Introduction to Financial Models
Lecture 01: Surprises & Paradoxes I

Simpson Paradox

	Women			Men		
	applied	accepted	%	applied	accepted	%
Computer Science	26	7	27	228	58	25
Economics	240	63	26	512	112	22
Engineering	164	52	32	972	252	26
Medicine	416	99	24	578	140	24
Veterinary medicine	338	53	16	180	22	12
Total	1184	274	23	2470	584	24

Table: Cambridge University Admission Data, 1996.

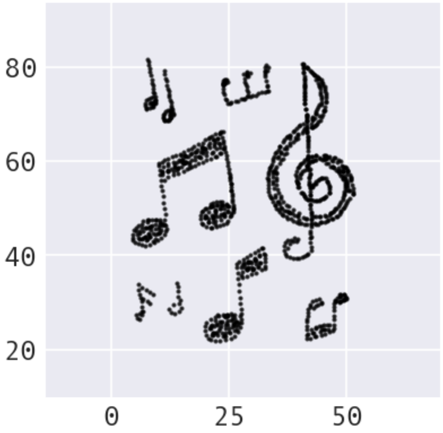
Data Morph: A Guided Tour

Milestones

- Anscombe, F., 1973. Anscombe's Quartet.
- Cairo, A., 2016. Datasaurus Dozen
- Matejka, J., Fitzmaurice, G., 2017. Same Stats, Different Graphs: Generating Datasets with Varied Appearance and Identical Statistics through Simulated Annealing. Website, Paper, Code, YouTube
- Molin, S., 2024. Data Morph: Moving Beyond the Datasaurus Dozen. Website, Code.

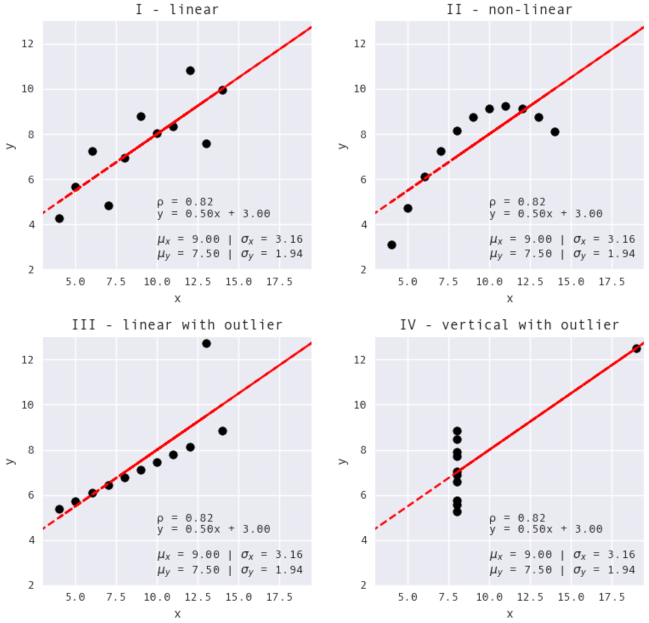
Let's play a game. I'm thinking of a distribution with the following summary statistics. Can you picture what a scatter plot of the data would look like?

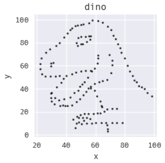
- X mean $= 30.37$
- Y mean $= 53.01$
- X standard deviation $= 13.44$
- Y standard deviation $= 15.53$
- Correlation coefficient $= 0.04$



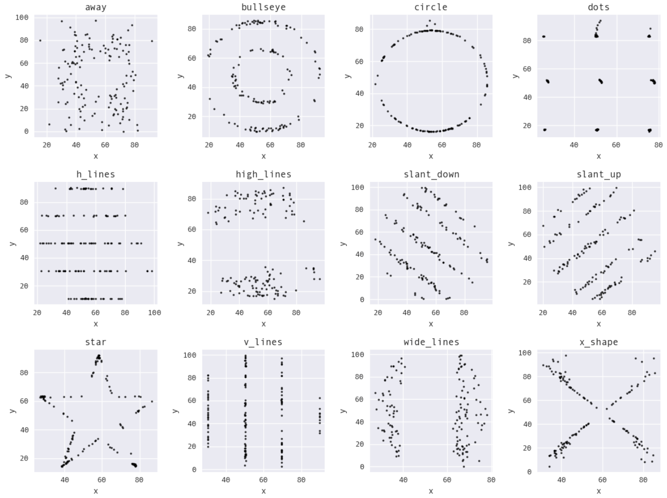
```
X Mean: 30.3685136
Y Mean: 53.0126900
X SD  : 13.4415721
Y SD  : 15.5344370
Corr. : +0.0396375
```

Anscombe's Quartet





$$\rho_p = -0.06$$
$$\mu_x = 54.26$$
$$\sigma_x = 16.71$$
$$\mu_y = 47.83$$
$$\sigma_y = 26.84$$



Data Morph

Demonstration: Datasaurus Dozen, Panda to Star, Happy Easter

How to Fit Any Dataset with a Single Parameter

The Core Result

Boué, L., 2019. Real Numbers, Data Science and Chaos: How to Fit any Dataset with a Single Parameter. arXiv, Code.

Main theorem: Any dataset can be fit using

$$f_\alpha(x) = \sin^2(2^{x\tau} \arcsin \sqrt{\alpha})$$

where:

- $\alpha \in \mathbb{R}$ is a single learned parameter
- $x \in [0, \cdots, n]$ takes integer values
- $\tau \in \mathbb{N}$ controls accuracy

Properties:

- Continuous and differentiable
- Arbitrary precision fit
- Single real-valued parameter

Demonstration: Animal Shapes

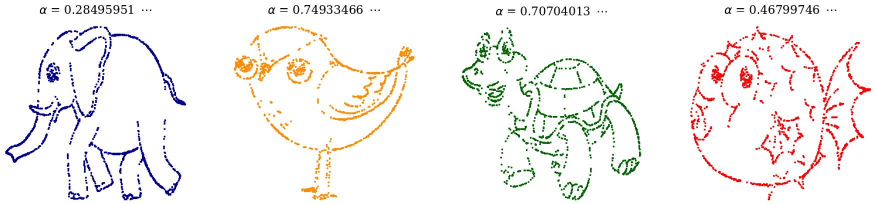
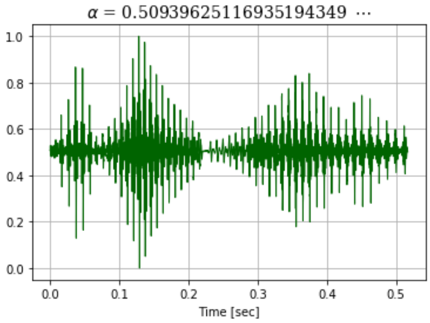


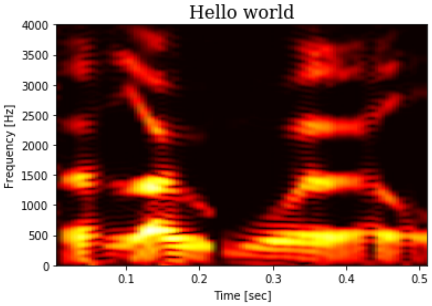
Figure: Generated using different values of α

- Each shape is a scatter plot (x, y)
- $x \in \mathbb{N}$ are integer values
- $y = f_\alpha(x)$ gives y-coordinates
- Different α values = different shapes

Audio Signal Example



(a) Waveform



(b) Spectrogram

Figure: "Hello world" audio signal

Processing:

- Sample at 11kHz
- Values determined by f_α
- Complex waveform from single α

Fixed-point Binary Representation

For $\alpha \in [0, 1]$:

$$\alpha = \sum_{n=1}^{+\infty} \frac{a_n}{2^n}$$

where $a_n \in \{0, 1\}$

$$\boxed{0.}\underbrace{\boxed{a_1}\boxed{a_2}\boxed{a_3}\boxed{a_4}\boxed{a_5}\boxed{a_6}\boxed{a_7}\boxed{a_8}\boxed{a_9}}_{\tau \text{ significant bits}}\cdots$$

In practice:

- Truncate to τ bits
- Error bound: $|\alpha - \alpha_{\text{approx}}| \leq \frac{1}{2^\tau}$

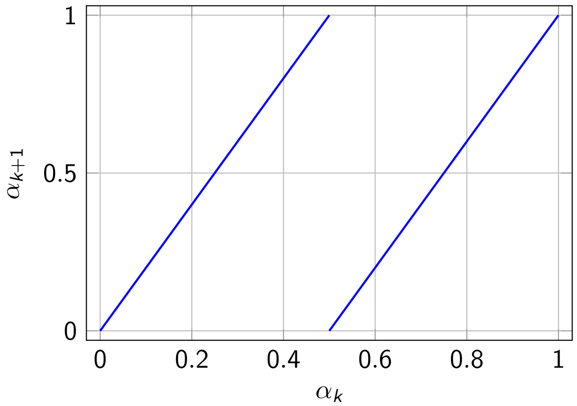
The Dyadic Transformation

Definition:

$$\mathcal{D}(\alpha_k) = 2\alpha_k \bmod 1$$

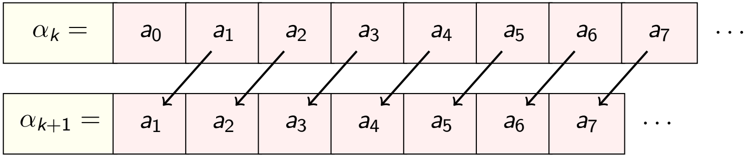
Properties:

- Maps [0,1] to itself
- Piecewise linear
- Exhibits chaos



Bit-Shift Property

In binary, $\mathcal{D}$ is a left shift:



$\alpha_k =$
a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 $\cdots$
$\alpha_{k+1} =$
a_1 a_2 a_3 a_4 a_5 a_6 a_7 $\cdots$

Key properties:

- Each iteration loses 1 bit
- After τ iterations, significant bits lost
- Shows sensitive dependence on initial conditions

Initial Implementation

Convert decimal to binary:

```
def decimalToBinary(decimalInitial):
    return reduce(lambda acc, _:
        [dyadicMap(acc[0]), acc[1] + ('0' if acc[0] < 0.5 else
            '1')],
        range(tau), [decimalInitial, ''])[1]
```

The dyadic map:

```
dyadicMap = lambda x: (2 * x) % 1
```

Encoding Strategy

Converting $\mathcal{X} = [x_0, \cdots, x_n]$ to α_0:

1. Convert each x_i to τ-bit binary
2. Concatenate all strings
3. Convert to decimal α_0

0.	x^0_{bin}	x^0_{bin}	x^0_{bin}	x^0_{bin}	x^0_{bin}	x^0_{bin}	x^0_{bin}	x^0_{bin}	x^1_{bin}	x^1_{bin}	x^1_{bin}	x^1_{bin}	x^1_{bin}	x^1_{bin}	x^1_{bin}	x^1_{bin}	$\cdots$

First τ bits encode x_0, next τ bits encode x_1, etc.

Historical Background

Origins:

- Population demographics model
- Studied by Robert May (1976)
- Canonical example of chaos

Definition:

$$z_{k+1} = \mathcal{L}(z_k) = rz_k(1 - z_k)$$

We focus on $r = 4$ case where:

- System is fully chaotic
- Maps [0,1] to itself
- No stable fixed points

Properties of the Logistic Map

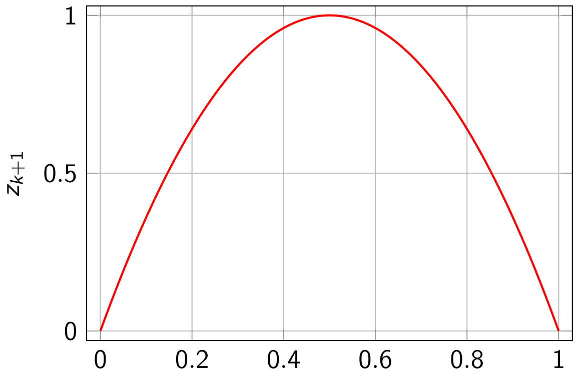
Mathematical structure:

$$\mathcal{L}(z_k) = 4z_k(1 - z_k)$$

Key features:

- Continuous and differentiable
- Maximum at $z = 1/2$
- Quadratic nonlinearity

Logistic Map for $r = 4$



Contrast with Dyadic Map

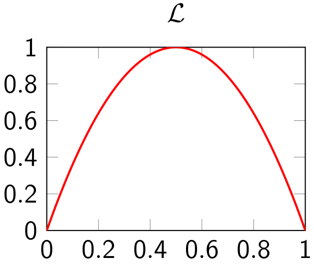
Comparison:

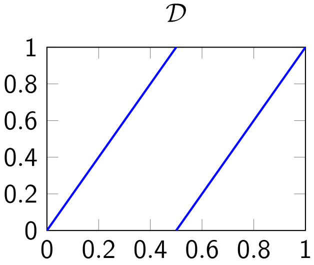
Logistic Map $\mathcal{L}$:

- Smooth
- Quadratic
- Continuous

Dyadic Map $\mathcal{D}$:

- Piecewise linear
- Uses modulo
- Discontinuous





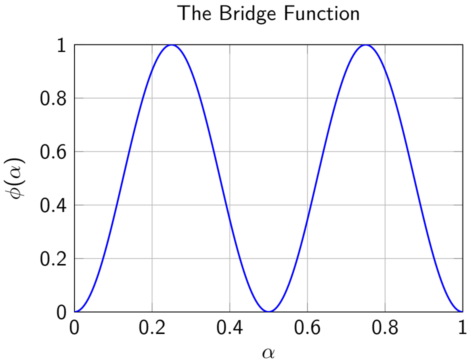
The Bridge Function ϕ

Definition:

$$\phi(\alpha) = \sin^2(2\pi\alpha)$$

Properties:

- Continuous and differentiable
- Maps [0,1] to [0,1]
- Periodic with period 1
- Has continuous inverse



The Inverse Bridge

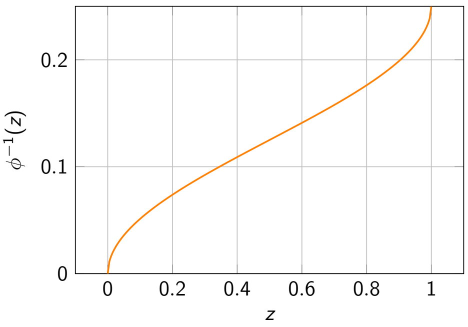
Definition:

$$\phi^{-1}(z) = \frac{\arcsin \sqrt{z}}{2\pi}$$

Properties:

- Also continuous
- Maps [0,1] to [0,1/4]
- Composition yields identity

The Inverse Bridge



The Conjugacy Relation

Note that

$$z_{k+1} = \mathcal{L}(z_k) = 4\phi(\alpha_k)(1-\phi(\alpha_k)) = 4\sin^2(2\pi\alpha_k)\cos^2(2\pi\alpha_k) = \sin^2(2\pi\cdot 2\alpha_k)$$

Key equation:

$$\mathcal{L}\circ\phi = \phi\circ\mathcal{D}$$

$$\begin{tikzcd}
\alpha_k \arrow[r, "\mathcal{D}"] \arrow[d, "\phi"'] & \alpha_{k+1} \arrow[d, "\phi"] \\
z_k \arrow[r, "\mathcal{L}"'] & z_{k+1}
\end{tikzcd}$$

Implications:

- Same dynamics in both spaces
- Can work in either representation
- Smooth version available

Deriving the Final Formula

Starting with:

$$z_k = \phi(\alpha_k) = \sin^2(2\pi\alpha_k)$$

From dyadic map:

$$\alpha_k = 2^{k\tau}\alpha_0 \bmod 1$$

Combining gives:

$$f_\alpha(x) = \sin^2(2^{x\tau} \arcsin\sqrt{\alpha})$$

This is our elegant final result!

Setting Up

Required imports and precision:

```python
from mpmath import mp, pi, sin, asin, sqrt
import numpy as np
from functools import reduce

# Set precision
mp.dps = 1000  # decimal digits
tau = 8        # bits per sample
```

Basic helper functions:

```python
# Dyadic map
dyadicMap = lambda x: (2 * x) % 1

# Bridge function
phi = lambda alpha: sin(2 * pi * alpha)**2
phiInv = lambda z: asin(sqrt(z)) / (2 * pi)
```

Binary Conversion

Converting between representations:

```
def decimalToBinary(decimalInitial):
    return reduce(lambda acc, _:
        [dyadicMap(acc[0]), acc[1] + ('0' if acc[0] < 0.5 else
            '1')],
        range(tau), [decimalInitial, ''])[1]

def binaryToDecimal(binaryInitial):
    return reduce(lambda acc, val:
        acc + int(val[1]) / 2**(val[0] + 1), enumerate(
            binaryInitial),
        mp.mpf(0.0))
```

Dataset Processing

Encoding the dataset:

```
# Convert dataset to binary
binaryInitial = ''.join(map(decimalToBinary, xs))
decimalInitial = binaryToDecimal(binaryInitial)

print('Binary initial:', binaryInitial[:50], '...')
print('Decimal initial:', float(decimalInitial))
```

The decoder function:

```
def logisticDecoder(k):
    return sin(2**(k*tau) * asin(sqrt(decimalInitial)))**2

# Recover all samples
decodedValues = [float(logisticDecoder(_)) for _ in range(len(
    xs))]
```

Error Checking

Verify theoretical bounds:

```python
# Maximum allowed error
maxError = pi / 2**(tau - 1)

# Check all errors
normalizedErrors = [abs(decoded - true) / maxError
    for decoded, true in zip(decodedValues, xs)
]

# Verify bounds
assert all(e <= 1.0 for e in normalizedErrors)
print('Maximum normalized error:', max(normalizedErrors))
```

Example output:

```
Maximum normalized error: 0.8732
All errors within theoretical bound
```

Animal Shape Example

Complete process:

1. Generate x-coordinates: $x \in [0, \ldots, n]$
2. Choose appropriate α value
3. Compute $y = f_\alpha(x)$ for each x
4. Plot resulting (x, y) pairs

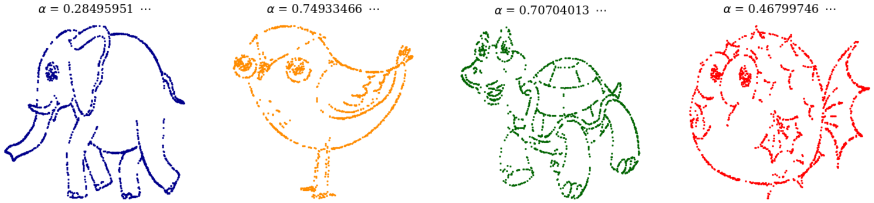
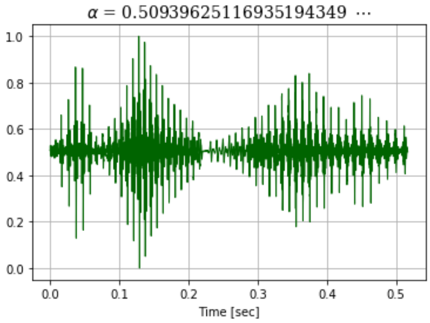


Figure: Different α values generate different shapes

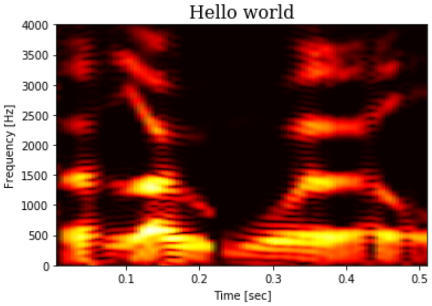
Audio Signal Generation

Process:

1. Choose sampling rate (11kHz)
2. Generate time points t_i
3. Compute $f_\alpha(i)$ for each t_i
4. Scale to audio range [-1,1]



(a) Time domain



(b) Frequency domain

Figure: "Hello world" audio encoding

Image Generation

CIFAR-10 process:

1. Generate 3072 values ($32\times32\times3$)
2. Reshape into RGB channels
3. Scale to [0,255] range
4. Stack into final image

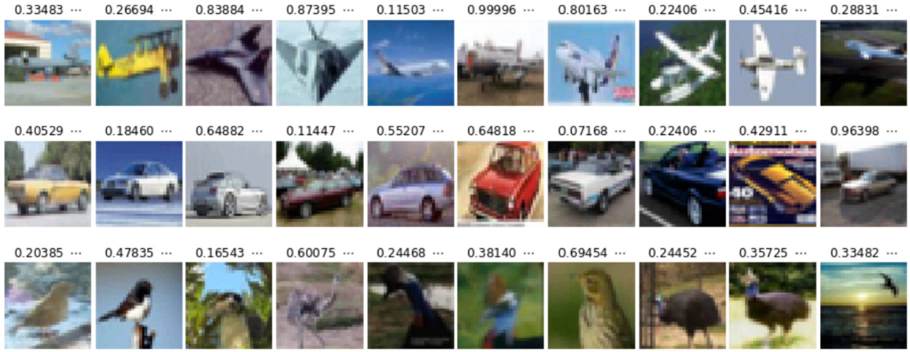


Figure: Generated CIFAR-10 style images

Generalization Analysis

Time series example:

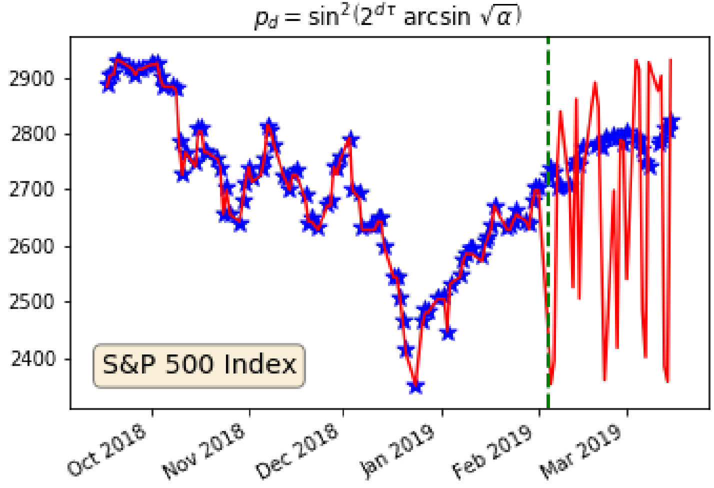


Figure: S&P 500 predictions showing no generalization