Algorithm

Homework 1

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Problem 1

1. $3n+1 = \mathrm{O}(\log(n))$

This statement is FALSE.

To show that $3n+1 \neq \mathrm{O}(\log(n))$, we assume there exist constant c, $n_0 > 0$ such that

$0 \leq 3n+1 \leq c\log n$ for all $n \geq n_0$.

For the left part,

$$0 \leq 3n+1 \Rightarrow n \geq -\frac{1}{3}$$

For the right part,

$$3n+1 \leq c\log n \Rightarrow \frac{3n+1}{\log n} \leq c \Rightarrow \frac{3n+1}{\log n} \leq c$$

But no constants is greater than all $\frac{3n+1}{\log n}$, and so the assumption leads to a contradiction.

2. $100^{3409}(n+34n^2) = o(n^{2.0000000001})$

This statement is TRUE.

To show that $100^{3409}(n+34n^2) = o(n^{2.0000000001})$, we must proof that $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$:

Given

$f(n) = 100^{3409}(n+34n^2)$,

$g(n) = n^{2.0000000001}$

Thus

$$\frac{f(n)}{g(n)} = \frac{100^{3409}(n+34n^2)}{n^{2.0000000001}} = 100^{3409}\frac{1+34n}{n^{1.0000000001}} \Rightarrow \lim_{n\to\infty} 100^{3409}\frac{1+34n}{n^{1.0000000001}} = 0$$

Which meets the definition of little o, thus the statement is TRUE.

3. $100^{3409}(n+34n^2)=o(n^{1.999999999999})$

This statement is FALSE

To show that $100^{3409}(n+34n^2)\neq o(n^{1.999999999999})$, we must proof that $\lim_{n\to\infty}\frac{f(n)}{g(n)}\neq 0$:

Given

$f(n)=100^{3409}(n+34n^2)$,

$g(n)=n^{1.999999999999}$

Thus

$$\frac{f(n)}{g(n)}=\frac{100^{3409}(n+34n^2)}{n^{1.999999999999}}=100^{3409}\frac{1+34n}{n^{0.999999999999}}\Rightarrow \lim_{n\to\infty}100^{3409}\frac{1+34n}{n^{1.999999999999}}=+\infty$$

Which does not meet the definition of little o, thus the statement is FALSE.

4. $e^n=\theta(e^{n^2})$

This statement is FALSE

To show that the statement is FALSE, we assume there exist constants c_1, c_2, $n_0 > 0$ such that:

$c_1e^{n^2}\leq e^n\leq c_2e^{n^2}$ for all $n>=n_0$

Then $c_1\leq\frac{e^n}{e^{n^2}}\leq c_2\Rightarrow c_1\leq e^{n-n^2}\leq c_2\Rightarrow c_1\leq\frac{1}{e^{n^2-n}}\leq c_2$

For the left part

$$c_1\leq\frac{1}{e^{n^2-n}}\Rightarrow e^{n^2-n}\leq\frac{1}{c_1}$$

But no constant is greater than all e^{n^2-n}, and so the assumption leads to a contradiction.

5. $e^n = \Theta(e^{3n})$

This statement is FALSE

To show that the statement is FALSE, we assume there exist constants c_1, c_2, $n_0 > 0$ such that:

$c_1 e^{3n} \le e^n \le c_2 e^{3n}$ for all n>=n_0

Then $c_1 \le \dfrac{e^n}{e^{3n}} \le c_2 \Rightarrow c_1 \le e^{n-3n} \le c_2 \Rightarrow c_1 \le \dfrac{1}{e^{3n-n}} \le c_2$

For the left part

$$c_1 \le \frac{1}{e^{3n-n}} \Rightarrow e^{3n-n} \le \frac{1}{c_1} \Rightarrow e^{2n} \le \frac{1}{c_1}$$

But no constant is greater than all e^{2n}, and so the assumption leads to a contradiction.

6. $e^n = \theta(e^{n+3})$

This statement is TRUE.

To show that the statement is TRUE, we must find there exist constants c_1, c_2, $n_0 > 0$ such that:

$c_1 e^{n+3} \le e^n \le c_2 e^{n+3}$ for all n>=n_0

Then $c_1 \le \dfrac{e^n}{e^{n+3}} \le c_2 \Rightarrow c_1 \le e^{n-n-3} \le c_2 \Rightarrow c_1 \le \dfrac{1}{e^3} \le c_2$

For all n, we can satisfy the definition with $c_1 = \dfrac{1}{20000}, c_2 = \dfrac{1}{e^3} + 1$

Thus the statement is TRUE.

7. $\log(n) = O(\log(n^{10000000000}))$

This statement is TRUE.

To show the statement is TRUE, we must find constants c, $n_0 > 0$ such that:

$0 \le \log(n) \le c(\log(n^{10000000000}))$ for all $n >= n_0$

For the left part, $0 \le \log(n) \Rightarrow n \ge 1$

For the right part,

$$\log(n) \le c(\log(n^{10000000000})) \Rightarrow \log(n) \le \log(n^{10000000000})^{c}$$
$$\Rightarrow 0 \le \log(\frac{n^{10000000000 \cdot c}}{n}) \Rightarrow 0 \le \log(n^{c \cdot 10000000000 - 1})$$

We can satisfy the right part with c = 1 and $n_0 = 2$.

Thus for c = 1 and $n_0 = 2$, we can prove the statement is TRUE.

8. $\log(n) = O((\log(n^{10000000000}))^{1.000001})$

The statement is TRUE.

To show that $\log(n) \ne O((\log(n^{10000000000}))^{1.000001})$, we need to find constants c, $n_0 > 0$ such that

$0 \le \log(n) \le c \cdot (\log(n^{10000000000}))^{1.000001}$ for all $n >= n_0$

For the left part, $0 \le \log(n) \Rightarrow n \ge 1$

For the right part,

$$\log(n) \le c \cdot (\log(n^{10000000000}))^{1.000001} \Rightarrow \frac{\log(n)}{\log(n^{10000000000}) \cdot (\log(n^{10000000000}))^{0.000001}} \le c$$
$$\Rightarrow \frac{\log(n)}{\log(n^{10000000000}) \cdot (\log(n^{10000000000}))^{0.000001}} \le c \Rightarrow \frac{1}{10000000000 \cdot (\log(n^{10000000000}))^{0.000001}} \le c$$

We can satisfy the statement with c = 100, $n_0 = 1$. Thus the statement is TRUE.

9. $\log(n) = o((\log(n^{10000000000}))^{1.000001})$

The statement is TRUE.
We got

$$f(n) = \log(n)$$

$$g(n) = (\log(n^{10000000000}))^{1.000001}$$

Thus

$$\frac{f(n)}{g(n)} = \frac{\log(n)}{(\log(n^{10000000000}))^{1.000001}} = \frac{\log(n)}{\log(n^{10000000000})\cdot\log(n^{10000000000})^{0.000001}}$$

$$= \frac{1}{\frac{\log(n^{10000000000})}{\log(n)}\cdot\log(n^{10000000000})^{0.000001}} = \frac{1}{10000000000\cdot\log(n^{10000000000})^{0.000001}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{1}{10000000000\cdot\log(n^{10000000000})^{0.000001}} = 0$$

Which is meet the definition of little o, thus the statement is TRUE.

10. $\log(n) = \Omega((\log(n^{10000000000}))^{1.000001})$

The statement is FALSE.

To show that $\log(n) \neq \Omega((\log(n^{10000000000}))^{1.000001})$, we can assume that there exist constants c, $n_0 > 0$ such that

$0 \le c \cdot (\log(n^{10000000000}))^{1.000001} \le \log(n)$ for all n > =n_0

For the left part,

$$0 \le c \cdot (\log(n^{10000000000}))^{1.000001} \Rightarrow \frac{1}{(\log(n^{10000000000}))^{1.000001}} \le c$$

For the right part,

$$c \cdot (\log(n^{10000000000}))^{1.000001} \le \log(n) \Rightarrow \frac{(\log(n^{10000000000}))^{1.000001}}{\log(n)} \le \frac{1}{c}$$

$$\Rightarrow \frac{(\log(n^{10000000000}))(\log(n^{10000000000}))^{0.000001}}{\log(n)} \le \frac{1}{c}$$

$$\Rightarrow \frac{(10000000000 \cdot \log(n))(\log(n^{10000000000}))^{0.000001}}{\log(n)} \le \frac{1}{c}$$

$$\Rightarrow 10000000000(\log(n^{10000000000}))^{0.000001} \le \frac{1}{c}$$

But no constant of $\frac{1}{c}$ is greater than all $10000000000(\log(n^{10000000000}))^{0.000001}$, and so the assumption leads to a contradiction.

11. $\log(n) = \Theta((\log(n^{10000000000}))^{1.000001})$

This statement is FALSE.

To show that $\log(n) \neq \Theta((\log(n^{10000000000}))^{1.000001})$, assume there exist c_1, c_2, $n_0 > 0$ such that:

$$c_1(\log(n^{10000000000}))^{1.000001} \leq \log(n) \leq c_2(\log(n^{10000000000}))^{1.000001}$$

For the left part,

$$c_1 \cdot (\log(n^{10000000000}))^{1.000001} \leq \log(n) \Rightarrow \frac{(\log(n^{10000000000}))^{1.000001}}{\log(n)} \leq \frac{1}{c_1}$$

$$\Rightarrow \frac{(\log(n^{10000000000}))(\log(n^{10000000000}))^{0.000001}}{\log(n)} \leq \frac{1}{c_1}$$

$$\Rightarrow \frac{(10000000000 \cdot \log(n))(\log(n^{10000000000}))^{0.000001}}{\log(n)} \leq \frac{1}{c_1}$$

$$\Rightarrow 10000000000(\log(n^{10000000000}))^{0.000001} \leq \frac{1}{c_1}$$

But no constant of $\frac{1}{c_1}$ is greater than all $10000000000(\log(n^{10000000000}))^{0.000001}$, and so the assumption leads to a contradiction.

12. $\log(\log(n)) = \Omega(1)$

The statement is TRUE.

We must find the constants c, n_0 > 0 such that

$0 \le c \cdot 1 \le \log(\log(n))$ for all n >=n_0

For the left part, we have $c \ge 0$

For the right part,

$c \cdot 1 \le \log(\log(n)) \Rightarrow 2^c \le \log(n)$

We can satisfy the definition with c = 0, n_0 = 2

Thus the statement is TRUE.

Problem 2

1. $\sum_{i=0}^{n}(i^2-45i)$

$$\sum_{i=0}^{n}(i^2-45i)=\sum_{i=0}^{n}i^2-\sum_{i=0}^{n}45i=\frac{n(n+1)(2n+1)}{6}-45(\frac{1}{2}n(n+1))$$
$$=\frac{n(n+1)(2n+1)}{6}-\frac{45}{2}n^2-\frac{45}{2}n=\frac{2}{6}n^3+\frac{3}{6}n^2+\frac{1}{6}n-\frac{135}{6}n^2-\frac{135}{6}n$$
$$=\frac{1}{3}n^3-\frac{132}{6}n^2-\frac{134}{6}n$$
$$=\frac{1}{3}n^3-22n^2-\frac{67}{3}n$$

2. $\sum_{i=0}^{n}3^i$

$$\sum_{i=0}^{n}3^i=1+3+3^2+\cdots+3^n=\frac{3^{n+1}-1}{3-1}$$

3. $\sum_{i=0}^{n}(\frac{1}{2i+1})$

As we know $\frac{1}{2i+1}<\frac{1}{2i}$

Thus $\sum_{i=0}^{n}(\frac{1}{2i+1})<1+\sum_{i=1}^{n}\frac{1}{2i}=1+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{i}$

Apply harmonic series:

$$\sum_{i=1}^{n}\frac{1}{i}\le\ln n+1$$

We got:

$$\sum_{i=0}^{n}(\frac{1}{2i+1})<1+\sum_{i=1}^{n}\frac{1}{2i}=1+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{i}\le 1+\frac{1}{2}(\ln n+1)=\frac{1}{2}\ln n+\frac{3}{2}$$

Problem 3

1. Prove by induction that for every natural number n there exists a larger natural number m such that n < m <= 3n and m is a power of 3.

n=1: $1 < m \le 3$, thus m = 3 = 3^1.

Assume n=k holds: $k < m \le 3k \Rightarrow m = 3^a$

Show n=k+1 holds: $k+1 < m \le 3(k+1) \Rightarrow m = 3^b$

Start from the left side:

(1) If $n = k+1 = m-1 \quad m = 3^b$

Then

$$m^b = m \cdot 3 = 3^{b+1} > m-1 = n$$

$$m^b = 3 \cdot 3^b = 3(k+1)$$

Given $k < m \le 3k$, we have

$$n = k+1 < m \le 3(k+1) = 3n$$

(2) If $n = k+1 < m-1 \quad m = 3^b$

Then

$$k+1 < m$$

Given $n < m \le 3(n+1)$, we have

$$n = k+1 < m < 3(k+2) = 3n$$

From (1) and (2), we know

$$n = k+1$$

$$n < m \le 3n$$

$$m = 3^b$$

2. $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)$

$(A \cup B) \cap (C \cup D)$

Apply Distributive law:

$= ((A \cup B) \cap C) \cup ((A \cup B) \cap D)$

Apply Distributive law again:

$= ((A \cap C) \cup (B \cap C)) \cup ((A \cap D) \cup (B \cap D))$

Apply Associative lay:

$= (A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)$

QED