

Homework #4  
Introduction to Algorithms/Algorithms 1  
600.363/463

**Due on:** Friday, April 13, 11:59 a.m. (NOON)

**Where to submit:** On blackboard, under student assessment

**Late submissions:** will NOT be accepted

**Format:** Please start each problem (1 through 18) on a new page.  
Please type your answers; handwritten assignments will not be accepted.

March 30, 2012

**1**

Show that if a graph's edges all have distinct weights, the MST is unique.

**2**

Give an algorithm for a maximum spanning tree.

**3**

Develop a version of Prim's algorithm or of Kruskal's algorithm for the minimum spanning forest for graphs that are not necessarily connected.

*Hint: Use the strongly connected components algorithm.*

**4**

Suppose you implement the disjoint-set data structure using union-by-rank but without path compression. Give a sequence of  $m$  union and find operations whose running time is  $\Omega(m \lg(n))$ .

## 5

Let  $T$  be a MST of a graph  $G$ . Given a connected subgraph  $H$  of  $G$ , show that  $T \cap H$  is contained in some MST of  $H$ .

## 6

For each of the following statements, either prove its correctness or give a counterexample:

1. If a graph  $G = (V, E)$  has more than  $|V| - 1$  edges and there is a unique heaviest edge, then  $G$  cannot be part of a MST.
2. If  $G$  has a cycle with a unique heaviest edge  $e$ , then  $e$  cannot be a part of the MST.
3. If the lightest edge  $e$  is unique, then  $e$  must be a part of every MST.

## 7

State whether the following statement is true or false, providing either a proof or a counterexample:

Adding a constant weight to the weight of every edge does not change the outcome for the single source shortest-path algorithm.

## 8

Is the path between two nodes in a MST necessarily the shortest path between them? Prove or give a counterexample.

## 9

Does either of Prim or Kruskal's algorithm work if we allow negative weights?

## 10

Modify Dijkstra's algorithm by using a *max-priority queue* and extracting *Max* every time. Will it produce paths with a maximum weight (single-source longest paths)?

## 11

Design a linear-time algorithm that takes a connected undirected graph and checks whether there is an edge that can be removed while still leaving the graph connected.

## 12

Prove that if an undirected graph has  $k$  components then it has at least  $|V| - k$  edges.

## 13

Suppose that for a graph  $G = (V, E)$ , every vertex  $v$  has a weight  $c$  in addition to the edge weights, where

$$c(v) \geq 0 \quad \text{for } v \in V$$

The weight of a path is now defined as the sum of all the weights (both edges and vertices, including the ends). Design an algorithm to solve this modified version of the single-source shortest paths problem.

## 14

Given a directed weighted graph  $G$  and a tree  $T$  with root  $s$ , check in a time  $O(V + E)$  that  $T$  is a single-source shortest path tree with source  $s$ .

## 15

Consider a directed graph whose only negative edges are those that leave  $s$ ; will Dijkstra's algorithm work when started at  $s$ ?

## 16

Let  $d$  be a degree of a vertex  $v$ . (Please refer to CLRS for a definition). Prove that:

1.

$$\sum_{a \in V} d(a) = 2|E|$$

2. the number of vertices with odd degree is even.

**17**

Prove that in any connected undirected graph, there exists a vertex whose removal leaves the graph connected.

**18 Bonus question**

Given a DAG, check if there is a path that touches every vertex exactly once.