# Symmetry Methods for Differential Equation

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# Merton's Equation

Passage in [Merton(1971)].

satisfying (5.40). From (5.28), the optimality equation for $J$ is

$$0 = \frac{(1-\gamma)^2}{\gamma} \exp(-\rho t) \left( \frac{\exp(\rho t) J_W}{\beta} \right)^{\gamma/(\gamma-1)} + J_t + \left[ \frac{(1-\gamma)\eta}{\beta} + rW \right] J_W - \frac{J_W^2}{J_{WW}} \frac{(\alpha - r)^2}{2\sigma^2} \tag{5.44}$$

subject to $J(W, T) = 0$.[17] The equations for the optimal consumption and portfolio

Solution of the above complicated-looking nonlinear PDE. Note that Merton did not mention HOW this was derived...

A solution[18] to (5.44) is

$$J(W,t) = \frac{\delta \beta^\gamma}{\gamma} \exp(-\rho t) \left( \frac{\delta\{1 - \exp[-(\rho - \gamma v)(T-t)/\delta]\}}{\rho - \gamma v} \right)^\delta \times \left( \frac{W}{\delta} + \frac{\eta}{\beta r}\{1 - \exp[-r(T-t)]\} \right)^\gamma \tag{5.47}$$

where $\delta \equiv 1 - \gamma$ and $v \equiv r + (\alpha - r)^2 / 2\delta\sigma^2$.

By means of

$$\delta = 1 - \gamma, \quad \mu = r + \frac{(\alpha - r)^2}{2\delta\sigma^2},$$

the equation becomes

$$\frac{\delta^2 \beta^{\frac{\gamma}{\delta}}}{\gamma} e^{-\frac{\rho}{\delta} t} J_W^{-\frac{\gamma}{\delta}} + J_t + \left(\frac{\delta\eta}{\beta} + rW\right) J_W - \delta(\mu - r)\frac{J_W^2}{J_{WW}} = 0; \quad J(T, W) = 0.$$

The solution announced in [Merton(1971)] can be rearranged as

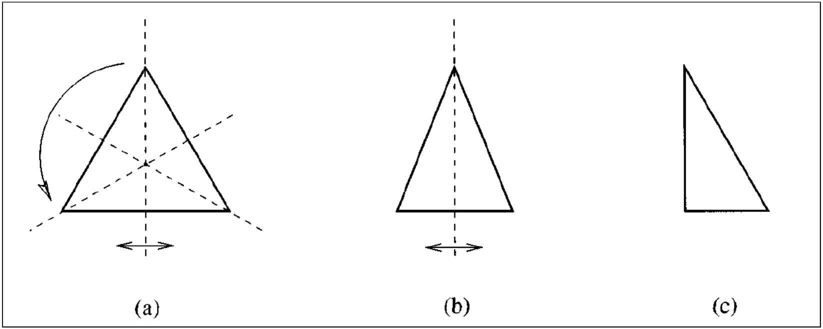
$$J(t, W) = \frac{\delta^{2\delta}\beta^{\gamma}}{\gamma} e^{-\rho t} \left(\frac{1 - e^{\frac{(\rho - \gamma\mu)(t - T)}{\delta}}}{\rho - \gamma\mu}\right)^{\delta} \left(W + \frac{\delta\eta}{\beta r}\left(1 - e^{r(t-T)}\right)\right)^{\gamma}$$

We will show that, in fact,

$$J(t, W) = \frac{\delta^{2\delta}\beta^{\gamma}}{\gamma} e^{-\rho t} \left(\frac{1 - e^{\frac{(\rho - \gamma\mu)(t - T)}{\delta}}}{\rho - \gamma\mu}\right)^{\delta} \left(W + \frac{\delta\eta}{\beta r} + \mathrm{const} \cdot e^{rt}\right)^{\gamma}$$

# Symmetries of Planar Objects

Symmetry: a transformation whose action leaves the object "unchanged"



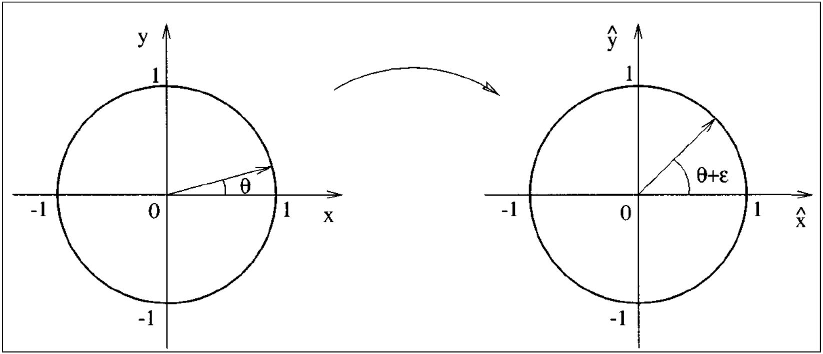
# Conditions for a Transformation to be a Symmetry

$$\Gamma : x \longmapsto \hat{x}(x)$$

1. $\Gamma$ preserves the structure.
2. $\Gamma$ is a diffeomorphism (a smooth invertible mapping whose inverse is smooth).
3. $\Gamma$ maps the object to itself.

# Object with Infinite Symmetries

$$\Gamma_\varepsilon : (x, y) \mapsto (\hat{x}, \hat{y}) = (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon)$$




(Example of an one-parameter Lie group)

# Symmetries of ODE

A diffeomorphism that maps the set of solutions of the DE to itself.
ODE in mind:

$$y^{(n)} = \omega(x, y, y', \ldots, y^{(n-1)}), \quad y^{(k)} \equiv \frac{\mathrm{d}^k y}{\mathrm{d}x^k}$$

Diffeomorphism $\Gamma$:

$$\Gamma : (x, y) \longmapsto (\hat{x}, \hat{y})$$

Prolongation of $\Gamma$:

$$\Gamma : (x, y, y', \ldots, y^{(n)}) \longmapsto (\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n)})$$

$$\hat{y}^{(k)} \equiv \frac{\mathrm{d}^k \hat{y}}{\mathrm{d}\hat{x}^k} = \frac{\mathrm{D}_x \hat{y}^{(k-1)}}{\mathrm{D}_x \hat{x}}, \quad \mathrm{D}_x = \partial_x + y'\partial_y + y''\partial y' + \cdots$$

Symmetry Condition:

$$\hat{y}^{(n)} = \omega\left(\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n-1)}\right) \quad \text{as} \quad y^{(n)} = \omega\left(x, y, y', \ldots, y^{(n-1)}\right)$$

# Linearized Symmetry Condition

Lie's Ansatz:

$$\hat{x} = x + \varepsilon\xi + \mathcal{O}(\varepsilon^2)$$
$$\hat{y} = y + \varepsilon\eta + \mathcal{O}(\varepsilon^2)$$
$$\hat{y}^{(k)} = y^{(k)} + \varepsilon\eta^{(k)} + \mathcal{O}(\varepsilon^2)$$

Substitute back and take $\varepsilon$ terms:

$$\eta^{(k)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \cdots + \eta^{(k-1)}\omega_{y^{(n-1)}}$$

Compute $\eta^{(k)}$ recursively:

$$\hat{y}^{(1)} = \frac{\mathrm{D}_x\hat{y}}{\mathrm{D}_x\hat{x}} = \frac{y' + \varepsilon\mathrm{D}_x\eta + \mathcal{O}(\varepsilon^2)}{1 + \varepsilon\mathrm{D}_x\xi + \mathcal{O}(\varepsilon^2)} = y' + \varepsilon(\mathrm{D}_x\eta - y'\mathrm{D}_x\xi) + \mathcal{O}(\varepsilon^2)$$
$$\eta^{(1)} = \mathrm{D}_x\eta - y'\mathrm{D}_x\xi$$

$$\hat{y}^{(k)} = \frac{y^{(k)} + \varepsilon \mathrm{D}_x \eta^{(k-1)} + \mathcal{O}(\varepsilon^2)}{1 + \varepsilon \mathrm{D}_x \xi + \mathcal{O}(\varepsilon^2)}$$

$$\eta^{(k)} = \mathrm{D}_x \eta^{(k-1)} - y^{(k-1)} \mathrm{D}_x \xi$$

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2$$

$$\eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3 + (\eta_y - 2\xi_x - 3\xi_y y') y''$$

$$\begin{aligned} \eta^{(3)} = \eta_{xxx} &+ (3\eta_{xxy} - \xi_{xxx}) y' + 3(\eta_{xyy} - \xi_{xxy}) y'^2 + (\eta_{yyy} - 3\xi_{xyy}) y'^3 \\ &- \xi_{yyy} y'^4 + 3\{\eta_{xy} - 2\xi_{xx} + (\eta_{yy} - 3\xi_{xy}) y' - 2\xi_{yy} y'^2\} y'' \\ &- 3\xi_y y''^2 + (\eta_y - 3\xi_x - 4\xi_y y') y''' \end{aligned}$$

# Infinitesimal Generator of the Symmetry Group

Original:

$$X = \xi \partial_x + \eta \partial_y$$

Order $k$ prolongation:

$$X^{(k)} = \xi \partial_x + \eta \partial_y + \eta^{(1)} \partial_{y'} + \cdots + \eta^{(k)} \partial_{y^{(k)}}$$

Symmetry Condition Rewrite:

$$X^{(n)} \left( y^{(n)} - \omega(x, y, y', \cdots y^{(n-1)}) \right) = 0$$

# Determining Equations for Second-Order ODE

Given the second-order ode

$$y'' = \omega(x, y, y')$$

Assume the linearized symmetry condition and $\eta^{(k)}$; substitute $y''$ with $\omega(x, y, y')$, we have

$$\begin{aligned}\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')\omega \\ = \xi\omega_x + \eta\omega_y + \{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}\omega_{y'}\end{aligned}$$

Although it looks intimidating, in fact it's rather easy to solve. Note that $\eta, \xi$ are independent of $y'$, it can be further decoupled into PDEs, which are called "determining equations". As an example, we try to solve

$$y'' = \frac{y'^2}{y} - y^2$$

Using the formula above,

$$\eta_{xx}+(2\eta_{xy}-\xi_{xx})y'+(\eta_{yy}-2\xi_{xy})y'^2-\xi_{yy}y'^3+(\eta_y-2\xi_x-3\xi_y y')\left(\frac{y'^2}{y}-y^2\right)$$
$$=\eta\left(-\frac{y'^2}{y^2}-2y\right)+\{\eta_x+(\eta_y-\xi_x)y'-\xi_y y'^2\}\frac{2y'}{y}$$

Comparing powers of $y'$, we have

$$\xi_{yy}+\frac{1}{y}\xi_y=0$$
$$\eta_{yy}-2\xi_{xy}-\frac{1}{y}\eta_y+\frac{1}{y^2}\eta=0$$
$$2\eta_{xy}-\xi_{xx}+3y^2\xi_y-\frac{2}{y}\eta_x=0$$
$$\eta_{xx}-y^2(\eta_y-2\xi_x)+2y\eta=0$$

From the first two equations we can readily get

$$\begin{aligned}\xi &= A(x)\log|y| + B(x)\\ \eta &= A'(x)y(\log|y|)^2 + C(x)y\log|y| + D(x)y\end{aligned}$$

Substitute into the third equation, we have

$$3A''(x)\log|y| + 3A(x)y + 2C'(x) - B''(x) = 0$$

hence $A(x) = 0, B''(x) = 2C'(x)$. Now the last equation becomes

$$C(x)y^2\log|y| + C''(x)y\log|y| + (2B'(x) - C(x) + D(x))y^2 + D''(x)y = 0$$

which splits into

$$C(x) = 0, \quad D(x) = -2B'(x), \quad D''(x) = 0$$

We finally get

$$\xi = c_1 + c_2 x, \eta = -2c_2 y \Longrightarrow X = c_1 X_1 + c_2 X_2, X_1 = \partial_x, X_2 = x\partial_x - 2y\partial_y$$

# Invariant Solution from Symmetry: Example I

The equation

$$y''' = -yy''$$

has two infinitesimal generator

$$X_1 = \partial_x, \quad X_2 = x\partial_x - y\partial_y.$$

Non-trivial closed form solution from $X_2$:

$$\frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}y}{y} \quad \Longrightarrow \quad y = \frac{c}{x}$$

# Invariant Solution from Symmetry: Example II

The equation

$$y''' = \frac{1}{y^3}$$

has two infinitesimal generator

$$X_1 = \partial_x, \quad X_2 = x\partial_x + \frac{3}{4}y\partial_y.$$

Non-trivial closed form solution from $X_2$:

$$\frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}y}{\frac{3}{4}y} \quad \Longrightarrow \quad y = cx^{\frac{3}{4}}$$

Substitute back, we have $c = \pm\left(\frac{64}{15}\right)^{\frac{1}{4}}$.

# Solving Merton's Equation: Goal

1. Construct infinitesimal generator of the form:

$$X = \eta_1(x_1, x_2, u_1)\partial x_1 + \eta_2(x_1, x_2, u_1)\partial x_2 + \Phi_1(x_1, x_2, u_1)\partial u_1$$

2. Derive invariant solution from the infinitesimal generator.

# Determining Equations of Merton's Equation

$$\frac{\partial \Phi_1}{\partial x_2}(\gamma - 1)e^{\frac{x_1 \gamma \rho}{\gamma - 1} - x_1 \rho} = 0 \tag{1}$$

$$\frac{\partial \Phi_1}{\partial x_2}(r - \alpha)^2 = 0 \tag{2}$$

$$\frac{\partial^2 \Phi_1}{\partial x_2^2}(r - \alpha)^2 = 0 \tag{3}$$

$$\frac{\partial \eta_1}{\partial u_1}(\gamma - 1)^3 e^{\frac{2x_1 \gamma \rho}{\gamma - 1} - 2x_1 \rho} = 0 \tag{4}$$

$$\frac{\partial \eta_1}{\partial x_2}(\gamma - 1)^3 e^{\frac{2x_1 \gamma \rho}{\gamma - 1} - 2x_1 \rho} = 0 \tag{5}$$

$$x_2 \frac{\partial \Phi_1}{\partial x_2}\beta r - \frac{\partial \Phi_1}{\partial x_2}\eta\gamma + \frac{\partial \Phi_1}{\partial x_2}\eta + \frac{\partial \Phi_1}{\partial x_1}\beta = 0 \tag{6}$$

$$\frac{\partial^2 \eta_1}{\partial u_1^2}(r - \alpha)^4 = 0 \tag{7}$$

# Determining Equations

$$\frac{\partial^2 \eta_1}{\partial u_1 \partial x_2}(r-\alpha)^4 = 0 \tag{8}$$

$$\frac{\partial^2 \eta_1}{\partial x_2^2}(r-\alpha)^4 = 0 \tag{9}$$

$$\frac{\partial^3 u_1}{\partial x_2^3}\frac{\partial \eta_1}{\partial u_1}(r-\alpha)^4 = 0 \tag{10}$$

$$\frac{\partial^3 u_1}{\partial x_2^3}\frac{\partial \eta_1}{\partial x_2}(r-\alpha)^4 = 0 \tag{11}$$

$$\beta^{-\frac{\gamma}{\gamma-1}-1}(\gamma-1)\left\{x_2\frac{\partial \eta_1}{\partial u_1}\beta r - \frac{\partial \eta_1}{\partial u_1}\eta\gamma + \frac{\partial \eta_1}{\partial u_1}\eta - \frac{\partial \eta_2}{\partial u_1}\beta\right\} e^{\frac{x_1\gamma\rho}{\gamma-1}-x_1\rho} = 0 \tag{12}$$

$$\frac{\partial \eta_1}{\partial u_1}(\gamma-1)(2\gamma-1)(r-\alpha)^2 e^{\frac{x_1\gamma\rho}{\gamma-1}-x_1\rho} = 0 \tag{13}$$

$$\frac{\partial \eta_1}{\partial x_2}(\gamma-2)(\gamma-1)(r-\alpha)^2 e^{\frac{x_1\gamma\rho}{\gamma-1}-x_1\rho} = 0 \tag{14}$$

# Determining Equations

$$\frac{\partial^2 \eta_1}{\partial u_1^2}(\gamma-1)^2(r-\alpha)^2 e^{\frac{x_1\gamma\rho}{\gamma-1}-x_1\rho}=0 \tag{15}$$

$$\frac{\partial^2 \eta_1}{\partial u_1 \partial x_2}(\gamma-1)^2(r-\alpha)^2 e^{\frac{x_1\gamma\rho}{\gamma-1}-x_1\rho}=0 \tag{16}$$

$$\frac{\partial^2 \eta_1}{\partial x_2^2}(\gamma-1)^2(r-\alpha)^2 e^{\frac{x_1\gamma\rho}{\gamma-1}-x_1\rho}=0 \tag{17}$$

$$(r-\alpha)^2\left\{x_2\frac{\partial \eta_1}{\partial u_1}\beta r-\frac{\partial \eta_1}{\partial u_1}\eta\gamma+\frac{\partial \eta_1}{\partial u_1}\eta-\frac{\partial \eta_2}{\partial u_1}\beta\right\}=0 \tag{18}$$

$$(r-\alpha)^2\left\{x_2\frac{\partial \eta_1}{\partial x_2}\beta r-\frac{\partial \eta_1}{\partial x_2}\eta\gamma+\frac{\partial \eta_1}{\partial x_2}\eta+\frac{\partial \eta_1}{\partial x_1}\beta\right\}=0 \tag{19}$$

$$(r-\alpha)^2\left\{x_2\frac{\partial^2 \eta_1}{\partial u_1^2}\beta r-\frac{\partial^2 \eta_1}{\partial u_1^2}\eta\gamma+\frac{\partial^2 \eta_1}{\partial u_1^2}\eta-\frac{\partial^2 \eta_2}{\partial u_1^2}\beta\right\}=0 \tag{20}$$

# Determining Equations

$$x_2^2\frac{\partial\eta_1}{\partial x_2}\beta^2r^2-2x_2\frac{\partial\eta_1}{\partial x_2}\beta\eta\gamma r+2x_2\frac{\partial\eta_1}{\partial x_2}\beta\eta r-x_2\frac{\partial\eta_2}{\partial x_2}\beta^2r+x_2\frac{\partial\eta_1}{\partial x_1}\beta^2r$$
$$+\eta_2\beta^2r+\frac{\partial\eta_1}{\partial x_2}\eta^2\gamma^2-2\frac{\partial\eta_1}{\partial x_2}\eta^2\gamma+\frac{\partial\eta_2}{\partial x_2}\beta\eta\gamma-\frac{\partial\eta_1}{\partial x_1}\beta\eta\gamma$$
$$+\frac{\partial\eta_1}{\partial x_2}\eta^2-\frac{\partial\eta_2}{\partial x_2}\beta\eta+\frac{\partial\eta_1}{\partial x_1}\beta\eta-\frac{\partial\eta_2}{\partial x_1}\beta^2=0 \quad (21)$$

$$\beta^{-\frac{\gamma}{\gamma-1}-1}(\gamma-1)\left\{\eta_1\beta\rho+2x_2\frac{\partial\eta_1}{\partial x_2}\beta\gamma r-x_2\frac{\partial\eta_1}{\partial x_2}\beta r-2\frac{\partial\eta_1}{\partial x_2}\eta\gamma^2\right.$$
$$+3\frac{\partial\eta_1}{\partial x_2}\eta\gamma-\frac{\partial\eta_2}{\partial x_2}\beta\gamma+\frac{\partial\eta_1}{\partial x_1}\beta\gamma$$
$$\left.-\frac{\partial\eta_1}{\partial x_2}\eta+\frac{\partial\Phi_1}{\partial u_1}\beta-\frac{\partial\eta_1}{\partial x_1}\beta\right\}e^{\frac{x_1\gamma\rho}{\gamma-1}-x_1\rho}=0 \quad (22)$$

# Determining Equations

$$
\begin{aligned}
(r-\alpha)^2 \Bigg\{ & 2x_2 \frac{\partial^2 \eta_1}{\partial u_1 \partial x_2} \beta r \sigma^2 + 2 \frac{\partial \eta_1}{\partial u_1} \beta r \sigma^2 - 2 \frac{\partial^2 \eta_1}{\partial u_1 \partial x_2} \eta \gamma \sigma^2 \\
& +2 \frac{\partial^2 \eta_1}{\partial u_1 \partial x_2} \eta \sigma^2 - 2 \frac{\partial^2 \eta_2}{\partial u_1 \partial x_2} \beta \sigma^2 + \frac{\partial^2 \Phi_1}{\partial u_1^2} \beta \sigma^2 \\
& -2 \frac{\partial \eta_1}{\partial u_1} \beta r^2 + 4 \frac{\partial \eta_1}{\partial u_1} \alpha \beta r - 2 \frac{\partial \eta_1}{\partial u_1} \alpha^2 \beta \Bigg\} = 0
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
(r-\alpha)^2 \Bigg\{ & x_2 \frac{\partial^2 \eta_1}{\partial x_2^2} \beta r \sigma^2 + 2 \frac{\partial \eta_1}{\partial x_2} \beta r \sigma^2 - \frac{\partial^2 \eta_1}{\partial x_2^2} \eta \gamma \sigma^2 \\
& +2 \frac{\partial^2 \eta_1}{\partial x_2^2} \eta \sigma^2 - \frac{\partial^2 \eta_2}{\partial x_2^2} \beta \sigma^2 + 2 \frac{\partial^2 \Phi_1}{\partial u_1 \partial x_2} \beta \sigma^2 \\
& - \frac{\partial \eta_1}{\partial x_2} \beta r^2 + 2 \frac{\partial \eta_1}{\partial x_2} \alpha \beta r - \frac{\partial \eta_1}{\partial x_2} \alpha^2 \beta \Bigg\} = 0
\end{aligned} \tag{24}
$$

# Solving Determining Equations

- From (1)(2)(3)
$$\Phi_1(x_1, u_1)$$
- From (4)
$$\eta_1(x_1, x_2)$$
- From (5)(7)(8)(9)(10)(11)(13)(14)(15)(16)(17)
$$\eta_1(x_1)$$
- From (6)
$$\Phi_1(u_1)$$
- From (12)(18)(20)
$$\eta_2(x_1, x_2)$$
- From (19)
$$\eta_1 = \text{const} \equiv c_3$$

# Solving Determining Equations

- From (23)

$$\frac{\partial^2 \Phi_1}{\partial u_1^2} = 0 \Longrightarrow \Phi(u_1) = c_1 u_1 + c_2$$

- From (24)

$$\frac{\partial^2 \eta_2}{\partial x_2^2} = 0 \Longrightarrow \eta_2(x_1, x_2) = f_1(x_1)x_2 + f_2(x_1)$$

- From (22)

$$\eta_1 \beta \rho - \frac{\partial \eta_2}{\partial x_2} \beta \gamma + \frac{\partial \Phi_1}{\partial u_1} \beta = 0$$

$$\Longrightarrow c_3 \rho - f_1(x_1)\gamma + c_1 = 0 \Longrightarrow f_1(x_1) = \frac{c_1 + c_3 \rho}{\gamma}$$

# Solving Determining Equations

From (21)

$$-x_2\frac{\partial\eta_2}{\partial x_2}\beta^2 r+\eta_2\beta^2 r+\frac{\partial\eta_2}{\partial x_2}\beta\eta\gamma-\frac{\partial\eta_2}{\partial x_2}\beta\eta-\frac{\partial\eta_2}{\partial x_1}\beta^2=0$$

$$\begin{aligned}-x_2\frac{c_1+c_3\rho}{\gamma}\beta^2 r+\left(\frac{c_1+c_3\rho}{\gamma}x_2+f_2(x_1)\right)\beta^2 r\\ -\left(\frac{c_1+c_3\rho}{\gamma}\right)\beta\delta\eta-f_2'(x_1)\beta^2=0\end{aligned}$$

$$\begin{aligned}&f_2(x_1)\beta r-\left(\frac{c_1+c_3\rho}{\gamma}\right)\delta\eta-f_2'(x_1)\beta=0\\ &\Longrightarrow f_2(x_1)=\left(c_4-\frac{(c_1+c_3\rho)\delta\eta}{\gamma\beta r}\right)e^{x_1 r}+\frac{(c_1+c_3\rho)\delta\eta}{\gamma\beta r}\end{aligned}$$

The infinitesimal generator is

$$X = c_3 \frac{\partial}{\partial x_1} + (c_1 u_1 + c_2) \frac{\partial}{\partial u_1} + \left\{ \frac{c_1 + c_3\rho}{\gamma} x_2 + \left( c_4 - \frac{(c_1 + c_3\rho)\delta\eta}{\gamma\beta r} \right) e^{x_1 r} + \frac{(c_1 + c_3\rho)\delta\eta}{\gamma\beta r} \right\} \frac{\partial}{\partial x_2}$$

$$A_1 = e^{x_1 r} \frac{\partial}{\partial x_2}, \quad A_2 = \frac{\partial}{\partial u_1}$$

$$A_3 = \frac{\rho}{\gamma} \left( x_2 + \frac{\delta\eta}{\beta r}(1 - e^{x_1 r}) \right) \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}$$

$$A_4 = \frac{1}{\gamma} \left( x_2 + \frac{\delta\eta}{\beta r}(1 - e^{x_1 r}) \right) \frac{\partial}{\partial x_2} + u_1 \frac{\partial}{\partial u_1}$$

$$[A_1, A_2] = 0 \qquad [A_1, A_3] = \left( \frac{\rho}{\gamma} - r \right) A_1 \qquad [A_1, A_4] = \frac{1}{\gamma} A_1$$

$$[A_2, A_3] = 0 \qquad [A_2, A_4] = A_2 \qquad [A_3, A_4] = -\frac{\delta\eta}{\gamma\beta} A_1$$

# Computing the Invariant Solution of $A_4$

$$\frac{\mathrm{d}x_2}{\frac{1}{\gamma}\left(x_2 + \frac{\delta\eta}{\beta r}(1 - e^{x_1 r})\right)} = \frac{\mathrm{d}u_1}{u_1}$$

$$\gamma \frac{\mathrm{d}x_2}{x_2 + \frac{\delta\eta}{\beta r}(1 - e^{x_1 r})} = \frac{\mathrm{d}u_1}{u_1}$$

$$\gamma \log\left(x_2 + \frac{\delta\eta}{\beta r}(1 - e^{x_1 r})\right) = \log u_1 + g_0(x_1)$$

$$u_1 = g_1(x_1) \cdot \left(x_2 + \frac{\delta\eta}{\beta r}(1 - e^{x_1 r})\right)^{\gamma}$$

# Solution of Merton's Equation: Completion

Let

$$\Psi(t, W) = W + \frac{\delta\eta}{\beta r} + \text{const} \cdot e^{rt}$$

Then

$$\frac{\partial\Psi}{\partial t} = r \cdot \text{const} \cdot e^{rt}, \; \frac{\partial\Psi}{\partial W} = 1$$

Let $J(t, W) = g(t)\Psi(t, W)^{\gamma}$, then

$$\begin{aligned} J_W &= g\,\gamma\,\Psi^{\gamma-1} \\ J_{WW} &= g\,\gamma(\gamma-1)\,\Psi^{\gamma-2} \\ J_t &= g'\,\Psi^{\gamma} + g\,\gamma\,\Psi^{\gamma-1}\,r \cdot \text{const} \cdot e^{rt} \end{aligned}$$

$$
\begin{aligned}
\left(\frac{\delta\eta}{\beta} + rW\right) J_W &= r\left(\Psi - \mathrm{const}\cdot e^{rt}\right)\cdot g\,\gamma\,\Psi^{\gamma-1} \\
&= r\,g\,\gamma\,\Psi^{\gamma-1}\left(\Psi - \mathrm{const}\cdot e^{rt}\right) \\
&= r\,g\,\gamma\,\Psi^{\gamma} - g\,\gamma\,\Psi^{\gamma-1}\,r\cdot\mathrm{const}\cdot e^{rt}
\end{aligned}
$$

$$
\frac{J_W^2}{J_{WW}} = \frac{\left(g\,\gamma\,\Psi^{\gamma-1}\right)^2}{g\,\gamma(\gamma-1)\,\Psi^{\gamma-2}} = -g\,\frac{\gamma}{\delta}\,\Psi^{\gamma}
$$

$$
\frac{\delta^2\beta^{\frac{\gamma}{\delta}}}{\gamma} e^{-\frac{\rho}{\delta}t} J_W^{-\frac{\gamma}{\delta}} = \frac{\delta^2\beta^{\frac{\gamma}{\delta}}}{\gamma}\, e^{-\frac{\rho}{\delta}t}\left(g\,\gamma\,\Psi^{\gamma-1}\right)^{-\frac{\gamma}{\delta}} = \delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\, e^{-\frac{\rho}{\delta}t}\, g^{-\frac{\gamma}{\delta}}\,\Psi^{\gamma}
$$

$$
\begin{aligned}
&\frac{\delta^2\beta^{\frac{\gamma}{\delta}}}{\gamma} e^{-\frac{\rho}{\delta}t} J_W^{-\frac{\gamma}{\delta}} + J_t + \left(\frac{\delta\eta}{\beta} + rW\right) J_W - \delta(\mu - r)\frac{J_W^2}{J_{WW}} \\
&= \delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\, g^{-\frac{\gamma}{\delta}}\,\Psi^{\gamma}\, e^{-\frac{\rho}{\delta}t} + g'\,\Psi^{\gamma} + r\,g\,\gamma\,\Psi^{\gamma} + \gamma\,(\mu - r)\,g\,\Psi^{\gamma} \\
&= \Psi^{\gamma}\left\{\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\, e^{-\frac{\rho}{\delta}t}\, g^{-\frac{\gamma}{\delta}} + g' + \gamma\,\mu\,g\right\}
\end{aligned}
$$

$$
g' + \gamma\,\mu\,g = -\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}\, e^{-\frac{\rho}{\delta}t}\, g^{-\frac{\gamma}{\delta}}
$$

Bernoulli equation

$$f' + \varphi f = \xi f^k.$$

Let $z = f^{1-k}$, then $z' = (1-k)f^{-k}f'$. The original equation can be written as $(1-k)f^{-k}f' + (1-k)\varphi f^{-k}f = (1-k)\xi$, so $z' + (1-k)\varphi z = (1-k)\xi$, a linear ode with solution

$$z = e^{-\int(1-k)\varphi} \int (1-k)\xi \, e^{\int(1-k)\varphi} + \text{const} \cdot e^{-\int(1-k)\varphi}$$

and $f = z^{\frac{1}{1-k}}$.

$$k = -\frac{\gamma}{\delta}; 1-k = \frac{1}{\delta}$$
$$(1-k)\varphi = \frac{\gamma\mu}{\delta}$$
$$(1-k)\xi = -\frac{1}{\delta}\delta^2\gamma^{-\frac{1}{\delta}}\beta^{\frac{\gamma}{\delta}}e^{-\frac{\rho}{\delta}t} \equiv -\frac{1}{\delta}\chi e^{-\frac{\rho}{\delta}t}$$

$$e^{\int(1-k)\varphi} = e^{\frac{\gamma\mu}{\delta}t}$$

$$\begin{aligned}
z &= e^{-\int(1-k)\varphi}\int(1-k)\xi\, e^{\int(1-k)\varphi} + \text{const}\cdot e^{-\int(1-k)\varphi} \\
&= e^{-\frac{\gamma\mu}{\delta}t}\int -\frac{1}{\delta}\chi e^{-\frac{\rho}{\delta}t}\cdot e^{\frac{\gamma\mu}{\delta}t}\,\mathrm{d}t + \text{const}\cdot e^{-\frac{\gamma\mu}{\delta}t} \\
&= e^{-\frac{\gamma\mu}{\delta}t}\cdot\frac{\chi}{\delta}\cdot\frac{\delta}{\rho-\gamma\mu}e^{-\frac{\rho-\gamma\mu}{\delta}t} + \text{const}\cdot e^{-\frac{\gamma\mu}{\delta}t} \\
&= \frac{\chi e^{-\frac{\rho}{\delta}t}}{\rho-\gamma\mu} + \text{const}\cdot e^{-\frac{\gamma\mu}{\delta}t}
\end{aligned}$$

$$0 = z(T) = \frac{\chi e^{-\frac{\rho}{\delta}T}}{\rho-\gamma\mu} + \text{const}\cdot e^{-\frac{\gamma\mu}{\delta}T} \Longrightarrow \text{const} = -\frac{\chi\, e^{-\frac{\rho-\gamma\mu}{\delta}T}}{\rho-\gamma\mu}$$

$$z(t) = \frac{\chi e^{-\frac{\rho}{\delta}t}\left(1-e^{\frac{(\rho-\gamma\mu)(t-T)}{\delta}}\right)}{\rho-\gamma\mu}$$

$$
\begin{aligned}
g(t) &= \left( \frac{\chi e^{-\frac{\rho}{\delta} t} \left( 1 - e^{\frac{(\rho - \gamma\mu)(t - T)}{\delta}} \right)}{\rho - \gamma\mu} \right)^{\delta} \\
&= (\delta^2 \gamma^{-\frac{1}{\delta}} \beta^{\frac{\gamma}{\delta}})^{\delta} \left( e^{-\frac{\rho}{\delta} t} \right)^{\delta} \left( \frac{1 - e^{\frac{(\rho - \gamma\mu)(t - T)}{\delta}}}{\rho - \gamma\mu} \right)^{\delta} \\
&= \frac{\delta^{2\delta} \beta^{\gamma}}{\gamma} e^{-\rho t} \left( \frac{1 - e^{\frac{(\rho - \gamma\mu)(t - T)}{\delta}}}{\rho - \gamma\mu} \right)^{\delta}
\end{aligned}
$$

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