GR 12 MATHS

PAPER 1: Notes & Summaries

BASIC ALGEBRA

Types of Numbers

- Algebraic Expressions
  Products; Factors and Fractions
- Algebraic Expressions & Equations compared
- Function Notation

EXPONENTS & SURDS

Notes and Worked Examples

Answers to Exercises

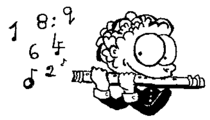
Compliments of

the ANSWER series... your key to exam success

Module 1: NUMBERS & FUNDAMENTAL CONCEPTS

TOPIC OUTLINE

- Types of numbers
- Algebraic expressions, fractions, equations and inequalities
- Non-standard number patterns



Types of Numbers

Numbers can be whole or fractional, positive or negative, rational or irrational, real or non-real (imaginary) . . .

Natural numbers, counting numbers & integers

It is important to know the names of the various sets of numbers and how they fit together.

We recall:

Natural numbers: $\mathbb{N} = \{1;\ 2;\ 3;\ \ldots\}$

Counting numbers: $\mathbb{N}_0 = \{0;\ 1;\ 2;\ 3;\ \ldots\}$

Integers: $\mathbb{Z} = \{\ldots;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3\ \ldots\}$

Fractions

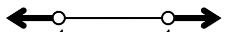
There are common fractions, e.g. $-\frac{1}{4}$; $\frac{2}{3}$; $\frac{5}{2}$ & decimal fractions, e.g. $-0{,}25$; $1{,}4$; $0{,}\dot{6}$

Proper fractions are fractions which lie between -1 and 1,

e.g. $-\frac{4}{5}$; $\frac{2}{3}$; . . . the numerator is < the denominator.

Improper fractions have values less than -1 or greater than 1,

e.g. $-\frac{3}{2}$; $\frac{11}{5}$ . . . the numerator is > the denominator



(Improper fractions can be converted to mixed fractions)

Rational and Irrational numbers

Rational numbers

Examples of rational numbers are:

integers: -7 ; 0 ; 20 & **fractions:** $\frac{1}{4} = 0{,}25$; $\frac{13}{10} = 1{,}1$; $\frac{0}{4} = 0$

NB: $\frac{5}{0}$ is undefined.

Division by zero is undefined.

Also: $\frac{2}{3} = 0{,}666\ldots\ (= 0{,}\dot{6})$; $\frac{9}{11} = 0{,}818181\ldots\ (= 0{,}\dot{8}\dot{1})$

Given a decimal fraction which is infinite, but recurring, it is possible to write it as a common fraction. See the following method:

How to write an infinite, RECURRING decimal as a fraction

Example: $0{,}\dot{3}$, i.e. 0,333..... *even though you know it equals $\frac{1}{3}$*

Let $x = 0{,}333\ldots\ldots$ . . . ❶

then $10x = 3{,}3333\ldots\ldots$ . . . ❷

❷ - ❶: $\therefore\ 9x = 3$

÷ 9: $\therefore\ x = \frac{3}{9} = \frac{1}{3}$

$\therefore\ \mathbf{0{,}\dot{3} = \frac{1}{3}}$ ✓ So, $0{,}\dot{3}$ is rational

This method clarifies the definition of a rational number.

The definition of a Rational number

A rational number is any number which can be written *as a fraction*

i.e. as $\frac{\text{an integer}}{\text{an integer}}$, or $\frac{\mathbf{a}}{\mathbf{b}}$ **where a & b $\in \mathbb{Z}$; b $\neq$ 0**

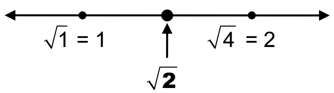
Irrational numbers

Examples of irrational numbers are:

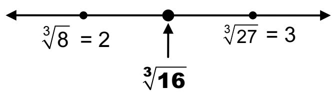
$\sqrt{2}$ = 1,414213562 . . . ; $\sqrt[3]{16}$ = 2,5198421 . . .

These decimals are infinite and non-recurring. Their values cannot be found exactly, but they do exist on the number line :

$\sqrt{2}$ lies between $\sqrt{1}$ and $\sqrt{4}$



$\sqrt[3]{16}$ lies between $\sqrt[3]{8}$ and $\sqrt[3]{27}$



What about π?

π is an irrational number because π = 3,1415926 . . . an infinite non-recurring decimal.

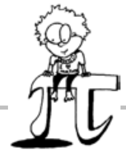
Note

3,14 and $\frac{22}{7}$ are very useful to use as approximate values of π, but:

$3{,}14 = \frac{314}{100}$ and $\frac{22}{7} = 3{,}142857\ 142857$ . . . infinite and recurring

$\therefore$ 3,14 and $\frac{22}{7}$ are both **rational** numbers.

But, π *itself* is **irrational**!



The definition of an Irrational number

An irrational number can only be written in number form with never-ending, non-repeating digits after the decimal comma. An irrational number cannot be written as a fraction.

Real and Imaginary numbers

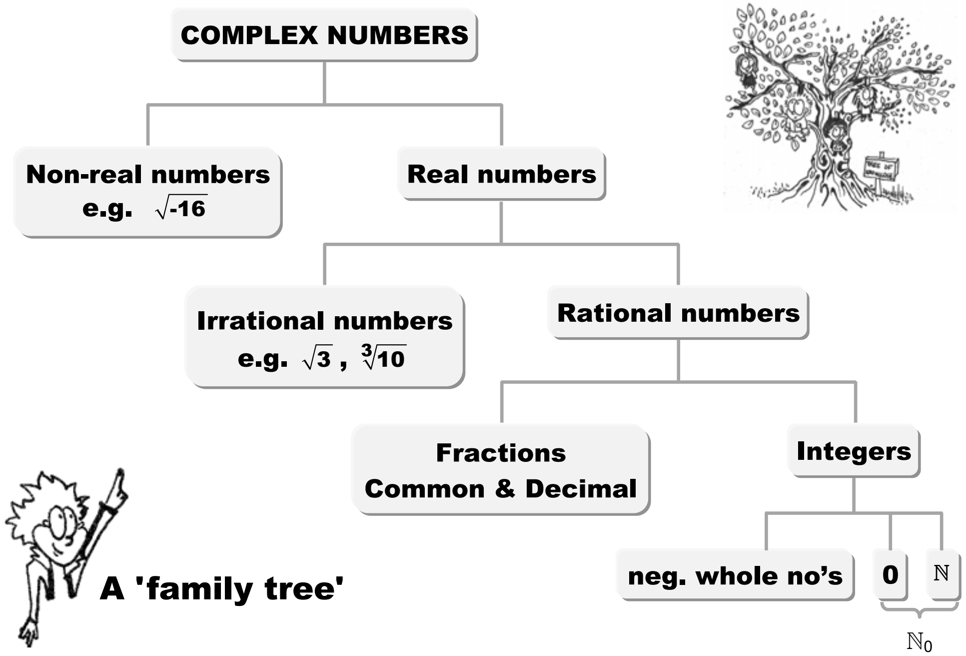
Real numbers

The number line consists of all the rational (**Q**) and irrational (**Q′**) numbers which together form the set of Real numbers (**R**).

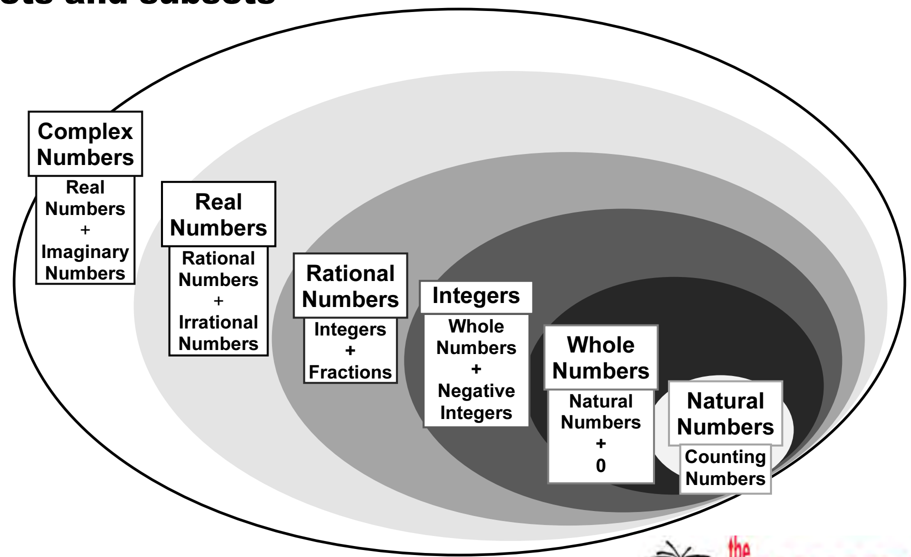
Imaginary numbers

$\sqrt{-25}$ is an example of an imaginary number. There is no number which, if squared, would equal -25. $\sqrt{-25}$ does not exist on the number line. The real numbers and the imaginary numbers together form the set of complex numbers.

Representations of the Number System



Sets and subsets



EXERCISE 1.1: Types of numbers

Answers on page A1.1

1. Classify the following numbers by placing a ✓ in the appropriate column(s):

	Complex	Non-real	Real	Irrational	Rational	Integer	$\mathbb{N}_0$	$\mathbb{N}$
10								
-4								
-3,2								
$4\frac{2}{3}$								
π								
$0,\dot{6}$								
$\sqrt{25}$								
$-\sqrt[5]{32}$								
$\sqrt{35}$								
$\sqrt{1\frac{11}{25}}$								
$\sqrt{-36}$								
$\sqrt[3]{-27}$								
$\frac{0}{6}$								
$\frac{6}{0}$								

2. Between which two integers does $\sqrt{48}$ lie?

3. Between which two integers do ALL PROPER FRACTIONS, negative and positive lie?

4. Express each of the following in interval notation (where possible) and represent each on a number line:

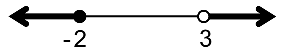
4.1 $\{x \mid x > -1;\ x \text{ a real number}\}$ 4.2 $\{x \mid -4 \leq x < 1;\ x \in \mathbb{R}\}$

4.3 $\{x \mid x < 3;\ x \in \mathbb{N}_0\}$ 4.4 $\{x \mid -2 < x \leq 2;\ x \in \mathbb{Z}\}$

5. Express the following in interval notation and algebraically:

5.1 

5.2 

6. Express the following algebraically: 

NB: In Q5.2: x is > -7 **AND** x is ≤ 4: . . . *1 piece on the no. line*
whereas, in Q6: x is ≤ -2 **OR** x is > 3: . . . *2 pieces on the no. line*

7. If x is an integer and lies in the interval [-4; 3), write down:
   7.1 the minimum value of x 7.2 the maximum value of x

8. In which intervals do ALL IMPROPER FRACTIONS, negative and positive, lie?

9. If $\frac{x - 1}{y + 3} = 0$, what can you say:
   (a) about x if y = 0? (b) about y if $x = 1$? (c) if $x = 1$ and y = -3?

10. If $(x - 1)(y + 3) = 0$ what can you say:
   (a) about x if y = -3? (b) about x if y = 2?

11. If M = $\sqrt{p - 3}$, for which values of p is M a real number?

12. If M = $\sqrt{\frac{2}{2x + 5}}$: (a) Show that M is a rational number if $x = 1,5$
   (b) Determine the values of x for which M is a real number.

13. If P = $\frac{(x - 2)(2x - 1)}{3 - x}$, for which value(s) of x will P be:
   (a) equal to zero (b) undefined

14. Given P = $\frac{\sqrt{3 - x}}{x^2 - 4}$: Determine the values of x for which
   (a) P = 0 (b) P is undefined (c) P is real (d) P is positive

15. Given $a^x = 1$, state the values of a if
   (a) $x = 0$ (b) $x \neq 0$

16. Given: $x = 4 + \sqrt{3k}$
   Write **only one** of the three words 'rational, irrational, non-real' alongside (a), (b) and (c), that describes x best for the given value of k:

k	x is . . .
-6	(a)
1	(b)
12	(c)

Algebraic Expressions, Fractions and Equations

Algebraic Expressions

▶ Products: The distributive property

a(b + c) = ab + ac

monomial × binomial

(a + b)(c + d)
= (a + b)c + (a + b)d
= ac + bc + ad + bd

binomial × binomial

(a + b)(c + d + e)
= **ac + ad + ae + bc + bd + be**

binomial × trinomial

- a **linear** expression × a **linear** expression = a **quadratic** expression
  $(x - 1) \times (x + 5) = x^2 + 4x - 5$
- a **linear** expression × a **quadratic** expression = a **cubic** expression:
  $(x + 3) \times (x^2 - x + 2) = x^3 + 2x^2 - x + 6$

How? $(x + 3)(x^2 - x + 2) = x^3 \mathbf{- x^2} + 2x \mathbf{+ 3x^2} - 3x + 6$

$= x^3 + \mathbf{2x^2} - x + 6$

Perfect squares: $(a + b)^2 = a^2 + 2ab + b^2$; $(a - b)^2 = a^2 - 2ab + b^2$

whereas: $(a + b)(a - b) = a^2 - b^2$ . . . **diff. between squares**

Trinomials

Trinomials are extremely important. See how quickly and accurately you can factorise the following trinomials.

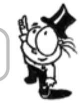
1. $x^2 - 5x - 6$ = ( )( )
2. $x^2 + 5x - 6$ = ( )( )
3. $x^2 - 5x + 6$ = ( )( )
4. $x^2 + 5x + 6$ = ( )( )
5. $x^2 + 10x - 24$ = ( )( )
6. $x^2 - 10x - 24$ = ( )( )
7. $x^2 + 10x + 24$ = ( )( )
8. $x^2 - 10x + 24$ = ( )( )
9. $8x^2 + 10x + 3$ = ( )( )
10. $8x^2 - 10x + 3$ = ( )( )
11. $8x^2 + 10x - 3$ = ( )( )
12. $8x^2 - 10x - 3$ = ( )( )

Answers

1. $(x - 6)(x + 1)$	2. $(x + 6)(x - 1)$	3. $(x - 2)(x - 3)$	4. $(x + 2)(x + 3)$
5. $(x + 12)(x - 2)$	6. $(x - 12)(x + 2)$	7. $(x + 6)(x + 4)$	8. $(x - 6)(x - 4)$
9. $(4x + 3)(2x + 1)$	10. $(4x - 3)(2x - 1)$	11. $(4x - 1)(2x + 3)$	12. $(4x + 1)(2x - 3)$

▶ Factorisation

Note: Multiplying and factorising are reverse processes



The Five Ways to Factorise

▶ **Common Factor** . . . *always look out for this! – no matter how many terms*

e.g. $2x^3 + 10x = 2x(x^2 + 5)$

▶ **The Difference between Two Squares** . . . *appropriate when there are 2 terms*

e.g. $4x^2 - 16 = 4(x^2 - 4) = 4(x + 2)(x - 2)$

▶ **Trinomial** . . . *appropriate when there are 3 terms*

e.g. $x^2 - 7x + 12 = (x - 3)(x - 4)$

▶ **Grouping** . . . *appropriate when there are 4 (or 5) terms*

e.g. $3x^2 - 12 - 4xy + 8y$
$= 3(x^2 - 4) - 4y(x - 2)$
$= 3(x + 2)(x - 2) - 4y(x - 2)$
$= (x - 2)[3(x + 2) - 4y]$
$= (x - 2)(3x + 6 - 4y)$

▶ **Sum & Difference of Cubes** . . . *appropriate when there are 2 terms*

e.g. $8x^3 + 27y^3 = (2x + 3y)(4x^2 - 6xy + 9y^2)$
$x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1)$

The Most Fundamental Algebra

– PASTE THIS ON YOUR WALL –

$(x + y)^2 = (x + y)(x + y)$
$= x^2 + xy + xy + y^2$
$= x^2 + 2xy + y^2$

$(x - y)^2 = (x - y)(x - y)$
$= x^2 - xy - xy + y^2$
$= x^2 - 2xy + y^2$

The same signs in the brackets cause a 'doubling up' of the middle term.

whereas $(x + y)(x - y) = x^2 - xy + xy - y^2$
$= x^2 - y^2$

The different signs in the brackets make the middle terms fall away.

Know these products!!!

And backwards . . .
$x^2 + 2xy + y^2 = (x + y)^2$
$x^2 - 2xy + y^2 = (x - y)^2$
and $x^2 - y^2 = (x + y)(x - y)$

NOW, THE NEXT 'LEVEL':

$(x + y)^3 = (x + y)(x + y)(x + y)$
$= (x + y)(x + y)^2$
$= (x + y)(x^2 + 2xy + y^2)$ ◄
$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$
$= x^3 + 3x^2y + 3xy^2 + y^3$

$(x - y)^3 = (x - y)(x - y)(x - y)$
$= (x - y)(x - y)^2$
$= (x - y)(x^2 - 2xy + y^2)$ ◄
$= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3$
$= x^3 - 3x^2y + 3xy^2 - y^3$

So, what product will give us an answer of just $x^3 + y^3$ or $x^3 - y^3$, with NO MIDDLE TERMS?

We know that **a linear** × **a quadratic = a cubic!**

So: $(x + y)(\ ?\ \ ?\ \ ?\ ) = x^3 + y^3$
$\therefore (x + y)(x^2\ ?\ xy + y^2) = x^3 + y^3$
$\therefore (x + y)(x^2 - xy + y^2) = x^3 + y^3$

& $(x - y)(\ ?\ \ ?\ \ ?\ ) = x^3 - y^3$
$\therefore (x - y)(x^2\ ?\ xy + y^2) = x^3 - y^3$
$\therefore (x - y)(x^2 + xy + y^2) = x^3 - y^3$

& backwards . . .
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ **SUM OF CUBES**
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ **DIFF. OF CUBES**

Algebraic Fractions

the ANSWER series... your key to exam success

N

1

Signs of a fraction

A fraction has 3 signs: $\pm \dfrac{\pm}{\pm}$

Very important!

The following fractions are EQUAL to each other:

$-\dfrac{8}{-4}$ $+\dfrac{-8}{-4}$ $-\dfrac{-8}{4}$ $+\dfrac{+8}{+4}$ . . . **they all equal + 2**

You can change any 2 signs to keep the value the same.

e.g. $-\dfrac{2-a}{5} = \dfrac{a-2}{5}$; $+\dfrac{x-y}{y-x} = -\dfrac{x-y}{x-y} = -1$

'**=**' means 'has the same value as'

Simplifying fractions

$\dfrac{a+b}{a}$ cannot be simplified

never cancel terms

$\dfrac{a^2 + ab}{a} = \dfrac{\cancel{a}(a+b)}{\cancel{a}} = a + b$

only cancel factors

Note: A fraction $\dfrac{a}{b}$ is only defined if $b \neq 0$.

Multiplying and dividing fractions

$\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$; $a \times \dfrac{b}{c} = \dfrac{ab}{c}$

$\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c}$; $\dfrac{2\frac{1}{4}}{1\frac{1}{3}} \times \dfrac{12}{12} = \dfrac{27}{16}$

Equivalent fractions:

$\dfrac{a}{b} = \dfrac{2a}{2b} = \dfrac{3a}{3b}$, etc.

$\therefore \dfrac{1}{4} = \dfrac{3}{12}$; $\dfrac{1}{3} = \dfrac{4}{12}$

(see examples below)

Adding and subtracting fractions

In order to add or subtract fractions, they need to be written over the l.c.m.

e.g.

$\dfrac{1}{4} + \dfrac{1}{3}$
$= \dfrac{3+4}{12}$
$= \dfrac{7}{12}$ ◄

$\dfrac{5}{4x} + \dfrac{3}{x} - \dfrac{2}{5}$
$= \dfrac{25 + 60 - 8x}{20x}$
$= \dfrac{85 - 8x}{20x}$ ◄

$\dfrac{3a}{4} - \dfrac{2a-3}{3}$
$= \dfrac{9a - 4(2a-3)}{12}$
$= \dfrac{9a - 8a + 12}{12}$
$= \dfrac{a+12}{12}$ ◄

MODULE 1: NUMBERS & FUNDAMENTAL CONCEPTS

Algebraic Expressions and Equations

It is extremely important to differentiate between expressions and equations. Observe the following examples.

Expressions

- Simplify: $\frac{x}{2} + 3$

$\frac{x}{2} + 3$

$= \frac{x+6}{2}$ ➤

Do not multiply.

- Multiply: $(x-2)(x+5)$

$(x-2)(x+5)$ . . . *factors*

$= x^2 - 2x + 5x - 10$

$= x^2 + 3x - 10$ . . . *terms*

↑ Note the = signs.

The value of the expression must not change.

- Factorise: $x^2 - 9$

$x^2 - 9$ . . . *terms*

$= (x+3)(x-3)$ . . . *factors*

Note the equal signs (=) down the left.

- Factorise: $-2x^2 + 14x - 24$

$-2x^2 + 14x - 24$ . . . *a trinomial*

$= -2(x^2 - 7x + 12)$

$= \mathbf{-2}(x-4)(x-3)$

Keep the value; so, keep the **-2**

Equations

- Solve for x: $\frac{x}{2} + 3 = 0$

$\frac{x}{2} + 3 = 0$

$\times 2)$ $\therefore x + 6 = 0$

$\therefore x = -6$ . . . *a solution*

Do multiply.

- Solve for x: $(x-2)(x+5) = 0$

Don't multiply. You need the factors for a zero product:

$(x-2)(x+5) = 0$

$\therefore x - 2 = 0$ or $x + 5 = 0$

$\therefore x = 2$ $\therefore x = -5$

logic

↑ Note the $\therefore$ signs.

- Solve for x: $x^2 - 9 = 0$

Method 1: $(x+3)(x-3) = 0$

$\therefore x + 3 = 0$ or $x - 3 = 0$

$\therefore x = -3$ $\therefore x = 3$

Method 2:

$x^2 - 9 = 0$ ➡ $x^2 = 9$

$\therefore x = \pm 3$

- Solve for x: $-2x^2 + 14x - 24 = 0$

$-2x^2 + 14x - 24 = 0$

$\div$ **(-2)** $\therefore x^2 - 7x + 12 = 0$

$\therefore (x-4)(x-3) = 0$

$\therefore x = 4$ or $x = 3$

Logic allows you to **DIVIDE** both sides of the equation by **-2**.

- Factorise: $3x^2 - 6x$

$3x^2 - 6x$ . . . *terms*

$= \mathbf{3x}(x-2)$ . . . *factors*

The expression was transformed by taking out a **common factor**, and **keeping it**!

- Solve for x: $3x^2 - 6x = 0$

$3x^2 - 6x = 0$

Divide only **by 3**, not $3x$

$\div$ **3)** $\therefore x^2 - 2x = 0$

$\therefore x(x-2) = 0$

$\therefore x = 0$ **or** $x - 2 = 0$

$\therefore x = 2$

If we had divided by x, we would've lost this solution!

- Evaluate: $x^2 - x - 6$

(a) If $x = -3$ & (b) If $x = -2$

(a) If $x = -3$: $x^2 - x - 6$

$= (-3)^2 - (-3) - 6$

$= 9 + 3 - 6$

$= 6$ ➤

(b) If $x = -2$: $x^2 - x - 6$

$= (-2)^2 - (-2) - 6$

$= 4 + 2 - 6$

$= 0$ ➤

Here we are finding the value of the expression for various values of x

- Given the equation: $x^2 - x - 6 = 0$

(a) Is -3 a root? (b) Is -2 a root?

(a) If $x = -3$: $x^2 - x - 6 \neq 0$ (*see lhs*)

$\therefore$ No, it is not a root

(b) If $x = -2$: $x^2 - x - 6$ does $= 0$ (*see lhs*)

$\therefore$ Yes, it is a root

Here we are testing the truth of the statement that says:

$x^2 + x - 6$ must $= 0$

A root is a value of x that makes this statement true.

Is there another value of x which would make the expression $x^2 - x - 6$ have a value of 0?

- Solve the equation $x^2 - x - 6 = 0$:

$\therefore (x-3)(x+2) = 0$

$\therefore x = 3$ or $x = -2$

So, which is the 'other root'? Why?

Check the root $x = 3$, i.e. check that $x = 3$ makes the equation true:

When $x = 3$: $x^2 - x - 6$ **the expression**

$= 3^2 - 3 - 6$

$= 9 - 9$

$= 0$ **the equation**

$\therefore$ **The statement**: $x^2 - x - 6$ **= 0** is true when $x = 3$

$\therefore$ 3 *is* the 'other root'

Algebraic Equations

Linear equations

Solve the following equations for x by inspection:

1. $x + 3 = 7$ 2. $3 - x = 5$ 3. $3x = 1$
4. $5x = 0$ 5. $x + 2 = x$ 6. $2(x + 5) = 2x + 10$

Answers

1. $x = 4$ 2. $x = -2$ 3. $x = \frac{1}{3}$
4. $x = 0$ 5. no solution 6. true for all x

Linear inequalities

Solve for x and show the solutions on a number line:

1. $x + 3 \leq 7$ 2. $2x + 5 > -1$ 3. $-5x \geq x - 18$
4. $-2 \leq 2x + 4 < 3$ 5. $2x < 6$ or $-x < -5$

Answers

1. $x \leq 4$

2. $2x > -6$

$\therefore\ x > -3$

3. $-6x \geq -18$

$\div (-6)\quad \therefore\ x \leq 3$

4. Subtract 4: $\therefore\ -6 \leq 2x < -1$

Divide by 2: $\therefore\ -3 \leq x < -\frac{1}{2}$

Number line:

5. $x < 3$ or $x > 5$

Quadratic equations

Solve for x and describe the nature of the roots (type of number). Say whether there are 2 roots, 1 root or no roots.

1. $x^2 + x - 6 = 0$ 2. $x(x + 4) = 0$ 3. $x^2 = 7$
4. $(x - 3)^2 = 0$ 5. $x^2 = -25$

Answers

1. $x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$\therefore\ x + 3 = 0$ or $x - 2 = 0$

$\therefore\ x = -3$ or $x = 2$

> Remember the **principle** of solving quadratic equations:
> **If a × b = 0, then . . .**
> **either a = 0 or b = 0 (or both)**
> This is referred to as a **zero product**.

The nature of the roots: Real, rational, integral. There are 2 roots.

2. $x(x + 4) = 0$

$\therefore\ x = 0$ or $x + 4 = 0$

$\therefore\ x = -4$

The nature of the roots: Real, rational and integral. There are 2 roots.

3. $x^2 = 7$

$\therefore\ x = \pm\sqrt{7}$

> $+\sqrt{7}$ and $-\sqrt{7}$ are additive inverses:
> $(+\sqrt{7}) + (-\sqrt{7}) = 0$

The nature of the roots: Real and irrational. There are 2 roots.

4. $(x - 3)^2 = 0$ Note: perfect square = 0

$\therefore\ x - 3 = 0 \quad \therefore\ x = 3$

The nature of the roots: Real, rational and integral. There is only 1 root.

5. $x^2 = -25$

$\therefore\ x = \pm\sqrt{-25}$

The nature of the roots: imaginary. There are no roots.

EXERCISE 1.2: Equations

Answers on page A1.2

1. Solve for x in each of the following equations and then describe the 'nature of the roots'. Choose whichever of the following words are applicable when describing the roots: real, imaginary, rational, irrational and integral.

1.1 $2x + 18 = 10$ 1.2 $\frac{2}{3}x = 1$

2. Solve for x in each of the following quadratic equations and then describe the nature of the roots (real, non-real, rational, irrational). Mention the number of roots.

2.1 $x^2 - 9 = 0$ 2.2 $(3x - 2)(x + 1) = 0$ 2.3 $x^2 + 8x + 16 = 0$
2.4 $x^2 - 7 = 0$ 2.5 $x(2x + 1) = 0$ 2.6 $x^2 + 9 = 0$

Function Notation and Algebraic Expressions

We use **f(x)** instead of **y** when referring to a straight line graph,

e.g. instead of **y** $= 2x + 1$, we write **f(x)** $= 2x + 1$.

- Consider the graph $y = x^2 - x - 6$ and use the symbol $f(x)$ instead of y:
  $f(x) = x^2 - x - 6$

- Determine f(-1), i.e. the value of $f(x)$ when $x = -1$:
  $f(-1) = (-1)^2 - (-1) - 6 = 1 + 1 - 6 = -4$

- Without substituting, write down the value of $f(-2)$ and $f(3)$:
  $f(-2)$ and $f(3)$ both equal 0.
  Reason: We proved that -2 & 3 were the roots of the equation $x^2 - x - 6 = 0$.
  (See page 1.6)

Algebra knowledge and confidence is crucial as it appears throughout all sections of Maths.

These notes include excerpts from various **Answer Series** publications.

We hope you find them useful as you build your strong foundation.

We wish you all the best.

The Answer Series Team

NOTES

the ANSWER series... your key to exam success

Module 2: EXPONENTS AND SURDS

EXPONENTS

The definition of a power, a^m

m is the 'instruction' for how many times a must $\times$ by itself.

$a^m = a \times a \times a \ldots$ for m factors of a, $m \in \mathbb{N}$; **a** is the base and **m** is the exponent

e.g. the power → $\mathbf{64 = 2^6}$ ← the exponent, ← the base $\ldots\ 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$

The Laws of Exponents

The multiplication of powers

1. $a^m \times a^n = a^{m+n}$

e.g. $x^3 \times x^2 = x \times x \times x \times x \times x = x^5$;

$2^x \times 2^y = 2^{x+y}$; $s^p \times s^q = s^{p+q}$

When we multiply powers with the same bases, we add the exponents.

The division of powers

2. $\dfrac{a^m}{a^n} = a^{m-n}$

e.g. $\dfrac{x^5}{x^2} = \dfrac{x \times x \times x \times \cancel{x} \times \cancel{x}}{\cancel{x} \times \cancel{x}} = x^3$;

$\dfrac{2^x}{2^y} = 2^{x-y}$; $\dfrac{s^p}{s^q} = s^{p-q}$

When we divide powers with the same bases, we subtract the exponents.

The power of a power

3. $(a^m)^n = a^{mn}$

e.g. $(x^2)^3 = x \times x \times x \times x \times x \times x = x^6$;

$(5^a)^b = 5^{ab}$; $(s^p)^q = s^{pq}$

In equations: $x^3 = 8$

$\therefore\ x = \sqrt[3]{8} = \mathbf{2}$

When we find the power of a power, we multiply the exponents.

The root of a power

4. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

e.g. $\sqrt{x^6} = x^{\frac{6}{2}} = x^3$;

$\because\ \sqrt{x^6} = \sqrt{x \times x \times x \times x \times x \times x} = x^3$

$\sqrt[3]{2^{12}} = 2^{\frac{12}{3}} = 2^4$; $\sqrt[a]{5^b} = 5^{\frac{b}{a}}$

In equations: $\sqrt[3]{x} = 2$

$\therefore\ x = 2^3 = \mathbf{8}$

When we find the root of a power, we divide the exponents.

The power of a product

5. $(ab)^n = a^n b^n$

e.g. $(x^2y^3)^2 = (x^2)^2(y^3)^2 = x^4y^6$;

$(2x^2)^5 = (2)^5(x^2)^5 = 32x^{10}$

The exponent of the product is the exponent of each factor.

The power of a quotient

6. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, $b \neq 0$

e.g. $\left(\dfrac{p}{q}\right)^6 = \dfrac{p^6}{q^6}$; $\left(\dfrac{-2}{x}\right)^5 = \dfrac{(-2)^5}{(x)^5} = \dfrac{-32}{x^5}$

$\left(\dfrac{3x^3}{y^2}\right)^3 = \dfrac{(3)^3(x^3)^3}{(y^2)^3} = \dfrac{27x^9}{y^6}$

The exponent of the quotient is the exponent of the factors in the numerator and the denominator.

Prime factors

$6 = 2 \times 3$; $12 = 2^2 \times 3$; $18 = 2 \times 3^2$; $54 = 2 \times 3^3$; $50 = 2 \times 5^2$

Further examples, applying laws 1 to 6

1. ♦ $x^5 . x^7 = x^{12}$ ♦ $y^3 . y = y^4$ ♦ $x^a . x^b = x^{a+b}$ ♦ $a^{2b} . a^{4b} = a^{6b}$

   ♦ $y^{2n} . y^5 = y^{2n+5}$ ♦ $z^{2x} . z^{3x} = z^{5x}$ ♦ $x^2x^3x^4 = x^9$ ♦ $(x+y)^3(x+y)^4 = (x+y)^7$

2. ♦ $9^7 \div 9^5 = 9^2$ ♦ $y^8 \div y^3 = y^5$ ♦ $a^{3n} \div a^n = a^{2n}$ ♦ $x^{n+1} \div x^n = x$

   ♦ $y^{3n+4} \div y^{n+1} = y^{2n+3}$ ♦ $(x+y)^5 \div (x+y)^2 = (x+y)^3$

3. ♦ $(y^3)^5 = y^{15}$ ♦ $(2^4)^3 = 2^{12}$ ♦ $(3^x)^x = 3^{x^2}$ ♦ $(3^x)^2 = 3^{2x}$

4. ♦ $\sqrt{x^8} = x^{\frac{8}{2}} = x^4$ ♦ $\sqrt[3]{y^{15}} = y^5$ ♦ $\sqrt[4]{r^4} = r$ ♦ $\sqrt[3]{a^3b^6} = ab^2$

5. ♦ $(ab)^4 = a^4b^4$ ♦ $(x^2y)^7 = x^{14}y^7$ ♦ $(-5a)^3 = -125a^3$

   ♦ $(-2x^2y^3)^3 = -8x^6y^9$ ♦ $\left(\dfrac{1}{2}x^2\right)^4 = \dfrac{1}{16}x^8$ ♦ $(-3y^3)^2 = 9y^6$

6. ♦ $\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$ ♦ $\left(\dfrac{2x^2}{3y^3}\right)^3 = \dfrac{8x^6}{27y^9}$ ♦ $\left(-\dfrac{3c}{d^3}\right)^4 = \dfrac{81c^4}{d^{12}}$

The laws reversed

1. $\mathbf{a^{m+n} = a^m . a^n}$

e.g. $2^{x+3} = 2^x.2^3$; $2^{x-1} = 2^x.2^{-1}$

2. $\mathbf{a^{m-n} = \frac{a^m}{a^n}}$

e.g. $3^{2-a} = \frac{3^2}{3^a}$

3. $\mathbf{a^{mn} = (a^m)^n}$ **or** $\mathbf{(a^n)^m}$

e.g. $3^{2x} = (3^2)^x$ or $(3^x)^2$;

$x^{\frac{1}{2}} = \left(x^{\frac{1}{4}}\right)^2$; $x^{\frac{2}{3}} = \left(x^{\frac{1}{3}}\right)^2$

4. $\mathbf{\frac{a^m}{a^n} = \sqrt[n]{a^m}}$ **or** $\mathbf{(\sqrt[n]{a})^m}$

e.g. $8^{\frac{2}{3}} = \sqrt[3]{8^2}$ or $8^{\frac{2}{3}} = (\sqrt[3]{8})^2$

$= \sqrt[3]{64}$ $\quad = 2^2$

$= 4$ $\quad = 4$

5. $\mathbf{a^n b^n = (ab)^n}$

e.g. $5^x.3^x = (5.3)^x = 15^x$

6. $\mathbf{\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n}$

e.g. $\frac{6^n}{2^n} = \left(\frac{6}{2}\right)^n = 3^n$

Expressions to be simplified

Worked Example 1

1. $64^{\frac{2}{3}}$
2. $3^{m+2} - 8.3^m - 3^m$
3. $(\sqrt[3]{a}.\sqrt{b})^6$
4. $\sqrt{9^3}$
5. $\frac{2x^3}{(2x)^3}$
6. $\left(\frac{2ab^3}{c^2}\right)^4$
7. $\sqrt{a^{3n}}.\left(a^{\frac{n}{4}}\right)^2$

Answers

1. $\left(64^{\frac{1}{3}}\right)^2 = 4^2 = \mathbf{16}$ ◄

or: $(2^6)^{\frac{2}{3}} = 2^4 = 16$

2. $3^m.3^2 - 8.3^m - 3^m$

$= 3^m(9 - 8 - 1)$

$= \mathbf{0}$ ◄

3. $\left(a^{\frac{1}{3}}.b^{\frac{1}{2}}\right)^6$

$= \left(a^{\frac{1}{3}}\right)^6.\left(b^{\frac{1}{2}}\right)^6$

$= \mathbf{a^2.b^3}$ ◄

4. $\sqrt{9^3} = (\sqrt{9})^3 = 3^3 = \mathbf{27}$ ◄

or: $9^{\frac{3}{2}} = (3^2)^{\frac{3}{2}} = 3^3 = 27$

5. $\frac{2x^3}{8x^3}$

$= \mathbf{\frac{1}{4}}$ ◄

6. $\frac{(2ab^3)^4}{(c^2)^4}$

$= \mathbf{\frac{16a^4b^{12}}{c^8}}$ ◄

7. $a^{\frac{3n}{2}}.a^{\frac{n}{2}}$

$= a^{\frac{3n+n}{2}}$

$= \mathbf{a^{2n}}$ ◄

Factor type expressions

8. $\frac{8^{n+3}.32^{-n-1}.6^{2n}}{9^n}$

9. $\frac{25^n.36^{n+1}}{81.30^{2n}}$

Answers

8.

$= \frac{2^{3n+9}.2^{-5n-5}.2^{2n}.\cancel{3^{2n}}}{\cancel{3^{2n}}}$

$= 2^{3n+9-5n-5+2n}$

$= 2^4$

$= \mathbf{16}$ ◄

Use prime factors and the laws.

9. $\frac{(5^2)^n.(2^2.3^2)^{n+1}}{3^4.(2.3.5)^{2n}}$

$= \frac{\cancel{5^{2n}}.2^{2n+2}.3^{2n+2}}{3^4.2^{2n}.3^{2n}.\cancel{5^{2n}}}$

$= 2^{2n+2-2n}.3^{2n+2-4-2n}$

$= 2^2.3^{-2}$

$= \mathbf{\frac{4}{9}}$ ◄

Remember: $3^{-2} = \frac{1}{3^2}$

Term type expressions

10. $\frac{2^{x+3} - 2.2^x}{2^x}$

11. $\frac{6^{2x-1} - 36^x}{6^{2x}}$

Answers

10. $\frac{2^x.2^3 - 2.2^x}{2^x}$

$= \frac{\cancel{2^x}(8 - 2)}{\cancel{2^x}} = \mathbf{6}$

Factorise, e.g. common factor.

11. $\frac{\frac{6^{2x}}{6} - 6^{2x}}{6^{2x}} = \frac{\cancel{6^{2x}}\left(\frac{1}{6} - 1\right)}{\cancel{6^{2x}}} = \mathbf{-\frac{5}{6}}$ ◄

12. Express the following as a power of 2 : (a) $(\sqrt{8})^6$ (b) $\sqrt{8^6}$

13. Determine the value of $\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right) - (x - y)^{\frac{1}{2}}$ if $x = 100$ and $y = 64$.

14. Write down the values of a, b and c : $64^a = 8$; $64^b = 4$; $64^c = 2$

Answers

12. (a) $(\sqrt{8})^6 = \left[(2^3)^{\frac{1}{2}}\right]^6$

$= \mathbf{2^9}$ ◄

(b) $\sqrt{8^6} = \left[(2^3)^6\right]^{\frac{1}{2}}$

$= \mathbf{2^9}$ ◄

13. $\left(100^{\frac{1}{2}} - 64^{\frac{1}{2}}\right) - (100 - 64)^{\frac{1}{2}}$

$= (10 - 8) - 36^{\frac{1}{2}}$

$= 2 - 6$

$= \mathbf{-4}$ ◄

14. $\mathbf{a = \frac{1}{2}}$ ◄

$\mathbf{b = \frac{1}{3}}$ ◄

$\mathbf{c = \frac{1}{6}}$ ◄

Equations to be solved

In some equations, x is in the base ; in others, x is in the exponent.

Worked Example 2

x in the base

1. $x^{\frac{1}{3}} = 4$
2. $3x^{\frac{1}{5}} = 6$
3. $x^{\frac{3}{2}} = 8$
4. $2x^{\frac{3}{4}} = 54$
5. $2x^2 = 18$
6. $x^{\frac{2}{3}} = 25$
7. $16x^{\frac{2}{3}} = 1$
8. $16x^4 + 1 = 0$
9. $x - \sqrt{x} - 6 = 0$

Answers

> **NB:** Powers and roots are opposite operations.
> In equations: If you have the power, take the root.
> If you have the root, raise to the power.

1. $\sqrt[3]{x} = 4$
$\therefore\ x = 4^3$
$= \mathbf{64}$ ◀
or: $\left(x^{\frac{1}{3}}\right)^3 = 4^3$

2. $x^{\frac{1}{5}} = 2$
$\therefore\ \sqrt[5]{x} = 2$
$\therefore\ x = 2^5$
$\therefore\ \boldsymbol{x = 32}$ ◀

3. $\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}$
$\therefore\ x^{\frac{1}{2}} = 2$ . . . $\sqrt[3]{\ }$
$\therefore\ \sqrt{x} = 2$
$\therefore\ x = 2^2$ . . . $(\ )^2$
$= \mathbf{4}$ ◀

4. $x^{\frac{3}{4}} = 27$
$\therefore\ x^{\frac{1}{4}} = 3$ . . . $\sqrt[3]{\ }$
$\therefore\ \boldsymbol{x = 81}$ ◀ . . . $(\ )^4$

5. $x^2 = 9$
$\therefore\ \boldsymbol{x = \pm 3}$ ◀ . . . $\sqrt{\ }$

Because the given power is **even**, *there are 2 possible answers.*

6. $x^{\frac{2}{3}} = 25$
$\therefore\ x^{\frac{1}{3}} = \pm 5$ . . . $\sqrt{\ }$
$\therefore\ \boldsymbol{x = \pm 125}$ ◀ . . . $(\ )^3$

7. $x^{\frac{2}{3}} = \frac{1}{16}$
$\therefore\ x^{\frac{1}{3}} = \pm\frac{1}{4}$ . . . $\sqrt{\ }$
$\therefore\ x = \left(\pm\frac{1}{4}\right)^3$ . . . $(\ )^3$
$\therefore\ \boldsymbol{x = \pm\frac{1}{64}}$ ◀

8. $16x^4 = -1$
$\therefore\ \boldsymbol{x^4 = -\frac{1}{16}}$ ◀
No solution.

An even power cannot be negative.

9. **NB:** $\sqrt{x}\,.\,\sqrt{x} = x$
$\therefore\ (\sqrt{x})^2 - \sqrt{x} - 6 = 0$
$\therefore\ (\sqrt{x} - 3)(\sqrt{x} + 2) = 0$
$\therefore\ \sqrt{x} = 3$
$\therefore\ \boldsymbol{x = 9}$ ◀

$\sqrt{x} \neq -2$
$\because\ \sqrt{x} \geq 0$

x in the exponent

- The logic we use: **If $a^m = a^n$, then m = n**
  i.e. If powers are equal and their bases are the same, then their exponents are equal.
- Apply the laws when necessary.

10. $9^{2x-3} = 27$
11. $25^x = \sqrt{125}$
12. $7^{\frac{1}{x}} = 49$
13. $\dfrac{8^x}{2^{x-1}} = 32$
14. $3.2^x = 24$
15. $p^x.p^{x+1} = p^9$

- And, when there is more than 1 term on one side, it is necessary to factorise.

16. $5^{x+1} + 5^x = 30$
17. $3^{x+1} - 3^x = 54$
18. (a) Solve the equation: $a^2 - a - 12 = 0$, and **hence**:
(b) solve the equation: $2^{2x} - 2^x - 12 = 0$
19. (a) Solve the equation: $a + \frac{9}{a} = 10$, and **hence**:
(b) solve the equation: $3^x + 3^{2-x} = 10$

Answers

10. $(3^2)^{2x-3} = 3^3$
$\therefore\ 3^{4x-6} = 3^3$
$\therefore\ 4x - 6 = 3$
$\therefore\ 4x = 9$
$\therefore\ \boldsymbol{x = \frac{9}{4}}$ ◀

11. $(5^2)^x = (5^3)^{\frac{1}{2}}$
$\therefore\ 5^{2x} = 5^{\frac{3}{2}}$
$\therefore\ 2x = \frac{3}{2}$
$\therefore\ \boldsymbol{x = \frac{3}{4}}$ ◀

12. $7^{\frac{1}{x}} = 7^2$
$\therefore\ \frac{1}{x} = 2$
$\therefore\ \boldsymbol{x = \frac{1}{2}}$ ◀

13. $\dfrac{2^{3x}}{2^{x-1}} = 2^5$
$\therefore\ 2^{3x-(x-1)} = 2^5$
$\therefore\ 2x + 1 = 5$
$\therefore\ 2x = 4$
$\therefore\ \boldsymbol{x = 2}$ ◀

14. Divide by 3:
$\therefore\ 2^x = 8$
$\therefore\ \boldsymbol{x = 3}$ ◀
NB: $3.2^x \neq (3.2)^x$

15. $p^{x+x+1} = p^9$
$\therefore\ 2x + 1 = 9$
$\therefore\ 2x = 8$
$\therefore\ \boldsymbol{x = 4}$ ◀

16. $5^x.5 + 5^x = 30$
$\therefore\ 5^x(5 + 1) = 30$
$\therefore\ 5^x = 5$
$\therefore\ \boldsymbol{x = 1}$ ◀

17. $3^x.3 - 3^x = 54$
$\therefore\ 3^x(3 - 1) = 54$
$\therefore\ 3^x = 27$
$\therefore\ \boldsymbol{x = 3}$ ◀

Answers

Compare

18. (a) $a^2 - a - 12 = 0$

$\therefore (a-4)(a+3) = 0$

$\therefore$ **a = 4 or -3**

(b) $(2^x)^2 - (2^x) - 12 = 0$

$\therefore (2^x - 4)(2^x + 3) = 0$

$\therefore 2^x = 4$

$\therefore$ $\boldsymbol{x = 2}$

$2^x \neq -3$ because $2^x > 0$ for all x

Compare

19. (a) $a + \frac{9}{a} = 10$

$\times a)$ $\therefore a^2 + 9 = 10a$

$\therefore a^2 - 10a + 9 = 0$

$\therefore (a-1)(a-9) = 0$

$\therefore$ **a = 1 or 9**

(b) $3^x + \frac{3^2}{3^x} = 10$

$\times 3^x)$ $\therefore (3^x)^2 + 9 = 10.(3^x)$

$\therefore (3^x)^2 - 10.(3^x) + 9 = 0$

$\therefore (3^x - 1)(3^x - 9) = 0$

$\therefore 3^x = 1$ or $3^x = 9$

$\therefore \boldsymbol{x = 0}$ $\therefore \boldsymbol{x = 2}$

Rational Exponents

The laws of exponents have been explored for natural numbers. These laws can be applied for all rational numbers.

Consider the meaning of exponents which are

- zero
- negative
- fractional

The meaning of a zero exponent

What does $\mathbf{a^0}$ mean?

We know: $\frac{a^3}{a^3} = \mathbf{1}$

& **Law 2:** $\frac{a^3}{a^3} = a^{3-3} = \mathbf{a^0}$

$\longrightarrow$ $\boxed{\therefore \mathbf{a^0 = 1}}$

Also, $\mathbf{a^0} \times \mathbf{a^3} = a^{0+3} = \mathbf{a^3}$ ➡ $a^0 = 1$ . . . *identity element*

e.g. $5^0 = 1$; $5.2^0 = 5 \times 1 = 5$; $(5 \times 2)^0 = 10^0 = 1$;

$2^{x-3} = 1$ ➡ $x = 3$ . . . *only* $2^{\mathbf{0}} = 1$;

$2^x = 5^x$ ➡ $x = 0$ . . . *no other way that a power of 2 = a power of 5*

The meaning of a negative exponent

What does $\mathbf{a^{-n}}$ mean?

We know: $\frac{2^4}{2^7} = \frac{\mathbf{1}}{\mathbf{2^3}}$

& **Law 2:** $\frac{2^4}{2^7} = 2^{4-7} = \mathbf{2^{-3}}$

$\longrightarrow$ $\boxed{\therefore \mathbf{a^{-n} = \frac{1}{a^n}} \text{ or } \left(\frac{\mathbf{1}}{\mathbf{a}}\right)^{\mathbf{n}}}$

Also, $2^{-3} \times \mathbf{2^3} = 2^{-3+3} = 2^0 = \mathbf{1}$ ➡ $2^{-3} = \frac{1}{2^3}$ . . . *multiplicative inverse*

e.g. $a^{-5} = \frac{1}{a^5}$; $\frac{1}{a^{-5}} = a^5$; $\frac{a^{-1}}{b^{-2}} = \frac{b^2}{a}$

$3^{-1} = \frac{1}{3}$; $\left(\frac{1}{5}\right)^{-1} = 5$; $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

The meaning of a fractional exponent

What does $\mathbf{a^{\frac{1}{q}}}$ mean?

We know: $\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2} \times 2} = x$ $\therefore x^{\frac{1}{2}} = \sqrt{x}$

& **Law 3:** $\left(x^{\frac{1}{3}}\right)^3 = x^{\frac{1}{3} \times 3} = x$ $\therefore x^{\frac{1}{3}} = \sqrt[3]{x}$

$\longrightarrow$ $\boxed{\therefore \mathbf{a^{\frac{1}{q}} = \sqrt[q]{a}}}$

e.g. $9^{\frac{1}{2}} = (3^2)^{\frac{1}{2}} = 3$; $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^{3 \times \frac{1}{3}} = 2^1$

$\therefore 9^{\frac{1}{2}}$ means $\sqrt{9}$ $\therefore 8^{\frac{1}{3}}$ means $\sqrt[3]{8}$

And $\mathbf{a^{\frac{p}{q}}}$? *p is the power ; q is the root ; either can be found 1st*

$8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = 64^{\frac{1}{3}} = 4$ . . . *the power 1st, then the root*

or, $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$ . . . *the root 1st, then the power*

$\therefore 8^{\frac{2}{3}} = \sqrt[3]{8^2}$ or $8^{\frac{2}{3}} = (\sqrt[3]{8})^2$

$\boxed{\therefore \mathbf{a^{\frac{p}{q}} = \sqrt[q]{a^p} \text{ or } (\sqrt[q]{a})^p}}$

e.g. Write in surd form:

$a^{\frac{2}{3}} = \sqrt[3]{a^2}$; $a^{\frac{1}{4}} = \sqrt[4]{a}$; $x^{\frac{b}{c}} = \sqrt[c]{x^b}$

Write with fractional exponents:

$\sqrt{p^3} = p^{\frac{3}{2}}$; $\sqrt[5]{a} = a^{\frac{1}{5}}$; $\sqrt{x^2} = x^{\frac{2}{2}} = x$

▶ Popular powers to know

Powers of 2

$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$

Powers of 3

$3^2 = 9$
$3^3 = 27$
$3^4 = 81$

Powers of 4

$4^2 = 16$
$4^3 = 64$

Powers of 5

$5^2 = 25$
$5^3 = 125$

Note: $\boxed{64 = 2^6 \quad \& \quad 64 = 4^3 \quad \& \quad 64 = 8^2}$

▶ Perfect Squares and Square Roots

$1^2 = 1 \times 1 = 1$	$\therefore \sqrt{1} = 1$	$8^2 = 8 \times 8 = 64$	$\therefore \sqrt{64} = 8$
$2^2 = 2 \times 2 = 4$	$\therefore \sqrt{4} = 2$	$9^2 = 9 \times 9 = 81$	$\therefore \sqrt{81} = 9$
$3^2 = 3 \times 3 = 9$	$\therefore \sqrt{9} = 3$	$10^2 = 10 \times 10 = 100$	$\therefore \sqrt{100} = 10$
$4^2 = 4 \times 4 = 16$	$\therefore \sqrt{16} = 4$	$11^2 = 11 \times 11 = 121$	$\therefore \sqrt{121} = 11$
$5^2 = 5 \times 5 = 25$	$\therefore \sqrt{25} = 5$	$12^2 = 12 \times 12 = 144$	$\therefore \sqrt{144} = 12$
$6^2 = 6 \times 6 = 36$	$\therefore \sqrt{36} = 6$	$13^2 = 13 \times 13 = 169$	$\therefore \sqrt{169} = 13$
$7^2 = 7 \times 7 = 49$	$\therefore \sqrt{49} = 7$	$14^2 = 14 \times 14 = 196$	$\therefore \sqrt{196} = 14$

EXERCISE 2.1: EXPONENTIAL EXPRESSIONS

Answers on page A2.1

Simplify the following expressions:

1. (a) 5^{-1} (b) 3.2^0 (c) $3^{-1}.2^0$ (d) $\left(\frac{1}{2}\right)^{-3}$ (e) $\left(\frac{1}{4}\right)^{-2}$ (f) $2^{-1} - 2^{-2}$
   (g) 8^{-1} (h) $8^{\frac{1}{3}}$ (i) $8^{\frac{2}{3}}$ (j) $8^{-\frac{2}{3}}$ (k) 16^{-1} (l) $16^{\frac{1}{2}}$
   (m) $16^{-\frac{1}{2}}$ (n) $16^{-\frac{3}{2}}$ (o) $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$ (p) $(-2)^{-3}$ (q) $(3^{-1} + 2^{-1})^{-1}$
   (r) $2a^{-1}$ (s) $(2a)^{-1}$ (t) $\frac{(3a)^{-2}}{3a^{-2}}$ (u) $\sqrt[4]{81^3}$ (v) $\sqrt[5]{32^2}$ (w) $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$
   (x) $(-64)^{\frac{2}{3}}$ (y) $\left(-\frac{2}{3}\right)^{-3}$ (z) $\sqrt[3]{(a+b)^6}$

2. (a) $\frac{2^0.3^{-2}}{5^{-1}}$ (b) $125^{\frac{2}{3}} + 16^{\frac{3}{4}}$ (c) $6^0 + \sqrt{2^4} + (0,5)^{-2}$
   (d) $\left(16^{\frac{1}{4}} + 32^{-\frac{2}{5}}\right)^{\frac{1}{2}}$ (e) $\left(-\frac{1}{64x^6}\right)^{-\frac{2}{3}}$ (f) $(9ab^2)^{\frac{1}{2}} \times 4a^{\frac{3}{2}}b^{-4} \div 6a^2b^{-2}$

3. (a) $\frac{4^{n+1}.8^{2n}}{16^{2n-1}}$ (b) $\frac{25^k \times 5^{k-1}}{5^{3k-1}}$ (c) $\frac{9^{n-1}.27^{3-2n}}{81^{2-n}}$
   (d) $\frac{8^{n-3}.32^{-n+1}.6^{2n}}{9^n}$ (e) $\frac{4^{n-3}.10^{n+2}}{8^{n-1}.5^{1+n}}$ (f) $\frac{12^{n+1}.27^{n-2}}{18^{2n-1}}$

4. (a) $\frac{3^{n+1} - 3^n}{3^{n-1}}$ (b) $\frac{2.3^x - 3^{x-1}}{5.3^x}$ (c) $\frac{2^{x+3} - 2.2^x}{2^{x+1}.8}$
   (d) $\frac{3^{x+1} - 3^{x-1}}{3.3^{x-2}}$ (e) $\frac{9^x + 3^{2x+1}}{18^x.2^{-x}}$ (f) $\frac{a^{x+3} + 2a^{x+2}}{a^x + 2a^{x-1}}$

5. (a) $\sqrt{\frac{15^x.3^x}{9^{x+1}.5^{x-2}}}$ (b) $\sqrt{\frac{2^{x+3} - 2^{x+1}}{2.2^x} + 1}$ (c) $\frac{9^{2a} - 3^{3a}}{27^a}$

6. Given: $A = \frac{3^n - 4}{6^n - 2^{n+2}}$. (a) Simplify A (b) Hence, determine $\sqrt[n]{A}$

EXERCISE 2.2: EXPONENTIAL EQUATIONS

Answers on page A2.2

Solve for x:

1. (a) $x^{-1} = 0,2$ (b) $x^{-2} = \frac{1}{9}$ (c) $x^{\frac{1}{3}} = 4$ (d) $x^{-\frac{1}{5}} = 2$
   (e) $x^{\frac{3}{4}} = 8$ (f) $2x^{\frac{3}{5}} = 54$ (g) $x^{\frac{2}{3}} = 16$ (h) $x^{-\frac{3}{2}} = 64$
   (i) $8x^{\frac{5}{4}} = 256$ (j) $2.5^x = 0,4$

2. (a) $x - 3x^{\frac{1}{2}} = 0$ (b) $x^{\frac{1}{2}} - x^{\frac{1}{4}} - 6 = 0$ (c) $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 12 = 0$ (d) $x - 5\sqrt{x} = 0$

3. (a) $13^x = 1$ (b) $49^{-x} = 7$ (c) $9^{2x-3} = 27$ (d) $\left(\frac{1}{2}\right)^{x-3} = 4$
   (e) $2^x = 0,125$ (f) $(0,2)^x = 25$ (g) $2^{-x} = 32$ (h) $4^x = \frac{1}{64}$
   (i) $3.2^x = 24$ (j) $3.9^{x+1} = 27^{-x}$ (k) $125^{3x-2} = 25^{4x+10}$ (l) $2^{x(x-3)} = 0,25$

4. (a) $3^{x+1} - 3^x = 54$ (b) $3.2^x + 2^{x-2} = 26$ (c) $2^{2x} + 2^x - 6 = 0$

5. (a) $16.\left(\frac{1}{2}\right)^{x-1} = 1$ (b) $2^x \times 3^{x+1} = 108$ (c) $3a^{\frac{2}{3}} = 48a^{-\frac{2}{3}}$

6. (a) $\frac{a^x.\sqrt[x]{a^2}}{a^{5-x}} = 1$ (b) $12^x \times 4 = 36 \times 4^x$ (c) $a^x - a^{3x^2} = 0$

7. Express as a power of 4: (a) 8 (b) $\sqrt{32}$

SURDS

The definition of a surd

Surds are irrational numbers and cannot be determined exactly.

e.g. $\sqrt{2}$; $\sqrt[3]{10}$; $\sqrt{27}$ . . . *all infinite non-recurring decimals*

▶ Restrictions

▸ If n is even: $\sqrt[n]{p}$ is only defined if $p \geq 0$.
e.g. $\sqrt{p}$, $\sqrt[4]{p}$, . . . are only defined if $p \geq 0$

> $\sqrt{-9}$ is imaginary, but, $\sqrt[3]{-8} = -2$

▸ $\sqrt{p}$ is read as $+\sqrt{p}$, by definition.
Even though $p^2 = 25$ ➡ $p = \pm 5$, $\sqrt{25} = +\sqrt{25} = +5$ only

▶ The 'root of a power': $\sqrt[q]{a^p} = a^{\frac{p}{q}}$ . . . *as introduced on page 2.1*

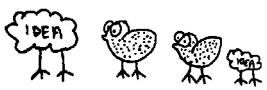
This law enables one to convert between exponents and surds.

▸ Squaring and taking the square root

If we square a number: $9^2 = 81$ and then take the square root: $\sqrt{81} = 9$,
we notice that the 2 operations cancel each other, and, could do so in any order:

$\therefore \sqrt{9^2} = 9$, but also $(\sqrt{9})^2 = 9$

$\therefore (\sqrt{2})^2 = 2$ [Also: $(\sqrt[3]{4})^3$ or $\sqrt[3]{4^3} = 4$]



▸ The order of taking the m^th^ power and the n^th^ root

Law 1: $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$ . . . **Proof:** $\sqrt[n]{a^m} = a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m$

e.g. $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$ & $(\sqrt[3]{8})^2 = 2^2 = 4$ $\therefore$ Choose whichever order is convenient.

▶ Multiplying and dividing surds

Law 2: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ . . . **Proof:** $\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}$

Law 3: $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ . . . **Proof:** $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}} = \sqrt[n]{\dfrac{a}{b}}$

e.g. $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \times 8} = \sqrt{16} = 4$; $(\sqrt{5})^2 = 5$; $\dfrac{\sqrt{27}}{\sqrt{3}} = \sqrt{\dfrac{27}{3}} = \sqrt{9} = 3$

Worked Example 1

Simplify:

(a) $(\sqrt{25})^2$ (b) $\sqrt{3^2}$ (c) $(\sqrt{5})^2$ (d) $\sqrt{2}\sqrt{3}$

(e) $\sqrt{12}\sqrt{3}$ (f) $\dfrac{\sqrt{20}}{\sqrt{5}}$ (g) $2\sqrt{3} \cdot 5\sqrt{3}$ (h) $\dfrac{\sqrt{x^9}}{\sqrt{x}}$

Answers

(a) **25** ◂ (b) **3** ◂ (c) **5** ◂ (d) $\sqrt{\mathbf{6}}$ ◂

(e) $\sqrt{36} = \mathbf{6}$ ◂ (f) $\sqrt{4} = \mathbf{2}$ ◂ (g) $10.3 = \mathbf{30}$ ◂ (h) $\sqrt{\dfrac{x^9}{x}} = \sqrt{x^8} = \boldsymbol{x^4}$ ◂

▶ Simplifying surds

Apply the reverse of law 2: $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

e.g. $\sqrt{8} = \sqrt{\mathbf{4} \times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$ $\sqrt[3]{54} = \sqrt[3]{\mathbf{27} \times 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3 \cdot \sqrt[3]{2}$

$\sqrt{48} = \sqrt{\mathbf{16} \times 3} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$ $\sqrt{x^5} = \sqrt{\mathbf{x^4} \cdot x} = \sqrt{x^2} \cdot \sqrt{x} = x \cdot \sqrt{x}$

▶ Adding and Subtracting surds

Only like surds can be added or subtracted.

e.g. $\sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$ . . . $x + 3x = 4x$

$4\sqrt{2} + \sqrt{3} - \sqrt{2} - 2\sqrt{3} = 3\sqrt{2} - \sqrt{3}$ . . . $4x + y - x - 2y = 3x - y$

Worked Example 2

Simplify: $(\sqrt{3} + \sqrt{12})^2$

Answer:

$(\sqrt{3} + \sqrt{4 \times 3})^2 = (\sqrt{3} + 2\sqrt{3})^2 = (3\sqrt{3})^2 = 9.3 = \mathbf{27}$ ◂

▶ Rationalising the denominator

The denominators in the following examples are irrational. Observe the techniques to rationalise the denominators, while preserving the value of the expressions:

- $\boxed{\dfrac{5}{\sqrt{2}}}$ $\dfrac{5}{\sqrt{2}} = \dfrac{5}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{5\sqrt{2}}{2}$
- $\boxed{\dfrac{2}{\sqrt{2}}}$ $\dfrac{\mathbf{2}}{\sqrt{\mathbf{2}}} = \dfrac{2}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{\mathbf{2}}$ just as $\dfrac{9}{3} = 3$
- $\boxed{\dfrac{1}{\sqrt{3} + \sqrt{2}}}$ $\dfrac{1}{\sqrt{3} + \sqrt{2}} = \dfrac{1}{\sqrt{3} + \sqrt{2}} \times \dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \dfrac{\sqrt{3} - \sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} = \dfrac{\sqrt{3} - \sqrt{2}}{3 - 2} = \sqrt{3} - \sqrt{2}$

Note the use of: $(a + b)(a - b) = a^2 - b^2$, the difference of squares.

Surd Equations

A surd equation is one that contains a surd. The surd needs to be on its own on one side of the equation. **Squaring** both sides of the equation then eliminates the radical sign $(\sqrt{\ })$, but also results in the need to **check** the solutions.

Checking the solutions of a surd equation

Remember: $\sqrt{x}$ is defined as $+\sqrt{x}$ $\quad \therefore \sqrt{x}$ can only be positive or zero.

Worked Examples

Compare these two equations while solving for x:

(1) $3 + \sqrt{x-1} = x$ $\qquad$ (2) $3 - \sqrt{x-1} = x$

Answers

Surd alone on one side (the line to be checked)

(1) $\sqrt{x-1} = x - 3$ ← $\qquad$ (2) $-\sqrt{x-1} = x - 3$ ←

Square both sides of the equation

$\therefore (\sqrt{x-1})^2 = (x-3)^2$ $\qquad$ $\therefore (-\sqrt{x-1})^2 = (x-3)^2$

$\therefore x - 1 = x^2 - 6x + 9$

$\therefore 0 = x^2 - 7x + 10$

$\therefore (x-2)(x-5) = 0$

$\therefore x = 2$ or $x = 5$

Now, **check** these answers in the line marked ← above

(1)

For $x = 2$: LHS $= \sqrt{2-1} = \sqrt{1} = 1$

RHS $= 2 - 3 = -1$

$\therefore$ **Reject** $x = 2$ . . . LHS $\neq$ RHS

For $x = 5$: LHS $= \sqrt{5-1} = \sqrt{4} = 2$

RHS $= 5 - 3 = 2$

$\therefore$ **Accept** $x = 5$ . . . LHS = RHS

(2)

For $x = 2$: LHS $= -\sqrt{2-1} = -\sqrt{1} = -1$

RHS $= 2 - 3 = -1$

$\therefore$ **Accept** $x = 2$ . . . LHS = RHS

For $x = 5$: LHS $= -\sqrt{5-1} = -\sqrt{4} = -2$

RHS $= 5 - 3 = 2$

$\therefore$ **Reject** $x = 5$ . . . LHS $\neq$ RHS

The solution

$\therefore$ **Only $x = 5$** $\qquad$ $\therefore$ **Only $x = 2$**

These 2 examples illustrate why 2 solutions appeared, but 1 had to be rejected in each case.

EXERCISE 2.3: SURDS

No calculator to be used. *Answers on page A2.4*

Simplify the following expressions:

1. (a) $\sqrt{21}\,.\,\sqrt{60}\,.\,\sqrt{35}$ $\qquad$ (b) $6\sqrt{45} \div 3\sqrt{5}$ $\qquad$ (c) $(\sqrt{2}+2)^2 - 2\sqrt{8}$

   (d) $(\sqrt{50} - \sqrt{162})^2$ $\qquad$ (e) $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$ $\qquad$ (f) $(2\sqrt{3}+\sqrt{7})(2\sqrt{3}-\sqrt{7})$

2. (a) $\dfrac{\sqrt{8}+\sqrt{8}}{\sqrt{2}\,.\,\sqrt{8}}$ $\qquad$ (b) $\dfrac{\sqrt{75}+\sqrt{48}}{\sqrt{12}}$ $\qquad$ (c) $\dfrac{\sqrt{27}-\sqrt{18}}{\sqrt{8}-\sqrt{12}}$

   (d) $\dfrac{\sqrt{98}-\sqrt{50}}{\sqrt{2}}$ $\qquad$ (e) $\sqrt{2+\sqrt{3}}\,.\,\sqrt{2-\sqrt{3}}$ $\qquad$ (f) $\dfrac{2\sqrt{8}-\sqrt{50}}{(1-\sqrt{2})(1+\sqrt{2})}$

3. Arrange in descending order:

   (a) $4\sqrt{5}$; $6\sqrt{2}$; $5\sqrt{3}$ $\qquad$ (b) $\sqrt[3]{5}$; $\sqrt{3}$; $\sqrt[6]{26}$

4. Rationalise the denominators:

   (a) $\dfrac{2}{\sqrt{3}}$ $\qquad$ (b) $\dfrac{\sqrt{6}-\sqrt{8}}{\sqrt{2}}$ $\qquad$ (c) $\dfrac{1}{\sqrt{5}-\sqrt{3}}$

5. Refer to the figure.

   A rectangle has sides of $\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$.

   Calculate:

   (a) the length of the diagonal, leaving your answer in simplest surd form.

   (b) the area of the rectangle.

6. Solve for x:

   (a) $\sqrt{x-4} = 5$ $\qquad$ (b) $\sqrt{x-4} = -5$ $\qquad$ (c) $\sqrt{2-x} = x$ $\qquad$ (d) $\sqrt{x+1} = x - 1$

   (e) $\sqrt{1-x} = 2x - 1$ $\qquad$ (f) $\sqrt{2x} - x + 4 = 0$ $\qquad$ (g) $2x - 3\sqrt{x} - 2 = 0$

7. Calculate the exact value of: $\dfrac{\sqrt{10^{2009}}}{\sqrt{10^{2011}} - \sqrt{10^{2007}}}$ (Show ALL calculations.)

8. Simplify:

   8.1 $(1+\sqrt{2x^2})^2 - \sqrt{8x^2}$ $\qquad$ 8.2 $\left(\dfrac{\sqrt{y}+\sqrt{y^3}}{\sqrt{y}} - 1\right)^2$, $y > 0$

9. If $\dfrac{14}{\sqrt{63}-\sqrt{28}} = a\sqrt{b}$ determine, without using a calculator, the value(s) of a and b if a and b are integers.

10. Without using a calculator, show that $\dfrac{9-\sqrt{54}}{6\sqrt{2}}$ is equal to $\dfrac{3\sqrt{2}-2\sqrt{3}}{4}$.

A ANSWERS TO MODULE EXERCISES

1 EXERCISE 1.1: Types of numbers

Questions on page 1.3

1.

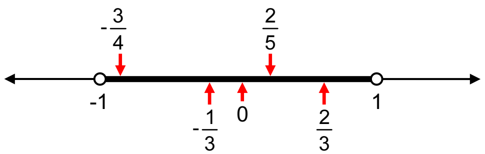
	Complex	Non-real	Real	Irrational	Rational	Integer	$\mathbb{N}_0$	$\mathbb{N}$
10	✓		✓		✓	✓	✓	✓
-4	✓		✓		✓	✓		
-3,2	✓		✓		✓			
$4\frac{2}{3}$	✓		✓		✓			
π	✓		✓	✓				
$0,\dot{6}$	✓		✓		✓			
$\sqrt{25}$	✓		✓		✓	✓	✓	✓
$-\sqrt[5]{32}$	✓		✓		✓	✓		
$\sqrt{35}$	✓		✓	✓				
$\sqrt{1\frac{11}{25}}$	✓		✓		✓			
$\sqrt{-36}$	✓	✓						
$\sqrt[3]{-27}$	✓		✓		✓	✓		
$\frac{0}{6}$	✓		✓		✓	✓	✓	
$\frac{6}{0}$	*Division by zero is undefined!*							

2. $\sqrt{36} < \sqrt{48} < \sqrt{49}$

$\therefore\ 6 < \sqrt{48} < 7$

$\therefore\ \sqrt{48}$ lies between 6 and 7

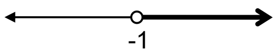
3. Between -1 and 1



A very important example of this (which includes ±1) is:

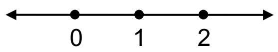
For ALL values of θ: $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$

4.1 $(-1; \infty)$

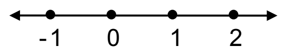


4.2 $[-4; 1)$

4.3



4.4



5. Interval notation is only used for x real, i.e. for $x \in \mathbb{R}$

5.1 $[-3; \infty)$; $x \geq -3$; $x \in \mathbb{R}$

5.2 $(-7; 4]$; $-7 < x \leq 4$; $x \in \mathbb{R}$

6. $x \leq -2$ or $x > 3$; $x \in \mathbb{R}$ . . .

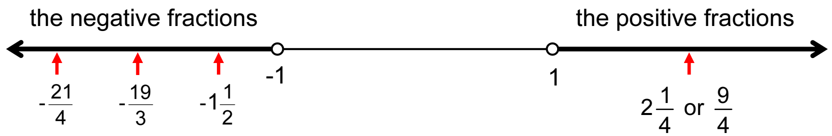
There are two separate intervals indicated separately.
$\therefore$ **Either** $x \leq -2$ or $x > 3$

7. x an integer and $x \in [-4; 3)$ ➡ $x = -4; -3; -2; -1; 0; 1$ or 2

7.1 The minimum value of x is -4 . . . *the smallest*

7.2 The maximum value of x is 2 . . . *the biggest*

8. The negative improper (and mixed) fractions lie in the interval $(-\infty; -1)$, i.e. $x < -1$
& the positive improper (and mixed) fractions lie in the interval $(1; \infty)$), i.e. $x > 1$



9. (a) $x = 1$ ◀ . . . a fraction only equals 0 when the numerator = 0

(b) y can be any real number except -3 ◀ . . . *division by 0 is undefined*

(c) if y = -3, the fraction is undefined ◀ . . . *the denominator = 0*

10. (a) x can be any real number ◀ . . . *any number* $\times 0 = 0$

(b) $x = 1$ ◀ . . . *If* $y = 2$, *then* $y + 3 \neq 0$. *So,* $x - 1$ *must be 0.*

11. $p - 3 \geq 0$ . . . $\sqrt{\text{a negative number}}$ *is imaginary*

$\therefore\ p \geq 3$ ◀

12. (a) If $x = 1{,}5$: M $= \sqrt{\dfrac{2}{2(1{,}5)+5}} = \sqrt{\dfrac{2}{8}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$

$\therefore$ M is rational because it can be written as $\dfrac{a}{b}$ where a and b are integers and b $\neq$ 0.

(b) M is real when $\dfrac{2}{2x+5}$ is positive (or 0)

- This fraction will never be 0 because the numerator $\neq$ 0.
- It will be positive when $2x + 5 > 0$ . . . $\dfrac{+}{+} = +$

$\therefore\ 2x > -5$

$\therefore\ \boldsymbol{x > -\dfrac{5}{2}}$ ◄

---

13. (a) P = 0 when $(x - 2)(2x - 1) = 0$ . . . *when the numerator = 0*

$\therefore$ when $x - 2 = 0$ or when $2x - 1 = 0$

$\therefore\ \boldsymbol{x = 2}$ ◄ $\quad\therefore\ 2x = 1$

$\therefore\ x = \dfrac{1}{2}$ ◄

(b) P is undefined when $3 - x = 0$ . . . *denominator = 0*

$\therefore\ \boldsymbol{x = 3}$ ◄

---

14. (a) P = 0 when:

$\sqrt{3-x} = 0$

$\therefore\ 3 - x = 0$ . . . *only* $\sqrt{0} = 0$

$\therefore\ \boldsymbol{x = 3}$ ◄

(b) P is undefined when:

$x^2 - 4 = 0$

$\therefore\ (x + 2)(x - 2) = 0$ . . . or $x^2 = 4$

$\therefore\ x + 2 = 0$ or $x - 2 = 0$ $\qquad \therefore\ x = \pm 2$ ◄

$\therefore\ \boldsymbol{x = -2}$ or $\boldsymbol{x = 2}$ ◄

(c) P is real when $3 - x > 0$ . . . $\sqrt{\textit{a negative number}}$ *is imaginary*

$\therefore\ -x > -3$

$\times(-1)$ $\quad\therefore\ x < 3$; BUT, $x \neq 2$ and $x \neq -2$ . . . *See 14(b)*

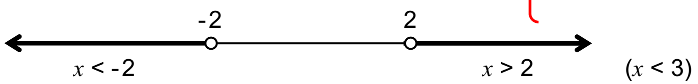
$\therefore\ \boldsymbol{x < 3\ ;\ x \neq \pm 2}$ ◄

(d) $\sqrt{3-x}$ is always positive . . . $+\sqrt{\ }$

$\therefore$ The fraction is positive when $x^2 - 4$ is positive

$\therefore$ When x^2 is greater than 4

Try values of $x < -2$ or $x > 2$ and evaluate x^2



$\therefore\ x < -2$ or $x > 2$; BUT, $x < 3$ . . . *See 14(c)*

$\therefore\ \boldsymbol{x < -2}$ or $\boldsymbol{2 < x < 3}$ ◄

---

15. (a) If $x = 0$, a can be any real number $\neq$ 0 $\qquad$ (b) If $x \neq 0$, then a = 1

---

16. (a) imaginary $\qquad$ (b) irrational $\qquad$ (c) rational

---

EXERCISE 1.2: Equations

Questions on page 1.7

1.1 By inspection, the solution (or 'root') is: $x = -4$ . . . *root is real, rational and integral*

We don't always need to solve an equation *formally*!
In this example, by inspection, we say: What plus 18 = 10? . . . -8; then
2 × what = -8? . . . -4

1.2 By inspection, the root is: $x = \dfrac{3}{2}$ . . . *this root is real and rational*

NB: Remember the **principle** of solving quadratic equations!

2.1 $(x + 3)(x - 3) = 0$

$\therefore\ x + 3 = 0$ or $x - 3 = 0$

$\therefore\ x = -3$ or $x = 3$ . . . *roots are real and rational*

This quadratic equation has 2 roots

Or: $x^2 = 9$ ➡ $x = \pm 3$ . . . *Take* $\sqrt{\ }$ *on both sides*

2.2 $3x - 2 = 0$ or $x + 1 = 0$

$\therefore\ 3x = 2$ $\qquad \therefore\ x = -1$

$\therefore\ x = \dfrac{2}{3}$ . . . *roots are real and rational*

This quadratic equation has 2 roots

2.3 $(x + 4)^2 = 0$

$\therefore\ x + 4 = 0$

$\therefore\ x = -4$ . . . *root is real and rational*

This quadratic equation has only 1 root

2.4 $(x + \sqrt{7})(x - \sqrt{7}) = 0$ . . . *Compare to Q 2.1!*

$\therefore\ x + \sqrt{7} = 0$ or $x - \sqrt{7} = 0$

$\therefore\ x = -\sqrt{7}$ or $x = \sqrt{7}$ . . . *roots are real and irrational*

This quadratic equation has 2 roots

Or: $x^2 = 7$ ➡ $x = \pm\sqrt{7}$ . . . *Take* $\sqrt{\ }$ *on both sides*

2.5 $x = 0$ or $x = -\dfrac{1}{2}$ . . . *roots are real and rational*

This quadratic equation has 2 roots

2.6 $x^2 = -9$ ➡ *we cannot factorise* $x^2 + 9$!

This quadratic equation has no roots

$\therefore\ x = \pm\sqrt{-9}$ . . . *roots are imaginary, because* $\sqrt{\textit{negative number}}$ *is imaginary.*

A

1

MODULE 1: NUMBERS & FUNDAMENTAL CONCEPTS ANSWERS

EXERCISE 2.1: Exponential Expressions

Questions on page 2.5

Note: The methods below promote an understanding of the meaning of each exponent.

1.

(a) $\frac{1}{5}$ (b) $3 \times 1 = 3$ (c) $\frac{1}{3} \times 1 = \frac{1}{3}$

(d) $2^3 = 8$ (e) $4^2 = 16$ (f) $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(g) $\frac{1}{8}$ (h) 2 (i) $2^2 = 4$

(j) $\left[\left(\frac{1}{8}\right)^{\frac{1}{3}}\right]^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ (k) $\frac{1}{16}$

(l) 4 (m) $\frac{1}{4}$

(n) $\left[\left(\frac{1}{16}\right)^{\frac{1}{2}}\right]^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$ (o) $\left[\left(\frac{27}{8}\right)^{\frac{1}{3}}\right]^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

(p) $\left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$ (q) $\left(\frac{1}{3} + \frac{1}{2}\right)^{-1} = \left(\frac{5}{6}\right)^{-1} = \frac{6}{5}$

(r) $\frac{2}{a}$ (s) $\frac{1}{2a}$

(t) $\frac{a^2}{3.(3a)^2} = \frac{a^2}{27a^2} = \frac{1}{27}$ (u) $(\sqrt[4]{81})^3 = 3^3 = 27$

(v) $(\sqrt[5]{32})^2 = 2^2 = 4$ (w) $(4)^{\frac{1}{2}} = 2$

(x) $(-4)^2 = 16$ (y) $\left(-\frac{3}{2}\right)^3 = -\frac{27}{8}$ (z) $(a+b)^{\frac{6}{3}} = (a+b)^2$

2.

(a) $\frac{1.5}{3^2} = \frac{5}{9}$ (b) $5^2 + 2^3 = 25 + 8 = 33$

(c) $1 + 4 + 4 = 9$ (d) $\left(2 + \left(\frac{1}{2}\right)^2\right)^{\frac{1}{2}} = \left(\frac{9}{4}\right)^{\frac{1}{2}} = \frac{3}{2}$

(e) $(-64x^6)^{\frac{2}{3}} = (-4x^2)^2 = 16x^4$

(f) $3a^{\frac{1}{2}}.b \times 4a^{\frac{3}{2}}.b^{-4} \div 6a^2b^{-2}$

$= 2a^{\frac{1}{2}+\frac{3}{2}-2}.b^{1-4+2}$

$= 2a^0b^{-1}$

$= \mathbf{\frac{2}{b}}$ . . . $a^0 = 1$

3.

(a) $\frac{(2^2)^{n+1}.(2^3)^{2n}}{(2^4)^{2n-1}}$

$= \frac{2^{2n+2}.2^{6n}}{2^{8n+4}}$

$= 2^{2n+2+6n-(8n+4)}$

$= 2^{-2}$

$= \mathbf{\frac{1}{4}}$

(b) $\frac{(5^2)^k \times 5^{k-1}}{5^{3k-1}}$

$= 5^{2k+k-1-(3k-1)}$

$= 5^0$

$= \mathbf{1}$

(c) $\frac{(3^2)^{n-1}.(3^3)^{3-2n}}{(3^4)^{2-n}}$

$= \frac{3^{2n-2}.3^{9-6n}}{3^{8n-4n}}$

$= 3^{2n-2+9-6n-(8-4n)}$

$= 3^{-1}$

$= \mathbf{\frac{1}{3}}$

(d) $\frac{(2^3)^{n-1}.(2^5)^{-n+1}.(2.3)^{2n}}{(3^2)^n}$

$= \frac{2^{3n-9}.2^{-5n+5}.2^{2n}.\cancel{3^{2n}}}{\cancel{3^{2n}}}$

$= 2^{3n-9-5n+5+2n}$

$= 2^{-4}$

$= \mathbf{\frac{1}{16}}$

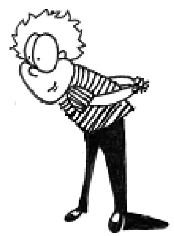
(e) $\frac{(2^2)^{n-3}.(2.5)^{n+2}}{(2^3)^{n-1}.5^{1+n}}$

$= \frac{2^{2n-6}.2^{n+2}.5^{n+2}}{2^{3n-3}.5^{1+n}}$

$= 2^{2n-6+n+2-3n+3}.5^{n+2-1-n}$

$= 2^{-1}.5$

$= \mathbf{\frac{5}{2}}$



(f) $\frac{(2^2.3)^{n+1}.(3^3)^{n-2}}{(2.3^2)^{2n-1}}$

$= \frac{2^{2n+2}.3^{n+1}.3^{3n-6}}{2^{2n-1}.3^{4n-2}}$

$= 2^{2n+2-(2n-1)}.3^{n+1+3n-6-(4n-2)}$

$= 2^3.3^{-3}$

$= \mathbf{\frac{8}{27}}$

4.

(a) $\frac{3^n.3 - 3^n}{3^n.3^{-1}}$

$= \frac{\cancel{3^n}(3-1)}{\cancel{3^n}.\frac{1}{3}}$

$= 2 \times \frac{3}{1}$

$= \mathbf{6}$

(b) $\frac{2.3^x - 3^x.3^{-1}}{5.3^x}$

$= \frac{\cancel{3^x}\left(2 - \frac{1}{3}\right)}{5.\cancel{3^x}}$

$= \frac{5}{3} \times \frac{1}{5}$

$= \mathbf{\frac{1}{3}}$

(c) $\frac{2^x.2^3 - 2.2^x}{2^x.2.8}$

$= \frac{\cancel{2^x}(8-2)}{\cancel{2^x}.16}$

$= \frac{6}{16}$

$= \mathbf{\frac{3}{8}}$

(d) $\frac{3^x.3 - 3^x.3^{-1}}{3.3^x.3^{-2}}$

$= \frac{\cancel{3^x}\left(3 - \frac{1}{3}\right)}{\cancel{3^x}.3^{-1}}$

$= \frac{8}{\cancel{3}}.\cancel{3}$

$= \mathbf{8}$

(e) $\frac{(3^2)^x + 3^{2x}.3^1}{(2.3^2)^x.2^{-x}}$

$= \frac{3^{2x} + 3^{2x}.3}{2^x.3^{2x}.2^{-x}}$

$= \frac{\cancel{3^{2x}}(1+3)}{\cancel{3^{2x}}.2^{x-x}}$

$= \mathbf{4}$ . . . $2^0 = 1$

(f) $\frac{a^x.a^3 + 2a^x.a^2}{a^x + 2a^x.a^{-1}}$

$= \frac{\cancel{a^x}.a^2(a+2)}{\cancel{a^x}\left(1 + \frac{2}{a}\right)} \times \frac{\mathbf{a}}{\mathbf{a}}$

$= \frac{a^3\cancel{(a+2)}}{\cancel{(a+2)}}$

$= \mathbf{a^3}$

the ANSWER series... your key to exam success

5.

(a) $\sqrt{\dfrac{(3.5)^x.3^x}{(3^2)^{x+1}.5^{x-2}}}$

$= \sqrt{\dfrac{3^x.5^x.3^x}{3^{2x+2}.5^{x-2}}}$

$= \sqrt{3^{x+x-2x-2}.5^{x-x+2}}$

$= \sqrt{3^{-2}.5^2}$

$= 3^{-1}.5$

$= \mathbf{\dfrac{5}{3}}$ $\blacktriangleleft$

(b) $\sqrt{\dfrac{2^x.2^3 - 2^x.2^1}{2.2^x} + 1}$

$= \sqrt{\dfrac{\cancel{2^x}(8-2)}{2.\cancel{2^x}} + 1}$

$= \sqrt{3+1}$

$= \sqrt{4}$

$= \mathbf{2}$ $\blacktriangleleft$

(c) $\dfrac{(3^2)^{2a} - 3^{3a}}{(3^3)^a}$

$= \dfrac{3^{4a} - 3^{3a}}{3^{3a}}$

$= \dfrac{\cancel{3^{3a}}(3^a - 1)}{\cancel{3^{3a}}}$

$= \mathbf{3^a - 1}$ $\blacktriangleleft$

6.

(a) $A = \dfrac{3^n - 4}{2^n.3^n - 2^n.2^2} = \dfrac{\cancel{(3^n - 4)}}{2^n\cancel{(3^n - 4)}} = \dfrac{1}{2^n} = \mathbf{\left(\dfrac{1}{2}\right)^n}$ $\blacktriangleleft$

(b) $\therefore\ \sqrt[n]{A} = A^{\frac{1}{n}} = \left[\left(\dfrac{1}{2}\right)^n\right]^{\frac{1}{n}} = \mathbf{\dfrac{1}{2}}$ $\blacktriangleleft$

EXERCISE 2.2: Exponential Equations

Questions on page 2.5

1.

(a) $\dfrac{1}{x} = \dfrac{1}{5}$

$\therefore\ \mathbf{x = 5}$ $\blacktriangleleft$

(b) $x^2 = 9$

$\therefore\ \mathbf{x = \pm 3}$ $\blacktriangleleft$

(c) $\left(x^{\frac{1}{3}}\right)^3 = \mathbf{4^3}$

$\therefore\ \mathbf{x = 64}$ $\blacktriangleleft$

(d) $x^{\frac{1}{5}} = \dfrac{1}{2}$

$\therefore\ \left(x^{\frac{1}{5}}\right)^5 = \left(\dfrac{1}{2}\right)^5$

$\therefore\ \mathbf{x = \dfrac{1}{32}}$ $\blacktriangleleft$

(e) $x^{\frac{3}{4}} = 8$

$\therefore\ x^{\frac{1}{4}} = 2 \quad \ldots \sqrt[3]{\ }$

$\therefore\ \mathbf{x = 16}$ $\blacktriangleleft$ $\quad \ldots\ (\ )^4$

(f) $\div 2)$ $x^{\frac{3}{5}} = 27$

$\therefore\ x^{\frac{1}{5}} = 3$

$\therefore\ \mathbf{x = 243}$ $\blacktriangleleft$

(g) $x^{\frac{2}{3}} = 16$ **2 ← an even number**

$\therefore\ x^{\frac{1}{3}} = \pm 4 \quad \ldots \sqrt{\ }$

$\therefore\ \mathbf{x = \pm 64}$ $\blacktriangleleft$ $\quad \ldots\ (\ )^3$

(h) $x^{+\frac{3}{2}} = \dfrac{1}{64} \quad \ldots\ (\ )^{-1}$

$\therefore\ x^{\frac{1}{2}} = \dfrac{1}{4} \quad \ldots \sqrt[3]{\ }$

$\therefore\ \mathbf{x = \dfrac{1}{16}}$ $\blacktriangleleft$ $\quad \ldots\ (\ )^2$

(i) $\div 8)$ $x^{\frac{5}{4}} = 32$

$\therefore\ x^{\frac{1}{4}} = 2 \quad \ldots \sqrt[5]{\ }$

$\therefore\ \mathbf{x = 16}$ $\blacktriangleleft$ $\quad \ldots\ (\ )^4$

(j) $5^x = 0{,}2$

$\therefore\ 5^x = 5^{-1} \quad \ldots\ 0{,}2 = \dfrac{1}{5} = 5^{-1}$

$\therefore\ \mathbf{x = -1}$ $\blacktriangleleft$

2.

(a) $x^{\frac{1}{2}}\left(x^{\frac{1}{2}} - 3\right) = 0$

$\therefore\ x^{\frac{1}{2}} = 3 \quad \ldots\ x^{\frac{1}{2}} > 0$

$\therefore\ \mathbf{x = 9}$ $\blacktriangleleft$

(b) $\left(x^{\frac{1}{4}}\right)^2 - \left(x^{\frac{1}{4}}\right) - 6 = 0$

$\therefore\ \left(x^{\frac{1}{4}} - 3\right)\left(x^{\frac{1}{4}} + 2\right) = 0$

$\therefore\ x^{\frac{1}{4}} = 3 \quad \ldots\ x^{\frac{1}{4}} > 0$

$\therefore\ \mathbf{x = 81}$ $\blacktriangleleft$

(c) $\left(x^{\frac{1}{3}} - 4\right)\left(x^{\frac{1}{3}} + 3\right) = 0$

$\therefore\ x^{\frac{1}{3}} = 4$ or -3

$\therefore\ \mathbf{x = 64}$ or $\mathbf{-27}$ $\blacktriangleleft$

(d) $(\sqrt{x})^2 - 5\sqrt{x} = 0$

$\therefore\ \sqrt{x}(\sqrt{x} - 5) = 0$

$\therefore\ \sqrt{x} = 5 \quad \ldots\ \sqrt{x} > 0$

$\therefore\ \mathbf{x = 25}$ $\blacktriangleleft$

3.

(a) $13^x = 13^0$

$\therefore\ \mathbf{x = 1}$ $\blacktriangleleft$

(b) $(7^2)^{-x} = 7$

$\therefore\ 7^{-2x} = 7^1$

$\therefore\ -2x = 1$

$\therefore\ \mathbf{x = -\dfrac{1}{2}}$ $\blacktriangleleft$

(c) $(3^2)^{2x-3} = 3^3$

$\therefore\ 3^{4x-6} = 3^3$

$\therefore\ 4x - 6 = 3$

$\therefore\ 4x = 9$

$\therefore\ \mathbf{x = \dfrac{9}{4}}$ $\blacktriangleleft$

(d) $(2^{-1})^{x-3} = 4$

$\therefore\ 2^{-x+3} = 2^2$

$\therefore\ -x + 3 = 2$

$\therefore\ -x = -1$

$\therefore\ \mathbf{x = 1}$ $\blacktriangleleft$

(e) $2^x = \dfrac{1}{8}$

$\therefore\ 2^x = \dfrac{1}{2^3}$

$\therefore\ 2^x = 2^{-3}$

$\therefore\ \mathbf{x = -3}$ $\blacktriangleleft$

(f) $\left(\dfrac{1}{5}\right)^x = 5^2$

$\therefore\ (5^{-1})^x = 5^2$

$\therefore\ 5^{-x} = 5^2$

$\therefore\ \mathbf{x = -2}$ $\blacktriangleleft$

(g) $2^{-x} = 2^5$

$\therefore\ \mathbf{x = -5}$ $\blacktriangleleft$

(h) $4^x = 4^{-3}$

$\therefore\ \mathbf{x = -3}$ $\blacktriangleleft$

(i) $\div 3)$ $\therefore\ 2^x = 8$

$\therefore\ \mathbf{x = 3}$ $\blacktriangleleft$

(j) $3^1.(3^2)^{x+1} = (3^3)^{-x}$

$\therefore\ 3^1.3^{2x+2} = 3^{-3x}$

$\therefore\ 3^{1+2x+2} = 3^{-3x}$

$\therefore\ 2x + 3 = -3x$

$\therefore\ 5x = -3$

$\therefore\ \mathbf{x = -\dfrac{3}{5}}$ $\blacktriangleleft$

(k) $(5^3)^{3x-2} = (5^2)^{4x+10}$

$\therefore\ 5^{9x-6} = 5^{8x+20}$

$\therefore\ 9x - 6 = 8x + 20$

$\therefore\ \mathbf{x = 26}$ $\blacktriangleleft$

(l) $\therefore\ 2^{x^2 - 3x} = 2^{-2} \quad \ldots\ \boxed{0{,}25 = \dfrac{1}{4} = \dfrac{1}{2^2} = 2^{-2}}$

$\therefore\ x^2 - 3x = -2$

$\therefore\ x^2 - 3x + 2 = 0$

$\therefore\ (x - 1)(x - 2) = 0$

$\therefore\ \mathbf{x = 1}$ **or** $\mathbf{2}$ $\blacktriangleleft$

4.

(a) $3^x.3^1 - 3^x = 54$

$\therefore\ 3^x(3-1) = 54$

$\therefore\ 3^x.2 = 54$

$\therefore\ 3^x = 27$

$\therefore\ \boldsymbol{x = 3}$

(b) $3.2^x + 2^x.2^{-2} = 26$

$\therefore\ 2^x\left(3+\frac{1}{4}\right) = 26$

$\therefore\ 2^x.\frac{13}{4} = 26$

$\times\frac{4}{13})\quad \therefore\ 2^x = 8$

$\therefore\ \boldsymbol{x = 3}$

(c) $(2^x)^2 - (2^x) - 6 = 0$

$\therefore\ (2^x-3)(2^x+2) = 0$

$\therefore\ \boldsymbol{2^x = 3}$

$\boxed{2^x > 0 \text{ for all } x\in\mathbb{R}}$

5.

(a) $2^4.(2^{-1})^{x-1} = 1$

$\therefore\ 2^4.2^{-x+1} = 1$

$\therefore\ 2^{4-x+1} = 1$

$\therefore\ 5 - x = \mathbf{0}$ ... $a^0 = 1$

$\therefore\ \boldsymbol{x = 5}$

(b) $2^x\times 3^x\times 3^1 = 108$

$\therefore\ (2.3)^x\times 3 = 108$

$\div 3)\quad \therefore\ 6^x = 36$

$\therefore\ \boldsymbol{x = 2}$

(c) $3a^{\frac{2}{3}} = 48a^{-\frac{2}{3}}$

$\div 3a^{-\frac{2}{3}})\quad \therefore\ \frac{\cancel{3}a^{\frac{2}{3}}}{\cancel{3}a^{-\frac{2}{3}}} = \frac{48\cancel{a^{-\frac{2}{3}}}}{3\cancel{a^{-\frac{2}{3}}}}$

$\therefore\ a^{\frac{2}{3}+\frac{2}{3}} = 16$

$\therefore\ a^{\frac{4}{3}} = 16$

$\therefore\ a^{\frac{1}{3}} = \pm 2$ ... $\sqrt[4]{\ }$

$\therefore\ \boldsymbol{a = \pm 8}$ ... $(\ )^3$

6. (a) $\frac{a^x.a^{\frac{2}{x}}}{a^{5-x}} = 1$

$\therefore\ a^{x+\frac{2}{x}-5+x} = 1$

$\therefore\ 2x - 5 + \frac{2}{x} = \mathbf{0}$

$\times x)\quad \therefore\ 2x^2 - 5x + 2 = 0$

$\therefore\ (2x-1)(x-2) = 0$

$\therefore\ \boldsymbol{x = \frac{1}{2}}$ **or 2**

(b) $\div(4^x\times 4)$ ... *to get x on the left*

$\therefore\ \frac{12^x}{4^x} = \frac{36}{4}$

$\therefore\ \left(\frac{12}{4}\right)^x = 9$

$\therefore\ 3^x = 9$

$\therefore\ \boldsymbol{x = 2}$

(c) $a^x - a^{3x^2} = 0$

$\therefore\ a^x = a^{3x^2}$

$\therefore\ x = 3x^2$

$\therefore\ 0 = 3x^2 - x$

$\therefore\ x(3x-1) = 0$

$\therefore\ \boldsymbol{x = 0}$ **or** $\boldsymbol{x = \frac{1}{3}}$

7.

(a) Let $4^x = 8$

$\therefore\ (2^2)^x = 2^3$

$\therefore\ 2x = 3$

$\therefore\ x = \frac{3}{2}$

$\therefore\ \mathbf{8 = 4^{\frac{3}{2}}}$

(b) Let $4^x = \sqrt{32}$

$\therefore\ 2^{2x} = 2^{\frac{5}{2}}$

$\therefore\ 2x = \frac{5}{2}$

$\therefore\ 4x = 5$

$\therefore\ \boldsymbol{x = \frac{5}{4}}$

EXERCISE 2.3: Surds

Questions on page 2.7

1. (a) $\sqrt{3\times 7}.\sqrt{2^2\times 3\times 5}.\sqrt{5\times 7}$

$= \sqrt{3^2\times 2^2\times 5^2\times 7^2}$

$= 3\times 2\times 5\times 7$

$= \mathbf{210}$

(b) $\frac{6\sqrt{45}}{3\sqrt{5}}$

$= 2\sqrt{\frac{45}{5}}$

$= 2\sqrt{9}$

$= 2.3$

$= \mathbf{6}$

(c) $(\sqrt{2})^2 + 2.2\sqrt{2} + 2^2 - 2.\sqrt{4\times 2}$

$= 2 + 4\sqrt{2} + 4 - 2.\sqrt{4}\sqrt{2}$

$= 6 + 4\sqrt{2} - 4\sqrt{2}$

$= \mathbf{6}$

(d) $(\sqrt{25\times 2} - \sqrt{81\times 2})^2$

$= (\sqrt{25}\sqrt{2} - \sqrt{81}\sqrt{2})^2$

$= (5\sqrt{2} - 9\sqrt{2})^2$

$= (-4\sqrt{2})^2$

$= 16.2$

$= \mathbf{32}$

(e) $(\sqrt{5})^2 - (\sqrt{3})^2$

$= 5 - 3$

$= \mathbf{2}$

(f) $(2\sqrt{3})^2 - (\sqrt{7})^2$

$= 4.3 - 7$

$= \mathbf{5}$

perfect squares

2. $\sqrt{8} = \sqrt{\mathbf{4}\times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$;

$\sqrt{75} = \sqrt{\mathbf{25}\times 3} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$;

$\sqrt{48} = \sqrt{\mathbf{16}\times 3} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$;

$\sqrt{12} = \sqrt{\mathbf{4}\times 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$;

$\sqrt{27} = \sqrt{\mathbf{9}\times 3} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$;

$\sqrt{18} = \sqrt{\mathbf{9}\times 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$;

$\sqrt{98} = \sqrt{\mathbf{49}\times 2} = \sqrt{49}\sqrt{2} = 7\sqrt{2}$;

$\sqrt{50} = \sqrt{\mathbf{25}\times 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

(a) $\frac{2\sqrt{2}+2\sqrt{2}}{\sqrt{2}.2\sqrt{2}}$

$= \frac{\cancel{4}\sqrt{2}}{\cancel{2.2}}$

$= \boldsymbol{\sqrt{2}}$

(b) $\frac{5\sqrt{3}+4\sqrt{3}}{2\sqrt{3}}$

$= \frac{9\cancel{\sqrt{3}}}{2\cancel{\sqrt{3}}}$

$= \mathbf{4\frac{1}{2}}$

(c) $\frac{3\sqrt{3}-3\sqrt{2}}{2\sqrt{2}-2\sqrt{3}}$

$= \frac{3(\sqrt{3}-\sqrt{2})}{2(\sqrt{2}-\sqrt{3})}$

$= \frac{-3\cancel{(\sqrt{2}-\sqrt{3})}}{2\cancel{(\sqrt{2}-\sqrt{3})}}$

$= \mathbf{-\frac{3}{2}}$

(d) $\frac{7\sqrt{2}-5\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{2}}{\sqrt{2}}$

$= \mathbf{2}$

(e) $\sqrt{(2+\sqrt{3})(2-\sqrt{3})}$
$= \sqrt{2^2 - (\sqrt{3})^2}$
$= \sqrt{4-3}$
$= \sqrt{1}$
$= \mathbf{1}$ ◄

(f) $\dfrac{2.2\sqrt{2} - 5\sqrt{2}}{1^2 - (\sqrt{2})^2}$
$= \dfrac{4\sqrt{2} - 5\sqrt{2}}{1-2}$
$= \dfrac{-\sqrt{2}}{-1}$
$= \sqrt{2}$ ◄

3. (a) Write as a single surd.

$4\sqrt{5} = \sqrt{16}\sqrt{5} = \sqrt{16\times 5} = \sqrt{80}$
$6\sqrt{2} = \sqrt{36}\sqrt{2} = \sqrt{36\times 2} = \sqrt{72}$
$5\sqrt{3} = \sqrt{25}\sqrt{3} = \sqrt{25\times 3} = \sqrt{75}$
$\therefore$ **Desc.: $4\sqrt{5}$; $5\sqrt{3}$; $6\sqrt{2}$** ◄

(b) Express all as $\sqrt[6]{\ }$:

$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{5^2} = \sqrt[6]{25}$
$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}$
$\therefore$ **Desc.: $\sqrt{3}$; $\sqrt[6]{26}$; $\sqrt[3]{5}$** ◄

4. (a) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$
$= \dfrac{\mathbf{2\sqrt{3}}}{\mathbf{3}}$ ◄

(b) $\dfrac{\sqrt{2}\sqrt{3} - 2\sqrt{2}}{\sqrt{2}}$
$= \dfrac{\sqrt{2}(\sqrt{3}-2)}{\sqrt{2}}$
$= \mathbf{\sqrt{3} - 2}$ ◄

(c) $\dfrac{1}{\sqrt{5}-\sqrt{3}} = \dfrac{1}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$
$= \dfrac{\sqrt{5}+\sqrt{3}}{(\sqrt{5})^2 - (\sqrt{3})^2}$
$= \dfrac{\sqrt{5}+\sqrt{3}}{5-3} = \dfrac{\mathbf{\sqrt{5}+\sqrt{3}}}{\mathbf{2}}$ ◄

5. (a) (The diagonal)^2 $= (\sqrt{5}+\sqrt{3})^2 + (\sqrt{5}-\sqrt{3})^2$
$= 5 + 2\sqrt{5}\sqrt{3} + 3 + 5 - 2\sqrt{5}\sqrt{3} + 3$
$= 16$
$\therefore$ **The diagonal = 4 units** ◄

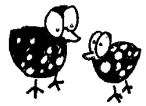
(b) The area of the rectangle $= (\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$
$= (\sqrt{5})^2 - (\sqrt{3})^2$
$= 5 - 3$
$= \mathbf{2 \text{ units}^2}$ ◄

6. (a) $(\sqrt{x-4})^2 = 5^2$
$\therefore x - 4 = 25$
$\therefore \mathbf{x = 29}$ ◄

(b) $(\sqrt{x-4}) = -5$
$\therefore$ **No solution** ◄
. . . $\sqrt{x-4} \geq 0$ *always*

(c) $\sqrt{2-x} = x$ ⬅ checking line
$\therefore 2 - x = x^2$
$\therefore 0 = x^2 + x - 2$
$\therefore (x+2)(x-1) = 0$
$\therefore x = -2$ or 1
But, $x \geq 0$ . . . see checking line
$\therefore$ **Only $x = 1$** ◄

(d) $\sqrt{x+1} = x - 1$ ⬅ checking line
$(\sqrt{x+1})^2 = (x-1)^2$
$\therefore x + 1 = x^2 - 2x + 1$
$\therefore 0 = x^2 - 3x$
$\therefore x(x-3) = 0$
$\therefore x = 0$ or 3
But, $x \neq 0$: LHS = 1; RHS = -1
$\therefore$ **Only $x = 3$** ◄ . . . LHS = RHS = 2



(e) $\sqrt{1-x} = 2x - 1$ ⬅ checking line
$\therefore (\sqrt{1-x})^2 = (2x-1)^2$
$\therefore 1 - x = 4x^2 - 4x + 1$
$\therefore 0 = 4x^2 - 3x$
$\therefore x(4x-3) = 0$
$\therefore x = 0$ or $\frac{3}{4}$
But, $x \neq 0$: LHS = 1; RHS = -1
$\therefore$ **Only $x = \frac{3}{4}$** ◄ . . . LHS = RHS = $\frac{1}{2}$

(f) $\sqrt{2x} = x - 4$ ⬅ checking line
$\therefore 2x = x^2 - 8x + 16$
$\therefore 0 = x^2 - 10x + 16$
$\therefore (x-2)(x-8) = 0$
$x = 2$ or 8
But, $x \neq 2$: LHS = 2; RHS = -2
$\therefore$ **Only $x = 8$** ◄ . . . LHS = RHS = 4

(g) $2(\sqrt{x})^2 - 3\sqrt{x} - 2 = 0$
$\therefore (2\sqrt{x}+1)(\sqrt{x}-2) = 0$
$\therefore \sqrt{x} = -\frac{1}{2}$ or 2
But, $\sqrt{x}$ cannot equal $-\frac{1}{2}$ because $\sqrt{x} \geq 0$ by definition
$\therefore$ Only $\sqrt{x} = 2$
$\therefore \mathbf{x = 4}$ ◄

7. Let $10^{2007} = x$; then the expression is:

$\dfrac{\sqrt{x\times 10^2}}{\sqrt{x\times 10^4} - \sqrt{x}} = \dfrac{\sqrt{x}.10}{\sqrt{x}.10^2 - \sqrt{x}} = \dfrac{10\sqrt{x}}{\sqrt{x}(100-1)} = \dfrac{\mathbf{10}}{\mathbf{99}}$ ◄

8.1 $(1 + x\sqrt{2})^2 - x.2\sqrt{2}$
$= 1 + 2\sqrt{2}x + 2x^2 - 2\sqrt{2}.x$
$= \mathbf{2x^2 + 1}$ ◄

8.2 $\left(\dfrac{\sqrt{y} + y\sqrt{y}}{\sqrt{y}} - 1\right)^2$
$= \left(\dfrac{\sqrt{y}(1+y)}{\sqrt{y}} - 1\right)^2$
$= (1 + y - 1)^2$
$= \mathbf{y^2}$ ◄

9. $\sqrt{63} = \sqrt{9\times 7} = \sqrt{9}\sqrt{7} = 3\sqrt{7}$
& $\sqrt{28} = \sqrt{4\times 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

The denominator must be rationalised

$\therefore$ The expression
$= \dfrac{14}{3\sqrt{7} - 2\sqrt{7}} = \dfrac{14}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{14\sqrt{7}}{7} = 2\sqrt{7}$
$\therefore$ **a = 2** and **b = 7** ◄

10. $\sqrt{54} = \sqrt{9\times 6} = 3\sqrt{6}$

Rationalising the denominator

$\therefore$ The expression $= \dfrac{9 - 3\sqrt{6}}{6\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}.\cancel{3}(3-\sqrt{6})}{{}_2\cancel{6}.2}$
$= \dfrac{\sqrt{2}(3 - \sqrt{2}\sqrt{3})}{4}$
$= \dfrac{3\sqrt{2} - \mathbf{2}\sqrt{3}}{4}$ . . . $\mathbf{\sqrt{2}\sqrt{2} = 2}$

the ANSWER series ... your key to exam success