Elementary Methods in Exotic Option Pricing Under Black-Scholes Framework

July 9, 2020

Gaussian Shift Theorem (GST)

Theorem

Let $Z \sim N(0, 1; R)$, $c \in \mathbb{R}^n$ and $F(Z)$ a measurable scalar function of Z with finite mean. Then

$$\mathrm{E}\left\{e^{c'Z}F(Z)\right\} = e^{\frac{1}{2}c'Rc}\mathrm{E}\left\{F(Z+Rc)\right\} \tag{1}$$

Special cases:

$$\mathrm{E}\left\{e^{cZ}F(Z)\right\} = e^{\frac{1}{2}c^2}\mathrm{E}\left\{F(Z+c)\right\} \quad \text{(1-d)}$$

$$\mathrm{E}\left\{e^{aX+bY}F(X,Y)\right\} = e^{\frac{1}{2}c^2}\mathrm{E}\left\{F(X+a', Y+b')\right\} \quad \text{(2-d)}$$

$$\text{where } c^2 = a^2 + b^2 + 2\rho ab,\ a' = a + \rho b,\ b' = b + \rho a$$

Proof of Gaussian Shift Theorem.

Use $\frac{1}{2}(z-Rc)'R^{-1}(z-Rc) = \frac{1}{2}z'R^{-1}z + \frac{1}{2}c'Rc - c'z$, we have

$$
\begin{aligned}
\mathrm{E}\left\{e^{c'Z}F(Z)\right\} &= \int_{-\infty}^{\infty} e^{c'z}F(z)\frac{e^{-\frac{1}{2}z'R^{-1}z}}{(2\pi)^{\frac{m}{2}}\sqrt{\det R}}\,\mathrm{d}z \\
&= e^{\frac{1}{2}c'Rc}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(z-Rc)'R^{-1}(z-Rc)}}{(2\pi)^{\frac{m}{2}}\sqrt{\det R}}F(z)\,\mathrm{d}z \\
&= e^{\frac{1}{2}c'Rc}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}u'R^{-1}u}}{(2\pi)^{\frac{m}{2}}\sqrt{\det R}}F(u+Rc)\,\mathrm{d}u \\
&= e^{\frac{1}{2}c'Rc}\,\mathrm{E}\left\{F(Z+Rc)\right\}
\end{aligned}
$$

$\square$

Definition (First Order Binary Option)

A derivative on an underlying asset is called an up/down type first order binary option with payoff $f(x)$ and exercise price $\xi > 0$ if at expiry T it pays

$$V_\xi^s(x, T) = f(x)\,\mathbb{1}(sx > s\xi)$$

where $s = \pm$ for the up/down type respectively.

Example (First Order Asset/Bond Binaries)

- The asset binary $A_\xi^\pm(x, t)$: $f(x) = x$, at expiry T
  $A_\xi^+(x, T) = x\,\mathbb{1}(x > \xi)$, $A_\xi^-(x, T) = x\,\mathbb{1}(x < \xi)$.
- The bond binary $B_\xi^\pm(x, t)$: $f(x) = 1$, at expiry T
  $B_\xi^+(x, T) = \mathbb{1}(x > \xi)$, $B_\xi^-(x, T) = \mathbb{1}(x < \xi)$.

BS Prices of First Order Asset/Bond Binaries

Observation (simple algebra): If $X = xe^{\left(r-\frac{1}{2}\sigma^2\right)\tau+\sigma\sqrt{\tau}Z}$, where $\tau = T - t$ and $Z \sim N(0,1)$, then $X \lessgtr k \Longleftrightarrow Z \lessgtr -d_k^-(x,\tau)$, where $d_k^{\pm}(x,\tau) = \frac{\log\left(\frac{x}{k}\right)+\left(r\pm\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$. Then

$$
\begin{aligned}
B_\xi^+(x,t) &= e^{-r\tau}\mathrm{E}^Q\left\{\mathbb{1}\left(X > \xi\right)\right\}\\
&= e^{-r\tau}\mathrm{E}^Q\left\{\mathbb{1}\left(Z > -d_\xi^-(x,\tau)\right)\right\}\\
&= e^{-r\tau}\mathrm{E}^Q\left\{\mathbb{1}\left(Z < d_\xi^-(x,\tau)\right)\right\}\\
&= e^{-r\tau}\mathcal{N}(d_\xi^-(x,\tau)),
\end{aligned}
$$

and accordingly

$$B_\xi^-(x,t) = e^{-r\tau}\mathcal{N}(-d_\xi^-(x,\tau)).$$

$$
\begin{aligned}
A_\xi^+(x,t) &= e^{-r\tau}\mathsf{E}^Q\left\{X\,\mathbb{1}\left(X>\xi\right)\right\} \\
&= x\,e^{-\frac{1}{2}\sigma^2\tau}\mathsf{E}^Q\left\{e^{\sigma\sqrt{\tau}Z}\,\mathbb{1}\left(Z>-d_\xi^-(x,\tau)\right)\right\} \\
&= x\mathsf{E}^Q\left\{\mathbb{1}\left(Z+\sigma\sqrt{\tau}>-d_\xi^-(x,\tau)\right)\right\} \quad \text{(Using GST)} \\
&= x\mathsf{E}^Q\left\{\mathbb{1}\left(Z<\sigma\sqrt{\tau}+d_\xi^-(x,\tau)\right)\right\} \quad \text{(Using Symmetry of } Z\text{)} \\
&= x\mathcal{N}(d_\xi^+(x,\tau)),
\end{aligned}
$$

and accordingly

$$
A_\xi^-(x,t) = x\mathcal{N}(-d_\xi^+(x,\tau)).
$$

Gap Options

Gap options: strike price k is different from exercise price ξ. Payoff functions at T:

$$C(x,T) = (x-k)\,\mathbb{1}(x>\xi) \quad (2)$$
$$P(x,T) = (k-x)\,\mathbb{1}(x<\xi) \quad (3)$$

Prices:

$$C(x,t) = A_\xi^+(x,t) - kB_\xi^+(x,t) = x\mathcal{N}(d_\xi^+(x,\tau)) - ke^{-r\tau}\mathcal{N}(d_\xi^-(x,\tau))$$
$$P(x,t) = -A_\xi^-(x,t) + kB_\xi^-(x,t)$$
$$= -x\mathcal{N}(-d_\xi^+(x,\tau)) + k\,e^{-r\tau}\mathcal{N}(-d_\xi^-(x,\tau))$$

Definition (B-S Operator, Image Function, Image Operator)

The B-S operator $\mathcal{L}$ is

$$\mathcal{L}V(x,t) = \frac{\partial V}{\partial t} + rV - rx\frac{\partial V}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}$$

The corresponding B-S PDE is $\mathcal{L}V = 0$. For a function $F(x)$,

$$\widetilde{F}(x) = \left(\frac{b}{x}\right)^{\alpha} F\left(\frac{b^2}{x}\right), \quad \alpha = \frac{2r}{\sigma^2} - 1$$

is the *image of* $F(x)$ *with respect to* $x = b$ *and* $\mathcal{L}$, and the *image operator* $\mathcal{I}_b^{\mathcal{L}}$ is defined accordingly by

$$\widetilde{F}(x) = \mathcal{I}_b^{\mathcal{L}}\left\{F(x)\right\}.$$

Barrier Options: Types and Their Boundary Conditions

Table: Barrier Options and Their Boundary Conditions (BCs).

Type	PDE	Active Domain	BC at $x = b$	BC at $t = T$
D/O	$\mathcal{L}V_{\text{do}} = 0$	$x > b$	0	$f(x)$
U/O	$\mathcal{L}V_{\text{uo}} = 0$	$x < b$	0	$f(x)$
D/I	$\mathcal{L}V_{\text{di}} = 0$	$x > b$	$V_0(b, t)$	0
U/I	$\mathcal{L}V_{\text{ui}} = 0$	$x < b$	$V_0(b, t)$	0

In addition, there are call/put types; suppose the strike is set at $x = k$. Then there are different results for $k > b$ and $k < b$. Totally there are $4 \times 2 \times 2 = 16$ cases.

Theorem (Properties of the Image Function / Operator)

- $\mathcal{I}_b^{\mathcal{L}}\left\{\mathcal{I}_b^{\mathcal{L}}\left\{V(x,t)\right\}\right\} = V(x,t)$ *for any function* $V(x,t)$.
- *If* $\mathcal{L}V(x,t) = 0$, *then* $\mathcal{L}\widetilde{V}(x,t) = 0$.
- $V(b,t) = \widetilde{V}(b,t)$.
- *If* $x > b$ *(resp.* $x < b$*) is the active domain of* $V(x,t)$, *then* $x < b$ *(resp.* $x > b$*) is the active domain of* $\widetilde{V}(x,t)$.

Theorem (Method of Images (MoI) for the D/O Barrier Option)

Let $V^+(x,t)$ solve the problem

$$\mathcal{L}V^+(x,t) = 0, \quad V(x,T) = f(x)\,\mathbb{1}(x > b)$$

in $\{x > 0, t < T\}$. Then

$$V_{do}(x,t) \equiv V^+(x,t) - \widetilde{V}^+(x,t) \tag{4}$$

solves the problem for the D/O barrier option in $\{x > b, t < T\}$.

Note that the result relies on the existence and the uniqueness of the solution of BS PDE.

Proof.

- $\mathcal{L}V_{\text{do}}(x,t) = 0$: Evident.
- $V_{\text{do}}(b,t) = 0$: Evident from $V^+(b,t) = \widetilde{V}^+(b,t)$.
-

$$
\begin{aligned}
V_{\text{do}}(x,T) &= f(x)\,\mathbb{1}(x>b) - \mathcal{I}_b^{\mathcal{L}}\{(f(x)\,\mathbb{1}(x>b)\} \\
&= f(x)\,\mathbb{1}(x>b) - \widetilde{f}(x)\,\mathbb{1}\left(\frac{b^2}{x} > b\right) \\
&= f(x)\,\mathbb{1}(x>b) - \widetilde{f}(x)\,\mathbb{1}\,(x<b)
\end{aligned}
$$

Hence, in the active domain of D/O option $x > b$, $V_{\text{do}}(x,T) = f(x)$, as required.

□

Theorem (Image Function w.r.t Exponential Barrier)

If the barrier $b(t) \equiv B\,e^{\beta t}$, *then*

$$\widetilde{V}(x,t) = \mathcal{I}^{\mathcal{L}}_{b(t)}\left\{V(x,t)\right\} = \left(\frac{b(t)}{x}\right)^{\alpha'} V\left(\frac{b(t)^2}{x}, t\right),$$

where $\alpha' = \frac{2(r-\beta)}{\sigma^2} - 1$.

Proof of above uses the fact: If $u_r(x,t)$ solves B-S PDE, then $v_{r,s}(x,t) \equiv e^{st} u_{r+s}(xe^{-st}, t)$ solves B-S PDE for $s \in \mathbb{R}$. Furthermore, if $u_r(x,0)$ is independent of r, then $v_{r,s}(x,t) = u_r(x,t)$ holds for all $t > 0$ and $s \in \mathbb{R}$.

Example: Recovering Grosen/Jørgensen Solution

In Grosen/Jørgensen, they priced the value of the contingent claim with payoff function $\Psi(A_T)$ at expiry T

$$\Psi(A_T) = \delta \left(A_T - \alpha L_T^G\right)^+ + L_T^G - \left(L_T^G - A_T\right)^+$$

and an lower time-dependent exponential barrier $B_t = B\, e^{r_G t}$.

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