Survival Models

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Definition

- (x): A life aged x; $x \geqslant 0$.
- T_x: Future lifetime of (x), a random variable; $x + T_x$: age-at-death of (x).
- $F_x(t)$: The distribution function of T_x; $F_x(t) \equiv \mathrm{P}(T_x \leqslant t)$. (Should have been written as $F_{T_x}(t)$; an abbreviation.)
- $S_x(t)$: $S_x(t) \equiv 1 - F_x(t) = \mathrm{P}(T_x > t)$.
- $f_x(t)$: The probability density function of T_x; $f_x(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} F_x(t) = -\frac{\mathrm{d}}{\mathrm{d}t} S_x(t)$;

$F_x(t)$ represents the probability that (x) survives $\leqslant t$ years

$S_x(t)$ represents the probability that (x) survives $> t$ years

Important Postulate

$$\mathrm{P}(T_x \leqslant t) = \mathrm{P}(T_0 \leqslant x + t \,|\, T_0 > x) \tag{1}$$

Note that from the rules of conditional probabilities,

$$\mathrm{P}(T_0 \leqslant x + t \,|\, T_0 > x) = \frac{\mathrm{P}(x < T_0 \leqslant x + t)}{\mathrm{P}(T_0 > x)}$$

$$F_x(t) \equiv \mathrm{P}(T_x \leqslant t) = \mathrm{P}(T_0 \leqslant x + t \,|\, T_0 > x) = \frac{F_0(x+t) - F_0(x)}{S_0(x)}$$

Use of $S_x(t) = 1 - F_x(t)$,

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)} \quad \Longrightarrow \quad S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{2}$$

Note $S_x(t+u) = \frac{S_0(x+t+u)}{S_0(x)} = \frac{S_0(x+t)}{S_0(x)} \frac{S_0(x+t+u)}{S_0(x+t)} = S_x(t)\, S_{x+t}(u)$ and

Conditions

- $S_x(0) = 1$.
- $S_x(t) \to 0$ as $t \to \infty$.
- $S_x(t)$ is non-increasing in t.

Assumptions

- $S_x(t)$ is differentiable in t.
- $t \cdot S_x(t) \to 0$ as $t \to \infty$.
- $t^2 \cdot S_x(t) \to 0$ as $t \to \infty$.

Definition (μ_x — The force of mortality at age x)

$$\begin{aligned}
\mu_x &\equiv \lim_{\Delta x \to 0+} \frac{1}{\Delta x} \mathrm{P}(T_0 \leqslant x + \Delta x \mid T_0 > x) \\
&= \lim_{\Delta x \to 0+} \frac{1}{\Delta x} \mathrm{P}(T_x \leqslant \Delta x) \\
&= \lim_{\Delta x \to 0+} \frac{1}{\Delta x} (1 - S_x(\Delta x))
\end{aligned} \tag{3}$$

Interpretation: $\mu_x \, \mathrm{d}x \approx \mathrm{P}(T_0 \leqslant x + \mathrm{d}x \mid T_0 > x)$, "The probability that a life who has attained age x dies before attaining $x + \mathrm{d}x$".

Note that $S_x(\Delta x) = \frac{S_0(x+\Delta x)}{S_0(x)}$,

$$\begin{aligned}
\mu_x &= \lim_{\Delta x \to 0+} \frac{1}{\Delta x}(1 - S_x(\Delta x)) \\
&= \lim_{\Delta x \to 0+} \frac{1}{\Delta x}\left(1 - \frac{S_0(x+\Delta x)}{S_0(x)}\right) \\
&= \frac{1}{S_0(x)} \lim_{\Delta x \to 0+} \frac{S_0(x) - S_0(x+\Delta x)}{\Delta x} \\
&= -\frac{1}{S_0(x)} \frac{d}{dx} S_0(x) \qquad (4) \\
&= \frac{f_0(x)}{S_0(x)} \qquad (5)
\end{aligned}$$

Fix x, for variable t, $\mathrm{d}(x+t) = \mathrm{d}t$. Then

$$
\begin{aligned}
\mu_{x+t} &= -\frac{1}{S_0(x+t)}\frac{\mathrm{d}}{\mathrm{d}(x+t)}S_0(x+t) \\
&= -\frac{1}{S_0(x+t)}\frac{\mathrm{d}}{\mathrm{d}t}S_0(x+t) \\
&= -\frac{1}{S_0(x+t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(S_0(x)S_x(t)\right) \\
&= -\frac{S_0(x)}{S_0(x+t)}\frac{\mathrm{d}}{\mathrm{d}t}S_x(t) \\
&= -\frac{1}{S_x(t)}\frac{\mathrm{d}}{\mathrm{d}t}S_x(t) \qquad (6) \\
&= \frac{f_x(t)}{S_x(t)} \qquad (7)
\end{aligned}
$$

Fact: for a differentiable function $h(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\log h(x) = \frac{1}{h(x)}\frac{\mathrm{d}}{\mathrm{d}x}h(x)$$

So

$$\mu_x = -\frac{1}{S_0(x)}\frac{\mathrm{d}}{\mathrm{d}x}S_0(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\log S_0(x)$$

Integrate above, we have

$$\int_0^y \mu_x\,\mathrm{d}x = -\left(\log S_0(y) - \log S_0(0)\right)$$

Note that $\log S_0(0) = \log \mathrm{P}(T_0 > 0) = \log 1 = 0$, we have

$$S_0(y) = \exp\left\{-\int_0^y \mu_x\,\mathrm{d}x\right\} \tag{8}$$

$$
\begin{aligned}
S_x(t) &= \frac{S_0(x+t)}{S_0(x)} \\
&= \frac{\exp\left\{-\int_0^{x+t}\mu_v\,\mathrm{d}v\right\}}{\exp\left\{-\int_0^{x}\mu_v\,\mathrm{d}v\right\}} \\
&= \exp\left\{-\int_0^{x+t}\mu_v\,\mathrm{d}v + \int_0^{x}\mu_v\,\mathrm{d}v\right\} \\
&= \exp\left\{-\int_x^{x+t}\mu_v\,\mathrm{d}v\right\} = \exp\left\{-\int_0^{t}\mu_{x+v}\,\mathrm{d}v\right\} \qquad (9)
\end{aligned}
$$

Definition

- ${}_tp_x \equiv \mathrm{P}(T_x > t) = S_x(t)$, the probability that (x) survives to at least $x + t$.
- ${}_tq_x \equiv \mathrm{P}(T_x \leqslant t) = F_x(t)$, the probability that (x) dies before $x + t$.
- ${}_{u|t}q_x \equiv \mathrm{P}(u < T_x \leqslant u + t) = S_x(u) - S_x(u + t)$, the probability that (x) survives u years and then dies within t years.
- $\mathring{e}_x \equiv \mathrm{E}\,T_x$, the expected future lifetime of (x).

$$ {}_tp_x + {}_tq_x = 1 \tag{10}$$

$$ {}_{u|t}q_x = {}_up_x - {}_{u+t}p_x \tag{11}$$

$$ {}_{t+u}p_x = {}_tp_x \cdot {}_up_{x+t} \tag{12}$$

$$ \mu_x = -\frac{1}{{}_xp_0}\frac{\mathrm{d}}{\mathrm{d}x}{}_xp_0 \tag{13}$$

$$ \mu_{x+t} = -\frac{1}{{}_tp_x}\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x \implies \frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x = -{}_tp_x \cdot \mu_{x+t} \tag{14}$$

$$ \mu_{x+t} = \frac{f_x(t)}{S_x(t)} \implies f_x(t) = {}_tp_x \cdot \mu_{x+t} \tag{15}$$

$$ {}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}\,\mathrm{d}s\right\} \tag{16}$$

$$ F_x(t) = \int_0^t f_x(s)\,\mathrm{d}s \implies {}_tq_x = \int_0^t {}_sp_x \cdot \mu_{x+s}\,\mathrm{d}s \tag{17}$$

Note the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x = -{}_tp_x \cdot \mu_{x+t}$$
$$f_x(t) = {}_tp_x \cdot \mu_{x+t}$$

Use $t \cdot {}_tp_x \to 0$ as $t \to \infty$, we have

$$\begin{aligned}
\mathring{e}_x \equiv \mathrm{E}\,T_x = \int_0^\infty t \cdot f_x(t)\,\mathrm{d}t &= \int_0^\infty t \cdot {}_tp_x \cdot \mu_{x+t}\,\mathrm{d}t \\
&= \int_0^\infty t \cdot \left(-\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x\right)\,\mathrm{d}t \\
&= -\left\{ t \cdot {}_tp_x \Big|_0^\infty - \int_0^\infty {}_tp_x\,\mathrm{d}t \right\} \\
&= \int_0^\infty {}_tp_x\,\mathrm{d}t. \qquad (18)
\end{aligned}$$

By the same token, using $t^2 \cdot {}_tp_x \to 0$ as $t \to \infty$ we have

$$\begin{aligned} \mathrm{E}\,T_x^2 &= \int_0^\infty t^2 \cdot f_x(t)\,\mathrm{d}t \\ &= \int_0^\infty t^2 \cdot \left(-\frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x\right)\,\mathrm{d}t \\ &= -\left\{ t^2 \cdot {}_tp_x\Big|_0^\infty - \int_0^\infty 2t \cdot {}_tp_x\,\mathrm{d}t \right\} \\ &= \int_0^\infty 2t \cdot {}_tp_x\,\mathrm{d}t. \qquad (19) \end{aligned}$$

The variance of T_x, denoted by var T_x, is

$$\operatorname{var} T_x = \mathrm{E}\,T_x^2 - (\mathrm{E}\,T_x)^2$$

Definition

- $K_x \equiv \lfloor T_x \rfloor$, the integer part of T_x
- $e_x \equiv \mathrm{E}K_x$, the expectation of K_x: curtated expectation of life.

$$\begin{aligned} \mathrm{P}(K_x = k) &= \mathrm{P}(k \leqslant T_x < k+1) \\ &= {}_{k|}p_x = {}_kp_x - {}_{k+1}p_x = {}_kp_x - {}_kp_x \cdot p_{x+k} = {}_kp_x \cdot q_{x+k} \end{aligned}$$

$$\begin{aligned} e_x \equiv \mathrm{E}K_x &= \sum_{k=0}^{\infty} k \cdot \mathrm{P}(K_x = k) = \sum_{k=0}^{\infty} k \cdot \left({}_kp_x - {}_{k+1}p_x\right) \\ &= \left({}_1p_x - {}_2p_x\right) + 2\left({}_2p_x - {}_3p_x\right) + 3\left({}_3p_x - {}_4p_x\right) + \ldots = \sum_{k=1}^{\infty} {}_kp_x \end{aligned}$$

Definition

$$\mathrm{E}K_x^2 = \sum_{k=0}^{\infty} k^2 \cdot \mathrm{P}(K_x = k) = \sum_{k=0}^{\infty} k^2 \cdot ({}_kp_x - {}_{k+1}p_x)$$
$$= ({}_1p_x - {}_2p_x) + 4\,({}_2p_x - {}_3p_x) + 9\,({}_3p_x - {}_4p_x) + 16\,({}_4p_x - {}_5p_x) + \dots$$
$$= 2\sum_{k=1}^{\infty} k \cdot {}_kp_x - \sum_{k=1}^{\infty} {}_kp_x = 2\sum_{k=1}^{\infty} k \cdot {}_kp_x - e_x$$