Value at Risk / Expected Shortfall

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Risk Measure

In the following, the random variable X stands for the potential loss due to various risks.

Risk Measure

A risk measure is a mapping that assigns a value V_X to X.

Coherent Risk Measure

A risk measure V is said to be *coherent* if it satisfies the following properties for r.v.s. X, Y:

- (Subadditivity) $V_{X+Y} \leqslant V_X + V_Y$
- (Monotonicity) $X \leqslant Y \Longrightarrow V_X \leqslant V_Y$
- (Homogeneity) $V_{\lambda X} = \lambda V_X, \lambda > 0$
- (Translation Invariance) $V_{X+\mu} = V_X + \mu, \mu > 0$

Risk Measure (Cont.)

Subadditivity means that, the combined risk of several portfolios is lower than the sum of risks of those portfolios, as should happen with *portfolio diversification*.

Example

The expectation $\mathrm{E}X$ of a random variable X is a coherent risk measure.

Cumulative Distribution Function

Cumulative Distribution Function

The cumulative distribution function (cdf) F_X of a r.v. X is

$$F_X(x) \equiv \mathrm{P}(X \leqslant x)$$

Empirical Cumulative Distribution Function

$$F_N(x) \equiv \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{x_i \leqslant x\}}, \quad x \in \mathbb{R}$$

Value at Risk

The Value at Risk has two objectives:

- to provide a risk measure
- to determine an adequate level of capital reserves that matches the current level of risk

In other words, risk managing means determining a level V_X (of capital requirement) that will not be "too much" exceed by X. For example, setting V_X such that

$$P(X \leqslant V_X) \geqslant 0.95, \text{ i.e. } P(X > V_X) \leqslant 0.05$$

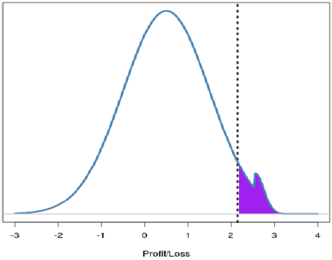
means that insolvency will occur with probability less than 5%.

Value at Risk

The Value at Risk (VaR) at the level $p \in (0, 1)$ (or p-quantile risk measure) is defined as

$$V_X^p \equiv \inf\{x \in \mathbb{R} : \mathrm{P}(X \leqslant x) \geqslant p\}$$

- X is lower than V_X^p with probability at least p.
- V_X^p can be negative, which indicates a profit.
- (Caveat) V_X^p does not contain any information on how large losses can be beyond V_X^p!



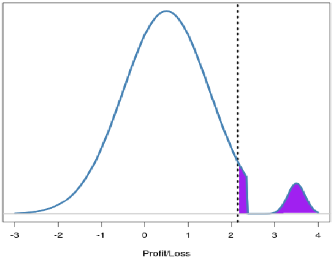


Figure: Two distributions having the same Value at Risk $V_X^{0.95} = 2.145$

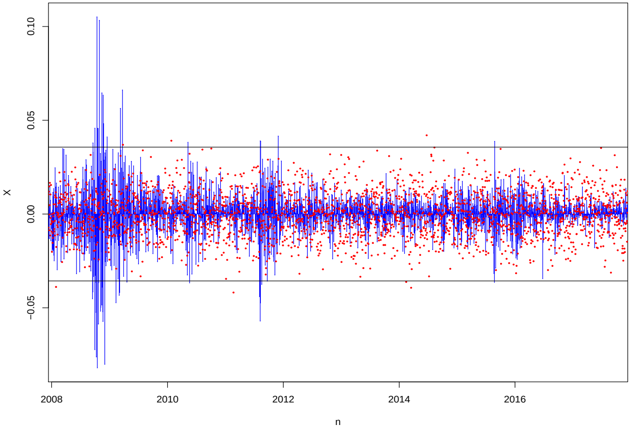


Figure: DJIA Market returns v.s. normalized Gaussian returns, which tends to underestimate the probabilities of extreme events.

The Tail Value at Risk

The Tail Value at Risk at the level p is defined by

$$\eta_X^p \equiv \frac{1}{1-p} \int_p^1 V_X^q \, \mathrm{d}q$$

The Conditional Tail Expectation

The Conditional Tail Expectation at the level p is defined by

$$\mathrm{CTE}_X^p \equiv \mathrm{E}\left\{X \,|\, X > V_X^p\right\} = \frac{\mathrm{E}\left\{X \mathbb{I}_{\{X > V_X^p\}}\right\}}{\mathrm{P}(X > V_X^p)}$$

The Expected Shortfall

The Expected Shortfall at the level p is defined by

$$E_X^p \equiv \frac{1}{1-p} \mathrm{E}\left\{X \mathbb{I}_{\{X \geqslant V_X^p\}}\right\} + \frac{V_X^p}{1-p}\left(1 - p - \mathrm{P}(X \geqslant V_X^p)\right)$$

Propositions

- V_X^p is coherent when X is Gaussian.
- If $\mathrm{P}(X = V_X^p) = 0$, then $\mathrm{CTE}_X^p = \eta_X^p$.
- $E_X^p = \eta_X^p$.
- E_X^p and η_X^p are coherent risk measures.

Example

Consider r.v. $X \in \{10, 100, 150\}$ with

$$P(X = 10) = 0.96, \ P(X = 100) = 0.03, \ P(X = 150) = 0.01.$$

Compute $V_X^{0.98}, \eta_X^{0.98}, \mathrm{CTE}_X^{0.98}, E_X^{0.98}$.

We have

$$\begin{aligned}
V_X^{0.98} &= 100, \\
\eta_X^{0.98} &= \frac{1}{0.02}\left((0.99 - 0.98) \times 100 + (1 - 0.99) \times 150\right) = 125, \\
\mathrm{CTE}_X^{0.98} &= \frac{1}{0.01} \times 150 \times 0.01 = 150, \\
E_X^{0.98} &= \frac{1}{0.02}(100 \times 0.03 + 150 \times 0.01) + \frac{100}{0.02}(0.02 - (0.03 + 0.01)) \\
&= 125.
\end{aligned}$$

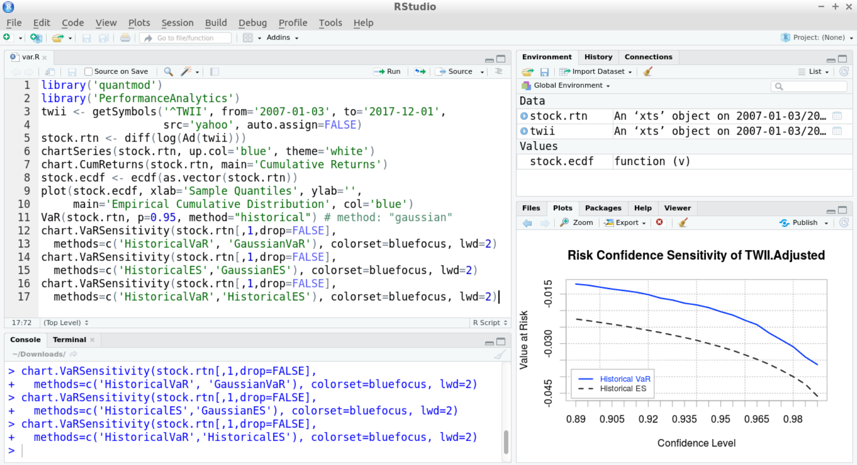


Figure: R Program / RStudio in Action

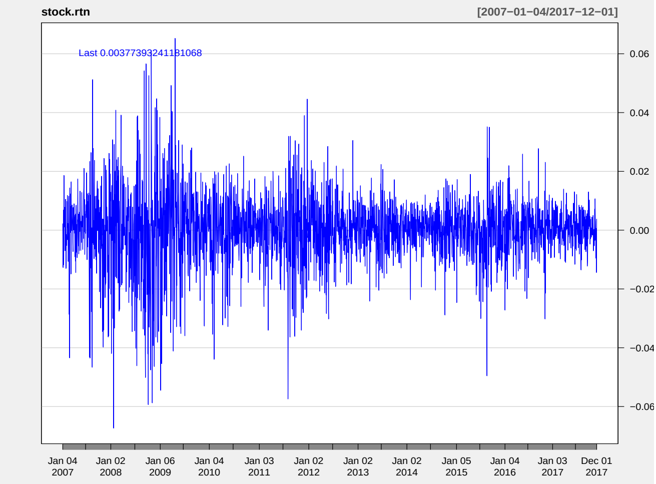


Figure: TWII stock returns

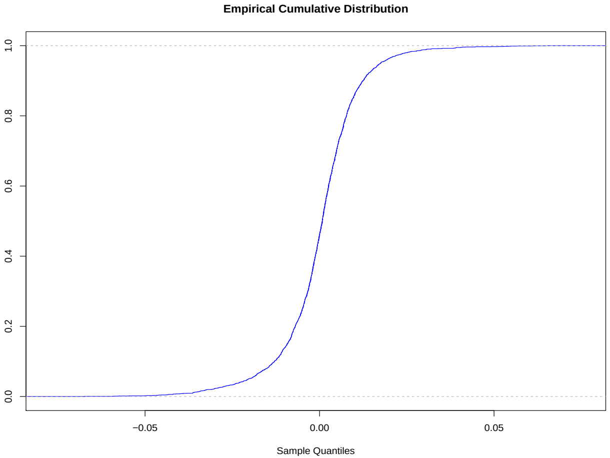


Figure: Empirical cumulative distribution function of TWII returns

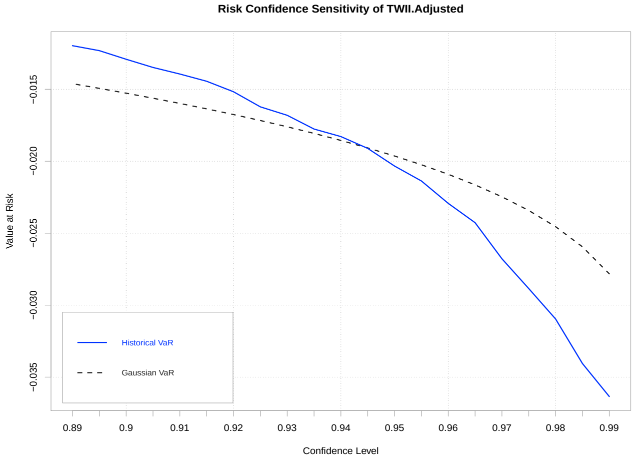


Figure: Historical v.s. Gaussian estimates of Value at Risk

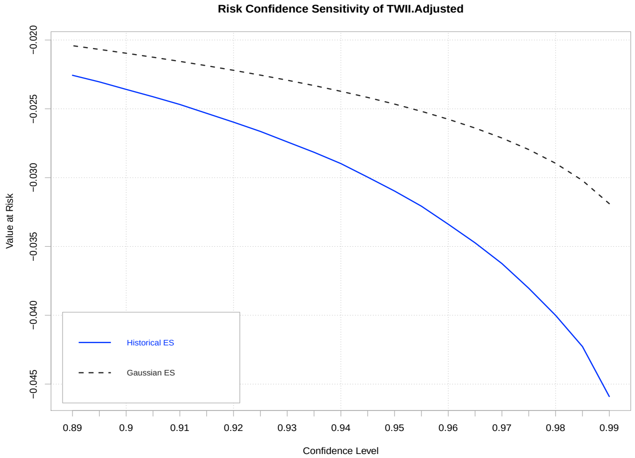


Figure: Historical v.s. Gaussian estimates of Expected Shortfall

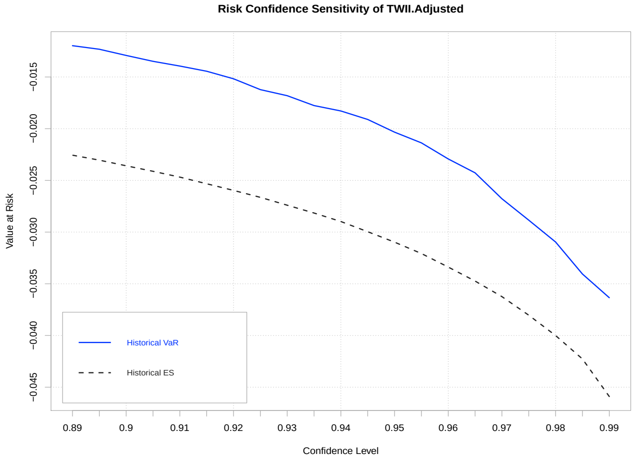


Figure: Historical Value at Risk v.s. Historical Expected Shortfall