

# Portfolio Choice

# Utility

# Preferences and Utility

- Let  $\Gamma$  be a sample space of possible outcomes for gambles with random payoffs
- Let  $\mathbb{P}$  be a set of probabilities on  $\Gamma$ , where  $\mathcal{A} \in \mathbb{P}$  satisfies:
  - (i)  $0 \leq \mathcal{A}(G) \leq 1$ , for all  $G \subseteq \Gamma$
  - (ii)  $\mathcal{A}(\Gamma) = 1$
  - (iii) For disjoint events  $\{G_i\}_i$ :  $\mathcal{A}(\bigcup_i G_i) = \sum_i \mathcal{A}(G_i)$
- $\mathcal{A} \in \mathbb{P}$  is a *gamble* - probability distribution of the outcome
- $\mathbb{P}$  is closed under convex combinations:  $p\mathcal{A} + (1-p)\mathcal{B} \in \mathbb{P} \forall 0 \leq p \leq 1$
- The gamble  $p\mathcal{A} + (1-p)\mathcal{B}$  corresponds to tossing a coin with probability  $p$  of “heads”, choosing  $\mathcal{A}$  for heads and  $\mathcal{B}$  for tails
- By induction, for  $p_i \geq 0$  with  $\sum_{i=1}^k p_i = 1$ :  $p_1\mathcal{A}_1 + \dots + p_k\mathcal{A}_k \in \mathbb{P}$
- An investor has a preference relation  $\succ$  on  $\mathbb{P}$
- $\mathcal{A} \succ \mathcal{B}$  means “ $\mathcal{A}$  is preferred to  $\mathcal{B}$ ”
- Define the indifference relation  $\sim$  on  $\mathbb{P}$  by setting  $\mathcal{A} \sim \mathcal{B}$  when  $\mathcal{A} \not\succ \mathcal{B}$  and  $\mathcal{B} \not\succ \mathcal{A}$ ;  $\mathcal{A} \sim \mathcal{B}$  means “investor is indifferent between  $\mathcal{A}$  and  $\mathcal{B}$ ”
- The relations  $\succ$  and  $\sim$  satisfy rational axioms as follows.

# Rational Axioms

- 1 (Completeness) For any  $\mathcal{A}, \mathcal{B} \in \mathbb{P}$  exactly one of the following holds:
  - (i)  $\mathcal{A} \succ \mathcal{B}$
  - (ii)  $\mathcal{B} \succ \mathcal{A}$
  - (iii)  $\mathcal{A} \sim \mathcal{B}$
- 2 (Equivalence Relation) The relation  $\sim$  is an equivalence relation on  $\mathbb{P}$ :
  - (i)  $\mathcal{A} \sim \mathcal{A}$  for all  $\mathcal{A} \in \mathbb{P}$
  - (ii) If  $\mathcal{A} \sim \mathcal{B}$  and  $\mathcal{B} \sim \mathcal{C}$  then  $\mathcal{A} \sim \mathcal{C}$
  - (iii) If  $\mathcal{A} \sim \mathcal{B}$  then  $\mathcal{B} \sim \mathcal{A}$
- 3 (Transitivity of Preference) For any  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$ , if  $\mathcal{A} \succ \mathcal{B}$  and  $\mathcal{B} \succ \mathcal{C}$  then  $\mathcal{A} \succ \mathcal{C}$ .
- 4 (Mixed Transitivity) For any  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$ ,
  - (i) If  $\mathcal{A} \succ \mathcal{B}$  and  $\mathcal{B} \sim \mathcal{C}$  then  $\mathcal{A} \succ \mathcal{C}$
  - (ii) If  $\mathcal{A} \sim \mathcal{B}$  and  $\mathcal{B} \succ \mathcal{C}$  then  $\mathcal{A} \succ \mathcal{C}$
- 5 (Independence – Indifference) For any  $\mathcal{A}, \mathcal{C} \in \mathbb{P}$  and  $p \in [0, 1]$ , if  $\mathcal{A} \sim \mathcal{C}$  and  $\mathcal{B} \in \mathbb{P}$  then  $p\mathcal{A} + (1 - p)\mathcal{B} \sim p\mathcal{C} + (1 - p)\mathcal{B}$ .
- 6 (Independence – Preference) For any  $\mathcal{A}, \mathcal{C} \in \mathbb{P}$  and  $p \in (0, 1]$ , if  $\mathcal{A} \succ \mathcal{C}$  and  $\mathcal{B} \in \mathbb{P}$  then  $p\mathcal{A} + (1 - p)\mathcal{B} \succ p\mathcal{C} + (1 - p)\mathcal{B}$ .
- 7 (Continuity) For any  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$ , if  $\mathcal{A} \succ \mathcal{C} \succ \mathcal{B}$  then there exists  $p \in [0, 1]$  with  $p\mathcal{A} + (1 - p)\mathcal{B} \sim \mathcal{C}$ .

# Uniqueness of Probability Values

## Lemma

Suppose that  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$  with  $\mathcal{A} \succ \mathcal{C} \succ \mathcal{B}$  and  $p\mathcal{A} + (1-p)\mathcal{B} \sim \mathcal{C}$ , then  $0 < p < 1$  and  $p$  is unique.

## Proof

- Trivially  $p \neq 0, 1$ ; Suppose  $p$  is not unique:  $\exists q$  with  $q\mathcal{A} + (1-q)\mathcal{B} \sim \mathcal{C}$
- WLOG assume  $q < p$ , so  $0 < p - q < 1 - q$
- Note that  $\mathcal{B} = \left(\frac{p-q}{1-q}\right)\mathcal{B} + \left(\frac{1-p}{1-q}\right)\mathcal{B}$  and  $\mathcal{A} \succ \mathcal{B}$
- By Independence – Preference Axiom  $\left(\frac{p-q}{1-q}\right)\mathcal{A} + \left(\frac{1-p}{1-q}\right)\mathcal{B} \succ \mathcal{B}$
- However  $p\mathcal{A} + (1-p)\mathcal{B} = q\mathcal{A} + (1-q)\left(\left(\frac{p-q}{1-q}\right)\mathcal{A} + \left(\frac{1-p}{1-q}\right)\mathcal{B}\right)$
- By Independence – Preference Axiom again  $p\mathcal{A} + (1-p)\mathcal{B} \succ q\mathcal{A} + (1-q)\mathcal{B}$
- But this contradicts that both expressions are indifferent to  $\mathcal{C}$

# Existence of Utility Function

## Theorem

*There exists a real-valued function  $f : \mathbb{P} \rightarrow \mathbb{R}$  with*

$$f(\mathcal{A}) > f(\mathcal{B}) \quad \text{if and only if} \quad \mathcal{A} \succ \mathcal{B},$$

*and*

$$f(p\mathcal{A} + (1-p)\mathcal{B}) = pf(\mathcal{A}) + (1-p)f(\mathcal{B})$$

*for any  $\mathcal{A}, \mathcal{B} \in \mathbb{P}$  and  $0 \leq p \leq 1$ . Furthermore,  $f$  is unique up to affine transformations.*

# Proof of Utility Function Existence (Part 1)

## Proof

- If  $\mathcal{A} \sim \mathcal{B}$  for all  $\mathcal{A}, \mathcal{B} \in \mathbb{P}$ , take  $f(\mathcal{A}) \equiv 0$
- Otherwise,  $\exists$  a pair  $\mathcal{C}, \mathcal{D} \in \mathbb{P}$  with  $\mathcal{C} \succ \mathcal{D}$
- For any  $\mathcal{A} \in \mathbb{P}$ , five possibilities:

$$\textcircled{a} \quad \mathcal{A} \succ \mathcal{C}$$

$$\textcircled{c} \quad \mathcal{C} \succ \mathcal{A} \succ \mathcal{D}$$

$$\textcircled{e} \quad \mathcal{D} \succ \mathcal{A}$$

$$\textcircled{b} \quad \mathcal{A} \sim \mathcal{C}$$

$$\textcircled{d} \quad \mathcal{A} \sim \mathcal{D}$$

- Define  $f(\mathcal{C}) = 1$  and  $f(\mathcal{D}) = 0$
- For case (a):  $\exists$  unique  $p \in (0, 1)$  with  $p\mathcal{A} + (1-p)\mathcal{D} \sim \mathcal{C}$ ; define  $f(\mathcal{A}) = \frac{1}{p}$
- For case (b): set  $f(\mathcal{A}) = 1$
- For case (c):  $\exists$  unique  $q \in (0, 1)$  with  $q\mathcal{C} + (1-q)\mathcal{D} \sim \mathcal{A}$ ; define  $f(\mathcal{A}) = q$
- For case (d): set  $f(\mathcal{A}) = 0$
- For case (e):  $\exists$  unique  $r \in (0, 1)$  with  $r\mathcal{C} + (1-r)\mathcal{A} \sim \mathcal{D}$ ; define  $f(\mathcal{A}) = \frac{-r}{1-r}$

# Proof of Utility Function Existence (Part 2)

## Proof (continued)

- To verify  $f$  satisfies the conditions requires checking 15 cases: 5 where both  $\mathcal{A}$  and  $\mathcal{B}$  are in the same case, and 10 where they're in different cases
- Consider one example: both  $\mathcal{A}$  and  $\mathcal{B}$  satisfy case (c):  $\mathcal{C} \succ \mathcal{A} \succ \mathcal{D}$  and  $\mathcal{C} \succ \mathcal{B} \succ \mathcal{D}$
- We have  $f(\mathcal{A}) = q_1$  and  $f(\mathcal{B}) = q_2$  where  $\mathcal{A} \sim q_1 \mathcal{C} + (1 - q_1) \mathcal{D}$  and  $\mathcal{B} \sim q_2 \mathcal{C} + (1 - q_2) \mathcal{D}$
- When  $q_1 = q_2$ :  $\mathcal{A} \sim \mathcal{B}$  and condition is satisfied
- When  $q_1 > q_2$ :  $q_1 \mathcal{C} + (1 - q_1) \mathcal{D} \succ q_2 \mathcal{C} + (1 - q_2) \mathcal{D}$ , so  $\mathcal{A} \succ \mathcal{B}$  and  $f(\mathcal{A}) > f(\mathcal{B})$  as required
- Similarly when  $q_1 < q_2$ :  $\mathcal{B} \succ \mathcal{A}$
- For linearity, let  $p \in (0, 1)$  and apply Independence – Indifference Axiom:

$$\begin{aligned} p \mathcal{A} + (1 - p) \mathcal{B} &\sim p(q_1 \mathcal{C} + (1 - q_1) \mathcal{D}) + (1 - p)(q_2 \mathcal{C} + (1 - q_2) \mathcal{D}) \\ &\sim (pq_1 + (1 - p)q_2)\mathcal{C} + (p(1 - q_1) + (1 - p)(1 - q_2))\mathcal{D} \end{aligned}$$



# Proof of Utility Function Existence (Part 3)

## Proof (continued)

- From definition of  $f$ :  
$$f(p\mathcal{A} + (1-p)\mathcal{B}) = pq_1 + (1-p)q_2 = pf(\mathcal{A}) + (1-p)f(\mathcal{B})$$
  - To verify  $f$  is unique up to affine transformations, suppose  $g$  also satisfies the conditions
  - Since  $\mathcal{C} \succ \mathcal{D}$ , we have  $g(\mathcal{C}) > g(\mathcal{D})$
  - Define  $\beta = g(\mathcal{D})$  and  $\alpha = g(\mathcal{C}) - g(\mathcal{D}) > 0$
  - For an  $\mathcal{A}$  in case (c) with  $f(\mathcal{A}) = q$ , we have  $\mathcal{A} \sim q\mathcal{C} + (1-q)\mathcal{D}$
  - Therefore  $g(\mathcal{A}) = g(q\mathcal{C} + (1-q)\mathcal{D}) = qg(\mathcal{C}) + (1-q)g(\mathcal{D}) = q(\alpha + \beta) + (1-q)\beta = q\alpha + \beta = \alpha f(\mathcal{A}) + \beta$
  - The other cases follow similarly, proving  $g(\mathcal{A}) = \alpha f(\mathcal{A}) + \beta$  for all  $\mathcal{A} \in \mathbb{P}$
- 
- For an investor with consistent preferences, there exists a function  $f$ , unique up to affine transformations, which quantifies preference ordering
  - For  $p_i \geq 0$  with  $\sum_{i=1}^k p_i = 1$ :  $f\left(\sum_{i=1}^k p_i \mathcal{A}_i\right) = \sum_{i=1}^k p_i f(\mathcal{A}_i)$

# Expected Utility

- In finance, an investor faces investments yielding random payoffs
- Let  $\Omega$  be a probability space with measure  $P$
- Let  $\mathcal{X}$  be the set of real-valued random variables on  $\Omega$
- For  $X \in \mathcal{X}$ , let  $P^X$  be the probability distribution on  $\mathbb{R}$  induced by  $X$
- Take  $\Gamma = \mathbb{R}$  and  $\mathbb{P} = \{P^X : X \in \mathcal{X}\}$
- If  $X$  takes values  $\{x_1, \dots, x_m\}$ , then:

$$P^X(\{x\}) = \begin{cases} P(X = x) & \text{for } x \in \{x_1, \dots, x_m\}, \\ 0 & \text{otherwise.} \end{cases}$$

- Define a utility function  $v : \mathbb{R} \rightarrow \mathbb{R}$  by  $v(x) = f(P^x)$ , where  $P^x$  assigns probability 1 to value  $x$
- Then  $f(P^X) = \sum_{i=1}^m f(P^{x_i}) P(X = x_i) = \sum_{i=1}^m v(x_i) P(X = x_i) = E\{v(X)\}$
- This gives us  $E\{v(X)\} > E\{v(Y)\} \iff X \succ Y$

# Risk Attitudes

- Consider an investment with outcome described by random variable  $X$  on  $(\Omega, P)$
- Investor has utility function  $v : \mathbb{R} \rightarrow \mathbb{R}$  and prefers higher expected utility
- Let  $E_P$  denote expectation with respect to  $P$
- An investor is *risk averse* when  $E_P v(X) \leq v(E_P X)$  for all random variable  $X$  and all probabilities  $P$
- This is equivalent to  $v$  being concave
- Given  $x, y \in \mathbb{R}$  and  $\lambda \in [0, 1]$  with  $P(X = x) = \lambda$ ,  $P(X = y) = 1 - \lambda$ :  
 $\lambda v(x) + (1 - \lambda)v(y) \leq v(\lambda x + (1 - \lambda)y)$
- Risk aversion implies preferring certain outcome  $\mu$  to random investment with mean  $\mu$
- An investor is *risk neutral* when  $E_P v(X) = v(E_P X)$  for all  $P$  and  $X$
- Risk neutrality corresponds to  $v$  being affine (linear)
- An investor is *risk preferring* when  $E_P v(X) > v(E_P X)$ , corresponding to  $v$  being convex

# Risk Premiums

- *Compensatory risk premium*  $\alpha$  is the amount that must be added to make an investor indifferent between a risky investment and a certain outcome:

$$E\{v(\alpha + X)\} = v(\mu) \quad \text{with} \quad \mu = E X$$

- *Insurance risk premium*  $\beta$  is the amount an investor would pay to avoid risk:

$$E\{v(X)\} = v(\mu - \beta) \quad \text{with} \quad \mu = E X$$

- The insurance premium  $\beta$  satisfies: if  $X$  and  $Y$  have same mean  $\mu$  and  $v$  is strictly increasing, then  $X \succ Y \iff \beta_X < \beta_Y$
- Using the Taylor expansion of  $E\{v(X)\}$  about  $\mu = E X$ , we have  $E\{v(X)\} = E \left\{ v(\mu) + (X - \mu)v'(\mu) + \frac{(X - \mu)^2}{2}v''(\mu) + \dots \right\} = v(\mu) + \frac{\text{var } X}{2}v''(\mu) + \dots$
- Expanding  $v(\mu - \beta)$  and equating yields  $\beta \approx \frac{1}{2} \left( \frac{-v''(\mu)}{v'(\mu)} \right) \text{var } X$
- $-\frac{v''(\mu)}{v'(\mu)}$  is the *Arrow-Pratt absolute risk aversion* at  $\mu$
- $-\frac{E\{v''(X)\}}{E\{v'(X)\}}$  is the *global absolute risk aversion* for investment  $X$

# HARA Utility Functions

- *Hyperbolic absolute risk aversion* (HARA) functions have form:

$$v(x) = \frac{1-\gamma}{\gamma} \left( \frac{ax}{1-\gamma} + b \right)^\gamma$$

for constants  $a, b, \gamma$  with  $\frac{ax}{1-\gamma} + b \geq 0$  (usually  $b \geq 0$ )

- Arrow-Pratt risk aversion for HARA functions:

$$-\frac{v''(x)}{v'(x)} = \left( \frac{x}{1-\gamma} + \frac{b}{a} \right)^{-1}$$

- Special cases (possibly with affine transformations):

- Ⓐ *Quadratic*:  $v(x) = x - \frac{1}{2}\theta x^2$ ; take  $\gamma = 2$ ,  $a = \sqrt{\theta}$ ,  $ab = 1$
- Ⓑ *Exponential*:  $v(x) = -e^{-ax}$ ; let  $\gamma \rightarrow -\infty$ . Has absolute risk aversion  $a$
- Ⓒ *Power*:  $v(x) = x^\gamma$  with  $\gamma > 0$ . Strictly concave only when  $\gamma < 1$ . Case  $\gamma = 1$  gives risk-neutral utility
- Ⓓ *Logarithmic*:  $v(x) = \ln x$ . Follows from HARA as  $\gamma \rightarrow 0$ , using l'Hôpital's rule:  
$$\frac{x^\gamma - 1}{\gamma} \rightarrow \ln x$$

## Mean-Variance Analysis

# Classical PO: Mean-Variance (MV) Criterion

- Assets evolve from time 0 to time 1 for one period
- $s$ : number of risky assets
- $\mathbf{S}_0 \equiv (S_{1,0}, S_{2,0}, \dots, S_{s,0})^\top \neq \mathbf{0}$ : the constant price vector at time 0
- $\mathbf{S}_1 \equiv (S_{1,1}, S_{2,1}, \dots, S_{s,1})^\top$ : the random price vector at time 1
- $\mathbf{x} \equiv (x_1, x_2, \dots, x_s)^\top$ : the proportion vector of the time-0 wealth invested in each asset;  $\sum_{i=1}^s x_i = 1$ .
- $\mathbf{R} \equiv (R_1, R_2, \dots, R_s)^\top$ : the random vector representing the rate of return on the assets;  $R_i = \frac{S_{i,1}}{S_{i,0}}$
- $w$ : the (constant) wealth at time 0

# Classical PO: Mean-Variance (MV) Criterion

- $W$ : the (random) wealth at time 1;  $W = \left( \sum_{i=1}^s x_i R_i \right) w = \mathbf{x}^\top \mathbf{R} w$   
(For asset  $S_i$ ,  $\frac{x_i w}{S_{i,0}}$  denotes the “quantity” allocated at time 0; so at time 1 this part of wealth becomes  $\frac{x_i w}{S_{i,0}} S_{i,1} = x_i R_i w$ )
- $\mathbf{r} \equiv \mathbb{E} \mathbf{R} = (r_1, r_2, \dots, r_s)^\top$ : the (constant) mean vector of  $\mathbf{R}$ ;  $r_i = \mathbb{E} R_i$
- $\mathbf{V} \equiv \text{cov} \mathbf{R} \equiv \mathbb{E}\{(\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top\}$ : the (constant) covariance matrix of  $\mathbf{R}$ ;  $\mathbf{V}$  is symmetric positive definite  $s \times s$  matrix
- $\mathbb{E} W = \mathbb{E}\{\mathbf{x}^\top \mathbf{R}\} = \mathbf{x}^\top \mathbf{r} = \mu$
- $\sigma^2 = \text{var} W = \text{var}\{\mathbf{x}^\top \mathbf{R}\} = \mathbb{E}\{\mathbf{x}^\top (\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top \mathbf{x}\} = \mathbf{x}^\top \mathbf{V} \mathbf{x}$
- “For some fixed mean rate of return  $\mu = \mathbb{E}\{\mathbf{x}^\top \mathbf{R}\}$ , try to minimize the variance  $\sigma^2 = \text{var}\{\mathbf{x}^\top \mathbf{R}\}$  of the return over portfolios  $\mathbf{x}$ ”



# MV: All Risky Assets

$$\min_{\mathbf{x}} \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} \quad \text{s.t.} \quad \begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \quad \mathbf{e} \equiv \underbrace{(1, 1, \dots, 1)^\top}_{s \text{ items}}$$

- $\mathbf{V}$  is symmetric, positive definite, so  $\mathbf{V}^{-1}$  also is
- Set  $\mathcal{L} \equiv \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} + \lambda (1 - \mathbf{x}^\top \mathbf{e}) + \nu (\mu - \mathbf{x}^\top \mathbf{r})$  with Lagrange multipliers  $\lambda, \nu$
- By  $\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \mathbf{V} \mathbf{x} - \lambda \mathbf{e} - \nu \mathbf{r} = 0 \implies \mathbf{x} = \lambda \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{V}^{-1} \mathbf{r}$   
 $\implies \mathbf{x}^\top = \lambda \mathbf{e}^\top (\mathbf{V}^{-1})^\top + \nu \mathbf{r}^\top (\mathbf{V}^{-1})^\top = \lambda \mathbf{e}^\top \mathbf{V}^{-1} + \nu \mathbf{r}^\top \mathbf{V}^{-1}$
- Substitute into  $\begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \implies \begin{cases} \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = 1 \\ \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \mu \end{cases}$

- Set  $\alpha = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e}$ ,  $\beta = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r}$ ,  $\gamma = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r}$ ,  $\delta \equiv \alpha\gamma - \beta^2$ , then

$$\begin{cases} \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = 1 \\ \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \mu \end{cases}$$

becomes

$$\begin{cases} \lambda\alpha + \nu\beta = 1 \\ \lambda\beta + \nu\gamma = \mu \end{cases}$$

Solutions:  $\lambda = \frac{\gamma - \beta\mu}{\delta}$ ,  $\nu = \frac{\alpha\mu - \beta}{\delta}$

- If  $\mathbf{r} \neq c\mathbf{e}$ ,  $c \in \mathbb{R}$ , then from the positive-definiteness of  $\mathbf{V}^{-1}$

$$\begin{aligned} &(\mathbf{r} - c\mathbf{e})^\top \mathbf{V}^{-1} (\mathbf{r} - c\mathbf{e}) > 0 \\ \implies &\mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} - c \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} - c \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + c^2 \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} > 0 \\ \implies &\gamma - 2c\beta + c^2\alpha > 0 \\ \implies &-\delta = \beta^2 - \gamma\alpha < 0 \end{aligned}$$

- The relation of  $\sigma$  with  $\mu$ :

$$\begin{aligned}\sigma^2 &= \mathbf{x}^\top \mathbf{V} \mathbf{x} = \mathbf{x}^\top \mathbf{V} (\lambda \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{V}^{-1} \mathbf{r}) = \lambda (\mathbf{x}^\top \mathbf{e}) + \nu (\mathbf{x}^\top \mathbf{r}) \\ &= \lambda + \nu \mu = \frac{\gamma - \beta \mu}{\delta} + \frac{\alpha \mu - \beta}{\delta} \mu = \frac{\alpha \mu^2 - 2\beta \mu + \gamma}{\delta} \\ \Rightarrow \frac{\sigma^2}{\left(\frac{1}{\sqrt{\alpha}}\right)^2} - \frac{\left(\mu - \frac{\beta}{\alpha}\right)^2}{\left(\frac{\sqrt{\delta}}{\alpha}\right)^2} &= 1\end{aligned}$$

- Recall the standard form of hyperbola  $(x, y)$

$$\text{equation: } \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\text{asymptotes: } (y-k) = \pm \frac{b}{a}(x-h)$$

- Here we have  $(\sigma, \mu)$  with  $a = \frac{1}{\sqrt{\alpha}}$ ,  $b = \frac{\sqrt{\delta}}{\alpha}$ ,  $h = 0$ ,  $k = \frac{\beta}{\alpha}$ , the asymptotes

$$\text{are } \left(\mu - \frac{\beta}{\alpha}\right) = \pm \frac{\frac{\sqrt{\delta}}{\alpha}}{\frac{1}{\sqrt{\alpha}}} \sigma \Rightarrow \mu = \frac{\beta}{\alpha} \pm \sqrt{\frac{\delta}{\alpha}} \sigma$$

- Global minimum-variance portfolio  $\mathbf{x}_g$ 
  - First find  $\mu_g$  that minimizes  $\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$ : By differentiation

$$2\alpha\mu_g - 2\beta = 0 \implies \mu_g = \frac{\beta}{\alpha}$$

$$\bullet \lambda_g = \frac{\gamma - \beta\mu_g}{\delta} = \frac{\gamma - \beta\frac{\beta}{\alpha}}{\delta} = \frac{\gamma\alpha - \beta^2}{\alpha\delta} = \frac{1}{\alpha}$$

$$\nu_g = \frac{\alpha\mu_g - \beta}{\delta} = \frac{\beta - \beta}{\delta} = 0$$

$$\text{so } \mathbf{x}_g = \lambda_g \mathbf{V}^{-1}\mathbf{e} + \nu_g \mathbf{r}^\top \mathbf{V}^{-1} = \frac{1}{\alpha} \mathbf{V}^{-1}\mathbf{e}$$

- Diversified portfolio: define  $\mathbf{x}_d \equiv \frac{1}{\beta} \mathbf{V}^{-1}\mathbf{r}$ , then the expected return

$$\mu_d = \mathbf{x}_d^\top \mathbf{r} = \frac{1}{\beta} \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \frac{\gamma}{\beta}$$

- $\mathbf{x} = \lambda \mathbf{V}^{-1}\mathbf{e} + \nu \mathbf{V}^{-1}\mathbf{r} = \lambda \alpha \mathbf{x}_g + \nu \beta \mathbf{x}_d$ , so **every portfolio is the convex combination of  $\mathbf{x}_g$  and  $\mathbf{x}_d$** : note that  $\lambda\alpha + \nu\beta = 1$  (constraint  $\mathbf{x}^\top \mathbf{e} = 1$ ) !

## Theorem (Mutual Fund Theorem)

Any minimum-variance portfolio is equivalent to investing in the convex combination of  $\mathbf{x}_g$  and  $\mathbf{x}_d$ .

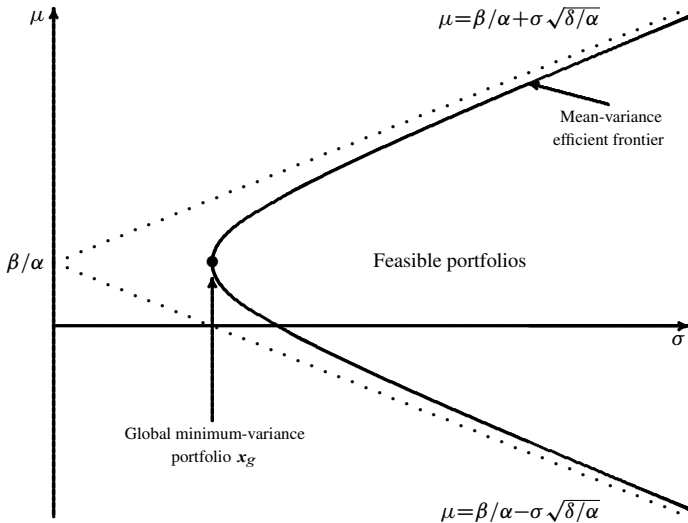


Figure: The Case of All Risky Assets

## Theorem

Diversified portfolio  $\mathbf{x}_d$  is the portfolio that maximize  $s(\mathbf{x}) \equiv \frac{\mathbf{x}^\top \mathbf{r}}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$ .

## Proof

- Maximize  $s(\mathbf{x}) \equiv \text{maximize } \log(s(\mathbf{x}))$  s.t.  $\mathbf{x}^\top \mathbf{e} = 1$
- Change of variable:  $\mathbf{x}^\top \mathbf{r} = \mu \implies \log(s(\mathbf{x})) = \log \frac{\mu}{\sqrt{\frac{\alpha \mu^2 - 2\beta \mu + \gamma}{\delta}}} \equiv f(\mu)$

with  $\mu > 0$

- $f'(\mu) = \frac{\gamma - \beta \mu}{\mu \left( \alpha \left( \mu - \frac{\beta}{\alpha} \right)^2 + \frac{\delta}{\alpha} \right)} = 0$  at  $\mu = \frac{\gamma}{\beta} = \mu_d$

- The covariance between the return of the global minimum-variance portfolio and other minimum-variance portfolio is constant:

$$\begin{aligned} \text{cov}(\mathbf{x}_g^\top \mathbf{R}, \mathbf{x}^\top \mathbf{R}) &= \mathbf{x}_g^\top \mathbf{V} \mathbf{x} = \mathbf{x}_g^\top \mathbf{V} (\lambda \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{V}^{-1} \mathbf{r}) = \lambda \mathbf{x}_g^\top \mathbf{e} + \nu \mathbf{x}_g^\top \mathbf{r} \\ &= \frac{\lambda}{\alpha} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \frac{\nu}{\alpha} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} = \frac{\lambda \alpha + \nu \beta}{\alpha} = \frac{1}{\alpha} \end{aligned}$$

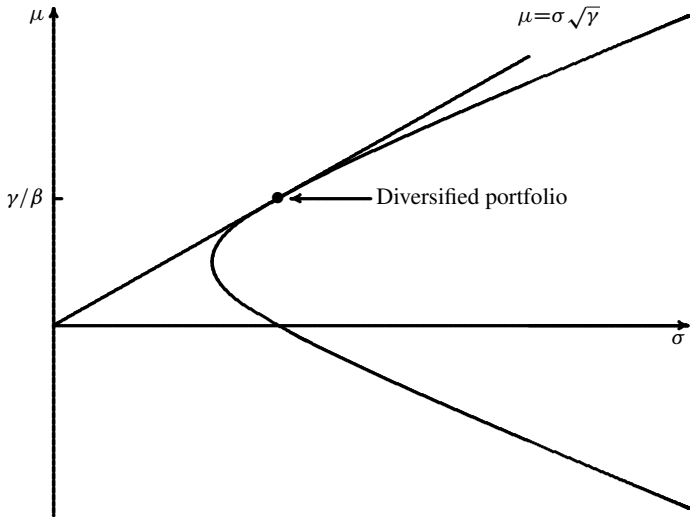


Figure: The Diversified Portfolio

# MV: All But One Risky Assets

WLOG add riskless asset 0 with constant return  $r_0$ ; the portfolio becomes  $(x_0, x_1, x_2, \dots, x_s)^\top$

$$\min_{x_0, \mathbf{x}} \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} \quad \text{s.t.} \quad \begin{cases} x_0 + \mathbf{x}^\top \mathbf{e} = 1 \\ x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \quad \mathbf{e} \equiv \underbrace{(1, 1, \dots, 1)^\top}_{s \text{ items}}$$

- Set  $\bar{\mathcal{L}} \equiv \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} + \bar{\lambda} (1 - x_0 - \mathbf{x}^\top \mathbf{e}) + \bar{\nu} (\mu - x_0 r_0 - \mathbf{x}^\top \mathbf{r})$  with Lagrange multipliers  $\bar{\lambda}, \bar{\nu}$
- By  $\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{x}} = \mathbf{V} \mathbf{x} - \bar{\lambda} \mathbf{e} - \bar{\nu} \mathbf{r} = 0 \implies \mathbf{x} = \bar{\lambda} \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{V}^{-1} \mathbf{r}$ ,  
so  $\mathbf{x}^\top = \bar{\lambda} \mathbf{e}^\top (\mathbf{V}^{-1})^\top + \bar{\nu} \mathbf{r}^\top (\mathbf{V}^{-1})^\top = \bar{\lambda} \mathbf{e}^\top \mathbf{V}^{-1} + \bar{\nu} \mathbf{r}^\top \mathbf{V}^{-1}$
- By  $\frac{\partial \bar{\mathcal{L}}}{\partial x_0} = -\bar{\lambda} - \bar{\nu} r_0 = 0 \implies \bar{\nu} = -\frac{\bar{\lambda}}{r_0}$



- $\begin{cases} x_0 + \mathbf{x}^\top \mathbf{e} = 1 \\ x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \implies \begin{cases} x_0 + \bar{\lambda} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = 1 \\ x_0 r_0 + \bar{\lambda} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + \bar{\nu} \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \mu \end{cases}$
- Set  $\alpha = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e}$ ,  $\beta = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r}$ ,  $\gamma = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r}$ ,  $\delta \equiv \alpha\gamma - \beta^2$ , the above becomes

$$\begin{cases} x_0 + \bar{\lambda}\alpha + \bar{\nu}\beta = x_0 + \bar{\lambda}\alpha - \frac{\bar{\lambda}}{r_0}\beta = 1 \\ x_0 r_0 + \bar{\lambda}\beta + \bar{\nu}\gamma = x_0 r_0 + \bar{\lambda}\beta - \frac{\bar{\lambda}}{r_0}\gamma = \mu \end{cases}$$

with solutions  $x_0 = \frac{\alpha\mu r_0 - \beta r_0 + \gamma - \beta\mu}{\epsilon^2}$ ,  $\bar{\lambda} = \frac{(r_0 - \mu)r_0}{\epsilon^2}$ ,

$\bar{\nu} = -\frac{r_0 - \mu}{\epsilon^2}$ , where  $\epsilon^2 = \alpha r_0^2 - 2\beta r_0 + \gamma = \alpha \left(r_0 - \frac{\beta}{\alpha}\right)^2 + \frac{\delta}{\alpha}$

- The relation of  $\sigma$  with  $\mu$

$$\begin{aligned} \sigma^2 &= \mathbf{x}^\top \mathbf{V} \mathbf{x} = \mathbf{x}^\top \mathbf{V} (\bar{\lambda} \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{V}^{-1} \mathbf{r}) = \bar{\lambda} (\mathbf{x}^\top \mathbf{e}) + \bar{\nu} (\mathbf{x}^\top \mathbf{r}) \\ &= \bar{\lambda} (1 - x_0) + \bar{\nu} (\mu - x_0 r_0) = \bar{\lambda} + \bar{\nu} \mu = \frac{(\mu - r_0)^2}{\epsilon^2} \end{aligned}$$

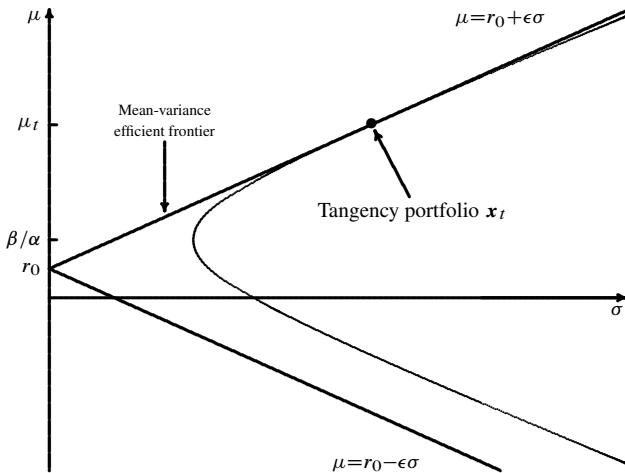


Figure: The Case of All But One Risky Assets

## Property

If  $r_0 < \frac{\beta}{\alpha}$ , then  $\mu = r_0 + \epsilon\sigma$  touches the hyperbola  $\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$  at  $\left(\frac{\epsilon}{\beta - \alpha r_0}, \frac{\gamma - \beta r_0}{\beta - \alpha r_0}\right)$

## Proof

On  $\sigma - \mu$  plane the slope of the tangent is obtained by implicit differentiation of

$$\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta} \text{ w.r.t } \sigma \text{ (let } \mu \equiv \mu(\sigma)\text{): } 2\sigma = \frac{2\alpha\mu\mu' - 2\beta\mu'}{\delta} \Rightarrow$$

$$\mu' = \frac{\delta\sigma}{\alpha\mu - \beta}. \text{ The tangent line is } \mu = r_0 + \epsilon\sigma \text{ with slope } \epsilon, \text{ so } \epsilon = \frac{\delta\sigma}{\alpha\mu - \beta} \Rightarrow$$

$$\delta\sigma = \alpha\mu\epsilon - \beta\epsilon \Rightarrow \delta\sigma = \alpha\epsilon(r_0 + \epsilon\sigma) - \beta\epsilon \Rightarrow (\delta - \alpha\epsilon^2)\sigma = \epsilon(\alpha r_0 - \beta). \text{ Note}$$

$$\text{that } \epsilon^2 = \alpha r_0^2 - 2\beta r_0 + \gamma = \alpha\left(r_0 - \frac{\beta}{\alpha}\right)^2 + \frac{\delta}{\alpha}, \text{ so } \sigma = \frac{\epsilon(\alpha r_0 - \beta)}{\delta - \alpha\epsilon^2} =$$

$$\frac{\epsilon(\alpha r_0 - \beta)}{-\alpha^2\left(r_0 - \frac{\beta}{\alpha}\right)^2} = \frac{\epsilon}{\beta - \alpha r_0}, \mu = r_0 + \epsilon\frac{\epsilon}{\beta - \alpha r_0} = \frac{\beta r_0 - \alpha r_0^2 + \epsilon^2}{\beta - \alpha r_0} = \frac{\gamma - \beta r_0}{\beta - \alpha r_0}.$$

- Define the tangency portfolio

$$\mathbf{x}_t = \frac{1}{\beta - \alpha r_0} \mathbf{V}^{-1}(\mathbf{r} - r_0 \mathbf{e}) = \frac{\beta}{\beta - \alpha r_0} \mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mathbf{x}_g$$

- $\mathbf{x} = \bar{\lambda} \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{V}^{-1} \mathbf{r} = \bar{\nu} \mathbf{V}^{-1}(\mathbf{r} - r_0 \mathbf{e}) \equiv (1 - x_0) \mathbf{x}_t$

- $\mathbf{e}^\top \mathbf{x}_t = \frac{\beta}{\beta - \alpha r_0} \mathbf{e}^\top \mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mathbf{e}^\top \mathbf{x}_g = \frac{\beta}{\beta - \alpha r_0} - \frac{\alpha r_0}{\beta - \alpha r_0} = 1$

- $\mu_t = \mathbf{x}_t^\top \mathbf{r} = \mathbf{r}^\top \mathbf{x}_t = \frac{\beta}{\beta - \alpha r_0} \mathbf{r}^\top \mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mathbf{r}^\top \mathbf{x}_g$   
 $= \frac{\beta}{\beta - \alpha r_0} \mu_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mu_g = \frac{\gamma - \beta r_0}{\beta - \alpha r_0}$  for  $\mu_d = \frac{\gamma}{\beta}$ ,  $\mu_g = \frac{\beta}{\alpha}$

## Theorem

Tangency portfolio  $\mathbf{x}_t$  is the portfolio that maximize  $s(\mathbf{x}) \equiv \frac{\mathbf{x}^\top \mathbf{r} - r_0}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$ .

## Proof

- Maximize  $s(\mathbf{x}) \equiv \text{maximize } \log(s(\mathbf{x}))$  s.t.  $\mathbf{x}^\top \mathbf{e} = 1$
- Change of variable  $\mathbf{x}^\top \mathbf{r} = \mu \implies \log(s(\mathbf{x})) = \log \frac{\mu - r_0}{\sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}} \equiv f(\mu)$  with  $\mu > r_0$
- $f'(\mu) = \frac{(\gamma - \beta r_0) - (\beta - \alpha r_0)\mu}{(\mu - r_0)(\alpha\mu^2 - 2\beta\mu + \gamma)} = 0$  at  $\mu = \frac{\gamma - \beta r_0}{\beta - \alpha r_0} = \mu_t$ .

# Mean-Variance Pricing Equation

- $\mathbf{V} = \mathbb{E} \{ (\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top \} = \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top - \mathbf{R} \mathbf{r}^\top - \mathbf{r} \mathbf{R}^\top + \mathbf{r} \mathbf{r}^\top \} = \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top - \mathbf{R} \mathbf{r}^\top \}$
- $\text{cov}(R_i, \mathbf{x}_t^\top \mathbf{R}) = \mathbb{E} \{ (R_i - r_i)(\mathbf{x}_t^\top \mathbf{R} - \mathbf{x}_t^\top \mathbf{r}) \} = \mathbb{E} \{ R_i \mathbf{x}_t^\top \mathbf{R} - R_i \mathbf{x}_t^\top \mathbf{r} - r_i \mathbf{x}_t^\top \mathbf{R} + r_i \mathbf{x}_t^\top \mathbf{r} \} = \mathbb{E} \{ R_i \mathbf{x}_t^\top \mathbf{R} - R_i \mathbf{x}_t^\top \mathbf{r} \} = \mathbb{E} \{ R_i \mathbf{R}^\top \mathbf{x}_t - R_i \mathbf{r}^\top \mathbf{x}_t \}$
- $\mathbf{V} \mathbf{x}_t = \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top \mathbf{x}_t - \mathbf{R} \mathbf{r}^\top \mathbf{x}_t \}$
- $(\mathbf{V} \mathbf{x}_t)_i = \frac{1}{\beta - \alpha r_0} (r_i - r_0);$
- $\text{var}(\mathbf{x}_t^\top \mathbf{R}) = \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \cdot (\mathbf{x}_t^\top \mathbf{R})^\top \} - (\mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \})^2 = \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \mathbf{R}^\top \mathbf{x}_t \} - \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \} \mathbb{E} \{ \mathbf{R}^\top \mathbf{x}_t \} = \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \mathbf{R}^\top \mathbf{x}_t \} - \mathbf{x}_t^\top \mathbf{r} \mathbf{r}^\top \mathbf{x}_t = \mathbf{x}_t^\top \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top - \mathbf{r} \mathbf{r}^\top \} \mathbf{x}_t = \mathbf{x}_t^\top \mathbf{V} \mathbf{x}_t = \frac{\mu_t - r_0}{\beta - \alpha r_0}.$
- $\beta_{i,t} = \frac{\text{cov}(R_i, \mathbf{x}_t^\top \mathbf{R})}{\text{var}(\mathbf{x}_t^\top \mathbf{R})} = \text{cor}(R_i, \mathbf{x}_t^\top \mathbf{R}) \sqrt{\frac{\text{var } R_i}{\text{var}(\mathbf{x}_t^\top \mathbf{R})}}; \text{ define}$   
 $\beta_t \equiv (\beta_{1,t}, \beta_{2,t}, \dots, \beta_{s,t})^\top$
- $\beta_t = \frac{1}{\mu_t - r_0} (\mathbf{r} - r_0 \mathbf{e}) \implies \mathbf{r} = r_0 \mathbf{e} + (\mu_t - r_0) \beta_t$

# Mean-Variance Analysis and Expected Utility

- Define  $f(\sigma, \mu) = \mathbb{E} v(W)$  where  $W = (x_0 r_0 + \mathbf{x}^\top \mathbf{R})w$ ,  $\sigma^2 = \mathbf{x}^\top \mathbf{V} \mathbf{x}$ ,  $\mu = x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mathbf{x}^\top (\mathbf{r} - r_0 \mathbf{e})$
- Assume that  $\frac{\partial f}{\partial \sigma} < 0$ ,  $\frac{\partial f}{\partial \mu} > 0$  with  $x_0 + \mathbf{x}^\top \mathbf{e} = 1$ , perform 
$$\max_{\mathbf{x}} f\left(\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}, r_0 + \mathbf{x}^\top (\mathbf{r} - r_0 \mathbf{e})\right)$$
- $$\frac{\partial f}{\partial \mathbf{x}} = \frac{1}{\sigma} \frac{\partial f}{\partial \sigma} \mathbf{V} \mathbf{x} + \frac{\partial f}{\partial \mu} (\mathbf{r} - r_0 \mathbf{e}) = 0 \implies \mathbf{x} = -\frac{\sigma \frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial \sigma}} \mathbf{V}^{-1} (\mathbf{r} - r_0 \mathbf{e}) \propto \mathbf{x}_t$$
- Example:
  - For quadratic utility  $v(x) = ax + bx^2$  where  $a, b \in \mathbb{R}$ ,  $b \leq 0$ :  
 $\mathbb{E} v(W) = \mathbb{E} v((x_0 r_0 + \mathbf{x}^\top \mathbf{R})w) = aw\mu + bw^2(\mu^2 + \sigma^2) = f(\sigma, \mu)$
  - For normally distributed returns  $\mathbf{R} \sim N(\mathbf{r}, \mathbf{V})$ ,  $\mathbf{x}^\top \mathbf{R} \sim N(\mathbf{x}^\top \mathbf{r}, \mathbf{x}^\top \mathbf{V} \mathbf{x})$ :  
 $\mathbb{E} v(W) = \mathbb{E} v((\mu + \sigma Y)w)$ , where  $Y \sim N(0, 1)$

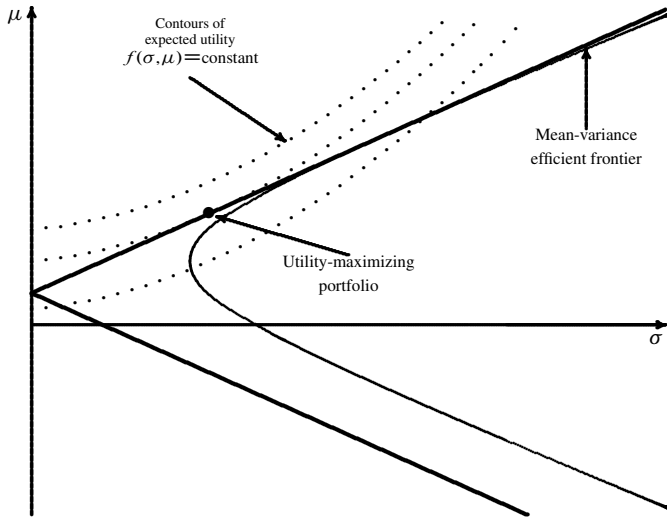


Figure: Determining the Utility-Maximizing Portfolio



# Equilibrium: The Capital-Asset Pricing Model

- Investors indexed by  $j \in \mathcal{J}$ , each with proportions of wealth  $x_{0,j}$  and  $\mathbf{x}_j = (x_{1,j}, x_{2,j}, \dots, x_{s,j})^\top$
- When each investor  $j$  has the utility function as above, the optimal  $\mathbf{x}_j \propto \mathbf{x}_t$   
 $\implies \mathbf{x}_j = (1 - x_{0,j}) \mathbf{x}_t \quad \forall j \in \mathcal{J}$
- The total value of the demand for risky asset  $i$ :  

$$\sum_{j \in \mathcal{J}} w_j x_{i,j} = \left( \sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) (\mathbf{x}_t)_i$$
- Market portfolio* of risky assets  $\mathbf{x}_m$ :  

$$(\mathbf{x}_m)_i \equiv \frac{\text{The total value of the supply of risky asset } i}{\text{The total value of the supply of all risky assets}}; \quad \mathbf{x}_m^\top \mathbf{e} = 1$$
- In equilibrium  $(\mathbf{x}_m)_i = \frac{\left( \sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) (\mathbf{x}_t)_i}{\sum_{j \in \mathcal{J}} \sum_{k=1}^s w_j x_{k,j}} =$   

$$\frac{\left( \sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) (\mathbf{x}_t)_i}{\left( \sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) \sum_{k=1}^s (\mathbf{x}_t)_k} = (\mathbf{x}_t)_i, \text{ since } \mathbf{x}_t^\top \mathbf{e} = 1$$
- $\mathbf{r} = r_0 \mathbf{e} + (\mu_m - r_0) \boldsymbol{\beta}_m$ ,  $\boldsymbol{\beta}_m \equiv (\beta_{1,m}, \beta_{2,m}, \dots, \beta_{s,m})^\top$ ,  

$$\beta_{i,m} = \frac{\text{cov}(R_i, \mathbf{x}_m^\top \mathbf{R})}{\text{var}(\mathbf{x}_m^\top \mathbf{R})} \text{ — capital-asset-pricing equation}$$

## Problems and Solutions

# Problem 1

Suppose that an investment  $X$  has either (i) the uniform distribution  $U[0, 2\mu]$  or (ii) the exponential distribution with  $E X = \mu$ , and the investor has a utility function which is either (a) logarithmic,  $v(x) = \log x$  (b) power form,  $v(x) = x^\theta$ . Show that both the compensatory risk premium and the investment risk premium are proportional to  $\mu$  in all 4 possible cases.

# Problem 1 Solution

- For distributions (i)(ii) of  $X$ , the random variable  $Y \equiv \frac{X}{\mu}$  does not depend on  $\mu$ , so  $E v(X + \alpha) = v(\mu)$  for the compensatory risk premium  $\alpha$  reduces to  $E v(Y + c) = v(1)$  in cases (a)(b) when  $\alpha = c\mu$ . For the insurance risk premium when  $\beta = d\mu$ ,  $d$  is the solution of  $E v(Y) = v(1 - d)$ .
- For case (i)(a),

$$E v(Y + c) = \int_0^2 \frac{\log(y + c)}{2} dy = \frac{1}{2}((2 + c) \log(2 + c) - c \log c - 2), \text{ and}$$

$v(1) = \log 1 = 0$ , so  $\alpha = c\mu$  where  $c$  is the unique positive root of  $(2 + c) \log(2 + c) - c \log c - 2 = 0$ . Using `rmaxima`

```
find_root((2 + x) * log(2 + x) - x * log(x) - 2, x, 0.01, 20);
```

we have  $c = 0.176965531$ . For the insurance premium  $\beta = d\mu$ ,

$$E \log Y = \log 2 - 1 = \log(1 - d), \text{ so } d = 1 - \frac{2}{e} = 0.264.$$

## Problem 2

An investor has a utility function  $v(x) = \sqrt{x}$  and is considering three investments with random outcomes  $X, Y, Z$ . Here  $X$  has the uniform distribution  $U[0, a]$ ,  $Y$  has the gamma distribution  $\Gamma(\gamma, \lambda)$  with probability density function  $\frac{e^{-\lambda y} \lambda^\gamma y^{\gamma-1}}{\Gamma(\gamma)}$  for  $y > 0$ , where  $\gamma > 0$ ,  $\lambda > 0$ , and  $Z$  is log-normal, i.e.  $Z \sim N(\nu, \sigma^2)$ . The parameter of the distributions are such that  $E X = E Y = E Z = \mu$  and  $\text{var } X = \text{var } Y = \text{var } Z$ . Recall that the gamma function  $\Gamma(\gamma) = \int_0^\infty u^{\gamma-1} e^{-u} du$  that satisfies  $\Gamma(\gamma + 1) = \gamma \Gamma(\gamma)$  and  $\Gamma(1/2) = \sqrt{\pi}$ . Determine the investor's preference ordering of  $X, Y, Z$  for all values of  $\mu$ .

## Problem 2 Solution I

- $X \sim U[0, a] \implies \mathbb{E} X = \frac{a}{2}, \text{ var } X = \frac{a^2}{12}$
- $Y \sim \Gamma(\gamma, \lambda) \implies \mathbb{E} Y = \frac{\gamma}{\lambda}, \text{ var } Y = \frac{\gamma}{\lambda^2}$
- $Z \sim \text{lognormal}(\nu, \sigma^2) \implies \mathbb{E} Z = e^{\nu + \frac{\sigma^2}{2}}, \text{ var } Z = e^{2\nu + \sigma^2}(e^{\sigma^2} - 1)$  by the formula  $\mathbb{E} e^{\theta W} = e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}}$  for  $W \sim N(\mu, \sigma^2)$

$$\begin{aligned}\mathbb{E} e^{\theta W} &= e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \text{ as } W \sim N(\mu, \sigma^2): \sqrt{2\pi}\sigma \mathbb{E} e^{\theta W} = \int_{-\infty}^{\infty} e^{\theta x} \cdot e^{-\frac{1}{2} \frac{x^2 - 2\mu x + \mu^2}{\sigma^2}} dx \\&= \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{x^2 - 2(\mu + \theta\sigma^2)x + \mu^2}{\sigma^2}} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x - (\mu + \theta\sigma^2))^2 + \mu^2 - (\mu + \theta\sigma^2)^2}{\sigma^2}} dx \\&= \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x - (\mu + \theta\sigma^2))^2 - 2\mu\theta\sigma^2 - (\theta\sigma^2)^2}{\sigma^2}} dx = e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x - (\mu + \theta\sigma^2))^2}{\sigma^2}} dx \\&= e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi}\sigma \cdot e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \text{ by } \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.\end{aligned}$$

The conditions  $\mathbb{E} X = \mathbb{E} Y = \mathbb{E} Z = \mu$  and  $\text{var } X = \text{var } Y = \text{var } Z$  imply

- $a = 2\mu$ , so that  $\text{var } X = \frac{\mu^2}{3}$ .

## Problem 2 Solution II

- $EY = \frac{\gamma}{\lambda} = \mu$ , so that  $\text{var } Y = \frac{\gamma}{\lambda^2} = \frac{\mu^2}{\gamma} = \text{var } X = \frac{\mu^2}{3} \implies \gamma = 3$
- $EZ = e^{\nu + \frac{\sigma^2}{2}} = \mu$ ,  $\text{var } Z = e^{2\nu + \sigma^2}(e^{\sigma^2} - 1) = \mu^2(e^{\sigma^2} - 1) = \text{var } X = \frac{\mu^2}{3}$   
 $\implies \sigma^2 = \log \frac{4}{3}$ .
- $E\sqrt{X} = \int_0^{2\mu} \frac{\sqrt{x}}{2\mu} dx = \frac{2^{\frac{3}{2}}}{3} \sqrt{\mu} \approx 0.943\sqrt{\mu}$
- $E\sqrt{Y} = \int_0^\infty \sqrt{y} \frac{1}{2} e^{-\lambda y} \lambda^3 y^2 dy = \frac{\Gamma(\frac{7}{2})}{2\sqrt{\lambda}} = \frac{\Gamma(\frac{7}{2})}{2\sqrt{3}} \sqrt{\mu} = \frac{5\sqrt{3}\pi}{16} \sqrt{\mu} \approx 0.959\sqrt{\mu}$
- $E\sqrt{Z} = e^{\frac{\nu}{2} + \frac{\sigma^2}{8}} = e^{-\frac{\sigma^2}{8}} \sqrt{\mu} = \left(\frac{3}{4}\right)^{\frac{1}{8}} \sqrt{\mu} \approx 0.965\sqrt{\mu}$

So  $Z \succ Y \succ X$ .

## Problem 3

Suppose that an investor has the utility function  $v(x) = 1 - e^{-ax}$  with  $a > 0$ , and the outcome of an investment is a random variable  $X$  with mean  $\mu$ , finite variance and finite moment-generating function  $\psi(a) = E\{e^{-aX}\}$  for  $a > 0$ . Show that

- ① The compensatory risk premium and the insurance risk premium have the same value  $\alpha$ , and express  $\alpha$  in terms of  $\mu$  and the moment generating function  $\psi$ .
- ② Both the Arrow-Pratt and global risk aversions are  $a$ .
- ③ As  $a \downarrow 0$ ,  $\alpha = \frac{a}{2} \text{var } X + \mathcal{O}(a)$ . Under what circumstances is it true that  $\alpha = \frac{a}{2} \text{var } X$  for  $a > 0$ ?
- ④  $\psi''\psi - (\psi')^2 \geq 0$  and hence  $\alpha$  is an increasing function of  $a$ . This shows that the more risk-averse the investor is, the higher the value of the premium that is required.



# Problem 3 Solution I

- ①
- The compensatory risk premium  $\alpha$  solves  $E\{v(\alpha + X)\} = v(\mu) \implies E\{1 - e^{-a(\alpha + X)}\} = 1 - e^{-a\mu} \implies 1 - e^{-a\alpha} E\{e^{-aX}\} = 1 - e^{-a\mu} \implies -a\alpha + \ln \psi(a) = -a\mu \implies \alpha = \mu + \frac{1}{a} \ln(\psi(a))$
  - The insurance risk premium  $\beta$  solves  $E v(X) = v(\mu - \beta) \implies E\{1 - e^{-aX}\} = 1 - e^{-a(\mu - \beta)} \implies 1 - \psi(a) = 1 - e^{-a(\mu - \beta)} \implies \ln \psi(a) = -a(\mu - \beta) \implies \beta = \mu + \frac{1}{a} \ln(\psi(a))$

So  $\alpha = \beta = \mu + \frac{1}{a} \ln(\psi(a))$ .

- ② Note that  $v'(x) = ae^{-ax}$ ,  $v''(x) = -a^2e^{-ax}$ , the Arrow-Pratt absolute risk aversion is  $-\frac{v''(\mu)}{v'(\mu)} = \frac{a^2e^{-a\mu}}{ae^{-a\mu}} = a$ , the global absolute risk aversion is  $-\frac{E\{v''(X)\}}{E\{v'(X)\}} = \frac{a^2 E\{e^{-aX}\}}{a E\{e^{-aX}\}} = a$ .

## Problem 3 Solution II

- ③ From  $\alpha(a) = \mu + \frac{1}{a} \ln(\psi(a)) \implies \psi(a) = e^{a(\alpha(a)-\mu)}$ ; Differentiation yields  $\psi'(a) = e^{a(\alpha(a)-\mu)}(\alpha(a) - \mu + a\alpha'(a))$ ,  $\psi''(a) = e^{a(\alpha(a)-\mu)}(2\alpha'(a) + a\alpha''(a)) + e^{a(\alpha(a)-\mu)}(\alpha(a) - \mu + a\alpha'(a))^2$ . Note that  $\psi(0) = 1$ ,  $\psi'(0) = -\mu$ ,  $\psi''(0) = \mathbb{E} X^2$ ; setting  $a = 0$  yields  $\psi'(0) = e^{0(\alpha(0)-\mu)}(\alpha(0) - \mu + 0\alpha'(0)) \implies -\mu = \alpha(0) - \mu \implies \alpha(0) = 0$ ,  $\psi''(0) = e^{0(\alpha(0)-\mu)}(2\alpha'(0) + 0\alpha''(0)) + e^{0(\alpha(0)-\mu)}(\alpha(0) - \mu + 0\alpha'(0))^2 \implies \mathbb{E} X^2 = 2\alpha'(0) + \mu^2 \implies \alpha'(0) = \frac{1}{2}(\mathbb{E} X^2 - \mu^2) = \frac{1}{2} \text{var } X$ . For small  $a > 0$ , the Taylor expansion of  $\alpha(a) = \alpha(0) + a\alpha'(0) + \mathcal{O}(a^2) = \frac{a}{2} \text{var } X + \mathcal{O}(a^2)$ . When  $\alpha(a) = \frac{a}{2} \text{var } X$  exactly for  $a > 0$ , then  $\psi(a) = \mathbb{E} \{e^{-aX}\} = e^{-a\mu + \frac{a^2}{2} \text{var } X}$ , which is true only when  $X$  is normally distributed.
- ④  $\psi''\psi - (\psi')^2 = \mathbb{E} \{X^2 e^{-aX}\} \mathbb{E} \{e^{-aX}\} - (\mathbb{E} \{X e^{-aX}\})^2 \geq 0$  by the Cauchy-Schwarz inequality applied to random variables  $A = X e^{-\frac{a}{2}X}$  and  $B = e^{-\frac{a}{2}X}$ . To see that  $\alpha$  is increasing  $\frac{d\alpha}{da} = \frac{1}{a^2} \left( \frac{a\psi'}{\psi} - \ln(\psi) \right) \equiv \frac{1}{a^2} f(a)$ , but  $f(0) = 0$  and  $f' = \frac{a(\psi''\psi - (\psi')^2)}{\psi^2} \geq 0$  and the conclusion follows.

## Problem 4

Consider a one-period investment model in which there are only two risky assets. The returns on these assets have means 3, 4 respectively and variances 2, 3 respectively with the covariance between the returns being 2.

- 1 Calculate the mean-variance efficient frontier and the minimum-variance portfolio in terms of the mean return.
- 2 Calculate the mean return of the global minimum-variance portfolio and of diversified portfolio.
- 3 Suppose that in addition to the two risky assets there is a riskless asset with return  $3/2$ . Find the minimum-variance portfolio in terms of the mean return and hence calculate the mean return of the tangency portfolio.

## Problem 4 Solution I

The inverse matrix of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is  $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ , so if  $V = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$ ,  
 $V^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$ .  $\alpha = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} = \frac{1}{2}$ ,  $\beta = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} = \frac{3}{2}$ ,  $\gamma = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \frac{11}{2}$ ,  
 $\delta = \alpha\gamma - \beta^2 = \frac{1}{2}$ .

$$\textcircled{1} \min_{x_1, x_2} (x_1 \quad x_2) \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \min_{x_1, x_2} 2x_1^2 + 4x_1x_2 + 3x_2^2 \quad \text{s.t.} \\ \begin{cases} x_1 + x_2 = 1 \\ 3x_1 + 4x_2 = \mu \end{cases} \quad \text{From constraints } x_1 = 4 - \mu, x_2 = \mu - 3, \text{ so the} \\ \text{mean-variance efficient frontier is } \sigma^2 = \mu^2 - 6\mu + 11.$$

$$\textcircled{2} \mu_g \text{ is the root of } \frac{d\sigma^2}{d\mu} = 0, \text{ so } 2\mu_g - 6 = 0 \implies \mu_g = 3. \quad \mu_d = \frac{\gamma}{\beta} = \frac{11}{3}.$$

## Problem 4 Solution II

③ Now the problem is  $\min_{x_0, x_1, x_2} 2x_1^2 + 4x_1x_2 + 3x_2^2$  s.t.

$$\begin{cases} x_0 + x_1 + x_2 = 1 \\ \frac{3}{2}x_0 + 3x_1 + 4x_2 = \mu \end{cases}. \text{ Form the Lagrangian}$$

$$\mathcal{L} = 2x_1^2 + 4x_1x_2 + 3x_2^2 + \lambda(1 - x_0 - x_1 - x_2) + \nu(\mu - \frac{3}{2}x_0 - 3x_1 - 4x_2).$$

By solving  $\frac{\partial \mathcal{L}}{\partial x_0} = 0$ ,  $\nu = -\frac{2\lambda}{3}$ . From  $\frac{\partial \mathcal{L}}{\partial x_1} = 0$  and  $\frac{\partial \mathcal{L}}{\partial x_2} = 0$  we have

$$4x_1 + 4x_2 - \lambda - 3\nu = 0 \text{ and } 4x_1 + 6x_2 - \lambda - 4\nu = 0; \text{ so } x_1 = \frac{\lambda}{12}, x_2 = -\frac{\lambda}{3}.$$

Substitute into the constraints yields  $\lambda = \frac{12(3 - 2\mu)}{17}$ , and so  $x_0 = \frac{26 - 6\mu}{17}$ ,

$$x_1 = \frac{3 - 2\mu}{17}, x_2 = -\frac{4(3 - 2\mu)}{17}. \text{ The tangency portfolio corresponds to}$$

$$x_0 = 0 \text{ or } \mu_t = \frac{13}{3}.$$

## Problem 5

Suppose that  $v$  is concave,  $X \sim N(\mu, \sigma^2)$  and  $f(\sigma, \mu) = \mathbb{E} v(X)$ .

- 1 Show that  $\frac{\partial f}{\partial \mu} > 0$  when  $v$  is strictly increasing, and  $\frac{\partial f}{\partial \sigma} \leq 0$ . Hence show in the context of mean-variance analysis that, when all returns are jointly normally distributed, an investor maximizing the expected utility of his final wealth will choose a mean-variance-efficient optimal portfolio.
- 2 Show that  $f$  is concave in  $\mu$  and  $\sigma$ . Deduce that this optimal portfolio corresponds to a point in the  $(\sigma, \mu)$  plane where an indifference contour is tangent to the efficient frontier.

## Problem 5 Solution I

Note that, when  $W \sim N(\mu, \sigma^2)$ , then for  $a, b \in \mathbb{R}$ ,  $a \neq 0$ , the random variable  $Z = aW + b \sim N(a\mu + b, a^2\sigma^2)$ . So if  $X \sim N(\mu, \sigma^2)$ ,  $X$  can be written as  $X = \mu + \sigma Y$  where  $Y \sim N(0, 1)$ . Moreover  $E\{f(X)(X - \mu)\} = \sigma^2 E\{f'(X)\}$  for any differentiable  $f$  and if both sides of the equation are finite: first note that for the standard normal density function  $\phi$ ,  $\phi'(y) = -y\phi(y)$ , so

$$\begin{aligned} E\{f(X)(X - \mu)\} &= \sigma E\{f(\mu + \sigma Y)Y\} = \int_{-\infty}^{\infty} yf(\mu + \sigma y)\phi(y) dy = \\ &= - \int_{-\infty}^{\infty} f(\mu + \sigma y) d\phi(y) = -\sigma f(\mu + \sigma y)\phi(y) \Big|_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} f'(\mu + \sigma y)\phi(y) dy = \\ &= \sigma^2 \int_{-\infty}^{\infty} f'(\mu + \sigma y)\phi(y) dy = \sigma^2 E\{f'(X)\}. \end{aligned}$$

- ④ Note that  $\frac{\partial f}{\partial \mu} = E\{v'(\mu + \sigma Y)\} > 0$  when  $v' > 0$ ,  $\frac{\partial f}{\partial \sigma} = E\{Yv'(\mu + \sigma Y)\} = \sigma E\{v''(\mu + \sigma Y)\} \leq 0$  by the concavity of  $v$  ( $v'' < 0$ ). When returns are normally distributed, the wealth created by each portfolio is normally distributed; this shows that maximizing in  $\sigma$  for fixed  $\mu$  gives a value of  $(\sigma, \mu)$  on the efficient frontier.

## Problem 5 Solution II

② To see the concavity of  $f$ , note that  $\frac{\partial^2 f}{\partial \mu^2} = E\{v''(\mu + \sigma Y)\} \leq 0$ ,

$$\frac{\partial^2 f}{\partial \sigma^2} = E\{Y^2 v''(\mu + \sigma Y)\} \leq 0, \quad \frac{\partial^2 f}{\partial \mu \partial \sigma} = E\{Y v''(\mu + \sigma Y)\}, \text{ and then}$$

$$\frac{\partial^2 f}{\partial \mu^2} \frac{\partial^2 f}{\partial \sigma^2} \geq \left( \frac{\partial^2 f}{\partial \mu \partial \sigma} \right)^2 \text{ follows by applying the Cauchy-Schwarz inequality to}$$

the random variables  $A = Y \sqrt{-v''(\mu + \sigma Y)}$  and  $B = \sqrt{-v''(\mu + \sigma Y)}$ . This shows that the  $2 \times 2$  matrix of second derivatives has non-positive diagonal entries and a non-negative determinant which is sufficient for the matrix to be negative semi-definite: it is clear that the quadratic form

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2cxy + by^2 = a \left( x + \frac{c}{a} y \right)^2 + \left( \frac{ab - c^2}{a} \right) y^2$$

$\leq 0$  if  $a, b \leq 0$  and  $c^2 - ab \geq 0$ . The fact that  $f$  is concave means that sets of the form  $\{(\sigma, \mu) : f(\sigma, \mu) \geq c\}$  are convex which gives the last statement.



## Problem 6

Suppose that an investor has a concave utility function  $v$ . The investor seeks to maximize  $E\{v(W)\}$  where  $W = (x_0 r_0 + \mathbf{x}^\top \mathbf{R}) w$  is his final wealth.

- ① Show that, when  $\bar{W}$  is his optimal final wealth, then

$$E\{v'(\bar{W})(R_j - r_0)\} = 0, \quad \forall j = 1, 2, \dots, s.$$

- ② Show that, when  $\mathbf{R}$  has a multivariate normal distribution, then

$$r_j - r_0 = \alpha \operatorname{cov}(\bar{W}, R_j), \quad \forall j = 1, 2, \dots, s, \text{ where } \alpha = -\frac{E\{v''(\bar{W})\}}{E\{v'(\bar{W})\}}.$$

- ③ Now suppose that the market is determined by investors  $i = 1, 2, \dots, n$ , where investor  $i$  has concave utility  $v_i$ , initial wealth  $w_i$ , optimal final wealth  $\bar{W}_i$  and global risk aversion  $\alpha_i$ . With the normality assumption, show that

$$E M - r_0 = \bar{w} \bar{\alpha} \operatorname{var} M, \text{ where } M = \frac{\sum_{i=1}^n \bar{W}_i}{\sum_{i=1}^n w_i} \text{ is the market rate of return,}$$

$$\bar{w} = \frac{\sum_{i=1}^n w_i}{n} \text{ is the average initial wealth of investors, and } \bar{\alpha} \text{ is the harmonic mean of } \alpha_i.$$

## Problem 6 Solution I

- ① The objective function to maximize is

$$f(\mathbf{x}) = \mathbb{E} \left\{ v \left( w \left( r_0 + \sum_{j=1}^s x_j (R_j - r_0) \right) \right) \right\}$$

where  $\mathbf{x} = (x_1, \dots, x_s)^\top$  and we have used  $x_0 + \sum_{j=1}^s x_j = 1$ . The first-order conditions give

$$\frac{\partial f}{\partial x_j} = w \mathbb{E} \{ v'(\bar{W})(R_j - r_0) \} = 0, \quad \forall j = 1, 2, \dots, s$$

- ② Since  $r_j = \mathbb{E} R_j$  and the fact that  $\bar{W}$  and  $R_j$  have a joint normal distribution we have that

$$\begin{aligned} 0 &= \mathbb{E} \{ v'(\bar{W})(R_j - r_0) \} = \mathbb{E} \{ v'(\bar{W})(R_j - r_j) \} + \mathbb{E} \{ v'(\bar{W}) \} (r_j - r_0) \\ &= \text{cov}(v'(\bar{W}), R_j) + \mathbb{E} \{ v'(\bar{W}) \} (r_j - r_0) \\ &= \mathbb{E} \{ v''(\bar{W}) \} \text{cov}(\bar{W}, R_j) + \mathbb{E} \{ v'(\bar{W}) \} (r_j - r_0) \end{aligned}$$

## Problem 6 Solution II

where the last equality uses  $\text{cov}(f(X), Y) = E\{f'(X)\} \text{cov}(X, Y)$  for jointly normal  $X, Y$ ; so

$$r_j - r_0 = \alpha \text{cov}(\bar{W}, R_j)$$

Note that for random variables  $X$  and  $Y$  and constant  $a$ ,  $\text{cov}(X, Y + a) = \text{cov}(X, Y)$  and  $\text{cov}(aX, Y) = a \text{cov}(X, Y)$ . Now for each  $i$

$$\frac{1}{\alpha_i}(r_j - r_0) = \text{cov}(\bar{W}_i, R_j)$$

and summing this on  $i$  yields

$$\left(\sum_{i=1}^n \frac{1}{\alpha_i}\right)(r_j - r_0) = \left(\sum_{i=1}^n w_i\right) \text{cov}(M, R_j)$$

Divide through by  $n$  and multiply by  $\bar{\alpha}$ , where  $\frac{1}{\bar{\alpha}} = \frac{\sum_{i=1}^n \frac{1}{\alpha_i}}{n}$ , to obtain

$$E R_j - r_0 = \bar{w} \bar{\alpha} \text{cov}(M, R_j)$$

## Problem 6 Solution III

- ⑤ When  $\bar{x}_{ij}$  is the optimal proportion invested by investor  $i$  in asset  $j$  then

$$\bar{W}_i = w_i \left( r_0 + \sum_{j=1}^s \bar{x}_{ij} (R_j - r_0) \right)$$

which when summed on  $i$  gives

$$(M - r_0) \left( \sum_{i=1}^n w_i \right) = \sum_{i=1}^n \sum_{j=1}^s w_i \bar{x}_{ij} (R_j - r_0) \quad (2)$$

Take the expectation in (2) yields

$$(\mathbb{E} M - r_0) \left( \sum_{i=1}^n w_i \right) = \sum_{i=1}^n \sum_{j=1}^s w_i \bar{x}_{ij} (\mathbb{E} R_j - r_0) \quad (3)$$

## Problem 6 Solution IV

Multiply (1) by  $w_i \bar{x}_{ij}$ , sum on  $i$  and  $j$  yields

$$\begin{aligned}\sum_{i=1}^n \sum_{j=1}^s w_i \bar{x}_{ij} (E R_j - r_0) &= \bar{w} \bar{\alpha} \operatorname{cov} \left( M, \sum_{i=1}^n \sum_{j=1}^s w_i \bar{x}_{ij} R_j \right) \\ &= \bar{w} \bar{\alpha} \operatorname{cov} \left( M, \sum_{i=1}^n w_i \left( r_0 + \sum_{j=1}^s \bar{x}_{ij} (R_j - r_0) \right) \right) \\ &= \bar{w} \bar{\alpha} \operatorname{cov} \left( M, \sum_{i=1}^n \bar{W}_i \right) \quad (4)\end{aligned}$$

Comparing (3) (4)

$$E M - r_0 = \bar{w} \bar{\alpha} \operatorname{cov} \left( M, \frac{\sum_{i=1}^n \bar{W}_i}{\sum_{i=1}^n w_i} \right) = \bar{w} \bar{\alpha} \operatorname{cov} (M, M) = \bar{w} \bar{\alpha} \operatorname{var} M$$

which we have used properties  $\operatorname{cov}(X, Y + a) = \operatorname{cov}(X, Y)$  and  $\operatorname{cov}(aX, Y) = a \operatorname{cov}(X, Y)$  extensively. This shows that the risk premium for the market is proportional to  $\bar{\alpha}$  which is a measure of the risk aversion in the economy.

## Problem 7

Consider an investor with the utility function  $v(x) = 1 - e^{-ax}$ ,  $a > 0$ , who is faced with a riskless asset with return  $r_0$  and  $s$  risky assets with returns  $\mathbf{R} \sim N(\mathbf{r}, \mathbf{V})$ .

- 1 Show that when he seeks to maximize the expected utility of his final wealth, he will hold the risky assets in the same proportion as the tangency portfolio.
- 2 Show that when  $\beta > \alpha r_0$ , the more risk averse that he is, the smaller amount of his wealth that he invests in the risky assets.

# Problem 7 Solution

- ① Suppose that the investor's initial wealth is  $w > 0$  and that he wishes to minimize  $E e^{-aW}$  where

$$W = w \left( r_0 + \sum_{j=1}^s x_j (R_j - r_0) \right) = w (r_0(1 - \mathbf{x}^\top \mathbf{e}) + \mathbf{x}^\top \mathbf{R})$$

and  $\mathbf{x} = (x_1, \dots, x_s)^\top$ ,  $\mathbf{e} = (1, \dots, 1)^\top$ ,  $x_0 = 1 - \mathbf{x}^\top \mathbf{e}$ . Note that  $\mathbf{x}^\top \mathbf{R} \sim N(\mathbf{r}^\top \mathbf{x}, \mathbf{x}^\top \mathbf{V} \mathbf{x})$ , so

$$E e^{-aW} = \exp \left\{ -aw r_0(1 - \mathbf{x}^\top \mathbf{e}) - aw \mathbf{r}^\top \mathbf{x} + \frac{1}{2} a^2 w^2 \mathbf{x}^\top \mathbf{V} \mathbf{x} \right\}$$

It amounts to minimize  $\frac{1}{2} aw \mathbf{x}^\top \mathbf{V} \mathbf{x} - \mathbf{x}^\top (\mathbf{r} - r_0 \mathbf{e})$  for which the minimum occurs when  $\mathbf{x} = \frac{1}{aw} \mathbf{V}^{-1} (\mathbf{r} - r_0 \mathbf{e})$ , and the conclusion follows.

- ② The amount of his wealth invested in the risky assets is  $(\mathbf{x}^\top \mathbf{e})w = \frac{\beta - \alpha r_0}{a}$ , which decreases in  $a > 0$  when  $\beta > \alpha r_0$ .

## Problem 8

Consider an investor with  $\mathbf{R} = (R_1, R_2, \dots, R_s)^\top$  where  $R_i$ s are independent random variables with  $R_i$  having gamma distribution,  $E R_i = r_i$  and  $\text{var } R_i = \sigma_i^2$ . Suppose that he has the utility function  $v(x) = 1 - e^{-ax}$ ,  $a > 0$ , and he seeks to maximize the expected utility of his final wealth.

- 1 Show that he divides his wealth between the diversified portfolio and a second portfolio which should be identified; determine the amounts that he invests in each.
- 2 If he may invest in a risky asset with return  $r_0$ , show that he will again divide his wealth between these two portfolios and the riskless asset; determine the amounts that he invests in each.
- 3 Give a necessary and sufficient condition, expressed in terms of the parameters  $r_i$ ,  $i = 0, 1, 2, \dots, s$  and  $\sigma_i^2$ ,  $i = 1, 2, \dots, s$ , that he is long in the risky assets.



## Problem 8 Solution I

- ① When  $R_i \sim \Gamma(\gamma_i, \lambda_i)$ ,  $E R_i = r_i = \frac{\gamma_i}{\lambda_i}$  and  $\text{var } R_i = \sigma_i^2 = \frac{\gamma_i}{\lambda_i^2} \implies \gamma_i = \frac{r_i^2}{\sigma_i^2}$ ,  $\lambda_i = \frac{r_i}{\sigma_i^2}$ . For  $\phi + \lambda_i > 0$ , note that

$$\begin{aligned} E e^{-\phi R_i} &= \int_0^\infty \frac{e^{-\phi x} e^{-\lambda_i x} \lambda_i^{\gamma_i} x^{\gamma_i-1}}{\Gamma(\gamma_i)} dx \\ &= \left( \frac{\lambda_i}{\phi + \lambda_i} \right)^{\gamma_i} \int_0^\infty \frac{e^{-\phi x} e^{-\lambda_i x} (\phi + \lambda_i)^{\gamma_i} x^{\gamma_i-1}}{\Gamma(\gamma_i)} dx = \left( \frac{\lambda_i}{\phi + \lambda_i} \right)^{\gamma_i} \end{aligned}$$

because the integrand in the latter integral is a probability density function, and so the value of the integral is 1. The investor wishes to solve the constrained optimization problem

$$\text{maximize} \quad E \left\{ 1 - e^{-aw(\mathbf{x}^\top \mathbf{R})} \right\} \quad \text{subject to} \quad \mathbf{x}^\top \mathbf{e} = 1$$

## Problem 8 Solution II

which is equivalent to minimizing

$$\mathbb{E} \left\{ e^{-aw(\mathbf{x}^\top \mathbf{R})} \right\} = \prod_{i=1}^s \mathbb{E} \left\{ e^{-awx_i R_i} \right\} = \prod_{i=1}^s \left( \frac{\lambda_i}{awx_i + \lambda_i} \right)^{\gamma_i}$$

subject to the constraint. Taking logarithms, we need to

$$\text{maximize} \quad \sum_{i=1}^s \gamma_i \ln(awx_i + \lambda_i) \quad \text{subject to} \quad \sum_{i=1}^s x_i = 1$$

Maximizing the Lagrangian

$$\mathcal{L} = \sum_{i=1}^s \gamma_i \ln(awx_i + \lambda_i) + \theta \left( 1 - \sum_{i=1}^s x_i \right)$$

## Problem 8 Solution III

in  $x_i$  gives  $x_i = \frac{\gamma_i}{\theta} - \frac{\lambda_i}{aw}$ . Substituting back into the constraint shows that the Lagrange multiplier is given as

$$\theta = \frac{\sum_{j=1}^s \gamma_j}{1 + \frac{1}{aw} \sum_{j=1}^s \lambda_j}$$

from which it follows that the optimal portfolio may be expressed as

$$\mathbf{x} = \left(1 + \frac{1}{aw} \sum_{j=1}^s \lambda_j\right) \bar{\mathbf{x}} - \left(\frac{1}{aw} \sum_{j=1}^s \lambda_j\right) \mathbf{x}_d$$

where the two portfolios  $\bar{\mathbf{x}}$  and  $\mathbf{x}_d$  are

$$(\bar{\mathbf{x}})_i = \frac{\gamma_i}{\sum_{j=1}^s \gamma_j} = \frac{\frac{r_i^2}{\sigma_i^2}}{\sum_{j=1}^s \frac{r_j^2}{\sigma_j^2}} \text{ and } (\mathbf{x}_d)_i = \frac{\lambda_i}{\sum_{j=1}^s \lambda_j} = \frac{\frac{r_i}{\sigma_i^2}}{\sum_{j=1}^s \frac{r_j}{\sigma_j^2}}$$

## Problem 8 Solution IV

with the latter portfolio being the diversified portfolio. As his initial wealth is  $w$ , the investor invests the amount  $w + \frac{1}{a} \sum_{j=1}^s \lambda_j$  in  $\bar{\mathbf{x}}$  and the amount

$-\frac{1}{a} \sum_{j=1}^s \lambda_j$  in the diversified portfolio. Note that in the case when the random variables  $R_i$  have exponential distributions, then  $\gamma_i = 1$ , or  $r_i^2 = \sigma_i^2$ , for each  $1 \leq i \leq s$ , so that the portfolio  $\bar{\mathbf{x}}$  is just the uniform portfolio

$\bar{\mathbf{x}} = \left( \frac{1}{s}, \dots, \frac{1}{s} \right)^\top$  which apportions wealth equally between the  $s$  risky assets.

- ② When there is a riskless asset, set  $x_0 = 1 - \mathbf{x}^\top \mathbf{e}$  and we wish to minimize

$$\begin{aligned} \mathbb{E} \left\{ e^{-aw(r_0(1-\mathbf{x}^\top \mathbf{e}) + \mathbf{x}^\top \mathbf{R})} \right\} &= e^{awr_0(\sum_{j=1}^s x_j - 1)} \prod_{i=1}^s \mathbb{E} \left\{ e^{-awx_i R_i} \right\} \\ &= e^{awr_0(\sum_{j=1}^s x_j - 1)} \prod_{i=1}^s \left( \frac{\lambda_i}{awx_i + \lambda_i} \right)^{\gamma_i} \end{aligned}$$

## Problem 8 Solution V

in  $\mathbf{x} = (x_1, \dots, x_s)^\top$ , which is equivalent to maximizing

$$\sum_{i=1}^s \gamma_i \ln(awx_i + \lambda_i) - awr_0 \sum_{i=1}^s x_i$$

Deduce that for  $1 \leq i \leq s$ , the optimal  $x_i = \frac{1}{aw} \left( \frac{\gamma_i}{r_0} - \lambda_i \right)$ , and the optimal investment in the risky assets is determined by

$$\mathbf{x} = \left( \frac{1}{awr_0} \sum_{j=1}^s \gamma_j \right) \bar{\mathbf{x}} - \left( \frac{1}{aw} \sum_{j=1}^s \lambda_j \right) \mathbf{x}_d$$

- ③ The investor is long in the particular risky asset  $i$  when  $x_i > 0$ , which is true if and only if  $r_i > r_0$ ; he is long overall in risky assets if and only if

$$\sum_{j=1}^s x_j > 0 \text{ which is equivalent to the condition that } \frac{1}{r_0} > \frac{\sum_{j=1}^s \frac{r_j}{\sigma_j^2}}{\sum_{j=1}^s \frac{r_j^2}{\sigma_j^2}}.$$