Monte-Carlo Method in American Option Pricing

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Motivation

Financial contracts in mind that allow to exercise early before expiry:

- Early surrender of life insurance contracts
- Formosa bond: callable

How to price / hedge (Computing sensitivities)?

Pricing contracts with early exercise feature is very hard, if not impossible. Anaytical (closed-form) solutions are rare, should be numerically solved.

In a complete and arbitrage-free market

Pricing Formula of European Option

$$E_0 = \mathrm{E}^Q\{d_{0,T}\, g_T(X_T)\}$$

where

- g_t — nonnegative payoff function
- $(X_t)_{0\leqslant t\leqslant T}$ — underlying stochastic process
- $d_{s,t}$ — nonnegative $\mathcal{F}((X_u)_{s\leqslant u\leqslant t})$-measurable discount factors satisfying $d_{0,t} = d_{0,s} \cdot d_{s,t}$ for $s < t$

Pricing Formula of American Option

$$V_0 = \sup_{\tau \in \mathcal{T}([0,T])} \mathrm{E}^Q\{d_{0,\tau}\, g_\tau(X_\tau)\}$$

Stopping Time

A stopping time $\tau \in \mathcal{T}([0, T])$ is a measurable function of $(X_t)_{0 \leqslant t \leqslant T}$ with values in $[0, T]$ and has the property that $\forall r \in [0, T]$, the event $\{\tau \leqslant r\}$ is contained in the $\sigma-$ algebra $\mathcal{F}_r = \mathcal{F}((X_t)_{0 \leqslant t \leqslant r})$.

Reduction From Continuous Time to Discrete Time

Set $f_\tau \equiv d_{0,\tau}\, g_\tau$, the discrete time formula is

$$V_0 = \sup_{\tau \in \mathcal{T}(0,\dots,T)} \mathrm{E}^Q\{f_\tau(X_\tau)\}$$

- $\{X_0, X_1, \dots X_T\}$ — underlying discrete time stochastic process
- $\mathcal{T}(0, \dots, T)$ — all stopping times with value in $0, \dots, T$

The Aim

The optimal stopping time τ^* such that

$$V_0 = \sup_{\tau \in \mathcal{T}(0,\dots,T)} \mathrm{E}^Q\{f_\tau(X_\tau)\} = \mathrm{E}\{f_{\tau^*}(X_{\tau^*})\}$$

Essential Notions

$$q_t(x) = \sup_{\tau \in \mathcal{T}(t+1,\ldots,T)} \mathrm{E}\{f_\tau(X_\tau) \mid X_t = x\} \text{ as } 0 \leqslant t < T;\ q_T(x) = 0$$

$$v_t(x) = \sup_{\tau \in \mathcal{T}(t,t+1,\ldots,T)} \mathrm{E}\{f_\tau(X_\tau) \mid X_t = x\}$$

$$\tau_t^* = \inf\{s \geqslant t+1 : q_s(X_s) \leqslant f_s(X_s)\}, \quad t \in \{-1, 0, \ldots, T-1\}$$

Interpretation

- $q_t(x)$: Given $X_t = x$, the value of the option at t without selling.
- $v_t(x)$: Given $X_t = x$, the value we get in the mean if we sell the option optimally after $t-1$.

Theorem

The optimal stopping time τ^* is

$$\tau^* = \tau^*_{-1} = \inf\{s \in \{0, 1, \ldots, T\} : q_s(X_s) \leqslant f_s(X_s)\}$$

N.B. Books about American option usually state this result without proof; those with proofs are incomplete at best. Complete proof is given in Kohler M. "A Review on Regression-Based Monte Carlo Method for Pricing American Options".

Representations of $q_t(x)$

$$
\begin{aligned}
q_t(x) &= \mathrm{E}\{f_{\tau_t^*}(X_{\tau_t^*})|X_t = x\} \\
&= \mathrm{E}\{\max\left(f_{t+1}(X_{t+1}), q_{t+1}(X_{t+1})\right)|X_t = x\} \\
&= \mathrm{E}\{\Theta_{t+1,t+w+1}^{(w)}|X_t = x\}
\end{aligned}
$$

where

$$
\begin{aligned}
&\Theta_{t+1,t+w+1}^{(w)} = \\
&\qquad \sum_{s=t+1}^{t+w+1} f_s(X_s) \cdot 1_{\{f_{t+1}(X_{t+1})<q_{t+1}(X_{t+1}),\ldots,f_{s-1}(X_{s-1})<q_{s-1}(X_{s-1}),f_s(X_s)\geqslant q_s(X_s)\}} \\
&\qquad\qquad + q_{t+w+1}(X_{t+w+1}) \cdot 1_{\{f_{t+1}(X_{t+1})<q_{t+1}(X_{t+1}),\ldots,f_{t+w+1}(X_{t+w+1})<q_{t+w+1}(X_{t+w+1})\}}
\end{aligned}
$$

The exact value of $q_t(x)$ is hard to compute because of the conditional expectation. Insight: use regression estimates to approximately compute!

Excursion: Variance Reduction by Control Variate

If we want to compute $E(X)$, suppose we can find another r.v. Y 'close' to X with known $E(Y)$. Then r.v.

$$Z = X + E(Y) - Y$$

satisfies $E(Z) = E(X) + E(Y) - E(Y) = E(X)$. In this context, Y is the control variate.