Introduction to Financial Models
Lecture 03: Surprises & Paradoxes III

St. Petersberg Paradox

The Expected Utility Hypothesis

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Definition

The agent prefers the r.v. X to r.v. Y iff

$$\mathrm{E}\, U(X) > \mathrm{E}\, U(Y)$$

where E is the expectation operator, $U : \mathbb{R} \mapsto \mathbb{R}$ is the agent's utility function.

Allais Paradox

- Game A

$$X = \begin{cases} 101 & \text{prob. } 0.33 \\ 100 & \text{prob. } 0.66 \\ 0 & \text{prob. } 0.01 \end{cases} \qquad Y = 100 \ \text{ with prob. } 1$$

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Mostly prefer Y to X: from the Expected Utility Hypothesis

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Mostly prefer Y to X: from the Expected Utility Hypothesis

$$\begin{aligned} U(100) &> 0.33 \cdot U(101) + 0.66 \cdot U(100) + 0.01 \cdot U(0) \\ &\implies 0.34 \cdot U(100) > 0.33 \cdot U(101) + 0.01 \cdot U(0) \qquad (1) \end{aligned}$$

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- Game B

$$X = \begin{cases} 100 & \text{prob. } 0.34 \\ 0 & \text{prob. } 0.66 \end{cases} \qquad Y = \begin{cases} 101 & \text{prob. } 0.33 \\ 0 & \text{prob. } 0.67 \end{cases}$$

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Mostly prefer Y to X: from the Expected Utility Hypothesis

$$U(100) > 0.33 \cdot U(101) + 0.66 \cdot U(100) + 0.01 \cdot U(0)$$
$$\Longrightarrow \ 0.34 \cdot U(100) > 0.33 \cdot U(101) + 0.01 \cdot U(0) \quad (1)$$

- Game B

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Mostly prefer Y to X: from the Expected Utility Hypothesis

$$\begin{aligned} 0.33 \cdot U(101) + 0.67 \cdot U(0) > 0.34 \cdot U(100) + 0.66 \cdot U(0) \\ \implies 0.33 \cdot U(101) + 0.01 \cdot U(0) > 0.34 \cdot U(100) \end{aligned} \tag{2}$$

Ellsberg Paradox

- Given an urn with 30 balls of colors red, yellow, and black

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- There are 10 red balls; total 20 yellow / black balls, but the number of each type unknown
- The agent estimates the probability of drawing yellow as p where $0 < p < \frac{2}{3}$
- A single ball is drawn from the urn

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

Mostly prefer X to Y: from the Expected Utility Hypothesis

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

Mostly prefer X to Y: from the Expected Utility Hypothesis

$$\frac{1}{3} \cdot U(100) + \frac{2}{3} \cdot U(0) > p \cdot U(100) + (1-p) \cdot U(0)$$
$$\implies \left(\frac{1}{3} - p\right) \cdot U(100) > \left(\frac{1}{3} - p\right) \cdot U(0) \quad (3)$$

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

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$$\implies \left(\frac{1}{3} - p\right) \cdot U(100) > \left(\frac{1}{3} - p\right) \cdot U(0) \quad (3)$$

- Game B

$$X = \begin{cases} 100 & \text{if red or black} \\ 0 & \text{if yellow} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow or black} \\ 0 & \text{if red} \end{cases}$$

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

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$$\frac{1}{3} \cdot U(100) + \frac{2}{3} \cdot U(0) > p \cdot U(100) + (1-p) \cdot U(0)$$
$$\implies \left(\frac{1}{3} - p\right) \cdot U(100) > \left(\frac{1}{3} - p\right) \cdot U(0) \quad (3)$$

- Game B

$$X = \begin{cases} 100 & \text{if red or black} \\ 0 & \text{if yellow} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow or black} \\ 0 & \text{if red} \end{cases}$$

  Mostly prefer Y to X: from the Expected Utility Hypothesis

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$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

Mostly prefer X to Y: from the Expected Utility Hypothesis

$$\frac{1}{3} \cdot U(100) + \frac{2}{3} \cdot U(0) > p \cdot U(100) + (1-p) \cdot U(0)$$
$$\implies \left(\frac{1}{3} - p\right) \cdot U(100) > \left(\frac{1}{3} - p\right) \cdot U(0) \quad (3)$$

- Game B

$$X = \begin{cases} 100 & \text{if red or black} \\ 0 & \text{if yellow} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow or black} \\ 0 & \text{if red} \end{cases}$$

Mostly prefer Y to X: from the Expected Utility Hypothesis

$$\frac{2}{3} \cdot U(100) + \frac{1}{3} \cdot U(0) > (1-p) \cdot U(100) + p \cdot U(0)$$
$$\implies \left(\frac{1}{3} - p\right) \cdot U(0) > \left(\frac{1}{3} - p\right) \cdot U(100) \quad (4)$$