GR 12 MATHS

Analytical Geometry THEORY

CONTENTS:

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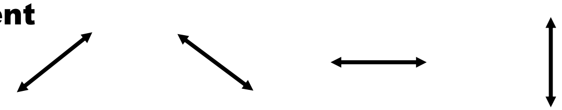
Gr 12 Maths Paper 2 – Analytical Geometry THEORY

THE STRAIGHT LINE

The Gradient of a line: Values & Applications

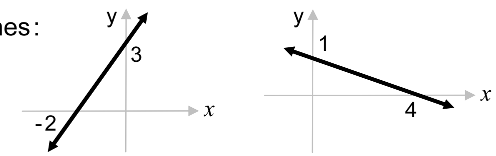
▶ Values of the gradient

- Consider these lines:



The gradients are: **POSITIVE** **NEGATIVE** **ZERO** **UNDEFINED**

- Consider these lines:



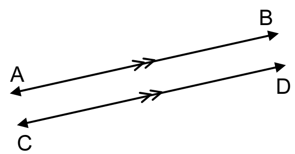
The gradients are: $+\frac{3}{2}$ $-\frac{1}{4}$

The gradient of a line $= \pm \frac{\textbf{RISE}}{\textbf{RUN}}$ $\left(or \pm \frac{vertical}{horizontal} \right)$

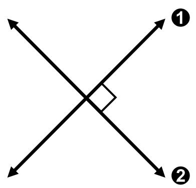
▶ Parallel lines

Parallel lines have **equal** gradients.

AB || CD ⬌ $m_{AB} = m_{CD}$



▶ Perpendicular lines



In the figure, line ❶ is perpendicular to line ❷.

If the gradient of line ❶ is $+\frac{\mathbf{2}}{\mathbf{3}}$,

then the gradient of line ❷ will be $-\frac{\mathbf{3}}{\mathbf{2}}$

So: $\mathbf{m_❶} \times \mathbf{m_❷} = \left(+\frac{2}{3}\right)\left(-\frac{3}{2}\right) = \mathbf{-1}$

i.e. The product of the gradients of ⊥ lines is -1.

line ❶ ⊥ line ❷ ⬌ $m_❶ \times m_❷ = -1$ or $m_❶ = -\frac{1}{m_❷}$

▶ Collinear points

Three points A, B & C are collinear if the gradients of **A**B & **A**C are equal.

$m_{\mathbf{A}B} = m_{\mathbf{A}C}$ ⬌ A, B & C are collinear

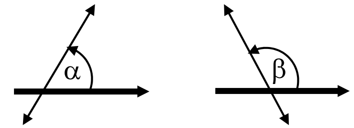
Note: Point A is common

also: $m_{AB} = m_{BC}$, **where point B is common,** $m_{AC} = m_{BC}$, **where pt C is common**

The Inclination of a line

Angles α and β alongside are angles of inclination.

The Inclination of a line is the angle which it makes with the positive direction of the *x*-axis.

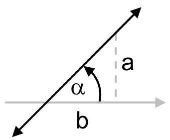


α acute . . . gradient of line is positive

β obtuse . . . gradient of line is negative

The gradient of a line is also the tan of the ∠ of inclination, i.e. **m = tan** α or **tan** β.

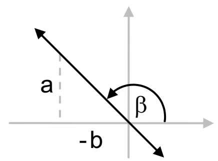
If a and b represent positive lengths, then:



$m = \frac{a}{b}$

and

$\tan \alpha = \frac{opp}{adj} = \frac{a}{b}$

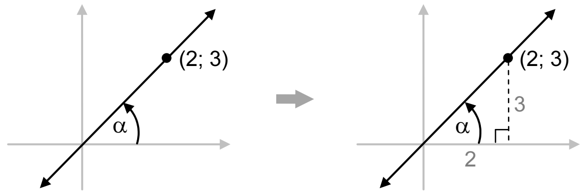


$m = -\frac{a}{b}$

and

$\tan \beta = -\frac{a}{b}$

e.g. **Acute angle α:**



The gradient = $\frac{3}{2}$

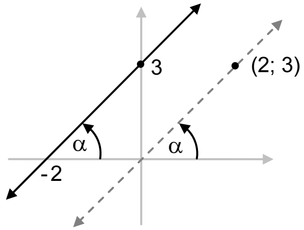
and

$\tan\alpha = \frac{3}{2}$

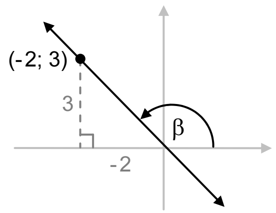
$\tan\alpha = \frac{3}{2}$ ➡ $\alpha = \tan^{-1}\frac{3}{2} = 56{,}31^\circ$ ⬅ **The inclination of the line**

In the sketch alongside, the two parallel lines both have a positive gradient = $+\frac{3}{2}$.

$\therefore$ They have the same inclination, α.
(The angles correspond.)



Obtuse angle β:



The gradient = $-\frac{3}{2}$ **and** $\tan\beta = -\frac{3}{2}$

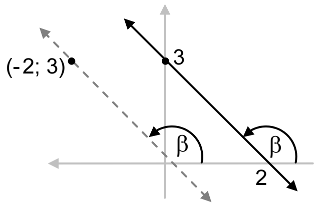
The gradient is NEGATIVE.
$\therefore$ β is obtuse

$\therefore$ $\beta = 180^\circ - 56{,}31^\circ$

$= 123{,}69^\circ$ ⬅ **The inclination of the line**

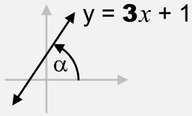
In the sketch alongside, the two parallel lines have a negative gradient = $-\frac{3}{2}$.

$\therefore$ They have the same inclination, β.
(The angles correspond.)

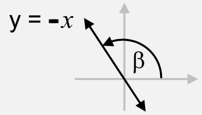


The gradient of the line is the tan of the angle of inclination
i.e. $\mathbf{m} = \tan\alpha$ or $\tan\beta$ as in $y = \mathbf{m}x + c$

e.g.



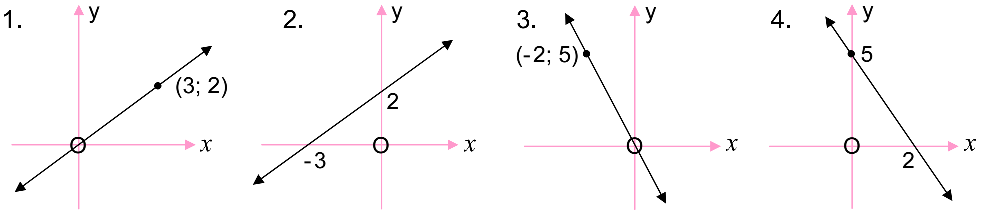
$m = 3$
$\therefore \tan\alpha = 3$
$\therefore \alpha \simeq 71{,}57^\circ$



$m = -1$
$\therefore \tan\beta = -1$
$\therefore \beta = 180^\circ - 45^\circ$
$= 135^\circ$

Worked Example

Write down (a) the gradient, and (b) the inclination of the following lines:



Note:
From the gradient, we can calculate the inclination, or from the inclination, we can calculate the gradient.

Answers

(a):	1. $\frac{2}{3}$	2. $\frac{2}{3}$	3. $-\frac{5}{2}$	4. $-\frac{5}{2}$
(b):	For both 1. & 2.: $33{,}69^\circ$		For both 3. & 4.: $180^\circ - 68{,}2^\circ = 111{,}8^\circ$	

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Graphs in general

3 Basic facts about graphs in general

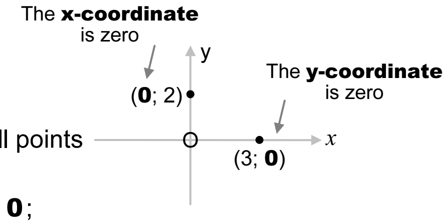
❶: Axis intercepts

Every point on the y-axis has $x = 0$.
Every point on the x-axis has $y = 0$.

❷: The equation

The **equation** of a graph is true for all points on the graph.

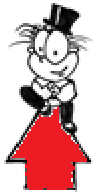
∴ The **equation** of the y-axis is $\mathbf{x = 0}$;
& the **equation** of the x-axis is $\mathbf{y = 0}$.



❸: Types of graph

Different **types/patterns** are indicated by various equations.

(See the variations of the equation of a line below.)



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Straight line graphs & their equations

Standard forms

There are 2 standard forms of the equation of a straight line:

- $\mathbf{y = mx + c}$**:** where **m** = the gradient & **c** = the y-intercept

  When m = 0: $\mathbf{y = c}$ . . . *a line || x-axis*

  When c = 0: $\mathbf{y = mx}$ . . . *a line through the origin*

  Also: $\mathbf{x = k}$ . . . *a line || y-axis* (see **Case 1** on p. 5.6)

- $\mathbf{y - y_1 = m(x - x_1)}$**:** where **m** = the gradient
  & $\mathbf{(x_1; y_1)}$ is a fixed point on the line.

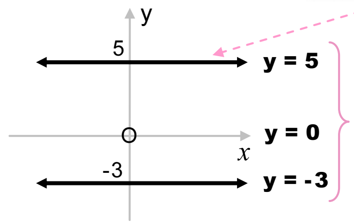
  This standard form will be explained on page 5.9.

Refer to The Answer Series Gr 11 Maths 3 in 1.

As with distance, midpoint and gradient, we will consider **equations** in the same **3 cases** as on page 5.6.

Case 1: Horizontal and Vertical lines

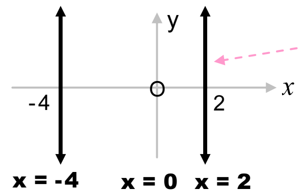
- **Horizontal lines:**
  Have the equation
  $\mathbf{y = c}$ . . . (m = 0)
  (i.e. y = a number)



EVERY point on this line has a y-coordinate equal to 5.

*These are the **EQUATIONS** of the lines.*

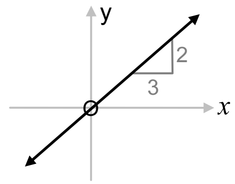
- **Vertical lines:**
  have the equation
  $\mathbf{x = k}$
  (i.e. x = a number)



EVERY point on this line has an x-coordinate equal to 2.

*These are the **EQUATIONS** of the lines.*

Case 2: Lines through the Origin



The y-intercept would always be zero.

∴ c = 0

Substitute in $y = mx + c$:

∴ $\mathbf{y = mx}$ ← the standard form of lines through the origin

∴ The equation of the line above is: $y = \frac{2}{3}x$ ◄

Case 3: "Other lines"

When lines are not parallel to the axes or through the origin, we consider:

$$\mathbf{y = mx + c} \quad \text{or} \quad \mathbf{y - y_1 = m(x - x_1)}$$

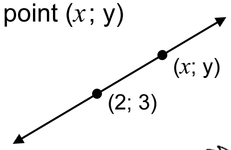
e.g. Substitute m = 5 and point (1; 6) in:

$y = mx + c$
∴ $6 = (5)(1) + c$
∴ $6 = 5 + c$
∴ $1 = c$
∴ $y = 5x + 1$ ◄

$y - y_1 = m(x - x_1)$
∴ $y - 6 = 5(x - 1)$
∴ $y - 6 = 5x - 5$
∴ $y = 5x + 1$ ◄

The Equation $y - y_1 = m(x - x_1)$: An Explanation

Given a fixed point, e.g. (2; 3), on a line, then, for **any** other point $(x; y)$ on the line, it is **true** that:

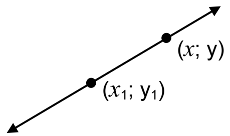


$$\frac{y-3}{x-2} = m \quad \ldots = \textit{the gradient of the line}$$

$$\therefore\ y - 3 = m(x - 2)$$

So, generally . . .

Given a **fixed** point $(x_1; y_1)$, then, for **any** point $(x; y)$ on the line, it is true that:



$$\frac{\mathbf{y - y_1}}{\mathbf{x - x_1}} = \mathbf{m} \quad \ldots = \textit{the gradient of the line}$$

$$\therefore\ \mathbf{y - y_1 = m(x - x_1)}$$

BE OPEN TO THIS ALTERNATIVE TO $y = mx + c$. It is a much quicker method!

Non-standard forms of the equation

e.g. (1) $3x - 4y = 12$ (2) $\frac{x}{3} + \frac{y}{5} = 1$

It is **not** always necessary to convert these equations into the standard form, $y = mx + c$.

The Dual-intercept method . . .

To sketch these graphs, one can determine the intercepts as follows.

For the **y-intercept, put x = 0**

(1) $3(0) - 4y = 12$

$\therefore\ y = -3$

(2) $\frac{0}{3} + \frac{y}{5} = 1$

$\therefore\ y = 5$

& for the **x-intercept, put y = 0**

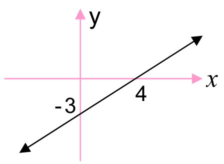
(1) $\therefore\ 3x - 4(0) = 12$

$\therefore\ x = 4$

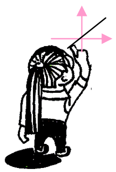
(2) $\frac{x}{3} + \frac{0}{5} = 1$

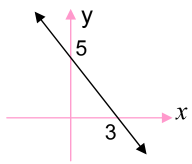
$\therefore\ x = 3$

$\therefore$ **The sketches:**



$\therefore\ m = \frac{3}{4}$ & $c = -3$





$\therefore\ m = -\frac{5}{3}$ & $c = 5$

The general form of the equation

ax + by + c = 0 is the **general form** of the equation of a straight line. This form is useful when finding the axis-intercepts and/or the gradient.

e.g. $2x + 3y + 6 = 0$

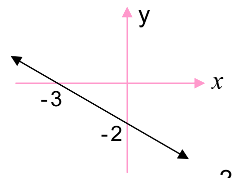
Again, the 'dual-intercept' method is useful.

y-int: Put **x = 0**, then $3y + 6 = 0$

$\therefore\ y = -2$

x-int: Put **y = 0**, then $2x + 6 = 0$

$\therefore\ x = -3$



$\therefore$ The intercepts are: (0; -2) and (-3; 0) & the gradient $= -\frac{2}{3}$

$2x + 3y + 6 = 0$, the *general* form, converts to the *standard* form, $y = -\frac{2}{3}x - 2$.

Finding the equation of a line . . .

❶ Given m and c:

1.1 **Given:** A line has a gradient of -2 and cuts the y-axis at 3.

Method: Substitute $\mathbf{m} = -2$ & $\mathbf{c} = 3$ in $y = \mathbf{m}x + \mathbf{c}$.

Equation: $y = -2x + 3$

The 'gradient-intercept' method

1.2 **Given:** A line || to the line $y = -x + 2$, passes through the point (0; 4)

Method: Substitute $\mathbf{m} = -1$ & $\mathbf{c} = 4$ in $y = \mathbf{m}x + \mathbf{c}$.

Equation: $y = -x + 4$

❷ Given m and a point:

2.1 **Given:** A line has a gradient of 3 and passes through the point (1; 6).

Method: Substitute $m = 3$ & (1; 7) in:

$\mathbf{y = mx + c}$

$\therefore\ 7 = (3)(1) + c$

$\therefore\ 4 = c$

Equation: $y = 3x + 4$

or

$\mathbf{y - y_1 = m(x - x_1)}$

$\therefore\ y - 7 = 3(x - 1)$

$\therefore\ y - 7 = 3x - 3$

$\therefore\ y = 3x + 4$

2.2 **Given:** A line passes through point (-2; 4) and is perpendicular to line y = 2x + 5.

Method: Substitute m = $-\frac{1}{2}$ & (-2; 4) in:

y = mx + c	or	**y - y₁ = m(x - x₁)**
$\therefore\ 4 = \left(-\frac{1}{2}\right)(-2) + c$		$\therefore\ y - 4 = -\frac{1}{2}(x + 2)$
$\therefore\ 4 = 1 + c$		$\therefore\ y - 4 = -\frac{1}{2}x - 1$
$\therefore\ 3 = c$		$\therefore\ y = -\frac{1}{2}x + 3$ ◄

Equation: $y = -\frac{1}{2}x + 3$ ◄

A quick method!

❸ Given (2 points):

3.1 **Given:** A line passes through the points (-3; 1) and (4; -6).

Method:

- The gradient of the line,
  $m = \frac{-6-1}{4-(-3)} = \frac{-7}{7} = -1$

> Determine m (the gradient) from the 2 points, then substitute m and either one of the 2 points, i.e. revert to the above method.

- Substitute m = -1 and a point, say (-3; 1):

y = mx + c	or	**y - y₁ = m(x - x₁)**
$\therefore\ 1 = (-1)(-3) + c$		$\therefore\ y - 1 = (-1)(x + 3)$
$\therefore\ 1 = 3 + c$		$\therefore\ y - 1 = -x - 3$
$\therefore\ -2 = c$		$\therefore\ y = -x - 2$ ◄

Equation: $y = -x - 2$ ◄

3.2 **Given:** A line passes through points (-3; -2) and (-3; 5).

No 'method' needed!

NB: The x-coordinates are the same! . . . *Draw a sketch!*

$\therefore$ The line is parallel to the y-axis

$\therefore$ Calculating m is 'not possible' . . . *The gradient is undefined!*

Equation: $x = -3$ ◄

Remember to sketch the situation and think before being lead blindly by formulae and rote methods.

Facts about Points on Graphs and Points of Intersection

FACT ❶

If a point lies on a graph, the equation is true for its coordinates, i.e. **the coordinates of the point SATISFY the equation** . . . so substitute!

and, conversely,

If a point (i.e. its coordinates) **satisfies the equation of a graph** (i.e. "makes it true"), **then it lies on the graph.** [See Q1 in Exercise 5.2 on p 5.11 in The Answer Series Mathematics Grade 11 3 in 1.]

FACT ❷

The POINT(S) OF INTERSECTION of two graphs:

The coordinates of the point(s) of intersection of two graphs "obey the conditions" of both graphs,

i.e. they **SATISFY BOTH EQUATIONS SIMULTANEOUSLY**.

They are found

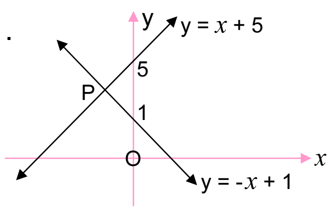
- **"algebraically"** by solving the 2 equations (see below), or
- **"graphically"** by reading the coordinates from the graph.

THESE 2 FACTS ARE CRUCIAL !!

Worked Example

Find the points of intersection of the 2 lines . . .

$y = x + 5$ & $y = -x + 1$



Answer

At the point of intersection, P

$x + 5 = -x + 1$ . . . *(both = y)*

$\therefore\ 2x = -4$

$\therefore\ x = -2$

& $y = x + 5 = 3$ or $y = -x + 1 = 3$

$\therefore$ The point of intersection, P is (-2; 3)

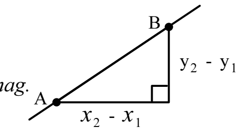
REVISION OF FORMULAE

Consider two points A(x_1; y_1) and B(x_2; y_2):

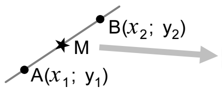
DISTANCE FORMULA

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ ... *Pythag.*

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

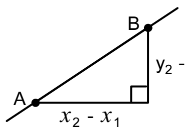


MIDPOINT



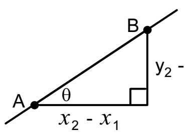
$\left(\dfrac{x_1 + x_2}{2} ; \dfrac{y_1 + y_2}{2}\right)$

GRADIENT



$m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Also **NOTE**



$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\therefore \tan\theta = m$

FACTS ABOUT GRADIENTS

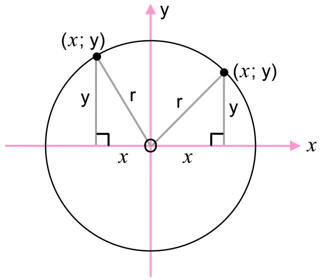
- **|| lines:** equal gradients
- **⊥ lines:** gradients neg. inv. of each other, i.e. $\mathbf{m_1 \times m_2 = -1}$
- For points A, B and C to be **collinear**: $\mathbf{m_{AB} = m_{AC} = m_{BC}}$

CIRCLES

Circles with the origin as centre:

True of **any** point (x; y) on a circle with centre **(0 ; 0)** and radius r is that:

$\boxed{\mathbf{x^2 + y^2 = r^2}}$

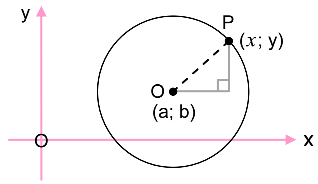


(The Theorem of Pythagoras !)

Circles with any given centre:

True of any point (x; y) on a circle with centre (a ; b) and radius r is that:

$\boxed{\mathbf{(x - a)^2 + (y - b)^2 = r^2}}$

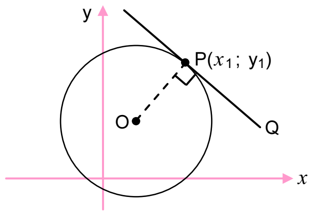


Distance formula !
(i.e. the Theorem of Pythagoras again)

A Tangent to a ⊙

is **perpendicular** to the **radius** of the ⊙ at the **POINT** of **contact**.

Therefore, to find the **equation** of a tangent we usually use **"m and 1 point"** in the straight line equation
$y - y_1 = m(x - x_1)$



$m_{OP} = 2$

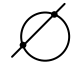
$\therefore m_{PQ} = -\frac{1}{2}$

($\because$ radius OP ⊥ tangent PQ)

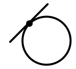
NOTE:

A line and a circle (or parabola!) either

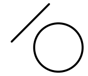
❶ "cut" (twice!) *[secant]* (2 points in common)



or ❷ "touch" (once!) *[tangent]* (1 point in common)



or ❸ don't cut or touch (no points in common)



and if we substitute $y = mx + c$ into the equation of the ⊙ there will either be 2 solutions, 1 solution or no solutions for x, resulting in one of the above scenarios

Converting from

general form $Ax^2 + Bx + Cy^2 + Dy + E = 0$
to
standard form $(x - a)^2 + (y - b)^2 = r^2$
(using completion of squares)

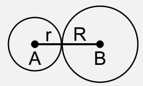
e.g.

$$\begin{aligned} x^2 - 6x + y^2 + 8y - 25 &= 0 \\ \therefore x^2 - 6x + y^2 + 8y &= 25 \\ \therefore x^2 - 6x \mathbf{+ 3^2} + y^2 + 8y \mathbf{+ 4^2} &= 25 \mathbf{+ 9 + 16} \\ \therefore (x - 3)^2 + (y + 4)^2 &= 50 \end{aligned}$$

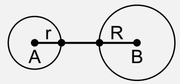
i.e. a ⊙ with centre (3; -4) & radius, $r = \sqrt{50}$ $(= 5\sqrt{2})$ units

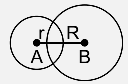
An interesting fact . . .

When 2 ⊙'s touch, the **distance between their centres = the sum of their radii** (& vice versa)
i.e. AB = r + R



$\therefore$ for AB > r + R and AB < r + R





FINAL ADVICE
Use your common sense & ALWAYS DRAW A PICTURE !!!