rho_j=\partial\rho/\partial\theta_j,\ L_{ijk}=\partial^3 l/\partial\theta_i\partial\theta_j\partial\theta_k$, $\sigma_{ij}=(i,j)th$ element of $[I_{ij}]^{-1}\Big|_{\theta=\hat\theta}$ with $I_{ij}=[-\partial^2 l/\partial\theta_i\partial\theta_j]$. Thus we have

$$\begin{aligned}E[h(\theta)\mid x,y]=h(\theta)+(h_1b_1+h_2b_2+h_3b_3+b_4+b_5)+0.5\,(a_1(h_1\sigma_{11}+h_2\sigma_{12}+h_3\sigma_{13})\\ +a_2(h_1\sigma_{21}+h_2\sigma_{22}+h_3\sigma_{23})+a_3(h_1\sigma_{31}+h_2\sigma_{32}+h_3\sigma_{33}))\,,\end{aligned}$$

where

$$\begin{aligned}
&b_i=(\rho_1\sigma_{i1}+\rho_2\sigma_{i2}+\rho_3\sigma_{i3}),\quad i=1,2,3,\\
&b_4=h_{12}\sigma_{12}+h_{13}\sigma_{13}+h_{23}\sigma_{23},\\
&b_5=\tfrac{1}{2}(h_{11}\sigma_{11}+h_{22}\sigma_{22}+h_{33}\sigma_{33}),\\
&a_1=L_{111}\sigma_{11}+2L_{121}\sigma_{12}+2L_{131}\sigma_{13}+2L_{231}\sigma_{23}+L_{221}\sigma_{22}+L_{331}\sigma_{33},\\
&a_2=L_{112}\sigma_{11}+2L_{122}\sigma_{12}+2L_{132}\sigma_{13}+2L_{232}\sigma_{23}+L_{222}\sigma_{22}+L_{332}\sigma_{33},\\
&a_3=L_{113}\sigma_{11}+2L_{123}\sigma_{12}+2L_{133}\sigma_{13}+2L_{233}\sigma_{23}+L_{223}\sigma_{22}+L_{333}\sigma_{33}.
\end{aligned}$$

Here we take $(\theta_1,\theta_2,\theta_3)\equiv(\eta_1,\eta_2,\xi)$ and $h(\theta)=R_{s,k}$ and also note that $\rho_1=-\frac{q_1}{\xi}(\eta_1-p_1),\ \rho_2=-\frac{q_2}{\xi}(\eta_1-p_2),\ \rho_3=\frac{q_1(\eta_1-p_1)^2}{2\xi^2}+\frac{q_2(\eta_2-p_2)^2}{2\xi^2}+\frac{q_3}{\xi^2}-\frac{(q_3+1)}{\xi}$,

$$L_{11}=\frac{\partial^2 l}{\partial\eta_1^2}=-\frac{nk}{\xi}+\frac{1}{\xi}\sum_{i=1}^{n}\sum_{j=1}^{k}R_j\frac{\phi(z_{ij})(z_{ij}(1-\Phi(z_{ij}))-\phi(z_{ij}))}{(1-\Phi(z_{ij}))^2},$$

$$L_{22}=\frac{\partial^2 l}{\partial\eta_2^2}=-\frac{n}{\xi}+\frac{1}{\xi}\sum_{i=1}^{n}S_i\frac{\phi(w_i)(w_i(1-\Phi(w_i))-\phi(w_i))}{(1-\Phi(w_i))^2},$$

$$L_3=\frac{\partial l}{\partial\xi}=\frac{1}{2\xi}A(\xi),$$

where

$$A(\xi)=\Big[-n(k+1)+\sum_{i=1}^{n}w_i^2+\sum_{i=1}^{n}S_i\frac{w_i\phi(w_i)}{1-\Phi(w_i)}+\sum_{i=1}^{n}\sum_{j=1}^{k}z_{ij}^2+\sum_{i=1}^{n}\sum_{j=1}^{k}R_j\frac{z_{ij}\phi(z_{ij})}{1-\Phi(z_{ij})}\Big],$$

$$\begin{aligned}L_{33}\;=\;&\frac{\partial^2 l}{\partial\xi^2}=-\frac{1}{2\xi^2}A(\xi)+\frac{1}{2\xi^2}\Big[-\sum_{i=1}^{n}w_i^2-\sum_{i=1}^{n}\sum_{j=1}^{k}z_{ij}^2\\
&+0.5\sum_{i=1}^{n}\sum_{j=1}^{k}R_j\{(-z_{ij}\phi(z_{ij})+z_{ij}^3\phi(z_{ij}))(1-\Phi(z_{ij}))-z_{ij}^2\phi^2(z_{ij})\}(1-\Phi(z_{ij}))^{-2}\\
&+0.5\sum_{i=1}^{n}S_i\{(-w_i\phi(w_i)+w_i^3\phi(w_i))(1-\Phi(w_i))-w_i^2\phi^2(w_i)\}(1-\Phi(w_i))^{-2}\Big],\end{aligned}$$

$L_{12}=L_{21}=0,$

$$
\begin{aligned}
L_{13} &= L_{31} = \frac{\partial^2 l}{\partial \eta_1 \partial \xi} = -\frac{1}{\xi^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{k} z_{ij} - \frac{1}{2\xi^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \frac{\phi(z_{ij})}{1-\Phi(z_{ij})} \\
&\quad + \frac{1}{2\xi^{3/2}} \Big[\sum_{i=1}^{n} \sum_{j=1}^{k} R_j z_{ij} \phi(z_{ij}) \{ z_{ij}(1-\Phi(z_{ij})) - \phi(z_{ij}) \} (1-\Phi(z_{ij}))^{-2} \Big],
\end{aligned}
$$

$$
\begin{aligned}
L_{23} &= L_{32} = \frac{\partial^2 l}{\partial \eta_2 \partial \xi} = -\frac{1}{\xi^{3/2}} \sum_{i=1}^{n} w_i - \frac{1}{2\xi^{3/2}} \sum_{i=1}^{n} S_i \frac{\phi(w_i)}{1-\Phi(w_i)} \\
&\quad + \frac{1}{2\xi^{3/2}} \Big[\sum_{i=1}^{n} S_i w_i \phi(w_i) \{ w_i(1-\Phi(w_i)) - \phi(w_i) \} (1-\Phi(w_i))^{-2} \Big],
\end{aligned}
$$

$$
\begin{aligned}
L_{111} &= \frac{\partial^3 l}{\partial \eta_1^3} = \frac{1}{\xi^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \Big\{ [z_{ij}\phi(z_{ij})(z_{ij}(1-\Phi(z_{ij})) - \phi(z_{ij})) - \phi(z_{ij})(1-\Phi(z_{ij}))](1-\Phi(z_{ij}))^2 \\
&\quad -2(1-\Phi(z_{ij}))\phi(z_{ij})[\phi(z_{ij})(z_{ij}(1-\Phi(z_{ij})) - \phi(z_{ij}))] \Big\} (1-\Phi(z_{ij}))^{-4},
\end{aligned}
$$

$$
\begin{aligned}
L_{222} &= \frac{\partial^3 l}{\partial \eta_2^3} = \frac{1}{\xi^{3/2}} \sum_{i=1}^{n} S_i \Big\{ [w_i\phi(w_i)(w_i(1-\Phi(w_i)) - \phi(w_i)) - \phi(w_i)(1-\Phi(w_i))](1-\Phi(w_i))^2 \\
&\quad -2(1-\Phi(w_i))\phi(w_i)[\phi(w_i)(w_i(1-\Phi(w_i)) - \phi(w_i))] \Big\} (1-\Phi(w_i))^{-4},
\end{aligned}
$$

$$
\begin{aligned}
L_{113} &= \frac{\partial^3 l}{\partial \eta_1^2 \partial \xi} = \frac{1}{\xi^2} \Big[nk - \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \frac{\phi(z_{ij})(z_{ij}(1-\Phi(z_{ij})) - \phi(z_{ij}))}{(1-\Phi(z_{ij}))^2} \Big] \\
&\quad + \frac{1}{2\xi} \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \Big\{ [z_{ij}^2\phi(z_{ij})(z_{ij}(1-\Phi(z_{ij})) - \phi(z_{ij})) - z_{ij}\phi(z_{ij})(1-\Phi(z_{ij}))](1-\Phi(z_{ij}))^2 \\
&\quad -2(1-\Phi(z_{ij}))\phi(z_{ij})z_{ij}[\phi(z_{ij})(z_{ij}(1-\Phi(z_{ij})) - \phi(z_{ij}))] \Big\} (1-\Phi(z_{ij}))^{-4},
\end{aligned}
$$

$$
\begin{aligned}
L_{223} &= \frac{\partial^3 l}{\partial \eta_2^2 \partial \xi} = \frac{1}{\xi^2} \Big[n - \sum_{i=1}^{n} S_i \frac{\phi(w_i)(w_i(1-\Phi(w_i)) - \phi(w_i))}{(1-\Phi(w_i))^2} \Big] \\
&\quad + \frac{1}{2\xi} \sum_{i=1}^{n} S_i \Big\{ [w_i^2\phi(w_i)(w_i(1-\Phi(w_i)) - \phi(w_i)) - w_i\phi(w_i)(1-\Phi(w_i))](1-\Phi(w_i))^2 \\
&\quad -2(1-\Phi(w_i))\phi(w_i)w_i[\phi(w_i)(w_i(1-\Phi(w_i)) - \phi(w_i))] \Big\} (1-\Phi(w_i))^{-4},
\end{aligned}
$$

$$\begin{aligned}
L_{133} &= L_{313} = L_{331} = \frac{\partial^3 l}{\partial \eta_1 \partial \xi^2} = \frac{3}{2\xi^{5/2}}\Big[\sum_{i=1}^{n}\sum_{j=1}^{k} z_{ij} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{\phi(z_{ij})}{1-\Phi(z_{ij})} \\
&\quad -\sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{z_{ij}\phi(z_{ij})(z_{ij}(1-\Phi(z_{ij}))-\phi(z_{ij}))}{(1-\Phi(z_{ij}))^2}\Big] \\
&\quad -\frac{1}{2\xi^{5/2}}\Big[-\sum_{i=1}^{n}\sum_{j=1}^{k} z_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j\{z_{ij}^2\phi(z_{ij})(1-\Phi(z_{ij})) - z_{ij}\phi(z_{ij})\}(1-\Phi(z_{ij}))^{-2}\Big] \\
&\quad +\frac{1}{2\xi^{5/2}}\Big[\Big\{[z_{ij}\phi(z_{ij})(z_{ij}^2-1)][z_{ij}(1-\Phi(z_{ij}))-\phi(z_{ij})] - z_{ij}^2\phi(z_{ij})(1-\Phi(z_{ij}))\Big\}(1-\Phi(z_{ij}))^2 \\
&\quad -2(1-\Phi(z_{ij}))z_{ij}\phi(z_{ij})[z_{ij}\phi(z_{ij})\{z_{ij}(1-\Phi(z_{ij}))-\phi(z_{ij})\}]\Big](1-\Phi(z_{ij}))^{-4},
\end{aligned}$$

$$\begin{aligned}
L_{233} &= L_{323} = L_{332} = \frac{\partial^3 l}{\partial \eta_2 \partial \xi^2} = \frac{3}{2\xi^{5/2}}\Big[\sum_{i=1}^{n} w_i + \frac{1}{2}\sum_{i=1}^{n} S_i \frac{\phi(w_i)}{1-\Phi(w_i)} \\
&\quad -\sum_{i=1}^{n} S_i \frac{w_i\phi(w_i)(w_i(1-\Phi(w_i))-\phi(w_i))}{(1-\Phi(w_i))^2}\Big] \\
&\quad -\frac{1}{2\xi^{5/2}}\Big[-\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} S_i\{w_i^2\phi(w_i)(1-\Phi(w_i)) - w_i\phi(w_i)\}(1-\Phi(w_i))^{-2}\Big] \\
&\quad +\frac{1}{2\xi^{5/2}}\Big[\Big\{[w_i\phi(w_i)(w_i^2-1)][w_i(1-\Phi(w_i))-\phi(w_i)] - w_i^2\phi(w_i)(1-\Phi(w_i))\Big\}(1-\Phi(w_i))^2 \\
&\quad -2(1-\Phi(w_i))w_i\phi(w_i)[w_i\phi(w_i)\{w_i(1-\Phi(w_i))-\phi(w_i)\}]\Big](1-\Phi(w_i))^{-4},
\end{aligned}$$

$$\begin{aligned}
L_{333} = {} & \frac{-n(k+1)}{\xi^3} + \sum_{i=1}^{n}\left(\frac{w_i}{\xi}\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{k}\left(\frac{z_{ij}}{\xi}\right)^2 + \frac{1}{4}\sum_{i=1}^{n} S_i\left[\left(\frac{w_i\phi(w_i)}{(\xi(1-\Phi(w_i)))^4}\right.\right. \\
\times & \left(-3\xi\,(1-\Phi(w_i))^3(w_i^2+2) + \frac{3}{2}\left(w_i^2\phi(w_i)(1-\Phi(w_i))\right)^2 + \frac{3\xi}{2}\left(w_i\phi(w_i)(1-\Phi(w_i))\right)^2\right. \\
+ & \left.\left.\frac{5\xi}{2}\left(w_i\phi(w_i)(1-\Phi(w_i))^2\right) - \frac{\xi}{2}\left(w_i^3\phi(w_i)(1-\Phi(w_i))\right)\left((1-\Phi(w_i)) + 2(\phi(w_i)^2)\right)\right)\right) \\
+ & \left.\frac{w_i(w_i^2-1)\phi(w_i)}{2\xi^2(1-\Phi(w_i))^2}\left(w_i^2(1-\Phi(w_i)) - 3(1-\Phi(w_i)) - w_i\phi(w_i)\right)\right] + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{k} R_j \\
\times & \left[\left(\frac{z_{ij}\phi(z_{ij})}{(\xi(1-\Phi(z_{ij})))^4}\left(-3\xi\,(1-\Phi(z_{ji}))^3(z_{ij}^2+2) + \frac{3}{2}\left(z_{ij}^2\phi(z_{ij})(1-\Phi(z_{ij}))\right)^2\right.\right.\right. \\
+ & \frac{3\xi}{2}\left(z_{ij}\phi(z_{ij})(1-\Phi(z_{ij}))\right)^2 + \frac{5\xi}{2}\left(z_{ij}\phi(z_{ij})(1-\Phi(z_{ij}))^2\right) \\
- & \left.\left.\frac{\xi}{2}\left(z_{ij}^3\phi(z_{ij})(1-\Phi(z_{ij}))\right)\left((1-\Phi(z_{ij})) + 2(\phi(z_{ij})^2)\right)\right)\right) \\
+ & \left.\frac{z_{ij}(z_{ij}^2-1)\phi(z_{ij})}{2\xi^2(1-\Phi(z_{ij}))^2}\left(z_{ij}^2(1-\Phi(z_{ij})) - 3(1-\Phi(z_{ij})) - z_{ij}\phi(z_{ij})\right)\right],
\end{aligned}$$

$l_{112} = l_{121} = l_{211} = l_{221} = l_{212} = l_{122} = 0$. Denote $\kappa = \frac{\eta_2-\eta_1}{\sqrt{\xi}}$. Then, we have

$$h_1 = \frac{\partial R_{s,k}}{\partial \eta_1} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}(j+k-i)}{\sqrt{\xi}}\int_{-\infty}^{\infty}\left[\Phi\left(y+\kappa\right)\right]^{j+k-i-1}\phi\left(y+\kappa\right)\phi(y)dy,$$

$$h_2 = \frac{\partial R_{s,k}}{\partial \eta_2} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j}(j+k-i)}{\sqrt{\xi}}\int_{-\infty}^{\infty}\left[\Phi\left(y+\kappa\right)\right]^{j+k-i-1}\phi\left(y+\kappa\right)\phi(y)dy,$$

$$h_3 = \frac{\partial R_{s,k}}{\partial \xi} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}(j+k-i)}{2\xi}\int_{-\infty}^{\infty}\left[\Phi\left(y+\kappa\right)\right]^{j+k-i-1}\phi\left(y+\kappa\right)\kappa\phi(y)dy,$$

$$\begin{aligned}h_{11} = \frac{\partial^2 R_{s,k}}{\partial \eta_1^2} &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j}(j+k-i)}{\xi}\int_{-\infty}^{\infty}[j+k-i-1)\phi(y+\kappa)-\Phi(y+\kappa)(y+\kappa)]\\ &\times\left[\Phi(y+\kappa)\right]^{j+k-i-2}\phi(y+\kappa)\phi(y)dy,\end{aligned}$$

$$\begin{aligned}h_{12} = \frac{\partial^2 R_{s,k}}{\partial \eta_1\partial\eta_2} &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}(j+k-i)}{\xi}\left[(j+k-i-1)\phi(y+\kappa)+(y+\kappa)\Phi(y+\kappa)\right]\\ &\times\left[\Phi(y+\kappa)\right]^{j+k-i-2}\phi(y+\kappa)\phi(y)dy = h_{21},\end{aligned}$$

$$\begin{aligned}h_{13} = \frac{\partial^2 R_{s,k}}{\partial \eta_1\partial\xi} &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j}(j+k-i)}{2\xi^{3/2}}\left[(j+k-i-1)\kappa\phi(y+\kappa)+\Phi(y+\kappa)\right.\\ &\times\left.(1-\kappa(y+\kappa))\right]\left[\Phi(y+\kappa)\right]^{j+k-i-2}\phi(y+\kappa)\phi(y)dy = h_{31},\end{aligned}$$

$$\begin{aligned}h_{22} = \frac{\partial^2 R_{s,k}}{\partial \eta_2^2} &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j}(j+k-i)}{\xi}\left[(j+k-i-1)\phi(y+\kappa)-(y+\kappa)\Phi(y+\kappa)\right]\\ &\times\left[\Phi(y+\kappa)\right]^{j+k-i-2}\phi(y+\kappa)\phi(y)dy,\end{aligned}$$

$$\begin{aligned}h_{23} = \frac{\partial^2 R_{s,k}}{\partial \eta_2\partial\xi} &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}(j+k-i)}{2\xi}\left[\Phi(y+\kappa)-\frac{(j+k-i-1)}{\sqrt{\xi}}\phi(y+\kappa)\right]\\ &\times\left[\Phi(y+\kappa)\right]^{j+k-i-2}\phi(y+\kappa)\kappa\phi(y)dy = h_{32},\end{aligned}$$

$$\begin{aligned}h_{33} = \frac{\partial^2 R_{s,k}}{\partial \xi^2} &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}(j+k-i)}{4\xi^{3}}\left[(j+k-i-1)\kappa\phi(y+\kappa)+\Phi(y+\kappa)\right.\\ &\times\left.\left(\kappa(y+\kappa)-\frac{3}{\xi}\right)\right]\left[\Phi(y+\kappa)\right]^{j+k-i-2}\phi(y+\kappa)\kappa\phi(y)dy.\end{aligned}$$

The required estimate of $R_{s,k}$ is now given as

$$\hat{R}^L_{s,k} = R_{s,k} + [h_1b_1 + h_2b_2 + b_4 + b_5] \\ + 0.5[a_1(h_1\sigma_{11} + h_2\sigma_{12}) + a_2(h_1\sigma_{21} + h_2\sigma_{22}) + a_3(h_1\sigma_{31} + h_2\sigma_{32})]_{(\hat{\eta_1},\hat{\eta_2},\hat{\xi})}. \tag{13}$$

It is interesting to note that Lindley approximation is very useful for obtaining point estimates. It may not provide intervals for unknown quantities. So next we discuss a widely used technique which is highly useful in Bayesian computations.

2.3.2 Importance sampling

This technique is widely used to generate samples from distribution under consideration. Generated data can then be used in Bayesian inference. Particularly credible interval of parametric function can be obtained. Note that intervals with minimum length is often a desirable criterion. Such intervals are termed as the HPD intervals. An HPD interval is an interval which has the smallest length among all credible intervals. Further some alternative point estimates of parameters can also be computed. The marginal distributions of η_1, η_2 and ξ, respectively, are of the following form

$$\eta_1 \mid \xi, \text{data} \sim N_{\eta_1|\xi}\Big(\frac{\sum_{i=1}^n \sum_{j=1}^k \ln x_{ij} + p_1q_1}{nk+q_1}, \frac{\xi}{nk+q_1}\Big), \tag{14}$$

$$\eta_2 \mid \xi, \text{data} \sim N_{\eta_2|\xi}\Big(\frac{\sum_{i=1}^n \ln y_i + p_2q_2}{n+q_2}, \frac{\xi}{n+q_2}\Big),$$

and (15)

$$\xi \mid \text{data} \sim IG_\xi\Big(\frac{n(k+1)}{2} + p_3, \frac{1}{2}\Big\{q_3 + q_1p_1^2 + \sum_{i=1}^n \sum_{j=1}^k (\ln x_{ij})^2 + \sum_{i=1}^n (\ln y_i)^2 + q_2p_2^2 \\ - \frac{(\sum_{i=1}^n \sum_{j=1}^k \ln x_{ij} + p_1q_1)^2}{nk+q_1} - \frac{(\sum_{i=1}^n \ln y_i + p_2q_2)^2}{n+q_2}\Big\}\Big), \tag{16}$$

with the additional term given as

$$\psi(\eta_1,\eta_2,\xi) = \prod_{i=1}^n \prod_{j=1}^k \Big[1 - \Phi\Big(\frac{\ln x_{ij} - \eta_1}{\sqrt{\xi}}\Big)\Big]^{R_j} \prod_{i=1}^n \Big[1 - \Phi\Big(\frac{\ln y_i - \eta_2}{\sqrt{\xi}}\Big)\Big]^{S_i}.$$

Now, to generate samples for η_1, η_2 and ξ and consequently for $R_{s,k}$, the following steps of algorithm are followed and then Bayesian interval estimate for $R_{s,k}$ is obtained.
Step 1: Consider initial guess $(\eta_1^{(0)}, \eta_2^{(0)}, \xi^{(0)})$.
Step 2: Take $t = 1$
Step 3: Generate $\eta_1^{(t)}, \eta_2^{(t)}$ and $\xi^{(t)}$ from (14), (15) and (16), respectively.
Step 4: Update $t = t + 1$.
Step 5: Repeat step 2 to step 4 N times to simulate $\{(\eta_{11}, \eta_{21}, \xi_1), (\eta_{12}, \eta_{22}, \xi_2), \ldots, (\eta_{1N}, \eta_{2N}, \xi_N)\}$.
Step 6: Compute $\phi_i = \phi(\eta_{1i}, \eta_{2i}, \xi_i);\ i = 1, 2, \ldots, N$.
Step 7: Calculate $\psi_i = \psi(\eta_{1i}, \eta_{2i}, \xi_i);\ i = 1, 2, \ldots, N$. where

$$\psi(\eta_1,\eta_2,\xi) = \prod_{i=1}^n \prod_{j=1}^k \Big[1 - \Phi\Big(\frac{\ln x_{ij} - \eta_1}{\sqrt{\xi}}\Big)\Big]^{R_j} \prod_{i=1}^n \Big[1 - \Phi\Big(\frac{\ln y_i - \eta_2}{\sqrt{\xi}}\Big)\Big]^{S_i}.$$

Step 8: Find Bayes estimates of $\phi(\eta_1, \eta_2, \xi)$ as

$$\hat{\phi}(\eta_1, \eta_2, \xi) = \frac{\frac{1}{N}\sum_{i=1}^{N}\phi_i\psi_i}{\frac{1}{N}\sum_{i=1}^{N}\psi_i} = \sum_{i=1}^{N}\nu_i\phi_i, \quad \text{where} \quad \nu_i = \frac{\psi_i}{\sum_{i=1}^{N}\psi_i}.$$

Step 9: To obtained credible interval of $\phi(\eta_1, \eta_2, \xi)$, we order ϕ_i for $i = 1, 2, \ldots, N$, say, $\phi_{(1)}, \phi_{(2)}, \ldots, \phi_{(N)}$ and record the corresponding ν_i as $(\nu_{(1)}, \ldots, \nu_{(N)})$. A $100(1-\alpha)\%$ credible interval can be obtained as $(\phi_{(j_1)}, \phi_{(j_2)})$ where j_1, j_2 such that,

$$j_1 < j_2, \quad j_1, j_2 \in \{1, \ldots, N\} \text{ and } \sum_{i=j_1}^{j_2}\nu_i \le 1-\alpha < \sum_{i=j_1}^{j_2+1}\nu_i. \tag{17}$$

The $100(1-\alpha)\%$ HPD can be obtained as $(\phi_{(j_1^*)}, \phi_{(j_2^*)})$, such that $\phi_{(j_2^*)} - \phi_{(j_1^*)} \le \phi_{(j_2)} - \phi_{(j_1)}$ and j_1^*, j_2^* satisfy (17) for all j_1, j_2 satisfying (17).

Next we consider estimation of the reliability when other parameter is known and common.

3 Estimation when η is common parameter

In previous section, we obtained estimates for the reliability when parameter ξ is taken to be unknown and common. Now we estimate this parametric function when η is unknown and common parameter. We note that expression for the reliability gets modified in this set up.

3.1 MLE

We now consider computation of the maximum likelihood estimator of SM parameter under the given censoring scheme. Suppose $X_1, X_2, \cdots, X_k$ follow $LN(\eta, \xi_1)$ distribution and Y follows $LN(\eta, \xi_2)$ distribution and both data are independently distributed. The multicomponent reliability $R_{s,k}$ with unknown parameters η, ξ_1 and ξ_2 is given as

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k}\binom{k}{i}\int_0^\infty\left[1-\Phi\left(\frac{\ln y-\eta}{\sqrt{\xi_1}}\right)\right]^i\left[\Phi\left(\frac{\ln y-\eta}{\sqrt{\xi_1}}\right)\right]^{k-i}\frac{1}{\sqrt{\xi_2}y}\phi\left(\frac{\ln y-\eta}{\sqrt{\xi_2}}\right)dy \\ &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}(-1)^j\int_0^\infty\left[\Phi\left(\frac{\ln y-\eta}{\sqrt{\xi_1}}\right)\right]^{j+k-i}\frac{1}{\sqrt{\xi_2}y}\phi\left(\frac{\ln y-\eta}{\sqrt{\xi_2}}\right)dy \\ &= \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}(-1)^j\int_{-\infty}^\infty\left[\Phi\left(\sqrt{\frac{\xi_2}{\xi_1}}\,y\right)\right]^{j+k-i}\phi(y)dy. \end{aligned} \tag{18}$$

Now suppose N systems each with K components are subjected to a test and n systems are observed with k components under the specified censoring. Then observed stress data are $\{X_{i1}, X_{i2}, \cdots, X_{ik}\}$, $i = 1, 2, \cdots, n$ which follow $LN(\eta, \xi_1)$ distribution. These failure times are obtained from scheme $\{K, k, R_1, R_2, \cdots, R_k\}$. In similar manner, the data $\{Y_1, Y_2, \cdots, Y_n\}$

are observed from $LN(\eta, \xi_2)$ distribution under scheme $\{N, n, S_1, S_2, \cdots, S_n\}$. Therefore, the likelihood becomes

$$L(\eta, \xi_1, \xi_2; x, y) = c_1 c_2^n \xi_1^{nk/2} \xi_2^{n/2} \prod_{i=1}^{n} \left[1 - \Phi\left(\frac{\text{ln}y_i - \eta}{\sqrt{\xi_2}}\right)\right]^{S_i} \frac{1}{y_i} \phi\left(\frac{\text{ln}y_i - \eta}{\sqrt{\xi_2}}\right)$$
$$\times \prod_{i=1}^{n} \prod_{j=1}^{k} \left[1 - \Phi\left(\frac{\text{ln}x_{ij} - \eta}{\sqrt{\xi_1}}\right)\right]^{R_j} \frac{1}{x_{ij}} \phi\left(\frac{\text{ln}x_{ij} - \eta}{\sqrt{\xi_1}}\right). \tag{19}$$

The log-likelihood is given as

$$\text{ln}L(\eta, \xi_1, \xi_2; x, y) \propto -\frac{nk}{2} \text{ln}\xi_1 - \frac{n}{2} \text{ln}\xi_2 - \sum_{i=1}^{n} \text{ln}y_i - \sum_{i=1}^{n} \sum_{j=1}^{k} \text{ln}x_{ij} + \sum_{i=1}^{n} \ln\left[\phi\left(\frac{\text{ln}y_i - \eta}{\sqrt{\xi_2}}\right)\right]$$
$$+ \sum_{i=1}^{n} S_i \ln\left[1 - \Phi\left(\frac{\text{ln}y_i - \eta}{\sqrt{\xi_2}}\right)\right] + \sum_{i=1}^{n} \sum_{j=1}^{k} \ln\left[\phi\left(\frac{\text{ln}x_{ij} - \eta}{\sqrt{\xi_1}}\right)\right]$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \ln\left[1 - \Phi\left(\frac{\text{ln}x_{ij} - eta}{\sqrt{\xi_1}}\right)\right]. \tag{20}$$

This is rewritten as

$$l(\eta, \xi_1, \xi_2; x, y) \propto -\frac{nk}{2} \text{ln}\xi_1 - \frac{n}{2} \text{ln}\xi_2 - \sum_{i=1}^{n} \text{ln}y_i - \sum_{i=1}^{n} \sum_{j=1}^{k} \text{ln}x_{ij} + \sum_{i=1}^{n} \ln\left[\phi\left(u_i\right)\right]$$
$$+ \sum_{i=1}^{n} S_i \ln\left[1 - \Phi\left(u_i\right)\right] + \sum_{i=1}^{n} \sum_{j=1}^{k} \ln\left[\phi\left(v_{ij}\right)\right] + \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \ln\left[1 - \Phi\left(v_{ij}\right)\right], \tag{21}$$

where u_i and v_{ij} are denoted as $u_i = \left(\frac{\text{ln}y_i - \eta}{\sqrt{\xi_2}}\right)$ and $v_{ij} = \left(\frac{\text{ln}x_{ij} - \eta}{\sqrt{\xi_1}}\right)$. The MLEs of the parameters η, ξ_1 and ξ_2 are computed from the likelihood equations as obtained below

$$\frac{\partial l}{\partial \eta} = \frac{1}{\sqrt{\xi_2}} \left[\sum_{i=1}^{n} u_i + \sum_{i=1}^{n} S_i \frac{\phi(u_i)}{[1 - \Phi(u_i)]}\right] + \frac{1}{\sqrt{\xi_1}} \left[\sum_{i=1}^{n} \sum_{j=1}^{k} v_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \frac{\phi(v_{ij})}{[1 - \Phi(v_{ij})]}\right], \tag{22}$$

$$\frac{\partial l}{\partial \xi_1} = \frac{1}{2\xi_1} \left[-nk + \sum_{i=1}^{n} \sum_{j=1}^{k} v_{ij}^2 + \sum_{i=1}^{n} \sum_{j=1}^{k} R_j \frac{\phi(v_{ij})}{[1 - \Phi(v_{ij})]}\right], \tag{23}$$

$$\frac{\partial l}{\partial \xi_2} = \frac{1}{2\xi_2} \left[-n + \sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} S_i \frac{\phi(u_i)}{[1 - \Phi(u_i)]}\right]. \tag{24}$$

The estimates of η, ξ_1 and ξ_2 denoted as $\hat{\eta}$, $\hat{\xi_1}$ and $\hat{\xi_2}$ respectively, are computed numerically by solving the above non-linear likelihood equations simultaneously. Using these estimates the maximum likelihood estimate of $R_{s,k}$ denoted as $\hat{R_{s,k}}$ is obtained. We evaluate these estimates numerically in simulation experiments.

3.2 Asymptotic intervals

Now we obtain ACI of the an SM reliability using large sample approximation. We compute these estimate under the given censoring scheme. The observed Fisher information matrix of θ as

$$I(\theta) = [I_{ij}]_{3\times 3} = \begin{pmatrix} -\frac{\partial^2 l}{\partial \eta^2} & -\frac{\partial^2 l}{\partial \eta \partial \xi_1} & -\frac{\partial^2 l}{\partial \eta \partial \xi_2} \\ -\frac{\partial^2 l}{\partial \xi_1 \partial \eta} & -\frac{\partial^2 l}{\partial \xi_1^2} & -\frac{\partial^2 l}{\partial \xi_1 \partial \xi_2} \\ -\frac{\partial^2 l}{\partial \xi_2 \partial \eta} & -\frac{\partial^2 l}{\partial \xi_2 \partial \xi_1} & -\frac{\partial^2 l}{\partial \xi_2^2} \end{pmatrix}.$$

We now note that MLE of reliability is approximately distributed like normal variable with mean $R_{s,k}$ and variance

$$\hat{V}(\hat{R}_{s,k}) = \Delta \hat{R}_{s,k}^T I^{-1}(\hat{\theta}) \Delta \hat{R}_{s,k},$$

where, $\Delta \hat{R}_{s,k} = \left(\frac{\partial R_{s,k}}{\partial \eta}, \frac{\partial R_{s,k}}{\partial \xi_1}, \frac{\partial R_{s,k}}{\partial \xi_2} \right)\Big|_{\theta=\hat{\theta}}$. Thus, $100(1-\alpha)\%$ confidence interval of $R_{s,k}$ is

$$\left(\hat{R}_{s,k} - z_{\alpha/2}\sqrt{\hat{V}(\hat{R}_{s,k})},\ \hat{R}_{s,k} + z_{\alpha/2}\sqrt{\hat{V}(\hat{R}_{s,k})} \right),$$

where $z_{\alpha/2}$ denotes $\alpha/2$th quantile of $N(0,1)$ distribution. We mention that coefficient of considered matrix $I_{ij},\ i,j = 1,2,3$ and also coefficient of $\Delta R_{s,k}$ are evaluated in subsection 3.3.1. We are not presenting these computations here for the sake of brevity.

3.3 Bayesian estimation

Now Bayes estimators of an SM parameter are investigated under SE loss function. It is assumed that η is a priori distributed as normal $N(p_1, q_1)$ distribution and ξ_1 and ξ_2 has independent inverse gamma $IG(p_2, q_2)$ and $IG(p_3, q_3)$ distributions respectively with density $IG(p_i, q_i) \propto \xi_i^{-(p_i+1)}\, e^{-q_i/\xi_i},\ \xi_i > 0,\ p_i > 0,\ q_i > 0,\ i = 2,3.$

The joint posterior of η_1, η_2 and ξ is expressed as

$$\begin{aligned} \pi(\eta, \xi_1, \xi_2 \mid x, y) \quad \propto \quad & IG_{\xi_1|\eta}\Big(\frac{nk}{2} + p_2,\, q_2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k} (\ln x_{ij} - \eta)^2 \Big) IG_{\xi_2|\eta}\Big(\frac{n}{2} + p_3, q_3 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k} (\ln y_i - \eta)^2 \Big) \\ & \frac{e^{-\frac{1}{2q_1}(\eta-p_1)^2} \prod_{i=1}^n \prod_{j=1}^k x_{ij}^{-1} \left[1 - \Phi\Big(\frac{\ln x_{ij}-\eta}{\sqrt{\xi_1}}\Big)\right]^{R_j} \prod_{i=1}^n y_i^{-1} \left[1 - \Phi\Big(\frac{\ln y_i - \eta}{\sqrt{\xi_2}}\Big)\right]^{S_i}}{\left[q_2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^k (\ln x_{ij} - \eta)^2\right]^{\frac{nk}{2}+p_2} \left[q_3 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^k (\ln y_i - \eta)^2\right]^{\frac{n}{2}+p_3}}. \end{aligned} \tag{25}$$

The Bayes estimator δ_B of multicomponent reliability against the squared error loss is given by the posterior mean of $R_{s,k}$, that is $\delta_B = E(R_{s,k} \mid x, y)$ where

$$E(R_{s,k} \mid x, y) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} R_{s,k}\, \pi(\eta, \xi_1, \xi_2 \mid x, y)\, d\eta d\xi_1 d\xi_2}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} \pi(\eta, \xi_1, \xi_2 \mid x, y)\, d\eta d\xi_1 d\xi_2}.$$

This estimator also appears in the form of integrals. This may not be simplified further as underlined distribution is not known properly. Thus here also we use previously discussed methods to obtain the required estimate of the reliability.

3.3.1 Lindley method

Here again we approximate posterior mean of $h(\theta)$ using the given posterior distribution. Here we take $(\theta_1, \theta_2, \theta_3) = (\eta, \xi_1, \xi_2)$ and also note that $h(\theta) = R_{s,k}$. Proceeding as earlier we find that various terms involved in approximating the posterior mean of $R_{s,k}$ are obtained as follow: $\rho_1 = -\left(\frac{\eta - p_1}{2q_1}\right)$, $\rho_2 = -\frac{p_2+1}{\xi_1} + \frac{q_2}{\xi_1^2}$ and $\rho_3 = -\frac{p_3+1}{\xi_2} + \frac{q_3}{\xi_2^2}$,

$$L_{11} = \frac{\partial^2 l}{\partial \eta^2} = -\frac{1}{\xi_1}\left[nk - \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{\phi(v_{ij})\{(1-\Phi(v_{ij}))v_{ij} - \phi(v_{ij})\}}{(1-\Phi(v_{ij}))^2}\right]$$
$$-\frac{1}{\xi_2}\left[n - \sum_{i=1}^{n} S_i \frac{\phi(u_i)\{(1-\Phi(u_i))u_i - \phi(u_i)\}}{(1-\Phi(u_i))^2}\right],$$

$$L_{22} = \frac{\partial^2 l}{\partial \xi_1^2} = -\frac{1}{2\xi_1^2}\left[-nk + 2\sum_{i=1}^{n}\sum_{j=1}^{k} v_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}\phi(v_{ij})}{(1-\Phi(v_{ij}))} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \right.$$
$$\left. \times \frac{v_{ij}^2\phi(v_{ij})\left[\phi(v_{ij}) - v_{ij}(1-\Phi(v_{ij}))\right]}{(1-\Phi(v_{ij}))^2}\right],$$

$$L_{33} = \frac{\partial^2 l}{\partial \xi_2^2} = -\frac{1}{2\xi_2^2}\left[-n + 2\sum_{i=1}^{n} u_i^2 + 2\sum_{i=1}^{n} S_i \frac{u_i\phi(u_i)}{(1-\Phi(u_i))} + \sum_{i=1}^{n} S_i \frac{u_i^2\phi(u_i)\left[\phi(u_i) - u_i(1-\Phi(u_i))\right]}{(1-\Phi(u_i))^2}\right],$$

$$L_{12} = \frac{\partial^2 l}{\partial \eta \partial \xi_1} = -\frac{1}{2\xi_1^{3/2}}\left[2\sum_{i=1}^{n}\sum_{j=1}^{k} v_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{\phi(v_{ij})(1-v_{ij}^2)}{(1-\Phi(v_{ij}))} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}(\phi(v_{ij}))^2}{(1-\Phi(v_{ij}))^2}\right],$$

$$L_{13} = \frac{\partial^2 l}{\partial \eta \partial \xi_2} = -\frac{1}{2\xi_2^{3/2}}\left[2\sum_{i=1}^{n} u_i + \sum_{i=1}^{n} S_i \frac{\phi(u_i)(1-u_i^2)}{(1-\Phi(u_i))} + \sum_{i=1}^{n} S_i \frac{u_i(\phi(u_i))^2}{(1-\Phi(u_i))^2}\right],$$

$$L_{23} = \frac{\partial^2 l}{\partial \xi_1 \partial \xi_2} = L_{32} = \frac{\partial^2 l}{\partial \xi_2 \partial \xi_1} = 0,$$

$$L_{111} = -\frac{1}{\xi_1^{3/2}}\sum_{i=1}^{n}\sum_{j=1}^{k} R_j\left[\frac{\phi(v_{ij})(1-v_{ij}^2)}{(1-\Phi(v_{ij}))} + 3v_{ij}\left[\frac{\phi(v_{ij})}{(1-\Phi(v_{ij}))}\right]^2 - 2\left[\frac{\phi(v_{ij})}{(1-\Phi(v_{ij}))}\right]^3\right]$$
$$-\frac{1}{\xi_2^{3/2}}\sum_{i=1}^{n} S_i\left[\frac{\phi(u_i)(1-u_i)^2}{(1-\Phi(u_i))} + 3u_i\left[\frac{\phi(u_i)}{(1-\Phi(u_i))}\right]^2 - 2\left[\frac{\phi(u_i)}{(1-\Phi(u_i))}\right]^3\right],$$

$$L_{222} = \frac{A^*}{\xi_1^3} + \frac{1}{2\xi_3^2}\left[2\sum_{i=1}^{n}\sum_{j=1}^{k} v_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}\phi(v_{ij})}{(1-\Phi(v_{ij}))^2}\left((1-\Phi(v_{ij}))(1-v_{ij}^2) + v_{ij}\phi(v_{ij})\right)\right.$$
$$\sum_{i=1}^{n}\sum_{j=1}^{k} R_j\left[\frac{v_{ij}^2\phi(v_{ij})}{(1-\Phi(v_{ij}))^3}\left((v_{ij}^2-2)(1-\Phi(v_{ij})) - v_{ij}\phi(v_{ij})(\phi(v_{ij}) - v_{ij}(1-\Phi(v_{ij})))\right)\right.$$
$$\left.\left. + \frac{v_{ij}^3\phi(v_{ij})}{(1-\Phi(v_{ij}))}\right]\right],$$

where

$$A^* = \left[-nk + 2\sum_{i=1}^{n}\sum_{j=1}^{k} v_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}\phi(v_{ij})}{(1-\Phi(v_{ij}))^2} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}^2\phi(v_{ij})}{(1-\Phi(v_{ij}))^2}\left(\phi(v_{ij}) - v_{ij}(1-\Phi(v_{ij}))\right)\right].$$

$$\begin{aligned} L_{333} &= \frac{B^*}{\xi_2^3} + \frac{1}{2\xi_2^3}\left[2\sum_{i=1}^{n} u_i^2 + \sum_{i=1}^{n} S_i \frac{u_i\phi(u_i)}{(1-\Phi(u_i))^2}\big((1-\Phi(u_i))(1-u_i^2) + u_i\phi(u_i)\big) - \sum_{i=1}^{n} S_i \frac{u_i^2\phi(u_i)}{(1-\Phi(u_i))^3}\right. \\ &\quad \left. \times \big((u_i^2-2)(1-\Phi(u_i)) - u_i\phi(u_i)(\phi(u_i) - u_i(1-\Phi(u_i)))\big) + \frac{u_i^2\phi(u_i)}{(1-\Phi(u_i))}\right], \end{aligned}$$

with

$$B^* = \left[-k + 2\sum_{i=1}^{n} u_i^2 + 2\sum_{i=1}^{n} S_i \frac{u_i\phi(u_i)}{(1-\Phi(u_i))} + \sum_{i=1}^{n} S_i \frac{u_i^2\phi(u_i)}{(1-\Phi(u_i))^2}\left(\phi(u_i) - u_i(1-\Phi(u_i))\right)\right].$$

$$\begin{aligned} L_{122} = L_{212} = L_{221} &= \frac{3C^*}{4\xi_1^{5/2}} + \frac{1}{4\xi_1^{5/2}}\left[2\sum_{i=1}^{n}\sum_{j=1}^{k} v_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}\phi(v_{ij})}{(1-\Phi(v_{ij}))^2}\big[v_{ij}(v_{ij}^3-3)(1-\Phi(v_{ij}))\right. \\ &\quad \left. -(v_{ij}^2-1)\phi(v_{ij})\big] - \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}(\phi(v_{ij}))^2}{(1-\Phi(v_{ij}))^3}\big((v_{ij}^2-2) - 2v_{ij}\phi(v_{ij})\big)\right], \end{aligned}$$

where

$$C^* = \left[2\sum_{i=1}^{n}\sum_{j=1}^{k} v_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{\phi(v_{ij})(1-v_{ij}^2)}{(1-\Phi(v_{ij}))} + \sum_{i=1}^{n}\sum_{j=1}^{k} R_j \frac{v_{ij}(\phi(v_{ij}))^2}{(1-\Phi(v_{ij}))^2}\right],$$

$$\begin{aligned} L_{133} = L_{313} = L331 &= \frac{3D^*}{4\xi_2^{5/2}} + \frac{1}{4\xi_2^{5/2}}\left[2\sum_{i=1}^{n} u_i + \sum_{i=1}^{n} S_i \frac{u_i\phi(u_i)}{(1-\Phi(u_i))^2}\big(u_i(u_i^3-3)(1-\Phi(u_i))\right. \\ &\quad \left. -(u_i^2-1)\phi(u_i)\big) - \sum_{i=1}^{n} S_i \frac{u_i(\phi(u_i))^2}{(1-\Phi(u_i))^3}\big((u_i^2-1) - 2u_i\phi(u_i)\big)\right], \end{aligned}$$

and

$$D^* = \left[2\sum_{i=1}^{n} u_i + \sum_{i=1}^{n} S_i \frac{(1-u_i^2)\phi(u_i)}{(1-\Phi(u_i))} + \sum_{i=1}^{n} \frac{u_i(\phi(u_i))^2}{(1-\Phi(u_i))^2}\right],$$

$$L_{123} = L_{213} = L_{231} = L_{132} = L_{312} = L_{321} = L_{233} = L_{323} = L_{332} = L_{223} = L_{232} = L_{322} = 0.$$

Suppose $\chi = \sqrt{\frac{\xi_2}{\xi_1}}y$, then we have the following expressions

$$h_1 = \frac{\partial R_{s,k}}{\partial \eta} = 0,$$

$$h_2 = \frac{\partial R_{s,k}}{\partial \xi_1} = \sum_{i=1}^{n}\sum_{j=1}^{k}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}\sqrt{\xi_2}(j+k-i)}{2\xi_1^{3/2}}\int_{-\infty}^{\infty}[\Phi(\chi)]^{j+k-i-1}\phi\left(\chi\right)\phi(y)ydy,$$

$$h_3 = \frac{\partial R_{s,k}}{\partial \xi_2} = \sum_{i=1}^{n}\sum_{j=1}^{k}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j}(j+k-i)}{2\sqrt{\xi_1\xi_2}}\int_{-\infty}^{\infty}[\Phi(\chi)]^{j+k-i-1}\phi\left(\chi\right)\phi(y)ydy,$$

$$\begin{aligned}h_{22} = \frac{\partial^2 R_{s,k}}{\partial \xi_1^2} \quad = \quad & \sum_{i=1}^{n}\sum_{j=1}^{k}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j}\sqrt{\xi_2}(j+k-i)}{4\xi_1^{3}}\int_{-\infty}^{\infty}[\Phi(\chi)]^{j+k-i-2}\phi(\chi) \\ & \left[\frac{1}{\sqrt{\xi_1}}(\xi_2 y^2 - 3\xi_1)\Phi(\chi) - (j+k-i-1)\sqrt{\xi_2}\, y\,\phi(\chi)\right]\phi(y)ydy,\end{aligned}$$

$$\begin{aligned}h_{33} = \frac{\partial^2 R_{s,k}}{\partial \xi_2^2} \quad = \quad & \sum_{i=1}^{n}\sum_{j=1}^{k}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}(j+k-i)}{4\xi_1\sqrt{\xi_2}}\int_{-\infty}^{\infty}[\Phi(\chi)]^{j+k-i-2}\phi(\chi) \\ & \left[\frac{1}{\sqrt{\xi_1}}(\xi_1\xi_2^{-1} + y^2)\Phi(\chi) - (j+k-i-1)\, y\,\phi(\chi)\right]\phi(y)ydy,\end{aligned}$$

$$\begin{aligned}h_{23} = \frac{\partial^2 R_{s,k}}{\partial \xi_1 \xi_2} \quad = \quad & \sum_{i=1}^{n}\sum_{j=1}^{k}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}(j+k-i)}{4\xi_1^{3/2}}\int_{-\infty}^{\infty}[\Phi(\chi)]^{j+k-i-2}\phi(\chi) \\ & \left[\frac{1}{\xi_1\sqrt{\xi_2}}(\xi_1 - \xi_2 y^2)\Phi(\chi) + (j+k-i-1)\, y\,\phi(\chi)\right]\phi(y)ydy.\end{aligned}$$

The estimate of reliability is now given as

$$\hat{R}^L_{s,k} = R_{s,k} + (h_2 b_2 + h_3 b_3 + b_4 + b_5) + 0.5\left[a_1(h_2\sigma_{12} + h_3\sigma_{13}) + a_2(h_2\sigma_{22} + h_3\sigma_{23}) + a_3(h_2\sigma_{32} + h_3\sigma_{33})\right].$$

The Lindley method gives the point estimates only and could not provide the interval estimates. Therefore, MCMC technique is used to generate samples for the model parameters (η_1, η_2, ξ) and desired intervals are obtained for the considered parametric function.

3.3.2 MCMC method

In this subsection, using Metropolis–Hastings (M-H) sampling we obtain Bayes estimate of the unknown parametric function. The marginal distributions of η, ξ_1 and ξ_2 are of the following form

$$\begin{aligned}\xi_1 \mid \eta, \text{data} &\sim IG_{\xi_1|\eta}\Big(\frac{nk}{2} + p_2,\; q_2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}\left(\ln x_{ij} - \eta\right)^2\Big), \\ \xi_2 \mid \eta, \text{data} &\sim IG_{\xi_2|\eta}\Big(\frac{n}{2} + p_3,\; q_3 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}\left(\ln y_i - \eta\right)^2\Big), \qquad (26)\end{aligned}$$

and

$$\pi(\eta \mid \xi_1, \xi_2, \text{data}) \propto \frac{e^{-\frac{1}{2q_1}(\eta-p_1)^2} \prod_{i=1}^{n} \prod_{j=1}^{k} x_{ij}^{-1} \left[1 - \Phi\left(\frac{\ln x_{ij}-\eta}{\sqrt{\xi_1}}\right)\right]^{R_j} \prod_{i=1}^{n} y_i^{-1} \left[1 - \Phi\left(\frac{\ln y_i-\eta}{\sqrt{\xi_2}}\right)\right]^{S_i}}{\left[q_2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}(\ln x_{ij}-\eta)^2\right]^{\frac{nk}{2}+p_2} \left[q_3 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}(\ln y_i-\eta)^2\right]^{\frac{n}{2}+p_3}}. \tag{27}$$

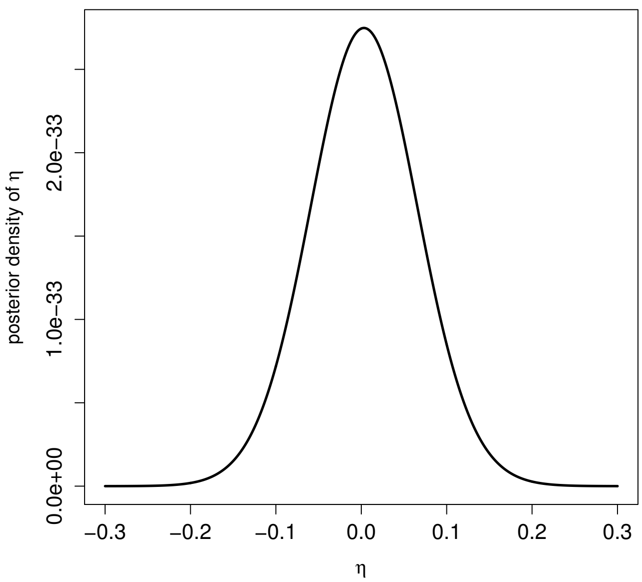

Figure 1: Posterior density of η.

Samples for ξ_1 and ξ_2 can be obtained from their posterior densities. Note that associated distribution of η is of unknown form. Accordingly required samples cannot be obtained in a straightforward manner. We try to apply M-H algorithm for this purpose by assuming a normal proposal density. We have plotted posterior of η in Figure 1 which suggests that a normal distribution may be used for observing required data. Thus required samples can be simulated under this framework. Particularly credible intervals can be evaluated from following steps.
Step 1: Consider a guess $(\eta^{(0)}, \xi_1^{(0)}, \xi_2^{(0)})$.
Step 2: Set $t = 1$.
Step 3: Obtain $\xi_1^{(t)}$ from $IG_{\xi_1|\eta}\left(\frac{nk}{2} + p_2,\ q_2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}(\ln x_{ij}-\eta)^2\right)$ distribution.
Step 4: Obtain $\xi_2^{(t)}$ from $IG_{\xi_2|\eta}\left(\frac{n}{2} + p_3,\ q_3 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}(\ln y_i-\eta)^2\right)$ distribution.
Step 5: Obtain $\eta^{(t)}$ from $\pi(\eta \mid \xi_1^{(t-1)}, \xi_2^{(t-1)}, \text{data})$ using $N(\gamma^{(t-1)}, \sigma_\eta)$ variable.
Step 6: Evaluate $R_{s,k}^{(t)}$ at $(\eta_1^{(t)}, \eta_2^{(t)}, \xi^{(t)})$.
Step 7: Update $t = t + 1$.
Step 8: Repeat step 3 to step 7, to obtain data $R_{s,k}^{(t)},\ t = 1, 2, \ldots, N$.
Also required estimate of an SM reliability is obtained as

$$\hat{R}_{s,k}^{MC} = \frac{1}{N - N_0} \sum_{t=N_0+1}^{N} R_{s,k}^{(t)}.$$

where N_0 is the burn-in period.

Step 9: We first order $R_{s,k}^1, \ldots, R_{s,k}^N$ as $R_{s,k}^1 < \cdots < R_{s,k}^N$ and construct $100(1-\alpha)\%$ credible interval by removing $\left[(N-N_0) * \frac{\alpha}{2}\right]$ sorted $R_{s,k}$ values from both the tails. Furthermore, to construct the HPD interval for $R_{s,k}$, now consider $I_j = \left[R_{s,k}^{(j)}, R_{s,k}^{(j+[(N-N_0)\times(1-\alpha)])}\right]$, for $j = N, N+1, \ldots, (N-N_0)-[(N-N_0)\times(1-\alpha)]$, where $[.]$ is greatest integer function. The interval with smallest length is considered as the $100(1-\alpha)\%$ HPD interval of $R_{s,k}$.

4 Simulation results

We now compare the performance of discussed classical and Bayesian methods namely MLE, Lindley and MCMC estimates of the SM reliability. Intervals are also evaluated. We have considered comparisons of point estimates based on estimated mean square error (MSE) values. On the other hand, the comparison of performance of intervals is based on average interval lengths and corresponding coverage probabilities. For the computational purpose, we use R statistical software. In our study, we consider two different cases. In first case, the parameter ξ is unknown and common. In this case we assume two different sets of parameter values for evaluating estimates of reliability. The assignment is like $\theta_1 = (\eta_1, \eta_2, \xi) = (0.01, 0.01, 1.5)$ and $\theta_2 = (\eta_1, \eta_2, \xi) = (-0.01, -0.01, 1.5)$. We also take $K = 10, k = 6,\ N = 15, n = 10$. Here K is number of components in SM. Among these only k failure times are observed. The other $(K-k)$ components are like censored units. Similarly if N number of systems are put on a test and n failure times are to be observed then $(N-n)$ units considered as censored units. We evaluate bias and risk values of estimators of reliability by employing various schemes. In this regard we denote censoring schemes as $R_1 = (4, 0^{*5}),\ R_2 = (1^{*4}, 0^{*2}),\ R_3 = (0^{*5}, 4)$ and $S_1 = (5, 0^{*9}),\ S_2 = (1^{*5}, 0^{*5}),\ S_3 = (0^{*9}, 5)$. Also s is assigned as $s = 1, 2$. This suggests that an SM parameter is evaluated for the case where at least one component survives or at least two components properly functions under some specific stress force. For Bayesian estimation of parametric function, we set hyper parameters values as $p_0 = 0.01, q_0 = 1, p_1 = 0.01, q_1 = 1, p_2 = 3, q_2 = 3$ for the first set of parameter θ_1 and $p_0 = -0.01, q_0 = 1, p_1 = -0.01, q_1 = 1, p_2 = 3, q_2 = 3$ for parameter θ_2. We have tabulated the point estimates along with their MSEs and 95% asymptotic and HPD intervals length along with associated coverage probabilities (CPs) with respects to the first set of parameter values θ_1 in Table 1 and for second set of parameters θ_2 in Table 2.

We also compute classical as well as Bayesian estimates of reliability for the second case when η is common and unknown. Here also we consider two sets of parameters $\theta_1 = (\eta, \xi_1, \xi_2) = (-0.01, 0.5, 0.5)$ and $\theta_2 = (\eta, \xi_1, \xi_2) = (0.01, 0.25, 0.5)$. The values of $K,\ k,\ N$ and n are set as $K = 10,\ k = 6,\ N = 25,\ n = 10$. For this case also, estimates of reliability are computed based on three different censoring schemes like $R_1 = (4, 0^{*5}),\ R_2 = (1^{*4}, 0^{*2}),\ R_3 = (0^{*2}, 2^{*2}, 0^{*2})$ and $S_1 = (5, 0^{*9}),\ S_2 = (1^{*5}, 0^{*5}),\ S_3 = (0^{*4}, 3, 2, 0^{*4})$. Furthermore, the estimates are obtained by assuming two different values of $s = 1, 2$, which means, estimates are computed when at least one component or at least two components survive at a given stress force. For Bayesian estimates, we set the hyperparameters value as $p_1 = -0.02, q_1 = 2, p_2 = 3, q_2 = 1, p_3 = 3, q_3 = 1$ for the first set of parameters θ_1 and are set as $p_1 = 0.03, q_1 = 3, p_2 = 3, q_2 = 1, p_3 = 3, q_3 = 0.5$ for θ_2. The estimated values, MSEs, interval lengths and corresponding CPs of $R_{s,k}$ under classical as well as Bayesian methods are tabulated in Table 3 for first set of parameters θ_1 and for the second set of parameters θ_2 the results are tabulated in Table 4. In Tables 1 - 4, the MSEs of each of the estimates are tabulated just below corresponding estimated values and the CPs of each of the intervals are tabulated below the corresponding interval lengths.

From the Tables 1-4, we observed that Bayes estimates having lesser MSE values than classical estimates. Thus Bayes estimates of reliability perform better than the MLE. On comparing Lindley and MCMC methods, it is observed that MCMC procedure shows better performance than the Lindley method. This holds for both the set of parameter values and for the considered censoring schemes $(R_i, S_i),\ i = 1, 2, 3$. Furthermore, from Tables 1-4, on comparing the interval estimates based on their lengths, we observed that HPD interval have smaller lengths than the asymptotic intervals and the coverage probabilities for asymptotic intervals is lower than the nominal level where as HPD intervals perform quite good in terms of their coverage probabilities.

Table 1: Estimation of $R_{s,k}$ when parameter ξ is common and unknown..

		Estimation			Interval length	
$(s,\, \theta_i,\, R_{s,k})$	CS	MLE	Lindley	MCMC	Avg CI Length	HPD Length
(1, θ_1, 0.857143)	(R_1, S_1)	0.834311	0.840336	0.840815	0.250610	0.228775
		0.006128	0.005164	0.000289	0.825	0.996
	(R_1, S_2)	0.867660	0.859806	0.856005	0.212133	0.207283
		0.004633	0.004264	0.000127	0.877	0.989
	(R_1, S_3)	0.876673	0.876580	0.864362	0.197108	0.189418
		0.004456	0.004050	0.000386	0.686	0.988
	(R_2, S_1)	0.807438	0.831395	0.837649	0.251863	0.240131
		0.006639	0.005391	0.002478	0.820	0.994
	(R_2, S_2)	0.829605	0.865507	0.853343	0.212549	0.219205
		0.004954	0.004425	0.000765	0.727	0.987
	(R_2, S_3)	0.874431	0.861410	0.846646	0.197654	0.189844
		0.00497	0.004173	0.000116	0.682	0.987
	(R_3, S_1)	0.803681	0.831966	0.838815	0.250056	0.213503
		0.006661	0.00532	0.002864	0.819	0.992
	(R_3, S_2)	0.816612	0.866299	0.853351	0.219273	0.211196
		0.004989	0.004386	0.001649	0.727	0.984
	(R_3, S_3)	0.875093	0.832735	0.861407	0.219641	0.196403
		0.004758	0.004137	0.000602	0.681	0.983
(2, θ_1, 0.714286)	(R_1, S_1)	0.683995	0.691482	0.698376	0.371489	0.350916
		0.013234	0.013083	0.000535	0.825	0.993
	(R_1, S_2)	0.739104	0.722646	0.723541	0.331660	0.321544
		0.011857	0.011213	0.000085	0.762	0.987
	(R_1, S_3)	0.804937	0.698189	0.713884	0.333388	0.266857
		0.010751	0.010557	0.008227	0.815	0.991
	(R_2, S_1)	0.636475	0.679445	0.694370	0.370917	0.350018
		0.013998	0.012018	0.006072	0.827	0.984
	(R_2, S_2)	0.670150	0.735873	0.719473	0.330775	0.324481
		0.012452	0.011676	0.001965	0.754	0.992
	(R_2, S_3)	0.632007	0.762270	0.710252	0.332262	0.307356
		0.011427	0.114002	0.002315	0.805	0.990
	(R_3, S_1)	0.742913	0.693251	0.713828	0.369682	0.356521
		0.013963	0.004169	0.000067	0.822	0.921
	(R_3, S_2)	0.645175	0.741732	0.726123	0.329059	0.321010
		0.011484	0.010107	0.004794	0.771	0.988
	(R_3, S_3)	0.753921	0.684945	0.737629	0.311819	0.309681
		0.011917	0.010290	0.000874	0.730	0.957

Table 2: Estimation of $R_{s,k}$ when parameter ξ is common and unknown.

		Estimation			Interval length	
$(s,\, \theta_i,\, R_{s,k})$	CS	MLE	Lindley	MCMC	Avg CI Length	HPD Length
$(1,\, \theta_2,\, 0.857143)$	(R_1, S_1)	0.835805	0.840367	0.842953	0.248422	0.229208
		0.006012	0.005096	0.000287	0.825	0.933
	(R_1, S_2)	0.869593	0.859787	0.857639	0.210082	0.207682
		0.004606	0.004199	0.000126	0.731	0.965
	(R_1, S_3)	0.90968	0.842997	0.854321	0.217268	0.154694
		0.004491	0.004321	0.002763	0.797	0.999
	(R_2, S_1)	0.807616	0.833944	0.841482	0.248174	0.240125
		0.006341	0.005214	0.002461	0.821	0.981
	(R_2, S_2)	0.826789	0.868442	0.855606	0.209640	0.200145
		0.004848	0.004303	0.000928	0.727	0.974
	(R_2, S_3)	0.884228	0.840563	0.852874	0.216777	0.190155
		0.004783	0.004464	0.000738	0.791	0.999
	(R_3, S_1)	0.793019	0.835205	0.839008	0.249413	0.212484
		0.006524	0.005269	0.004122	0.815	0.923
	(R_3, S_2)	0.787832	0.867364	0.857120	0.2311757	0.209450
		0.004836	0.004814	0.004395	0.720	0.911
	(R_3, S_3)	0.824600	0.851493	0.852325	0.271290	0.216402
		0.004789	0.004521	0.001069	0.785	0.969
$(2,\, \theta_2,\, 0.714286)$	(R_1, S_1)	0.743697	0.731109	0.701623	0.369860	0.190953
		0.012992	0.004141	0.000287	0.838	0.947
	(R_1, S_2)	0.760329	0.744741	0.742225	0.329751	0.206308
		0.011906	0.06456	0.000933	0.762	0.999
	(R_1, S_3)	0.807466	0.745994	0.717294	0.331585	0.236635
		0.010669	0.004351	0.000691	0.815	0.912
	(R_2, S_1)	0.743352	0.717082	0.699791	0.328059	0.175111
		0.013616	0.004077	0.000713	0.755	0.954
	(R_2, S_2)	0.759674	0.740826	0.723458	0.328056	.0.175121
		0.012336	0.006354	0.000128	0.757	0.977
	(R_2, S_3)	0.772015	0.745443	0.715334	0.329739	0.229529
		0.011222	0.004276	0.000340	0.807	0.986
	(R_3, S_1)	0.744032	0.696503	0.713749	0.367969	0.202219
		0.013804	0.004281	0.000371	0.823	0.989
	(R_3, S_2)	0.760741	0.738941	0.714172	0.326961	0.104045
		0.012349	0.006675	0.000308	0.747	0.990
	(R_3, S_3)	0.746399	0.731758	0.713366	0.327963	0.208713
		0.011256	0.004533	0.000310	0.797	0.985

Table 3: Estimation of $R_{s,k}$ when parameter η is common and unknown.

$(s,\,\theta_i,\,R_{s,k})$	CS	Estimation			Interval length	
		MLE	Lindley	MCMC	Avg CI Length	HPD Length
(1, θ_1, 0.612014)	(R_1, S_1)	0.629836	0.623958	0.619021	0.124812	0.116323
		0.003331	0.002166	0.001495	0.818	0.926
	(R_1, S_2)	0.628705	0.623636	0.617539	0.132295	0.123486
		0.003663	0.002245	0.001665	0.817	0.931
	(R_1, S_3)	0.606716	0.60720	0.613752	0.204518	0.132938
		0.005204	0.004471	0.000526	0.795	0.947
	(R_2, S_1)	0.630685	0.623798	0.618977	0.125141	0.110864
		0.003321	0.002179	0.001381	0.818	0.955
	(R_2, S_2)	0.630574	0.624068	0.618606	0.128467	0.111719
		0.003408	0.002817	0.001395	0.819	0.922
	(R_2, S_3)	0.621376	0.613541	0.608808	0.202222	0.132923
		0.004504	0.004924	0.002554	0.975	0.926
	(R_3, S_1)	0.630405	0.623729	0.618977	0.125361	0.112383
		0.003317	0.002184	0.001421	0.829	0.967
	(R_3, S_2)	0.630259	0.623988	0.618604	0.128631	0.113179
		0.003407	0.002191	0.001446	0.919	0.925
	(R_3, S_3)	0.625073	0.613224	0.611024	0.141503	0.136457
		0.004546	0.003261	0.0021196	0.875	0.933
(2, θ_1, 0.511685)	(R_1, S_1)	0.508208	0.518753	0.517041	0.109612	0.095258
		0.002325	0.001005	0.000702	0.834	0.919
	(R_1, S_2)	0.507866	0.519102	0.516992	0.112904	0.095292
		0.002294	0.001021	0.000693	0.836	0.977
	(R_1, S_3)	0.498071	0.518518	0.513801	0.168206	0.116807
		0.003511	0.003011	0.000751	0.910	0.928
	(R_2, S_1)	0.508091	0.518662	0.517674	0.109707	0.091391
		0.002244	0.000932	0.000703	0.834	0.954
	(R_2, S_2)	0.507918	0.518978	0.517619	0.112883	0.091277
		0.002262	0.000938	0.000700	0.931	0.954
	(R_2, S_3)	0.499689	0.503688	0.518398	0.165574	0.116692
		0.003317	0.003025	0.000752	0.810	0.992
	(R_3, S_1)	0.507562	0.517821	0.510524	0.118261	0.077221
		0.002314	0.001023	0.000757	0.846	0.971
	(R_3, S_2)	0.507319	0.516948	0.511041	0.120795	0.079832
		0.002348	0.00101	0.000820	0.884	0.983
	(R_3, S_3)	0.503269	0.513027	0.510738	0.123755	0.1115979
		0.003078	0.001043	0.001427	0.827	0.935

Table 4: Estimation of $R_{s,k}$ when parameter η is common and unknown.

		Estimation			Interval length	
$(s,\ \theta_i,\ \ R_{s,k})$	CS	MLE	Lindley	MCMC	Avg CI Length	HPD Length
(1, θ_2, 0.578615)	(R_1, S_1)	0.615321	0.584965	0.579459	0.138776	0.132674
		0.003076	0.002845	0.001928	0.847	0.991
	(R_1, S_2)	0.613835	0.585523	0.580945	0.142683	0.135809
		0.003054	0.002875	0.001933	0.852	0.929
	(R_1, S_3)	0.621745	0.584330	0.580894	0.147771	0.135428
		0.003358	0.002930	0.002930	0.944	0.965
	(R_2, S_1)	0.613484	0.584610	0.578875	0.139238	0.136644
		0.003128	0.002867	0.000339	0.875	0.977
	(R_2, S_2)	0.611813	0.584977	0.579388	0.142886	0.140504
		0.003111	0.002860	0.000650	0.851	0.927
	(R_2, S_3)	0.621044	0.583642	0.580146	0.147801	0.136363
		0.003436	0.002934	0.000307	0.933	0.999
	(R_3, S_1)	0.590409	0.583039	0.565137	0.150541	0.147197
		0.006299	0.003185	0.001460	0.851	0.983
	(R_3, S_2)	0.588497	0.583728	0.565887	0.153217	0.140395
		0.005257	0.003152	0.001562	0.860	0.964
	(R_3, S_3)	0.610973	0.582297	0.567070	0.157350	0.154006
		0.003599	0.003125	0.001234	0.748	0.955
(2, θ_2, 0.485917)	(R_1, S_1)	0.505229	0.493663	0.480363	0.108822	0.100941
		0.002058	0.0012963	0.000517	0.802	0.946
	(R_1, S_2)	0.505774	0.493927	0.480469	0.111793	0.103788
		0.002068	0.001305	0.000530	0.805	0.983
	(R_1, S_3)	0.510460	0.494882	0.479451	0.110773	0.107857
		0.002277	0.001563	0.000559	0.797	0.936
	(R_2, S_1)	0.503370	0.493350	0.479539	0.113188	0.100977
		0.002127	0.001308	0.000501	0.795	0.964
	(R_2, S_2)	0.502149	0.493671	0.480083	0.115363	0.103660
		0.002101	0.001315	0.000513	0.802	0.984
	(R_2, S_3)	0.509852	0.494478	0.479104	0.111602	0.107980
		0.002309	0.001544	0.000537	0.795	0.925
	(R_3, S_1)	0.487026	0.478702	0.484717	0.145297	0.105849
		0.002215	0.001651	0.000230	0.805	0.971
	(R_3, S_2)	0.487291	0.482979	0.479028	0.148176	0.110986
		0.002207	0.001739	0.000238	0.809	0.919
	(R_3, S_3)	0.501817	0.487712	0.478273	0.125871	0.114470
		0.002401	0.001502	0.000238	0.802	0.998

4.1 Real Data

Now we investigate one numerical example in support of considered estimation problem based on lognormal distribution. The data contain time-between-failures measured in seconds. Complete description about this failure data is reported in treatise Lyu [38]. We assign $s = 1$ and $k = 4$ to consider a 1-out-of-4:G system. Now take Y_1 as second failure time. Then $X_{1k},\ k = 1, 2, \ldots, 4$ are similar data of observations numbered from 3 to 6. Next take Y_2 as failure time of 7th observation. Then $X_{2k},\ k = 1, 2, \ldots, 4$, are similar data of observations from 8 to 11. This iteration up to 36th failures yielded $n = 7$ data for Y. The data is quite popular among researchers. For further results on this data, we refer to Nadar and Kizilaslan [8], Jha et al.

[39]. Now strength-stress data (X, Y) are as follows:

$$X = \begin{pmatrix} 4 & 25 & 3 & 186 \\ 36 & 4 & 78 & 53 \\ 4 & 30 & 30 & 14 \\ 5 & 42 & 205 & 2 \\ 91 & 9 & 5 & 10 \\ 49 & 44 & 129 & 1 \\ 1 & 32 & 224 & 34 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 10 \\ 4 \\ 1 \\ 49 \\ 1 \\ 103 \\ 9 \end{pmatrix}.$$

We verify fitting of these data to lognormal distributions. These data are also fitted to unit Gompertz and Kumarswamy distributions. In this regard, required MLEs of parameters corresponding competing models are computed. Then estimated Kolmogorov-Smirnov (K-S) statistic along with p-values are evaluated in Table 5. As a further illustration, the empirical distribution plot overlaid with theoretical distribution plot, the probability–probability (P-P) and the quantile–quantile (Q-Q) plots are presented in Figure 2. Smaller K-S estimate is an indication of good fit. Thus tabulated estimates indicate that lognormal model fit these data very good. The various plots also suggest good fit to the data sets. We now compute MLEs and Bayes estimates of SM Parameter along with 95% asymptotic and HPD intervals. Estimation is considered using progressive censored data recorded from three censoring schemes by considering $s = 1$. This implied that inferences are obtained for reliability when at least one strength component of the system exceeds stress force. The considered censoring schemes are listed below:

Scheme-I : $R = (1, 0^{*2}),\ S = (2, 0^{*4}),\ \ (N = 7,\ K = 4,\ n = 5,\ k = 3,\ s = 1)$
Scheme-II: $R = (0^{*2}, 1),\ S = (0^{*3}, 3),\ \ (N = 7,\ K = 4,\ n = 4,\ k = 3,\ s = 1)$
Scheme-II: $R = (1, 0^{*2}),\ S = (2, 0, 2),\ \ (N = 7,\ K = 4,\ n = 3,\ k = 3,\ s = 1)$
Corresponding to first scheme, the time to failure strength-stress data are as follow:

$$X = \begin{pmatrix} 4 & 49 & 42 \\ 36 & 1 & 9 \\ 4 & 25 & 44 \\ 5 & 4 & 32 \\ 91 & 30 & 3 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 10 \\ 4 \\ 1 \\ 49 \\ 1 \end{pmatrix}$$

Similarly for the second censoring, the observed time failure strength-stress data are as follow:

$$X = \begin{pmatrix} 4 & 91 & 4 \\ 36 & 49 & 30 \\ 4 & 1 & 42 \\ 5 & 25 & 9 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 10 \\ 4 \\ 1 \\ 49 \end{pmatrix}$$

For the third censoring, observed time failure strength-stress data are given below:

$$X = \begin{pmatrix} 4 & 5 & 1 \\ 36 & 91 & 25 \\ 4 & 49 & 4 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 10 \\ 4 \\ 1 \end{pmatrix}$$

First when parameter ξ is unknown and common, then we provide estimate of SM parameter based on censored data. In fact all estimates are presented in Table 6. Associated asymptotic and HPD intervals are listed as well. Similarly, we evaluate the reliability when η is common and unknown. The corresponding estimates are listed in Table 7. We see that length of HPD intervals are smaller than asymptotic intervals. This holds for both cases. In fact results are quite consistence with simulation experiments as well.

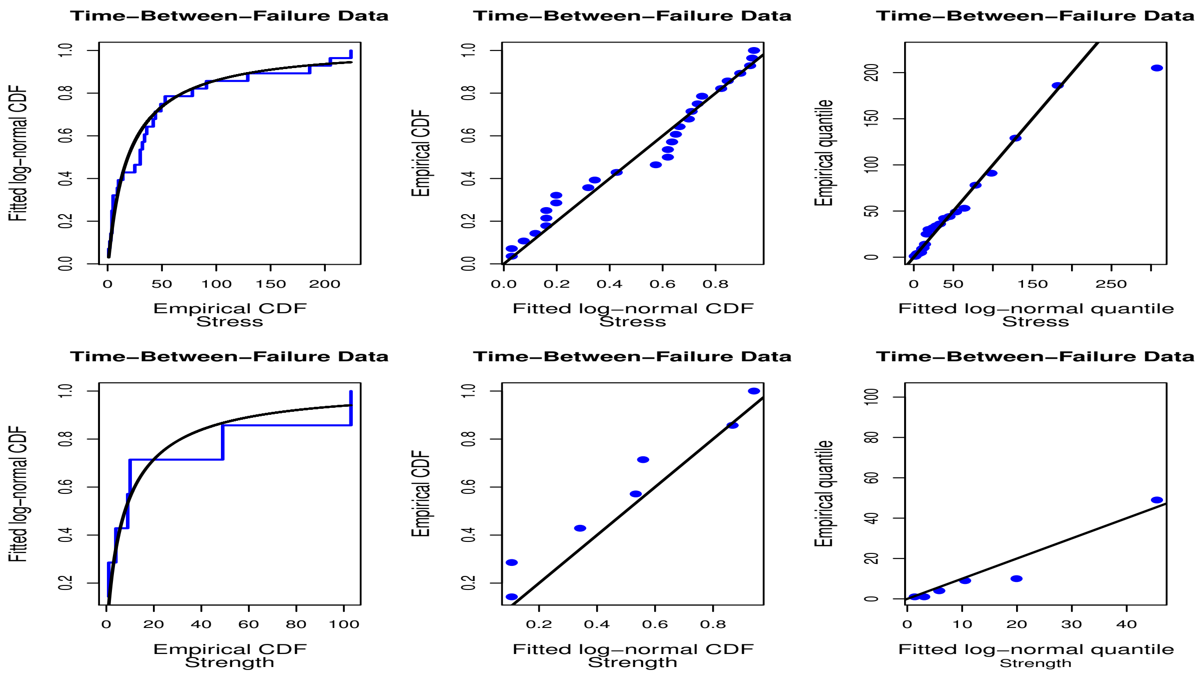

Figure 2: Empirical distribution and fitted lognormal Distribution, P–P and Q–Q plots under time between failure data.

Table 5: Goodness fit for Time-between-failure data

	Data X				Data Y			
Distribution	$\hat{\alpha}$	$\hat{\xi}$	K-S	p-value	$\hat{\alpha}$	$\hat{\xi}$	K-S	p-value
Lognormal	2.9277	2.4135	0.1554	0.5080	2.0589	2.7147	0.1799	0.9772
Unit Gompertz	0.0716	0.5784	0.1590	0.4801	0.0251	0.6606	0.1937	0.9600
Kumaraswamy	1.2104	11.3256	0.2445	0.0702	1.4708	45.2556	0.4513	0.1155
Lindley	0.0406	$\cdots$	0.3189	0.0067	0.0762	$\cdots$	0.5111	0.0516
Gumble	20.3385	18.6102	0.1795	0.3275	12.3740	8.5180	0.3024	0.5438
Generalized Exponential	0.6365	0.0151	0.1324	0.7099	0.5273	0.0250	0.2622	0.7214
Quasy Lindley	0.0223	11.0365	0.2238	0.1207	0.0418	15.1664	0.3893	0.2391
Generalized Logistic	636.6803	0.9683	0.6063	0.0000	645.1165	2.1353	0.5959	0.0138

Table 6: Estimation of $R_{s,k}$ for Time-between-failure data when ξ is common parameter.

	Estimates			Interval	
Scheme	MLE	Lindley	MCMC	ACI	HPD
I	0.891589	0.804700	0.811448	(0.752167, 1.031010)[0.278843]	(0.803764, 0.819436)[0.015672]
II	0.715007	0.597014	0.734846	(0.457582, 0.972433)[0.514851]	(0.712609, 0.760469)[0.047860]
III	0.879992	0.756394	0.764193	(0.706440, 1.053544)[0.347104]	(0.750682, 0.777424)[0.026742]

Table 7: Estimation of $R_{s,k}$ for Time-between-failure data when η is common parameter.

	Estimates			Interval	
Scheme	MLE	Lindley	MCMC	ACI	HPD
I	0.526548	0.493175	0.544626	(0.439766, 0.613329)[0.173563]	(0.538007, 0.549805)[0.011798]
II	0.488740	0.481505	0.564421	(0.394477, 0.583004)[0.188527]	(0.562582, 0.565374)[0.002792]
III	0.565398	0.522942	0.552311	(0.452451, 0.678345)[0.225894]	(0.544765, 0.559155)[0.014390]

5 Concluding remarks

We have considered estimation of the reliability in a multicomponent SM by applying classical and Bayesian approaches. We have obtained various estimates of this parametric function when stress and strength components follow lognormal distributions and data are observed under progressive type-II censoring. Inferences are obtained by sequentially considering one parameter common and unknown. Expressions for reliability are evaluated under these cases. For both the cases, estimates of parametric function are derived using likelihood and Bayesian methods. Particularly point and interval estimates are evaluated for the reliability. Simulation experiments are conducted to examine the behavior of all the estimates by taking into account various progressive censoring schemes. We observed that performance of Bayes procedures is reasonably better than corresponding classical estimates. Analysis of a real life numerical example is also discussed in support of the proposed estimates. This study can be extended to some other censoring schemes such as adaptive progressive hybrid and generalized progressive hybrid censoring schemes and can also be extended to multiple dependent competing failure models.

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