The Binomial Model

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The One Period Model

Definition

- time t: $t = 0, 1$
- (deterministic) bond B_t

$$B_0 = 1$$
$$B_1 = 1 + R$$

- (stochastic) stock S_t

$$S_0 = s > 0$$
$$S_1 = \begin{cases} s \cdot u & \text{with prob. } p_u \\ s \cdot d & \text{with prob. } p_d \end{cases} \equiv s\,Z \; (u > d \text{ and } p_u + p_d = 1)$$

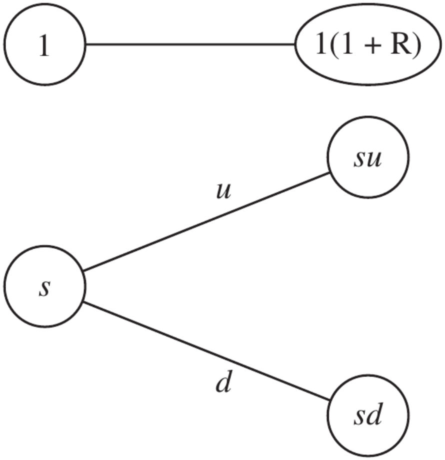


Figure: Asset Dynamics of One Period Model

Definition

The value process V_t^h of the portfolio $h = (x, y)$, $x, y \in \mathbb{R}$ at time t is

$$V_t^h = x\,B_t + y\,S_t$$

In details:

$$V_0^h = x + y\,s$$
$$V_1^h = x(1 + R) + y\,s\,Z$$

Definition

Arbitrage portfolio h: $V_0^h = 0$, $V_1^h > 0$ with prob. 1.

Theorem

The one period model is arbitrage free iff $u \geqslant 1 + R \geqslant d$.

Proof.

$(\Longrightarrow)$

- Suppose $u \geqslant 1 + R \geqslant d$ does not hold, then $u < 1 + R$ or $d > 1 + R$.
- $s(1 + R) > s\,u$ and a priori $s(1 + R) > s\,d$.
- Consider $h = (s, -1)$.
- $V_0^h = s \cdot 1 + (-1) \cdot s = 0$, $V_1^h = s(1 + R) - s \cdot Z > 0$.

$\square$

Theorem

The one period model is arbitrage free iff $u \geqslant 1 + R \geqslant d$.

Proof.

$(\Longleftarrow)$

- Arbitrage $h = (x, y)$: $V_0^h = 0$.
- $x + s \cdot y = 0 \Rightarrow x = -s \cdot y$.
- $$V_1^h = \begin{cases} ys[u - (1+R)] & Z = u \\ ys[d - (1+R)] & Z = d \end{cases}$$
- let $y > 0$; $V_1^h > 0 \Rightarrow u > 1 + R, d > 1 + R$.

$\square$

Theorem

No arbitrage $\iff u \geqslant 1+R \geqslant d$

Observation: $u \geqslant 1+R \geqslant d$ is equivalent to

$$\exists q_u, q_d \geqslant 0,\ q_u + q_d = 1 \quad \text{s.t.} \quad 1+R = q_u \cdot u + q_d \cdot d$$

Define new probability measure Q and expectation E^Q s.t.

$$Q(Z=u) = q_u, Q(Z=d) = q_d$$

$$\frac{1}{1+R}\mathrm{E}^Q[S_1] = \frac{1}{1+R}[q_u \cdot s\,u + q_d \cdot s\,d] = \frac{1}{1+R} \cdot s(1+R) = s$$

Risk-Neutral / Martingale Measure

A measure Q which satisfies

$$S_0 = \frac{1}{1+R} \mathrm{E}^Q[S_1]$$

is called a risk-neutral / martingale measure.

Martingale Probabilities

The martingale probabilities are given by

$$q_u = \frac{(1+R)-d}{u-d}$$
$$q_d = \frac{u-(1+R)}{u-d}$$

Contingent Claims

Definition

A *contingent claim* X is of the form $X = \Phi(Z)$; Z stochastic with *contract function* $\Phi(\cdot)$. The *price* of X at time t is denoted by $\Pi(t; X)$.

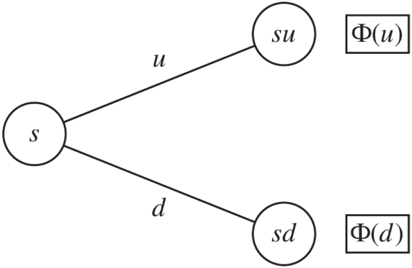


Figure: A Contingent Claim

Example (European Call Option with Strike K)

Assume $s\,u > K > s\,d$. At $t = 1$ exercise the option if $S_1 > K$; pay K to get the stock and sell at $s\,u$, thus making net profit $s\,u - K$. Do nothing if $S_1 < K$. We have

$$X = \begin{cases} s\,u - K, & Z = u \\ 0 & Z = d \end{cases}$$

and

$$\Phi(u) = s\,u - K$$
$$\Phi(d) = 0$$

Definition

A contingent claim X is said to be *reachable* if there exists a portfolio h such that $V_1^h = X$ with probability 1; this portfolio h is called a *hedging* or *replicating* portfolio. If all claims can be replicated we say the market is *complete*.

Pricing Principle

If a claim X is reachable with replicating portfolio h, then the "reasonable" price process of X is given by

$$\Pi(t; X) = V_t^h, \quad t = 0, 1.$$

Theorem

An arbitrage free one period model is complete.

Proof.

Fixed any $\Phi(\cdot)$, show that $\exists h = (x, y)$ s.t.

$$V_1^h = \begin{cases} \Phi(u) & Z = u, \\ \Phi(d) & Z = d. \end{cases}$$

$$\implies \quad x(1+R) + s\,u\,y = \Phi(u), \quad x(1+R) + s\,d\,y = \Phi(d).$$

Solve for x, y:

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\,\Phi(u)}{u-d}, \quad y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}.$$

$\square$

From Pricing Principle ($\Pi(t; X) = V_t^h,\ t = 0, 1$) we have

$$\begin{aligned}
\Pi(0; X) &= V_0^h = x + s\,y \\
&= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\,\Phi(u)}{u-d} + s \cdot \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d} \\
&= \frac{1}{1+R} \left\{ \frac{(1+R) - d}{u-d} \Phi(u) + \frac{u - (1+R)}{u-d} \Phi(d) \right\} \\
&= \frac{1}{1+R} \{ q_u \Phi(u) + q_d \Phi(d) \} \equiv \frac{1}{1+R} \mathrm{E}^Q[X]
\end{aligned}$$

Collecting above, we state the

The Risk Neutral Valuation Principle

If the one period binomial model is arbitrage-free, then $\Pi(0; X)$, the arbitrage-free price of a contingent claim X is

$$\Pi(0; X) = \frac{1}{1+R} \mathrm{E}^Q[X]$$

The Multiperiod Model

Definition

- time t: $t = 0, 1, 2, \ldots, T$.
- (deterministic) bond B_t

$$B_0 = 1, \quad B_{n+1} = (1+R)B_n$$

- (stochastic) stock S_t

$$S_0 = s > 0, \quad S_{n+1} = Z_n S_n$$

where $Z_0, Z_1, Z_2, \ldots, Z_{T-1}$ are iid with

$$\mathrm{P}(Z_n = u) = p_u, \quad \mathrm{P}(Z_n = d) = p_d$$

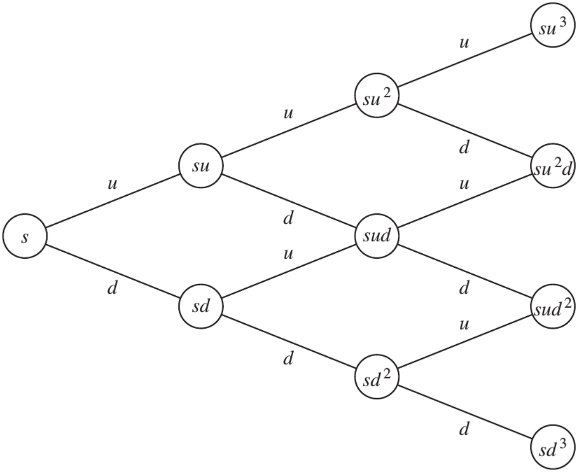


Figure: Asset Dynamics of Multiperiod Model: 'Recombining' Tree

Definition

The portfolio process $h_t \equiv (x_t, y_t)$; The value process V_t^h of portfolio h_t at time t is

$$V_t^h = x_t \, B_t + y_t \, S_t$$

(x_t is the amount of money which we invest in the bank at time $t-1$ and keep until t.)

Definition

Self-financing portfolio $h_t = (x_t, y_t)$:

$$x_t \, B_t + y_t \, S_t = x_{t+1} \, B_t + y_{t+1} \, S_t \quad \forall t = 0, 1, \ldots, T-1$$

Definition

Arbitrage: there exists a self-financing portfolio h_t with

$$V_0^h = 0, \quad \mathrm{P}(V_T^h \geqslant 0) = 1, \quad \mathrm{P}(V_T^h > 0) > 0.$$

Definition

A contingent claim X is said to be *reachable* if there exists a self-financing portfolio h such that $V_T^h = X$ with probability 1; this portfolio h is called a *hedging* or *replicating* portfolio. If all claims can be replicated we say the market is *complete*.

Pricing Principle

If a claim X is reachable with replicating (and self-financing) portfolio h, then the "reasonable" price process of X is given by

$$\Pi(t; X) = V_t^h, \quad t = 0, 1, 2, \dots T.$$

Theorem

An arbitrage-free multiperiod model is complete.

Example

Given $T = 3, S_0 = 80, K = 80, u = 1.5, d = 0.5, p_u = 0.6, p_d = 0.4, R = 0$ (European Call Option), check this multiperiod model is complete.

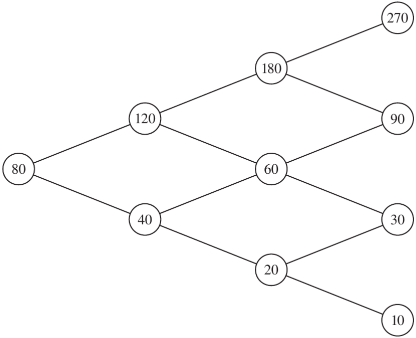
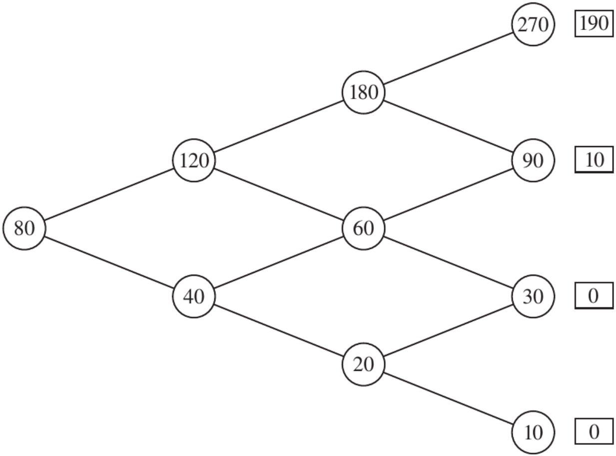
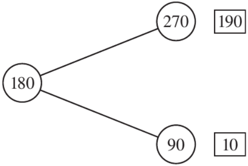


Figure: Asset Dynamics of the Example

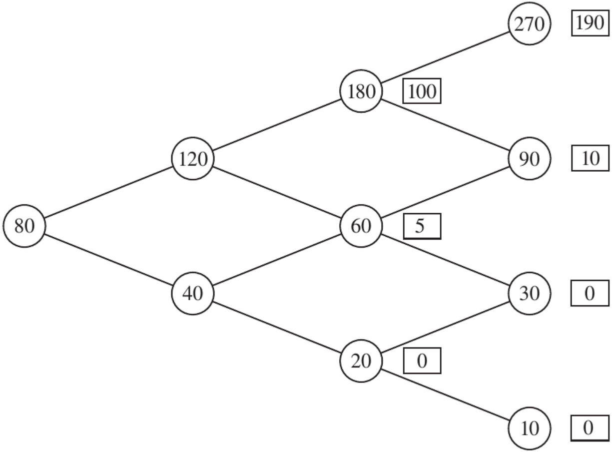


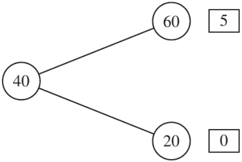
80
120
40
180
60
20
270
90
30
10
190
10
0
0



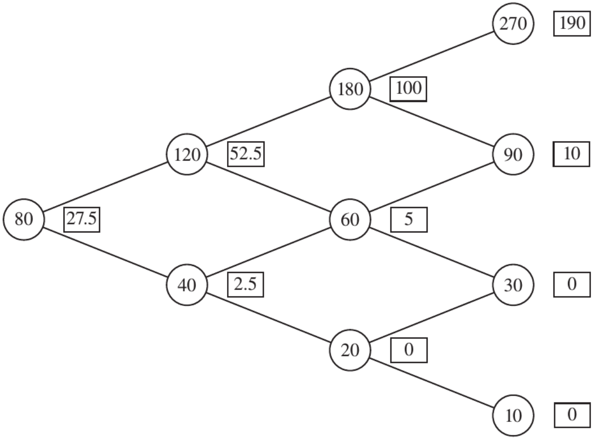
180
270
90
190
10

(Risk Neutral Valuation: $\Pi(0;X) \equiv \frac{1}{1+R}\mathrm{E}^{Q}[X] = \frac{1}{1+R}\left\{q_u\Phi(u) + q_d\Phi(d)\right\}$)



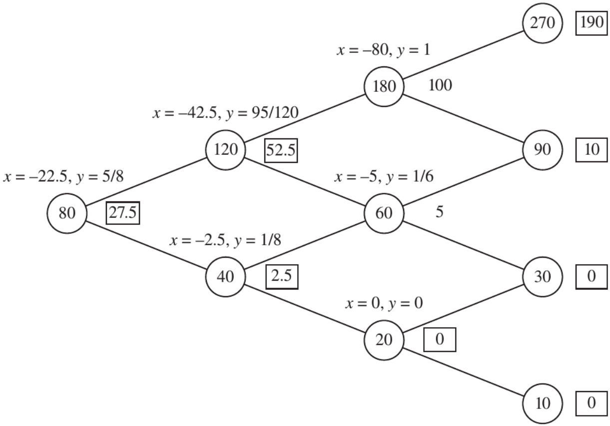


The Completed $\Pi(t; X)$



Replicating $h_t = (x_t, y_t)$

(The one period model formula $x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}$, $\quad y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}$)



Theorem (Binomial Algorithms)

Given a contingent claim $X = \Phi(S_T)$; let $V_t(k)$ denotes the value of the replicating portfolio at node (t, k), then $V_t(k)$ is computed via

$$V_T(k) = \Phi(s\, u^k\, d^{T-k}), \quad V_t(k) = \frac{1}{1+R}\{q_u\, V_{t+1}(k+1) + q_d\, V_{t+1}(k)\}$$

The martingale probabilities q_u, q_d are given by

$$q_u = \frac{(1+R)-d}{u-d}, \quad q_d = \frac{u-(1+R)}{u-d}.$$

The replicating portfolio $h_t = (x_t, y_t)$ is given by

$$x_t(k) = \frac{1}{1+R} \cdot \frac{u\, V_t(k) - d\, V_t(k+1)}{u-d}$$
$$y_t(k) = \frac{1}{S_{t-1}} \cdot \frac{V_t(k+1) - V_t(k)}{u-d}$$

The arbitrage-free price of a contingent claim X at $t = 0$ is given by

$$\Pi(0; X) = \frac{1}{(1+R)^T} \cdot \mathrm{E}^Q[X].$$

More precisely,

$$\Pi(0; X) = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^{T} \binom{T}{k} q_u^k \, q_d^{T-k} \Phi(s\, u^k\, d^{T-k})$$

- Note that, for big T the formula can't be directly used because of the binomial coefficient...
- Black-Scholes formula is the limit form! Proofs in Lamberton and Lapeyre [2007, pp.26–31]

Algorithmic Considerations

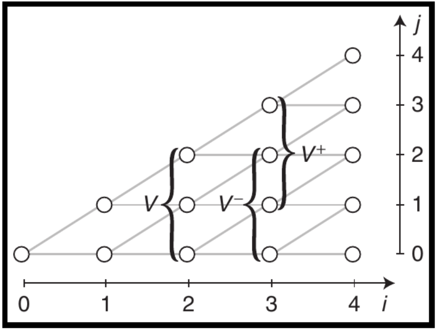


Figure: Vector Update

Python Code Illustration: Common Parts

```
import numpy as np

S0 = 80; r = 0; K = 80; u = 1.5; d = 0.5;
q = (1 - d) / (u - d); M = 3;
df = 1  # discount factor per time interval

# exhibit stock paths
S = np.zeros((M + 1, M + 1), dtype=np.float)
S[0, 0] = S0
for j in range(1, M + 1, 1):
    for i in range(j + 1):
        S[i, j] = S[0, 0] * (u ** (j - i)) * (d ** i)
```

Python Codes: Traditional Loops

```
# inner values: traditional loops
iv = np.zeros((M + 1, M + 1), dtype=np.float); z = 0
for j in range(0, M + 1, 1):
    for i in range(z + 1):
        iv[i, j] = round(max(S[i, j] - K, 0), 8)
    z += 1

# present values: traditional loops
pv = np.zeros((M + 1, M + 1), dtype=np.float)
pv[:, M] = iv[:, M]
z = M + 1
for j in range(M - 1, -1, -1):
    z -= 1
    for i in range(z):
        pv[i, j] = ( q       * pv[i, j + 1] +
                    (1 - q) * pv[i + 1, j + 1]) * df
```

Python Codes: Vectorized Loops

```python
# present values: vectorized loops

# pv initialized with iv in one line!!
pv = np.maximum(S - K, 0)

z = 0
for j in range(M - 1, -1, -1):
    pv[0:M - z, j] = (q       * pv[0:M - z, j + 1] +
                      (1 - q) * pv[1:M - z + 1, j + 1]) * df
    z += 1
```

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