GR 12 MATHS

DIFFERENTIAL CALCULUS

QUESTIONS and ANSWERS

DIFFERENTIAL CALCULUS

QUESTIONS

FINDING THE LIMIT

1. Determine $\lim_{x \to 4} (x + 4)$ (2)

2. Consider the function $f(x) = \dfrac{x^2 - 16}{x - 4}$

   2.1 Is there a value for $f(4)$? (2)

   2.2 Simplify the fraction, $f(x)$, and state the restriction on this expression for $f(x)$. (3)

   2.3 Now determine $\lim_{x \to 4} \dfrac{x^2 - 16}{x - 4}$ (2)

   2.4 On separate systems of axes, draw sketches of

   (a) $g(x) = x + 4$ (b) $f(x) = \dfrac{x^2 - 16}{x - 4}$ (4)

   2.5 Write down the domain of these functions. (2)

3. Calculate

   3.1 $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5}$ 3.2 $\lim_{x \to -1} \dfrac{x^2 - 4x - 5}{x + 1}$ (3)(3)

   3.3 $\lim_{x \to 0} \dfrac{x^2 - 2x}{x}$ (3)

FINDING THE DERIVATIVE

from first principles:

4. If $f(x) = x^2$, determine

   4.1 $f(x + h)$ 4.2 $f(x + h) - f(x)$

   4.3 $\dfrac{f(x + h) - f(x)}{h}$ 4.4 $\lim_{x \to 0} \dfrac{f(x + h) - f(x)}{h}$

   4.5 $f'(x)$, the gradient of a tangent to the function

   4.6 $f'(3)$, the gradient of the tangent to the function at $x = 3$

   4.7 the gradient of the tangent to f when $x = 3$ (7)

5.1 Given the function $f(x) = 3x^2$

   5.1.1 Determine $\dfrac{f(x + h) - f(x)}{h}$ (3)

   5.1.2 Hence determine

   (a) $f'(x)$ (b) $f'(-1)$ (4)

5.2.1 Use first principles to determine the derivative of f if

(a) $f(x) = x^2 + 5$ (b) $f(x) = x^3$ (c) $f(x) = -2x^2$

(d) $f(x) = -5x$ (e) $f(x) = x^2 + 3x + 2$ (f) $f(x) = \dfrac{3}{x}$ (6×4)

5.2.2 Determine in each case in 5.2.1 the derivative of $f(x)$ at the point where $x = -1$. (6×2)

5.3 If $g(x) = -2x^3$:

   5.3.1 determine the derivative, $g'(x)$, from first principles. (6)

   5.3.2 calculate the value of $g'(-2)$. (2)

   5.3.3 find the coordinates of the point(s) on the curve of g where the gradient of the tangent is equal to -6. (4)

   5.3.4 determine the average gradient of the curve of g, between the points $(1; -2)$ and $(2; -16)$. (3)

5.4 Given the functions defined by the following equations:

   (a) $y = 3x$ (b) $y = 5$ (c) $y = -2x$

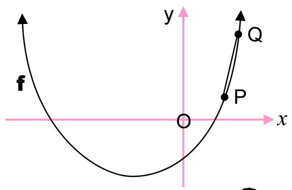
   5.4.1 Write down the gradients of each of these functions. (3)

   5.4.2 Use first principles to find the derivative of each of these functions. (6)

   5.4.3 Compare your answers to 5.4.1 and 5.4.2. Can you explain what you find? (2)

5.5 Given: $f(x) = 1 - 2x^2$. Find $f'(x)$ from first principles. (5)

5.6 The diagram alongside shows the graph of $y = f(x)$.



$P(x; f(x))$ & $Q(x + h; f(x + h))$ are points on the graph.

The gradient of the straight line through P and Q is given by

$m = \dfrac{f(x + h) - f(x)}{(x + h) - x}$.

   5.6.1 What line has a gradient given by $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$? (2)

   5.6.2 Calculate the gradient of PQ in terms of h and x if $f(x) = x^2 + 2x$. (4)

   5.6.3 Hence, determine the value of $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$. (2)

   5.6.4 Hence, find the value of

   (a) $f'(x)$ (b) $f'(2)$ (c) $\lim_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$ (3)

FINDING THE DERIVATIVE

using the rules:

6. Determine $f'(x)$

   6.1 $f(x) = x^2$ 6.2 $f(x) = x^3$ 6.3 $f(x) = x^4$

   6.4 $f(x) = x^n$ 6.5 $f(x) = x$ 6.6 $f(x) = 5$

   6.7 $f(x) = -3x$ 6.8 $f(x) = 3x^2$ 6.9 $f(x) = x^2 + x^3$

   6.10 $f(x) = 5x^2 - x - 3$ 6.11 $f(x) = \dfrac{1}{x}$ 6.12 $f(x) = \sqrt{x}$ (24)

7. Determine $\dfrac{dy}{dx}$ for the following:

   7.1 (a) $y = 3x$ (b) $y = 5$ (c) $y = -2x$

   Compare these answers to 5.4.1 and 5.4.2. Explain. (4)

   7.2 $y = 5 - x$ 7.3 $y = \dfrac{2}{x}$ 7.4 $y = \sqrt{x} + x^2$ (1)(2)(2)

   7.5 $y = (x + 1)^2$ 7.6 $y = (x + 1)(x - 1)$ (3)(2)

   7.7 $y = x + 1$ 7.8 $y = x - 1$ (1)(1)

   Write down your observation re 7.6 → 7.8

   7.9 $y = 2x.\sqrt[3]{8x^6}$ 7.10 $y = \dfrac{5x^2 - 4x}{x}$ (3)(3)

   7.11 $y = 3x^2 + 6x + \dfrac{2}{x^2}$ 7.12 $y = \left(x + \dfrac{2}{x}\right)\left(x - \dfrac{2}{x}\right)$ (3)(3)

   7.13 $y = \dfrac{x^2 - 3}{x}$ 7.14 $y = 3ax + a^2$ (3)(2)

   7.15 $x = 2y + 4x^2$ 7.16 $y = (x + 2)^3$ (3)(4)

8. Determine

   8.1 $\dfrac{d}{dx}\left(\dfrac{1}{3}x^3 + 2\sqrt{x}\right)$ 8.2 $D_x\,[x(x + 6)]$ (3)(3)

   8.3 $f'(2)$ if $f(x) = x^3 + x^{-2}$ 8.4 $\dfrac{ds}{dt}$ if $s = ut + \dfrac{1}{2}at^2$ (3)(2)

   8.5 the gradient of the tangent to the curve defined by $y = x^2 + 6x - 7$ at the point where $x = 2$. (2)

   8.6 $\dfrac{dy}{dt}$ if $y = \dfrac{t^2 - 1}{2t + 2}$ (4)

FINDING THE AVERAGE GRADIENT

9.1 The average gradient of the curve of $y = x^2$ between the point $x = 1$ and $x = 5$ is:

A 6 B 3 C 24 D 4 (2)

9.2 Find the average gradient of the curve $y = 2x^2 - 2$ between the points with x-values $x = 1$ and $x = 3$. (4)

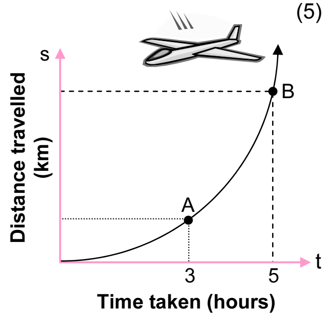
9.3 Given: $g(x) = 2x^2$

Determine the average gradient of g between the points with $x = -3$ and $x = 5$. (5)

9.4 I'm testing my new glider and find that the distance(s) covered in km can be expressed in terms of the time (t) taken in hours by the equation

$s = 4t^2$.

The graph alongside shows this relationship.



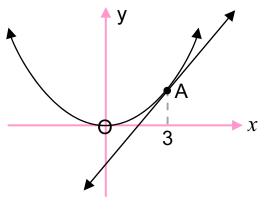
   9.4.1 What are the coordinates of A and B? (4)

9.4.2 What is the average speed from $t = 3$ to $t = 5$, i.e. from the 3rd to the 5th hour? (3)

TANGENTS

Finding the gradient/equation

10.1 Alongside is the graph $y = x^2$ and the tangent to this graph at $x = 3$.



10.1.1 Write down the coordinates of A. (2)

10.1.2 Find

(a) the **gradient** of the tangent at A. (2)

(b) the **equation** of the tangent at A. (2)

10.1.3 Find the coordinates of the point on this graph where the gradient of the tangent is:

(a) -6 (b) 10 (2)(2)

10.2 Given: $f(x) = 2x^2 - 6x$. Calculate

10.2.1 the average gradient between the points with $x = 2$ and $x = 5$. (4)

10.2.2 the gradient of the tangent to the curve where $x = 3$. (3)

10.3 Given: $f(x) = x^3 - 3x^2 + 4$
Calculate the gradient of the tangent to the curve f at the point (3; 4) and use it to find the equation of this tangent. (5)

10.4 If $g(x) = 3x^2 - 2x + 5$

Determine:

10.4.1 the equation of the tangent to the curve of g at the point (-2; 21). (4)

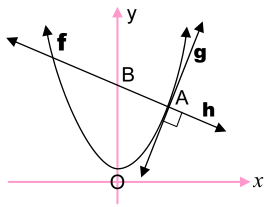
10.4.2 the x-coordinate of the point on the curve of g where the gradient of the tangent to the curve is equal to $\frac{1}{2}$. (2)

10.5 Given: $g(x) = x^2 + 3x - 4$
Determine the coordinates of the point on the curve of g where the gradient of the tangent is -3. (5)

10.6 Determine the value of p if the line given by $y = -2x + p$ is a tangent to the function given by $y = -x^2 + 4x + 5$. (6)

10.7 Calculate the equation of the tangent to the curve $y = -2x^3 + 3x^2 + 32x + 15$ at the point (-2; -21). (6)

10.8 The graphs of the parabola $f(x) = 3x^2 + 2$, and the straight lines g and h are represented in the sketch.



g is a tangent to **f** at the point A and **h** is perpendicular to **g** at A.

The coordinates of A are (1; 5).

10.8.1 Determine the equation of the tangent **g**. (6)

10.8.2 Determine the length of OB, if B is the intercept of **h** on the y-axis. (3)

10.9.1 Find the gradient of the straight line through the points where $x = 2$ and $x = 2 + h$ on the curve $f(x) = -x^2 + 1$. (4)

10.9.2 Hence find: (a) the gradient and (b) the equation of the tangent to the curve where $x = 2$. (6)

CURVESKETCHING

Be sure to have done the section on 3rd degree Polynomials!

At the turning points, the gradient of the curve is zero, i.e. at the turning points, the derivative, $f'(x) = 0$.

11.1 Given: $f(x) = x^3 - 12x + 16$

11.1.1 Show that $x - 2$ is a factor of $f(x)$. (2)

11.1.2 Factorise $f(x)$ completely. (2)

11.1.3 Determine the x- and y-intercepts of the graph of f. (2)

11.1.4 Sketch the graph of f, clearly labelling all the intercepts on the axes, the coordinates of the turning points and the coordinates of the point of inflection. (6)

At the point of inflection, the 2nd derivative, $f''(x) = 0$.

11.2 Consider the function f defined by $f(x) = x^3 - 6x^2$.

11.2.1 Find the intercepts of the graph of f with the axes. (4)

11.2.2 Write down the coordinates of the turning points of the graph of f. (5)

11.2.3 Draw a neat sketch of the graph of f. (3)

11.3 Given: $f(x) = x^3 - 5x^2 + 7x - 3 = (x - 1)^2(x - 3)$
Draw a neat sketch graph of f. Show all the intercepts on the axes as well as the coordinates of the turning points and the point of inflection on your graph. (14)

11.4 Given: $f(x) = x^3 - 4x^2 - 3x + 18$

11.4.1 Determine the value of f(3). (2)

11.4.2 Hence or otherwise determine the values of the x-intercepts of the graph of $f(x)$. (5)

11.4.3 Determine the y-intercept. (1)

11.4.4 Determine $f'(x)$ and $f''(x)$. (3)

11.4.5 Determine the coordinates of the turning points and the point of inflection. (6)

11.4.6 Sketch the graph of $f(x)$ showing clearly the turning points, the point of inflection and the intercepts on the axes. (6)

11.5 Given: $f(x) = -x^3 + 5x^2 + 8x - 12$

11.5.1 Use the factor theorem to show that $x + 2$ is a factor of f. Hence solve $f(x) = 0$. (6)

11.5.2 Determine the stationary points for the graph of f. (5)

11.5.3 Draw a neat sketch graph of f. Clearly show all intercepts and stationary points. (3)

11.5.4 Determine the equation of the tangent to the graph of f at $x = -2$. (4)

11.6 Now consider $g(x) = x^3 - 5x^2 - 8x + 12$.

11.6.1 Compare $g(x)$ to $f(x)$ in 11.5.
Under what transformation is the graph of g the image of the graph of f? (2)

Write down

11.6.2 the solution to the equation $g(x) = 0$. (7)

11.6.3 the coordinates of the turning points of the curve of g. (7)

And now

11.6.4 sketch the graph of g. Indicate clearly the coordinates of the intercepts with the axes, the turning points and the point of inflection. (6)

11.7 Given: $f(x) = -2x^3 + kx^2 + 4x - 3$

11.7.1 Given that $2x - 1$ is a factor of $f(x)$, show that $k = 5$. (3)

11.7.2 Hence solve the equation $-2x^3 + 5x^2 + 4x - 3 = 0$. (5)

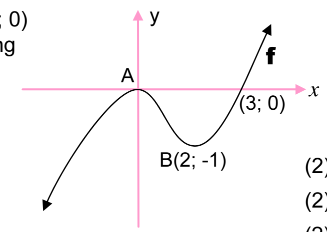
11.7.3 Calculate the coordinates of the turning point(s) of the graph defined by $f(x) = -2x^3 + 5x^2 + 4x - 3$. (6)

11.8 Given: $x^3 - x^2 - x + 10 = 0$

11.8.1 Solve the equation for x. (5)

11.8.2 Hence draw the graph of the function given by $f(x) = x^3 - x^2 - x + 10$. Clearly indicate the coordinates of the intercepts on the axes and the turning points on your graph. (7)

11.9 In the adjoining figure A(0; 0) and B(2; -1) are the turning points of the graph of f.



Determine the values of x in each of the following:

11.9.1 $f(x) = 0$ (2)

11.9.2 $f(x) < 0$ (2)

11.9.3 $f'(x) \leq 0$ (2)

11.9.4 $x.f'(x) \geq 0$ (3)

12.1.1 Use the information below to draw a neat sketch graph of the function $f(x) = ax^3 + bx^2 + cx + d$ for $x\,;\, y \in \mathbb{R}$.

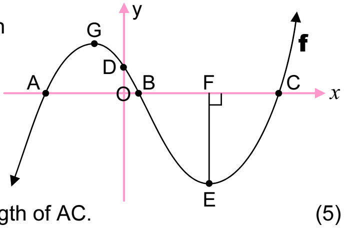
Indicate clearly intercepts with the axes as well as the coordinates of the turning points on your sketch.
(No calculations are necessary.)

$f(0) = 3$ $f(-3) = 0$ $f'(-2) = f'(1) = 0$
$f(-2) = 5$ $f(1) = 1$ (6)

12.1.2 Use the graph to determine for which values of x will $x\,.\,f(x) < 0$. (2)

12.1.3 If this graph is reflected in the y-axis, write down the coordinates of the local minimum turning point. (2)

13.1 The accompanying graph represents the function: $f(x) = x^3 - 4x^2 - 11x + 30$ with G and E the turning points and A, B, C and D the intercepts on the axes.



13.1.1 Calculate the length of AC. (5)

13.1.2 Calculate the length of EF. (6)

13.1.3 Determine the gradient of the curve at B. (3)

13.1.4 A line with gradient equal to -11 is a tangent to the curve of f. Determine the x-coordinates of the possible points of contact. (4)

13.1.5 Determine the values of x if $-7.f'(x) > 0$. (2)

13.2 A function f is defined by $y = px^3 + 5x^2 - qx - 3$.
The graph of f has a turning point at (-2; 9).
Calculate the values of p and q. (6)

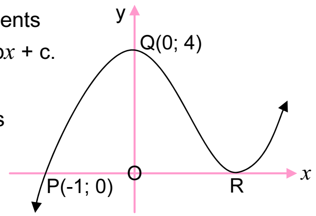
13.3.1 Find the coordinates of the turning points in terms of c, of the curve defined by $f(x) = 2x^3 - 24x + c$.
State with respect to each point, whether the point is a local minimum or a local maximum value. (8)

13.3.2 For what values of c does $f(x) = 0$ have three real roots? (3)

13.4 The figure alongside represents the curve of $y = x^3 + ax^2 + bx + c$.

Q(0; 4) is a turning point and a local minimum occurs at R.

P is the point (-1; 0).

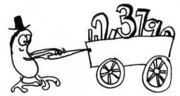


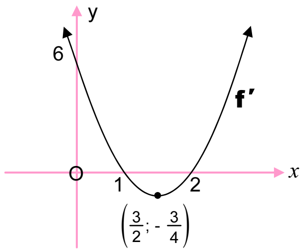
13.4.1 Calculate the values of a, b and c. (7)

13.4.2 Determine the coordinates of R. (3)

13.4.3 Use the sketch to write down the value(s) of d for which the equation $x^3 + ax^2 + bx + c = d$ will always have three real roots. (2)

13.5 The graph of the parabola $y = f'(x)$ is shown.





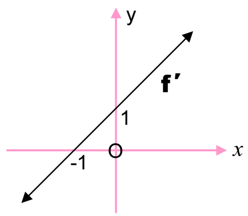
13.5.1 Write down the x-coordinate of the local minimum of $y = f(x)$. (2)

13.5.2 For which value of x will $f(x)$ be decreasing? (2)

13.5.3 What is the gradient of the tangent to f when $x = 0$? (2)

13.5.4 At which value of x will there be a tangent to f parallel to the one in 13.5.3? (2)

13.6 The graph of $y = f'(x)$ is shown.



13.6.1 Explain why $f(x)$ is a quadratic function (having the form $ax^2 + bx + c$). (2)

13.6.2 What is the value of $f'(-1)$? (1)

13.6.3 Use the graph to solve the inequalities:
(a) $f'(x) < 0$ (b) $f'(x) > 0$ (1)(1)

13.6.4 (a) For which values of x is $f(x)$ decreasing? (1)
(b) For which values of x is $f(x)$ increasing? (1)

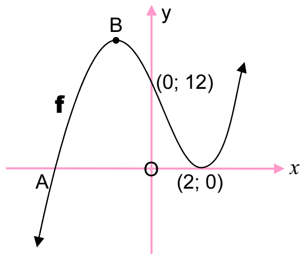
13.6.5 Does $f(x)$ have a maximum or minimum turning point? Justify your answer. (2)

13.6.6 Write down the equation of the axis of symmetry of $f(x)$. (2)

13.7 The sketch graph alongside shows the curve of $f(x) = x^3 - x^2 - 8x + 12$.

The curve has a y-intercept at (0; 12) and turning points at (2; 0) and B.

The point A is an x-intercept.



13.7.1 Calculate the coordinates of A. (5)

13.7.2 Calculate the x-coordinate of B. (4)

13.7.3 Write the values of x for which $f'(x) > 0$. (3)

13.7.4 If $k < 0$, how many real roots will the equation $x^3 - x^2 - 8x + 12 = k$ have? (2)

13.7.5 Calculate the coordinates of the point of inflection. (4)

A NOTE ABOUT THE POINT OF INFLECTION

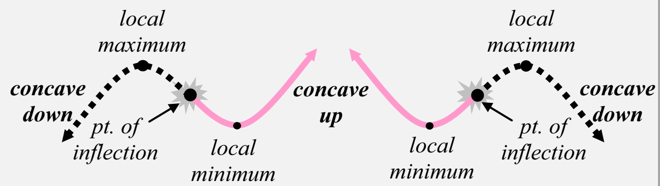
The second derivative is the derivative of the first derivative.

The **second derivative** is **zero** at the point of inflection, the point at which the **concavity** of a curve changes.

CONCAVITY

At the point of inflection the concavity of a curve changes from 'up' to 'down' or vice versa.

+ cubic **- cubic**



f″(x) = 0 at the point of inflection

f″(x) < 0 where the curve is 'concave down'
. . . the turning pt. will be a local maximum

f″(x) > 0 where the curve is 'concave up'
. . . therefore the turning pt. will be a local minimum

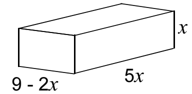
A Useful Tip

The x-coordinate of the point of inflection is the midpoint between the x-coordinates of the turning points.
This fact provides an alternative method!

PRACTICAL APPLICATIONS MAXIMUM & MINIMUM

14.1 Refer to the figure.
A rectangular box has the following dimensions:

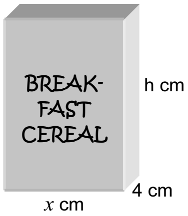
Length $5x$ units
Breadth $(9 - 2x)$ units
Height x units



14.1.1 Find the volume of the box in terms of x. (2)

14.1.2 Find the value of x for which the box will have a maximum volume. (4)

14.2 A cereal box has the shape of a rectangular prism as shown in the diagram below.



The box has a volume of 480 cm^3, a breadth of 4 cm and a length of x cm.

14.2.1 Show that the total surface area of the box (in cm^2) is given by:

$A = 8x + 960x^{-1} + 240$ (5)

14.2.2 Determine the value of x for which the total surface area is a minimum.

Round the answer off to the nearest cm. Hence find the dimensions of this "ideal" box. (5)

14.3 A community was given 1 200 metres of fencing to enclose an area of land that would be used as a play park. To use the available space efficiently, the park had to be rectangular in shape.

14.3.1 Suppose the length of the park is x metres, prove that the area, A, of the park can be represented by the following formula:

$A = 600x - x^2$ (2)

14.3.2 Hence determine the dimensions of the park which would ensure a maximum enclosed area. (4)

14.3.3 Now calculate the maximum area of the park. (1)

14.4 A factory has x employees and makes a profit of P rand per week. The relation between the profit and number of employees is expressed in the formula

$$P = -2x^3 + 600x + 1\,000.$$

Calculate:

14.4.1 the number of employees, x, for the factory to make a maximum profit. (4)

14.4.2 the maximum profit. (2)

14.5 The height to which a plant grows during the first six months is given by the following function:

$f(x) = 36x - 3x^2;\ 0 \leq x \leq 6$

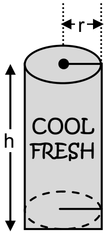
where x is the age of the plant in months and $f(x)$ the height in centimetres above the ground after x months.

14.5.1 What height would the plant reach at the end of 3 months? (2)

14.5.2 At the end of how many months will the plant reach its maximum height? (3)

14.5.3 Hence calculate the maximum height to which the plant will grow. (2)

14.6 A tin of Cool Fresh has a capacity of 450 mℓ.



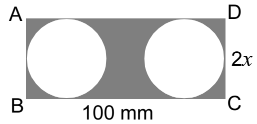
14.6.1 Express height, h, of the tin Cool Fresh in terms of radius, r, of the tin. (2)

14.6.2 Prove that the total surface area in terms of the radius is

$A = 2\pi r^2 + \dfrac{900}{r}$ (2)

14.6.3 Determine the radius such that the total surface area will be a minimum. (4)

14.7 Two circles with diameter $2x$ each are cut out of the rectangle ABCD as shown in the accompanying figure. BC = 100 mm and CD = $2x$ mm. $\left(\pi = \dfrac{22}{7}\right)$

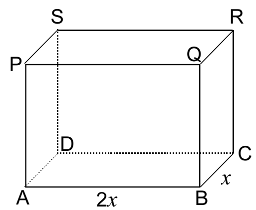


14.7.1 Show that the area of the shaded region is given by $A(x) = 200x - \dfrac{44}{7}x^2$. (2)

14.7.2 For which value of x will the shaded area be a maximum? (3)

14.7.3 Now calculate the maximum area. (1)

14.8 The sketch shows a rectangular box whose base ABCD has AB = $2x$ metres and BC = x metres.



The volume of the box is 24 cubic metres. Material to cover the top PQRS costs R25 per square metre. Material to cover the base ABCD and the four vertical sides costs R20 per square metre.

14.8.1 Show that:

(a) the height (h) of the box is given by $h = 12x^{-2}$. (2)

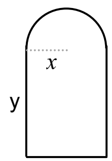
(b) the total cost (C) in rands is given by $C = 90x^2 + 1\,440x^{-1}$. (3)

14.8.2 Hence determine:

(a) the dimensions that will make the cost a minimum. (5)

(b) the minimum cost. (2)

14.9 The figure represents a hallway mirror. The semicircle has a radius of x metres and the rectangle has a length of y metres. The total area is 8 m^2.



14.9.1 Find expressions for:

(a) the total area (A) of the mirror in terms of π, x and y. (2)

(b) the perimeter (P) of the mirror in terms of π, x and y. (2)

14.9.2 If $P = \dfrac{8}{x} + \left(\dfrac{\pi}{2} + 2\right)x$, calculate to two decimal digits the value of x that will make the perimeter (P) a minimum. (5)

15. A stone is thrown upwards from the roof of a building, 35 metres above ground level. It moves according to the formula, $s = 35 + 30t - 5t^2$, where s is its height in metres above the ground after t seconds. Determine:

Hint: Draw a sketch of s vs t.

15.1 its height above the ground after 2 seconds. (2)

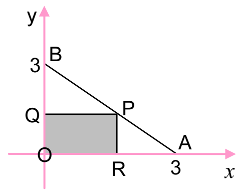
15.2 its speed after 2 seconds. (2)

15.3 the time taken to reach its maximum height. (2)

15.4 the maximum height above the roof of the building. (2)

15.5 the time taken to reach the ground. (2)

16.1 Rectangle PQOR is inscribed in ΔAOB where O is the origin. The coordinates of A, B and P are (3; 0), (0; 3) and (x; y) respectively.



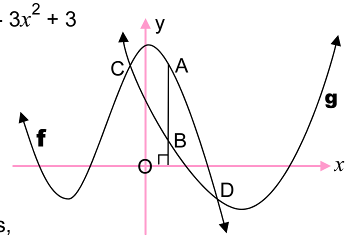
Show that:

16.1.1 $y = -x + 3$ (3)

16.1.2 the area of the rectangle PQOR is $-x^2 + 3x$ (2)

16.1.3 the maximum area of rectangle PQOR is half the area of ΔAOB (2)

16.2 The graphs of $f(x) = -x^3 - 3x^2 + 3$ and $g(x) = x^2 - 6x + 2$ are shown alongside.



A is any point on the graph of f between the points of intersection of f and g.

AB is parallel to the y-axis, with B on the graph of g.

16.2.1 Determine AB in terms of x. (1)

16.2.2 Find the maximum value of AB if it is known that the x-coordinate of A is greater than -1. (7)

16.3 In order to reduce the temperature in a room from 28ºC, a cooling system is allowed to operate for 10 minutes. The room temperature, T after t minutes is given in ºC by the formula:

$$T = 28 - 0{,}008t^3 - 0{,}16t \quad \text{where } t \in [0; 10]$$

16.3.1 At what rate (rounded off to TWO decimal places) is the temperature falling when t = 4 minutes? (4)

16.3.2 Calculate the lowest room temperature reached during the 10 minutes for which the cooling system operates. (4)

DIFFERENTIAL CALCULUS

ANSWERS

FINDING THE LIMIT

1. $\lim\limits_{x \to 4} (x + 4) = 4 + 4 = \mathbf{8}$ ◄

2.1 No; f(4) has a zero in the denominator ◄

2.2 $f(x) = \dfrac{(x+4)(x-4)}{x-4} = x + 4$, **IF** $x \neq 4$ ◄

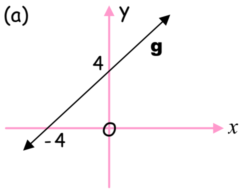
2.3 $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4}$

$= \lim\limits_{x \to 4} (x + 4)$

$= 4 + 4$

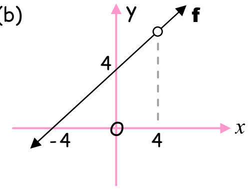
$= \mathbf{8}$ ◄

2.4 (a)



$g(x) = x + 4$

(b)



$f(x) = \dfrac{x^2 - 16}{x - 4} = x + 4;\ x \neq 4$

2.5 (a) domain of g: $x \in \mathbb{R}$ ◄

(b) domain of f: $x \neq 4$; $x \in \mathbb{R}$ ◄

3.1 $\lim\limits_{x \to 5} \dfrac{(x+5)(x-5)}{x-5}$

$= \lim\limits_{x \to 5} (x + 5)$

$= 5 + 5$

$= \mathbf{10}$ ◄

3.2 $\lim\limits_{x \to -1} \dfrac{(x-5)(x+1)}{x+1}$

$= \lim\limits_{x \to -1} (x - 5)$

$= -1 - 5$

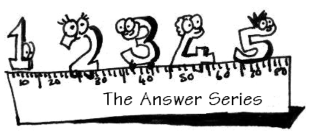
$= \mathbf{-6}$ ◄

3.3 $\lim\limits_{x \to 0} \dfrac{x(x-2)}{x}$

$= \lim\limits_{x \to 0} (x - 2)$

$= 0 - 2$

$= \mathbf{-2}$ ◄



FINDING THE DERIVATIVE from first principles:

Expansion of binomials	Pascal's Δ
$(a+b)^0 = \mathbf{1}$	1
$(a+b)^1 = \mathbf{1}a + \mathbf{1}b$	1 1
$(a+b)^2 = \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2$	1 2 1
$(a+b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3$	1 3 3 1

4.1 $f(x+h) = (x+h)^2 = \mathbf{x^2 + 2xh + h^2}$ ◄

4.2 $f(x+h) - f(x) = (x^2 + 2xh + h^2) - x^2$

$= \mathbf{2xh + h^2}$ ◄

4.3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2}{h}$

$= \mathbf{2x + h}$ ◄

4.4 $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} (2x + h)$

$= \mathbf{2x}$ ◄

4.5 $f'(x) = 2x$ ◄ . . . *what we found in 4.4!*

4.6 $f'(3) = 2(3) = \mathbf{6}$ ◄

4.7 Gradient of tangent at $x = 3$ **is** $f'(3) = 6$

5.1.1 $f(x) = 3x^2$

$\therefore f(x+h) = 3(x+h)^2$

$= 3(x^2 + 2xh + h^2)$ . . . *Pascal's Δ*

$= 3x^2 + 6xh + 3h^2$

$\therefore f(x+h) - f(x) = 6xh + 3h^2$

$\therefore \dfrac{f(x+h) - f(x)}{h} = \dfrac{6xh + 3h^2}{h}$

$= \mathbf{6x + 3h}$ ◄

5.1.2 (a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (6x + 3h)$

definition of a derivative

$= \mathbf{6x}$ ◄

(b) $\mathbf{f'(-1)} = 6(-1) = \mathbf{-6}$ ◄

5.2.1 (a) $f(x) = x^2 + 5$

$\therefore f(x+h) = (x+h)^2 + 5$

$= x^2 + 2xh + h^2 + 5$

$\therefore f(x+h) - f(x) = 2xh + h^2$

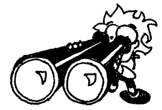
$\therefore \dfrac{f(x+h) - f(x)}{h} = \dfrac{2xh + h^2}{h}$

$= 2x + h$

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (2x + h)$

definition of a derivative

$= \mathbf{2x}$ ◄



(b) $f(x) = x^3$

$\therefore f(x+h) = (x+h)^3$

$= x^3 + 3x^2h + 3xh^2 + h^3$ . . . *Pascal's Δ*

$\therefore f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$

$\therefore \dfrac{f(x+h) - f(x)}{h} = \dfrac{3x^2h + 3xh^2 + h^3}{h}$

$= \mathbf{3x^2 + 3xh + h^2}$

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (3x^2 + 3xh + h^2)$

definition of a derivative

$= \mathbf{3x^2}$ ◄

(c) $f(x) = -2x^2$

$\therefore f(x+h) = -2(x+h)^2$

$= -2(x^2 + 2xh + h^2)$

$= -2x^2 - 4xh - 2h^2$

$\therefore f(x+h) - f(x) = -4xh - 2h^2$

$\therefore \dfrac{f(x+h) - f(x)}{h} = \dfrac{-4xh - 2h^2}{h}$

$= \mathbf{-4x - 2h}$

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (-4x - 2h)$

definition of a derivative

$= \mathbf{-4x}$ ◄

(d) $f(x) = -5x$

$\therefore f(x+h) = -5(x+h)$

$= -5x - 5h$

$\therefore f(x+h) - f(x) = -5h$

$\therefore \dfrac{f(x+h) - f(x)}{h} = \mathbf{-5}$

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (-5)$

definition of a derivative

$= \mathbf{-5}$ ◄

(e)

$$
\begin{aligned}
f(x) &= x^2 + 3x + 2\\
\therefore\ f(x+h) &= (x+h)^2 + 3(x+h) + 2\\
&= x^2 + 2xh + h^2 + 3x + 3h + 2\\
\therefore\ f(x+h) - f(x) &= 2xh + h^2 + 3h\\
\therefore\ \frac{f(x+h)-f(x)}{h} &= \mathbf{2x + h + 3}
\end{aligned}
$$

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}(2x + h + 3)$$

definition of a derivative $\quad = \mathbf{2x + 3}$ ◄

(f)

$$
\begin{aligned}
f(x) &= \frac{3}{x}\\
\therefore\ f(x+h) &= \frac{3}{x+h}\\
\therefore\ f(x+h) - f(x) &= \frac{3}{x+h} - \frac{3}{x}\\
&= \frac{3x - 3(x+h)}{x(x+h)}\\
&= \frac{-3h}{x^2 + xh}\\
\therefore\ \frac{f(x+h)-f(x)}{h} &= \mathbf{\frac{-3}{x^2 + xh}}
\end{aligned}
$$

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}\left(-\frac{3}{x^2 + xh}\right)$$

definition of a derivative $\quad = \mathbf{-\frac{3}{x^2}}$ ◄

5.2.2 (a) $f'(-1) = 2(-1) = -2$ (b) $f'(-1) = 3(-1)^2 = 3$

(c) $f'(-1) = -4(-1) = 4$ (d) $f'(-1) = -5$

(e) $f'(-1) = 2(-1) + 3 = 1$ (f) $f'(-1) = -\frac{3}{(-1)^2} = -3$

5.3.1

$$
\begin{aligned}
g(x) &= -2x^3\\
\therefore\ g(x+h) &= -2(x+h)^3\\
&= -2(x^3 + 3x^2h + 3xh^2 + h^3) \quad \ldots\ \textit{Pascal's } \Delta\\
&= -2x^3 - 6x^2h - 6xh^2 - 2h^3\\
\therefore\ g(x+h) - g(x) &= -6x^2h - 6xh^2 - 2h^3\\
\therefore\ \frac{g(x+h)-g(x)}{h} &= \mathbf{-6x^2 - 6xh - 2h^2}
\end{aligned}
$$

$$\boxed{g'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}(-6x^2 - 6xh - 2h^2)$$

definition of a derivative $\quad = \mathbf{-6x^2}$ ◄

OR:

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\\
&= \lim_{h\to 0}\frac{-2(x+h)^3 - (-2x^3)}{h}\\
&= \lim_{h\to 0}\frac{-2(x^3 + 3x^2h + 3xh^2 + h^3) + 2x^3}{h}\\
&= \lim_{h\to 0}\frac{-2x^3 - 6x^2h - 6xh^2 - 2h^3 + 2x^3}{h}\\
&= \lim_{h\to 0}\frac{-6x^2h - 6xh^2 - 2h^3}{h}\\
&= \lim_{h\to 0}(-6x^2 - 6xh - 2h^2)\\
&= \mathbf{-6x^2}
\end{aligned}
$$

◄

5.3.2 $\therefore\ g'(-2) = -6(-2)^2 = -6(4) = \mathbf{-24}$ ◄

5.3.3 $g'(x) = -6$ ➡ $-6x^2 = -6$

$\therefore\ x^2 = 1$

$\therefore\ x = \pm 1$

$g(1) = -2(1)^3 = -2$

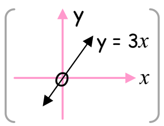
& $g(-1) = -2(-1)^3 = -2(-1) = 2$

$\therefore$ **The points (1; -2) & (-1; 2)** ◄

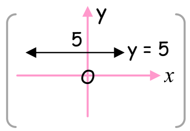
5.3.4 Average gradient $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{-16 - (-2)}{2 - 1}$

$= \mathbf{-14}$ ◄ . . . i.e. $\frac{g(2) - g(1)}{2 - 1}$

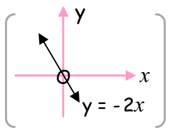
5.4.1 (a) grad. = 3



(b) grad. = 0



(c) grad. = -2



5.4.2 (a)

$$
\begin{aligned}
f(x) &= 3x\\
\therefore\ f(x+h) &= 3(x+h)\\
&= 3x + 3h\\
\therefore\ f(x+h) - f(x) &= 3h\\
\therefore\ \frac{f(x+h)-f(x)}{h} &= 3
\end{aligned}
$$

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}(3)$$

definition of a derivative $\quad = \mathbf{3}$ ◄

(b)

$$
\begin{aligned}
f(x) &= 5\\
\therefore\ f(x+h) &= 5\\
\therefore\ f(x+h) - f(x) &= 0\\
\therefore\ \frac{f(x+h)-f(x)}{h} &= 0
\end{aligned}
$$

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}(0)$$

definition of a derivative $\quad = \mathbf{0}$ ◄

(c)

$$
\begin{aligned}
f(x) &= -2x\\
\therefore\ f(x+h) &= -2(x+h)\\
&= -2x - 2h\\
\therefore\ f(x+h) - f(x) &= -2h\\
\therefore\ \frac{f(x+h)-f(x)}{h} &= \mathbf{-2}
\end{aligned}
$$

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}(-2)$$

definition of a derivative $\quad = \mathbf{-2}$ ◄

5.4.3 They're the same;

because the gradients of the lines **ARE** the derivatives!

GRADIENT IS THE DERIVATIVE IS THE GRADIENT IS . . .

5.5

$$
\begin{aligned}
f(x) &= 1 - 2x^2\\
\therefore\ f(x+h) &= 1 - 2(x+h)^2\\
&= 1 - 2(x^2 + 2xh + h^2)\\
&= 1 - 2x^2 - 4xh - 2h^2\\
\therefore\ f(x+h) - f(x) &= -4xh - 2h^2\\
\therefore\ \frac{f(x+h)-f(x)}{h} &= \mathbf{-4x - 2h}
\end{aligned}
$$

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}(-4x - 2h)$$

definition of a derivative $\quad = \mathbf{-4x}$ ◄

5.6.1 **The tangent to f at P** ◄

5.6.2

$$
\begin{aligned}
f(x) &= x^2 + 2x\\
\therefore\ f(x+h) &= (x+h)^2 + 2(x+h)\\
&= x^2 + 2xh + h^2 + 2x + 2h\\
\therefore\ f(x+h) - f(x) &= 2xh + h^2 + 2h\\
\therefore\ \frac{f(x+h)-f(x)}{h} &= \mathbf{2x + h + 2}
\end{aligned}
$$

$\therefore$ **Gradient of PQ = 2x + h + 2** ◄

average GRADIENT of CURVE PQ

5.6.3 Now, $f'(x) = \lim_{h \to 0} (2x + h + 2)$ . . . THE GRADIENT of P - *the derivative!*

$= \mathbf{2x + 2}$ ◄

5.6.4 (a) . . . *IS what we found in 5.6.3!* . . . $\mathbf{2x + 2}$ ◄

(b) $f'(2) = 2(2) + 2 = \mathbf{6}$ ◄

(c) . . . *IS what we found in 5.6.4(b)!* . . . **6** ◄

using the rules:

6.1 $2x$ ◄ 6.2 $3x^2$ ◄ 6.3 $4x^3$ ◄

6.4 nx^{n-1} ◄ 6.5 $1x^{1-1} = 1x^0 = \mathbf{1}$ ◄

6.6 $f(x) = 5x^0$ ➡ $f'(x) = 5.0x^{-1} = \mathbf{0}$ ◄

6.7 -3 ◄ 6.8 $6x$ ◄ 6.9 $2x + 3x^2$ ◄

6.10 $10x - 1$ ◄

6.11 $f(x) = x^{-1}$ ➡ $f'(x) = -x^{-2} = -\dfrac{1}{x^2}$ ◄

6.12 $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

➡ $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$

$= \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2} \times \dfrac{1}{x^{\frac{1}{2}}}$

$= \dfrac{1}{2\sqrt{x}}$ ◄

NB: What are the gradients of $f(x) = x$; $f(x) = 5$; $f(x) = -3x$?
answers: 1 ; 0 ; -3
Compare to 6.5 → 6.7 (AND to Q5.4)

7.1 (a) $\frac{dy}{dx} = \mathbf{3}$ ◄ (b) $\frac{dy}{dx} = \mathbf{0}$ ◄ (c) $\frac{dy}{dx} = \mathbf{-2}$ ◄

[They're the same because **the derivative is the gradient.**]
(See the sketches in Question 5.4.1)

7.2 $\frac{dy}{dx} = \mathbf{-1}$ ◄

7.3 $y = 2x^{-1}$ ➡ $\frac{dy}{dx} = -2x^{-2} = -2 \times \frac{1}{x^2} = -\dfrac{2}{x^2}$ ◄

7.4 $y = x^{\frac{1}{2}} + x^2$ ➡ $\frac{dy}{dx} = \dfrac{1}{2\sqrt{x}} + 2x$ ◄ . . . *see 6.12*

7.5 $y = x^2 + 2x + 1$ ➡ $\frac{dy}{dx} = \mathbf{2x + 2}$ ◄

7.6 $y = (x + 1)(x - 1)$

$\therefore y = x^2 - 1$ . . . *NB: First multiply*

$\therefore \frac{dy}{dx} = \mathbf{2x}$ ◄

7.7 $y = x + 1$ ➡ $\frac{dy}{dx} = \mathbf{1}$ ◄ 7.8 $y = x - 1$ ➡ $\frac{dy}{dx} = \mathbf{1}$ ◄

Important observation from 7.6, 7.7 & 7.8:
Note: $\frac{d}{dx}(x + 1)(x - 1) \neq \frac{d}{dx}(x + 1) \times \frac{d}{dx}(x - 1)$

7.9 $y = 2x . 2x^2 = 4x^3$ ➡ $\frac{dy}{dx} = \mathbf{12x^2}$ ◄

7.10 $y = \frac{5x^2}{x} - \frac{4x}{x} = 5x - 4$ ➡ $\frac{dy}{dx} = \mathbf{5}$ ◄

7.11 $y = 3x^2 + 6x + 2x^{-2}$ ➡ $\frac{dy}{dx} = 6x + 6 - 4x^{-3} = \mathbf{6x + 6 - \dfrac{4}{x^3}}$ ◄

7.12 $y = x^2 - \frac{4}{x^2} = x^2 - 4x^{-2}$ ➡ $\frac{dy}{dx} = 2x + 8x^{-3} = \mathbf{2x + \dfrac{8}{x^3}}$ ◄

7.13 $y = \frac{x^2}{x} - \frac{3}{x} = x - 3x^{-1}$ ➡ $\frac{dy}{dx} = 1 + 3x^{-2} = \mathbf{1 + \dfrac{3}{x^2}}$ ◄

7.14 $y = 3ax + a^2$

$\therefore \frac{dy}{dx} = 3a + 0$

$= \mathbf{3a}$ ◄

Note
x is the variable;
a is just a constant

7.15 $2y + 4x^2 = x$

$\therefore 2y = -4x^2 + x$

$\therefore y = -2x^2 + \frac{1}{2}x$

$\therefore \frac{dy}{dx} = -4x + \frac{1}{2}$

NB: First make y the subject, i.e. express y in terms of x.

7.16 $y = (x + 2)^3$

$\therefore y = (x + 2)(x + 2)(x + 2)$

$\therefore y = (x + 2)(x^2 + 4x + 4)$

$\therefore y = x^3 + 4x^2 + 4x + 2x^2 + 8x + 8$

$\therefore y = x^3 + 6x^2 + 12x + 8$

$\therefore \frac{dy}{dx} = \mathbf{3x^2 + 12x + 12}$ ◄

8.1 $\frac{dy}{dx}\left(\frac{1}{3}x^3 + 2x^{\frac{1}{2}}\right) = \frac{1}{3} . 3x^2 + 2 . \frac{1}{2}x^{-\frac{1}{2}} = \mathbf{x^2 + \dfrac{1}{\sqrt{x}}}$ ◄

8.2 $D_x[x^2 + 6x] = \mathbf{2x + 6}$ ◄

Note the various notations for the derivative.

8.3 $f'(x) = 3x^2 - 2x^{-3} = 3x^2 - \frac{2}{x^3}$

$\therefore f'(2) = 3(2)^2 - \frac{2}{2^3} = 12 - \frac{1}{4} = \mathbf{11\frac{3}{4}}$ ◄

8.4 $\frac{ds}{dt} = u + \frac{1}{2} . 2at = \mathbf{u + at}$ ◄

8.5 *Gradient of the curve* IS $\frac{dy}{dx} = 2x + 6$

$\therefore$ At $x = 2$, gradient $= 2(2) + 6$

$= \mathbf{10}$ ◄

8.6 $y = \dfrac{t^2 - 1}{2t + 2}$

$\therefore y = \dfrac{\cancel{(t+1)}(t - 1)}{2\cancel{(t+1)}}$

$\therefore y = \dfrac{t - 1}{2}$ . . . $t \neq -1$

$\therefore y = \frac{t}{2} - \frac{1}{2}$

$\therefore \frac{dy}{dx} = \mathbf{\frac{1}{2}}$ ◄

FINDING THE AVERAGE GRADIENT

9.1 **A** ◄ . . . $f(x) = x^2$ ➡ $f(5) = 5^2 = 25$ & $f(1) = (1)^2 = 1$

$\therefore$ Ave. grad. $= \dfrac{\mathbf{f(5) - f(1)}}{\mathbf{5 - 1}} = \frac{25 - 1}{4} = \frac{24}{4} = \mathbf{6}$ ◄

9.2 Let $f(x) = y = 2x^2 - 2$

$\therefore f(1) = 2(1)^2 - 2 = 0$

& $f(3) = 2(3)^2 - 2 = 16$

Ave. grad. between 1 & 3 is $\dfrac{\mathbf{f(3) - f(1)}}{\mathbf{3 - 1}} = \frac{16 - 0}{2} = \mathbf{8}$ ◄

9.3 $g(x) = 2x^2$

$\therefore g(-3) = 2(-3)^2 = 18$

& $g(5) = 2(5)^2 = 50$

Average gradient $= \dfrac{\mathbf{f(5) - f(-3)}}{\mathbf{5 - (-3)}} = \frac{50 - 18}{8} = \mathbf{4}$ ◄

9.4.1 $x_A = 3$ & $x_B = 5$

& $y_A = 4(3)^2$ & $y_B = 4(5)^2$

$= 36$ $= 100$

$\therefore$ A(3; 36) and B(5; 100)

9.4.2 **Average speed (= average gradient!)**

$= \dfrac{\textbf{Distance travelled}}{\textbf{time taken}} = \dfrac{\mathbf{f(5) - f(3)}}{\mathbf{5 - 3}}$

$= \frac{100 - 36}{5 - 3}$

$= \frac{64}{2}$

$= \mathbf{32}$ **km/h** ◄

the ANSWER series ... your key to exam success

TANGENTS
Finding the gradient/equation

10.1.1 A(3; 9)

10.1.2 (a) **The gradient of the tangent is the derivative!**

$\therefore$ Gradient of tangent $= 2x$

$\therefore$ Gradient of tangent at A $= 2(3)$ . . . *at $x = 3$*

$= \mathbf{6}\ (= m)$ ◄

(b) **Equation of tangent: $y = mx + c$**

Substitute m = 6 and point A(3; 9):

It is just a straight line!

$\therefore\ 9 = (6)(3) + c$

$\therefore\ c = -9$

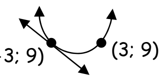
$\therefore$ Equation of the tangent: $\mathbf{y = 6x - 9}$ ◄

10.1.3 (a) Grad. of tangent = -6 ➡ the derivative, $2x = -6$

$\therefore\ x = -3$

Substitute $x = -3$ in $y = x^2$: $y = 9$

$\therefore$ The point is (-3; 9)



[Compare the tangent and sketch in Q10.1.2]

(b) Gradient of tangent = 10 ➡ $2x = 10$

$\therefore\ x = 5$

$\therefore$ The point is (5; 25)

10.2.1 **Average gradient** $= \dfrac{\mathbf{f(5) - f(2)}}{\mathbf{5 - 2}}$

$f(5) = 2(5)^2 - 6(5) = 50 - 30 = 20$

& $f(2) = 2(2)^2 - 6(2) = 8 - 12 = -4$

$\therefore$ Average gradient $= \dfrac{20 - (-4)}{5 - 2} = \dfrac{24}{3} = \mathbf{8}$ ◄

10.2.2 **Gradient of tangent to curve = $\mathbf{f'(x)}$** $= 4x - 6$

At $x = 3$, gradient of tangent to curve $= f'(3) = 4(3) - 6$

$= \mathbf{6}$ ◄

10.3 **Gradient of tangent to curve at x = $\mathbf{f'(x)}$** $= 3x^2 - 6x$

$\therefore$ **Gradient of tangent** at **(3; 4)** $= \mathbf{f'(3)} = 3(3)^2 - 6(3)$

$= 27 - 18$

$= \mathbf{9}$ ◄

$\therefore$ Subst. m = 9 & pt. (3; 4) in

$\mathbf{y = mx + c}$ . . .

$\therefore\ 4 = (9)(3) + c$

$\therefore\ 4 = 27 + c$

$\therefore\ -23 = c$

or: $\mathbf{y - y_1 = m(x - x_1)}$

$\therefore\ y - 4 = 9(x - 3)$

$\therefore\ y - 4 = 9x - 27$

$\therefore\ y = \mathbf{9x - 23}$

$\therefore$ Equation of the tangent is: $\mathbf{y = 9x - 23}$ ◄

10.4.1 **The GRADIENT of the tangent = $\mathbf{f'(x)}$** $= 6x - 2$

$\therefore$ The GRADIENT of the tangent at (**-2**; 21) $= f'(\mathbf{-2})$

$= 6(-2) - 2$

$= -14$

i.e. "m" = -14. Also point (-2; 21) on the line.

Substitute in $y = mx + c$:

$\therefore\ 21 = (-14)(-2) + c$

$\therefore\ 21 = 28 + c$

$\therefore\ -7 = c$

$\therefore$ **Equation of tangent: $\mathbf{y = -14x - 7}$** ◄

10.4.2 Gradient of tangent, $f'(x) = \dfrac{1}{2}$

➡ $6x - 2 = \dfrac{1}{2}$

$(\times 2)$ $\therefore\ 12x - 4 = 1$

$\therefore\ 12x = 5$

$\therefore\ \mathbf{x = \dfrac{5}{12}}$ ◄

10.5 $g(x) = x^2 + 3x - 4$

& **gradient of the tangent = $\mathbf{g'(x)}$** $= 2x + 3$

$2x + 3 = -3$

$\therefore\ 2x = -6$

$\therefore\ x = -3$

& $g(-3) = (-3)^2 + 3(-3) - 4$

$= 9 - 9 - 4$

$= -4$

$\therefore$ The point is **(-3; -4)** ◄

10.6 $y = -2x + p$ equation of tangent

➡ **gradient of tangent = -2**

i.e. **derivative,** $-2x + 4 = \mathbf{-2}$

$\therefore\ -2x = -6$

$\therefore\ x = 3$

$\therefore\ y = -3^2 + 4(3) + 5 = 8$

$\therefore$ Point of contact is (3; 8)

Now substitute point of contact:

(3; 8) in $y = -2x + p$

$\therefore\ 8 = -2(3) + p$

$\therefore\ 8 + 6 = p$

$\therefore\ \mathbf{p = 14}$ ◄

10.7 $y = -2x^3 + 3x^2 + 32x + 15$

Gradient of tangent, $\dfrac{dy}{dx} = -6x^2 + 6x + 32$

At $x = -2$, gradient of tangent $= -6(-2)^2 + 6(-2) + 32$

$= -24 - 12 + 32$

$= -4$

$\therefore$ Subst. m = -4 & (-2; -21) in

$\mathbf{y - y_1 = m(x - x_1)}$ . . .

or in $\mathbf{y = mx + c}$, etc.

$\therefore\ y + 21 = -4(x + 2)$

$\therefore\ y + 21 = -4x - 8$

$\therefore\ \mathbf{y = -4x - 29}$ ◄

10.8.1 Gradient of f, $f'(x) = 6x$

$\therefore$ At A, gradient $f'(1) = 6(1) = 6$

$\therefore$ Equation of g: m = 6 & point (1; 5)

$\therefore\ y - 5 = 6(x - 1)$

$\therefore\ \mathbf{y = 6x - 1}$ ◄

10.8.2 & for h, $m = -\dfrac{1}{6}$

$\therefore$ Equation of h: $y - 5 = -\dfrac{1}{6}(x - 1)$

$\therefore\ y = -\dfrac{1}{6}x + 5\dfrac{1}{6}$

On the y-axis, $x = 0$ $\therefore\ y = 5\dfrac{1}{6}$ by B

$\therefore$ **OB = $\mathbf{5\frac{1}{6}}$ units** ◄

10.9.1 $f(x) = -x^2 + 1$

$f(2) = -2^2 + 1 = -4 + 1 = -3$

& $f(2 + h) = -(2 + h)^2 + 1 = -(4 + 4h + h^2) + 1$

$= -4 - 4h - h^2 + 1$

$= -3 - 4h - h^2$

Gradient of the line

$= \dfrac{\mathbf{f(2 + h) - f(2)}}{\mathbf{h}}$ . . .

which = the **average** gradient of the **curve** of **f** from $x = 2$ to $x = 2 + h$

$= \dfrac{(-3 - 4h - h^2) - (-3)}{h}$

$= \dfrac{-4h - h^2}{h}$

$= \mathbf{-4 - h}$ ◄

10.9.2 (a) Gradient of the tangent to f at x,

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$\therefore\ f'(\mathbf{2}) = \lim\limits_{h \to 0} \dfrac{f(\mathbf{2} + h) - f(\mathbf{2})}{h}$

the gradient of the tangent to f at $x = \mathbf{2}$

$= \lim\limits_{h \to 0} \mathbf{(-4 - h)}$ . . . *from answer in 10.9.1*

$= \mathbf{-4}$ ◄

(b) Point of contact is (2; -3) . . . *f(2) = -3 in 10.9.1*

& m = -4 . . . *in 10.9.2 (a)*

$\therefore$ Equation of tangent: $y + 3 = -4(x - 2)$

$\therefore\ y + 3 = -4x + 8$

$\therefore\ \mathbf{y = -4x + 5}$ ◄

CURVESKETCHING

11.1.1 $f(x) = x^3 - 12x + 16$

$f(2)$ $= 2^3 - 12(2) + 16 = 8 - 24 + 16 = 24 - 24 = \mathbf{0}$

[$\therefore$ 2 is a root of $f(x) = 0$]

$\therefore$ **$(x - 2)$ is a factor of $f(x)$** ◄

11.1.2 $\therefore\ f(x) = (x - 2)(x^2 \ldots\ldots - 8)$ ... $\boxed{-2x^2 + 2x^2 = 0}$

$+ 2x^2$; $- 2x^2$

$= (x - 2)(x^2 + 2x - 8)$

$- 8x$; $- 4x$

Check: $-4x + (-8x) = -12x$ ✓

$= \mathbf{(x - 2)(x + 4)(x - 2)}$ ◄

11.1.3 **x-intercept: $x = 2$ or -4** ◄ & **y-intercept: $y = 16$** ◄

[$f(x) = 0$] [$f(0) = ?$]

11.1.4 **At the turning points, $f'(x) = 0$**

$\therefore\ 3x^2 - 12 = 0$

$\div 3)$ $\therefore\ x^2 - 4 = 0$

$\therefore\ (x + 2)(x - 2) = 0$

$\therefore\ x = -2$ or 2

Note: The x-coord. of the point of inflection is also the midpt. between the x-coords of the turning point.

$f(-2) = (-2)^3 - 12(-2) + 16 = -8 + 24 + 16 = 32$

& $f(2) = 0$ ... *see 11.1.1*

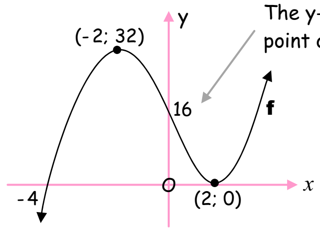
$\therefore$ Turning points are: **$(-2; 32)$ & $(2; 0)$**

At the point of inflection, $f''(x) = 0$

$\therefore\ 6x = 0$

$\therefore\ x = 0$

$\therefore$ Point of inflection is: **$(0; 16)$** ... *see 11.1.3*



11.2.1 **y-intercept: $y = 0$** ◄ **($x = 0$)**

x-intercept: $x^3 - 6x^2 = 0$ **($y = 0$)**

$\therefore\ x^2(x - 6) = 0$

$\therefore$ **$x = 0$ or 6** ◄

11.2.2 **At the turning pts, $f'(x) = 0$** ...

$\therefore\ 3x^2 - 12x = 0$

$\therefore\ 3x(x - 4) = 0$

$\therefore\ x = 0$ or 4

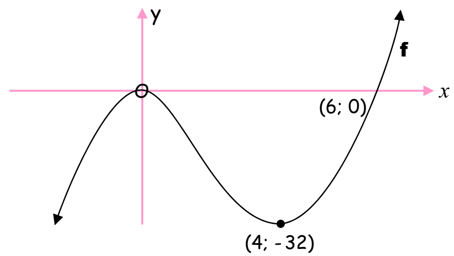
The derivative, *the gradient of the tangents,* **is zero** *at the turning points!*

$f(0) = 0$

& $f(4) = 4^3 - 6(4)^2 = 64 - 96 = -32$

$\therefore$ Turning points are: **$(0; 0)$ & $(4; -32)$** ◄

11.2.3



11.3

NB: *The expression has been given in:*
expanded form: nice for differentiating
and in factorised form: nice for finding the roots!

- $f(x) = x^3 - 5x^2 + 7x - 3$

y-intercept: $y = -3$ ... *when $x = 0$*

At the t. pts., $f'(x) = 0$

$\therefore\ 3x^2 - 10x + 7 = 0$

$\therefore\ (3x - 7)(x - 1) = 0$

$\therefore\ x = \frac{7}{3}$ or 1

$f\left(\frac{7}{3}\right) = -1{,}19$

& $f(1) = 0$

At the pt. of infl., $f''(x) = 0$

$\therefore\ 6x - 10 = 0$

$\therefore\ 6x = 10$

$\therefore\ x = \frac{5}{3}$

$f\left(\frac{5}{3}\right) = -0{,}59$

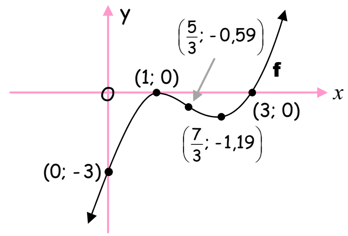
$\therefore$ Point of infl.: **$\left(\frac{5}{3}; -0{,}59\right)$** ◄

$\therefore$ Turning points: **$\left(\frac{7}{3}; -1{,}19\right)$ & $(1; 0)$** ◄

- $f(x) = (x - 1)^2(x - 3)$

x-intercepts: $x = 1$ ... *(1; 0) also a turning point!*

$x = 3$



11.4.1 $f(x) = x^3 - 4x^2 - 3x + 18$

$f(3)$ $= 3^3 - 4(3)^2 - 3(3) + 18 = 27 - 36 - 9 + 18 = \mathbf{0}$ ◄

11.4.2 $\therefore\ x - 3$ is a factor of $f(x)$

$\therefore\ f(x) = (x - 3)(x^2 \ldots\ldots - 6)$ ... $\boxed{-3x^2 - x^2 = -4x^2}$

$= (x - 3)(x^2 - x - 6)$ ...

Check: $-3x - 6x = 3x$ ✓

$= (x - 3)(x - 3)(x + 2)$

$= (x - 3)^2(x + 2)$

$\therefore\ f(x) = 0$ when $x = 3$ of -2

$\therefore$ x-intercepts: **$x = 3$ or $x = -2$** ◄

11.4.3 $f(0) = 18$ ➡ y-intercept: **$y = 18$** ◄

11.4.4 **$f'(x) = 3x^2 - 8x - 3$** ◄ **$f''(x) = 6x - 8$** ◄

11.4.5 **At the t. pts., $f'(x) = 0$**

$\therefore\ 3x^2 - 8x - 3 = 0$

$\therefore\ (3x + 1)(x - 3) = 0$

$\therefore\ x = -\frac{1}{3}$ of 3

$f\left(-\frac{1}{3}\right) \approx 18{,}52$

& $f(3) = 0$ ... *see 11.4.1*

At the pt. of infl., $f''(x) = 0$

$\therefore\ 6x - 8 = 0$

$\therefore\ 6x = 8$

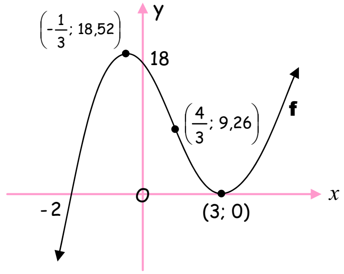
$\therefore\ x = \frac{4}{3}$

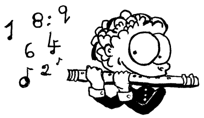
$f\left(\frac{4}{3}\right) = 9{,}26$

$\therefore$ Point of infl.: **$\left(\frac{4}{3}; 9{,}26\right)$** ◄

$\therefore$ Turning points: **$\left(-\frac{1}{3}; 18{,}52\right)$ & $(3; 0)$** ◄

11.4.6





11.5.1 $f(x) = -x^3 + 5x^2 + 8x - 12$

$\mathbf{f(-2)} = -(-2)^3 + 5(-2)^2 + 8(-2) - 12$
$= 8 + 20 - 16 - 12$
$= 28 - 28$
$\mathbf{= 0}$

$\therefore\ x + 2$ is a factor of $f(x)$

$\therefore\ f(x) = (x+2)(-x^2\ \ldots\ldots - 6)$ . . . $\boxed{-2x^2 + \mathbf{7x^2} = 5x^2}$

$= (x+2)(-x^2 + 7x - 6)$ . . . $\boxed{\text{Check: } -6x + 14x = 8x\ \checkmark}$

$= -(x+2)(x^2 - 7x + 6)$

$= -(x+2)(x-1)(x-6)$

$f(x) = 0 \ \Rightarrow\ x = \mathbf{-2;\ 1\ \text{or}\ 6}$ ◄

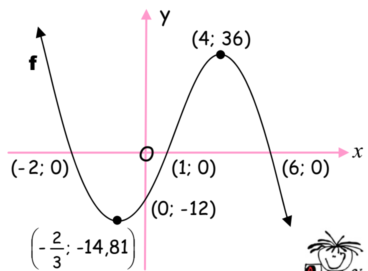
11.5.2 **At the stationary points, f′(x) = 0**

$\therefore\ -3x^2 + 10x + 8 = 0$
$\therefore\ 3x^2 - 10x - 8 = 0$
$\therefore\ (3x+2)(x-4) = 0$
$\therefore\ x = -\frac{2}{3}$ or 4

$f\left(-\frac{2}{3}\right) \doteq -14{,}81$
& $f(4) = 36$

$\therefore$ Stationary points: $\mathbf{\left(-\frac{2}{3};\ -14{,}81\right)}$ & **(4; 36)** ◄

11.5.3

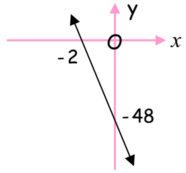


11.5.4 Gradient of tangent at x is $f'(x) = -3x^2 + 10x + 8$

$\therefore$ Gradient of tangent at -2 is $f'(-2)$
$= -3(-2)^2 + 10(-2) + 8$
$= -12 - 20 + 8$
$= -24$ $(= \mathbf{m})$

& Point of contact: $f(-2) = 0$ . . . *see roots in 11.5.1*
$\therefore\ (-2;\ 0)$

$\therefore$ Equation: $\mathbf{y = -24x - 48}$ ◄ . . .



$\boxed{\text{OR: } y - 0 = -24(x+2),\ \text{etc.}}$

11.6 $g(x) = x^3 - 5x^2 - 8x + 12$

11.6.1 Reflection in the x-axis
[The rule: (x; y) → (x; -y)]

11.6.2 $g(1) = 1 - 5 - 8 + 12 = 0$
$\therefore\ (x-1)$ is a factor of $g(x)$

$\therefore\ g(x) = (x-1)(x^2\ \ldots\ldots - 12)$ . . . $\boxed{-x^2 - \mathbf{4x^2} = -5x^2}$

$= (x-1)(x^2 - 4x - 12)$ . . . $\boxed{\text{Check: } 4x - 12x = -8x\ \checkmark}$

$= (x-1)(x-6)(x+2)$

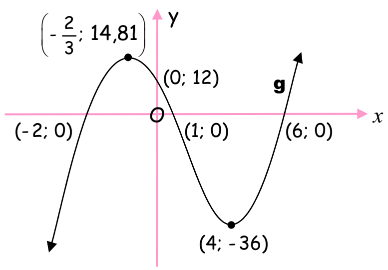
$g(x) = 0 \ \Rightarrow\ x = \mathbf{1;\ 6\ \text{or}\ -2}$ ◄

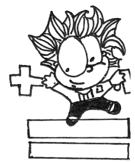
11.6.3 At the turning points, $g'(x) = 0$
$\therefore\ 3x^2 - 10x - 8 = 0$
$\therefore\ (3x+2)(x-4) = 0$
$\therefore\ x = -\frac{2}{3}$ or 4

$g\left(-\frac{2}{3}\right) \doteq 14{,}81$ & $g(4) = -36$

$\therefore$ The turning points are: $\mathbf{\left(-\frac{2}{3};\ 14{,}81\right)}$ & **(4; -36)** ◄

11.6.4





11.7.1 $f(x) = -2x^3 + kx^2 + 4x - 3$

$\mathbf{f\left(\frac{1}{2}\right)} = -2\left(\frac{1}{2}\right)^3 + k\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) - 3 \ \mathbf{= 0}$ . . . $\boxed{\because\ 2x - 1 \text{ is a factor}}$

$\therefore\ -2\left(\frac{1}{8}\right) + \frac{1}{4}k + 2 - 3 = 0$

$\times 4)$ $\therefore\ -1 + k + 8 - 12 = 0$
$\therefore\ k - 5 = 0$
$\therefore\ \mathbf{k = 5}$ ◄

11.7.2 $f(x) = -2x^3 + 5x^2 + 4x - 3$

$= (2x-1)(-x^2\ \ldots\ldots + 3)$ . . . $\boxed{+x^2 + \mathbf{4x^2} = 5x^2}$

$= (2x-1)(-x^2 + 2x + 3)$ . . . $\boxed{\text{Check: } -2x + 6x = 4x\ \checkmark}$

$= \mathbf{-(2x-1)(x^2 - 2x - 3)}$

$= -(2x-1)(x-3)(x+1)$

$\boxed{\begin{aligned} &\mathbf{-x^2 + 2x + 3} \\ = &\mathbf{-(x^2 - 2x - 3)} \end{aligned}}$

$f(x) = 0 \ \Rightarrow\ x = \frac{1}{2};\ 3\ \text{or}\ -1$ ◄

11.7.3 **At the turning points, f′(x) = 0**

$\therefore\ -6x^2 + 10x + 4 = 0$
$\div(-2)$ $\therefore\ 3x^2 - 5x - 2 = 0$
$\therefore\ (3x+1)(x-2) = 0$
$\therefore\ x = -\frac{1}{3}$ or 2

$f\left(-\frac{1}{3}\right) \doteq -3{,}70$ & $f(2) = 9$

$\therefore$ The turning points are: $\mathbf{\left(-\frac{1}{3};\ -3{,}70\right)}$ & **(2; 9)** ◄

11.8.1 Let $f(x) = x^3 - x^2 - x + 10$
$f(1) = 1 - 1 - 1 + 10 \neq 0$
$f(-1) = -1 - 1 + 1 + 10 \neq 0$
$f(2) = 8 - 4 - 2 + 10 \neq 0$
$\mathbf{f(-2)} = -8 - 4 + 2 + 10 \ \mathbf{= 0}$

$\therefore\ x + 2$ is a factor of $f(x)$

$\therefore\ f(x) = (x+2)(x^2\ \ldots\ldots + 5)$

$= (x+2)(x^2 - 3x + 5)$

$\therefore\ f(x) = 0 \ \Rightarrow\ x + 2 = 0$ or $x^2 - 3x + 5 = 0$

$\therefore\ \mathbf{x = -2}$ ◄ $\qquad \therefore\ x = \dfrac{3 \pm \sqrt{-11}}{2(1)}$

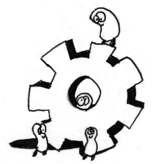
(non-real values)

11.8.2 **At the turning points, $f'(x) = 0$**

$\therefore\ 3x^2 - 2x - 1 = 0$

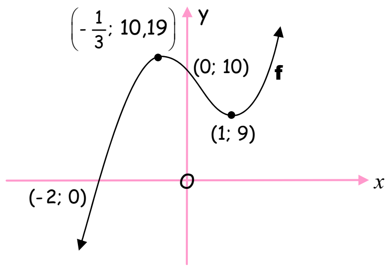
$\therefore\ (3x + 1)(x - 1) = 0$

$\therefore\ x = -\frac{1}{3}$ or 1



$f\left(-\frac{1}{3}\right) = 10{,}19$ & $f(1) = 9$

$\therefore$ The turning points are: $\left(-\frac{1}{3};\ 10{,}19\right)$ & $(1;\ 9)$



11.9.1 $\boldsymbol{x = 0}$ **or 3** ◀ [y = 0]

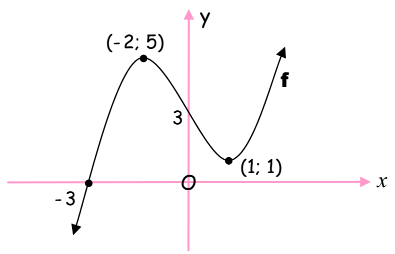
11.9.2 $\boldsymbol{x < 0}$ **or** $\boldsymbol{0 < x < 3}$ **OR** $\boldsymbol{x < 3\ ;\ x \neq 0}$ ◀
[y is negative]

11.9.3 $\boldsymbol{0 \leq x \leq 2}$ ◀ [Gradient is negative or 0]

11.9.4 $\boldsymbol{x = 0}$ **or** $\boldsymbol{x \geq 2}$ ◀

For these x's, x and the gradient are either 0 **or** they are both positive. (They're never both negative in this example.)

12.1.1



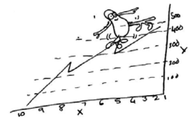
12.1.2 $\boldsymbol{-3 < x < 0}$ ◀ . . . *where x and $f(x)$ are opposite in sign*

12.1.3 **(-1; 1)** ◀

13.1.1 $f(x) = x^3 - 4x^2 - 11x + 30$

$\therefore\ f(1) \neq 0$

$f(-1) \neq 0$



$\mathbf{f(2)} = 2^3 - 4(2)^2 - 11(2) + 30$

$= 8 - 16 - 22 + 30$

$= 38 - 38$

$\mathbf{= 0}$

$\therefore\ x - 2$ is a factor of $f(x)$

$\therefore\ f(x) = (x - 2)(x^2 \ldots\ldots - 15)$ . . . $\boxed{-2x^2 - \mathbf{2x^2} = -4x^2}$

$= (x - 2)(x^2 - 2x - 15)$ . . . **Check:** $4x - 15x = -11x$ ✓

$= (x - 2)(x - 5)(x + 3)$

$f(x) = 0$ ➡ $x = 2\ ;\ 5$ or -3

$\therefore$ A(-3; 0) & C(5; 0) $\boxed{\text{and B(2; 0)}}$

$\therefore$ **AC = 8 units** ◀

13.1.2 At G & E, $f'(x) = 0$

$\therefore\ 3x^2 - 8x - 11 = 0$

$\therefore\ (3x - 11)(x + 1) = 0$

$\therefore\ x = \frac{11}{3}$ or -1

At E, $x = \frac{11}{3}$

$y = f\left(\frac{11}{3}\right) \simeq -14{,}81$

$\therefore$ **EF ≃ 14,81 units** ◀

13.1.3 Gradient of curve $= f'(x) = 3x^2 - 8x - 11$

$\therefore$ Gradient of curve at B(**2**; 0) $= f'(\mathbf{2}) = 3(\mathbf{2})^2 - 8(\mathbf{2}) - 11$

$= 12 - 16 - 11$

$= \mathbf{-15}$ ◀

13.1.4 Gradient $= -11$ ➡ $3x^2 - 8x - \cancel{11} = -\cancel{11}$

$\therefore\ 3x^2 - 8x = 0$

$\therefore\ x(3x - 8) = 0$

$\therefore\ \boldsymbol{x = 0}$ **or** $\boldsymbol{\frac{8}{3}}$ ◀

13.1.5 $-7(3x^2 - 8x - 11) > 0$ $\boxed{\div (-7)}$

➡ $3x^2 - 8x - 11 < 0$

$\therefore\ \boldsymbol{-1 < x < \frac{11}{3}}$ ◀ $\boxed{\text{gradient is neg. for these values of } x}$

13.2 Equation of f: $y = px^3 + 5x^2 - qx - 3$

(-2; 9) on p: $9 = p(-2)^3 + 5(-2)^2 - q(-2) - 3$

$\therefore\ 9 = -8p + 20 + 2q - 3$

$\therefore\ 8p - 2q = 8$

$\therefore\ 4p - q = 4$ . . . ❶

At a turning point, $f'(x) = 0$

$\therefore\ 3px^2 + 10x - q = 0$ by $x = -2$

$\therefore\ 3p(-2)^2 + 10(-2) - q = 0$

$\therefore\ 12p - 20 - q = 0$

$\therefore\ 12p - q = 20$ . . . ❷

❷ - ❶: $\therefore\ 8p = 16$

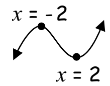
$\therefore\ \mathbf{p = 2}$ ◀

❶: $\therefore\ 8 - q = 4$

$\therefore\ -q = -4$

$\therefore\ \mathbf{q = 4}$ ◀

13.3.1 $f(x) = 2x^3 - 24x + c$ has shape:



"a" is positive

At the turning points, $f'(x) = 0$

$\therefore\ 6x^2 - 24 = 0$

$\therefore\ 6x^2 = 24$

$\therefore\ x^2 = 4$

$\therefore\ x = \pm 2$

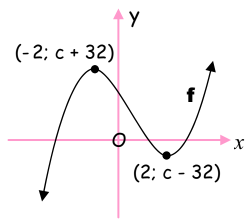
$f(2) = 2(2)^3 - 24(2) + c = 16 - 48 + c = c - 32$

& $f(-2) = 2(-2)^3 - 24(-2) + c = -16 + 48 + c = c + 32$

$\therefore$ Turning points are: **(-2; c + 32), the local maximum** ◀

& **(2; c - 32), the local minimum** ◀

13.3.2 The roots of $f(x) = 0$ are the x-intercepts of f.



There will be 3 real roots provided the maximum turning points are on or above the x-axis **and** the minimum turning point is on or below the x-axis.

$\therefore\ c + 32 \geq 0$ and $c - 32 \leq 0$

$\therefore\ c \geq -32$ $\qquad \therefore\ c \leq 32$

$\therefore\ \mathbf{-32 \leq c \leq 32}$ ◀

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13.4.1 $c = 4$ ◄ . . .

P(-1; 0) in $y = x^3 + ax^2 + bx + 4$
$\therefore\ 0 = -1 + a - b + 4$
$\therefore\ -a + b = 3$ . . . ❶

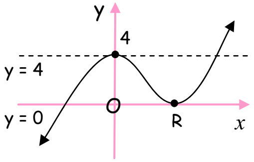
& Q a turning point
➡ $\frac{dy}{dx} = 3x^2 + 2ax + b = 0$ at $x = 0$ (& at R)
$\therefore\ \mathbf{b = 0}$ ◄ . . . ❷

❷ in ❶: $-a = 3$
$\therefore\ \mathbf{a = -3}$ ◄

13.4.2 $\frac{dy}{dx} = 3x^2 - 6x = 0$ at R (& at Q)
$\therefore\ 3x(x - 2) = 0$
$\therefore\ x = 2$ ($x = 0$ at Q!)

$\therefore$ **R(2; 0)** ◄

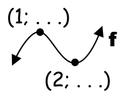
13.4.3 $\mathbf{0 \le d \le 4}$ ◄



Line y = d, || to x-axis, if d < 0 or > 4, will cut the curve once only ; therefore 1 real root & 2 imaginary roots!

13.5.1 $f'(x) = 0$ at $x = 1$ & $x = 2$:

x: 1 2
$f'(x)$: + - +
f: ↗ → ↘ → ↗ . . .



x-coordinate of local MINIMUM = 2 ◄

13.5.2 $\mathbf{1 < x < 2}$ ◄

13.5.3 **6** ◄ . . . $\boxed{f'(0) = 6}$

13.5.4 $\mathbf{x = 3}$ ◄ . . . $\boxed{f'(3) = f'(0) = 6}$

13.6.1 The degree of $f'(x)$ is always 1 less than that of $f(x)$.
Here $f'(x)$ is linear.
$\therefore$ **$f(x)$ is quadratic** ◄

13.6.2 $\mathbf{f'(-1) = 0}$ ◄

13.6.3 (a) $x < -1$ ◄ (b) $x > -1$ ◄

13.6.4 (a) $x < -1$ ◄ (b) $x > -1$ ◄

NB: See 13.6.3 & 13.6.4: the answers are the same. Why?

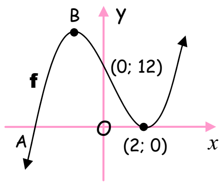
13.6.5 It has a minimum turning point.
$f'(x)$ does = 0 at $x = -1$ & $f'(x)$, the gradient of f, is negative left of $x = -1$, $\therefore$ f is decreasing & positive right of $x = -1$, $\therefore$ f is increasing. ◄

13.6.6 $\mathbf{x = -1}$ ◄

13.7.1 f **touches** the x-axis at $x = 2$
$\therefore\ f(x) = (x - 2)^2(\ ?\ )$
$= (x^2 - 4x + 4)(\ ?\ )$
$= (x^2 - 4x + 4)\mathbf{(x + 3)}$ . . . by inspection, see f(x)

$\therefore$ **A(-3; 0)** ◄ $\boxed{f(x) = 0 \text{ at A}}$

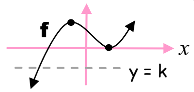
13.7.2 At B, $f'(x) = 0$
$\therefore\ 3x^2 - 2x - 8 = 0$
$\therefore\ (3x + 4)(x - 2) = 0$
$\therefore\ x = -\frac{4}{3}$ . . . *already have $x = 2$*
$\therefore\ x_B = -\frac{4}{3}$ ◄



13.7.3 $\mathbf{x < -\frac{4}{3}}$ or $\mathbf{x > 2}$ ◄ . . . $\boxed{f'(x) \text{ is the gradient of f!}}$

13.7.4 **1** ◄

$x^3 - x^2 - 8x + 12 = k$
➡ $f(x) = k$



Line y = k (if k < 0) will cut f once only.
[y = k || x-axis]

13.7.5 $f'(x) = 3x^2 - 2x - 8$
$\therefore\ f''(x) = 6x - 2$

$f''(x) = 0$ at the point of inflection
$\therefore\ 6x - 2 = 0$
$\therefore\ 6x = 2$
$\therefore\ x = \frac{1}{3}$

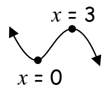
$\therefore\ f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 8\left(\frac{1}{3}\right) + 12$
$= \frac{250}{27}$
$= 9\frac{7}{27}$

$\therefore$ Point of inflection: $\left(\frac{1}{3};\ 9\frac{7}{27}\right)$ ◄

PRACTICAL APPLICATIONS

14.1.1 Volume = $(5x)(x)(9 - 2x)$
$\therefore\ \mathbf{V = (45x^2 - 10x^3)\ units^3}$ ◄

14.1.2 Maximum volume when $\frac{dV}{dx} = 0$



$\therefore\ 90x - 30x^2 = 0$
$\therefore\ 30x(3 - x) = 0$
$\therefore\ \mathbf{x = 3}$ ◄

14.2.1 Total surface area, $A = 2(4x) + (2x + 8).h$ . . . ❶
. . . $\boxed{(2 \times \text{base}) + (\text{perimeter of base} \times h)}$

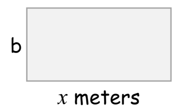
Now, $V = x \times 4 \times h = 480$
$\therefore\ h = \frac{480}{4x}$
$\therefore\ h = \frac{120}{x}$ . . . ❷

❷ in ❶: $\therefore\ A = 8x + (2x + 8)\frac{120}{x}$
$\therefore\ A = 8x + 240 + \frac{960}{x}$
$\therefore\ \mathbf{A = 8x + 960x^{-1} + 240}$ ◄

14.2.2 Minimum A occurs when $\frac{dA}{dx} = 0$
$\therefore\ 8 - 960x^{-2} = 0$
$\therefore\ 8 - \frac{960}{x^2} = 0$
$(\times x^2)$ $\therefore\ 8x^2 - 960 = 0$
$\therefore\ 8x^2 = 960$
$\therefore\ x^2 = 120$
$\therefore\ x = +\sqrt{120}$ $x > 0$ . . . it's a length!
$\therefore\ h = \frac{120}{\sqrt{120}}$ . . . see ❷ in 14.2.1
$= 10{,}95$
$\simeq 11$ cm

$\therefore$ **Dimensions: 11 cm × 4 cm × 11 cm** ◄ *(to the nearest cm)*

14.3.1 $2x + 2b = 1\,200$ . . . *perimeter*
$\therefore\ x + b = 600$
$\therefore\ b = 600 - x$



OR: $b = \frac{1\,200 - 2x}{2} = 600 - x$

Area $= x \times b = x(600 - x)$
$\therefore\ \mathbf{A = 600x - x^2}$ (metre^2) ◄

14.3.2 Maximum A when $\frac{dA}{dx} = 0$

$\therefore\ 600 - 2x = 0$

$\therefore\ -2x = -600$

$\therefore\ x = 300$

$\therefore\ b = 300$

$\therefore$ **300 m × 300 m** ◄ *. . . It happens to be a square!*

14.3.3 **Maximum A** $= 600(300) - 300^2 =$ **90 000 m²** ◄

14.4.1 $P = -2x^3 + 600x + 1\,000$

Maximum P when $\frac{dP}{dx} = 0$

$\therefore\ -6x^2 + 600 = 0$

$\therefore\ -6x^2 = -600$

$\therefore\ x^2 = 100$

$\therefore\ x = 10$ $\therefore$ **10 employees** ◄

14.4.2 **Maximum P** $= -2(10)^3 + 600(10) + 1\,000 =$ **5 000 rand** ◄

14.5.1 $f(3) = 36(3) - 3(3)^2 =$ **81 cm** ◄

14.5.2 Maximum when $f'(x) = 0$

$\therefore\ 36 - 6x = 0$

$\therefore\ -6x = -36$

$\therefore\ x = 6$ $\therefore$ **After 6 months** ◄

14.5.3 **Maximum height** $= f(6) = 36(6) - 3(6)^2 =$ **108 cm** ◄

14.6.1 Volume $= \pi r^2 h = 450$ (450 mℓ = 450 cm³)

$\therefore\ \mathbf{h = \frac{450}{\pi r^2}}$ ◄ *. . . in cm*

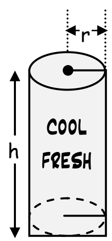
14.6.2 $A = 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r\left(\frac{450}{\pi r^2}\right)$

$= \mathbf{2\pi r^2 + \frac{900}{r}}$ ◄ *. . . in cm²*

14.6.3 $A = 2\pi r^2 + 900r^{-1}$

Minimum A when $\frac{dA}{dr} = 0$



$\therefore\ 4\pi r - 900r^{-2} = 0$

$\therefore\ 4\pi r - \frac{900}{r^2} = 0$

$(\times r^2)$ $\therefore\ 4\pi r^3 - 900 = 0$

$\therefore\ 4\pi r^3 = 900$

$\therefore\ r^3 = \frac{900}{4\pi}$

$\therefore$ **r ≃ 4,15 cm** ◄

14.7.1 $A(x) = (100 \times 2x) - 2\left(\frac{22}{7}x^2\right)$

$\therefore\ \mathbf{A(x) = 200x - \frac{44}{7}x^2}$ ◄

14.7.2 Maximum $A(x)$ when $A'(x) = 0$

$\therefore\ 200 - \frac{88}{7}x = 0$

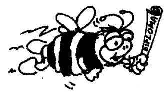
$\therefore\ -\frac{88}{7}x = -200$

$\times\left(-\frac{7}{88}\right)$ $\therefore\ \mathbf{x = 15{,}91}$ ◄

14.7.3 **Maximum A(x)** $= 200(15{,}91) - \frac{44}{7}(15{,}91)^2$

$\simeq$ **1 590,91 mm²** ◄

14.8.1 (a) $V = (2x)(x)(h) = 24$



$\therefore\ 2x^2 h = 24$

$\therefore\ h = \frac{24}{2x^2}$

$\therefore\ \mathbf{h = \frac{12}{x^2}}$ ◄

(b) Cost for the top $= (2x)(x)(25) = 50x^2$

Cost for the base $= (2x)(x)(20) = 40x^2$

Cost for the sides $=$ [perimeter of base × h] × R20

$= 6x \times \frac{12}{x^2} \times 20$

$= \frac{1\,440}{x}$

$\therefore$ **Total cost, C** $= 50x^2 + 40x^2 + 1\,440x^{-1}$

$= \mathbf{(90x^2 + 1\,440x^{-1})}$ **rand** ◄

14.8.2 (a) Minimum C when $\frac{dC}{dx} = 0$

$\therefore\ 180x - 1\,440x^{-2} = 0$

$\therefore\ 180x - \frac{1\,440}{x^2} = 0$

$(\times x^2)$ $\therefore\ 180x^3 - 1\,440 = 0$

$\therefore\ 180x^3 = 1\,440$

$(\div 180)$ $\therefore\ x^3 = 8$

$\therefore\ x = 2$

$\therefore$ length $= 2(2) = 4$ m, breadth $= 2$ m

& height $= \frac{12}{x^2} = \frac{12}{4} = 3$ m

(b) **Min. cost, C** $= 90(2)^2 + 1\,440(2)^{-1} = 360 + 720$

$=$ **R1 080** ◄

14.9.1 (a) $\mathbf{A = \left(2xy + \frac{1}{2}\pi x^2\right) m^2}$ ◄

(b) $P = 2y + 2x + \frac{1}{2}(2\pi x)$

$\therefore\ \mathbf{P = (2y + 2x + \pi x)\ m}$ ◄

14.9.2 $P = 8x^{-1} + \left(\frac{\pi}{2} + 2\right)x$

Minimum P when $\frac{dP}{dx} = 0$

$\therefore\ -8x^{-2} + \frac{\pi}{2} + 2 = 0$

$\therefore\ -\frac{8}{x^2} + \frac{\pi}{2} + 2 = 0$

$(\times 2x^2)$ $\therefore\ -16 + \pi x^2 + 4x^2 = 0$

$\therefore\ (\pi + 4)x^2 = 16$

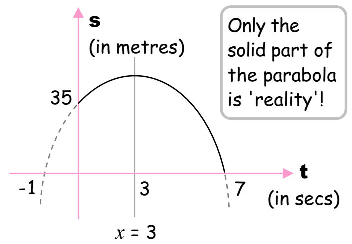
$\therefore\ x^2 = \frac{16}{\pi + 4} \simeq 2{,}24\ldots$

$\therefore\ \mathbf{x = 1{,}50}$ ◄

15. $s = -5t^2 + 30t + 35$

$\therefore\ s = -5(t^2 - 6t - 7)$

$\therefore\ s = -5(t + 1)(t - 7)$



Only the solid part of the parabola is 'reality'!

NB: The **height** is **s** metres; the **time** taken is **t** seconds; the **speed** (metres/sec) is the derivative, $\frac{ds}{dt}$.

15.1 Determine s when t = 2:

$\therefore\ s = -5(2 + 1)(2 - 7)$

$=$ **75 metres** ◄

OR: $s = -5(2)^2 + 30(2) + 35$

$=$ **75 metres** ◄

15.2 Speed $= \frac{ds}{dt} = -10t + 30$

When t = 2: Speed $= -10(2) + 30$

$=$ **10 m/s** ◄

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15.3 The max height (s) occurs for t halfway between the roots:

$\therefore\ t = \dfrac{-1+7}{2} = 3$

$\therefore$ **3 seconds** ◄ *. . . see sketch*

15.4 Maximum height (s) = $-5(3+1)(3-7)$ = **80 metres** ◄

OR: Max. s = $-5(3)^2 + 30(3) + 35$ = **80 metres** ◄

15.5 '. . . to reach the ground' ➨ the height, s = 0

s = 0 ➨ t = 7 ◄ . . . NB: Time cannot be negative $\therefore\ t \neq -1$

$\therefore$ **7 seconds** ◄ *. . . see sketch*

16.1.1 P(x; y) lies on line AB which has y-intercept,

c = 3 & m = −1

$\therefore$ equation $\boldsymbol{y = -x + 3}$ ◄ *. . . is true for P(x; y)!*

16.1.2 $\therefore$ If OR = x . . . $= x_p$

then PR = $-x + 3$ . . . $= y_p$ . . . P(x; $-x+3$)

& Area of rectangle PQOR = $x(-x+3)$

$\therefore\ \boldsymbol{A = -x^2 + 3x}$ ◄

16.1.3 **Maximum A when $\dfrac{dA}{dx} = 0$**

$\therefore\ -2x + 3 = 0$

$\therefore\ -2x = -3$

$\therefore\ x = \dfrac{3}{2}$

& Max. A = $-\left(\dfrac{3}{2}\right)^2 + 3\left(\dfrac{3}{2}\right) = -\dfrac{9}{4} + \dfrac{9}{2} = 2\dfrac{1}{4}$

& Area of ΔAOB = $\dfrac{1}{2}$OA.OB = $\dfrac{1}{2}(3)(3) = 4\dfrac{1}{2}$

$\therefore$ **Maximum A of rectangle PQOR**

= $\dfrac{1}{2}$ **area of ΔAOB** ◄

16.2.1 AB = f(x) − g(x)

$= (-x^3 - 3x^2 + 3) - (x^2 - 6x + 2)$

$= -x^3 - 3x^2 + 3 - x^2 + 6x - 2$

$= \boldsymbol{-x^3 - 4x^2 + 6x + 1}$ ◄

16.2.2 Maximum value of AB occurs when the derivative = 0

$\therefore\ -3x^2 - 8x + 6 = 0$

$\times(-1)\quad \therefore\ 3x^2 + 8x - 6 = 0$

$\therefore\ x = \dfrac{-8 \pm \sqrt{8^2 - 4(3)(-6)}}{2(3)}$

$= \dfrac{-8 \pm \sqrt{136}}{6}$

$= 0{,}61$ (or $-3{,}28$) $\quad (x > -1)$

$\therefore$ **Maximum AB** = $-(0{,}61)^3 - 4(0{,}61)^2 + 6(0{,}61) + 1$

= **2,94 units** ◄

16.3.1 $T(t) = 28 - 0{,}008t^3 - 0{,}16t \qquad t \in [0;\ 10]$

$T'(t) = -0{,}024t^2 - 0{,}16,$ *. . . the rate at any t*

$\therefore\ T'(4) = -0{,}024(4)^2 - 0{,}16,$ *. . . the rate at t = 4*

$= -0{,}544$°C per minute

$\therefore$ **The temperature is falling at a rate of 0,54°C/min.** ◄

16.3.2 **A LOCAL minimum t value would occur when**

$\boldsymbol{T'(t) = 0}$

$\therefore\ -0{,}024t^2 - 0{,}16 = 0$

$\therefore\ -0{,}024t^2 = 0{,}16$

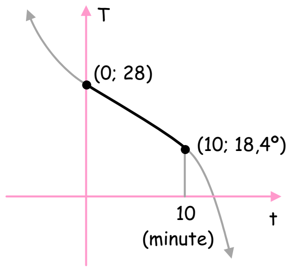
$\therefore\ t^2 = -6{,}\dot{6}$

There is no solution to this equation!

. . . $t^2 \geq 0$ for all t

$\therefore$ There are no turning points or stationary points.

BUT, consider the graph & in particular, the prescribed interval [0; 10] **- the SMALLEST value of T occurs at t = 10**



T(0) = 28

T(10) = 18,40

$\therefore$ **Minimum T = 18,4°C** ◄

(in the prescribed interval)

NOTES