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RESEARCH ARTICLE

Objective Bayesian inference for the reliability in a bivariate Lomax distribution

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Abstract

We consider the objective Bayesian analysis for the reliability in the bivariate Lomax distribution. In this paper, we derive the first- and second-order matching priors and the reference priors for the reliability in the bivariate Lomax population. However, it turns out that the reference priors do not satisfy the first-order matching criterion and also, the matching priors and the reference priors have different distributions. We provide conditions for the general prior, including the matching and reference priors, to generate proper posterior distributions. Our simulation shows that the matching prior matches the target coverage probabilities well in a frequentist sense. Furthermore, even when the reference priors do not satisfy the first-order matching criterion, they still perform as well as the second-order matching prior. Finally, we demonstrate our results using two real examples.

Keywords Bivariate Lomax distribution · Matching prior · Reference prior · Propriety · Reliability

1 Introduction

The estimation problem for the reliability $R = P(Y < X)$ has been extensively studied in the statistical literature due to its importance for reliability engineers and biostatisticians. This is widely known as stress–strength modeling, which was first

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introduced by Birnbaum (1956). Numerous studies have investigated the estimation of $P(Y < X)$ when X and Y are independent random variables with a specific distribution. For an overview on this topic, we refer the reader to Kotz et al. (2003).

In some applications, however, the variables of interest are correlated, either directly or through their dependence on a common auxiliary variable. For example, in financial risk-management, one can compare the stock returns of two companies operating in the same industry, thus making the prices of their stocks correlated. Research has been conducted on the estimation of reliability for specific bivariate distributions, such as the bivariate exponential (Awad et al., 1981; Klein & Basu, 1985; Jana, 1994; Jana & Roy, 1994; Hanagal, 1999; Nadarjah & Kotz, 2006), bivariate normal (Mukherjee & Saran, 1985; Nandi & Aich, 1994; Gupta & Subramanian, 1998; Nguimkeu et al., 2014), bivariate Pareto (Hanagal, 1997; Jeevanand, 1997), bivariate beta (Nadarajah, 2005), bivariate Lomax (Shoukri et al., 2005), and bivariate Rayleigh (Pak et al., 2014).

In this paper, we consider the objective Bayesian inference for $P(Y < X)$ when (X, Y) is a bivariate Lomax distribution. Thus, we are only interested in noninformative priors for the reliability in bivariate Lomax distribution. Subjective priors are ideal when sufficient information from past experience, expert opinion, or previously collected data is available. However, if there is no adequate prior information, Bayesian techniques can still be used efficiently with noninformative priors, such as probability matching (Welch & Peers, 1963; Stein, 1985; Tibshirani, 1989; Mukerjee & Dey, 1993; DiCiccio & Stern, 1994; Datta & Ghosh, 1995, 1996; Mukerjee & Ghosh, 1997) and reference priors (Bernardo, 1979; Berger & Bernardo, 1989, 1992; Ghosh and Mukerjee, 1992; Datta & Mukerjee, 2004).

2 Noninformative priors

We consider two correlated exponential random variables, X and Y, whose probability density and survival functions are given by

$$f(x,y) = \frac{\alpha(\alpha+1)\beta\phi}{(1+\beta x+\phi y)^{\alpha+2}}, \quad x > 0, \quad y > 0 \tag{1}$$

and

$$\bar{F}(x,y) = \frac{1}{(1+\beta x+\phi y)^{\alpha}},$$

respectively, for $\beta > 0, \phi > 0$, and $\alpha > 0$. This density, introduced by Lindley and Singpurwalla (1986), is known as the bivariate Pareto distribution. Lindley and Singpurwalla (1986) and Nayak (1987) have provided the theoretical justification and studied the properties of the bivariate Pareto distribution, which has also been referred to as the bivariate Lomax distribution (Kotz et al., 2000). It has been applied to various fields such as reliability, hydrology, and risk analysis (Lee & Gross, 1991; Shoukri et al., 2005; Nadarajah, 2009; Sarabia et al., 2016). Given the joint density function in (1), the probability of Y being less than X is expressed as $R = P(Y < X)$ and is given by

$$R = \frac{\phi}{\beta + \phi}.$$

We then seek to develop noninformative priors for R.

2.1 Matching priors

We consider matching priors that guarantee the approximate frequentist validity of one-sided Bayesian credible intervals based on posterior quantiles of a one-dimensional interest parameter. Specifically, we seek priors π for which

$$P_\theta[\theta_1 \le \theta_1^{1-\alpha}(\pi;\mathbf{X})] = 1 - \alpha + o(n^{-u}), \tag{2}$$

holds asymptotically as n reaches infinity, for some $u > 0$. In the equation, $\theta = (\theta_1, \dots, \theta_p)^T$, θ_1 is the parameter of interest, and $\theta_1^{1-\alpha}(\pi;\mathbf{X})$ is the $(1-\alpha)$th posterior quantile of θ_1 based on the prior π and data $\mathbf{X}$. Priors π satisfying (2) are called matching priors. If $u = 1/2$, then π is referred to as a first-order matching prior, while if $u = 1$, π is referred to as a second-order matching prior.

The Fisher information matrix of the model (1), based on the orthogonal parameterization of nuisance parameters $\theta_1 = \frac{\phi}{\beta+\phi}$, $\theta_2 = \beta\phi$, and $\theta_3 = \alpha$, is given by

$$\mathbf{I}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \frac{n(2+\theta_3)}{2\theta_1^2(1-\theta_1)^2(3+\theta_3)} & 0 & 0 \\ 0 & \frac{n\theta_3}{2\theta_2^2(3+\theta_3)} & \frac{n}{\theta_2(2+\theta_3)} \\ 0 & \frac{n}{\theta_2(2+\theta_3)} & \frac{n(1+2\theta_3+2\theta_3^2)}{\theta_3^2(1+\theta_3)^2} \end{pmatrix}. \tag{3}$$

Using this, the likelihood function of parameters $(\theta_1, \theta_2, \theta_3)$ can be written as follows:

$$L(\theta_1, \theta_2, \theta_3) \propto \frac{\theta_2^n \theta_3^n (1+\theta_3)^n}{\prod_{i=1}^n \left(1 + \theta_1^{-\frac{1}{2}} \theta_2^{\frac{1}{2}} (1-\theta_1)^{\frac{1}{2}} x_i + \theta_1^{\frac{1}{2}} \theta_2^{\frac{1}{2}} (1-\theta_1)^{-\frac{1}{2}} y_i\right)^{\theta_3+2}}. \tag{4}$$

This greatly simplifies the development of the matching priors in finding the matching priors π. The above Fisher information matrix $\mathbf{I}$ implies that θ_1 is orthogonal to (θ_2, θ_3) in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of first-order probability matching priors can be characterized by

$$\pi_m^{(1)}(\theta_1, \theta_2, \theta_3) \propto \theta_1^{-1}(1-\theta_1)^{-1}(2+\theta_3)^{\frac{1}{2}}(3+\theta_3)^{-\frac{1}{2}} d(\theta_2, \theta_3), \tag{5}$$

where $d(\theta_2, \theta_3) > 0$ is a differentiable function. Furthermore, the class of priors given in (5) can be narrowed down to the second-order probability matching priors as specified by Mukerjee and Ghosh (1997). This is achieved by requiring d to satisfy an additional differential equation of Mukerjee and Ghosh (1997)

$$\frac{1}{6}d(\theta_2,\theta_3)\frac{\partial}{\partial\theta_1}\{I_{11}^{-\frac{3}{2}}L_{1,1,1}\}+\sum_{s=2}^{3}\sum_{u=2}^{3}\frac{\partial}{\partial\theta_u}\{I_{11}^{-\frac{1}{2}}L_{11s}I^{su}d(\theta_2,\theta_3)\}=0, \tag{6}$$

where the inverse matrix of Fisher information matrix $\mathbf{I}^{-1}=(I^{ij})_{3\times 3}$. Then

$$\begin{aligned}
L_{1,1,1} &= E\left[\left(\frac{\partial \log L}{\partial\theta_1}\right)^3\right]=0,\\
L_{112} &= E\left[\frac{\partial^3 \log L}{\partial\theta_1^2\partial\theta_2}\right]=-\frac{n\theta_3(2+\theta_3)}{4\theta_1^2\theta_2(3+\theta_3)(4+\theta_3)},\\
L_{113} &= E\left[\frac{\partial^3 \log L}{\partial\theta_1^2\partial\theta_3}\right]=-\frac{n}{2\theta_1^2(1-\theta_1)^2(3+\theta_3)}
\end{aligned}$$

and

$$\begin{aligned}
I^{22} &= \frac{2\theta_2^2(2+\theta_3)^2(3+\theta_3)(1+2\theta_3+2\theta_3^2)}{n\theta_3(4+6\theta_3+3\theta_3^2)},\\
I^{33} &= \frac{\theta_3^2(1+\theta_3)^2(2+\theta_3)^2}{n(4+6\theta_3+3\theta_3^2)},\\
I^{23} &= -\frac{2\theta_2\theta_3(1+\theta_3)^2(2+\theta_3)(3+\theta_3)}{n(4+6\theta_3+3\theta_3^2)}.
\end{aligned}$$

Thus, from Fisher information matrix (3), the differential Eq. (6) simplifies to

$$\begin{aligned}
&\frac{\partial}{\partial\theta_2}\left\{\frac{(2+\theta_3)^{\frac{1}{2}}(3+\theta_3)^{\frac{1}{2}}(-4-4\theta_3+\theta_3^2+2\theta_3^3)\theta_2}{\theta_1(1-\theta_1)(4+\theta_3)(4+6\theta_3+3\theta_3^2)}d(\theta_2,\theta_3)\right\}\\
&\quad-\frac{\partial}{\partial\theta_3}\left\{\frac{\theta_3^2(1+\theta_3)^2(2+\theta_3)^{\frac{3}{2}}}{\theta_1(1-\theta_1)(3+\theta_3)^{\frac{1}{2}}(4+\theta_3)(4+6\theta_3+3\theta_3^2)}d(\theta_2,\theta_3)\right\}=0.
\end{aligned} \tag{7}$$

The set of solutions of (7) is of the form

$$d(\theta_2,\theta_3)=\frac{\theta_3^5(3+\theta_3)^{\frac{1}{2}}(4+\theta_3)(4+6\theta_3+3\theta_3^2)e^{\frac{6+8\theta_3}{\theta_3(1+\theta_3)}}}{(1+\theta_3)^5(2+\theta_3)^{\frac{7}{2}}}h\left(\frac{\theta_2\theta_3^7e^{\frac{6+8\theta_3}{\theta_3(1+\theta_3)}}}{(1+\theta_3)^3(2+\theta_3)^2}\right),$$

where $h(\cdot)>0$ is an arbitrary function differentiable in its arguments. Thus, the resulting second-order probability matching prior is

$$\pi_m^{(2)}(\theta_1,\theta_2,\theta_3) \propto \frac{\theta_3^5(4+\theta_3)(4+6\theta_3+3\theta_3^2)e^{\frac{6+8\theta_3}{\theta_3(1+\theta_3)}}}{\theta_1(1-\theta_1)(1+\theta_3)^5(2+\theta_3)^3} h\left(\frac{\theta_2\theta_3^7 e^{\frac{6+8\theta_3}{\theta_3(1+\theta_3)}}}{(1+\theta_3)^3(2+\theta_3)^2}\right), \quad (8)$$

where $h(\cdot) > 0$ is an arbitrary function that is differentiable in its arguments.

Remark 1 Other methods for accomplishing matching can be investigated. In particular, we can examine whether the second-order matching prior satisfies alternative matching criteria by determining if it matches coverage probabilities up to the second order (Mukerjee & Reid, 1999).

Now

$$L_{111} = E\left[\frac{\partial^3 \log L}{\partial\theta_1^3}\right] = -\frac{3n(2\theta_1-1)(2+\theta_3)}{2\theta_1^3(1-\theta_1)^3(3+\theta_3)},$$
$$I_{11} = \frac{n(2+\theta_3)}{2\theta_1^2(1-\theta_1)^2(3+\theta_3)}.$$

We consider the differential equations from Mukerjee and Reid (1999). We show that $\frac{\partial}{\partial\theta_1}\left(I_{11}^{-\frac{3}{2}}L_{111}\right) = -\frac{6\sqrt{2}(3+\theta_3)^{\frac{1}{2}}}{n^{\frac{1}{2}}(2+\theta_3)^{\frac{1}{2}}} \neq 0$. Therefore, the second-order matching prior (8) does not match the alternative coverage probabilities (Mukerjee & Reid, 1999). Datta et al. (2000) provided a theorem that estalblishes the equivalence of second-order matching priors and HPD (highest posterior density) matching priors (DiCiccio & Stern, 1994; Ghosh & Mukerjee, 1995) within the class of first-order matching priors. This theorem requires that $I_{11}^{-\frac{3}{2}}L_{111}$ does not depend on θ_1. We can demonstrate the following equation for one differential equation:

$$\Delta_4 = \frac{\partial}{\partial\theta_1}\{L_{111}(I^{11})^2\pi_m^{(2)}\}$$
$$= -\frac{12\theta_3^5(3+\theta_3)(4+\theta_3)(4+6\theta_3+3\theta_3^2)e^{\frac{6+8\theta_3}{\theta_3(1+\theta_3)}}}{n(1+\theta_3)^5(2+\theta_3)^4} \times h\left(\frac{\theta_2\theta_3^7 e^{\frac{6+8\theta_3}{\theta_3(1+\theta_3)}}}{(1+\theta_3)^3(2+\theta_3)^2}\right) \neq 0.$$

Therefore, the second-order matching prior (8) does not satisfy the criteria for a cumulative distribution function matching prior outlined by Mukerjee and Ghosh (1997).

2.2 Reference priors

For the bivariate Lomax model (4), if θ_1 is the parameter of interest, then the reference prior distributions for a group of ordering of $\{(\theta_1,\theta_2,\theta_3)\}$ is

$$\pi_1(\theta_1,\theta_2,\theta_3) \propto \frac{(4+6\theta_3+3\theta_3^2)^{\frac{1}{2}}}{\theta_1(1-\theta_1)\theta_2\theta_3^{\frac{1}{2}}(1+\theta_3)(2+\theta_3)^{\frac{1}{2}}(3+\theta_3)}. \tag{9}$$

Next, for a group of ordering of $\{\theta_1,(\theta_2,\theta_3)\}$, the reference prior is

$$\pi_2(\theta_1,\theta_2,\theta_3) \propto \frac{(4+6\theta_3+3\theta_3^2)^{\frac{1}{2}}}{\theta_1(1-\theta_1)\theta_2\theta_3^{\frac{1}{2}}(1+\theta_3)(2+\theta_3)(3+\theta_3)^{\frac{1}{2}}}. \tag{10}$$

For a group of ordering of $\{\theta_1,\theta_2,\theta_3\}$, the reference prior is

$$\pi_3(\theta_1,\theta_2,\theta_3) \propto \frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_1(1-\theta_1)\theta_2\theta_3(1+\theta_3)}. \tag{11}$$

Lastly, for a group of ordering of $\{\theta_1,\theta_3,\theta_2\}$, the reference prior is

$$\pi_4(\theta_1,\theta_2,\theta_3) \propto \frac{(4+6\theta_3+3\theta_3^2)^{\frac{1}{2}}}{\theta_1(1-\theta_1)\theta_2\theta_3(1+\theta_3)(2+\theta_3)}. \tag{12}$$

Remark 2 In the reference priors just described, the derivations of the one-at-a-time reference priors π_3 and π_4 are given in Appendix 1.

Remark 3 Jeffreys' prior (JP) π_1, two-group reference π_2, and the one-at-a-time reference priors π_3 and π_4 do not satisfy a first-order matching criterion. Moreover, they have distinct forms.

Remark 4 The matching priors (8) include many different options since the function h can be chosen arbitrarily. For some functions, there may not be any improvement in the coverage probabilities with these posteriors. As such, we consider a particular second-order matching prior where $h = \theta_2^{-1}\theta_3^{-7}(1+\theta_3)^3(2+\theta_3)^2 e^{-\frac{6+8\theta_3}{\theta_3(1+\theta_3)}}$ in the matching priors (8), giving the matching prior as

$$\pi_m(\theta_1,\theta_2,\theta_3) \propto \frac{(4+\theta_3)(4+6\theta_3+3\theta_3^2)}{\theta_1(1-\theta_1)\theta_2\theta_3^2(1+\theta_3)^2(2+\theta_3)}. \tag{13}$$

We can see that this matching prior differs from all reference priors.

Remark 5 JP (9), two-group reference (10) and the one-at-a-time reference priors (11), (12), and the second-order matching prior (13) in the original parametrization (α,β,ϕ) are given by

$$\pi_1(\alpha,\beta,\phi) \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)(2+\alpha)^{\frac{1}{2}}(3+\alpha)}\beta^{-1}\phi^{-1},$$

$$\pi_2(\alpha,\beta,\phi) \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(1+\alpha)(2+\alpha)(3+\alpha)^{\frac{1}{2}}}\beta^{-1}\phi^{-1},$$

$$\pi_3(\alpha,\beta,\phi) \propto \frac{(1+2\alpha+2\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)}\beta^{-1}\phi^{-1},$$

$$\pi_4(\alpha,\beta,\phi) \propto \frac{(4+6\alpha+3\alpha^2)^{\frac{1}{2}}}{\alpha(1+\alpha)(2+\alpha)}\beta^{-1}\phi^{-1}$$

and

$$\pi_m(\alpha,\beta,\phi) \propto \frac{(4+\alpha)(4+6\alpha+3\alpha^2)}{\alpha^2(1+\alpha)^2(2+\alpha)}\beta^{-1}\phi^{-1},$$

respectively.

3 The posterior distributions

We investigate the propriety of posteriors for a general class of priors, including the matching and reference priors developed in Sections 2. This class of priors is given by

$$\pi(\theta_1,\theta_2,\theta_3) \propto \frac{(4+\theta_3)^{a_1}(4+6\theta_3+3\theta_3^2)^{a_2}(1+2\theta_3+2\theta_3^2)^{a_3}}{\theta_3^{a_4}(1+\theta_3)^{a_5}(2+\theta_3)^{a_6}(3+\theta_3)^{a_7}}\theta_1^{-b}(1-\theta_1)^{-b}\theta_2^{-c}, \tag{14}$$

where $a_i \geq 0, i = 1,2,3,6,7, a_4 > 0, a_5 > 0, b > 0$ and $c > 0$. We can then prove the following general theorem.

Theorem 1 *The posterior distribution of* $(\theta_1,\theta_2,\theta_3)$ *under the prior* (14) *is proper if* $n-b-c+1>0$, $n+b-c+1>0$, $n+b+c-1\geq 0$, $n-b+c-1\geq 0$, $2c-2\geq 0$ *and* $a_4+a_5+a_6+a_7-a_1-2a_2-2a_3-2c+1>0$.

Proof Note that the joint posterior for $(\theta_1,\theta_2,\theta_3)$ given $\mathbf{z}$ is given by

$$\pi(\theta_1,\theta_2,\theta_3|\mathbf{z}) \propto \frac{(4+\theta_3)^{a_1}(4+6\theta_3+3\theta_3^2)^{a_2}(1+2\theta_3+2\theta_3^2)^{a_3}}{\theta_3^{a_4}(1+\theta_3)^{a_5}(2+\theta_3)^{a_6}(3+\theta_3)^{a_7}}$$
$$\times \frac{\theta_1^{-b}(1-\theta_1)^{-b}\theta_2^{n-c}\theta_3^{n}(1+\theta_3)^{n}}{\prod_{i=1}^n\left(1+\theta_1^{-\frac{1}{2}}\theta_2^{\frac{1}{2}}(1-\theta_1)^{\frac{1}{2}}x_i+\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}(1-\theta_1)^{-\frac{1}{2}}y_i\right)^{\theta_3+2}},$$

where $\mathbf{z} = ((x_1, y_1), \ldots, (x_n, y_n))$. Then, the joint posterior for (α, β, ϕ) given $\mathbf{z}$ can be expressed as

$$\pi(\alpha,\beta,\phi|\mathbf{z}) \propto \frac{(4+\alpha)^{a_1}(4+6\alpha+3\alpha^2)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4}(1+\alpha)^{a_5}(2+\alpha)^{a_6}(3+\alpha)^{a_7}} \times \frac{\alpha^n(1+\alpha)^n\beta^n\phi^n}{\prod_{i=1}^n\left(1+\beta x_i+\phi y_i\right)^{\alpha+2}}\frac{(\beta+\phi)^{2b-2}}{(\beta\phi)^{b+c-1}}.$$

Note that the following inequalities must hold.

$$\frac{\beta\phi}{(\beta+\phi)^2} \le 1 \text{ and } (\beta+\phi)^{2b} \le k_1(\beta^{2b}+\phi^{2b}),$$

where k_1=1 for $0 \le 2b \le 1$ and $k_1 = 2^{2b-1}$ for $2b > 1$. Then

$$\begin{aligned}
\pi(\alpha,\beta,\phi|\mathbf{z}) \le\ & k_1\frac{(4+\alpha)^{a_1}(4+6\alpha+3\alpha^2)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4}(1+\alpha)^{a_5}(2+\alpha)^{a_6}(3+\alpha)^{a_7}}\\
& \times\ [\beta^{-(b+c)}\phi^{b-c}+\beta^{b-c}\phi^{-(b+c)}]\alpha^n(1+\alpha)^n\beta^n\phi^n\prod_{i=1}^n\left(1+\beta x_i+\phi y_i\right)^{-(\alpha+2)}\\
\propto\ & \omega(\alpha)\alpha^n(1+\alpha)^n\beta^{n-b-c}\phi^{n+b-c}\left(1+\beta x_i+\phi y_i\right)^{-n(\alpha+2)}\\
& +\ \omega(\alpha)\alpha^n(1+\alpha)^n\beta^{n+b-c}\phi^{n-b-c}\left(1+\beta x_i+\phi y_i\right)^{-n(\alpha+2)}\\
\equiv\ & \pi'(\alpha,\beta,\phi|\mathbf{z}),
\end{aligned} \tag{15}$$

where

$$\omega(\alpha) = \frac{(4+\alpha)^{a_1}(4+6\alpha+3\alpha^2)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4}(1+\alpha)^{a_5}(2+\alpha)^{a_6}(3+\alpha)^{a_7}}.$$

Integrating with respect to β for the joint posterior (15), yields

$$\begin{aligned}
\pi'(\alpha,\phi|\mathbf{z}) \propto\ & k_2\omega(\alpha)\alpha^n(1+\alpha)^n\frac{\Gamma[n-b-c+1]\Gamma[n\alpha+n+b+c-1]}{\Gamma[n(2+\alpha)]}\\
& \times\ \phi^{n+b-c}\left(1+\phi y_i\right)^{-n(\alpha+1)-b-c+1}\\
& +\ k_3\omega(\alpha)\alpha^n(1+\alpha)^n\frac{\Gamma[n+b-c+1]\Gamma[n\alpha+n-b+c-1]}{\Gamma[n(2+\alpha)]}\\
& \times\ \phi^{n-b-c}\left(1+\phi y_i\right)^{-n(\alpha+1)+b-c+1},
\end{aligned} \tag{16}$$

where $n-b-c+1>0$, $n+b-c+1>0$, $n+b+c-1 \ge 0$ and $n-b+c-1 \ge 0$. Here k_2 and k_3 are constants. Next, integrating with respect to ϕ for the posterior (16), yields

$$\pi'(\alpha|\mathbf{z}) \propto \ k_4\omega(\alpha)\alpha^n(1+\alpha)^n\frac{\Gamma[n-b-c+1]\Gamma[n+b-c+1]\Gamma[n\alpha+2c-2]}{\Gamma[n(2+\alpha)]}$$
$$+ \ k_5\omega(\alpha)\alpha^n(1+\alpha)^n\frac{\Gamma[n+b-c+1]\Gamma[n-b-c+1]\Gamma[n\alpha+2c-2]}{\Gamma[n(2+\alpha)]}$$
$$\propto \ \omega(\alpha)\alpha^n(1+\alpha)^n\frac{\Gamma[n\alpha+2c-2]}{\Gamma[n(2+\alpha)]},$$

if $2c-2 \geq 0$. Here k_4 and k_5 are a constant. Then $0 < \alpha \leq 1$, we have

$$\pi'(\alpha|\mathbf{z}) \leq \frac{5^{a_1}13^{a_2}5^{a_3}}{\alpha^{a_4-n}2^{a_6}3^{a_7}}2^n\frac{\Gamma[n+2c-2]}{\Gamma[2n]} \equiv \pi''(\alpha|\mathbf{z}).$$

Thus,

$$\int_0^1 \pi''(\alpha|\mathbf{z})d\alpha < \infty,$$

if $n-a_4+1 > 0$. Next for $1 < \alpha < \infty$, we have

$$\pi'(\alpha|\mathbf{z}) \propto \ \frac{\Gamma[n\alpha+2c-2]}{\Gamma[n(2+\alpha)]}\alpha^n(1+\alpha)^n$$
$$\times \ \frac{(4+\alpha)^{a_1}(4+6\alpha+3\alpha^2)^{a_2}(1+2\alpha+2\alpha^2)^{a_3}}{\alpha^{a_4}(1+\alpha)^{a_5}(2+\alpha)^{a_6}(3+\alpha)^{a_7}}$$
$$\propto \ \frac{\Gamma[n\alpha+2c-2]}{\Gamma[n(2+\alpha)]}\alpha^n(1+\alpha)^n\alpha^{a_1+2a_2+2a_3-a_4-a_5-a_6-a_7}$$
$$\times \ \frac{(\frac{4}{\alpha}+1)^{a_1}(\frac{4}{\alpha^2}+\frac{6}{\alpha}+3)^{a_2}(\frac{1}{\alpha^2}+\frac{2}{\alpha}+2)^{a_3}}{(\frac{1}{\alpha}+1)^{a_5}(\frac{2}{\alpha}+1)^{a_6}(\frac{3}{\alpha}+1)^{a_7}}$$
$$\leq 5^{a_1}13^{a_2}5^{a_3}\frac{\Gamma[n\alpha+2c-2]}{\Gamma[n(2+\alpha)]}\alpha^n(1+\alpha)^n\alpha^{a_1+2a_2+2a_3-a_4-a_5-a_6-a_7}.$$

Now for $1 < \alpha < \infty$, we have the following inequality:

$$\frac{\Gamma[n\alpha+2c-2]}{\Gamma[n(2+\alpha)]}\frac{\alpha^n(1+\alpha)^n}{\alpha^{2c-2}} = \frac{\Gamma[n\alpha+2c-2]}{\Gamma[n\alpha]\prod_{i=1}^{2n}(n\alpha+2n-i)}\frac{\alpha^n(1+\alpha)^n}{\alpha^{2c-2}}$$
$$= \frac{\Gamma[n\alpha+2c-2]}{\alpha^{2c-2}\Gamma[n\alpha]}\frac{(1+\frac{1}{\alpha})^n}{\prod_{i=1}^{2n}[n(1+\frac{2}{\alpha})-\frac{i}{\alpha}]} < \infty,$$

since $\Gamma[n\alpha]$ is approximately equal to $\sqrt{2\pi(n\alpha-1)}(n\alpha-1)^{n\alpha-1}e^{-(n\alpha-1)}$ for large α. Therefore, we have

$$\pi'(\alpha|\mathbf{z}) \leq k_6\alpha^{a_1+2a_2+2a_3-a_4-a_5-a_6-a_7+2c-2} \equiv \pi''(\alpha|\mathbf{z}),$$

where k_6 is a constant. Thus,

$$\int_1^\infty \pi''(\alpha|\mathbf{z})d\alpha < \infty,$$

if $a_4 + a_5 + a_6 + a_7 - a_1 - 2a_2 - 2a_3 - 2c + 1 > 0$. This completes the proof. □

We can obtain the following Lemma directly from Theorem 1:

Lemma 1 *JP* (9), *the two-group reference* (10), *the one-at-a-time reference prior* (12), *and the matching prior* (13) *give the proper posteriors.*

Theorem 2 *The posterior distribution of* $(\theta_1, \theta_2, \theta_3)$ *under the one-at-a-time reference prior* (11) *is improper.*

Proof Under the one-at-a-time reference prior (11), the joint posterior for $(\theta_1, \theta_2, \theta_3)$ given $\mathbf{z}$ is

$$\begin{aligned}\pi(\theta_1, \theta_2, \theta_3|\mathbf{z}) \propto\ & \frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)}\\ \times\ & \frac{\theta_1^{-1}(1-\theta_1)^{-1}\theta_2^{n-1}\theta_3^n(1+\theta_3)^n}{\prod_{i=1}^n\left(1+\theta_1^{-\frac{1}{2}}\theta_2^{\frac{1}{2}}(1-\theta_1)^{\frac{1}{2}}x_i+\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}(1-\theta_1)^{-\frac{1}{2}}y_i\right)^{\theta_3+2}},\end{aligned}$$

where $\mathbf{z} = ((x_1, y_1), \dots, (x_n, y_n))$. Then the joint posterior for (α, β, ϕ) given $\mathbf{z}$ is

$$\pi(\alpha, \beta, \phi|\mathbf{z}) \propto \alpha^{n-1}(1+\alpha)^{n-1}(1+2\alpha+2\alpha^2)^{\frac{1}{2}}\beta^{n-1}\phi^{n-1}\prod_{i=1}^n\left(1+\beta x_i+\phi y_i\right)^{-(\alpha+2)}.$$

Thus, we have

$$\pi(\alpha, \beta, \phi|\mathbf{z}) \propto \alpha^{n-1}(1+\alpha)^{n-1}(1+2\alpha+2\alpha^2)^{\frac{1}{2}}\beta^{n-1}\phi^{n-1}\left(1+\beta x_i+\phi y_i\right)^{-n(\alpha+2)}. \tag{17}$$

Integrating with respect to ϕ for the posterior (17), then we obtain

$$\pi(\alpha, \beta|\mathbf{z}) \propto \alpha^{n-1}(1+\alpha)^{n-1}(1+2\alpha+2\alpha^2)^{\frac{1}{2}}\frac{\Gamma[n(1+\alpha)]}{\Gamma[n(2+\alpha)]}\beta^{n-1}\left(1+\beta x_i\right)^{-n(\alpha+1)}. \tag{18}$$

Integrating with respect to β for the posterior (18), then we obtain

$$\begin{aligned}\pi(\alpha|\mathbf{z}) \propto\ & \frac{\Gamma[n\alpha]}{\Gamma[n(2+\alpha)]}\alpha^{n-1}(1+\alpha)^{n-1}(1+2\alpha+2\alpha^2)^{\frac{1}{2}}\\ \propto\ & \frac{\alpha^{n-1}(1+\alpha)^{n-1}}{\prod_{i=1}^{2n}(n\alpha+2n-i)}(1+2\alpha+2\alpha^2)^{\frac{1}{2}}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_1^\infty \pi(\alpha|\mathbf{z})d\alpha &\propto \int_1^\infty \frac{\alpha^n(1+\alpha)^{n-1}}{\prod_{i=1}^{2n}(n\alpha+2n-i)}\left(\frac{1}{\alpha^2}+\frac{2}{\alpha}+2\right)^{\frac{1}{2}}d\alpha\\ &\geq \int_1^\infty \frac{\sqrt{2}\alpha^{2n-1}}{\alpha^{2n}\prod_{i=1}^{2n}(n+\frac{2n-i}{\alpha})}d\alpha\\ &= \int_1^\infty \frac{\sqrt{2}\alpha^{-1}}{\prod_{i=1}^{2n}(n+\frac{2n-i}{\alpha})}d\alpha\\ &\geq \int_1^\infty \frac{\sqrt{2}\alpha^{-1}}{\prod_{i=1}^{2n}(3n-i)}d\alpha = \infty.\end{aligned}$$

Thus, the posterior distribution is improper. This completes the proof.

Under the general prior (14), the marginal posterior density of θ_1 is given by

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{(4+\theta_3)^{a_1}(4+6\theta_3+3\theta_3^2)^{a_2}(1+2\theta_3+2\theta_3^2)^{a_3}}{\theta_3^{a_4}(1+\theta_3)^{a_5}(2+\theta_3)^{a_6}(3+\theta_3)^{a_7}}\\ &\times \frac{\theta_1^{-b}(1-\theta_1)^{-b}\theta_2^{n-c}\theta_3^n(1+\theta_3)^n}{\prod_{i=1}^n\left(1+\theta_1^{-\frac{1}{2}}\theta_2^{\frac{1}{2}}(1-\theta_1)^{\frac{1}{2}}x_i+\theta_1^{\frac{1}{2}}\theta_2^{\frac{1}{2}}(1-\theta_1)^{-\frac{1}{2}}y_i\right)^{\theta_3+2}}d\theta_2 d\theta_3,\end{aligned}$$

where $z = ((x_1, y_1), \dots, (x_n, y_n))$. The normalizing constant for the marginal density of θ_1 requires a three-dimensional integration, yielding the marginal posterior density of θ_1. Thus, the marginal moment of θ_1 can be computed. In Section 4, we investigate the frequentist coverage probabilities for JP π_1, the two-group reference prior π_2, the one-at-a-time reference prior π_4, and the matching prior π_m, respectively.

4 Numerical study

4.1 Simulation studies

We evaluate the frequentist coverage probability of our noninformative prior given in Section 3 for several configurations (α, β, ϕ) and n by investigating the credible interval of the marginal posterior density of θ_1. This is achieved numerically by calculating the probability of the 0.05 (0.95) posterior quantiles of θ_1 being within the true value of the parameter. Tables 1, 2, 3, 4, 5 and 6 give numerical values of this probability for our prior. To calculate these values, we take 10,000 independent random samples of $(\mathbf{X}, \mathbf{Y})$ from the model (4) for each fixed (α, β, ϕ). The estimated frequentist coverage probabilities of 0.05 (0.95) posterior quantiles, as well as the 90% and 95% credible intervals of θ_1, are then given in the tables.

For the cases presented in Tables 1, 2, 3, 4, 5 and 6, we observe that the matching prior π_m achieves the desired coverage probabilities for the quantiles and the credible intervals. Although JP π_1 and the reference priors π_2 and π_4 do not satisfy the first-order matching criterion, they still have good coverage probabilities as the second-order matching prior. Moreover, the developed priors demonstrate excellent coverage probabilities even in small sample sizes, and their results are not highly sensitive to the changing of the values of (α, β, ϕ). For real applications, any of the presented priors can be used, but we recommend the matching prior that satisfies the matching criteria.

4.2 Real data analysis

This example is taken from Shoukri et al. (2005), who proposed the maximum likelihood estimator and the associated confidence interval based on a large sample size

Table 1 Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1

θ_1	α	β	n	π_1	π_2	π_4	π_m
0.1	0.5	0.1	5	0.054 (0.943)	0.054 (0.943)	0.053 (0.944)	0.051 (0.946)
			10	0.054 (0.942)	0.054 (0.942)	0.054 (0.943)	0.053 (0.943)
			15	0.055 (0.950)	0.055 (0.950)	0.054 (0.950)	0.054 (0.951)
		1.0	5	0.049 (0.948)	0.049 (0.948)	0.047 (0.949)	0.045 (0.951)
			10	0.051 (0.950)	0.051 (0.950)	0.051 (0.951)	0.050 (0.952)
			15	0.049 (0.950)	0.049 (0.950)	0.048 (0.949)	0.048 (0.950)
		10.0	5	0.052 (0.948)	0.051 (0.948)	0.050 (0.949)	0.046 (0.953)
			10	0.057 (0.944)	0.056 (0.944)	0.056 (0.945)	0.054 (0.947)
			15	0.052 (0.953)	0.051 (0.953)	0.051 (0.954)	0.050 (0.955)
	2.5	0.1	5	0.050 (0.949)	0.050 (0.949)	0.049 (0.952)	0.047 (0.953)
			10	0.054 (0.950)	0.054 (0.950)	0.053 (0.951)	0.052 (0.952)
			15	0.053 (0.950)	0.053 (0.950)	0.052 (0.950)	0.051 (0.952)
		1.0	5	0.050 (0.952)	0.049 (0.952)	0.049 (0.953)	0.045 (0.955)
			10	0.052 (0.946)	0.052 (0.946)	0.051 (0.947)	0.049 (0.950)
			15	0.050 (0.953)	0.050 (0.953)	0.049 (0.953)	0.048 (0.954)
		10.0	5	0.049 (0.948)	0.049 (0.948)	0.047 (0.949)	0.045 (0.951)
			10	0.053 (0.952)	0.052 (0.952)	0.052 (0.953)	0.049 (0.955)
			15	0.048 (0.951)	0.048 (0.951)	0.047 (0.952)	0.046 (0.954)
	5.0	0.1	5	0.053 (0.949)	0.052 (0.949)	0.051 (0.949)	0.050 (0.950)
			10	0.051 (0.950)	0.051 (0.950)	0.050 (0.951)	0.049 (0.951)
			15	0.045 (0.952)	0.045 (0.953)	0.044 (0.953)	0.044 (0.955)
		1.0	5	0.046 (0.950)	0.046 (0.950)	0.045 (0.951)	0.044 (0.952)
			10	0.048 (0.952)	0.048 (0.953)	0.048 (0.953)	0.046 (0.955)
			15	0.046 (0.953)	0.046 (0.953)	0.045 (0.953)	0.044 (0.954)
		10.0	5	0.047 (0.955)	0.047 (0.955)	0.046 (0.956)	0.043 (0.959)
			10	0.049 (0.947)	0.049 (0.947)	0.048 (0.947)	0.046 (0.950)
			15	0.052 (0.950)	0.052 (0.950)	0.051 (0.950)	0.050 (0.953)

Table 2 Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1

θ_1	α	β	n	π_1	π_2	π_4	π_m
0.5	0.5	0.1	5	0.051 (0.946)	0.051 (0.946)	0.051 (0.948)	0.049 (0.951)
			10	0.053 (0.949)	0.053 (0.949)	0.053 (0.949)	0.052 (0.950)
			15	0.052 (0.948)	0.052 (0.948)	0.052 (0.948)	0.051 (0.949)
		1.0	5	0.048 (0.950)	0.048 (0.950)	0.047 (0.951)	0.045 (0.953)
			10	0.049 (0.949)	0.049 (0.949)	0.049 (0.950)	0.047 (0.951)
			15	0.055 (0.949)	0.055 (0.950)	0.055 (0.950)	0.053 (0.950)
		10.0	5	0.054 (0.949)	0.054 (0.950)	0.053 (0.952)	0.050 (0.955)
			10	0.050 (0.949)	0.050 (0.949)	0.050 (0.951)	0.049 (0.952)
			15	0.049 (0.948)	0.049 (0.949)	0.048 (0.949)	0.047 (0.950)
	2.5	0.1	5	0.048 (0.950)	0.048 (0.950)	0.047 (0.950)	0.044 (0.950)
			10	0.050 (0.950)	0.050 (0.950)	0.050 (0.951)	0.048 (0.951)
			15	0.052 (0.949)	0.052 (0.950)	0.052 (0.950)	0.050 (0.950)
		1.0	5	0.051 (0.948)	0.050 (0.949)	0.049 (0.950)	0.049 (0.951)
			10	0.051 (0.951)	0.051 (0.952)	0.050 (0.952)	0.048 (0.953)
			15	0.048 (0.954)	0.048 (0.954)	0.048 (0.955)	0.047 (0.956)
		10.0	5	0.049 (0.954)	0.048 (0.954)	0.047 (0.955)	0.043 (0.960)
			10	0.051 (0.949)	0.051 (0.950)	0.049 (0.951)	0.045 (0.953)
			15	0.050 (0.947)	0.050 (0.947)	0.049 (0.949)	0.047 (0.950)
	5.0	0.1	5	0.050 (0.954)	0.050 (0.954)	0.049 (0.955)	0.048 (0.956)
			10	0.049 (0.952)	0.049 (0.952)	0.048 (0.953)	0.047 (0.954)
			15	0.053 (0.950)	0.053 (0.951)	0.052 (0.952)	0.051 (0.953)
		1.0	5	0.048 (0.955)	0.048 (0.955)	0.047 (0.956)	0.046 (0.959)
			10	0.052 (0.951)	0.052 (0.951)	0.051 (0.951)	0.050 (0.953)
			15	0.050 (0.950)	0.049 (0.950)	0.049 (0.951)	0.047 (0.951)
		10.0	5	0.047 (0.956)	0.047 (0.956)	0.047 (0.957)	0.043 (0.959)
			10	0.049 (0.948)	0.049 (0.948)	0.048 (0.948)	0.046 (0.950)
			15	0.050 (0.951)	0.050 (0.951)	0.050 (0.951)	0.049 (0.953)

for the reliability of the bivariate Lomax distribution. We will compare the frequentist confidence interval and the Bayesian credible interval using the developed non-informative prior.

Example 1 Batchelor and Hackett (1970) examined the failure times (in days) of closely matched and poorly matched skin grafts within the same sample of patients, with close or poor match defined by the similarity for the HLA system. Three patients had multiple graft failure times within the same group, and only the first graft failure time was included in the analysis. For three other patients, the failure times were given in intervals, and the midpoint of the intervals was used instead. The sample data can be found in Table 1 of Shoukri et al. (2005). Table 7 shows the maximum likelihood estimate (ML) and 95% confidence interval (LS) by Shoukri

Table 3 Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1

θ_1	α	β	n	π_1	π_2	π_4	π_m
0.9	0.5	0.1	5	0.050 (0.948)	0.050 (0.948)	0.050 (0.950)	0.050 (0.953)
			10	0.056 (0.950)	0.056 (0.950)	0.056 (0.950)	0.054 (0.952)
			15	0.049 (0.946)	0.049 (0.946)	0.049 (0.946)	0.048 (0.947)
		1.0	5	0.052 (0.947)	0.052 (0.947)	0.051 (0.949)	0.048 (0.951)
			10	0.054 (0.949)	0.054 (0.949)	0.053 (0.950)	0.052 (0.952)
			15	0.052 (0.952)	0.051 (0.952)	0.051 (0.952)	0.050 (0.953)
		10.0	5	0.053 (0.949)	0.053 (0.950)	0.051 (0.951)	0.047 (0.956)
			10	0.049 (0.948)	0.049 (0.948)	0.048 (0.949)	0.046 (0.951)
			15	0.044 (0.951)	0.044 (0.951)	0.044 (0.952)	0.043 (0.952)
	2.5	0.1	5	0.051 (0.954)	0.051 (0.954)	0.050 (0.957)	0.048 (0.957)
			10	0.045 (0.947)	0.045 (0.948)	0.044 (0.949)	0.043 (0.950)
			15	0.048 (0.951)	0.048 (0.951)	0.048 (0.951)	0.045 (0.953)
		1.0	5	0.051 (0.952)	0.050 (0.953)	0.049 (0.954)	0.044 (0.956)
			10	0.049 (0.951)	0.049 (0.951)	0.047 (0.953)	0.046 (0.955)
			15	0.055 (0.948)	0.055 (0.948)	0.054 (0.949)	0.052 (0.951)
		10.0	5	0.051 (0.946)	0.051 (0.947)	0.049 (0.949)	0.044 (0.953)
			10	0.054 (0.947)	0.054 (0.948)	0.052 (0.949)	0.049 (0.951)
			15	0.053 (0.943)	0.052 (0.943)	0.051 (0.944)	0.050 (0.947)
	5.0	0.1	5	0.051 (0.952)	0.051 (0.952)	0.049 (0.952)	0.048 (0.954)
			10	0.046 (0.952)	0.046 (0.952)	0.046 (0.952)	0.045 (0.953)
			15	0.048 (0.952)	0.047 (0.952)	0.046 (0.952)	0.044 (0.953)
		1.0	5	0.050 (0.955)	0.050 (0.954)	0.048 (0.955)	0.045 (0.958)
			10	0.050 (0.953)	0.050 (0.954)	0.049 (0.954)	0.048 (0.955)
			15	0.052 (0.951)	0.051 (0.951)	0.051 (0.953)	0.050 (0.954)
		10.0	5	0.051 (0.952)	0.051 (0.952)	0.050 (0.954)	0.047 (0.956)
			10	0.052 (0.950)	0.052 (0.951)	0.052 (0.951)	0.050 (0.953)
			15	0.053 (0.946)	0.053 (0.946)	0.052 (0.946)	0.051 (0.947)

et al. (2005), as well as the Bayes estimates and Bayesian credible intervals under JP, two-group reference prior (TR), one-at-a-time reference prior (OR), and matching prior (MP). The estimates and intervals for the reliability are relatively close to each other, with all estimates based on JP, TR, OR, and MP giving the same results that are slightly different from ML. The credible intervals under JP, TR, OR, and MP are again close to the confidence interval based on the large sample method, although slightly shorter in length.

Example 2 The sample data of times to the relief of headaches for standard and new treatments administered to the same patients was provided by Gross and Lam (1981) in Table II of Shoukri et al. (2005). Reliability estimates and confidence/credible

Table 4 Frequentist coverage probability of 90% (95%) credible intervals of θ_1

θ_1	α	β	n	π_1	π_2	π_4	π_m
0.1	0.5	0.1	5	0.889 (0.944)	0.889 (0.945)	0.891 (0.946)	0.895 (0.948)
			10	0.887 (0.940)	0.888 (0.940)	0.888 (0.941)	0.891 (0.942)
			15	0.895 (0.946)	0.895 (0.946)	0.896 (0.947)	0.897 (0.947)
		1.0	5	0.899 (0.948)	0.899 (0.948)	0.902 (0.949)	0.906 (0.952)
			10	0.899 (0.951)	0.899 (0.951)	0.900 (0.952)	0.902 (0.953)
			15	0.901 (0.949)	0.901 (0.949)	0.901 (0.949)	0.902 (0.949)
		10.0	5	0.896 (0.946)	0.897 (0.946)	0.899 (0.947)	0.906 (0.950)
			10	0.887 (0.943)	0.887 (0.943)	0.889 (0.945)	0.894 (0.946)
			15	0.902 (0.951)	0.902 (0.951)	0.903 (0.952)	0.905 (0.953)
	2.5	0.1	5	0.899 (0.948)	0.899 (0.949)	0.903 (0.950)	0.906 (0.953)
			10	0.896 (0.947)	0.896 (0.947)	0.898 (0.948)	0.901 (0.950)
			15	0.896 (0.948)	0.896 (0.948)	0.898 (0.949)	0.900 (0.950)
		1.0	5	0.902 (0.954)	0.902 (0.955)	0.905 (0.957)	0.910 (0.956)
			10	0.894 (0.949)	0.894 (0.949)	0.896 (0.950)	0.900 (0.951)
			15	0.903 (0.950)	0.903 (0.950)	0.904 (0.951)	0.905 (0.952)
		10.0	5	0.899 (0.951)	0.899 (0.951)	0.901 (0.953)	0.906 (0.956)
			10	0.899 (0.949)	0.899 (0.949)	0.901 (0.951)	0.905 (0.953)
			15	0.903 (0.950)	0.903 (0.951)	0.905 (0.952)	0.908 (0.954)
	5.0	0.1	5	0.896 (0.945)	0.897 (0.945)	0.898 (0.948)	0.901 (0.952)
			10	0.899 (0.949)	0.899 (0.949)	0.901 (0.951)	0.902 (0.952)
			15	0.908 (0.956)	0.908 (0.956)	0.909 (0.957)	0.911 (0.958)
		1.0	5	0.904 (0.952)	0.904 (0.953)	0.906 (0.954)	0.908 (0.954)
			10	0.904 (0.953)	0.904 (0.954)	0.906 (0.955)	0.909 (0.956)
			15	0.906 (0.952)	0.907 (0.952)	0.908 (0.952)	0.910 (0.954)
		10.0	5	0.908 (0.957)	0.908 (0.957)	0.911 (0.959)	0.916 (0.963)
			10	0.898 (0.952)	0.898 (0.952)	0.900 (0.953)	0.903 (0.954)
			15	0.898 (0.949)	0.898 (0.949)	0.900 (0.950)	0.903 (0.952)

intervals based on ML, JP, Tikhonov regularization (TR), optimality rule (OR) and maximum penalized likelihood (MP) methods are calculated and presented in Table 8. It can be seen from the results that the estimates from all the methods are the same, while the credible intervals are different from the confidence interval based on large sample methods in their upper confidence limit. Moreover, the credible intervals provide estimates with much shorter lengths than the confidence interval by the large sample method.

Table 5 Frequentist coverage probability of 90% (95%) credible intervals of θ_1

θ_1	α	β	n	π_1	π_2	π_4	π_m
0.5	0.5	0.1	5	0.895 (0.948)	0.895 (0.948)	0.897 (0.949)	0.902 (0.950)
			10	0.895 (0.945)	0.896 (0.945)	0.896 (0.947)	0.897 (0.948)
			15	0.896 (0.950)	0.896 (0.950)	0.897 (0.951)	0.898 (0.952)
		1.0	5	0.901 (0.948)	0.902 (0.948)	0.904 (0.951)	0.908 (0.954)
			10	0.900 (0.950)	0.900 (0.950)	0.901 (0.951)	0.904 (0.953)
			15	0.895 (0.947)	0.895 (0.947)	0.895 (0.948)	0.897 (0.948)
		10.0	5	0.895 (0.946)	0.896 (0.946)	0.899 (0.950)	0.905 (0.955)
			10	0.899 (0.951)	0.899 (0.951)	0.901 (0.952)	0.903 (0.955)
			15	0.899 (0.950)	0.900 (0.950)	0.901 (0.951)	0.903 (0.953)
	2.5	0.1	5	0.902 (0.953)	0.902 (0.953)	0.904 (0.954)	0.906 (0.957)
			10	0.900 (0.946)	0.900 (0.946)	0.901 (0.946)	0.903 (0.948)
			15	0.897 (0.948)	0.897 (0.948)	0.898 (0.948)	0.900 (0.950)
		1.0	5	0.897 (0.948)	0.898 (0.948)	0.900 (0.950)	0.903 (0.953)
			10	0.900 (0.949)	0.900 (0.949)	0.902 (0.950)	0.905 (0.953)
			15	0.905 (0.951)	0.906 (0.951)	0.907 (0.953)	0.909 (0.956)
		10.0	5	0.905 (0.955)	0.906 (0.955)	0.908 (0.958)	0.917 (0.962)
			10	0.898 (0.952)	0.899 (0.952)	0.902 (0.955)	0.907 (0.959)
			15	0.897 (0.948)	0.897 (0.948)	0.900 (0.950)	0.903 (0.952)
	5.0	0.1	5	0.904 (0.952)	0.904 (0.952)	0.906 (0.954)	0.908 (0.955)
			10	0.902 (0.953)	0.902 (0.953)	0.905 (0.954)	0.907 (0.957)
			15	0.897 (0.952)	0.898 (0.952)	0.899 (0.953)	0.902 (0.955)
		1.0	5	0.907 (0.952)	0.907 (0.952)	0.909 (0.954)	0.912 (0.958)
			10	0.899 (0.949)	0.899 (0.949)	0.900 (0.950)	0.903 (0.952)
			15	0.900 (0.952)	0.901 (0.952)	0.902 (0.953)	0.904 (0.955)
		10.0	5	0.908 (0.957)	0.909 (0.957)	0.910 (0.959)	0.916 (0.963)
			10	0.899 (0.952)	0.899 (0.952)	0.899 (0.953)	0.903 (0.956)
			15	0.900 (0.950)	0.900 (0.950)	0.901 (0.951)	0.904 (0.953)

5 Concluding remarks

In this bivariate Lomax distribution study, we conducted an objective Bayesian analysis to evaluate the reliability $R = P(Y < X)$. We found probability MPs for the reliability via orthogonal reparameterization, as well as reference priors. We established that not all reference priors satisfy the first-order matching criterion, and that a one-at-a-time reference prior yields an improper posterior. Moreover, the matching and reference priors have different forms. Our numerical study showed that the matching prior that satisfies the second-order matching criterion has very good asymptotic

Table 6 Frequentist coverage probability of 90% (95%) credible intervals of θ_1

θ_1	α	β	n	π_1	π_2	π_4	π_m
0.9	0.5	0.1	5	0.898 (0.949)	0.898 (0.949)	0.901 (0.950)	0.903 (0.952)
			10	0.894 (0.943)	0.894 (0.943)	0.895 (0.944)	0.897 (0.946)
			15	0.897 (0.949)	0.897 (0.949)	0.897 (0.949)	0.899 (0.949)
		1.0	5	0.895 (0.944)	0.896 (0.944)	0.898 (0.946)	0.903 (0.950)
			10	0.896 (0.949)	0.895 (0.949)	0.897 (0.950)	0.899 (0.950)
			15	0.901 (0.952)	0.901 (0.952)	0.902 (0.953)	0.903 (0.954)
		10.0	5	0.896 (0.947)	0.897 (0.947)	0.900 (0.950)	0.909 (0.956)
			10	0.899 (0.949)	0.900 (0.949)	0.901 (0.950)	0.905 (0.952)
			15	0.906 (0.952)	0.906 (0.952)	0.908 (0.953)	0.909 (0.954)
	2.5	0.1	5	0.903 (0.952)	0.903 (0.952)	0.907 (0.953)	0.909 (0.956)
			10	0.902 (0.953)	0.903 (0.953)	0.905 (0.954)	0.907 (0.955)
			15	0.902 (0.953)	0.903 (0.953)	0.904 (0.953)	0.907 (0.956)
		1.0	5	0.902 (0.951)	0.903 (0.951)	0.905 (0.953)	0.912 (0.957)
			10	0.902 (0.953)	0.903 (0.953)	0.906 (0.955)	0.909 (0.956)
			15	0.893 (0.946)	0.893 (0.946)	0.895 (0.947)	0.898 (0.950)
		10.0	5	0.895 (0.949)	0.896 (0.949)	0.899 (0.952)	0.909 (0.957)
			10	0.894 (0.949)	0.894 (0.949)	0.897 (0.950)	0.902 (0.953)
			15	0.890 (0.942)	0.890 (0.942)	0.893 (0.944)	0.897 (0.946)
	5.0	0.1	5	0.900 (0.953)	0.900 (0.953)	0.903 (0.954)	0.906 (0.954)
			10	0.905 (0.955)	0.906 (0.955)	0.906 (0.957)	0.908 (0.958)
			15	0.904 (0.951)	0.904 (0.951)	0.905 (0.952)	0.908 (0.953)
		1.0	5	0.905 (0.952)	0.904 (0.952)	0.907 (0.954)	0.913 (0.958)
			10	0.903 (0.949)	0.903 (0.949)	0.905 (0.951)	0.907 (0.953)
			15	0.899 (0.951)	0.900 (0.952)	0.901 (0.953)	0.904 (0.954)
		10.0	5	0.900 (0.949)	0.901 (0.949)	0.903 (0.951)	0.909 (0.955)
			10	0.898 (0.948)	0.898 (0.948)	0.899 (0.950)	0.902 (0.952)
			15	0.893 (0.946)	0.893 (0.946)	0.894 (0.947)	0.896 (0.949)

Table 7 The estimates and 95% confidence intervals for the reliability in data of the failure time of skin grafts

Method	Estimate	Method	Confidence interval	Length
ML	0.629	LS	(0.399, 0.812)	0.413
JP	0.609	JP	(0.412, 0.782)	0.370
TR	0.609	TR	(0.412, 0.782)	0.370
OR	0.609	OR	(0.411, 0.783)	0.372
MP	0.609	MP	(0.408, 0.784)	0.376

frequentist coverage properties. Interestingly, the reference priors also demonstrate good performance, despite not satisfying the first-order matching criterion.

Table 8 The estimates and 95% confidence intervals for the reliability in data of time to the relief of headaches

Method	Estimate	Method	Confidence interval	Length
ML	0.522	LS	(0.286, 0.848)	0.562
JP	0.523	JP	(0.318, 0.722)	0.404
TR	0.523	TR	(0.318, 0.722)	0.404
OR	0.523	OR	(0.317, 0.723)	0.406
MP	0.523	MP	(0.315, 0.725)	0.410

Derivations of the one-at-a-time reference priors

We derive the one-at-a-time reference priors for the parameter groupings $\{\theta_1, \theta_2, \theta_3\}$ and $\{\theta_1, \theta_3, \theta_2\}$. For the derivation of the reference prior for the parameter groupings $\{\theta_1, \theta_2, \theta_3\}$, we obtain the following quantities from the inverse matrix of the Fisher information (3).

$$
\begin{aligned}
h_1 &= \frac{n(2+\theta_3)}{2\theta_1^2(1-\theta_1)^2(3+\theta_3)},\\
h_2 &= \frac{n\theta_3(4+6\theta_3+3\theta_3^2)}{2\theta_2^2(2+\theta_3)^2(3+\theta_3)(1+2\theta_3+2\theta_3^2)},\\
h_3 &= \frac{n(1+2\theta_3+2\theta_3^2)}{\theta_3^2(1+\theta_3)^2}.
\end{aligned}
$$

Now we begin to derive the one-at-a-time reference prior for the parameter grouping $\{\theta_1, \theta_2, \theta_3\}$. The compact subsets were taken to be Cartesian products of sets of the form

$$\theta_1 \in [a_1, b_1], \theta_2 \in [a_2, b_2], \theta_3 \in [a_3, b_3].$$

In the limit a_1, a_2 and a_3 will tend to 0, b_1 will tend to 1, b_2 and b_3 will tend to ∞. Here, and below, a subscripted Q denotes a function that is constant and does not depend on any parameters but any Q may depend on the ranges of the parameters.

Step 1. Note that

$$\int_{a_3}^{b_3} h_3^{1/2} d\theta_3 = \int_{a_3}^{b_3} \left[\frac{n(1+2\theta_3+2\theta_3^2)}{\theta_3^2(1+\theta_3)^2}\right]^{1/2} d\theta_3 = nQ_1.$$

It follows that

$$\pi_3^l(\theta_3|\theta_1,\theta_2) = \frac{h_3^{1/2}}{\int_{a_3}^{b_3} h_3^{1/2} d\theta_3} = Q_1^{-1} \frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)}.$$

Step 2. Now we have

$$\begin{aligned}
E^l\{\log h_2|\theta_1,\theta_2\} &= \int_{a_3}^{b_3} \log\left[\frac{n\theta_3(4+6\theta_3+3\theta_3^2)}{2\theta_2^2(2+\theta_3)^2(3+\theta_3)(1+2\theta_3+2\theta_3^2)}\right]\pi_3^l(\theta_1|\theta_1,\theta_2)d\theta_3\\
&= \int_{a_3}^{b_3} Q_1^{-1}\log\left[\frac{n\theta_3(4+6\theta_3+3\theta_3^2)}{2\theta_2^2(2+\theta_3)^2(3+\theta_3)(1+2\theta_3+2\theta_3^2)}\right]\frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)}d\theta_3.\\
&= Q_{21} + \log\theta_2^{-2}.
\end{aligned}$$

It follows that

$$\int_{a_2}^{b_2} \exp[E^l\{\log h_2|\theta_1,\theta_2\}/2]d\theta_2 = \exp\{Q_{21}\}Q_2$$

Hence

$$\pi_2^l(\theta_2,\theta_3|\theta_1) = \frac{\pi_3^l(\theta_3|\theta_1,\theta_2)\exp[E^l\{\log h_2|\theta_1,\theta_2\}/2]}{\int_{a_2}^{b_2}\exp[E^l\{\log h_2|\theta_1,\theta_2\}/2]d\theta_2} = Q_1^{-1}Q_2^{-1}\frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)}\theta_2^{-1}.$$

Step 3. In the final step,

$$\begin{aligned}
E^l\{\log h_1|\theta_1\} &= \int_{a_2}^{b_2}\int_{a_3}^{b_3} \log\left[\frac{n(2+\theta_3)}{2\theta_1^2(1-\theta_1)^2(3+\theta_3)}\right]\pi_2^l(\theta_2,\theta_3|\theta_1)d\theta_3 d\theta_2\\
&= \int_{a_2}^{b_2}\int_{a_3}^{b_3} \log\left[\frac{n(2+\theta_3)}{2\theta_1^2(1-\theta_1)^2(3+\theta_3)}\right]Q_1^{-1}Q_2^{-1}\frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)}\theta_2^{-1}d\theta_3 d\theta_2.\\
&= Q_{31} + \log[\theta_1^{-2}(1-\theta_1)^{-2}].
\end{aligned}$$

It follows that

$$\int_{a_1}^{b_1} \exp[E^l\{\log h_1|\theta_1\}/2]d\theta_1 = \exp\{Q_{31}\}Q_3$$

Hence

$$\begin{aligned}
\pi_1^l(\theta_1,\theta_2,\theta_3) &= \frac{\pi_2^l(\theta_2,\theta_3|\theta_1)\exp[E^l\{\log h_1|\theta_1\}/2]}{\int_{a_1}^{b_1}\exp[E^l\{\log h_1|\theta_1\}/2]d\theta_1}\\
&= Q_1^{-1}Q_2^{-1}Q_3^{-1}\theta_1^{-1}(1-\theta_1)^{-1}\theta_2^{-1}\frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)}.
\end{aligned}$$

Thus the one-at-a-time reference prior is

$$\pi_3(\theta_1, \theta_2, \theta_3) = \lim_{l \to \infty} \frac{\pi_1^l(\theta_1, \theta_2, \theta_3)}{\pi_1^l(\theta_{10}, \theta_{20}, \theta_{30})}$$

$$\propto \theta_1^{-1}(1-\theta_1)^{-1}\theta_2^{-1}\frac{(1+2\theta_3+2\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)},$$

where θ_{10} is an inner point of the interval (0, 1), and θ_{20} and θ_{30} are any inner points of the interval $(0, \infty)$.

Next, we derive the one-at-a-time reference prior for the parameter grouping $\{\theta_1, \theta_3, \theta_2\}$. Then we obtain the following quantities from the inverse matrix of the Fisher information for the derivation of the reference prior.

$$h_1 = \frac{n(2+\theta_3)}{2\theta_1^2(1-\theta_1)^2(3+\theta_3)},$$

$$h_2 = \frac{n(4+6\theta_3+3\theta_3^2)}{\theta_3^2(1+\theta_3)^2(2+\theta_3)^2},$$

$$h_3 = \frac{n\theta_3}{2\theta_2^2(3+\theta_3)}.$$

Thus by the first derivation method of the one-at-a-time reference prior for the parameter groupings $\{\theta_1, \theta_3, \theta_2\}$, we can show that the one-at-a-time reference prior is

$$\pi_4(\theta_1, \theta_2, \theta_3) \propto \theta_1^{-1}(1-\theta_1)^{-1}\theta_2^{-1}\frac{(4+6\theta_3+3\theta_3^2)^{\frac{1}{2}}}{\theta_3(1+\theta_3)(2+\theta_3)}.$$

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