On Merton's Optimal Consumption-Investment Problem: A Lie Group Analysis Approach

1. Introduction

In two pioneering papers Merton (1969, 1971) the Nobel Memorial Prize in Economic Sciences laureate Robert C. Merton formulated the prototypical optimal consumption-investment problem of continuous-time finance; the proposed solution is the tour de force application of stochastic calculus and optimal control techniques which culminated in the explicit integration of a complicated nonlinear Hamilton-Jacobi-Bellman (HJB) equation, albeit without intermediate derivation steps. Specifically, the HJB equation of the value function $u(t, w)$ (in this chapter we rename the dependent variable J to u) associated with the HARA utility function and the zero terminal condition (Merton (1971, (44)))

$$\frac{\delta^2\beta^{\frac{\gamma}{\delta}}}{\gamma}e^{-\frac{\rho}{\delta}t}u_w^{-\frac{\gamma}{\delta}} + u_t + \left(\frac{\delta\nu}{\beta} + rw\right)u_w - \delta(\mu - r)\frac{u_w^2}{u_{ww}} = 0; \quad u(T, w) = 0. \tag{1}$$

where

$$\delta = 1 - \gamma, \quad \mu = r + \frac{(\alpha - r)^2}{2\delta\sigma^2}$$

has the exact solution (Merton (1971, (47)))

$$u(t, w) = \frac{\delta^{2\delta}\beta^{\gamma}}{\gamma}e^{-\rho t}\left(\frac{1 - e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}}{\rho - \gamma\mu}\right)^{\delta}\left(w + \frac{\delta\nu}{\beta r}\left(1 - e^{r(t-T)}\right)\right)^{\gamma}, \tag{2}$$

the optimal fraction of wealth $q^\star$ invested in the risky asset and the optimal consumption rate $c^\star$ can be readily acquired by partial differentiations of this exact solution (Merton (1971, (45), (46)); see (10)). However, in Merton (1971) the result (2) is stated without hints as to the method of derivation.

As opposed to the ad hoc "guess but verify" approach of constructing exact solutions, there exists a systematic method, namely the Lie group analysis, which facilitates the understanding of the nature of the problem and in many occasions proposes the solution sought after. In a nutshell, Lie group analysis exploits the inherent symmetry structure of the equation to be solved both geometrically and algebraically and immensely simplifies the integration. The discipline has been developed for over a century; with the advent of modern computer algebra systems, the implementation of the procedure has been made easier than ever and deserves to be better known among researchers and professionals. One of the first applications of Lie group analysis in financial studies is the seminal paper

Gazizov and Ibragimov (1998) which investigates the symmetries and invariant solutions of the familiar Black-Scholes equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0$$

with constants A, B, and C, and the esoteric Jacob-Jones (Jacobs and Jones (1986)) equation

$$u_t - \frac{1}{2}A^2x^2u_{xx} - ABC\,xy\,u_{xy} - \frac{1}{2}B^2y^2u_{yy} \\ - \left(Dx\ln\frac{y}{x} + Ex^{\frac{3}{2}}\right)u_x - \left(Fy\ln\frac{G}{y} - Hy\,x^{\frac{1}{2}}\right)u_y + xu = 0$$

with constants A to H. Since then a series of papers adopt the Lie group analysis methodology for solving PDEs in finance and most of them originated from option pricing problems; among the few exceptions, the Bordag and Yamshchikov (2017) paper is the closest to the present note as both consider the reduction of the Hamilton-Jacobi-Bellman equation arising in portfolio optimization.

In this paper we begin by briefly reviewing the prototypical optimal consumption-investment problem proposed in Merton (1971) and supply some details which were glossed over or missing in the classic paper. Subsequently minimal facts of Lie group analysis needed for our investigation are introduced in a practical manner. Afterwards we perform the thorough analysis of the symmetry group admitted by Merton's equation (1) and derive a slightly different exact solution

$$u(t,w) = \frac{\delta^{2\delta}\beta^{\gamma}}{\gamma}e^{-\rho t}\left(\frac{1-e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}}{\rho-\gamma\mu}\right)^{\delta}\left(w + \frac{\delta\nu}{\beta r} + c_1\,e^{rt}\right)^{\gamma} + c_2, \quad c_1, c_2 \in \mathbb{R}. \tag{3}$$

which encompasses Merton's solution (2) as a particular case. This seemingly small discrepancy actually exposes the deficiency of Merton's treatment and is independent of the first documented arguments in Sethi and Taksar (1988). Finally, discussions conclude the paper.

2. Optimal Consumption-Investment Problem Revisited

Here we follow the heuristic approach of Sethi (2019) to delineate the problem. Let the wealth of an investor consists of a riskless asset p_0 with price dynamics

$$\mathrm{d}p_0 = p_0\,r\,\mathrm{d}t \tag{4}$$

and l risky assets $p_1, p_2, \ldots, p_l$ with price dynamics

$$\mathrm{d}p_i = p_i\left(\alpha_i\,\mathrm{d}t + \boldsymbol{e}_i\,\boldsymbol{S}\,\mathrm{d}\boldsymbol{Z}^{\top}\right), \qquad i = 1, 2, \ldots, l. \tag{5}$$

where r and α_i are constants with each $\alpha_i > r$, $\boldsymbol{e}_i = \left(0 \ldots 0\ 1\ 0 \ldots 0\right)$ is a $1 \times l$ row vector with 1 in the i-th place and 0 elsewhere, $\boldsymbol{S}$ is a $l \times l$ matrix such that $\boldsymbol{S}\boldsymbol{S}^{\top}$ is positive definite, and $\boldsymbol{Z}$ is the $1 \times l$ multidimensional standard Wiener process. Let

$\boldsymbol{q} \equiv \boldsymbol{q}(t) = \begin{pmatrix} q_1(t) & q_2(t) & \cdots & q_l(t) \end{pmatrix}$ be the $1 \times l$ weight vector such that the i-th component $q_i(t)$ is the fraction of wealth invested in the i-th asset at time t, $\sum_{i=0}^{l} q_i(t) = 1$ and $c \equiv c(t)$ be the consumption rate process. Set $1 \times l$ vectors $\boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_l \end{pmatrix}$ and $\mathbf{1}$ with all entries are 1, the wealth process w evolves as

$$\mathrm{d}w = \left(w\left(r + (\boldsymbol{\alpha} - r\mathbf{1})\,\boldsymbol{q}^\top\right) - c\right)\mathrm{d}t + w\,\boldsymbol{q}\,\boldsymbol{S}\,\mathrm{d}\boldsymbol{Z}^\top \tag{6}$$

To establish (6), let $n_i \equiv n_i(t)$ be the number of shares in i-th asset at time t and $0 < h \ll 1$, then

$$w(t) = \sum_{i=0}^{l} n_i(t-h)\,p_i(t)$$

and

$$\begin{aligned} -c(t)\,h &= \sum_{i=0}^{l} \left(n_i(t) - n_i(t-h)\right) p_i(t) \\ &= \sum_{i=0}^{l} \left(n_i(t) - n_i(t-h)\right)\left(p_i(t) - p_i(t-h)\right) + \left(n_i(t) - n_i(t-h)\right) p_i(t-h) \end{aligned}$$

On taking $h = \mathrm{d}t \to 0$, the above becomes

$$-c(t)\,\mathrm{d}t = \sum_{i=0}^{l} \mathrm{d}n_i\,\mathrm{d}p_i + \mathrm{d}n_i\,p_i$$

so from Itô lemma

$$\begin{aligned} \mathrm{d}w &= \sum_{i=0}^{l} n_i\,\mathrm{d}p_i + \mathrm{d}n_i\,p_i + \mathrm{d}n_i\,\mathrm{d}p_i = \sum_{i=0}^{l} n_i\,\mathrm{d}p_i - c\,\mathrm{d}t = \sum_{i=0}^{l} \frac{w\,q_i}{p_i}\,\mathrm{d}p_i - c\,\mathrm{d}t \\ &= \frac{w\,q_0}{p_0}\,\mathrm{d}p_0 + \sum_{i=1}^{l} \frac{w\,q_i}{p_i}\,\mathrm{d}p_i - c\,\mathrm{d}t = w\left(1 - \sum_{i=1}^{l} q_i\right)\frac{\mathrm{d}p_0}{p_0} + w\sum_{i=1}^{l} q_i\frac{\mathrm{d}p_i}{p_i} - c\,\mathrm{d}t \end{aligned}$$

Substitute (4), (5) back, (6) is readily obtained. Here in Merton's setup $l = 1$ and (6) becomes

$$\mathrm{d}w = \left(w\,(1-q)\,r + w\,q\,\alpha - c\right)\mathrm{d}t + w\,q\,\sigma\,\mathrm{d}Z$$

The optimal consumption-investment problem is to find c, q which maximize the functional

$$\mathsf{E}_0\left\{\int_0^T e^{-\rho\tau}\,U(c(\tau))\,\mathrm{d}\tau + B(w(T),T)\right\} \tag{7}$$

where T is a fixed terminal time, $\rho > 0$ is the discount factor, together with the utitlity function U of the investor and the bequest function B. Define the value function $u(t,w)$ as

$$u(t,w) = \max_{c,\,q}\,\mathsf{E}_t\left\{\int_t^T e^{-\rho\tau}\,U(c(\tau),\tau)\,\mathrm{d}\tau + B(w(T),T)\right\}$$

then by the principle of optimality

$$u(t,w) = \max_{c,\,q} \mathsf{E}_t \left\{ e^{-\rho t}\, U(c)\, \mathrm{d}t + u(t+\mathrm{d}t, w+\mathrm{d}w) \right\} \tag{8}$$

Applying Itô lemma to expand the term $u(t+\mathrm{d}t, w+\mathrm{d}w)$,

$$\begin{aligned} u(t+\mathrm{d}t, w+\mathrm{d}w) &= u(t,w) + u_t\,\mathrm{d}t + u_w\,\mathrm{d}w + \frac{1}{2} u_{ww}\,(\mathrm{d}w)^2 \\ &= u(t,w) + u_t\,\mathrm{d}t + u_w\left((w\,(1-q)\,r + w\,q\,\alpha - c)\,\mathrm{d}t + w\,q\,\sigma\,\mathrm{d}Z\right) \\ &\quad + \frac{1}{2} u_{ww} w^2 q^2 \sigma^2\,\mathrm{d}t \\ &= u(t,w) + u_w w\,q\,\sigma\,\mathrm{d}Z \\ &\quad + \left(u_t + u_w\,(w\,(1-q)\,r + w\,q\,\alpha - c) + \frac{1}{2} u_{ww} w^2 q^2 \sigma^2 \right) \mathrm{d}t \end{aligned}$$

Substitute back in (8) and note that $\mathsf{E}_t\left\{u_w w\,q\,\sigma\,\mathrm{d}Z\right\} = 0$, we have

$$0 = \max_{c,\,q} \left\{ e^{-\rho t}\, U(c) + u_t + u_w\,(w\,(1-q)\,r + w\,q\,\alpha - c) + \frac{1}{2} u_{ww} w^2 q^2 \sigma^2 \right\} \tag{9}$$

Differentiate (9) with respect to q and equate to zero, the first order condition gives the maximizer $q^\star$ as

$$-u_w w\,r + u_w w\,\alpha + u_{ww} w^2 \sigma^2 q^\star = 0 \quad \Longrightarrow \quad q^\star = -\frac{(\alpha - r) u_w}{\sigma^2 w u_{ww}} \tag{10}$$

If the utility function U belongs to the HARA (Hyperbolic Absolute Risk-Adverse) family, there exists constants γ, β, ν such that

$$U(c) = \frac{1-\gamma}{\gamma} \left(\frac{\beta\, c}{1-\gamma} + \nu \right)^{\gamma},$$

and $U'(c) = \beta \left(\frac{\beta\, c}{1-\gamma} + \nu \right)^{\gamma - 1}$; differentiate (9) with respect to c and equate to zero, the first order condition gives

$$e^{-\rho t}\, U'(c^\star) + u_w = 0 \quad \Longrightarrow \quad c^\star = \frac{1-\gamma}{\beta} \left(\frac{e^{\rho t} u_w}{\beta} \right)^{\frac{1}{\gamma-1}} - \frac{\nu(1-\gamma)}{\beta} \tag{11}$$

Substitute back the expressions (11), (10) of $c^\star$ and $q^\star$ respectively into (9), we arrive at the equation (1); the boundary condition is

$$u(T,w) = B(T, w(T)). \tag{12}$$

In Merton (1971), the bequest function B is taken to be zero for simplicity.

3. Rudiments of Lie Group Analysis

Here we state the essential facts of Lie group analysis directly related to our contruction of exact solutions. Our exposition follows Ibragimov (1999); further details can be found in Stephani (1989); Hydon (2000); Bluman and Kumei (1989); Anco and Bluman (2002); Bluman et al. (2010); Olver (1993); Ovsiannikov (1982). Olver (1993); Ovsiannikov (1982) provide rigorous treatment of the theoretical framework and its ramifications.

Symmetry Group. Infinitesimal Transform. Generators

For m independent variables denoted by $x = (x^1, x^2, \ldots, x^m) \equiv \{x^i\}$ and n dependent variables denoted by $u = (u^1, u^2, \ldots, u^n) \equiv \{u^\alpha\}$ (i.e. each u^α is a function of x), consider the invertible transformation $T_\varepsilon : (x, u) \mapsto (\overline{x}, \overline{u})$ consists of

$$\begin{aligned} \overline{x}^i &= \phi^i(x, u, \varepsilon), \qquad & \phi^i|_{\varepsilon=0} &= x^i, \qquad & i &= 1, 2, \ldots, m \\ \overline{u}^\alpha &= \psi^\alpha(x, u, \varepsilon), \qquad & \psi^\alpha|_{\varepsilon=0} &= u^\alpha, \qquad & \alpha &= 1, 2, \ldots, n. \end{aligned} \tag{13}$$

with smooth functions ϕ^i, ψ^α and parameter ε; such transformations $\{T_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ form an one-parameter local group G if

1. (Identity) $\mathsf{I} \in G$.
2. (Inverse) $T_\varepsilon^{-1} = T_{-\varepsilon} \in G$.
3. (Composition) $T_{\varepsilon_1} \circ T_{\varepsilon_2} \in G$ if T_{ε_1}, $T_{\varepsilon_2} \in G$.

For such an one-parameter local group G, using Taylor expansion with respect to ε in a neighborhood of $\varepsilon = 0$, equation (13) satisfied by T_ε become

$$\begin{aligned} \overline{x}^i &= x^i + \xi^i(x, u)\, \varepsilon + \mathcal{O}(\varepsilon^2), \quad & \xi^i(x, u) &= \left.\frac{\partial \phi^i(x, u, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad & i &= 1, 2, \ldots, m. \\ \overline{u}^\alpha &= u^\alpha + \eta^\alpha(x, u)\, \varepsilon + \mathcal{O}(\varepsilon^2), \quad & \eta^\alpha(x, u) &= \left.\frac{\partial \psi^\alpha(x, u, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad & \alpha &= 1, 2, \ldots, n. \end{aligned} \tag{14}$$

This is the *infinitesimal transformation* of G, and the *generator* of G, denoted by X, is defined as

$$X = \xi^i(x, u)\, \partial_{x^i} + \eta^\alpha(x, u)\, \partial_{u^\alpha} \tag{15}$$

where $\partial_{x^i} \equiv \frac{\partial}{\partial x^i}$ is the partial differential operator with respect to the variable x^i. We invoke and tacitly assume afterwards the repeated indices summation convention. For instance, $\xi^i(x, u)\, \partial_{x^i}$ should be understood as $\sum_{i=1}^m \xi^i(x, u)\, \partial_{x^i}$, and so on. Hereafter $u_{(k)}$ stands for the set of all derivatives of $u = \{u^\alpha\}$ with order $k \geqslant 1$, i.e. $u_{(1)} = \{u^\alpha_{i_1}\}$, $u_{(2)} = \{u^\alpha_{i_1 i_2}\}$, etc.

Prolongation

Given the transformation (13) of a group G which maps (x, u) to $(\overline{x}, \overline{u})$, transformations for derivatives can be obtained by means of the chain rule of the differential calculus; the extended transformation which maps $(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)})$ to $(\overline{x}, \overline{u}, \overline{u}_{(1)}, \overline{u}_{(2)}, \ldots, \overline{u}_{(k)})$ with integer $k \geqslant 1$ also forms a group and the extension process is called the *prolongation*.

To illustrate the extension process, we begin by first examine the simplest case $m = n = 1$. In this case, set $x \equiv x^1$, $y \equiv u^1$, the infinitesimal transformation (14) reads

$$\overline{x} = x + \varepsilon\, \xi(x, y) + \mathcal{O}(\varepsilon^2), \qquad \overline{y} = y + \varepsilon\, \eta(x, y) + \mathcal{O}(\varepsilon^2).$$

and the aim is to use this knowledge to get

$$\overline{y}^{(i)} = y^{(i)} + \varepsilon\,\eta_i + \mathcal{O}(\varepsilon^2), \quad i = 1, 2, \ldots \tag{16}$$

Define the total differentiation operator D_x as

$$\mathrm{D}_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots + y^{(k)}\partial_{y^{(k-1)}} + \cdots$$

Note that $\overline{x}$, $\overline{y}$ are functions of x, y; by the chain rule and (13),

$$\overline{y}^{(1)} = \frac{\mathrm{d}\overline{y}}{\mathrm{d}\overline{x}} = \frac{\psi_x\,\mathrm{d}x + \psi_y\,\mathrm{d}y}{\phi_x\,\mathrm{d}x + \phi_y\,\mathrm{d}y} = \frac{\psi_x + y'\psi_y}{\phi_x + y'\phi_y} = \frac{\mathrm{D}_x(\overline{y})}{\mathrm{D}_x(\overline{x})}$$

so

$$\begin{aligned}
\overline{y}^{(1)} = \frac{\mathrm{D}_x(\overline{y})}{\mathrm{D}_x(\overline{x})} &= \frac{y' + \varepsilon\,\mathrm{D}_x(\eta) + \mathcal{O}(\varepsilon^2)}{1 + \varepsilon\,\mathrm{D}_x(\xi) + \mathcal{O}(\varepsilon^2)} \\
&= \left(y' + \varepsilon\,\mathrm{D}_x(\eta) + \mathcal{O}(\varepsilon^2)\right)\left(1 - \varepsilon\,\mathrm{D}_x(\xi) + \mathcal{O}(\varepsilon^2)\right) \\
&= y' + \varepsilon\left(\mathrm{D}_x(\eta) - y'\mathrm{D}_x(\xi)\right) + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

and in view of (16),

$$\eta_1 = \mathrm{D}_x(\eta) - y'\,\mathrm{D}_x(\xi).$$

The process continues indefinitely as

$$\begin{aligned}
\overline{y}^{(k)} = \frac{\mathrm{D}_x(\overline{y}^{(k-1)})}{\mathrm{D}_x(\overline{x})} &= \frac{y^{(k)} + \varepsilon\,\mathrm{D}_x(\eta_{k-1}) + \mathcal{O}(\varepsilon^2)}{1 + \varepsilon\,\mathrm{D}_x(\xi) + \mathcal{O}(\varepsilon^2)} \\
&= \left(y^{(k)} + \varepsilon\,\mathrm{D}_x(\eta_{k-1}) + \mathcal{O}(\varepsilon^2)\right)\left(1 - \varepsilon\,\mathrm{D}_x(\xi) + \mathcal{O}(\varepsilon^2)\right) \\
&= y^{(k)} + \varepsilon\left(\mathrm{D}_x(\eta_{k-1}) - y^{(k)}\,\mathrm{D}_x(\xi)\right) + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

with

$$\eta_k = \mathrm{D}_x(\eta_{k-1}) - y^{(k)}\,\mathrm{D}_x(\xi)$$

Full expansions of η_1, η_2, and η_3 are

$$\eta_1 = \eta_x + (\eta_y - \xi_x)\,y' - \xi_y\,y'^2 \tag{17}$$

$$\eta_2 = \eta_{xx} + (2\,\eta_{xy} - \xi_{xx})\,y' + (\eta_{yy} - 2\,\xi_{xy})\,y'^2 - \xi_{yy}\,y'^3 + (\eta_y - 2\,\xi_x - 3\,\xi_y\,y')\,y'' \tag{18}$$

$$\begin{aligned}
\eta_3 = {} & \eta_{xxx} + (3\,\eta_{xxy} - \xi_{xxx})\,y' + 3\,(\eta_{xyy} - \xi_{xxy})\,y'^2 + (\eta_{yyy} - 3\,\xi_{xyy})\,y'^3 - \xi_{yyy}\,y'^4 \\
& + 3\left(\eta_{xy} - 2\,\xi_{xx} + (\eta_{yy} - 3\,\xi_{xy})\,y' - 2\,\xi_{yy}\,y'^2\right)y'' - 3\,\xi_y\,y''^2 + (\eta_y - 3\,\xi_x - 4\,\xi_y\,y')\,y'''
\end{aligned} \tag{19}$$

Starting from the original symmetry group with generator X,

$$X = \xi\,\partial_x + \eta\,\partial_y$$

the generator of the k-th extended group, denoted by $X_{(k)}$, is given by

$$X_{(k)} = \xi\,\partial_x + \eta\,\partial_y + \eta_1\,\partial_{y'} + \cdots + \eta_k\,\partial_{y^{(k)}}$$

For the general case, define the total differentiation operator $\mathrm{D}_i \equiv \mathrm{D}_{x^i}$ as

$$\mathrm{D}_i = \partial_{x^i} + u_i^\alpha \partial_{u^\alpha} + u_{i\,i_1}^\alpha \partial_{u_{i_1}^\alpha} + u_{i\,i_1 i_2}^\alpha \partial_{u_{i_1 i_2}^\alpha} + \cdots \tag{20}$$

By means of the chain rule,

$$\mathrm{D}_i = \left(\frac{\partial \phi^j}{\partial x^i} + u_i^\alpha \frac{\partial \phi^j}{\partial u^\alpha}\right)\overline{\mathrm{D}}_j = D_i(\phi^j)\,\overline{\mathrm{D}}_j \tag{21}$$

where $\overline{\mathrm{D}}_j \equiv \mathrm{D}_{\overline{x}^j}$ and the repeated indices summation convention is used. Note that

$$\overline{u}_i^\alpha = \overline{\mathrm{D}}_i(\overline{u}^\alpha), \quad \overline{u}_{ij}^\alpha = \overline{\mathrm{D}}_j(\overline{u}_i^\alpha) = \overline{\mathrm{D}}_j\,\overline{\mathrm{D}}_i(\overline{u}^\alpha).$$

By (21) and (13)

$$\overline{u}_j^\alpha\,\mathrm{D}_i(\phi^j) = \mathrm{D}_i(\psi^\alpha) \tag{22}$$

Assuming (14), we wish to get the prolongations

$$\overline{u}_{i_1\ldots i_s}^\alpha = u_{i_1\ldots i_s}^\alpha + \varepsilon\,\zeta_{i_1\ldots i_s}^\alpha + \mathcal{O}(\varepsilon^2), \quad s = 1, 2, \ldots \tag{23}$$

and each integer $1 \leqslant i_s \leqslant m$ and $1 \leqslant \alpha \leqslant n$. For $s = 1$, the first extended transformation reads

$$\overline{u}_i^\alpha = u_i^\alpha + \varepsilon\,\zeta_i^\alpha + \mathcal{O}(\varepsilon^2) \tag{24}$$

with $i \equiv i_1$ and ζ_i^α to be determined. From (14), $\phi^j = x^j + \varepsilon\,\xi^j$, $\psi^\alpha = u^\alpha + \varepsilon\,\eta^\alpha$, so $\mathrm{D}_i(\phi^j) = \delta_{ij} + \varepsilon\,\mathrm{D}_i(\xi^j)$, $\mathrm{D}_i(\psi^\alpha) = \mathrm{D}_i(u^\alpha) + \varepsilon\,\mathrm{D}_i(\eta^\alpha) = u_i^\alpha + \varepsilon\,\mathrm{D}_i(\eta^\alpha)$. Substitute into (22) and it becomes a system of linear equations

$$\overline{u}_i^\alpha + \varepsilon\,\mathrm{D}_i(\xi^j)\,\overline{u}_j^\alpha = u_i^\alpha + \varepsilon\,\mathrm{D}_i(\eta^\alpha), \quad 1 \leqslant i \leqslant m.$$

Set $\overline{\boldsymbol{u}} = \begin{pmatrix} \overline{u}_1^\alpha & \overline{u}_2^\alpha & \ldots & \overline{u}_m^\alpha \end{pmatrix}^\top$, $\boldsymbol{u} = \begin{pmatrix} u_1^\alpha & u_2^\alpha & \ldots & u_m^\alpha \end{pmatrix}^\top$, $\boldsymbol{\eta} = \begin{pmatrix} \mathrm{D}_1(\eta^\alpha) & \mathrm{D}_2(\eta^\alpha) & \ldots & \mathrm{D}_m(\eta^\alpha) \end{pmatrix}^\top$, and $(\boldsymbol{A})_{ij} = \mathrm{D}_i(\xi^j)$, this linear system can be rewritten as

$$(\boldsymbol{I} + \varepsilon \boldsymbol{A})\,\overline{\boldsymbol{u}} = \boldsymbol{u} + \varepsilon\boldsymbol{\eta} \tag{25}$$

For small ε, $(\boldsymbol{I} + \varepsilon\boldsymbol{A})^{-1} = (\boldsymbol{I} - \varepsilon\boldsymbol{A}) + \mathcal{O}(\varepsilon^2)$ and (25) becomes

$$\overline{\boldsymbol{u}} = (\boldsymbol{I} - \varepsilon\boldsymbol{A})\,(\boldsymbol{u} + \varepsilon\boldsymbol{\eta}) + \mathcal{O}(\varepsilon^2) = \boldsymbol{u} + \varepsilon\,(\boldsymbol{\eta} - \boldsymbol{A}\boldsymbol{u}) + \mathcal{O}(\varepsilon^2) \tag{26}$$

Comparing with (24) and spelling out the expression componentwise, we have

$$\zeta_i^\alpha = \mathrm{D}_i(\eta^\alpha) - u_j^\alpha\,\mathrm{D}_i(\xi^j) \tag{27}$$

Similarly, the second extended transformation reads

$$\overline{u}_{i_1 i_2}^\alpha = u_{i_1 i_2}^\alpha + \varepsilon\,\zeta_{i_1 i_2}^\alpha + \mathcal{O}(\varepsilon^2) \tag{28}$$

with $\zeta^\alpha_{i_1 i_2}$ to be determined. Applying (21) to $\overline{u}^\alpha_{i_1}$ with $i = i_2$,

$$D_{i_2}(\phi^j)\,\overline{\mathrm{D}}_j(\overline{u}^\alpha_{i_1}) = \mathrm{D}_{i_2}(\overline{u}^\alpha_{i_1}) \tag{29}$$

Note that $\mathrm{D}_{i_2}(\phi^j) = \delta_{i_2 j} + \varepsilon\,\mathrm{D}_{i_2}(\xi^j)$, $\overline{\mathrm{D}}_j(\overline{u}^\alpha_{i_1}) = \overline{u}^\alpha_{i_1 j}$, and $\mathrm{D}_{i_2}(\overline{u}^\alpha_{i_1}) = \mathrm{D}_{i_2}(u^\alpha_{i_1} + \varepsilon\,\zeta^\alpha_{i_1} + \mathcal{O}(\varepsilon^2)) = u^\alpha_{i_1 i_2} + \varepsilon\,\mathrm{D}_{i_2}(\zeta^\alpha_{i_1}) + \mathcal{O}(\varepsilon^2))$. Substitute into (29), it becomes a system of linear equations

$$\overline{u}^\alpha_{i_1 i_2} + \varepsilon\,\mathrm{D}_{i_2}(\xi^j)\,\overline{u}^\alpha_{i_1 j} = u^\alpha_{i_1 i_2} + \varepsilon\,\mathrm{D}_{i_2}(\zeta^\alpha_{i_1}), \qquad 1 \leqslant i_2 \leqslant m. \tag{30}$$

which resembles the system (25) previously solved; imitating the steps we have

$$\begin{aligned}\zeta^\alpha_{i_1 i_2} &= \mathrm{D}_{i_2}(\zeta^\alpha_{i_1}) - u^\alpha_{j\,i_1}\mathrm{D}_{i_2}(\xi^j)\\ &= \mathrm{D}_{i_2}\mathrm{D}_{i_1}(\eta^\alpha) - u^\alpha_j \mathrm{D}_{i_2}\mathrm{D}_{i_1}(\xi^j) - u^\alpha_{j\,i_1}\mathrm{D}_{i_2}(\xi^j)\end{aligned} \tag{31}$$

and the general expression reads

$$\zeta^\alpha_{i_1\ldots i_s} = \mathrm{D}_{i_s}(\zeta^\alpha_{i_1\ldots i_{s-1}}) - u^\alpha_{j\,i_1\ldots i_{s-1}}\mathrm{D}_{i_s}(\xi^j). \tag{32}$$

By writing $W^\alpha = \eta^\alpha - \xi^j u^\alpha_j$,

$$\zeta^\alpha_{i_1\ldots i_s} = \mathrm{D}_{i_1}\cdots\mathrm{D}_{i_s}(W^\alpha) - \xi^j u^\alpha_{j\,i_1\ldots i_s}, \quad s = 1, 2, \ldots \tag{33}$$

Starting from the original symmetry group with generator X,

$$X = \xi^i\,\partial_{x^i} + \eta^\alpha\,\partial_{u^\alpha}$$

the generator of the k-th extended group, denoted by $X_{(k)}$, is given by

$$X_{(k)} = \xi^i\,\partial_{x^i} + \eta^\alpha\,\partial_{u^\alpha} + \zeta^\alpha_{i_1}\,\partial_{u^\alpha_{i_1}} + \ldots + \zeta^\alpha_{i_1 i_2\ldots i_k}\,\partial_{u^\alpha_{i_1 i_2\ldots l_k}} \tag{34}$$

Determining Equations

The notion of prolongation is indispensable in view of the following theorem. Consider a typical system of (possibly nonlinear) equations

$$F_j(x) = 0, \quad j = 1, 2, \ldots, l$$

with $x \in \mathbb{R}^n$, $l < n$, and

$$\mathrm{rank}\left|\frac{\partial F_j(x)}{\partial x^i}\right| = l \qquad x \in V.$$

where $\frac{\partial F_j(x)}{\partial x^i}$ is the jacobian and $V \subseteq \mathbb{R}^n$ is the set of solutions. Given a transformation group G with $\overline{x} = \phi(x, \varepsilon)$ and the generator X, the system is said to be *invariant* with respect to G, or *admits* G, if

$$F_j(\overline{x}) = 0 \qquad \forall x \in V \text{ and } j = 1, 2, \ldots, l.$$

Theorem. The typical system of equations admits G if and only if

$$X F_j(x) = 0 \qquad \forall x \in V \text{ and } j = 1, 2, \ldots, l.$$

This theorem provides the key to the algorithmic determination of the symmetry group admitted by a given system of differential equations, which in turn generates exact solution candidates that can be exploited further. Specifically, for a typical system of differential equations

$$F_j\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0, \quad j = 1, 2, \ldots, l$$

and V its solution manifold, the condition that the system admits the symmetry group G with infinitesimal transform (14) is

$$X_{(k)} F_j\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0 \qquad \forall u \in V \text{ and } j = 1, 2, \ldots, l. \tag{35}$$

where $X_{(k)}$ is the generator of the k-th extended group of G given by (34); (35) will be referred to as *symmetry condition.* The resulting system of differential equations of (35) with unknown ξ's and η's are called *determining equations.* To solve the determining equations may seem daunting at first, but the overdetermined nature of the system often yield substantial simplifications; the whole solution process is best illustrated through detailed examples.

We consider the simplest case $m = n = 1$ first. Given a second-order ODE of the form

$$y'' = H(x, y, y')$$

the symmetry condition is

$$X_{(2)}\left(y'' - H(x, y, y')\right) = 0,$$

i.e. $X_{(2)} y'' = X_{(2)} H$. From

$$X_{(2)} = \xi\, \partial_x + \eta\, \partial_y + \eta_1\, \partial_{y'} + \eta_2\, \partial_{y''},$$

$X_{(2)} y'' = \eta_2$ and $X_{(2)} H = \xi\, H_x + \eta\, H_y + \eta_1\, H_{y'}$; substitute the expressions of η_1, η_2 in (17), (18) and note that $y'' = H$, the symmetry condition $X_{(2)} y'' = X_{(2)} H$ becomes

$$\begin{aligned} \eta_{xx} + (2\,\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} - 2\,\xi_{xy})\, y'^2 - \xi_{yy}\, y'^3 + (\eta_y - 2\,\xi_x - 3\,\xi_y\, y')\, H& \\ = \xi\, H_x + \eta\, H_y + \left(\eta_x + (\eta_y - \xi_x)\, y' - \xi_y\, y'^2\right) H_{y'}& \end{aligned} \tag{36}$$

As an example, we determine the symmetry group admitted by the second-order nonlinear ODE

$$y'' = \frac{y'^2}{y} - y^2$$

In this case $H(x, y, y') = \frac{y'^2}{y} - y^2$, so $H_x = 0, H_y = -\frac{y'^2}{y^2} - 2y, H_{y'} = \frac{2\,y'}{y}$. Applying (36), we have

$$\begin{aligned} \eta_{xx} + (2\,\eta_{xy} - \xi_{xx})\, y' + (\eta_{yy} - 2\,\xi_{xy})\, y'^2 - \xi_{yy}\, y'^3 + (\eta_y - 2\,\xi_x - 3\,\xi_y\, y')\left(\frac{y'^2}{y} - y^2\right)& \\ = \eta\left(-\frac{y'^2}{y^2} - 2y\right) + \left(\eta_x + (\eta_y - \xi_x)\, y' - \xi_y\, y'^2\right)\frac{2y'}{y}& \end{aligned}$$

Collecting all powers of y' and equating to zero, we have

$$\xi_{yy} + \frac{1}{y}\,\xi_y = 0 \tag{37}$$

$$\eta_{yy} - 2\,\xi_{xy} - \frac{1}{y}\,\eta_y + \frac{1}{y^2}\,\eta = 0 \tag{38}$$

$$2\,\eta_{xy} - \xi_{xx} + 3y^2\,\xi_y - \frac{2}{y}\,\eta_x = 0 \tag{39}$$

$$\eta_{xx} - y^2(\eta_y - 2\,\xi_x) + 2\,y\,\eta = 0 \tag{40}$$

From (37)

$$\xi = A(x)\,\log|y| + B(x)$$

with $A(x)$, $B(x)$ to be determined. From (38) and above we have

$$\eta = A'(x)\,y\,(\log|y|)^2 + C(x)\,y\,\log|y| + D(x)\,y$$

Substitute ξ, η into (39), we have

$$3\,A''(x)\,\log|y| + 3\,A(x)\,y + 2\,C'(x) - B''(x) = 0$$

and it holds for all y, so

$$A(x) = 0, \quad B''(x) = 2\,C'(x). \tag{41}$$

Now (40) becomes

$$C(x)\,y^2\,\log|y| + C''(x)\,y\,\log|y| + (2\,B'(x) - C(x) + D(x))\,y^2 + D''(x)\,y = 0$$

which splits into

$$C(x) = 0, \quad D(x) = -2\,B'(x), \quad D''(x) = 0 \tag{42}$$

From (41), (42)

$$\xi = c_1 + c_2\,x, \quad \eta = -2c_2\,y$$

so the generator X of the admitted symmetry group is

$$X = c_1X_1 + c_2X_2, \qquad X_1 = \partial_x, \quad X_2 = x\,\partial_x - 2y\,\partial_y. \tag{43}$$

Invariant Solution

For G is a symmetry group of a typical system of differential equations

$$F_j\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0, \quad j = 1, 2, \ldots, l$$

if

$$F_j\left(\overline{x}, \overline{u}, \overline{u}_{(1)}, \overline{u}_{(2)}, \ldots, \overline{u}_{(k)}\right) = 0, \quad j = 1, 2, \ldots, l$$

Under actions of the admitted symmetry group, transformed solutions still satisfy the system; those unaltered solutions are called the *invariant solution* and can be obtained by first computing the invariants of G via solving the PDE

$$X(I) = \xi^i(x,u)\,\partial_{x^i} I + \eta^\alpha(x,u)\,\partial_{u^\alpha} I = 0 \tag{44}$$

or equivalently integrating its characteristic system

$$\frac{\mathrm{d}x^1}{\xi^1(x,u)} = \frac{\mathrm{d}x^2}{\xi^2(x,u)} = \cdots = \frac{\mathrm{d}x^m}{\xi^m(x,u)} = \frac{\mathrm{d}u^1}{\eta^1(x,u)} = \frac{\mathrm{d}u^2}{\eta^2(x,u)} = \cdots = \frac{\mathrm{d}u^n}{\eta^n(x,u)} \tag{45}$$

Now the symmetry group has $m-1+n$ invariants

$$\lambda^1(x), \lambda^2(x), \ldots, \lambda^{m-1}(x), \quad \Phi^1(x,u), \Phi^2(x,u), \ldots, \Phi^n(x,u).$$

It can be shown that

$$\Phi^\alpha(x,u) = \Psi^\alpha(\lambda^1(x), \lambda^2(x), \ldots, \lambda^{m-1}(x))$$

for Ψ^α to be determined; solve for u and substitute the result back into the original system to get a new system comprised of Ψ's and λ's.

4. Symmetry Group Admitted by Merton's Equation

In this case $m = 2$, $n = 1$ and we set $t \equiv x^1$, $w \equiv x^2$, $u \equiv u^1$, and $\eta \equiv \eta^1$. X, the generator of the symmetry group admitted by Merton's equation (1)

$$\frac{\delta^2 \beta^{\frac{\gamma}{\delta}}}{\gamma} e^{-\frac{\rho}{\delta}t} u_w^{-\frac{\gamma}{\delta}} + u_t + \left(\frac{\delta\nu}{\beta} + rw\right) u_w - \delta(\mu - r)\frac{u_w^2}{u_{ww}} = 0; \quad u(T,w) = 0.$$

is

$$X = \xi^1(t,w,u)\,\partial_t + \xi^2(t,w,u)\,\partial_w + \eta(t,w,u)\,\partial_u \tag{46}$$

and $X_{(2)}$, the generator of the second extended group is

$$X_{(2)} = \xi^1\partial_t + \xi^2\partial_w + \eta\,\partial_u + \zeta_1\,\partial_{u_t} + \zeta_2\,\partial_{u_w} + \zeta_{11}\,\partial_{u_{tt}} + \zeta_{12}\,\partial_{u_{tw}} + \zeta_{22}\,\partial_{u_{ww}} \tag{47}$$

with

$$\begin{aligned} \zeta_1 &= \mathrm{D}_t(\eta) - u_t\mathrm{D}_t(\xi^1) - u_w\mathrm{D}_t(\xi^2) \\ \zeta_2 &= \mathrm{D}_w(\eta) - u_t\mathrm{D}_w(\xi^1) - u_w\mathrm{D}_t(\xi^2) \\ \zeta_{11} &= \mathrm{D}_t(\zeta_1) - u_{tt}\mathrm{D}_t(\xi^1) - u_{tw}\mathrm{D}_t(\xi^2) \\ \zeta_{12} &= \mathrm{D}_w(\zeta_1) - u_{tt}\mathrm{D}_w(\xi^1) - u_{tw}\mathrm{D}_w(\xi^2) \\ \zeta_{22} &= \mathrm{D}_w(\zeta_2) - u_{tw}\mathrm{D}_w(\xi^1) - u_{ww}\mathrm{D}_w(\xi^2) \end{aligned} \tag{48}$$

where the total differentiation opertors D_t, D_w are denoted by

$$\begin{aligned} \mathrm{D}_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tw}\partial_{u_w} + \cdots \\ \mathrm{D}_w &= \partial_w + u_w\partial_u + u_{tw}\partial_{u_t} + u_{ww}\partial_{u_w} + \cdots \end{aligned} \tag{49}$$

Expanding the terms by total differentiations, we have

$$
\begin{aligned}
\zeta_1 &= \eta_t + u_t\eta_u - u_t\xi^1_t - (u_t)^2\xi^1_u - u_w\xi^2_t - u_t u_w\xi^2_u \\
\zeta_2 &= \eta_w + u_w\eta_u - u_t\xi^1_w - u_t u_w\xi^1_u - u_w\xi^2_w - (u_w)^2\xi^2_u \\
\zeta_{11} &= \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + (u_t)^2\eta_{uu} - 2u_{tt}\xi^1_t - u_t\xi^1_{tt} \\
&\quad - 2(u_t)^2\xi^1_{tu} - 3u_t u_{tt}\xi^1_u - (u_t)^3\xi^1_{uu} - 2u_{tw}\xi^2_t - u_w\xi^2_{tt} \\
&\quad - 2u_t u_w\xi^2_{tu} - (u_w u_{tt} + 2u_t u_{tw})\xi^2_u - (u_t)^2 u_w\xi^2_{uu} \\
\zeta_{12} &= \eta_{tw} + u_w\eta_{tu} + u_t\eta_{wu} + u_{tw}\eta_u + u_t u_w\eta_{uu} - u_{tw}(\xi^1_t + \xi^2_w) \\
&\quad - u_t\xi^1_{tw} - u_{tt}\xi^1_w - u_t u_w(\xi^1_{tu} + \xi^2_{wu}) - (u_t)^2\xi^1_{wu} \\
&\quad - (2u_t u_{tw} + u_w u_{tt})\xi^1_u - (u_t)^2 u_w\xi^1_{uu} - u_w\xi^2_{tw} - u_{ww}\xi^2_t \\
&\quad - (u_w)^2\xi^2_{tu} - (2u_w u_{tw} + u_t u_{ww})\xi^2_u - u_t(u_w)^2\xi^2_{uu} \\
\zeta_{22} &= \eta_{ww} + 2u_w\eta_{wu} + u_{ww}\eta_u + (u_w)^2\eta_{uu} - 2u_{ww}\xi^2_w - u_w\xi^2_{ww} \\
&\quad - 2(u_w)^2\xi^2_{wu} - 3u_w u_{ww}\xi^2_u - (u_w)^3\xi^2_{uu} - 2u_{tw}\xi^1_w - u_t\xi^1_{ww} \\
&\quad - 2u_t u_w\xi^1_{wu} - (u_t u_{ww} + 2u_w u_{tw})\xi^1_u - u_t(u_w)^2\xi^1_{uu}
\end{aligned}
\tag{50}
$$

Aided by the open-source computer algebra system `Maxima` with subroutine `symmgrp2009` (Champagne et al. (1991); Hereman (1997)), we arrive at the set of determining equations:

$$\eta_w(\gamma-1)e^{\frac{t\gamma\rho}{\gamma-1}-t\rho} = 0 \tag{51}$$

$$\eta_w(r-\alpha)^2 = 0 \tag{52}$$

$$\eta_{ww}(r-\alpha)^2 = 0 \tag{53}$$

$$\xi^1_u(\gamma-1)^3 e^{\frac{2t\gamma\rho}{\gamma-1}-2t\rho} = 0 \tag{54}$$

$$\xi^1_w(\gamma-1)^3 e^{\frac{2t\gamma\rho}{\gamma-1}-2t\rho} = 0 \tag{55}$$

$$w\,\eta_w\,\beta r - \eta_w\,\nu\gamma + \eta_w\,\nu + \eta_t\,\beta = 0 \tag{56}$$

$$\xi^1_{uu}(r-\alpha)^4 = 0 \tag{57}$$

$$\xi^1_{uw}(r-\alpha)^4 = 0 \tag{58}$$

$$\xi^1_{ww}(r-\alpha)^4 = 0 \tag{59}$$

$$\xi^1_u(r-\alpha)^4 = 0 \tag{60}$$

$$\xi^1_w(r-\alpha)^4 = 0 \tag{61}$$

$$\beta^{-\frac{\gamma}{\gamma-1}-1}(\gamma-1)\left\{w\,\xi^1_u\,\beta r - \xi^1_u\,\nu\gamma + \xi^1_u\,\nu - \xi^2_u\,\beta\right\}e^{\frac{t\gamma\rho}{\gamma-1}-t\rho} = 0 \tag{62}$$

$$\xi^1_u\,(\gamma-1)(2\gamma-1)(r-\alpha)^2 e^{\frac{t\gamma\rho}{\gamma-1}-t\rho}=0 \tag{63}$$
$$\xi^1_w\,(\gamma-2)(\gamma-1)(r-\alpha)^2 e^{\frac{t\gamma\rho}{\gamma-1}-t\rho}=0 \tag{64}$$
$$\xi^1_{uu}\,(\gamma-1)^2(r-\alpha)^2 e^{\frac{t\gamma\rho}{\gamma-1}-t\rho}=0 \tag{65}$$
$$\xi^1_{uw}\,(\gamma-1)^2(r-\alpha)^2 e^{\frac{t\gamma\rho}{\gamma-1}-t\rho}=0 \tag{66}$$
$$\xi^1_{ww}\,(\gamma-1)^2(r-\alpha)^2 e^{\frac{t\gamma\rho}{\gamma-1}-t\rho}=0 \tag{67}$$

$$(r-\alpha)^2\left\{w\,\xi^1_u\,\beta r-\xi^1_u\,\nu\gamma+\xi^1_u\,\nu-\xi^2_u\,\beta\right\}=0 \tag{68}$$
$$(r-\alpha)^2\left\{w\,\xi^1_w\,\beta r-\xi^1_w\,\nu\gamma+\xi^1_w\,\nu+\xi^1_t\,\beta\right\}=0 \tag{69}$$
$$(r-\alpha)^2\left\{w\,\xi^1_{uu}\,\beta r-\xi^1_{uu}\,\nu\gamma+\xi^1_{uu}\,\nu-\xi^2_{uu}\,\beta\right\}=0 \tag{70}$$

$$\begin{aligned} w^2\,\xi^1_w\,\beta^2r^2-2w\,\xi^1_w\,\beta\nu\gamma r+2w\,\xi^1_w\,\beta\nu r-w\,\xi^2_w\,\beta^2 r+w\,\xi^1_t\,\beta^2 r\\ +\xi^2\beta^2 r+\xi^1_w\,\nu^2\gamma^2-2\,\xi^1_w\,\nu^2\gamma+\xi^2_w\,\beta\nu\gamma-\xi^1_t\,\beta\nu\gamma\\ +\xi^1_w\,\nu^2-\xi^2_w\,\beta\nu+\xi^1_t\,\beta\nu-\xi^2_t\,\beta^2=0 \end{aligned} \tag{71}$$

$$\begin{aligned} \beta^{-\frac{\gamma}{\gamma-1}-1}(\gamma-1)\left\{\xi^1\,\beta\rho+2w\,\xi^1_w\,\beta\gamma r-w\,\xi^1_w\,\beta r-2\,\xi^1_w\,\nu\gamma^2+3\,\xi^1_w\,\nu\gamma\right.\\ \left.\xi^2_w\,\beta\gamma+\xi^1_t\,\beta\gamma-\xi^1_w\,\nu+\eta_u\,\beta-\xi^1_t\,\beta\right\}e^{\frac{t\gamma\rho}{\gamma-1}-t\rho}=0 \end{aligned} \tag{72}$$

$$\begin{aligned} (r-\alpha)^2\left\{2\,w\,\xi^1_{uw}\,\beta r\sigma^2+2\,\xi^1_u\,\beta r\sigma^2-2\,\xi^1_{uw}\,\nu\gamma\sigma^2+2\,\xi^1_{uw}\,\nu\sigma^2\right.\\ \left.-2\,\xi^2_{uw}\,\beta\sigma^2+\eta_{uu}\,\beta\sigma^2-2\,\xi^1_u\,\beta r^2+4\,\xi^1_u\,\alpha\beta r-2\,\xi^1_u\,\alpha^2\beta\right\}=0 \end{aligned} \tag{73}$$

$$\begin{aligned} (r-\alpha)^2\left\{w\,\xi^1_{ww}\,\beta r\sigma^2+2\,\xi^1_w\,\beta r\sigma^2-\xi^1_{ww}\,\nu\gamma\sigma^2+2\,\xi^1_{ww}\,\nu\sigma^2\right.\\ \left.-\xi^2_{ww}\,\beta\sigma^2+2\eta_{uw}\,\beta\sigma^2-\xi^1_w\,\beta r^2+2\,\xi^1_w\,\alpha\beta r-\xi^1_w\,\alpha^2\beta\right\}=0 \end{aligned} \tag{74}$$

It is clear that most of the determining equations simply characterize the functional dependence of ξ^1, ξ^2, and η with respect to variables t, w, and u. Here we list step-by-step instructions on solving the overdetermined system — each of the step makes full use of all previously obtained results:

- From (51)(52)(53)
$$\eta\equiv\eta(t,u)$$
- From (54)
$$\xi^1\equiv\xi^1(t,w)$$

- From (55)(57)(58)(59)(60)(61)(63)(64)(65)(66)(67)

$$\xi^1 \equiv \xi^1(t)$$

- From (56)

$$\eta \equiv \eta(u)$$

- From (62)(68)(70)

$$\xi^2 \equiv \xi^2(t,w)$$

- From (69)

$$\xi^1 = \text{const} \equiv c_3$$

- From (73)

$$\eta_{uu} = 0 \quad \Longrightarrow \quad \eta(u) = c_1 u + c_2$$

- From (74)

$$\xi^2_{ww} = 0 \quad \Longrightarrow \quad \xi^2(t,w) = f_1(t)\,w + f_2(t)$$

- From (72)

$$\begin{aligned} \xi^1 \beta\rho - \xi^2_w\,\beta\gamma + \eta_u\,\beta = 0 \quad &\Longrightarrow \quad c_3\rho - f_1(t)\gamma + c_1 = 0 \\ &\Longrightarrow \quad f_1(t) = \frac{c_1 + c_3\rho}{\gamma} \end{aligned}$$

- From (71)

$$\begin{aligned} & -w\,\xi^2_w\,\beta^2 r + \xi^2\beta^2 r + \xi^2_w\,\beta\nu\gamma - \xi^2_w\,\beta\nu - \xi^2_t\,\beta^2 = 0 \\ \Longrightarrow \quad & -w\frac{c_1 + c_3\rho}{\gamma}\beta^2 r + \left(\frac{c_1 + c_3\rho}{\gamma} w + f_2(t)\right)\beta^2 r - \frac{c_1 + c_3\rho}{\gamma}\,\beta\delta\nu - f_2'(t)\beta^2 = 0 \\ \Longrightarrow \quad & f_2(t)\,\beta r - \frac{c_1 + c_3\rho}{\gamma}\,\delta\nu - f_2'(t)\,\beta = 0 \\ \Longrightarrow \quad & f_2(t) = \left(c_4 - \frac{(c_1 + c_3\rho)\,\delta\nu}{\gamma\beta r}\right) e^{rt} + \frac{(c_1 + c_3\rho)\,\delta\nu}{\gamma\beta r} \end{aligned}$$

Hence the generator X is

$$X = c_3\,\partial_t + (c_1 u + c_2)\,\partial_u + \left\{\frac{c_1 + c_3\rho}{\gamma} w + \left(c_4 - \frac{(c_1 + c_3\rho)\delta\nu}{\gamma\beta r}\right) e^{rt} + \frac{(c_1 + c_3\rho)\delta\nu}{\gamma\beta r}\right\}\partial_w$$

and its Lie algebra is spanned by

$$\begin{aligned} A_1 &= e^{rt}\partial_w, & A_3 &= \frac{\rho}{\gamma}\left(w + \frac{\delta\nu}{\beta r}(1 - e^{rt})\right)\partial_w + \partial_t \\ A_2 &= \partial_u, & A_4 &= \frac{1}{\gamma}\left(w + \frac{\delta\nu}{\beta r}(1 - e^{rt})\right)\partial_w + u\,\partial_u \end{aligned} \tag{75}$$

with non-zero commutation relations

$$\begin{aligned} [A_1, A_3] &= \left(\frac{\rho}{\gamma} - r\right) A_1, & [A_1, A_4] &= \frac{1}{\gamma}\,A_1, \\ [A_2, A_4] &= A_2, & [A_3, A_4] &= -\frac{\delta\nu}{\gamma\beta}\,A_1. \end{aligned} \tag{76}$$

5. Invariant Solution of Merton's Equation

It is clear that linear combination of A_1, A_2, and A_4 in (75) that is of the form

$$\frac{1}{\gamma}\left(w+\frac{\delta\nu}{\beta r}+c_1\,e^{rt}\right)\partial_w+(u+c_2)\,\partial_u,\qquad c_1,c_2\in\mathbb{R}$$

belongs to the Lie algebra admitted by Merton's equation. Integration of

$$\frac{\mathrm{d}w}{\frac{1}{\gamma}\left(w+\frac{\delta\nu}{\beta r}+c_1\,e^{rt}\right)}=\frac{\mathrm{d}u}{u+c_2}$$

suggests that

$$u=g(t)\cdot\left(w+\frac{\delta\nu}{\beta r}+c_1\,e^{rt}\right)^{\gamma}+c_2$$

with $g(t)$ to be determined, is an invariant solution. Let

$$\Psi(t,w)=w+\frac{\delta\nu}{\beta r}+c_1\,e^{rt}$$

then $\Psi_t=r\,c_1\,e^{rt}$, $\Psi_w=1$ and

$$u_w=g\,\gamma\,\Psi^{\gamma-1},\quad u_{ww}=g\,\gamma(\gamma-1)\,\Psi^{\gamma-2},\quad u_t=g'\,\Psi^{\gamma}+g\,\gamma\,\Psi^{\gamma-1}\,r\,c_1e^{rt}.$$

We now inspect each term of (1):

$$\frac{\delta^2\beta^{\frac{\gamma}{\delta}}}{\gamma}e^{-\frac{\rho}{\delta}t}u_w^{-\frac{\gamma}{\delta}}=\frac{\delta^2\beta^{\frac{\gamma}{\delta}}}{\gamma}\,e^{-\frac{\rho}{\delta}t}\left(g\,\gamma\,\Psi^{\gamma-1}\right)^{-\frac{\gamma}{\delta}}=\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\,e^{-\frac{\rho}{\delta}t}\,g^{-\frac{\gamma}{\delta}}\,\Psi^{\gamma}$$

$$\begin{aligned}\left(\frac{\delta\nu}{\beta}+rw\right)u_w&=r\left(\Psi-c_1e^{rt}\right)\cdot g\,\gamma\,\Psi^{\gamma-1}\\&=r\,g\,\gamma\,\Psi^{\gamma-1}\left(\Psi-c_1\,e^{rt}\right)\\&=r\,g\,\gamma\,\Psi^{\gamma}-g\,\gamma\,\Psi^{\gamma-1}\,r\,c_1e^{rt}\end{aligned}$$

$$\frac{u_w^2}{u_{ww}}=\frac{\left(g\,\gamma\,\Psi^{\gamma-1}\right)^2}{g\,\gamma(\gamma-1)\,\Psi^{\gamma-2}}=-g\,\frac{\gamma}{\delta}\,\Psi^{\gamma}$$

Now Merton's equation (1) becomes

$$\begin{aligned}&\frac{\delta^2\beta^{\frac{\gamma}{\delta}}}{\gamma}e^{-\frac{\rho}{\delta}t}u_w^{-\frac{\gamma}{\delta}}+u_t+\left(\frac{\delta\nu}{\beta}+rw\right)u_w-\delta(\mu-r)\frac{u_w^2}{u_{ww}}\\&=\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\,g^{-\frac{\gamma}{\delta}}\,\Psi^{\gamma}\,e^{-\frac{\rho}{\delta}t}+g'\,\Psi^{\gamma}+r\,g\,\gamma\,\Psi^{\gamma}+\gamma\,(\mu-r)\,g\,\Psi^{\gamma}\\&=\Psi^{\gamma}\left(\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\,e^{-\frac{\rho}{\delta}t}\,g^{-\frac{\gamma}{\delta}}+g'+\gamma\,\mu\,g\right)=0\end{aligned}$$

Equating the terms within parentheses to zero, we have

$$g' + \gamma\,\mu\,g = -\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\,e^{-\frac{\rho}{\delta}t}\,g^{-\frac{\gamma}{\delta}} \tag{77}$$

g satisfies the Bernoulli equation and the solution procedure runs as follows. To solve

$$g' + \varphi\,g = \xi\,g^k \tag{78}$$

where φ, ξ are functions and $k \neq 0,\,1$, let $\zeta = g^{1-k}$, then $\zeta' = (1-k)g^{-k}g'$. Then (78) turns into $(1-k)g^{-k}g' + (1-k)\,\varphi g^{-k}g = (1-k)\,\xi$, so $\zeta' + (1-k)\,\varphi\,\zeta = (1-k)\,\xi$, a linear ODE with solution

$$\zeta = e^{-\int(1-k)\varphi}\int(1-k)\xi\,e^{\int(1-k)\varphi} + c\,e^{-\int(1-k)\varphi}$$

and $g = \zeta^{\frac{1}{1-k}}$. Back to the solution of (77); note that $k = -\frac{\gamma}{\delta}$, so $1-k = \frac{1}{\delta}$, $(1-k)\varphi = \frac{\gamma\mu}{\delta}$, $e^{\int(1-k)\varphi} = e^{\frac{\gamma\mu}{\delta}t}$, and $(1-k)\xi = -\frac{1}{\delta}\,\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}e^{-\frac{\rho}{\delta}t} = -\frac{1}{\delta}\,\chi\,e^{-\frac{\rho}{\delta}t}$ by letting $\chi \equiv \delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}$. Now

$$\begin{aligned}
\zeta &= e^{-\int(1-k)\varphi}\int(1-k)\xi\,e^{\int(1-k)\varphi} + c\,e^{-\int(1-k)\varphi} \\
&= e^{-\frac{\gamma\mu}{\delta}t}\int -\frac{1}{\delta}\,\chi\,e^{-\frac{\rho}{\delta}t}\cdot e^{\frac{\gamma\mu}{\delta}t}\,\mathrm{d}t + c\,e^{-\frac{\gamma\mu}{\delta}t} \\
&= e^{-\frac{\gamma\mu}{\delta}t}\cdot\frac{\chi}{\delta}\cdot\frac{\delta}{\rho-\gamma\mu}e^{-\frac{\rho-\gamma\mu}{\delta}t} + c\,e^{-\frac{\gamma\mu}{\delta}t} \\
&= \frac{\chi\,e^{-\frac{\rho}{\delta}t}}{\rho-\gamma\mu} + c\,e^{-\frac{\gamma\mu}{\delta}t}
\end{aligned}$$

Determine the constant c using the boundary condition $\zeta(T) = 0$,

$$0 = \zeta(T) = \frac{\chi\,e^{-\frac{\rho}{\delta}T}}{\rho-\gamma\mu} + c\,e^{-\frac{\gamma\mu}{\delta}T} \implies c = -\frac{\chi\,e^{-\frac{\rho-\gamma\mu}{\delta}T}}{\rho-\gamma\mu}.$$

So

$$\zeta(t) = \frac{\chi e^{-\frac{\rho}{\delta}t}\left(1 - e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}\right)}{\rho-\gamma\mu},$$

$$\begin{aligned}
g(t) = \zeta(t)^{\frac{1}{1-k}} = \zeta(t)^{\delta} &= \left(\frac{\chi e^{-\frac{\rho}{\delta}t}\left(1 - e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}\right)}{\rho-\gamma\mu}\right)^{\delta} \\
&= (\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}})^{\delta}\left(e^{-\frac{\rho}{\delta}t}\right)^{\delta}\left(\frac{1 - e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}}{\rho-\gamma\mu}\right)^{\delta} = \frac{\delta^{2\delta}\beta^{\gamma}}{\gamma}e^{-\rho t}\left(\frac{1 - e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}}{\rho-\gamma\mu}\right)^{\delta}
\end{aligned}$$

and the invariant solution is

$$u(t,w) = \frac{\delta^{2\delta}\beta^{\gamma}}{\gamma}e^{-\rho t}\left(\frac{1 - e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}}{\rho-\gamma\mu}\right)^{\delta}\left(w + \frac{\delta\nu}{\beta r} + c_1\,e^{rt}\right)^{\gamma} + c_2, \quad c_1, c_2 \in \mathbb{R}. \tag{79}$$

6. Discussions and Conclusions

We now take a closer look at the invariant solution (79). Using the terminal condition $u(T, W(T)) = 0$, c_2 is identically 0. The solution (2) published in Merton (1971, (47)) is the special case of $c_1 = \frac{\delta\nu}{\beta\gamma} e^{-rT}$; any other $c_1 \in \mathbb{R}$ is equally suitable. Merton himself seemed unaware of the nonuniqueness of the equation with the only terminal condition as he wrote in Merton (1971, footnote 21, p.390) “the solution is unique” and referred to a theorem (Merton (1971, Theorem I, p.381)) — the so-called “verification theorem” — cited without proof for justification.

As pointed out in Sethi and Taksar (1988), the original problem formulation in Merton (1971) is flawed for not considering the possibility of bankruptcy: the wealth level may be negative during the investment period if not controlled. Sethi and Taksar (1988) scrutinized the solution and derived optimal rules in Merton (1971) for all feasible parameter ranges of the HARA utility function and found several questionable issues originated from the positive probability of being nonpositive.

One of the remedies proposed in Sethi and Taksar (1988) is to recast the problem along the lines of Karatzas et al. (1988), replacing (7) with

$$\mathsf{E}_0 \left\{ \int_0^{T_0} e^{-\rho\tau}\, U(c(\tau))\, \mathrm{d}\tau + P e^{-\rho T_0} \right\} \tag{80}$$

where $T_0 \equiv \inf\{t \geqslant 0 : w(t) = 0\}$ is the hitting time and P is the *natural payment* level to be specified beforehand. It is shown in Karatzas et al. (1988) that optimal policies are determined according to the value of P as P varies between $\rho^\star \equiv \frac{U(0)}{\rho}$ and

$$P^\star \equiv \rho^\star - \frac{U'(0)^{\lambda_- + 1}}{\rho\lambda_-} \int_0^\infty \frac{\mathrm{d}\theta}{U'(\theta)^{\lambda_-}},$$

where λ_- is the smaller root of the equation $\kappa\lambda^2 - (r - \rho - \kappa)\lambda - r = 0$, $\kappa = \frac{(\alpha - r)^2}{2\sigma^2}$. Another route suggested by Sethi and Taksar (1988) is to add a proper boundary condition at $u(t, 0)$, which is in line with our present pure mathematical reasoning.

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