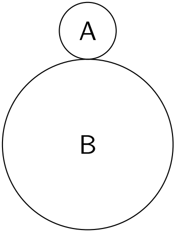
Introduction to Financial Models
Lecture 02: Surprises & Paradoxes II

1. Coin Rotation Paradox
2. Braess Paradox
3. The Voting Paradox
4. Arrow's Impossibility Theorem
5. St. Petersberg Paradox
6. Allais Paradox
7. Ellsberg Paradox

Coin Rotation Paradox

The 1982 SAT Question Everyone Got Wrong



The radius of circle A is $\frac{1}{3}$ of the radius of circle B. Circle A rolls around circle B one trip back to its starting point. How many times will circle A revolve in total?

(a) $\frac{3}{2}$ (b) 3 (c) 6 (d) $\frac{9}{2}$ (e) 9

Braess Paradox

Voting Paradox

- The Simplest Case

Voter	First preference	Second preference	Third preference
Voter 1	A	B	C
Voter 2	B	C	A
Voter 3	C	A	B

$$A > B > C > A$$

- A More Complicated Situation

Party	First preference	Second preference	Third preference
Left (3)	education	health	security
Center (4)	health	security	education
Right (5)	security	education	health

$$A > B > C > A$$

Arrow's Impossibility Theorem

St. Petersberg Paradox

The Expected Utility Hypothesis

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Definition

The agent prefers the r.v. X to r.v. Y iff

$$\mathrm{E}\, U(X) > \mathrm{E}\, U(Y)$$

where E is the expectation operator, $U : \mathbb{R} \mapsto \mathbb{R}$ is the agent's utility function.

Allais Paradox

- Game A

$$X = \begin{cases} 101 & \text{prob. } 0.33 \\ 100 & \text{prob. } 0.66 \\ 0 & \text{prob. } 0.01 \end{cases} \qquad Y = 100 \ \text{ with prob. } 1$$

- Game A

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Mostly prefer Y to X: from the Expected Utility Hypothesis

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Mostly prefer Y to X: from the Expected Utility Hypothesis

$$\begin{aligned} U(100) &> 0.33 \cdot U(101) + 0.66 \cdot U(100) + 0.01 \cdot U(0) \\ &\implies 0.34 \cdot U(100) > 0.33 \cdot U(101) + 0.01 \cdot U(0) \qquad (1) \end{aligned}$$

- Game A

$$X = \begin{cases} 101 & \text{prob. } 0.33 \\ 100 & \text{prob. } 0.66 \\ 0 & \text{prob. } 0.01 \end{cases} \qquad Y = 100 \text{ with prob. } 1$$

Mostly prefer Y to X: from the Expected Utility Hypothesis

$$\begin{aligned} U(100) > 0.33 \cdot U(101) + 0.66 \cdot U(100) + 0.01 \cdot U(0) \\ \implies 0.34 \cdot U(100) > 0.33 \cdot U(101) + 0.01 \cdot U(0) \end{aligned} \quad (1)$$

- Game B

$$X = \begin{cases} 100 & \text{prob. } 0.34 \\ 0 & \text{prob. } 0.66 \end{cases} \qquad Y = \begin{cases} 101 & \text{prob. } 0.33 \\ 0 & \text{prob. } 0.67 \end{cases}$$

- Game A

$$X = \begin{cases} 101 & \text{prob. } 0.33 \\ 100 & \text{prob. } 0.66 \\ 0 & \text{prob. } 0.01 \end{cases} \qquad Y = 100 \ \text{ with prob. } 1$$

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$$\implies 0.34 \cdot U(100) > 0.33 \cdot U(101) + 0.01 \cdot U(0) \quad (1)$$

- Game B

$$X = \begin{cases} 100 & \text{prob. } 0.34 \\ 0 & \text{prob. } 0.66 \end{cases} \qquad Y = \begin{cases} 101 & \text{prob. } 0.33 \\ 0 & \text{prob. } 0.67 \end{cases}$$

Mostly prefer Y to X: from the Expected Utility Hypothesis

$$0.33 \cdot U(101) + 0.67 \cdot U(0) > 0.34 \cdot U(100) + 0.66 \cdot U(0)$$
$$\implies 0.33 \cdot U(101) + 0.01 \cdot U(0) > 0.34 \cdot U(100) \quad (2)$$

Ellsberg Paradox

- Given an urn with 30 balls of colors red, yellow, and black

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- There are 10 red balls; total 20 yellow / black balls, but the number of each type unknown
- The agent estimates the probability of drawing yellow as p where $0 < p < \frac{2}{3}$
- A single ball is drawn from the urn

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

Mostly prefer X to Y: from the Expected Utility Hypothesis

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

Mostly prefer X to Y: from the Expected Utility Hypothesis

$$\frac{1}{3} \cdot U(100) + \frac{2}{3} \cdot U(0) > p \cdot U(100) + (1-p) \cdot U(0)$$
$$\implies \left(\frac{1}{3} - p\right) \cdot U(100) > \left(\frac{1}{3} - p\right) \cdot U(0) \quad (3)$$

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

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$$\implies (\frac{1}{3} - p) \cdot U(100) > (\frac{1}{3} - p) \cdot U(0) \quad (3)$$

- Game B

$$X = \begin{cases} 100 & \text{if red or black} \\ 0 & \text{if yellow} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow or black} \\ 0 & \text{if red} \end{cases}$$

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

Mostly prefer X to Y: from the Expected Utility Hypothesis

$$\frac{1}{3} \cdot U(100) + \frac{2}{3} \cdot U(0) > p \cdot U(100) + (1-p) \cdot U(0)$$
$$\implies \left(\frac{1}{3} - p\right) \cdot U(100) > \left(\frac{1}{3} - p\right) \cdot U(0) \quad (3)$$

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$$X = \begin{cases} 100 & \text{if red or black} \\ 0 & \text{if yellow} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow or black} \\ 0 & \text{if red} \end{cases}$$

Mostly prefer Y to X: from the Expected Utility Hypothesis

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

Mostly prefer X to Y: from the Expected Utility Hypothesis

$$\frac{1}{3} \cdot U(100) + \frac{2}{3} \cdot U(0) > p \cdot U(100) + (1-p) \cdot U(0)$$
$$\implies (\frac{1}{3} - p) \cdot U(100) > (\frac{1}{3} - p) \cdot U(0) \quad (3)$$

- Game B

$$X = \begin{cases} 100 & \text{if red or black} \\ 0 & \text{if yellow} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow or black} \\ 0 & \text{if red} \end{cases}$$

Mostly prefer Y to X: from the Expected Utility Hypothesis

$$\frac{2}{3} \cdot U(100) + \frac{1}{3} \cdot U(0) > (1-p) \cdot U(100) + p \cdot U(0)$$
$$\implies (\frac{1}{3} - p) \cdot U(0) > (\frac{1}{3} - p) \cdot U(100) \quad (4)$$