

# On Bayesian reliability analysis with informative priors and censoring

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In the statistical literature many methods have been presented to deal with censored observations, both within the Bayesian and non-Bayesian frameworks, and such methods have been successfully applied to, e.g., reliability problems. Also, in reliability theory it is often emphasized that, through shortage of statistical data and possibilities for experiments, one often needs to rely heavily on judgements of engineers, or other experts, for which means Bayesian methods are attractive. It is therefore important that such judgements can be elicited easily to provide informative prior distributions that reflect the knowledge of the engineers well. In this paper we focus on this aspect, especially on the situation that the judgements of the consulted engineers are based on experiences in environments where censoring has also been present previously. We suggest the use of the attractive interpretation of hyperparameters of conjugate prior distributions when these are available for assumed parametric models for lifetimes, and we show how one may go beyond the standard conjugate priors, using similar interpretations of hyperparameters, to enable easier elicitation when censoring has been present in the past. This may even lead to more flexibility for modelling prior knowledge than when using standard conjugate priors, whereas the disadvantage of more complicated calculations that may be needed to determine posterior distributions play a minor role due to the advanced mathematical and statistical software that is widely available these days. © 1996 Elsevier Science Limited.

## 1 INTRODUCTION

The Bayesian framework for statistics and decision theory<sup>1</sup> offers great opportunities for applications in reliability problems,<sup>2</sup> because of the possibility to take expert knowledge into account through the prior distribution for an assumed parametric model. Often a random variable of interest, such as lifetime of a unit, is modelled by a probability distribution that is assumed to belong to some common family of distributions, e.g., Weibull or gamma, and that is known up to a parameter, mostly one- or two-dimensional. When restricting to such a family, all information available is taken into account by a probability distribution for this parameter. Through Bayes' updating rule, this probability distribution for the parameter changes in the light of new information. In application of the Bayesian method, the first version of such a distribution for the parameter is

called the prior distribution, and usually this is either chosen in an attempt to represent as little information as possible (non-informative prior), or it is chosen such that it describes available expert knowledge reasonably well (informative prior). If further information in the form of data is gathered, e.g., failure or survival data, the prior is updated by multiplication with the likelihood of the data, giving the posterior distribution for the parameter.

Especially in practical reliability problems, the use of judgements of engineers may be important, since it may be the only source of information, or there may be only little additional data (e.g., historical or test data) that are relevant. Cooke<sup>3</sup> provides an explicit study of the role of expert judgement in practical decision problems, also reporting on practical aspects and applications. Coolen *et al.*<sup>4</sup> apply a Bayesian method for using expert judgement (of 11 experts) for reliability estimation for a critical heat-exchanger in a

chemical plant. Van Noortwijk *et al.*<sup>5</sup> use expert judgement for maintenance optimization, also within a Bayesian framework. The informative priors used by Coolen *et al.*<sup>4</sup> and Van Noortwijk *et al.*<sup>5</sup> are elicited by partitioning the time-axis into about 5 intervals and asking the engineers to state probabilities that lifetimes of specified equipment will end in each of these intervals. This is a simple method that looks quite logical, but it only provides a prior distribution for a parameter of interest in an indirect way. In this paper we suggest a possible alternative for elicitation of an informative prior, that should not be more difficult to use in practice than elicitation using a partition of the time-axis.

Although, theoretically, the Bayesian method can deal with every prior distribution, it is often advocated to use prior distributions of particular forms that enable updating by some simple calculations. In particular, priors that belong to a conjugate family<sup>6</sup> with regard to an assumed parametric model are attractive. This feature will be discussed in Section 3 of this paper, focusing on parametric models that belong to the exponential family, as many simple models that are commonly used in reliability theory do. A second advantage of conjugate priors will be emphasized, namely the possibility of relating expert knowledge to imaginary data. First, however, in Section 2, we briefly discuss some statistical methods to deal with censored data. In reliability problems, as, e.g., in medical statistics, lifetimes are often not actually observed, but they are only known to be greater than a certain value, for example the time when some system was replaced before it actually failed. There is a huge literature on this topic, and without trying to be exhaustive we will mention a few contributions. In the Bayesian methodology, censored data have an effect on the likelihood in updating that is quite well known. The main contribution of this paper is given in Section 4, where we suggest the possible generalization of well-known conjugate priors in order to elicit informative priors in situations where the judgements of consulted experts are based on experiences in environments where censoring has also been present. For example, if a safety rule has always forced some system to be replaced after 2 years, the engineers may not have any idea about the lifetime distribution for this system for values greater than 2 years. As far as we know this aspect has not been dealt with before in the reliability literature.

It is to be remarked that our proposed generalization of conjugate priors in the case of censoring can also be exploited in robust Bayesian analysis,<sup>7</sup> where inferences are robust with regard to the choice of the prior distribution, as well as in imprecise Bayesian reliability analysis.<sup>8</sup> We do not discuss these generalizations, which are quite straightforward, any further in this paper.

## 2 CENSORED DATA

Censoring of observations is a natural feature in many areas of applied statistics. In reliability theory, lifetimes of technical systems are often only known to exceed a certain value due to early replacement of the systems, major maintenance activities or other reasons. For example, in the study reported by Coolen *et al.*,<sup>4</sup> governmental rules asked for safety inspections of heat-exchangers once every two years, and until the moment of the study it had been usual policy in the firm to replace critical heat-exchangers at these moments of inspection. Therefore, actual lifetimes were only observed up to two years, which would lead to right-censoring of some of the observations (lifetimes exceeding two years). In Section 4 we will look at the possibility of getting simple informative priors in circumstances like this. In this section, we give a brief overview of some mainstream methods in statistics, not restricted to Bayesian statistics, for dealing with censored observations. Because of the enormous amount of literature on this topic, we do not intend to give anything near to an exhaustive overview, nor do we suggest that inclusion in this short overview is a judgement about the quality and usefulness of methods.

A major contribution to the (non-Bayesian) statistical theory for dealing with censored data was provided by Kaplan & Meier,<sup>9</sup> presenting their nonparametric 'product-limit' estimator for survivor functions under quite general right-censoring on the data. They showed that this estimator is the nonparametric maximum likelihood estimator, and this estimator has become a standard method described in almost all later textbooks on reliability or survival theory, e.g., Kalbfleisch & Prentice<sup>10</sup> or Lawless,<sup>11</sup> and also available in statistical packages such as S-Plus.<sup>12</sup> Many researchers have provided extended results to that of Kaplan & Meier, for example Turnbull,<sup>13</sup> who considered more general forms of censoring, and Kimber,<sup>14</sup> who discusses the estimation of quantiles in the presence of right-censored data. Efron<sup>15</sup> provided a clear intuition for Kaplan & Meier's estimator through a process of redistribution of probability mass. All such methods could be referred to as 'data-only' methods, in that they only use the statistical data and cannot be used to take subjective information into account. A recently developed theory that is very general, and also 'data-only', is the theory of counting processes,<sup>16</sup> with attractive possibilities for applications in reliability problems provided one has many observations, possibly censored, available. An interesting paper about practical problems in statistical analysis of reliability data is written by Ansell & Phillips,<sup>17</sup> who discuss so-called Total Time on Test (TTT) methods

for fitting models to right-censored data, and also mention many more interesting practical problems beyond the problem of censoring.

Within the Bayesian theory of statistics, dealing with censored data is quite straightforward through the adapted form of the likelihood function for a specified model, as presented in detail by, e.g., Martz & Waller.<sup>2</sup> This is also the area to which this paper contributes by discussing, in Section 4, an alternative easy method to elicit informative prior distributions in an environment with censoring. However, some interesting contributions to nonparametric Bayesian theory may also be mentioned. Mostly, these are attempts to create a Bayesian theory that is close to non-Bayesian nonparametric theories, such as Kaplan & Meier's estimator. A basic contribution was made by Ferguson,<sup>18</sup> using a Dirichlet process, and many authors have extended these ideas, e.g., Susarla & Van Ryzin<sup>19</sup> who also discuss the relation with Kaplan & Meier's estimator. Also, the method used by Van Noortwijk *et al.*<sup>5</sup> is related to Ferguson's work on the Dirichlet process and to a generalization of this process by Lochner,<sup>20</sup> and they actually aim at using judgements of experts to get informative priors for situations where there are not many observations available. These methods lead to a discrete estimate of the survivor function, just as the method of Kaplan & Meier does, and to overcome this problem (one may feel that the unknown survivor function should be a continuous function without discrete jumps) Berliner & Hill<sup>21</sup> suggest a different method to get a Bayesian nonparametric estimate of the survivor function, where, basically, the discrete probability mass at observed failure times is spread out over intervals between such failure times. Berliner & Hill's method can also be explained by a similar process as Efron<sup>15</sup> gave to explain the estimator of Kaplan & Meier, and Hill<sup>22</sup> gives a comparison of these two methods, both theoretically and empirically.

Generally, the Bayesian concept of statistics and decision making<sup>23-25</sup> is based on a theory of uncertainty, measured by subjective probabilities, as clearly described by De Finetti.<sup>26</sup> Particularly in applications of reliability theory, it seems often unavoidable to use judgements, and therefore subjective probability seems to be the logical tool to measure beliefs, and hence the Bayesian framework seems to be the logical concept for statistics here. This point of view is strongly defended by, e.g., Singpurwalla<sup>27</sup> and Singpurwalla & Soyer,<sup>28</sup> without considering aspects of practical elicitation. Goldstein<sup>29</sup> has developed an interesting theory for dealing with uncertainty, that can be regarded as extending ideas from De Finetti, and it would be interesting to see this theory applied to reliability problems, but there are no results or attempts in that direction known yet.

In the next section we restrict our attention to

Bayesian methodology for fully specified parametric models, and we consider the advantages of conjugate priors.

### 3 CONJUGATE PRIORS

Many of the commonly used parametric models in statistics and reliability theory have a similar form, referred to as the exponential family. A probability distribution for a random variable  $X \in \Omega \subseteq \mathbb{R}$ , labelled by parameter  $\theta \in \Theta \subseteq \mathbb{R}$ , belongs to the one-parameter exponential family if its probability density function (pdf) can be put into the form

$$p_X(x|\theta) = g(x)h(\theta)\exp\{t(x)\psi(\theta)\} \quad (1)$$

or equivalently if the likelihood of  $n$  independent uncensored observations  $\underline{x} = (x_1, x_2, \dots, x_n)$  from a population with this probability distribution is

$$L(\theta|\underline{x}) \propto \{h(\theta)\}^n \exp\left\{\sum_{i=1}^n t(x_i)\psi(\theta)\right\}. \quad (2)$$

For ease of notation we restrict the presentation to models that are labelled by a one-dimensional parameter, although the results are easily generalized to  $k$ -parameter exponential families, and we assume that  $\Omega$  does not depend on  $\theta$ .<sup>1</sup> The random variable  $X$  can be continuous or discrete. Some well-known members of this family<sup>23</sup> are the exponential, Weibull (with known shape parameter), Gamma (with either the shape or scale parameter known), normal (with either the mean or variance known), binomial, negative binomial and Poisson distributions. Of special interest for reliability theory are the exponential, Weibull and gamma distributions as models for lifetimes, whereas also the (negative) binomial distributions (for failure/no-failure data), the normal distributions (for their relation to lognormal distributions), and the Poisson distributions (related to Poisson processes) may be of interest.

From the likelihood eqn (2) it is clear that  $\left(n, \sum_{i=1}^n t(x_i)\right)$  are sufficient statistics of the observations  $\underline{x}$ .<sup>1</sup> If an assumed parametric model belongs to the exponential family, there is an unambiguous definition of a conjugate family of prior distributions in the Bayesian framework.<sup>6</sup> For the one-parameter exponential family, this family of priors is defined through pdf's for  $\theta \in \Theta$  that depend on hyperparameters  $\nu, \tau$

$$\pi(\theta|\nu, \tau) \propto \{h(\theta)\}^\nu \exp\{\tau\psi(\theta)\}. \quad (3)$$

The hyperparameters can be interpreted as sufficient statistics of imaginary observations, where  $\nu$

and  $\tau$  correspond to  $n$  and  $\sum_{i=1}^n t(x_i)$  in the likelihood eqn (2), respectively. So to model knowledge by the choice of a particular prior distribution, one can actually think about an imaginary data set consisting of  $v$  independent observations, with sufficient statistics  $(v, \tau)$  for the assumed parametric model within the one-parameter exponential family. Barnett<sup>30</sup> refers to this interpretation in terms of an 'equivalent prior sample'. Using this interpretation of the hyperparameters of conjugate priors, it is quite easy to elicit informative priors that represent judgements of engineers or other experts that are considered to be relevant. The engineers do not have to express their opinions about abstract parameters for an assumed model, but only need to think about actual observables. Geisser<sup>31</sup> provides sensible arguments for focusing on the use of observables in Bayesian analyses, but he hardly pays attention to elicitation of informative priors. There is, however, one possible problem in asking engineers to provide a sensible imaginary prior sample leading to a prior distribution, that is that their experience may be based on observing a process under censoring, e.g., replacement of systems after at most two years due to governmental safety rules. In such cases, it may be very hard for the engineers to give their opinions by use of a complete imaginary data set, including observations within the range that tends to be censored. This is what we will focus on in Section 4 of this paper. We should warn the reader for the interpretation of the imaginary prior data. These are only a means for elicitation of an informative prior, and only act to help an engineer expressing and quantifying his judgements. This elicitation process should take place before statistical data, to be used for the updating of the prior, are collected (or known to the engineer), and by no means should the imaginary data be regarded as playing an equal role to the actual data, so the imaginary data are certainly not to be interpreted as being exchangeable<sup>26,32</sup> with the real statistical data.

The possibility of rather simple elicitation is a useful and sensible argument for using a conjugate prior, although in the Bayesian literature it is overshadowed by the simple way in which such a prior can be updated if data  $\underline{x}$  become available. Combining the likelihood eqn (2) and the conjugate prior eqn (3) it follows that the posterior pdf is

$$\pi(\theta|\underline{x}, v, \tau) \propto \{h(\theta)\}^{v+n} \exp \left\{ \left( \tau + \sum_{i=1}^n t(x_i) \right) \psi(\theta) \right\}.$$

This posterior has the same functional form as the prior pdf, with  $(v, \tau)$  replaced by  $\left( v + n, \tau + \sum_{i=1}^n t(x_i) \right)$ . Although it is indeed attractive to be able to update a

prior in the light of new data by some simple calculations, this feature seems to play a less important role with the increasing possibilities of numerical updating with modern computing facilities and calculation techniques such as iterative Monte Carlo methods.<sup>33</sup> In the light of new calculation methods, Carlin & Gelfand<sup>34</sup> even propose a slightly altered definition of conjugacy, which is not relevant to us but supports the claim that easy updating is becoming a less important advantage of conjugate priors nowadays. The possible interpretation of the hyperparameters as sufficient statistics of an imaginary data set remains an important advantage of conjugate priors. In Section 4 we propose how, for simple elicitation, the advantages of generalized forms of conjugate priors can still be explored if the prior judgements are based on experience in environments where censoring has been present throughout.

#### 4 INFORMATIVE PRIORS AND CENSORING

Conjugacy of prior distributions with regard to some parametric model is based on the fact that the prior has the same functional form as the likelihood, where the likelihood is determined by sufficient statistics of

low dimension,  $\left( n, \sum_{i=1}^n t(x_i) \right)$  for the models in Section 3

where all data are uncensored. Also, if the data are subject to censoring it may be that there are sufficient statistics of low dimension for the entire data set, consisting of both censored and uncensored observations. Again, this can be exploited by the use of conjugate priors, with simple updating and interpretation of the hyperparameters. Furthermore, even if the actual functional form of the likelihood is not yet known at the time of elicitation of the informative prior distribution, if censoring of real observations could be dealt with by some simple sufficient statistics, this feature can also be used to simplify elicitation, which is of interest if the judgements of the engineers, or other experts, are, e.g., based on experiences with systems that have been subject to censoring in the past. We will explain this line of thought further in some examples. It needs to be remarked that the priors that we suggest to be used are generalizations to the standard conjugate priors, and in fact can have the same functional form as the likelihoods, but this is not necessary. By generalizing the standard conjugate priors, as discussed in Section 3, the functional form of the posterior, achieved by updating the prior in the light of new information, may be much more complicated as for the simple situation without any censoring as described in Section 3. However, most functional forms involved are nowadays easily dealt with by use of mathematical or statistical software,

and therefore this disadvantage is only small, especially when compared to the advantage of simple elicitation. It needs to be remarked that, although dealing with censored data has received much attention in the statistical literature (see Section 2), the aspect of elicitation of informative prior distributions in environments where censoring has played an important role in the experience of the consulted experts has been highly neglected in literature so far, we hope that the examples below serve as eye-openers to the possibilities with regard to this problem.

As a first example, let us consider an exponential distribution with right-censoring of type I. This means that, with  $n$  units starting at time 0 and assumed to fail independently at lifetimes  $X_i$ ,  $i = 1, \dots, n$ , we only observe  $\min(X_i, c)$  for all  $i$  and some known constant  $c$ .

#### 4.1 Example 1

Suppose that the lifetimes of  $n$  units are all independent and identically distributed, with an exponential distribution described by the pdf

$$p_X(x|\theta) = \theta \exp\{-\theta x\}.$$

It is easily seen that this can be regarded as a special case of the general form eqn (1) with  $g(x) = 1$ ,  $h(\theta) = \theta$ ,  $t(x) = x$  and  $\psi(\theta) = -\theta$ . If there is no censoring, we observe  $\underline{x} = \{x_1, \dots, x_n\}$ , then eqn (2) and eqn (3) give the likelihood

$$L(\theta|\underline{x}) \propto \theta^n \exp\left\{-\sum_{i=1}^n x_i \theta\right\}$$

and conjugate priors have the form

$$\pi(\theta|v, \tau) \propto \theta^v \exp\{-\tau\theta\}.$$

Data  $\underline{x}$  give a posterior of the same functional form as the prior, with  $(v, \tau)$  replaced by  $(v + n, \tau + \sum_{i=1}^n x_i)$ .

Next, suppose that we know that the experiment (all  $n$  units start at time 0) will stop after time  $c$  ( $0 < c < \infty$  known to us from the beginning), and that the data then will consist of  $m \leq n$  failure times, say  $x_i$ ,  $i = 1, \dots, m$ , and  $n - m$  survivors. Further, assume that it is regarded to be most likely that  $m < n$ . Since the model survivor function is

$$S_X(x|\theta) = \exp\{-\theta x\},$$

this leads to the likelihood

$$L(\theta|x_1, \dots, x_m, n, m) \propto \theta^m \exp\left\{-\left(\sum_{i=1}^m x_i + (n - m)c\right)\theta\right\}.$$

Sufficient statistics are  $(n, m, \sum_{i=1}^m x_i)$ , and this suggests informative priors of the form

$$\pi(\theta|v, \mu, \tau) \propto \theta^\mu \exp\{-[\tau + (v - \mu)c]\theta\}.$$

Even more generally, priors of such form with different values of  $c$  can be used if this fits better with the experience of the consulted engineers. For example, in the past, some systems may have been replaced after two years for safety reasons, whereas, in future, regulations may be relaxed to replacement after three years. The technical details for such a generalization are quite straightforward.

At first sight this does not seem to be of any use, since such priors lead to a family of priors that is actually equal to the family of priors related to the likelihood for uncensored observations only (this will not be true in general). But let us consider the possible interpretation of the hyperparameters, that need to be assigned in the elicitation process. The hyperparameter  $v$  can be interpreted as the size of an imaginary sample. For example, if we want the prior judgements to have an equal effect on the posterior distribution for  $\theta$  as the experimental data, we could ask the engineer to state his thoughts about the lifetimes of  $v = n$  units. Using a standard conjugate prior for  $\theta$  would require assessment of the sum of all imaginary failure times as the second hyperparameter. In the literature so far it is tacitly assumed that this is no problem for the consulted engineer, but if most of his knowledge is based on experience with systems that have always been replaced after time  $c$ , it may be very difficult to express useful judgements about imaginary failure times beyond  $c$ . Our suggested generalized prior allows the expert to consider the same time  $c$ , and to state  $\mu$ , the number of failures before time  $c$  in his imaginary data set, together with  $\tau$ , the sum of these failure times. This would only require explicit information about lifetimes smaller than  $c$ , of which the expert is supposed to have considerable knowledge. Once the hyperparameters have been assessed, and the experimental data become available, the prior  $\pi(\theta|v, \mu, \tau)$  can be updated

by replacing  $(v, \mu, \tau)$  by  $(v + n, \mu + m, \tau + \sum_{i=1}^m x_i)$ . It is to be remarked that, sometimes, for decisions for which the Bayesian analysis of the lifetime distribution could be performed, it may not be necessary to specify lifetime distributions beyond a certain fixed time, in which cases the tail of the lifetime distribution is not of any particular interest, see, e.g., age-replacement policies for systems<sup>35</sup> or the analysis of optimal condition-monitoring intervals as described by Coolen & Dekker.<sup>36</sup>

Example 1 shows the advantage of the interpretation of the hyperparameters for elicitation of informative priors in environments with censoring, even when the family of priors does not become larger than the standard conjugate family. In example 2 we consider a similar experiment on units with Gamma distributed lifetimes.

## 4.2 Example 2

Consider a similar experiment as discussed in example 1, but now suppose that the lifetimes have a Gamma distribution with known shape parameter  $\alpha$ , given by the pdf

$$p_X(x|\theta) = \theta(\theta x)^{\alpha-1} \exp\{-\theta x\}/\Gamma(\alpha),$$

with Gamma function

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du,$$

and let us assume that  $\alpha > 1$ . This can be written as eqn (1) with  $g(x) = x^{\alpha-1}/\Gamma(\alpha)$ ,  $h(\theta) = \theta^\alpha$ ,  $t(x) = x$  and  $\psi(\theta) = -\theta$ . The survivor function is

$$S_X(x|\theta) = 1 - (\Gamma_{\theta x}(\alpha)/\Gamma(\alpha)),$$

with incomplete Gamma function

$$\Gamma_y(z) = \int_0^y u^{z-1} e^{-u} du.$$

With right-censoring at time  $c$  the likelihood based on  $m$  failure times  $x_i$ ,  $i = 1, \dots, m$ , and  $n - m$  survivors, is

$$L(\theta|x_1, \dots, x_m, n, m) \propto \theta^{m\alpha} \exp\left\{-\sum_{i=1}^m x_i \theta\right\} [1 - (\Gamma_{\theta c}(\alpha)/\Gamma(\alpha))]^{n-m}.$$

Again, sufficient statistics are  $(n, m, \sum_{i=1}^m x_i)$ , suggesting the possible use of informative priors of the form

$$\pi(\theta|v, \mu, \tau) \propto \theta^{\mu\alpha} \exp\{-\tau\theta\} [1 - (\Gamma_{\theta c}(\alpha)/\Gamma(\alpha))]^{v-\mu}.$$

This family of priors is more general than the standard conjugate family, which occurs for  $v - \mu = 0$ , and therefore it offers more flexibility to model expert opinions, and the hyperparameters may again actually be easier to assess (similar arguments hold as in example 1). Updating is again simply performed by replacing  $(v, \mu, \tau)$  by  $(v + n, \mu + m, \tau + \sum_{i=1}^m x_i)$ , and although calculations are more difficult than for the uncensored case, because of the presence of the Gamma functions, this difficulty can be dealt with

easily by modern mathematical or statistical software,<sup>12</sup> where the necessary procedures for dealing with such functional forms of the likelihood are often already available. If it is most likely that the experimental data set will contain at least one survival of time  $c$ , our suggested more general class of priors does not even introduce any further calculation difficulties, in addition to those resulting from the likelihood.

Example 2 shows that the suggested family of informative priors, for a situation where censored observations are an important part of the experience of consulted engineers, can indeed be more general than the standard conjugate family. In general, if at the time at which the prior distribution has to be elicited (so before the data that are to be used for updating become available) it is most likely that there will be some censoring in the data set, and one knows the impact this will have on the form of the likelihood and the according sufficient statistics, then choosing a prior that is conjugate with respect to that particular likelihood may lead to more hyperparameters, which leads to more flexibility to fit the prior to the expert opinions. Because of the possible interpretation of these hyperparameters, the task to set these at values that represent the expert opinions can actually become simpler because the consulted engineers are not asked to consider possible lifetimes that fall outside their experience.

As a final example of this approach, example 3 discusses a test on Weibull distributed units, in which two different censoring mechanisms are active, right-censoring of type I together with interval censoring.

## 4.3 Example 3

Consider the following experiment. A large number  $n$  of items have independent and identically distributed lifetimes, all having a Weibull distribution with known shape parameter  $\beta > 1$ . This distribution is specified by the pdf

$$p_X(x|\theta) = \beta \theta^\beta x^{\beta-1} \exp\left\{-(\theta x)^\beta\right\},$$

which is a special case of eqn (1) with  $g(x) = \beta x^{\beta-1}$ ,  $h(\theta) = \theta^\beta$ ,  $t(x) = x^\beta$  and  $\psi(\theta) = -\theta^\beta$ . The corresponding survivor function is

$$S_X(x|\theta) = \exp\left\{-(\theta x)^\beta\right\}.$$

The experiment has to start some day at noon, and has to be terminated at noon the following day. In between, for practical reasons, the experimenters will not be able to see what happens between midnight and 6 am. Let us fix the time scale, in hours, as the

interval [0,24] Such an experiment will provide 3 types of data, (1) observed failure times during [0,12] and (18,24], (2) interval censored failure times within (12,18] and (3) survivors of the experiment. Without loss of generality we assume that there are  $l$  observed failure times  $x_1, \dots, x_l$ ,  $m - l$  interval censored failure times and  $n - m$  survivors. Suppose that, before the experiment, everyone involved agrees that it is likely that  $l < m < n$ . The likelihood based on these data is

$$L(\theta|x_1, \dots, x_l, l, m, n) \propto \theta^{l\beta} \exp \left\{ - \sum_{i=1}^l x_i^\beta \theta^\beta - (n - m)(24\theta)^\beta \right\} \times \left[ \exp \left\{ - (12\theta)^\beta \right\} - \exp \left\{ - (18\theta)^\beta \right\} \right]^{m-l}.$$

Sufficient statistics are  $(l, m, n, \sum_{i=1}^l x_i^\beta)$ . As related informative priors we can use

$$\pi(\theta|\lambda, \mu, \nu, \tau) \propto \theta^{\lambda\beta} \exp \left\{ - \tau\theta^\beta - (\nu - \mu)(24\theta)^\beta \right\} \times [\exp \{ - (12\theta)^\beta \} - \exp \{ - (18\theta)^\beta \}]^{\mu-\lambda}.$$

This only reduces to the standard family of priors if  $\mu - \lambda = 0$ , else it is a larger family. Values for the four hyperparameters can be set by letting the experts consider an imaginary experiment of exactly the same form before the actual experiment takes place. The choice of the imaginary sample size  $\nu$  can be varied, with regard to  $n$ , to change the weight of the expert opinions related to the real data that will become available. Elicitation of such hyperparameters may be of special interest if the experience leading to the prior judgements is based on earlier experiments of the same form, where it was also impossible to see failure times during the interval (12,18]. As before, the experts also don't have to think about failure times beyond  $t = 24$ . This makes it clear that the consulted experimenters do not have to think about possible lifetimes in intervals that they have never been able to observe explicitly. To assess a useful value for  $\tau$  the experts could actually provide  $\lambda$  (assuming that  $\lambda \in \mathbb{N}$ , which is not strictly necessary) likely imaginary failure times within [0,12] and (18,24], say  $y_1, \dots, y_\lambda$ , and then take

$$\tau = \sum_{i=1}^{\lambda} y_i^\beta.$$

After the experiment has been performed, the posterior is derived by replacing  $(\lambda, \mu, \nu, \tau)$  by

$$\left( \lambda + l, \mu + m, \nu + n, \tau + \sum_{i=1}^l x_i^\beta \right).$$

As is clear from the above examples, such forms of informative priors will be most useful when one

actually wants to take expert knowledge into consideration as a major source of information in decision processes, where, given the assumed parametric model for the lifetime of interest, the posterior distribution for  $\theta$  reflects as well the information from the experts as the experimental information. The posterior distribution derived in this way can be used in the well-known Bayesian framework for decision making.<sup>24,25</sup>

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