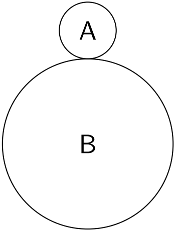
Introduction to Financial Models
Lecture 02: Surprises & Paradoxes II

Coin Rotation Paradox

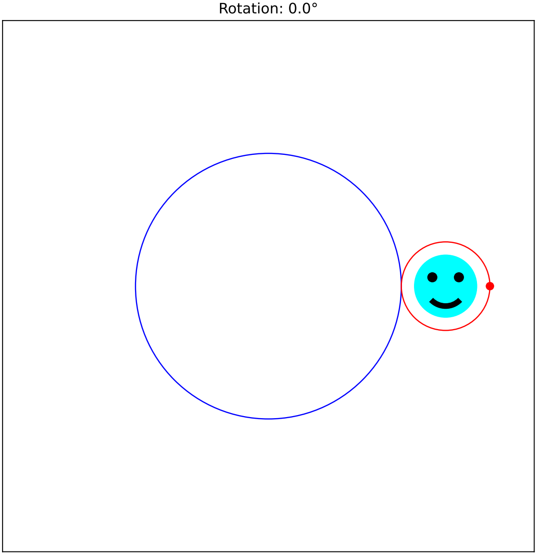
The 1982 SAT Question Everyone Got Wrong



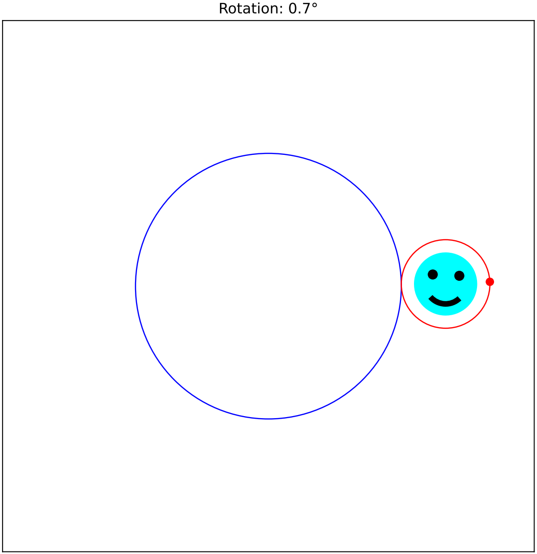
The radius of circle A is $\frac{1}{3}$ of the radius of circle B. Circle A rolls around circle B one trip back to its starting point. How many times will circle A revolve in total?

(a) $\frac{3}{2}$ (b) 3 (c) 6 (d) $\frac{9}{2}$ (e) 9

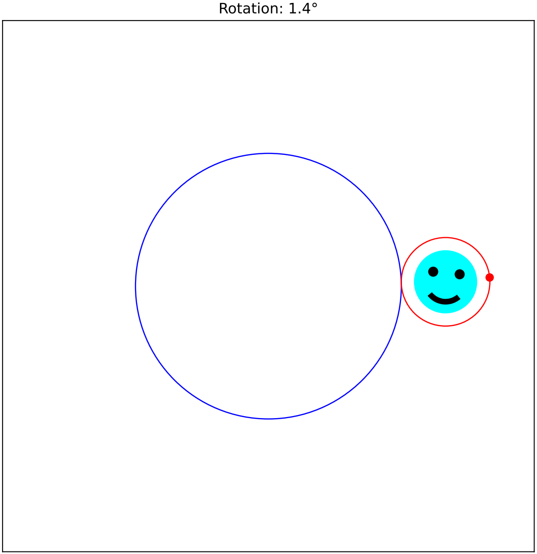
Animation of Coin Rotation Paradox



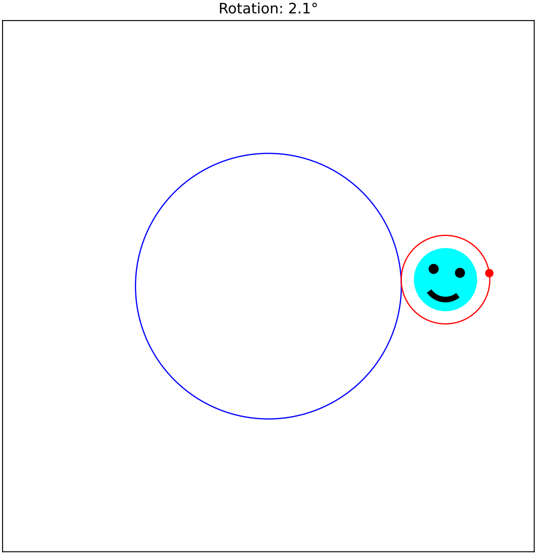
Animation of Coin Rotation Paradox



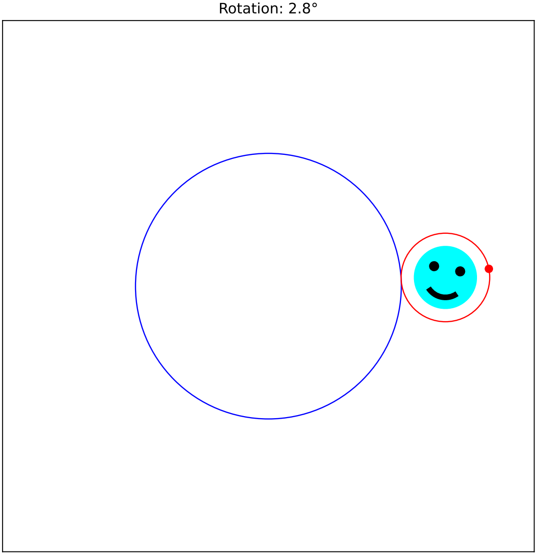
Animation of Coin Rotation Paradox



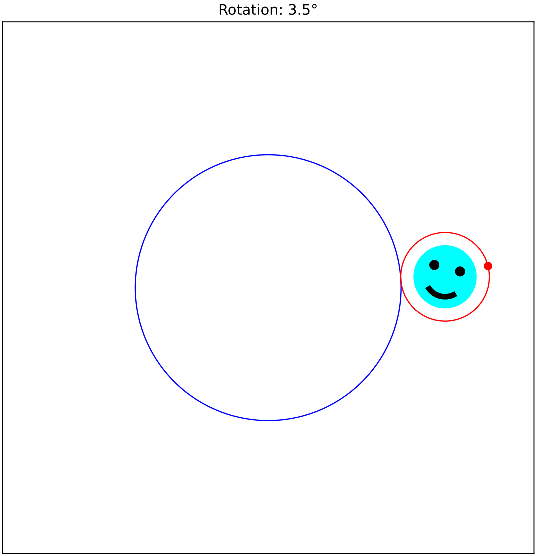
Animation of Coin Rotation Paradox



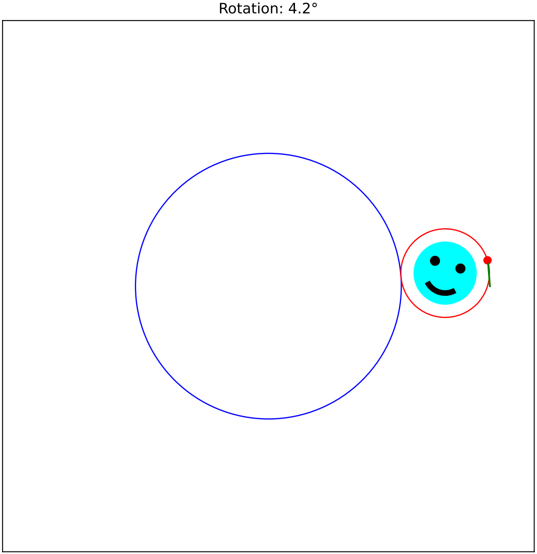
Animation of Coin Rotation Paradox



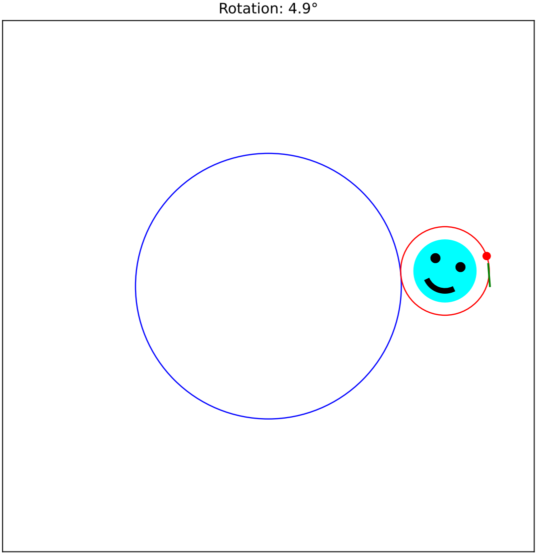
Animation of Coin Rotation Paradox



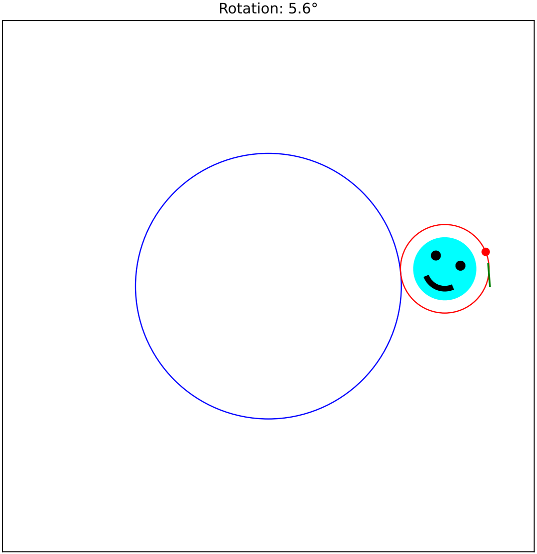
Animation of Coin Rotation Paradox



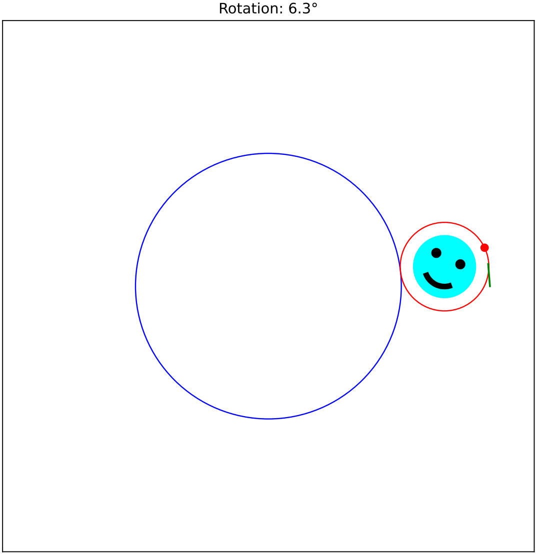
Animation of Coin Rotation Paradox



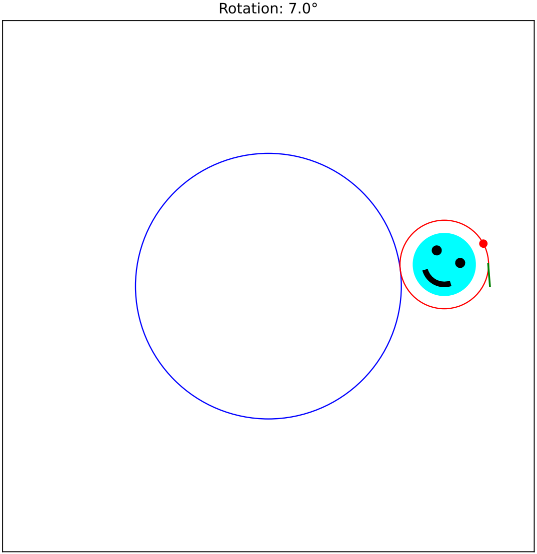
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

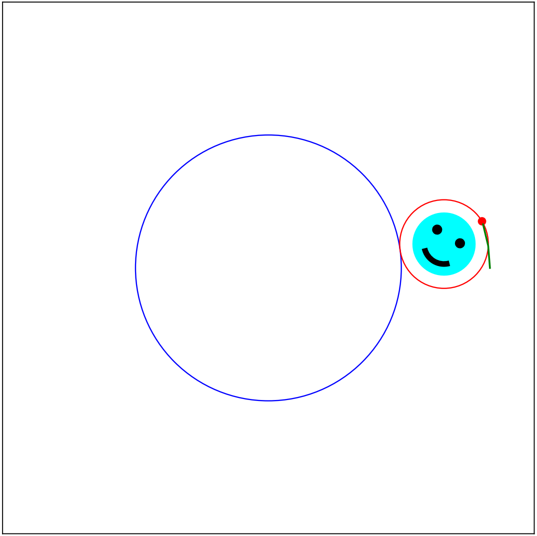


Animation of Coin Rotation Paradox

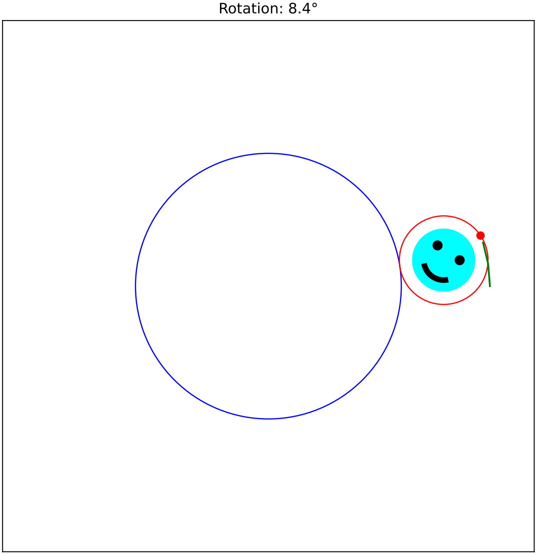


Animation of Coin Rotation Paradox

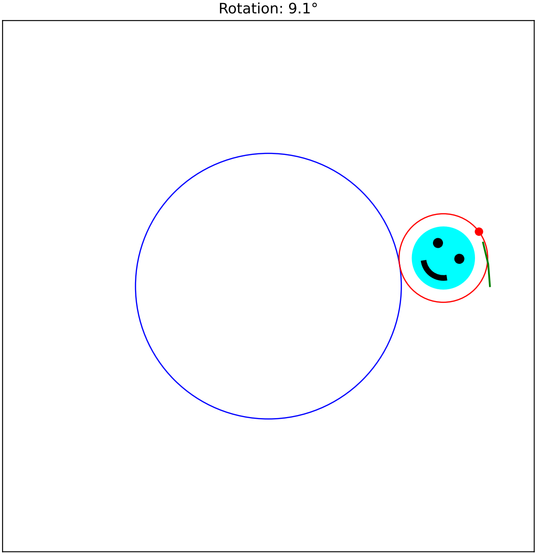
Rotation: 7.7°



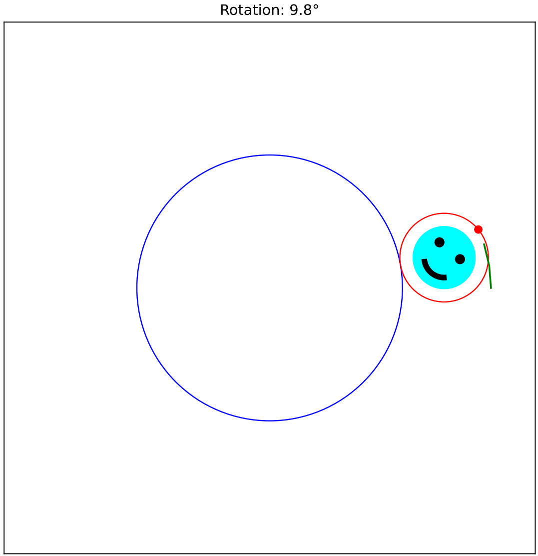
Animation of Coin Rotation Paradox



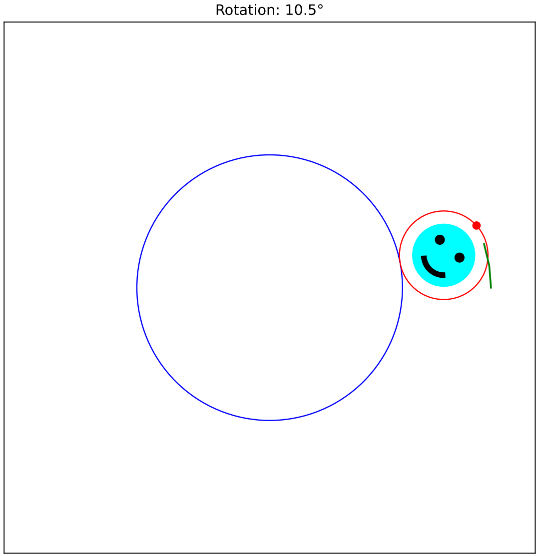
Animation of Coin Rotation Paradox



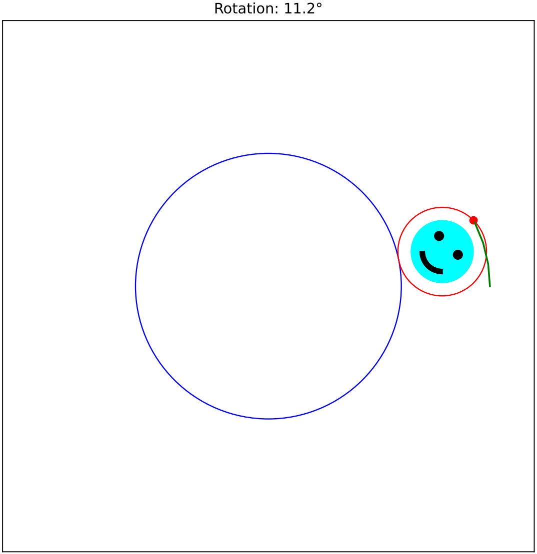
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

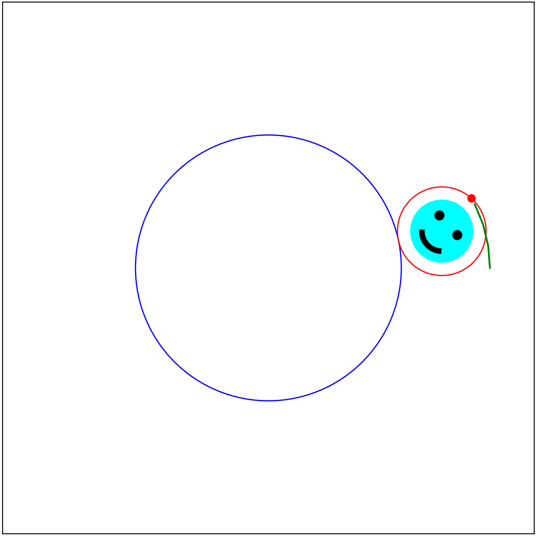


Animation of Coin Rotation Paradox

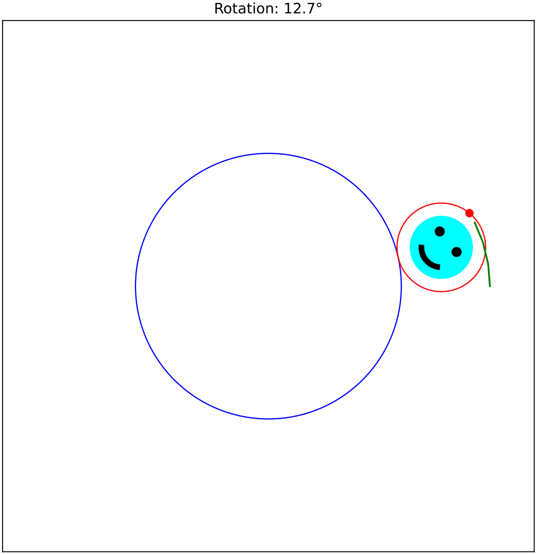


Animation of Coin Rotation Paradox

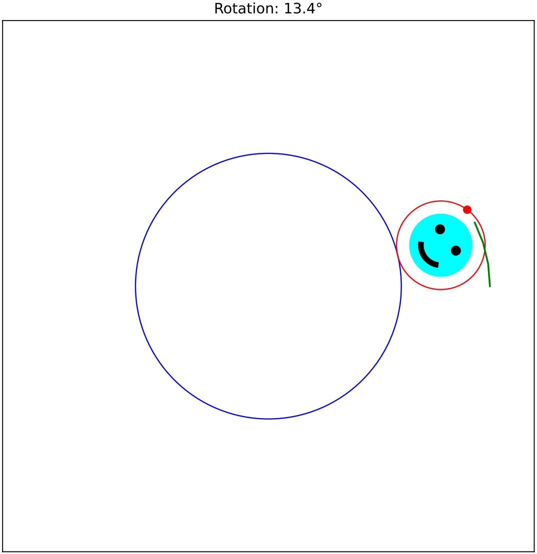
Rotation: 12.0°



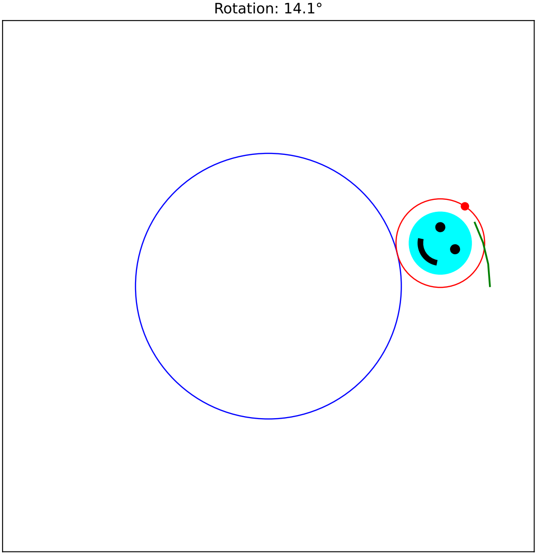
Animation of Coin Rotation Paradox



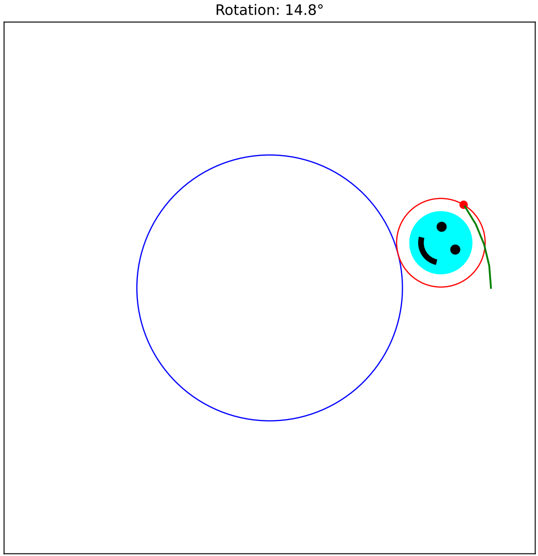
Animation of Coin Rotation Paradox



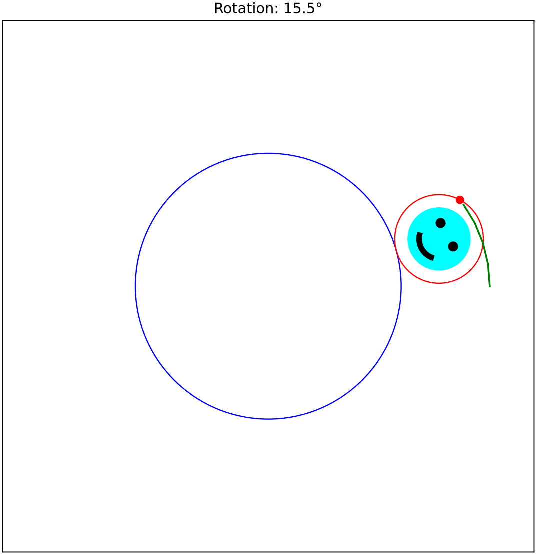
Animation of Coin Rotation Paradox



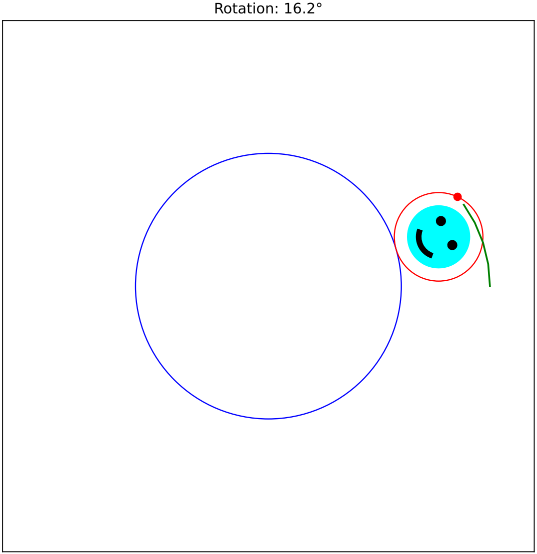
Animation of Coin Rotation Paradox



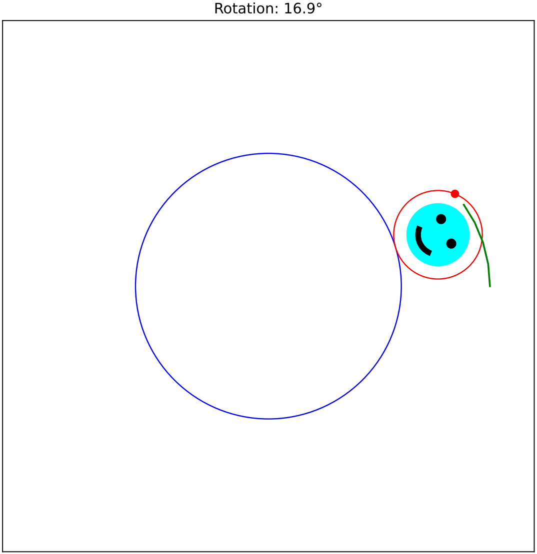
Animation of Coin Rotation Paradox



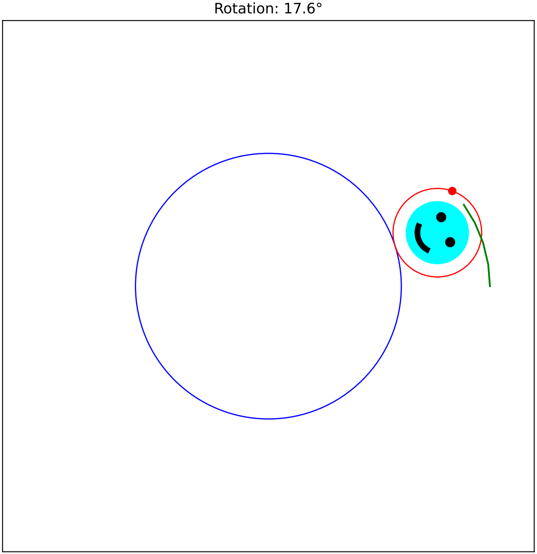
Animation of Coin Rotation Paradox



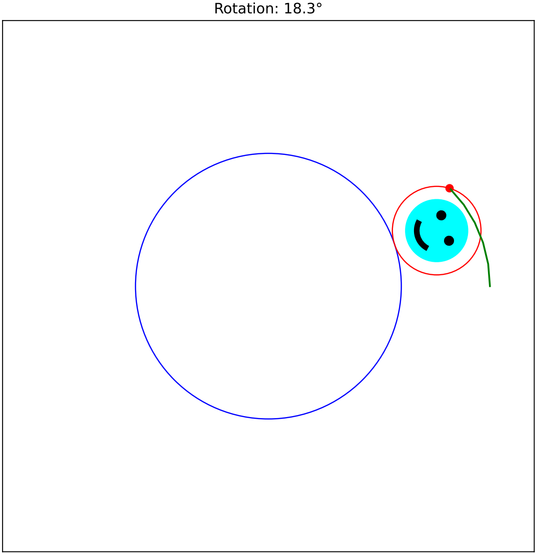
Animation of Coin Rotation Paradox



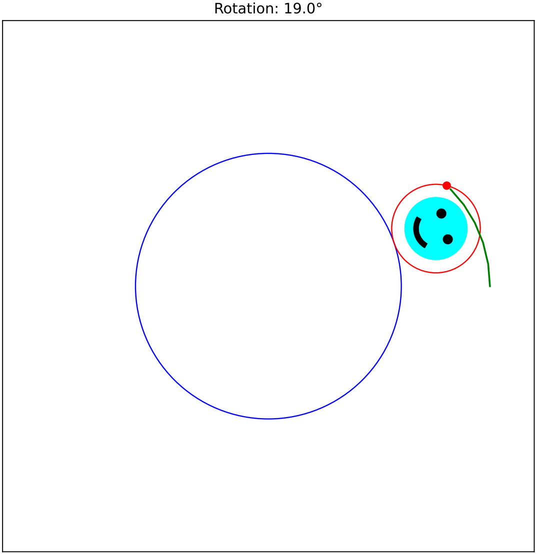
Animation of Coin Rotation Paradox



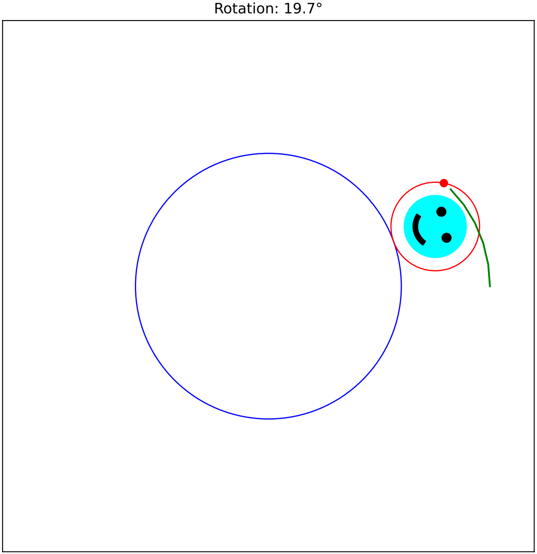
Animation of Coin Rotation Paradox



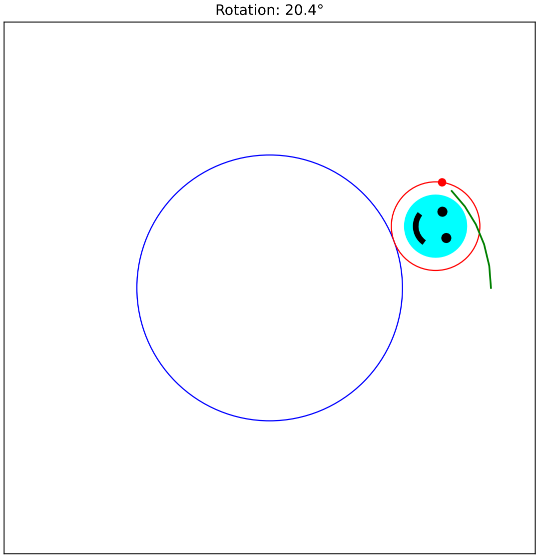
Animation of Coin Rotation Paradox



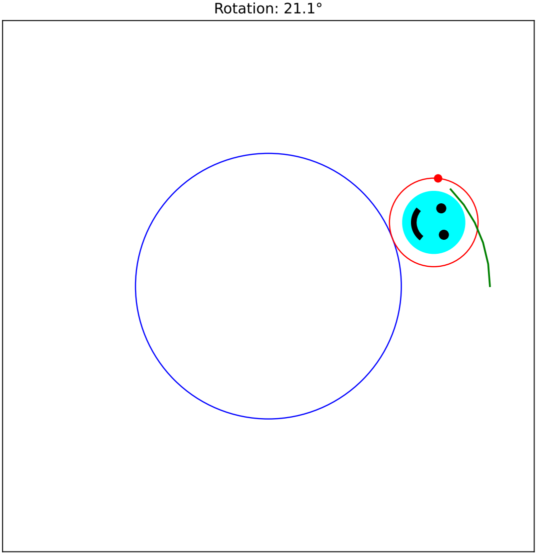
Animation of Coin Rotation Paradox



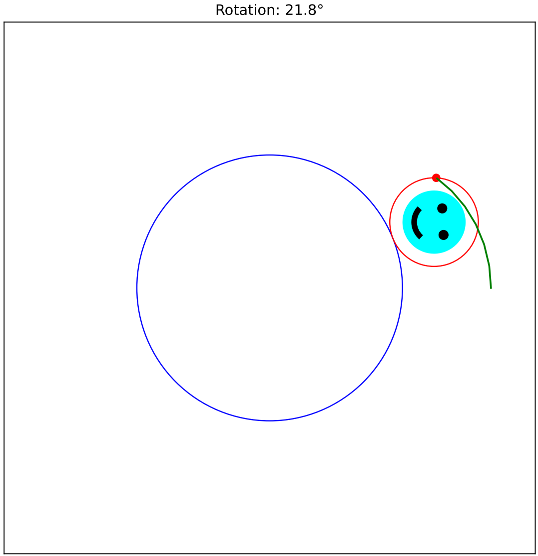
Animation of Coin Rotation Paradox



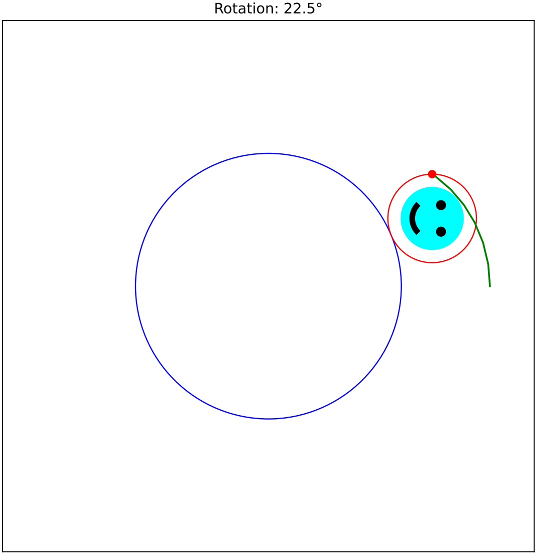
Animation of Coin Rotation Paradox



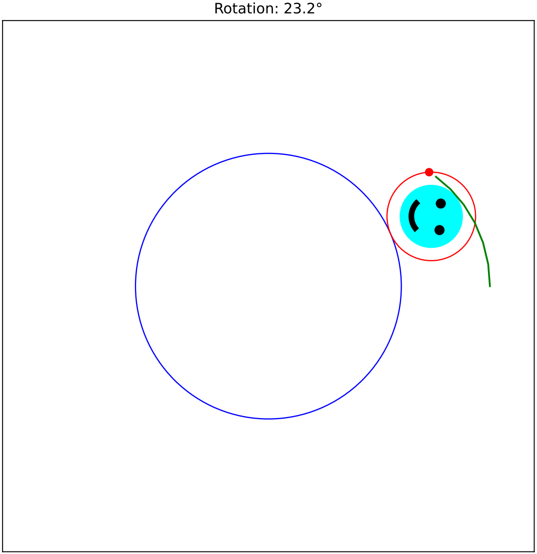
Animation of Coin Rotation Paradox



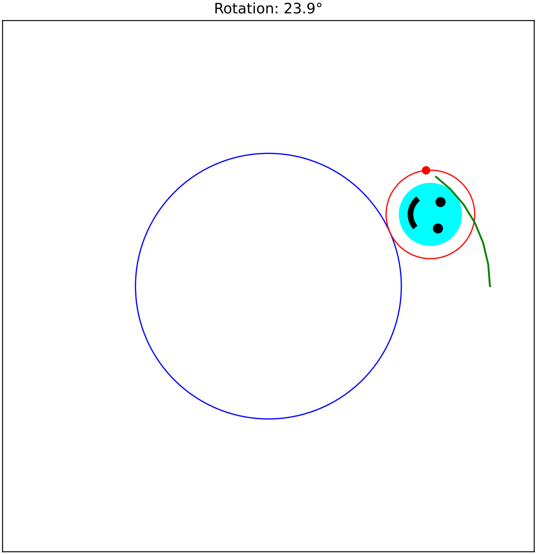
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

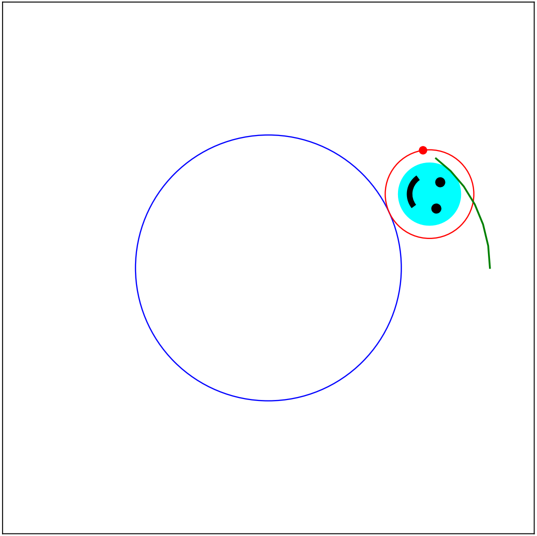


Animation of Coin Rotation Paradox

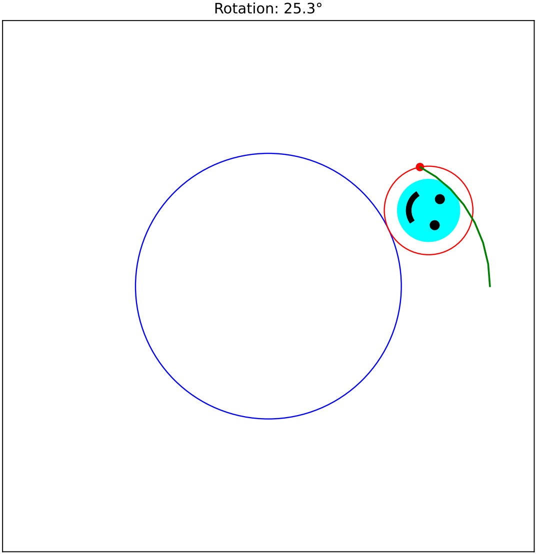


Animation of Coin Rotation Paradox

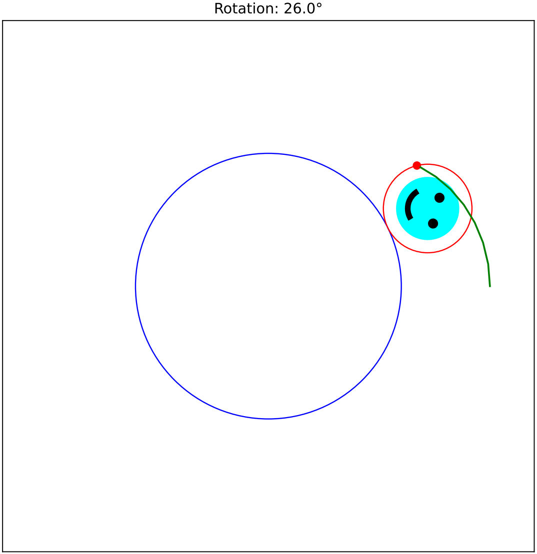
Rotation: 24.6°



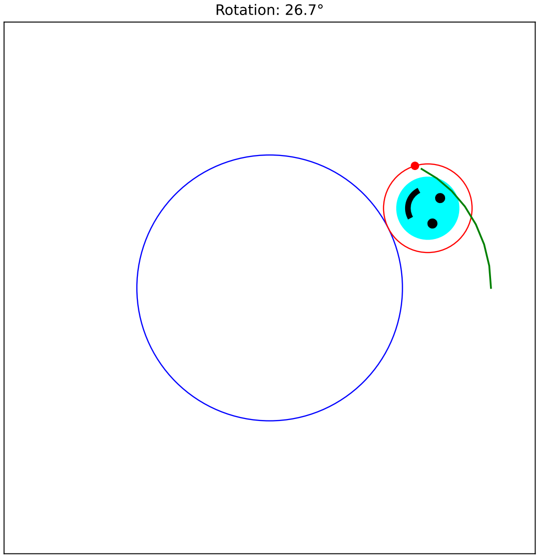
Animation of Coin Rotation Paradox



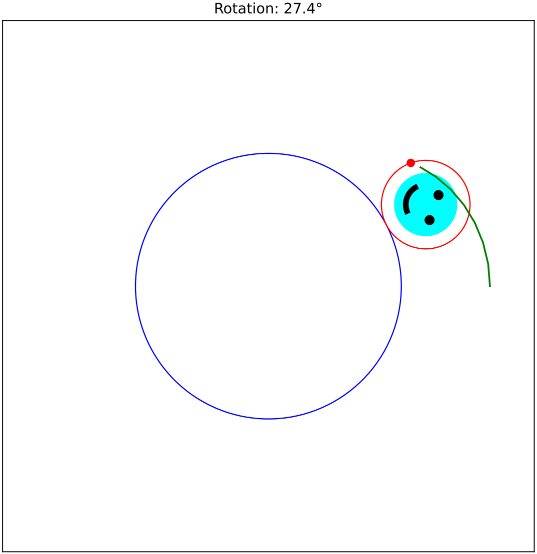
Animation of Coin Rotation Paradox



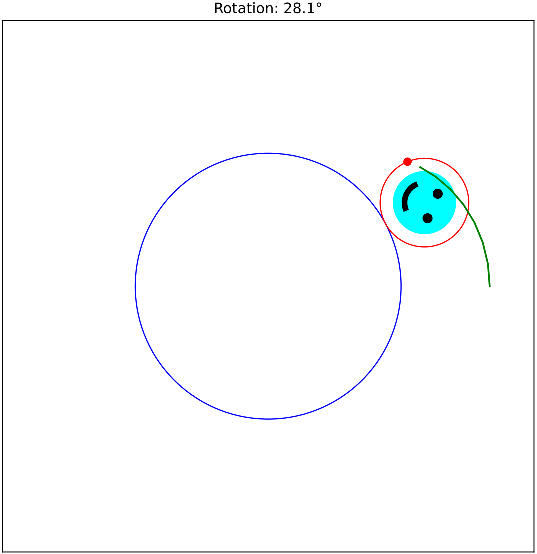
Animation of Coin Rotation Paradox



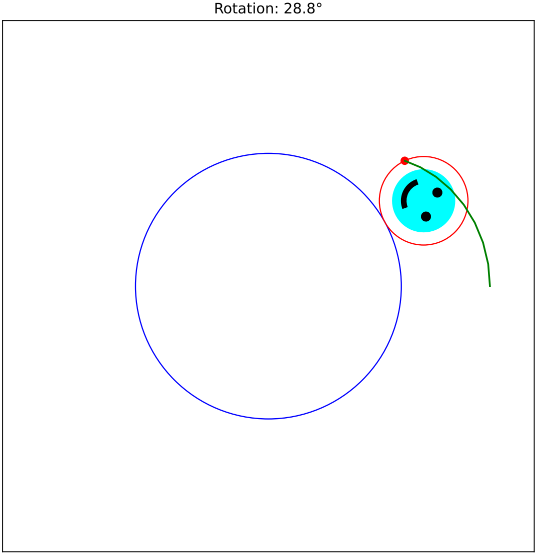
Animation of Coin Rotation Paradox



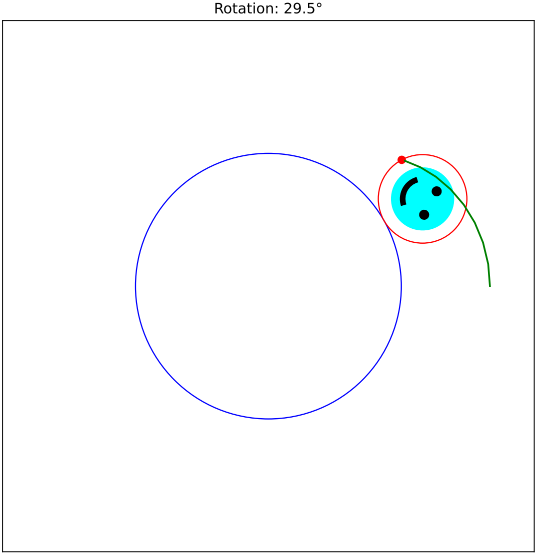
Animation of Coin Rotation Paradox



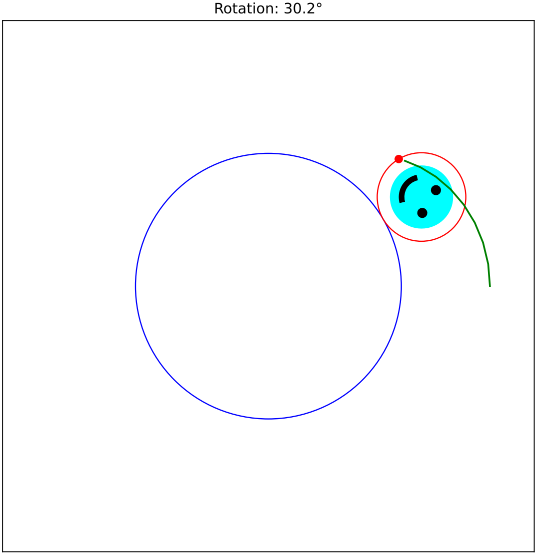
Animation of Coin Rotation Paradox



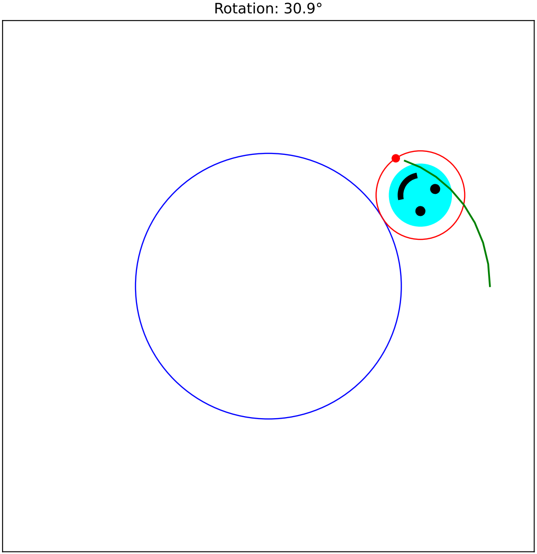
Animation of Coin Rotation Paradox



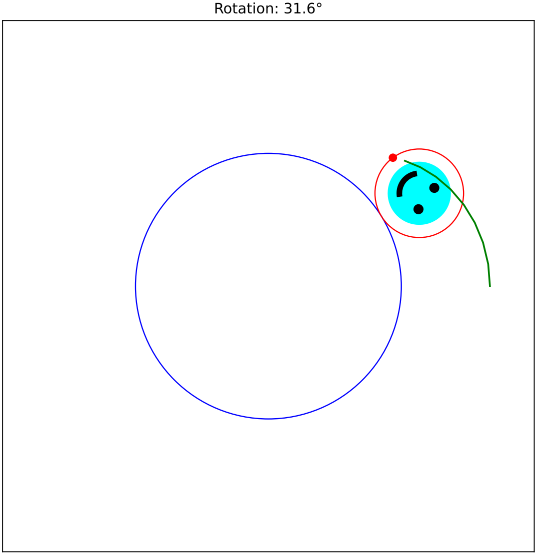
Animation of Coin Rotation Paradox



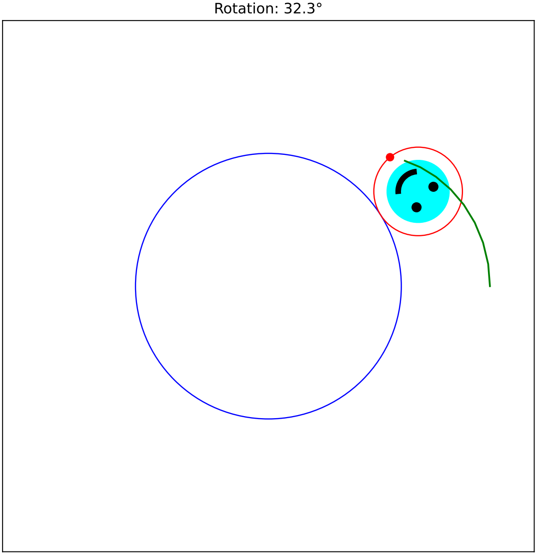
Animation of Coin Rotation Paradox



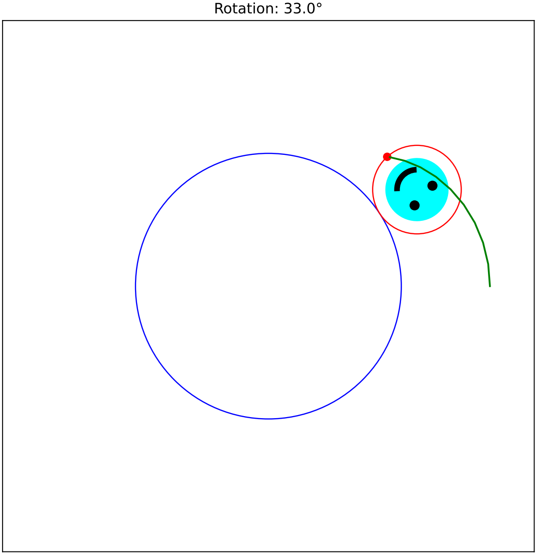
Animation of Coin Rotation Paradox



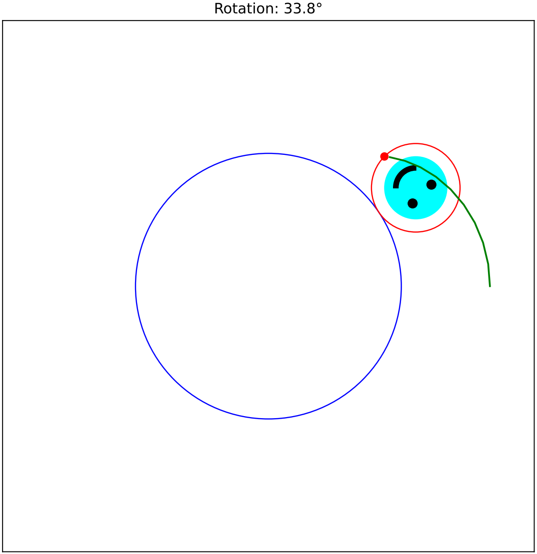
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

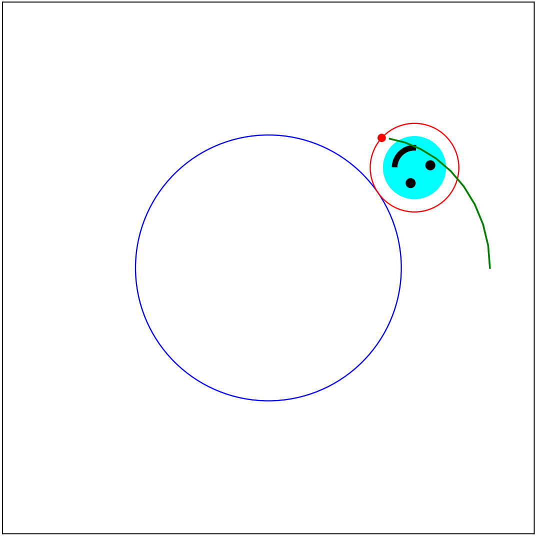


Animation of Coin Rotation Paradox

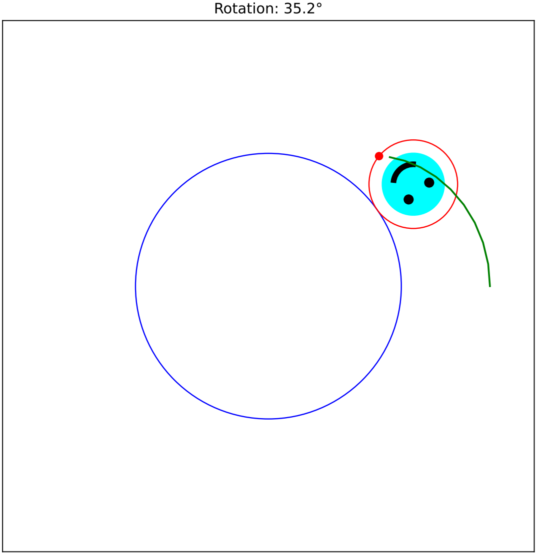


Animation of Coin Rotation Paradox

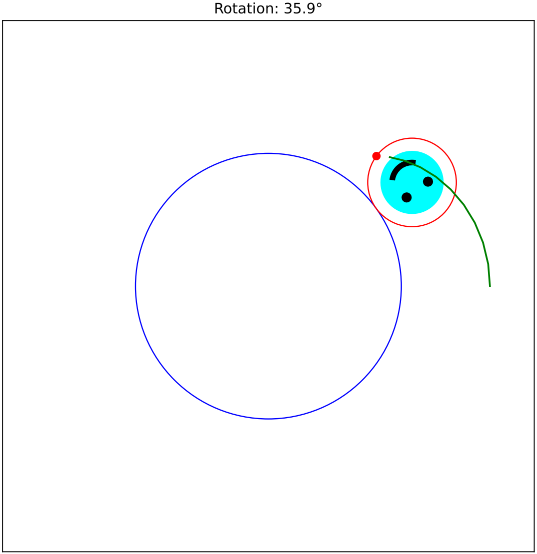
Rotation: 34.5°



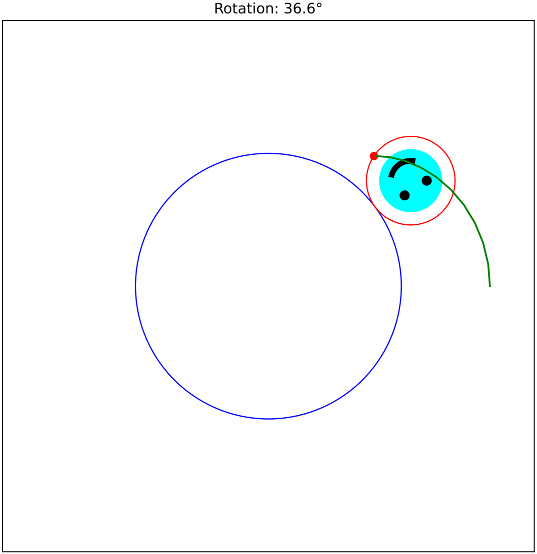
Animation of Coin Rotation Paradox



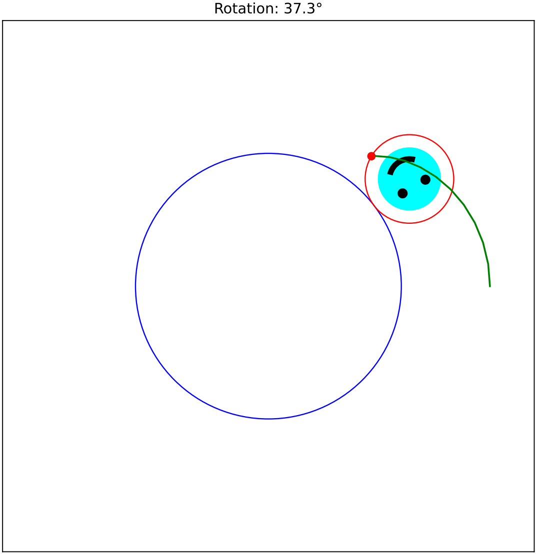
Animation of Coin Rotation Paradox



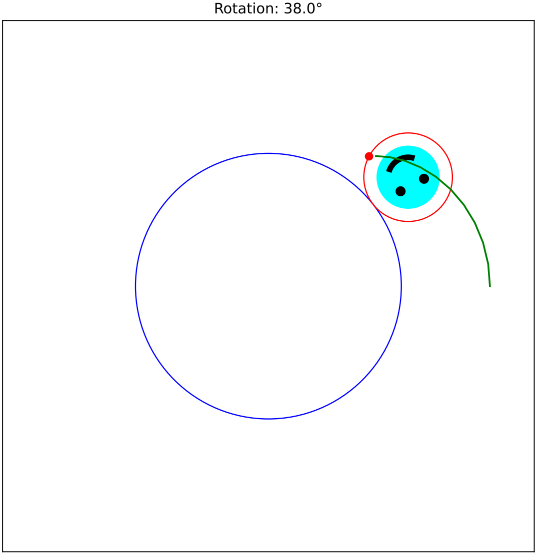
Animation of Coin Rotation Paradox



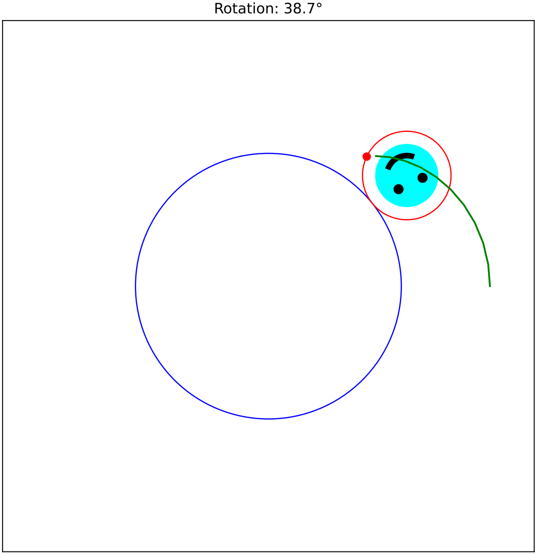
Animation of Coin Rotation Paradox



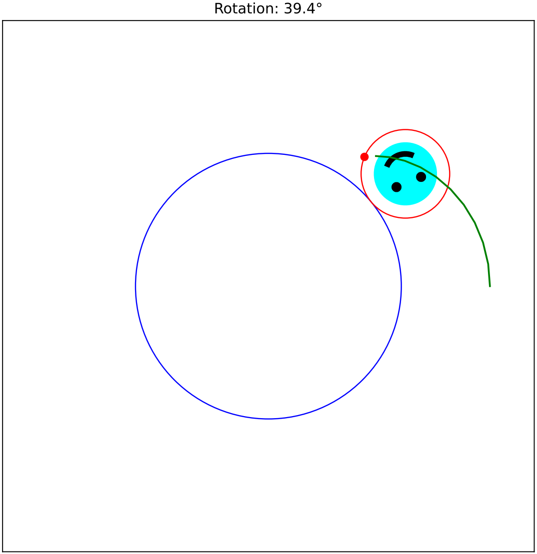
Animation of Coin Rotation Paradox



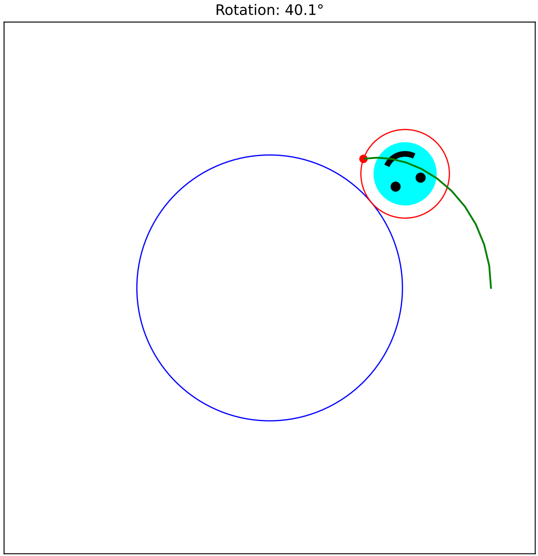
Animation of Coin Rotation Paradox



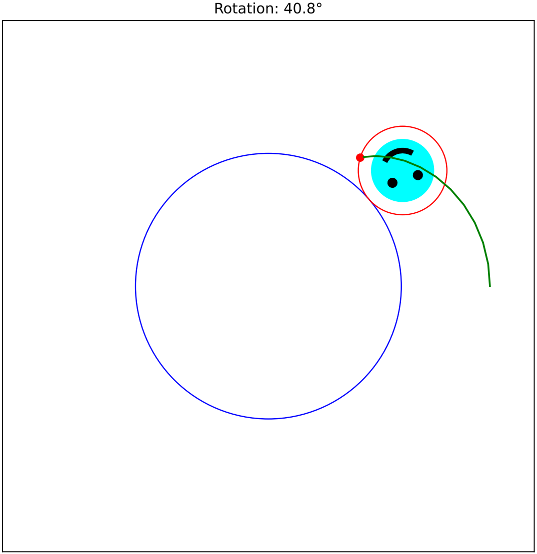
Animation of Coin Rotation Paradox



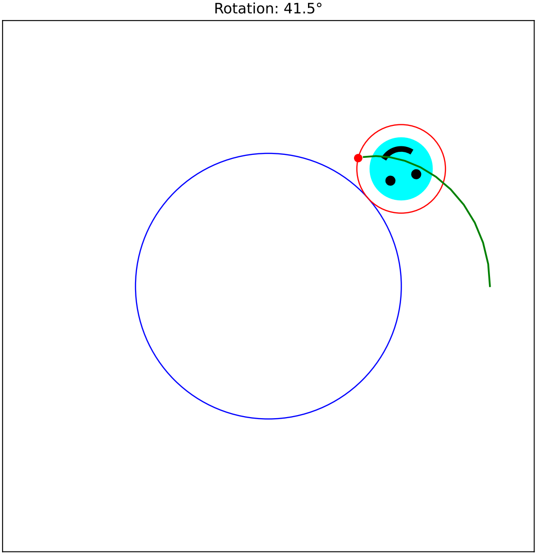
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

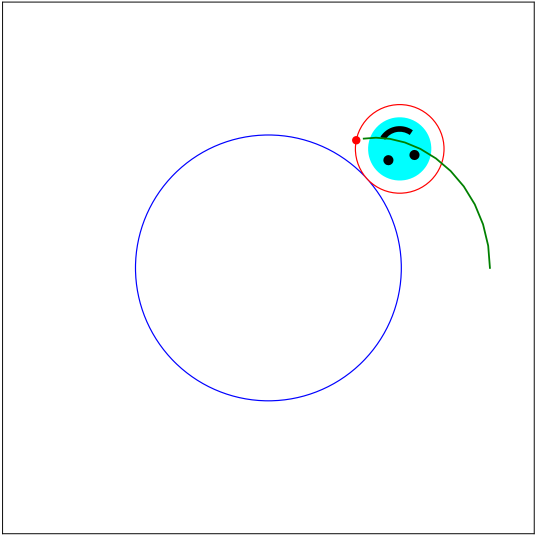


Animation of Coin Rotation Paradox

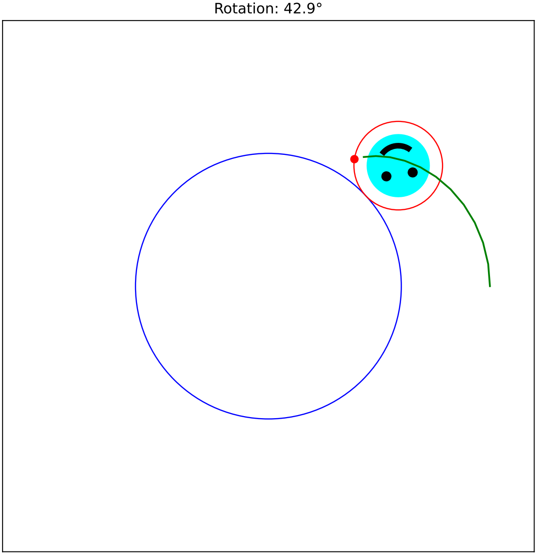


Animation of Coin Rotation Paradox

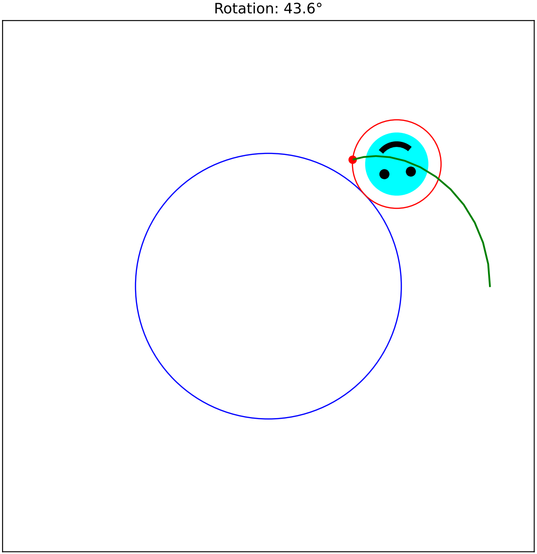
Rotation: 42.2°



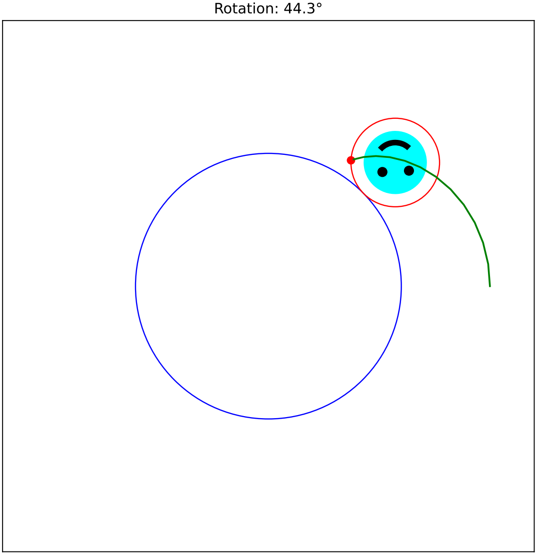
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

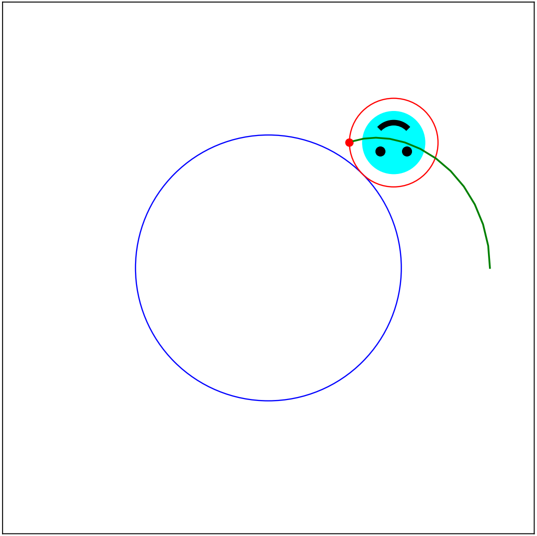


Animation of Coin Rotation Paradox

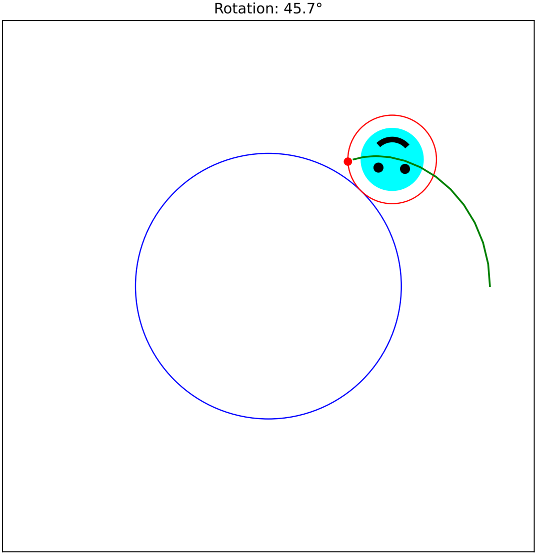


Animation of Coin Rotation Paradox

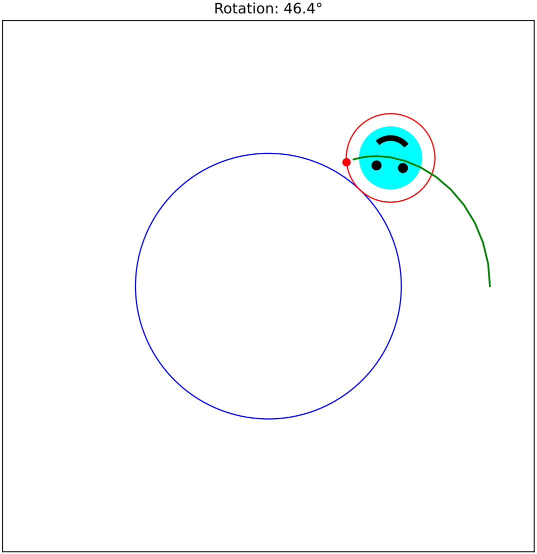
Rotation: 45.0°



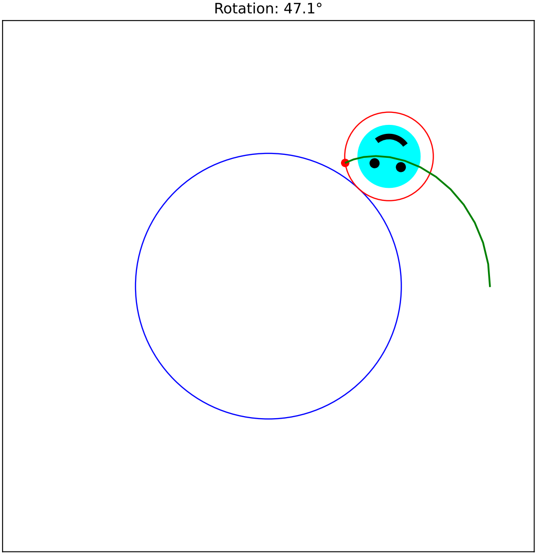
Animation of Coin Rotation Paradox



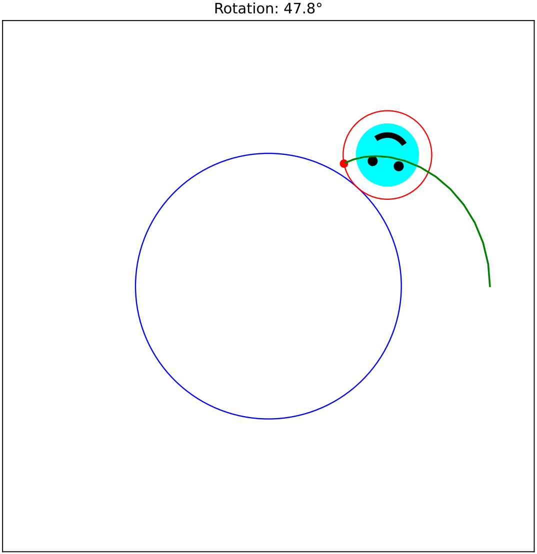
Animation of Coin Rotation Paradox



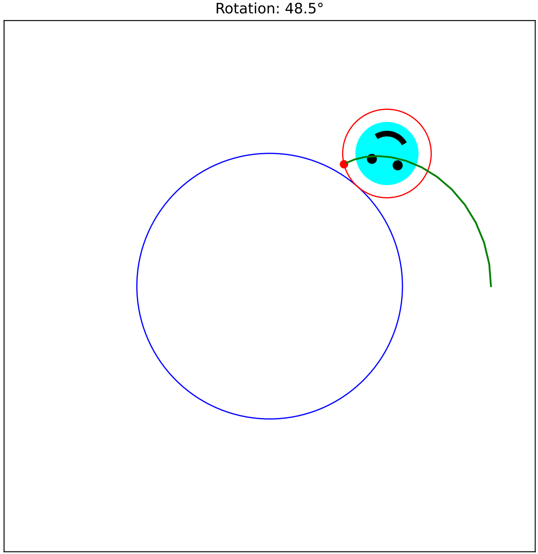
Animation of Coin Rotation Paradox



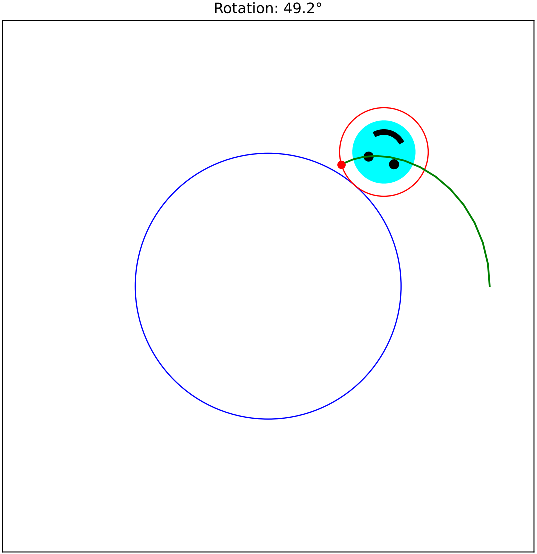
Animation of Coin Rotation Paradox



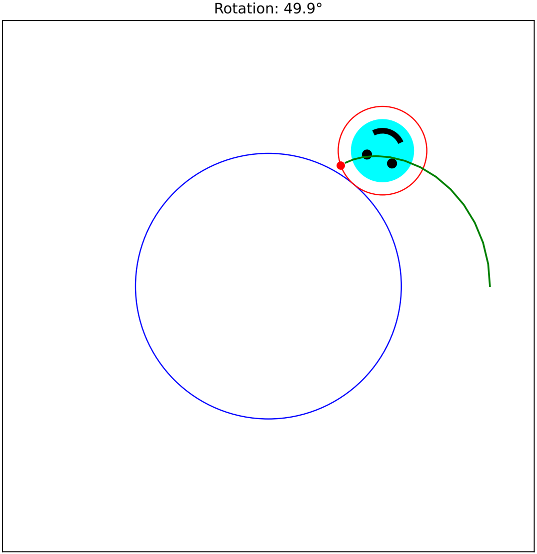
Animation of Coin Rotation Paradox



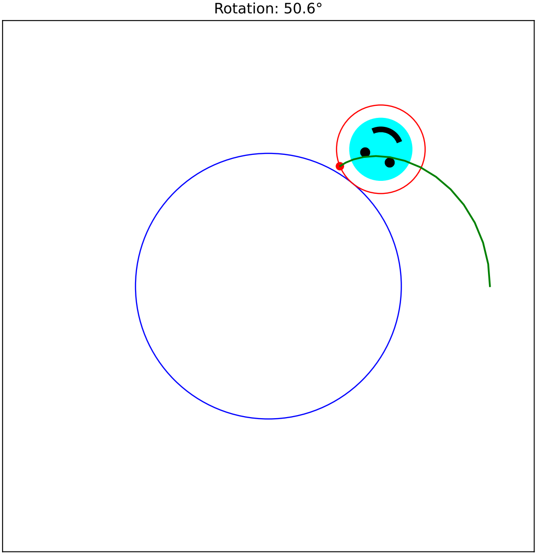
Animation of Coin Rotation Paradox



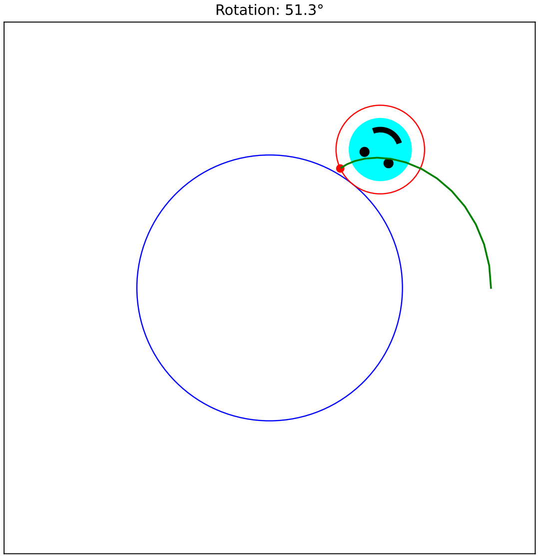
Animation of Coin Rotation Paradox



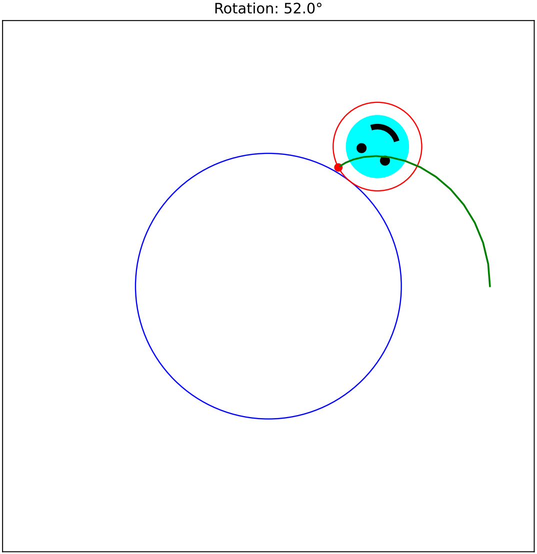
Animation of Coin Rotation Paradox



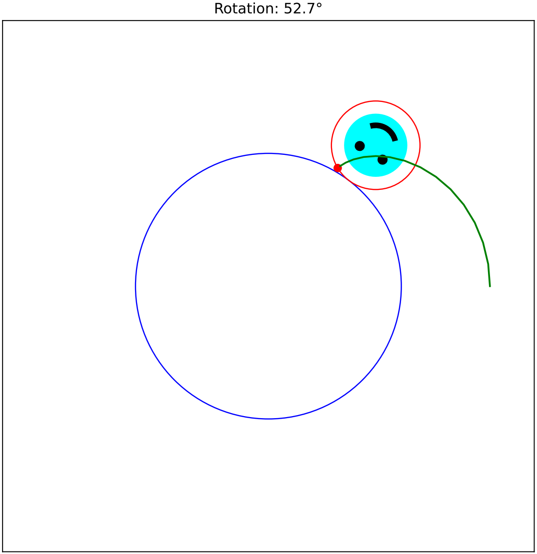
Animation of Coin Rotation Paradox



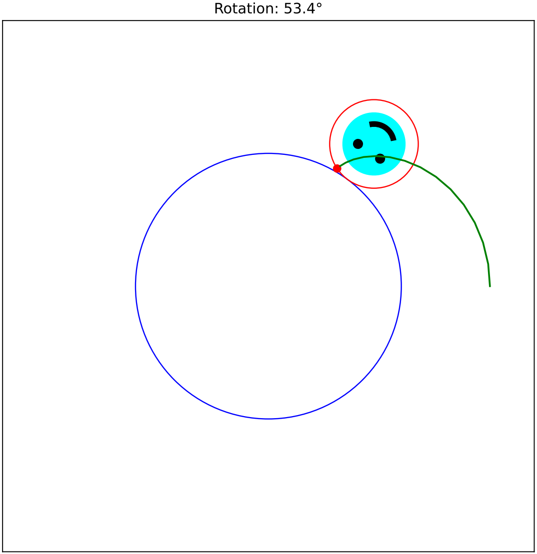
Animation of Coin Rotation Paradox



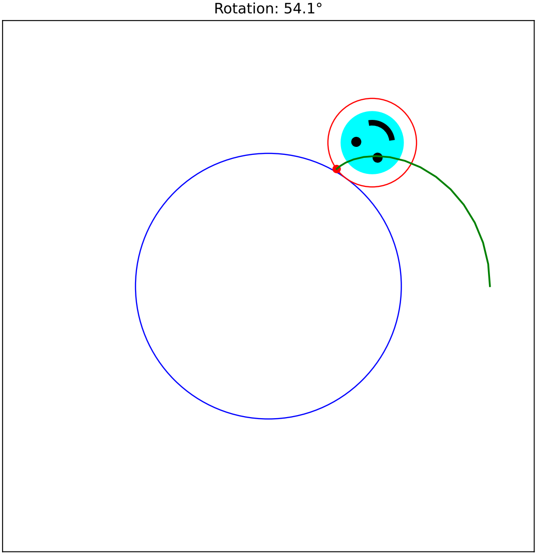
Animation of Coin Rotation Paradox



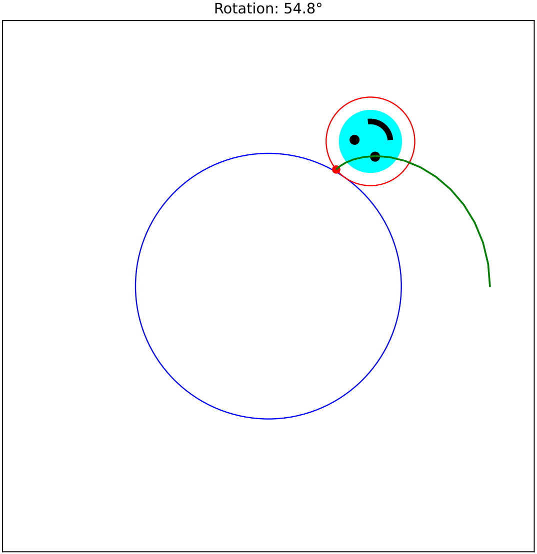
Animation of Coin Rotation Paradox



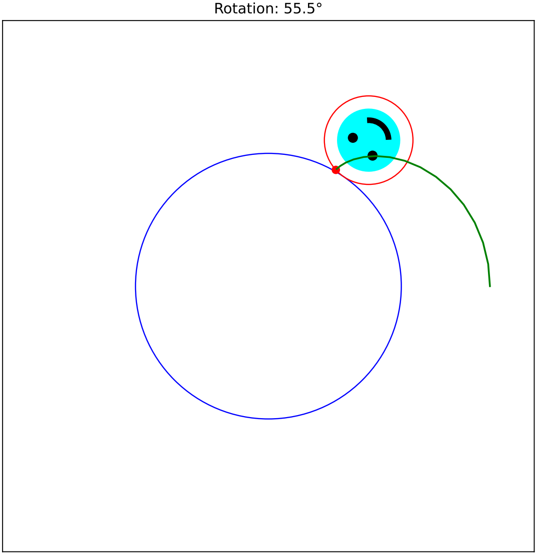
Animation of Coin Rotation Paradox



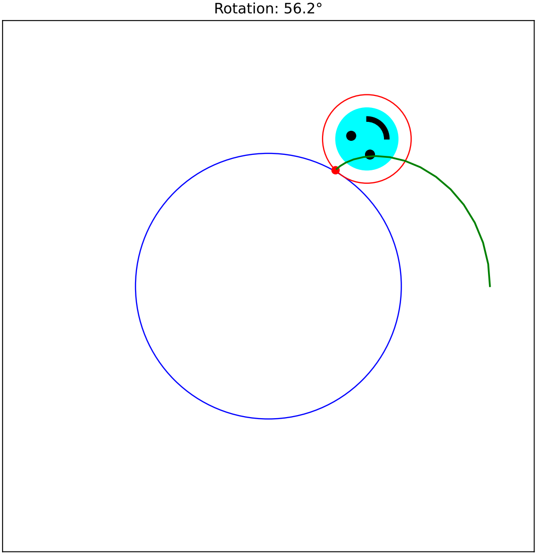
Animation of Coin Rotation Paradox



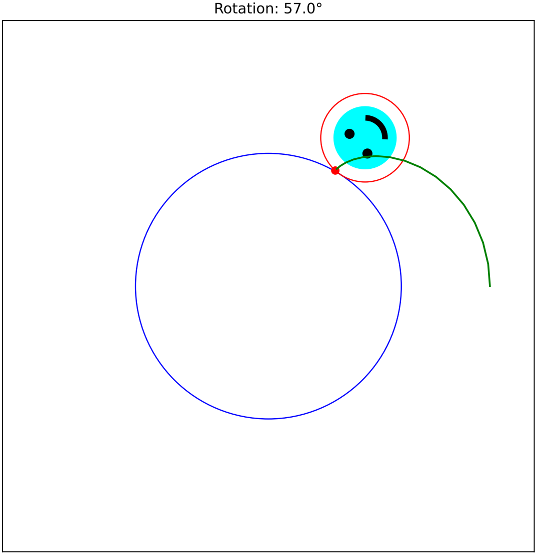
Animation of Coin Rotation Paradox



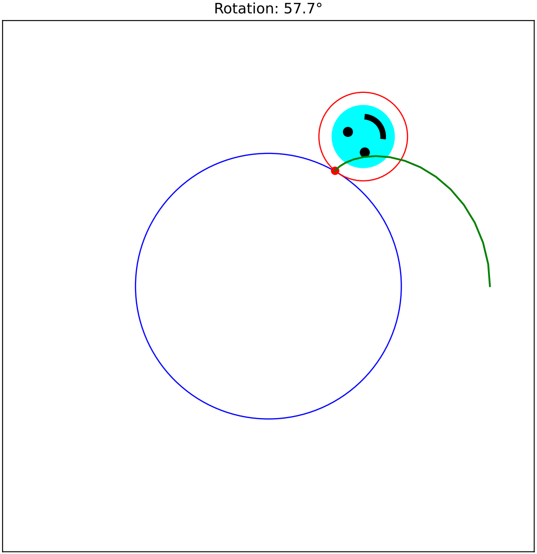
Animation of Coin Rotation Paradox



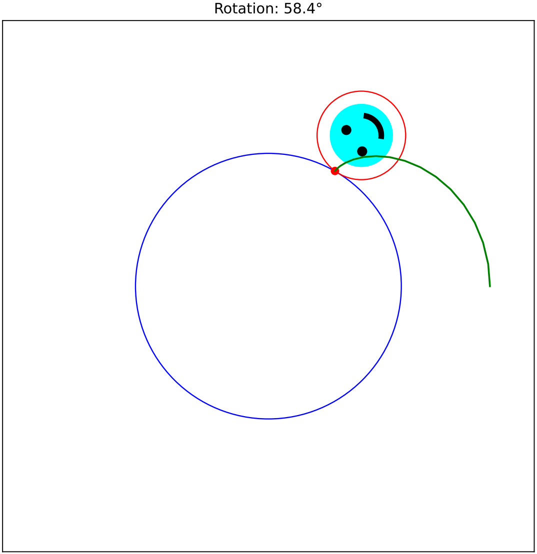
Animation of Coin Rotation Paradox



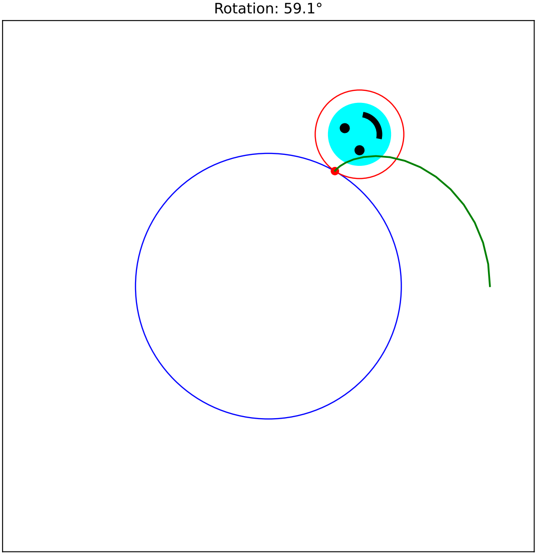
Animation of Coin Rotation Paradox



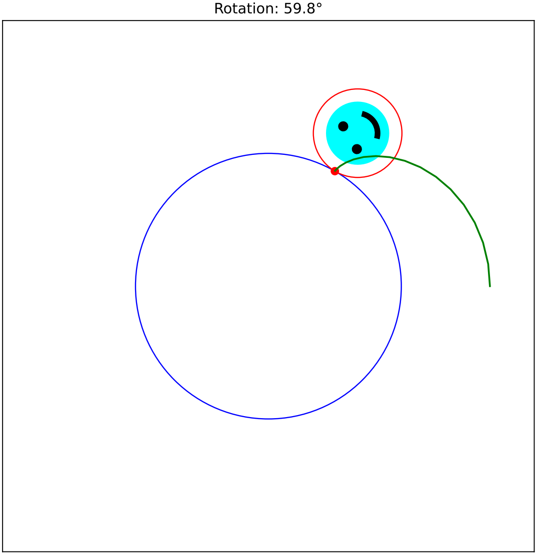
Animation of Coin Rotation Paradox



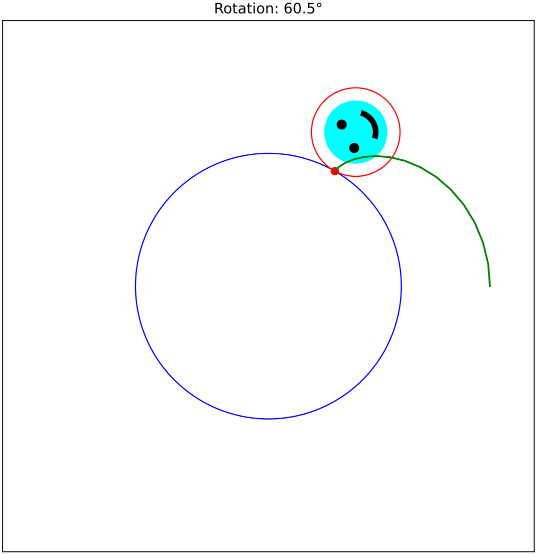
Animation of Coin Rotation Paradox



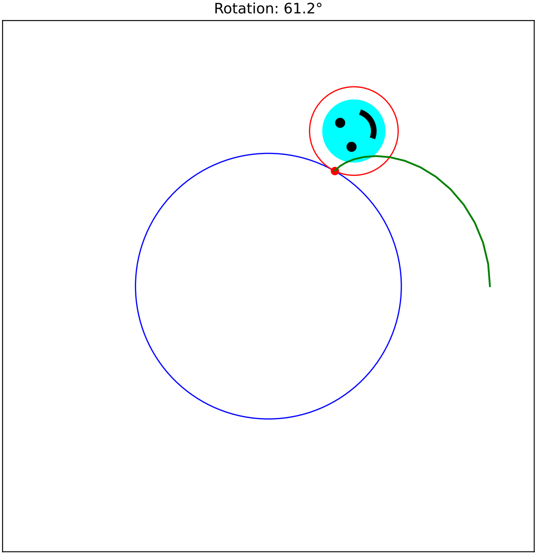
Animation of Coin Rotation Paradox



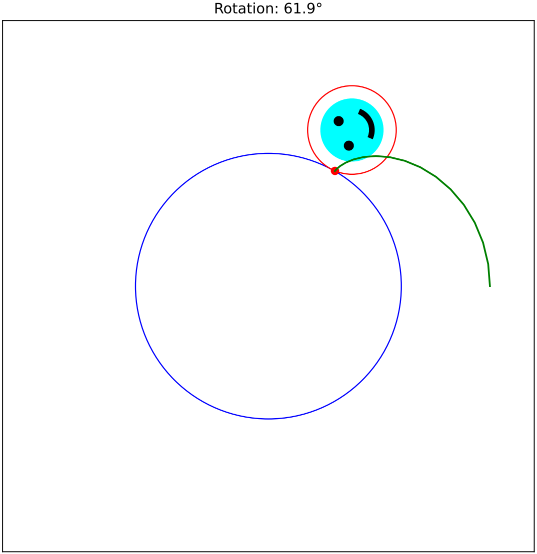
Animation of Coin Rotation Paradox



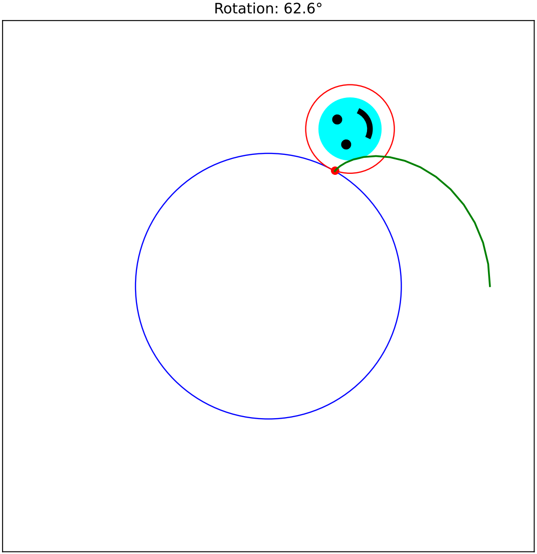
Animation of Coin Rotation Paradox



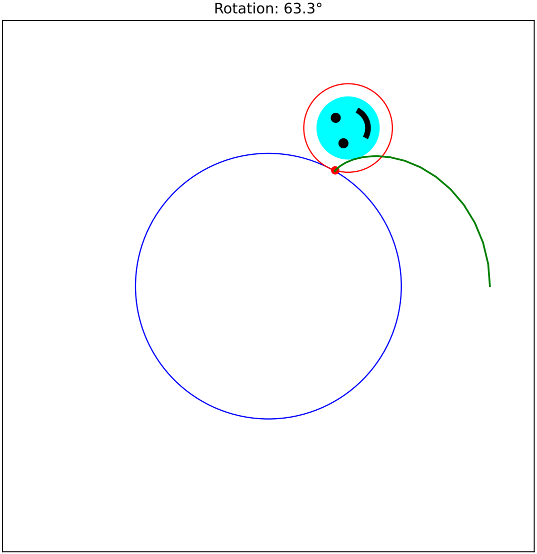
Animation of Coin Rotation Paradox



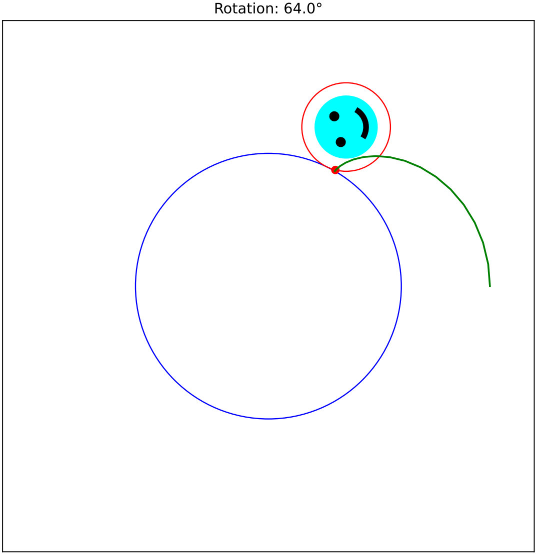
Animation of Coin Rotation Paradox



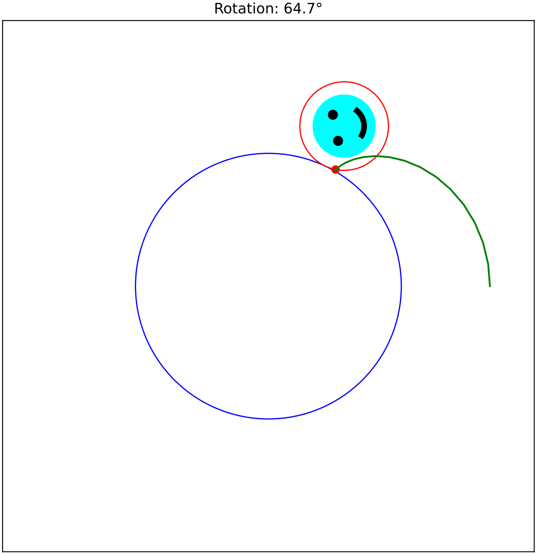
Animation of Coin Rotation Paradox



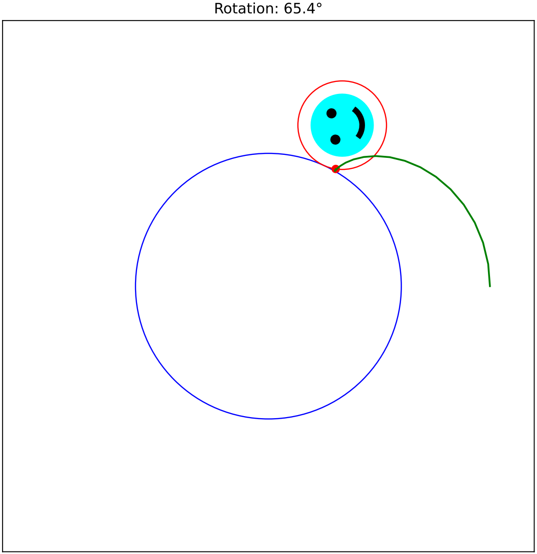
Animation of Coin Rotation Paradox



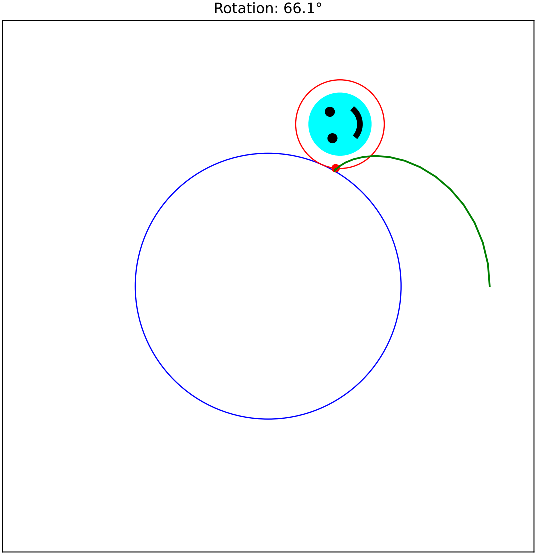
Animation of Coin Rotation Paradox



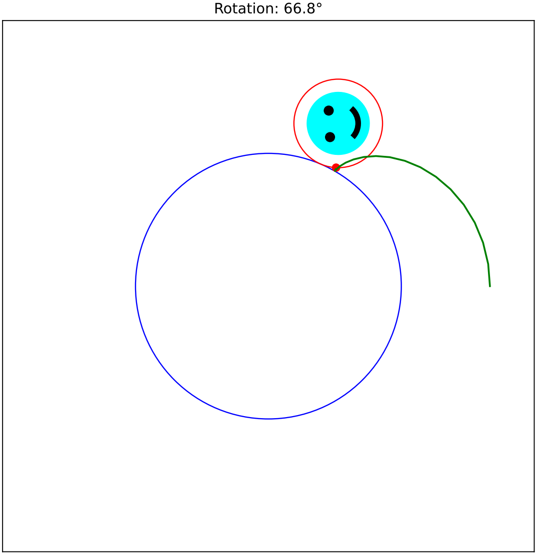
Animation of Coin Rotation Paradox



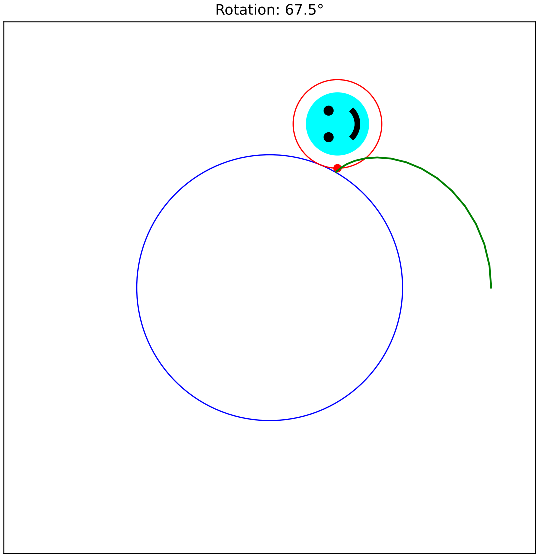
Animation of Coin Rotation Paradox



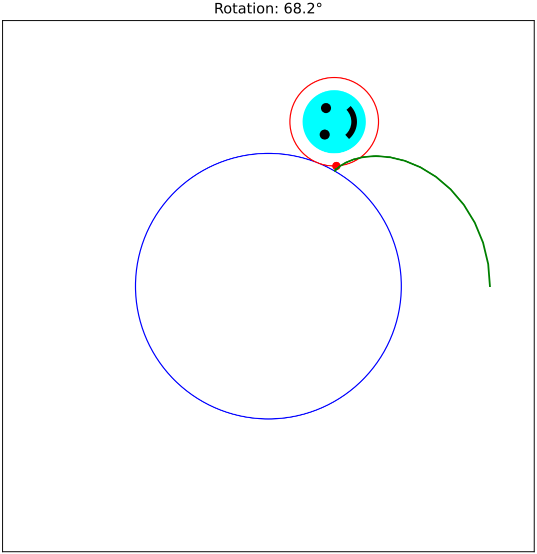
Animation of Coin Rotation Paradox



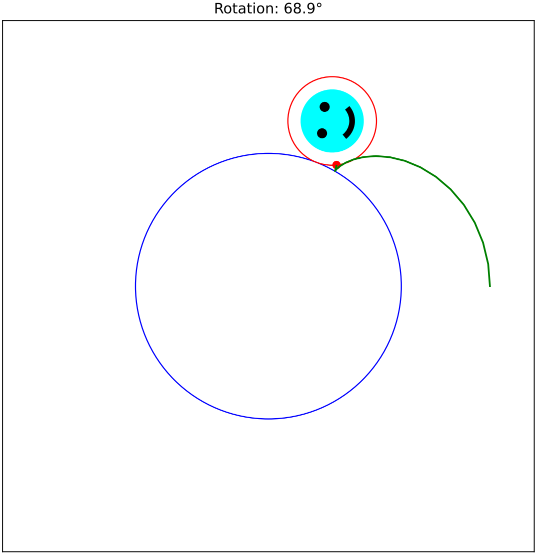
Animation of Coin Rotation Paradox



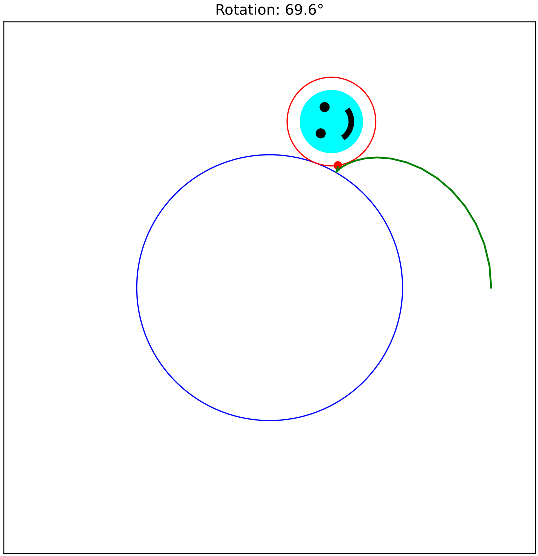
Animation of Coin Rotation Paradox



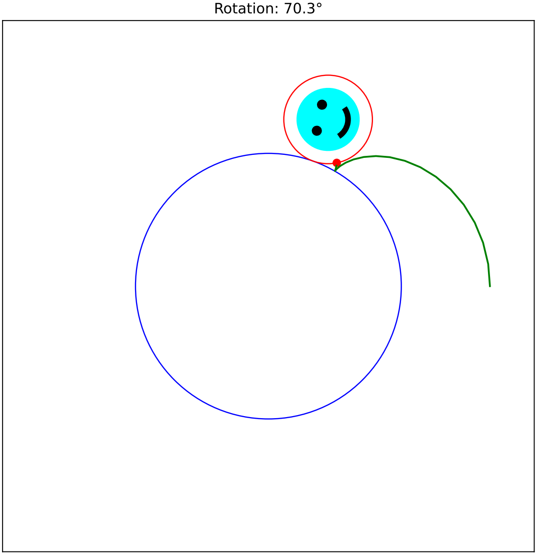
Animation of Coin Rotation Paradox



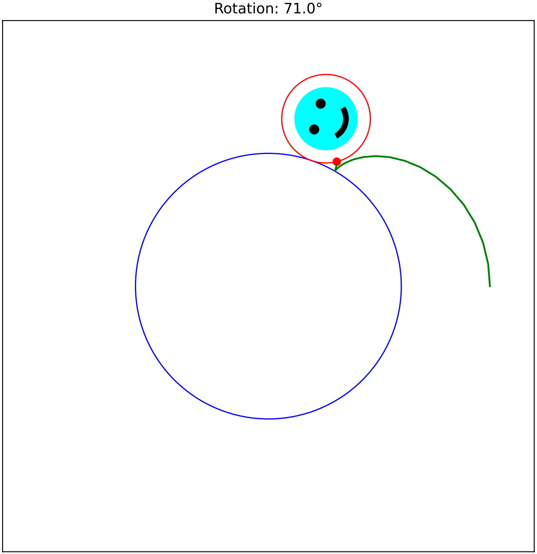
Animation of Coin Rotation Paradox



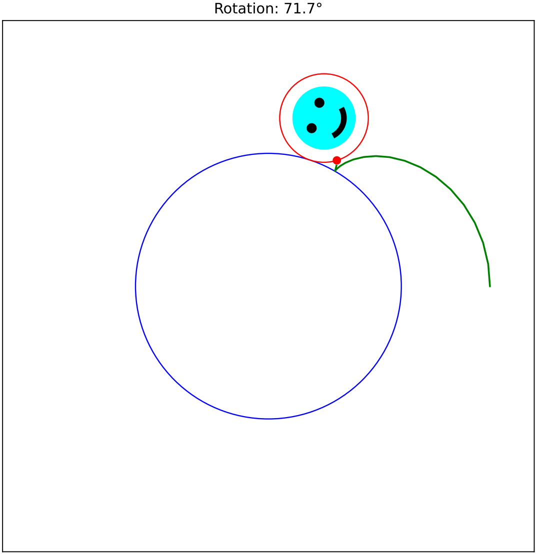
Animation of Coin Rotation Paradox



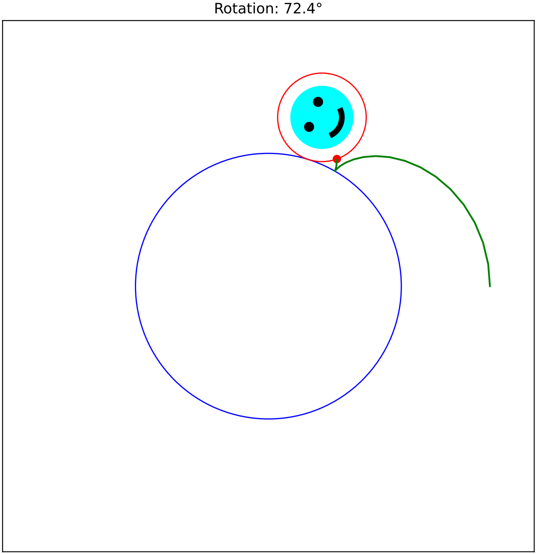
Animation of Coin Rotation Paradox



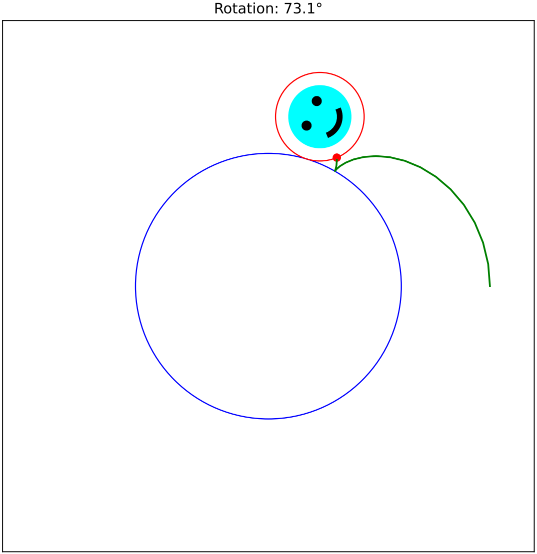
Animation of Coin Rotation Paradox



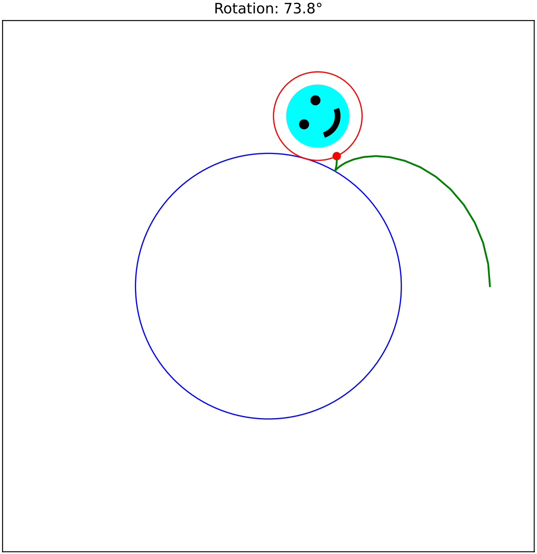
Animation of Coin Rotation Paradox



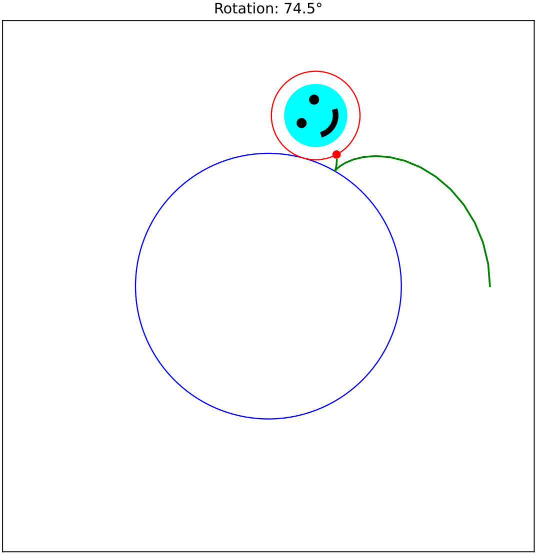
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

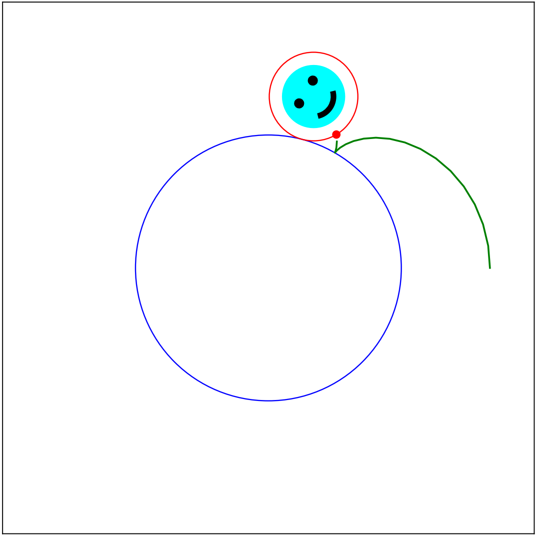


Animation of Coin Rotation Paradox



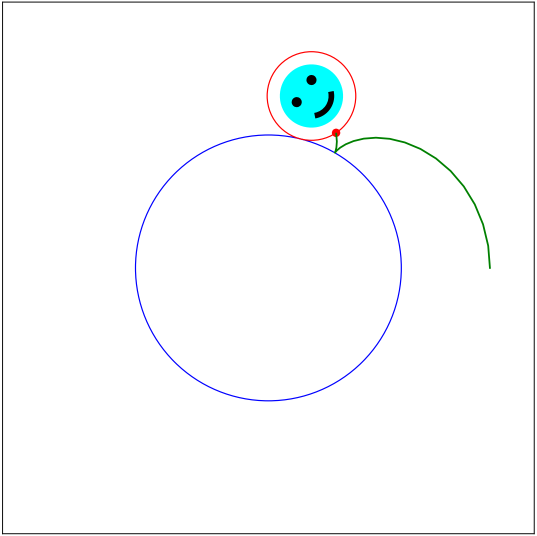
Animation of Coin Rotation Paradox

Rotation: 75.2°

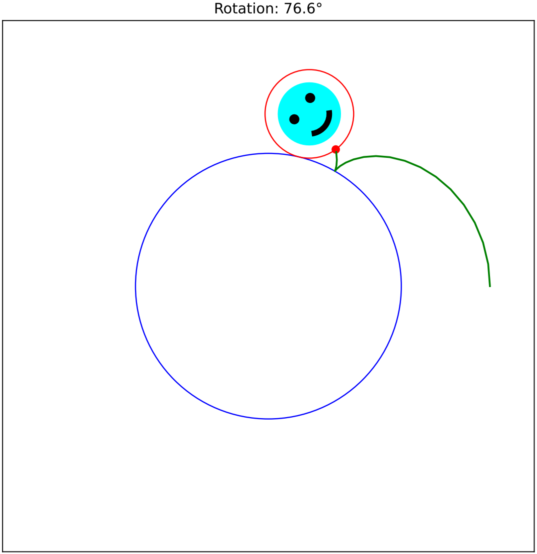


Animation of Coin Rotation Paradox

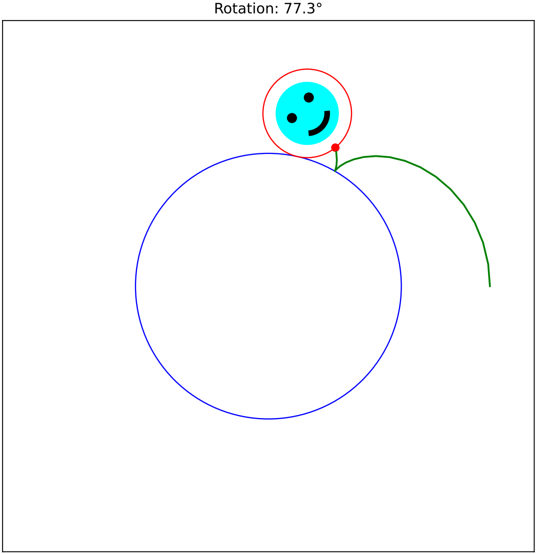
Rotation: 75.9°



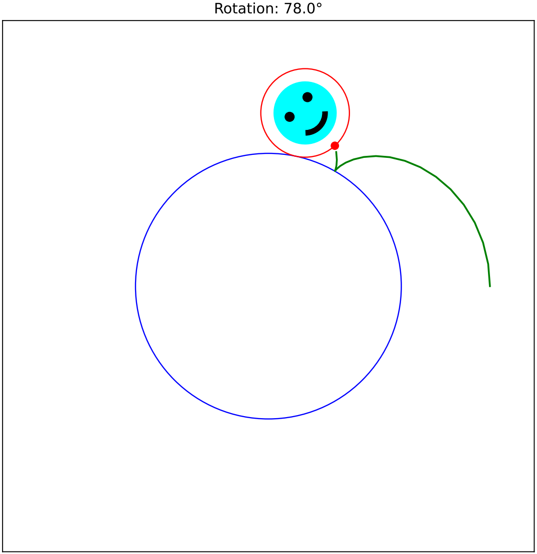
Animation of Coin Rotation Paradox



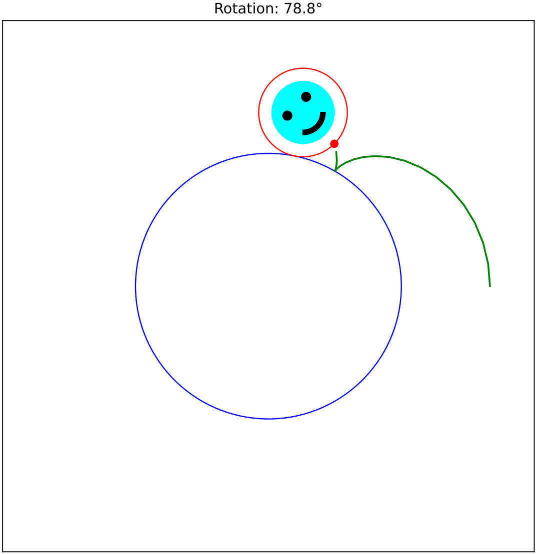
Animation of Coin Rotation Paradox



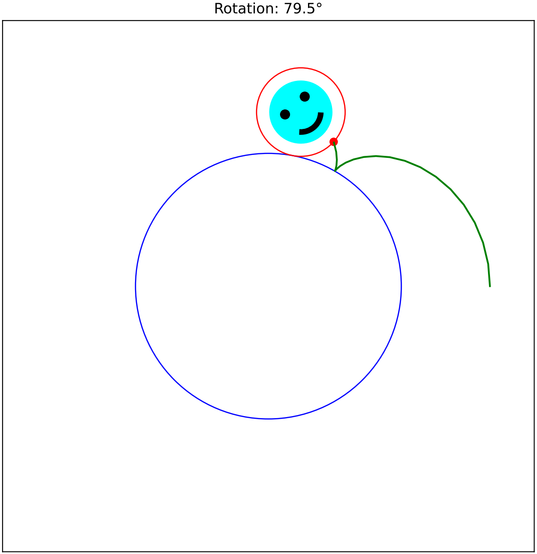
Animation of Coin Rotation Paradox



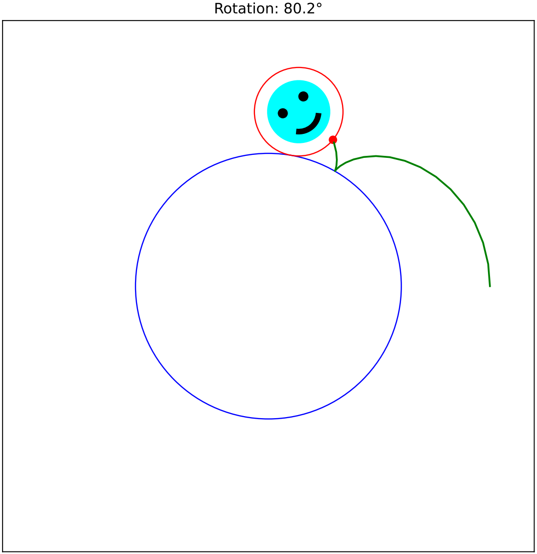
Animation of Coin Rotation Paradox



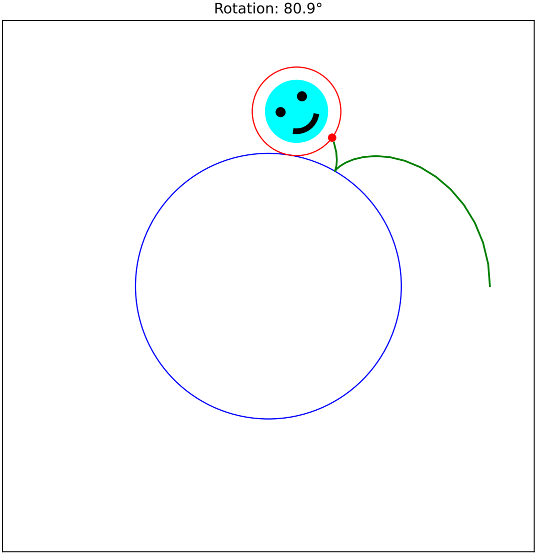
Animation of Coin Rotation Paradox



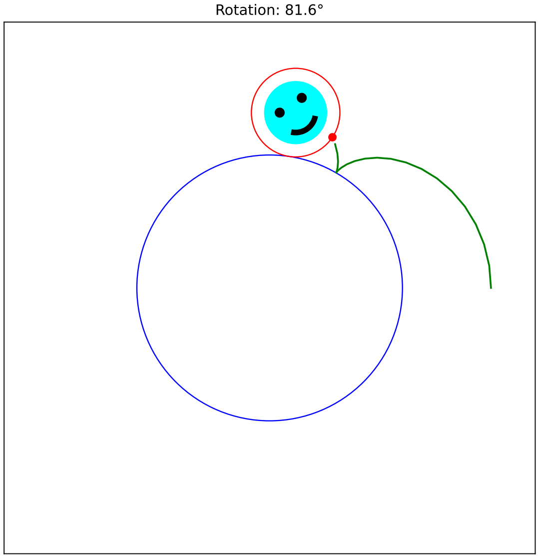
Animation of Coin Rotation Paradox



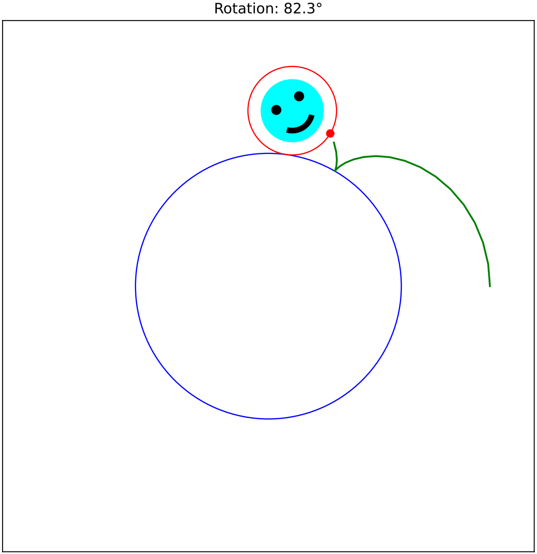
Animation of Coin Rotation Paradox



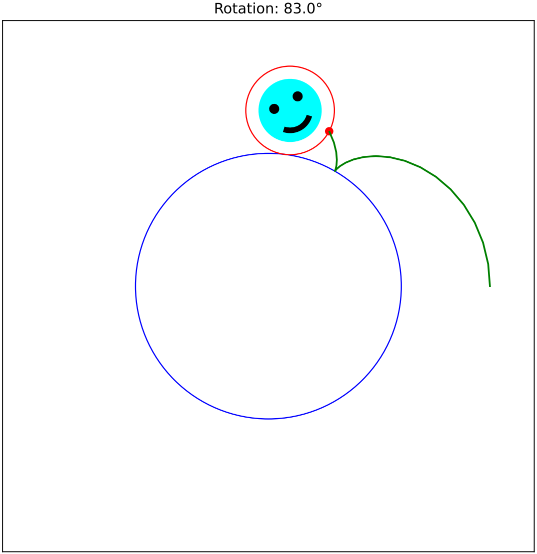
Animation of Coin Rotation Paradox



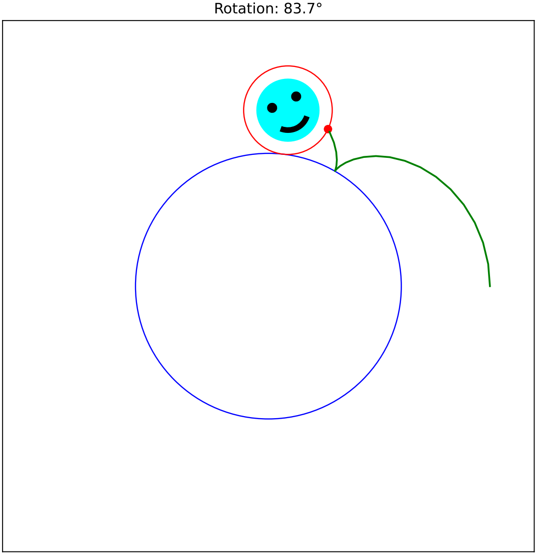
Animation of Coin Rotation Paradox



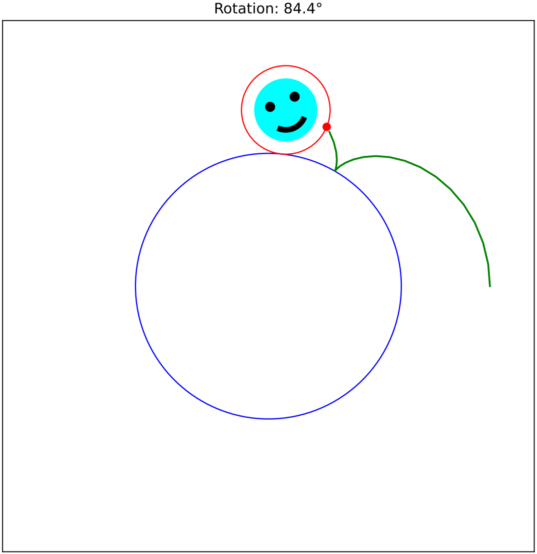
Animation of Coin Rotation Paradox



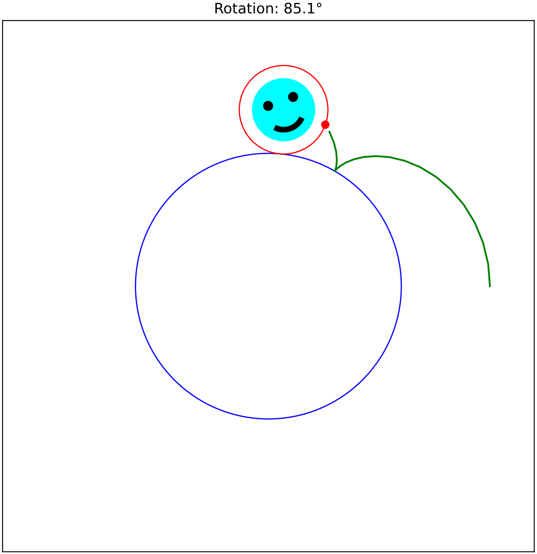
Animation of Coin Rotation Paradox



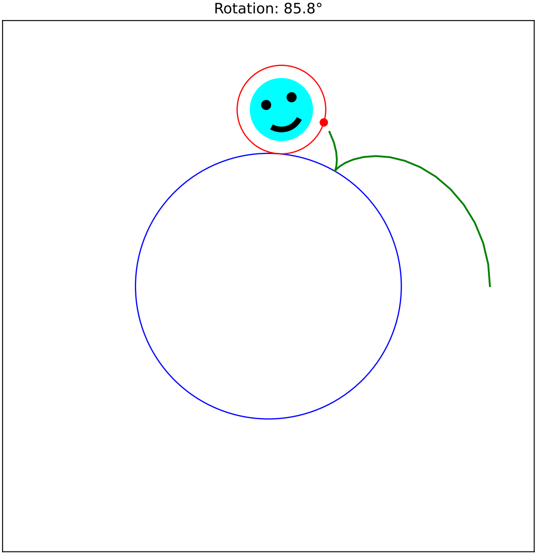
Animation of Coin Rotation Paradox



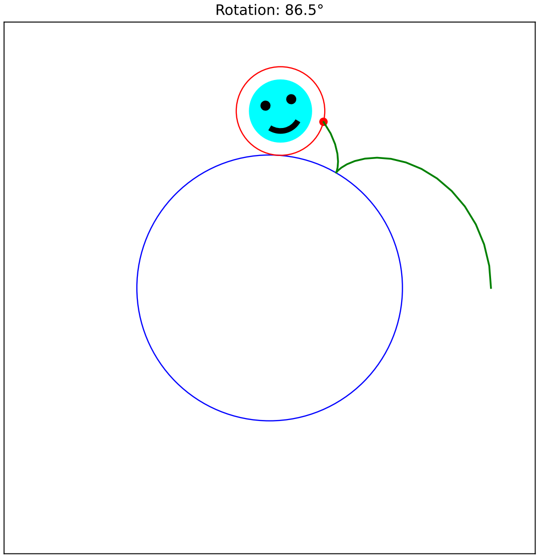
Animation of Coin Rotation Paradox



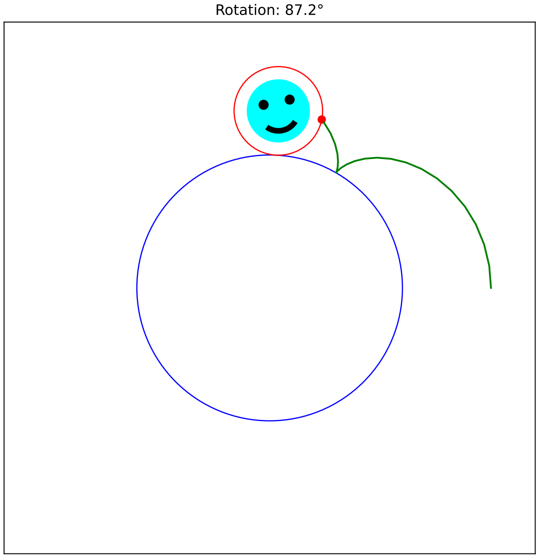
Animation of Coin Rotation Paradox



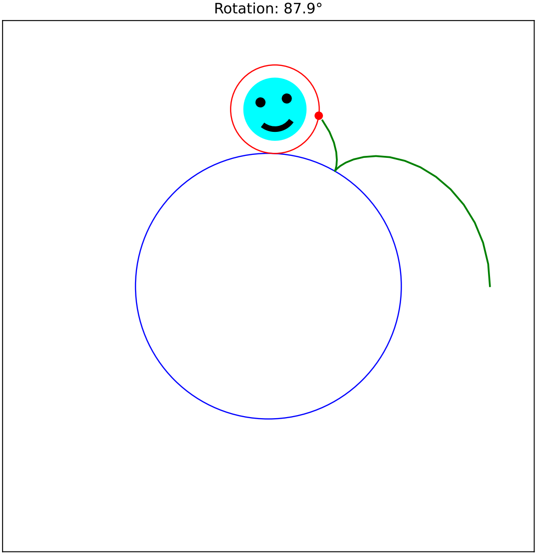
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

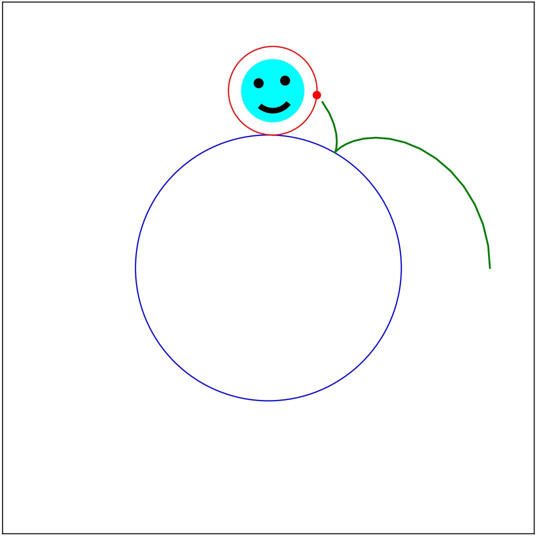


Animation of Coin Rotation Paradox



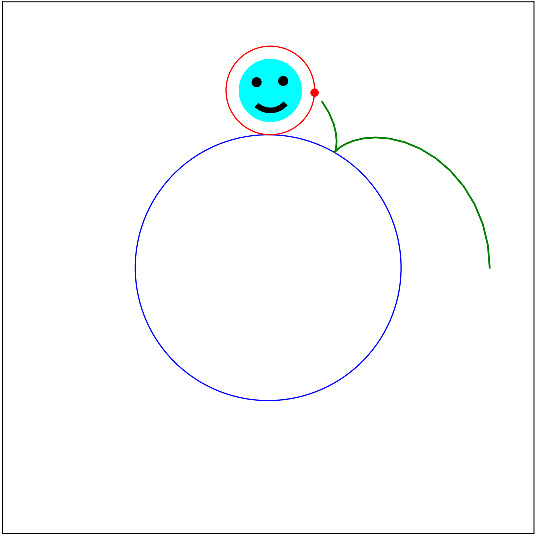
Animation of Coin Rotation Paradox

Rotation: 88.6°

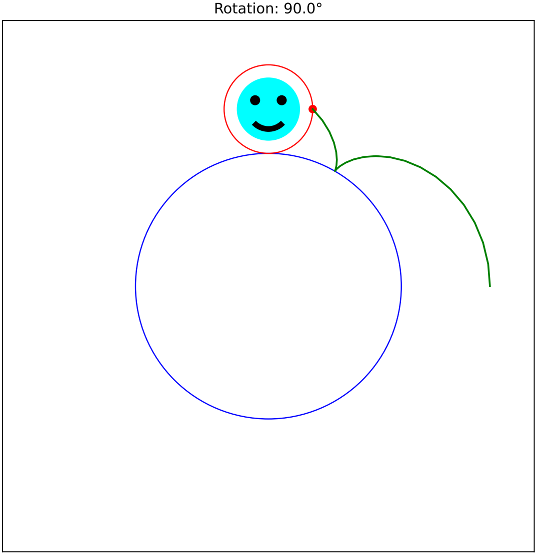


Animation of Coin Rotation Paradox

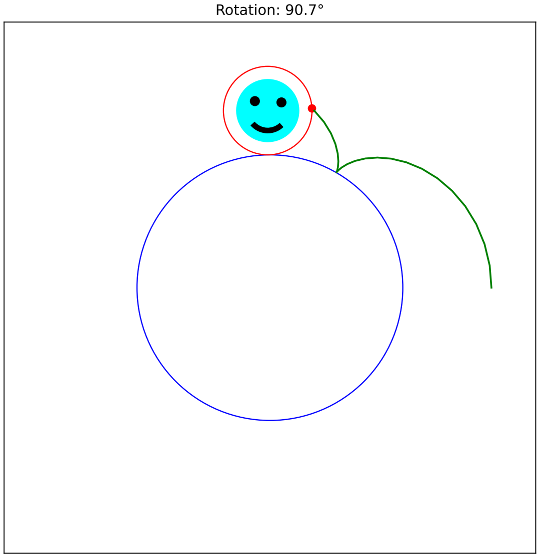
Rotation: 89.3°



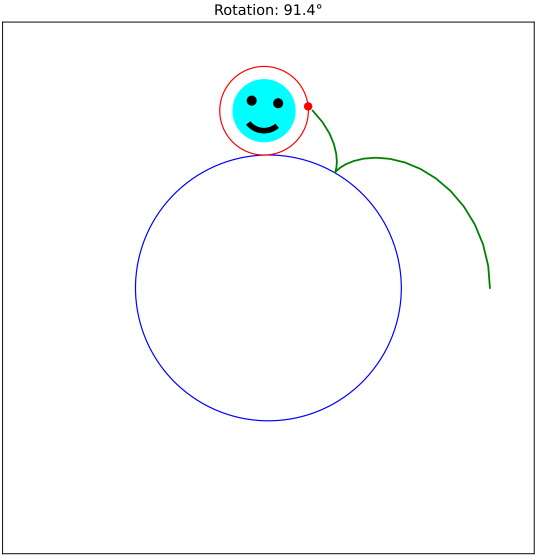
Animation of Coin Rotation Paradox



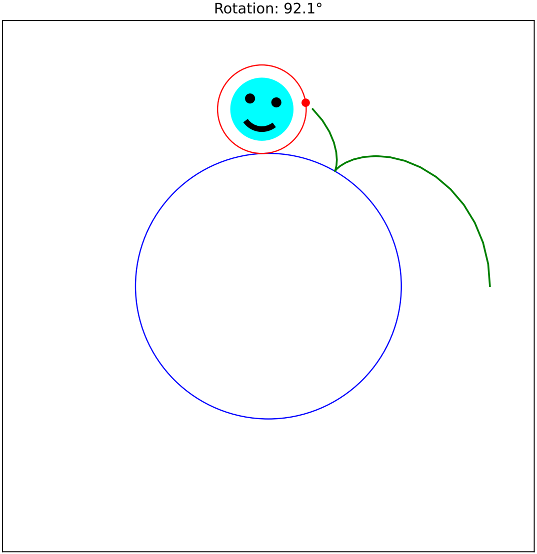
Animation of Coin Rotation Paradox



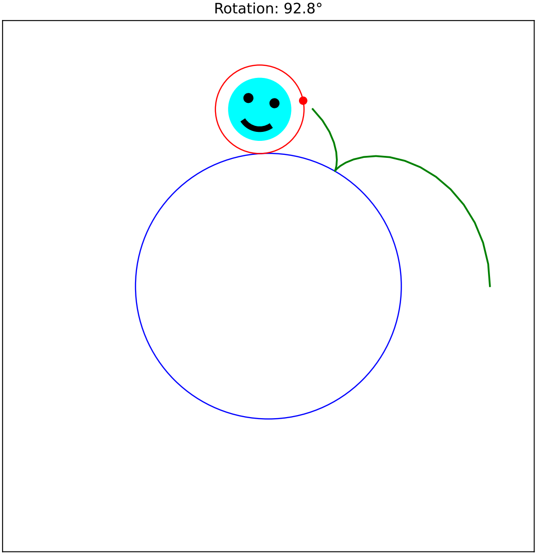
Animation of Coin Rotation Paradox



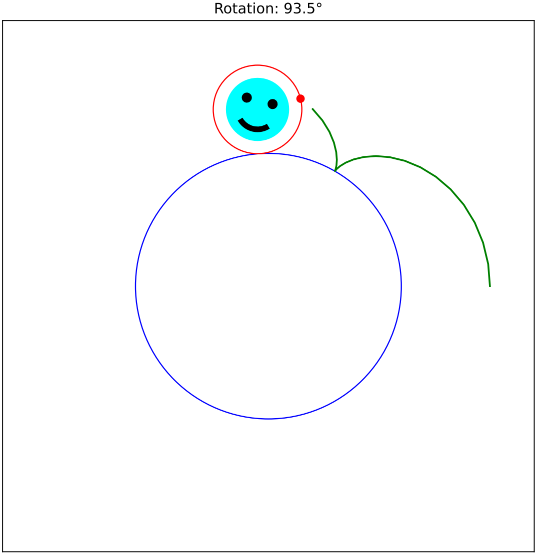
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

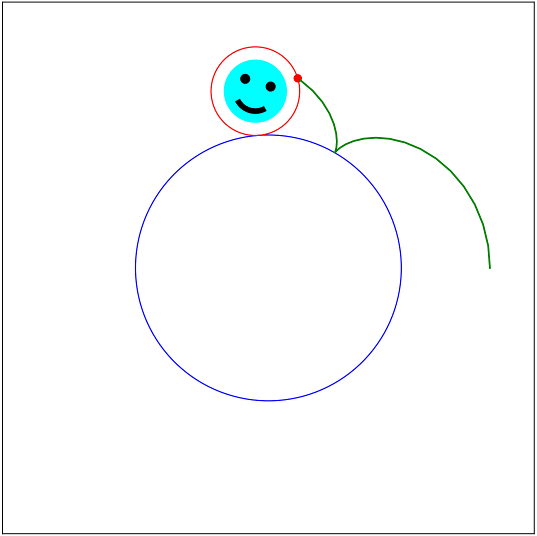


Animation of Coin Rotation Paradox

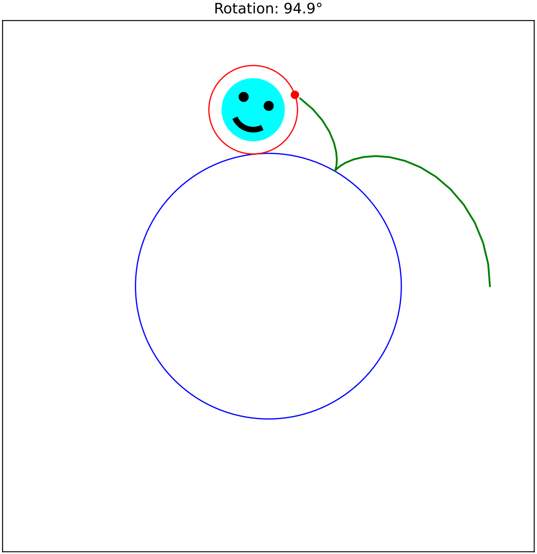


Animation of Coin Rotation Paradox

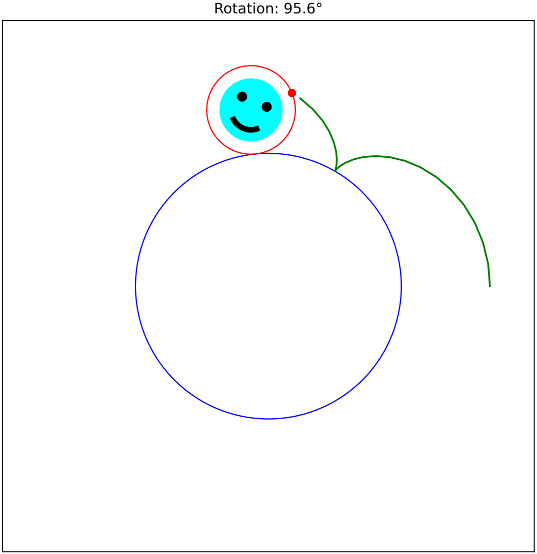
Rotation: 94.2°



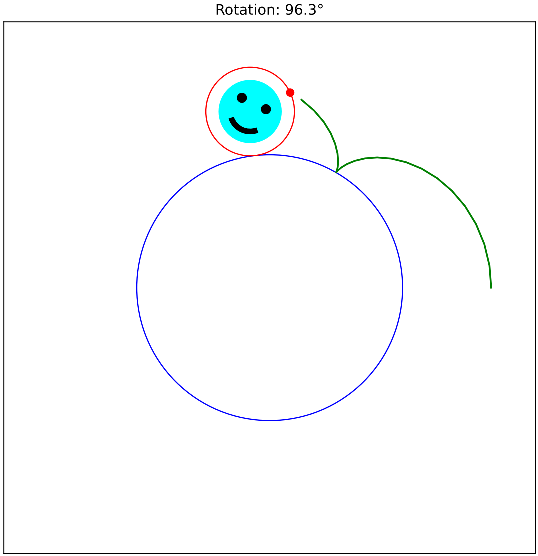
Animation of Coin Rotation Paradox



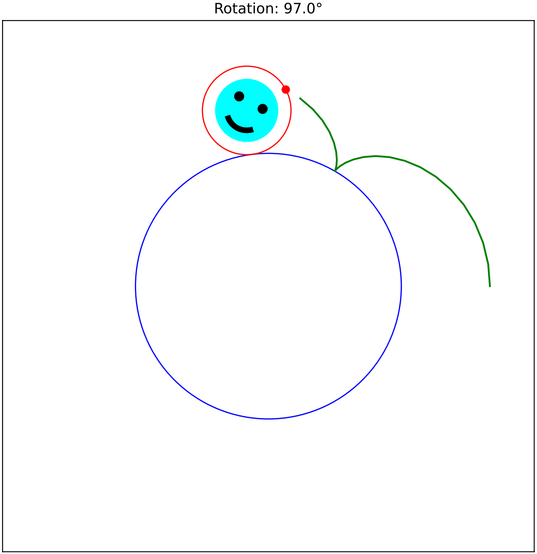
Animation of Coin Rotation Paradox



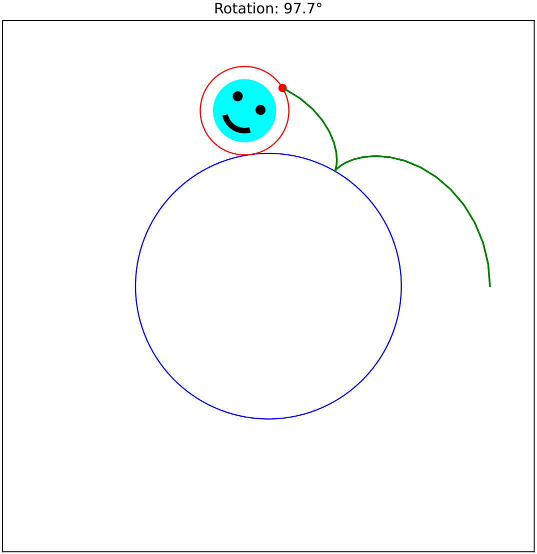
Animation of Coin Rotation Paradox



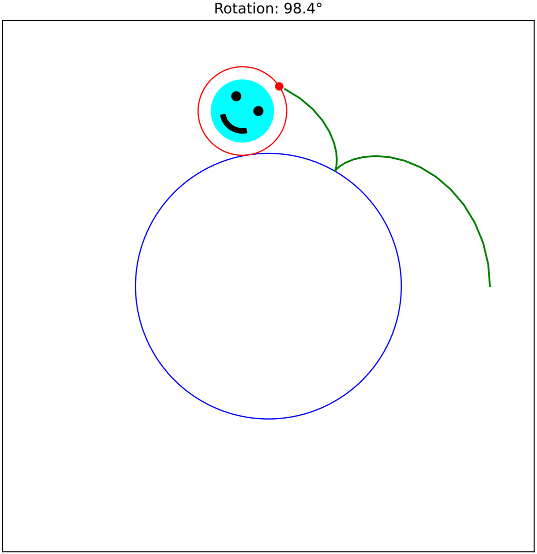
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

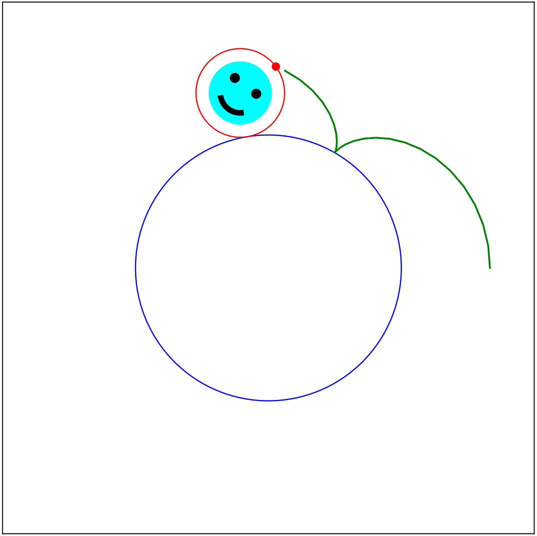


Animation of Coin Rotation Paradox

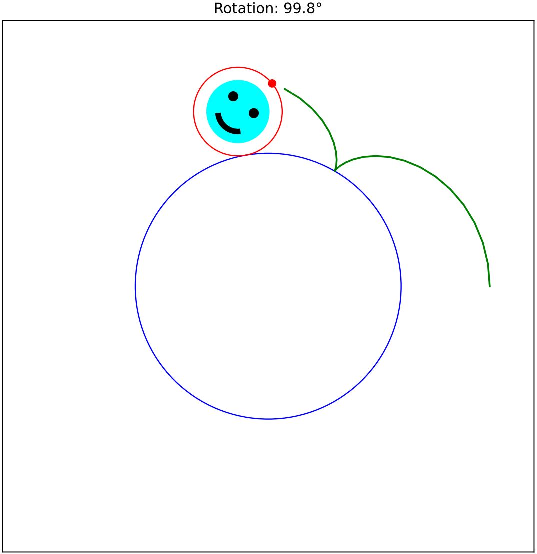


Animation of Coin Rotation Paradox

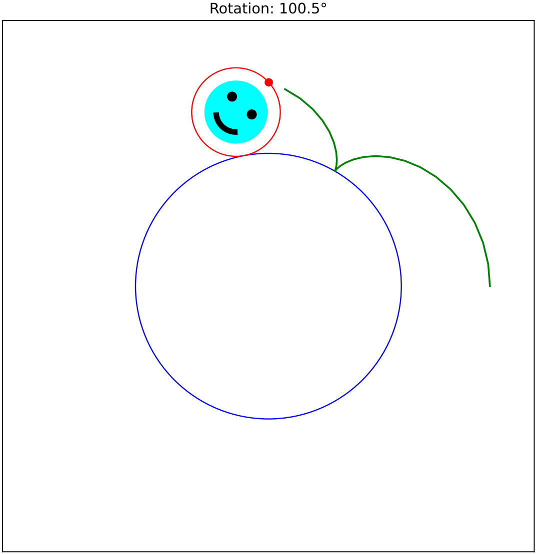
Rotation: 99.1°



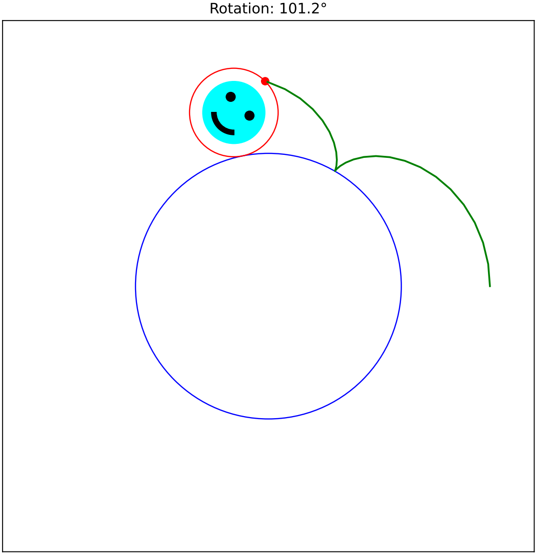
Animation of Coin Rotation Paradox



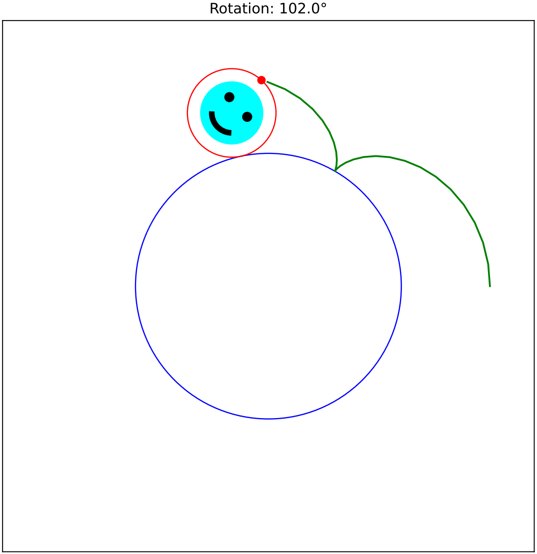
Animation of Coin Rotation Paradox



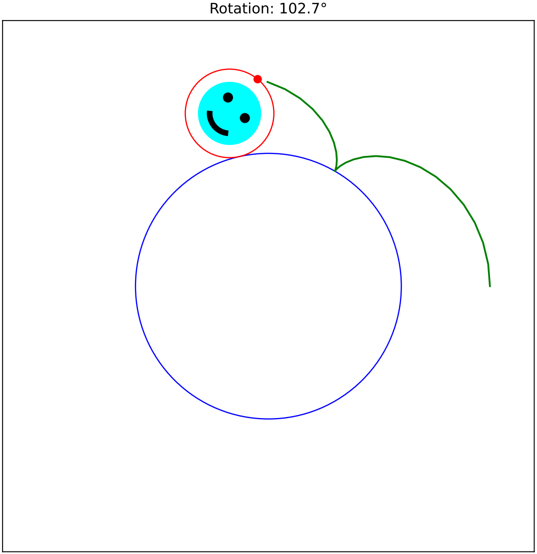
Animation of Coin Rotation Paradox



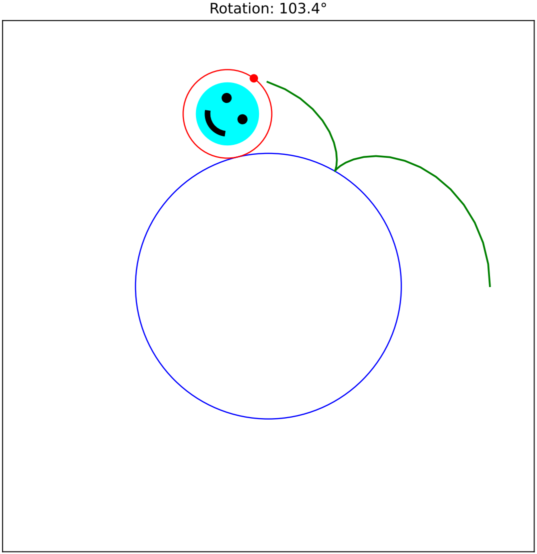
Animation of Coin Rotation Paradox



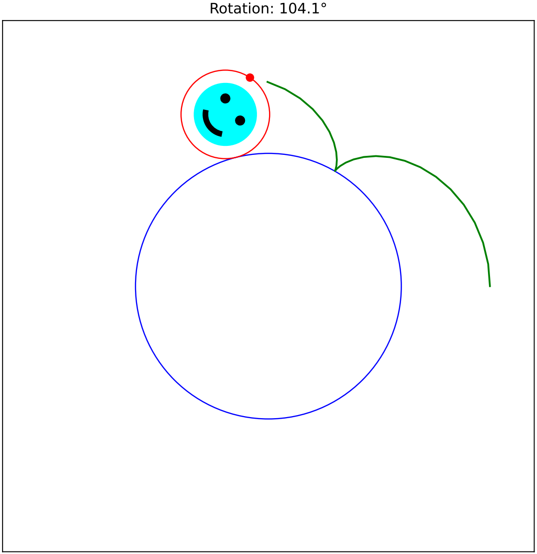
Animation of Coin Rotation Paradox



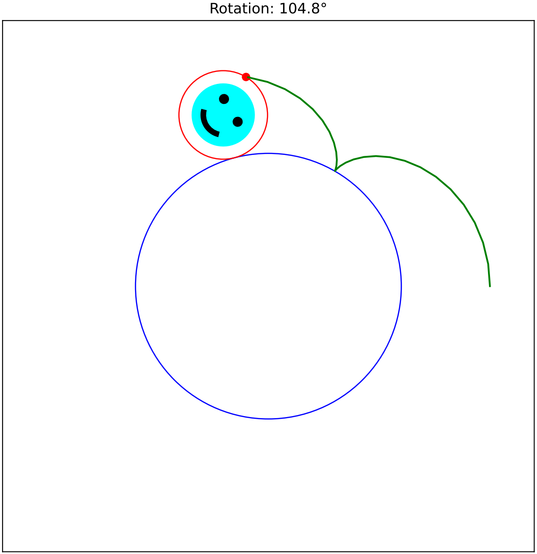
Animation of Coin Rotation Paradox



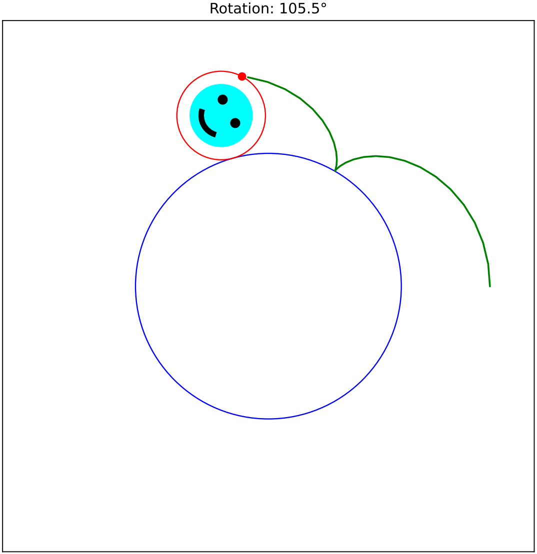
Animation of Coin Rotation Paradox



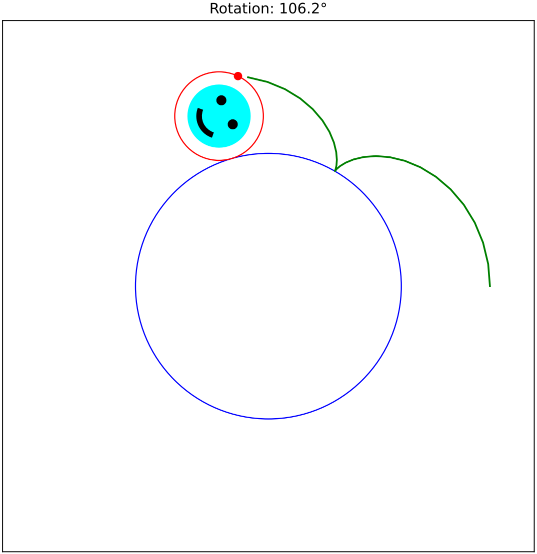
Animation of Coin Rotation Paradox



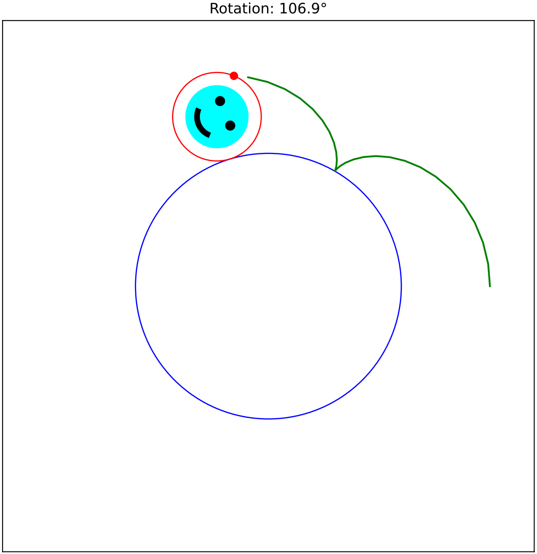
Animation of Coin Rotation Paradox



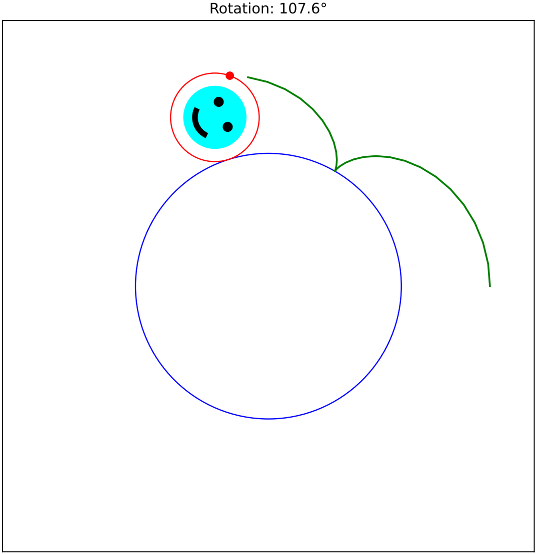
Animation of Coin Rotation Paradox



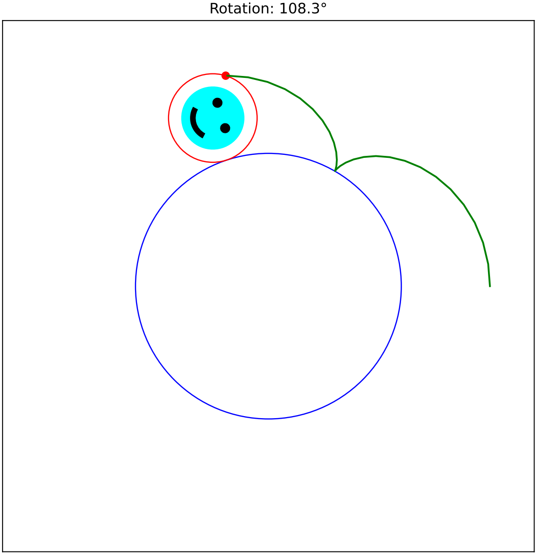
Animation of Coin Rotation Paradox



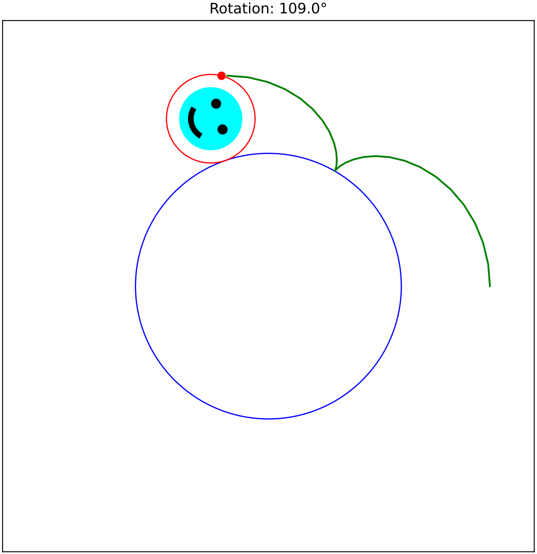
Animation of Coin Rotation Paradox



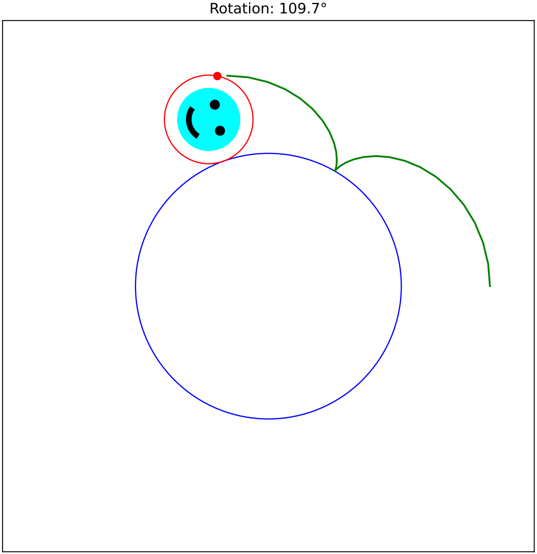
Animation of Coin Rotation Paradox



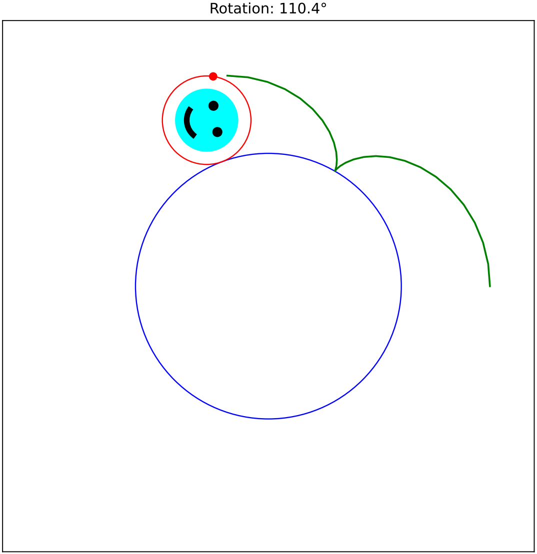
Animation of Coin Rotation Paradox



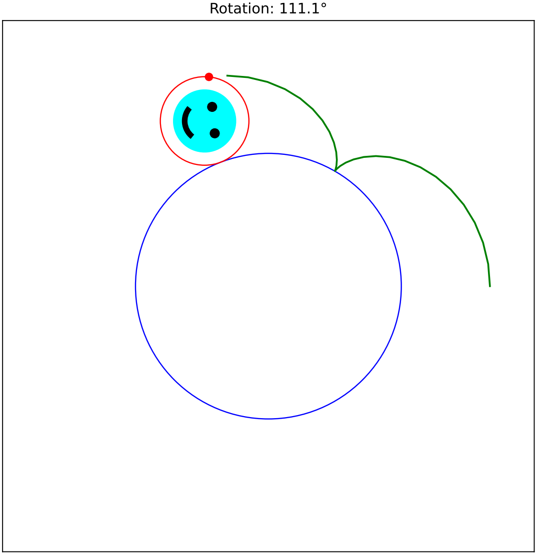
Animation of Coin Rotation Paradox



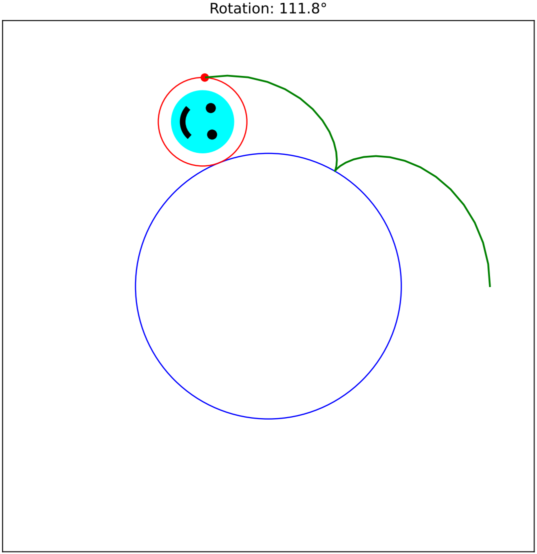
Animation of Coin Rotation Paradox



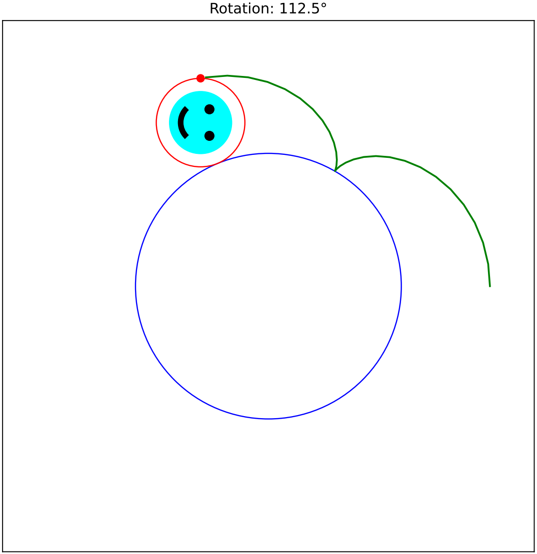
Animation of Coin Rotation Paradox



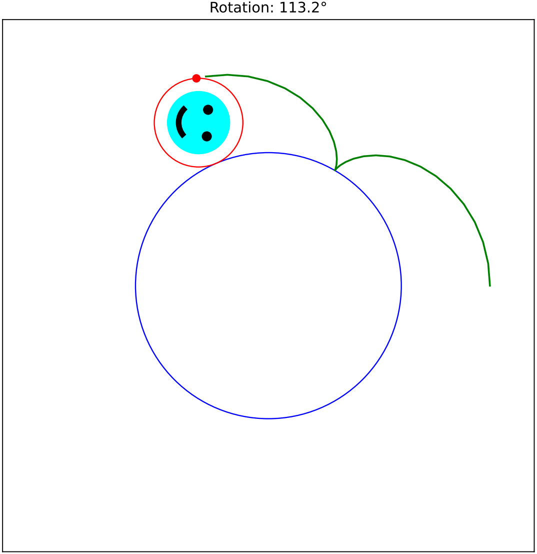
Animation of Coin Rotation Paradox



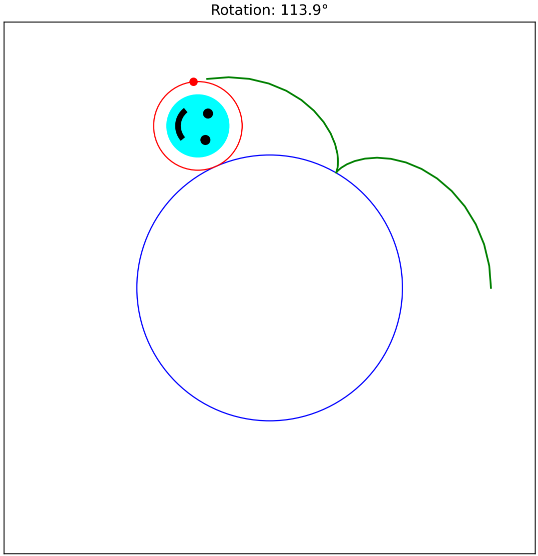
Animation of Coin Rotation Paradox



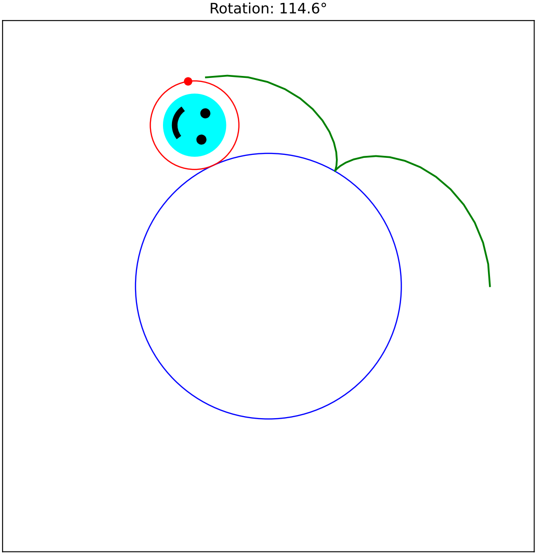
Animation of Coin Rotation Paradox



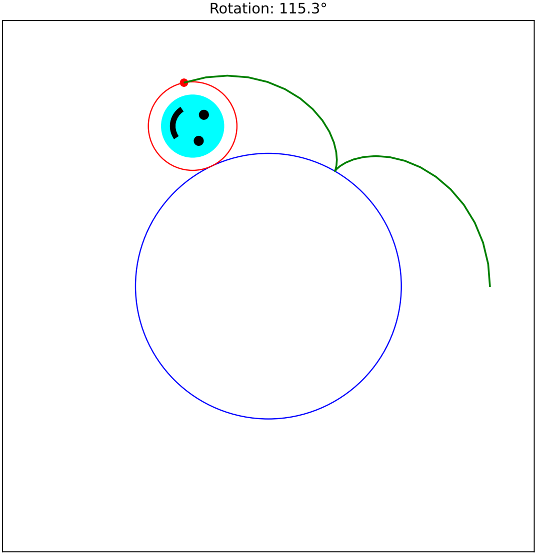
Animation of Coin Rotation Paradox



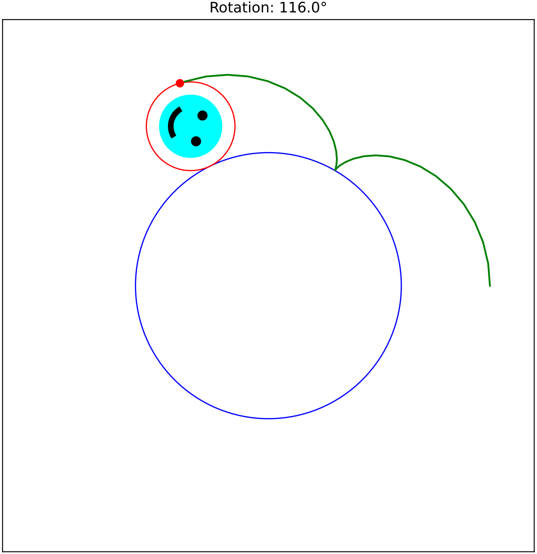
Animation of Coin Rotation Paradox



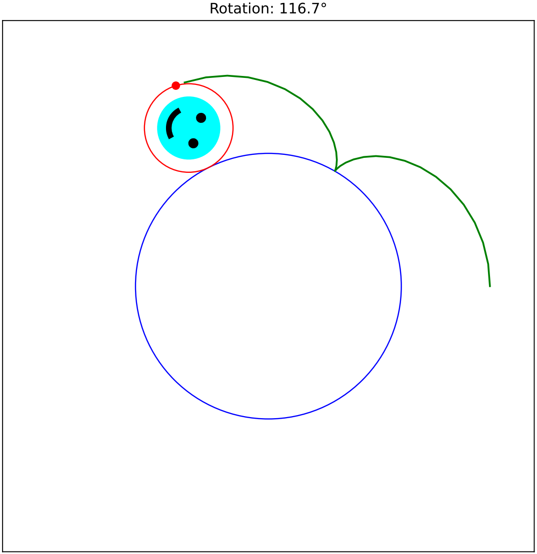
Animation of Coin Rotation Paradox



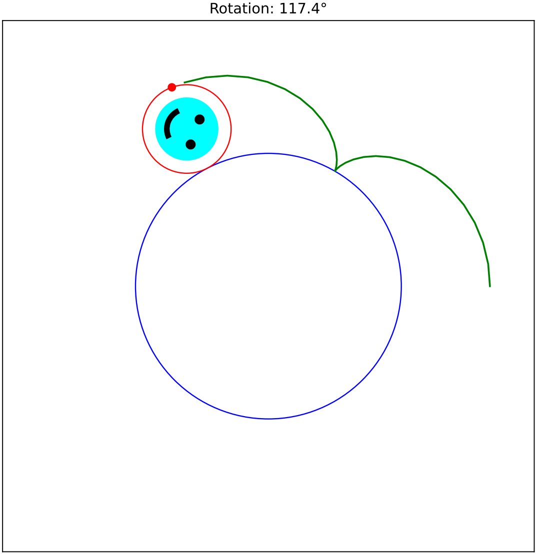
Animation of Coin Rotation Paradox



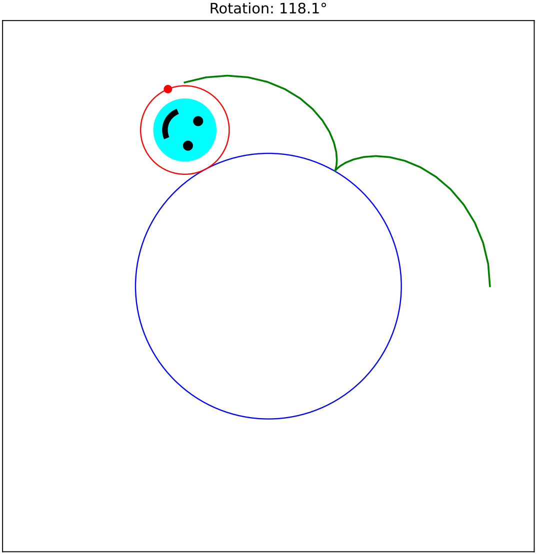
Animation of Coin Rotation Paradox



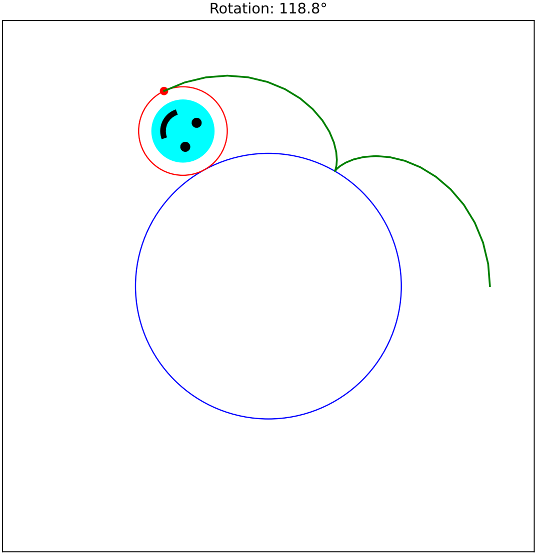
Animation of Coin Rotation Paradox



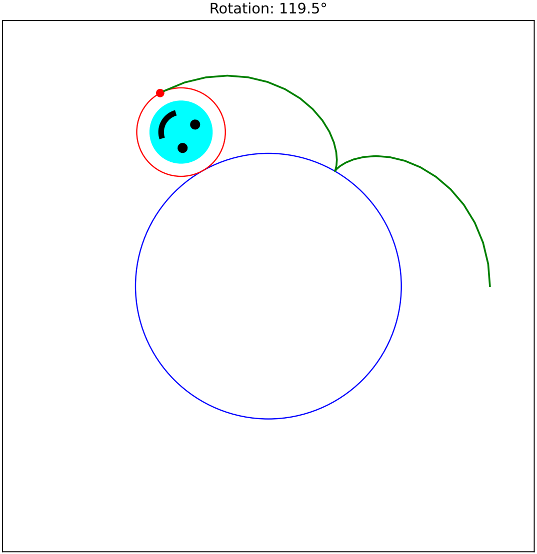
Animation of Coin Rotation Paradox



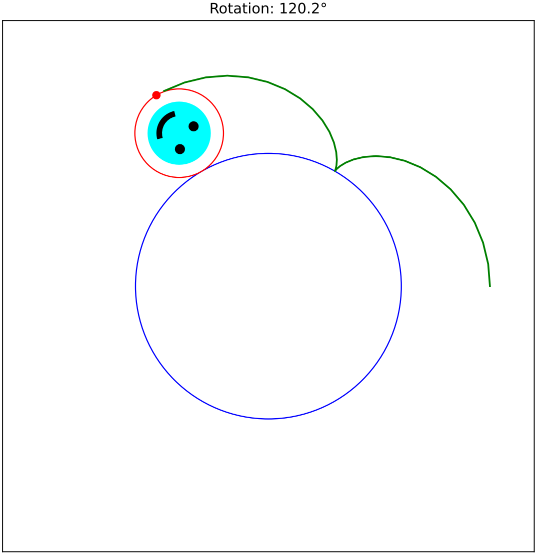
Animation of Coin Rotation Paradox



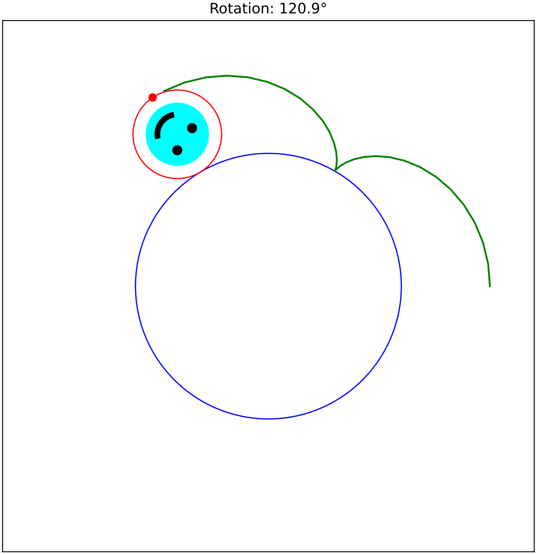
Animation of Coin Rotation Paradox



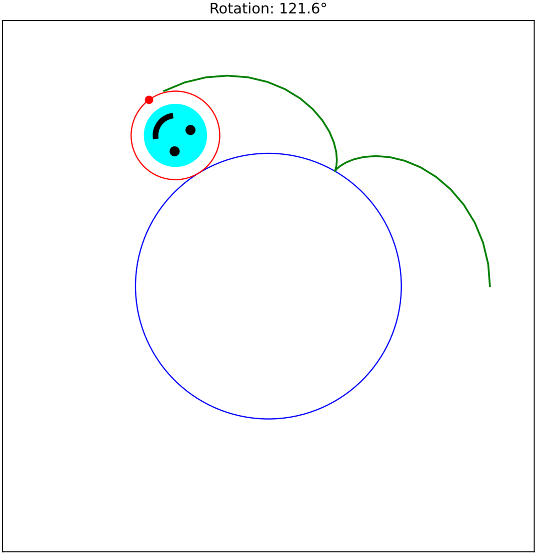
Animation of Coin Rotation Paradox



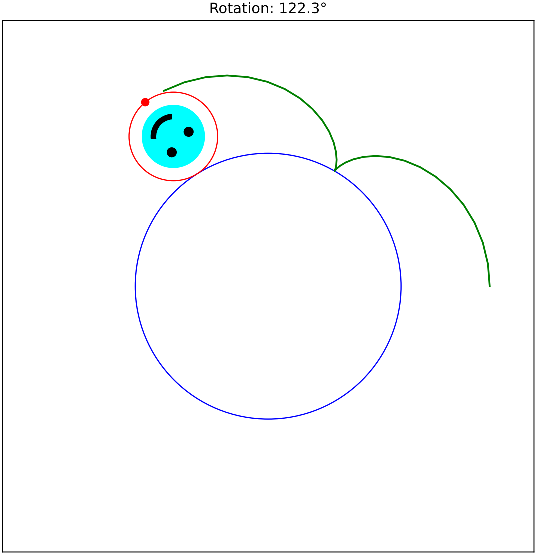
Animation of Coin Rotation Paradox



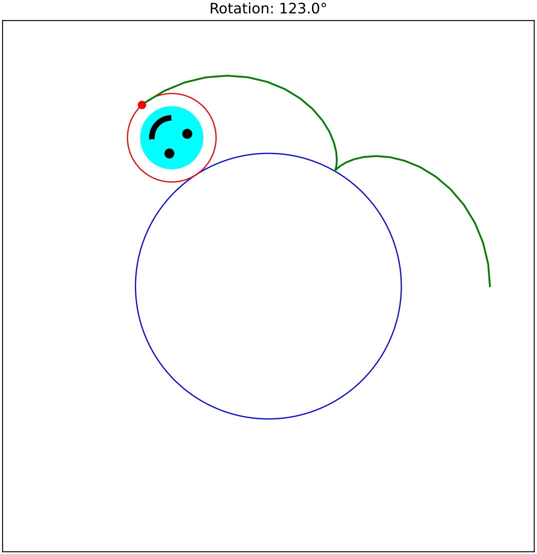
Animation of Coin Rotation Paradox



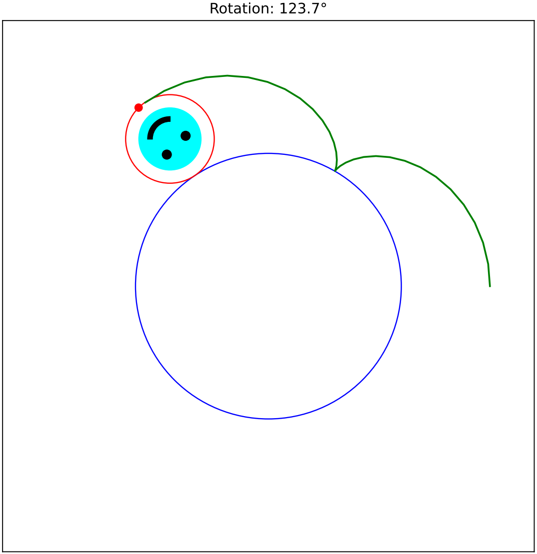
Animation of Coin Rotation Paradox



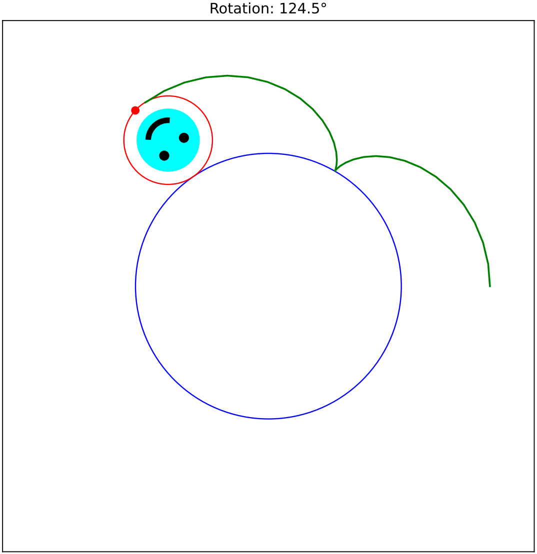
Animation of Coin Rotation Paradox



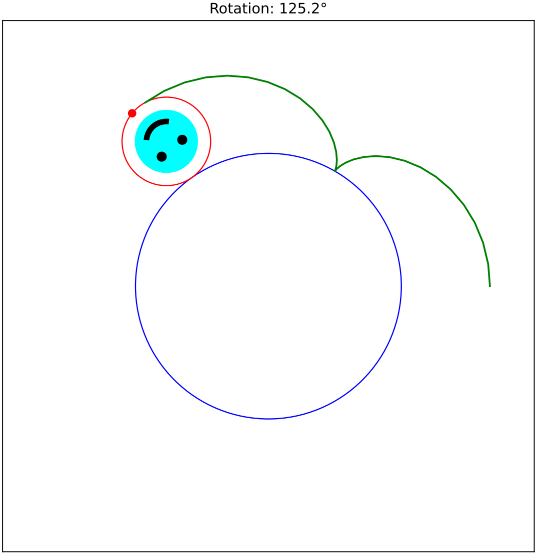
Animation of Coin Rotation Paradox



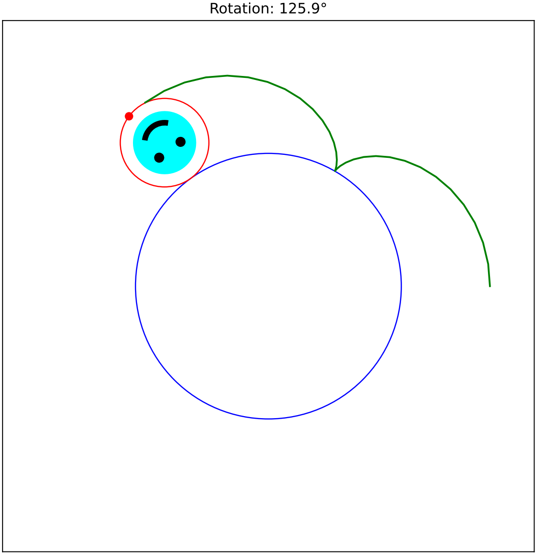
Animation of Coin Rotation Paradox



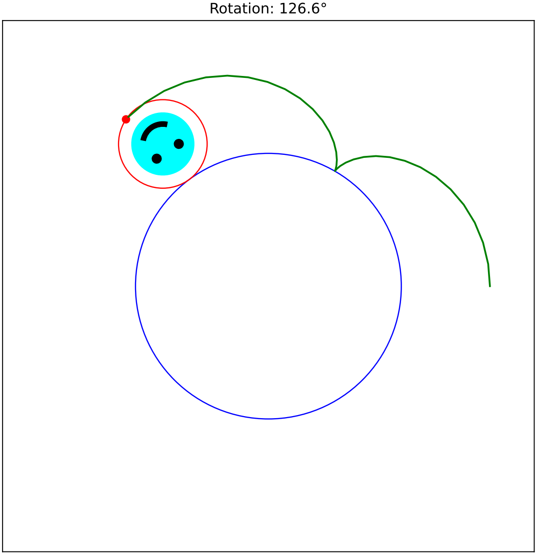
Animation of Coin Rotation Paradox



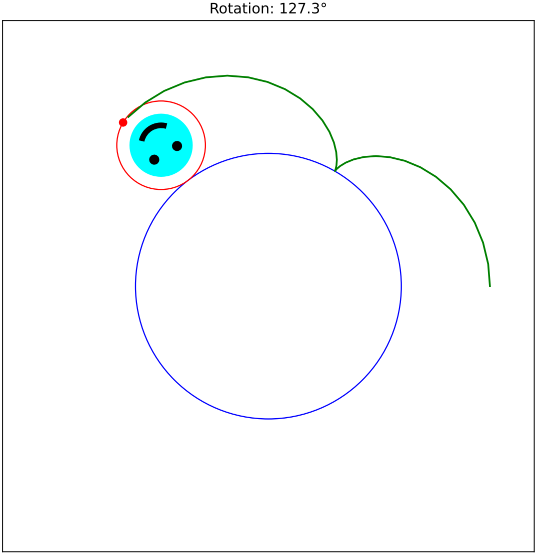
Animation of Coin Rotation Paradox



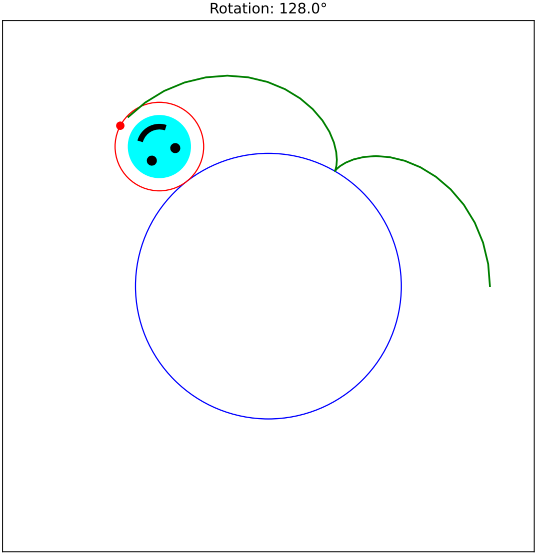
Animation of Coin Rotation Paradox



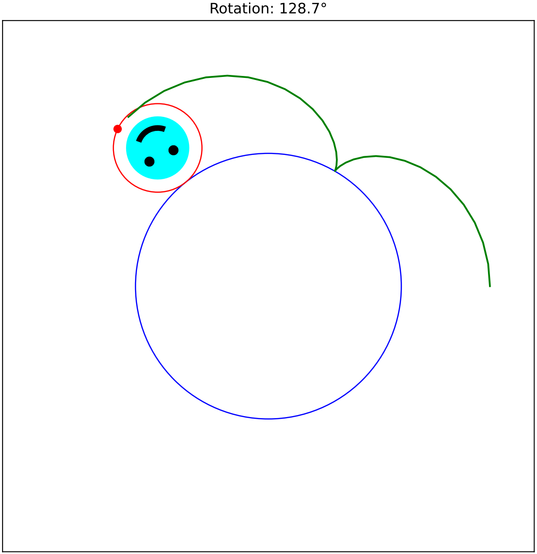
Animation of Coin Rotation Paradox



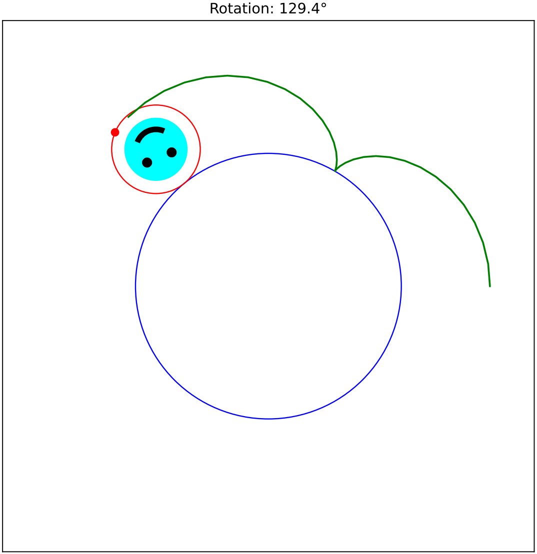
Animation of Coin Rotation Paradox



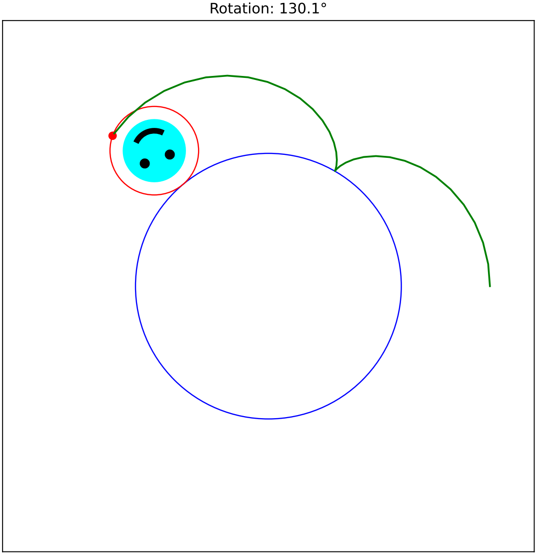
Animation of Coin Rotation Paradox



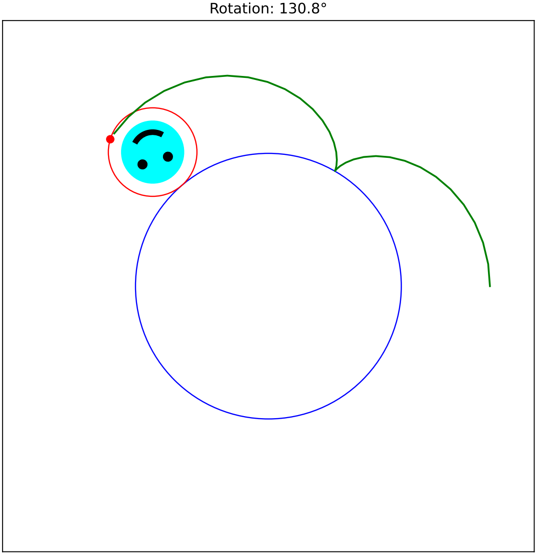
Animation of Coin Rotation Paradox



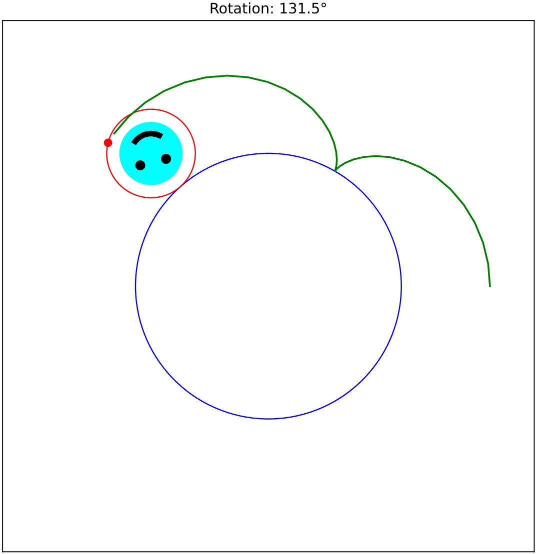
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

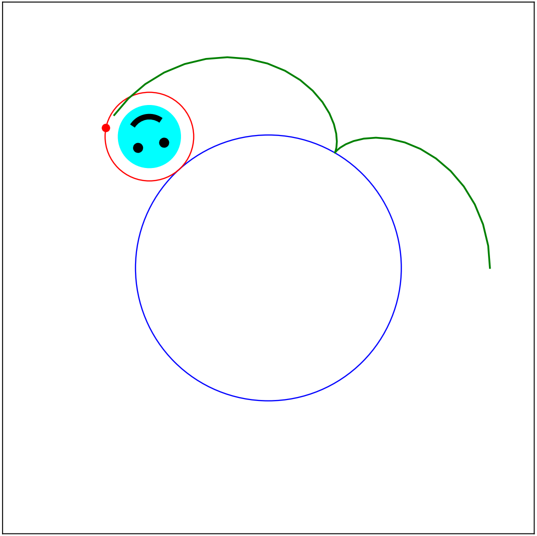


Animation of Coin Rotation Paradox

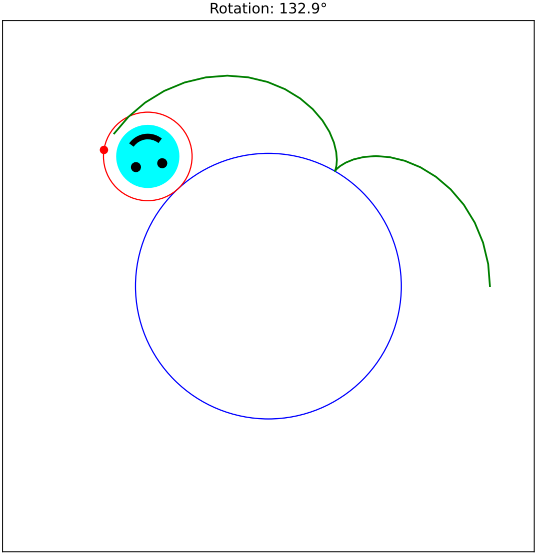


Animation of Coin Rotation Paradox

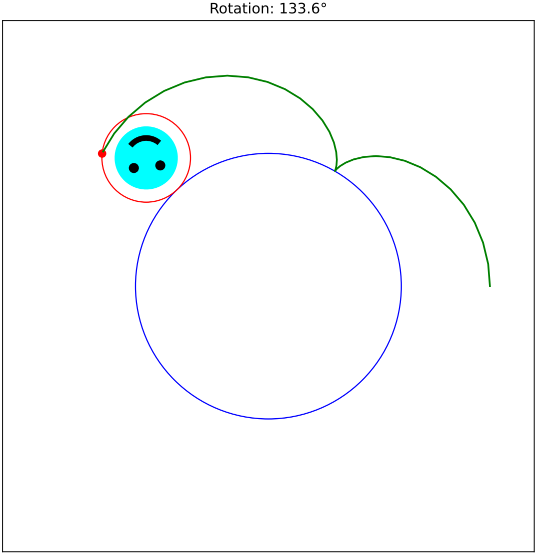
Rotation: 132.2°



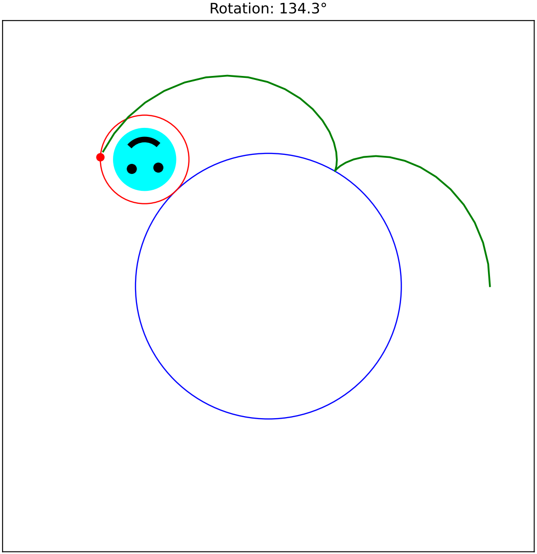
Animation of Coin Rotation Paradox



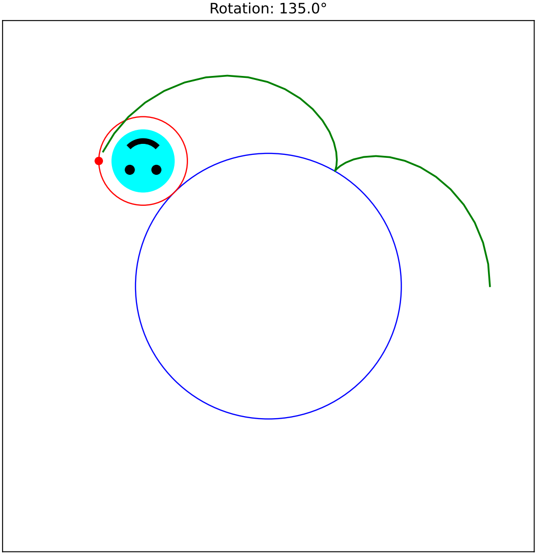
Animation of Coin Rotation Paradox



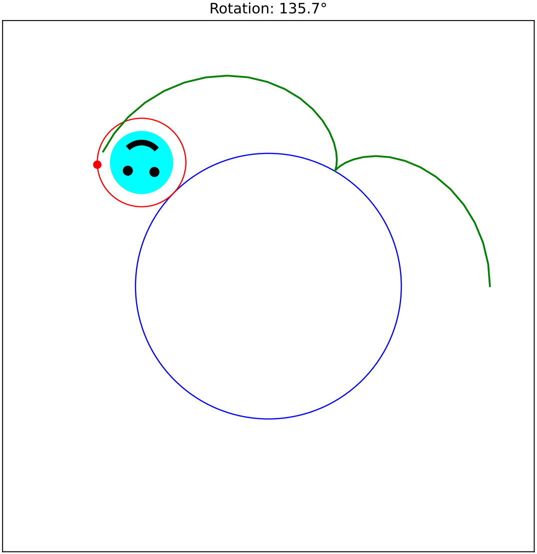
Animation of Coin Rotation Paradox



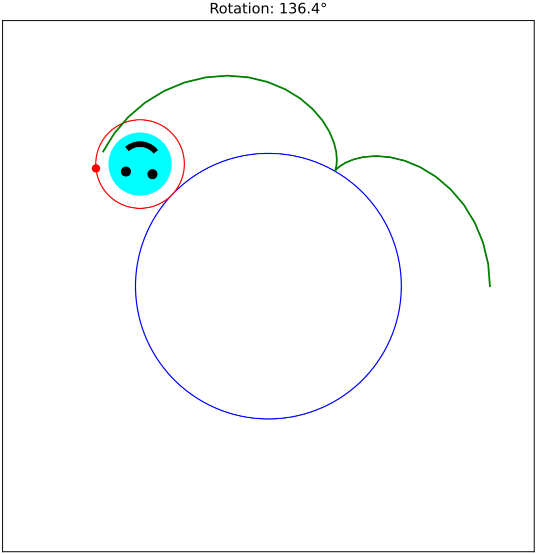
Animation of Coin Rotation Paradox



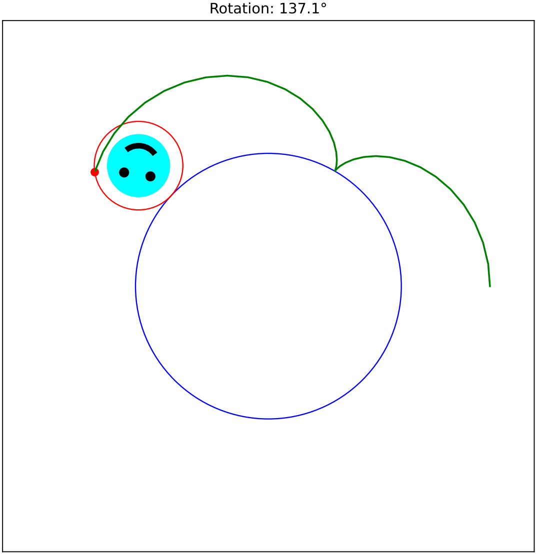
Animation of Coin Rotation Paradox



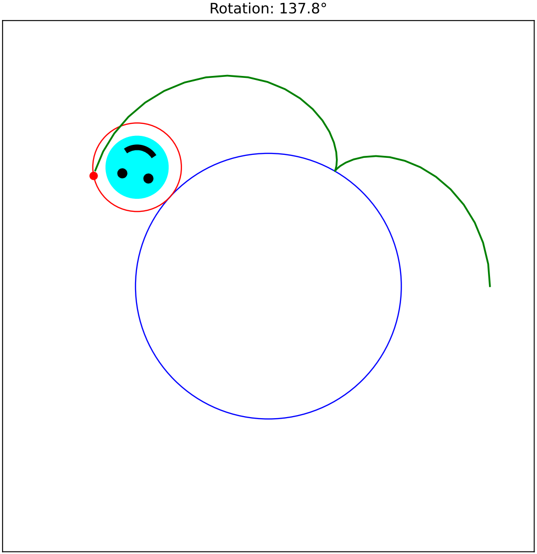
Animation of Coin Rotation Paradox



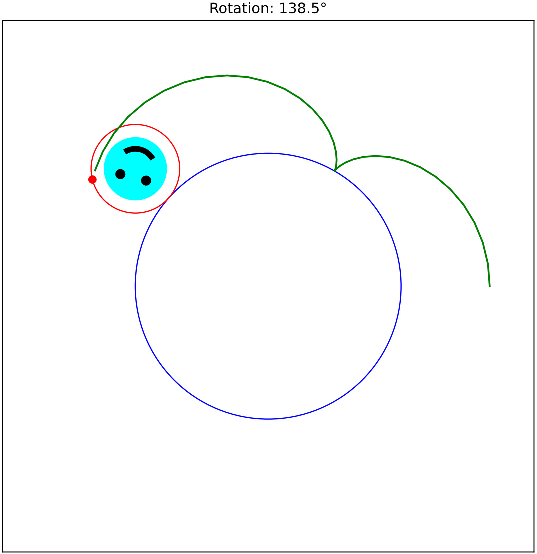
Animation of Coin Rotation Paradox



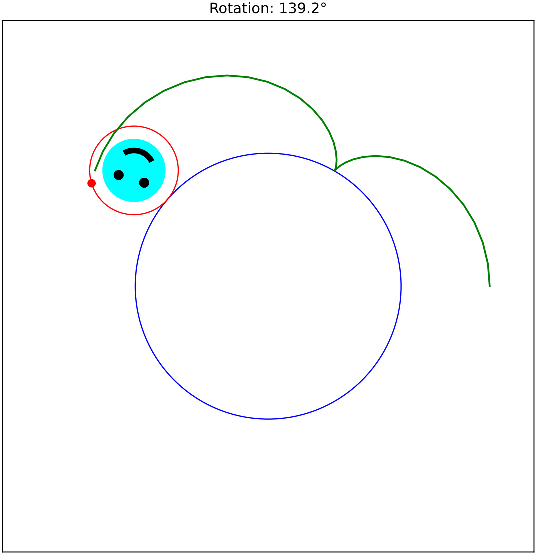
Animation of Coin Rotation Paradox



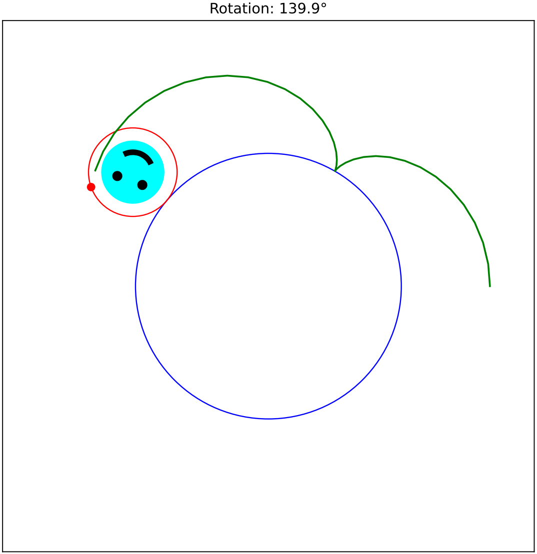
Animation of Coin Rotation Paradox



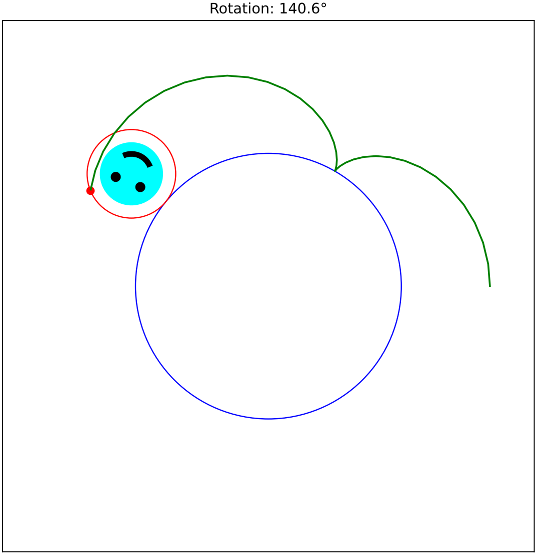
Animation of Coin Rotation Paradox



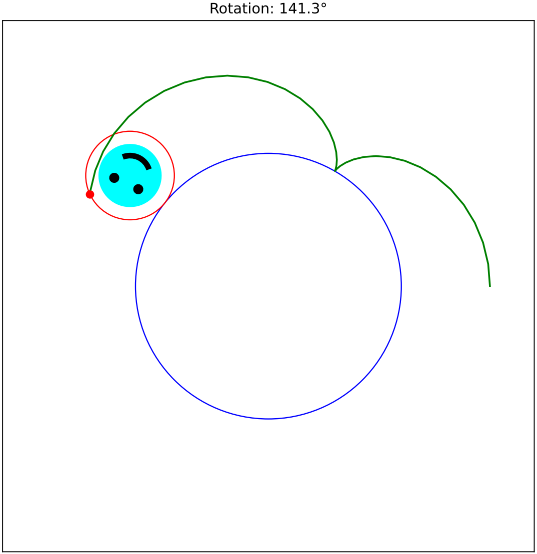
Animation of Coin Rotation Paradox



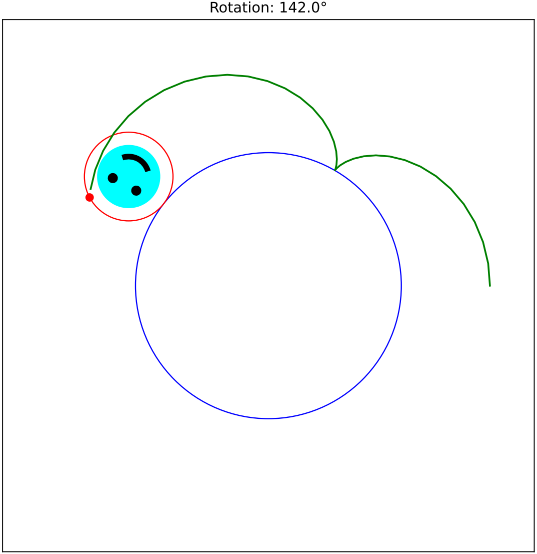
Animation of Coin Rotation Paradox



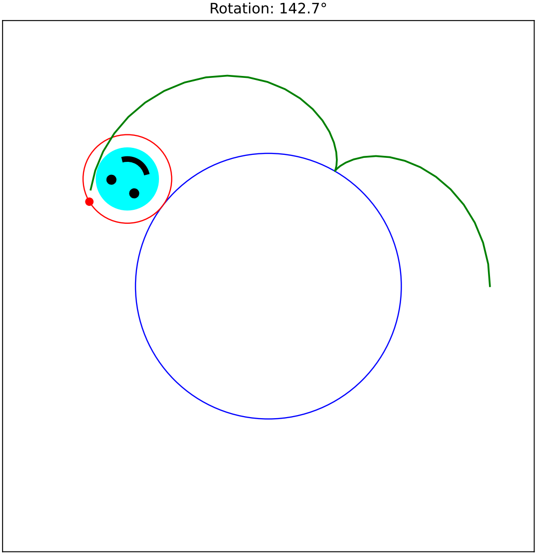
Animation of Coin Rotation Paradox



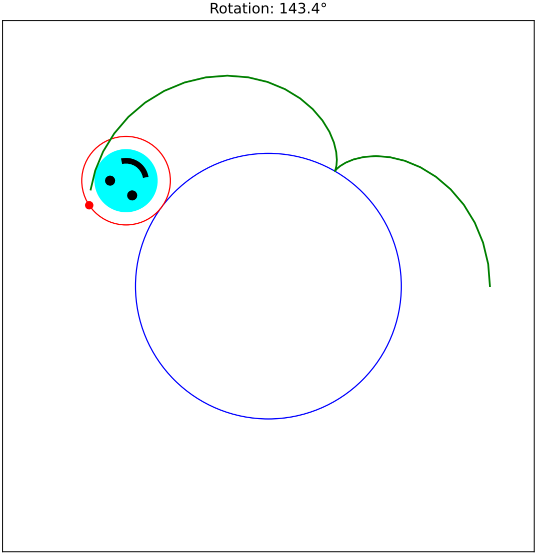
Animation of Coin Rotation Paradox



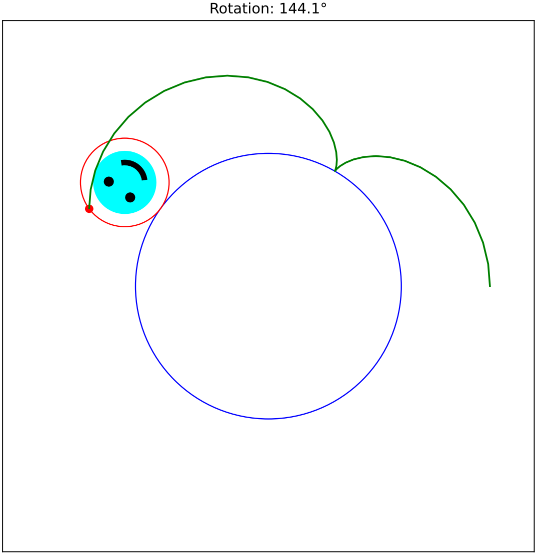
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

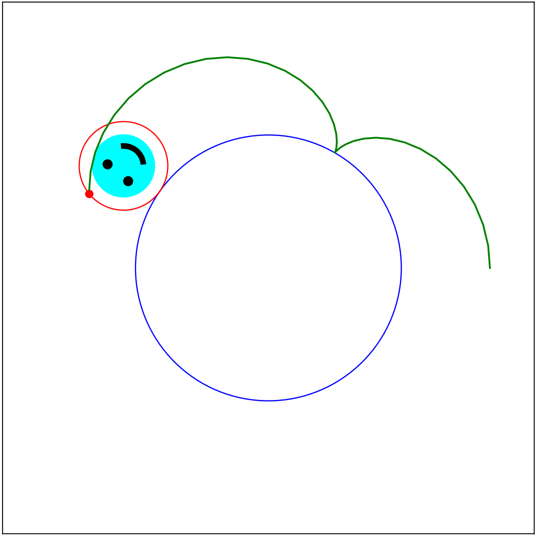


Animation of Coin Rotation Paradox

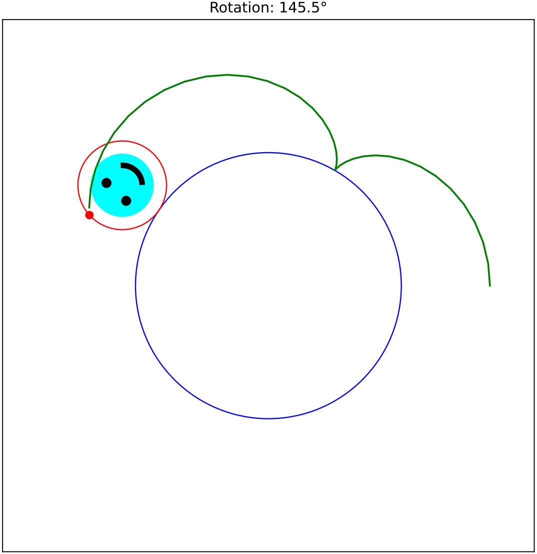


Animation of Coin Rotation Paradox

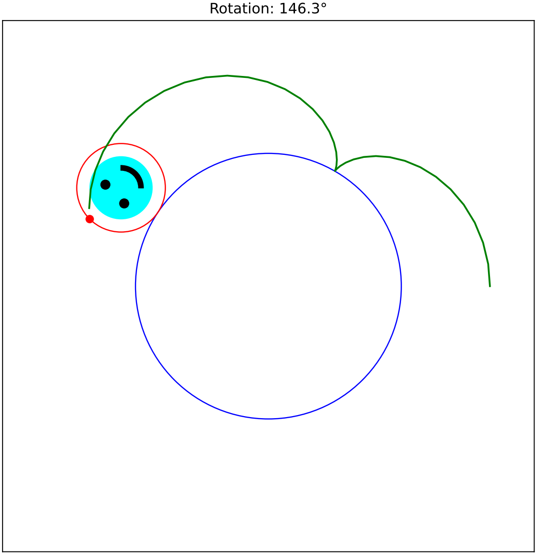
Rotation: 144.8°



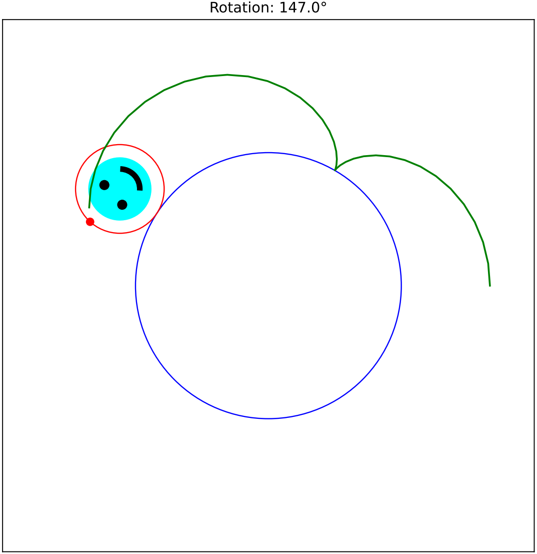
Animation of Coin Rotation Paradox



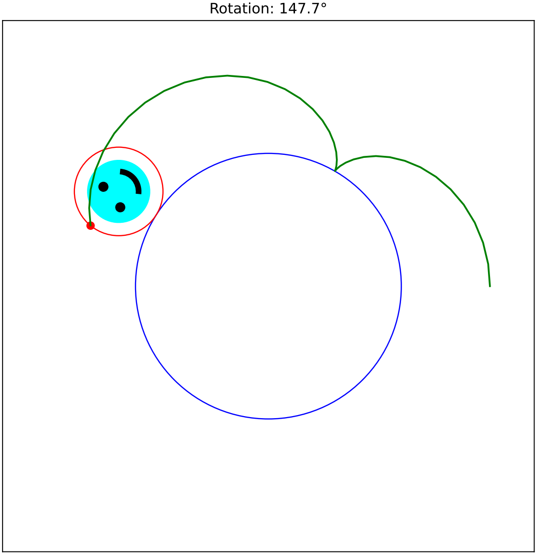
Animation of Coin Rotation Paradox



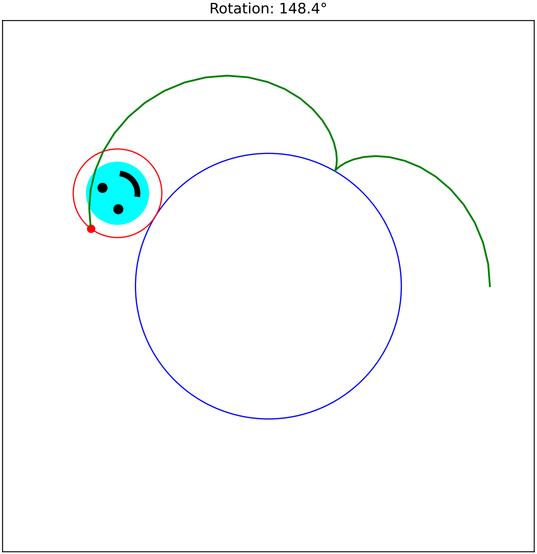
Animation of Coin Rotation Paradox



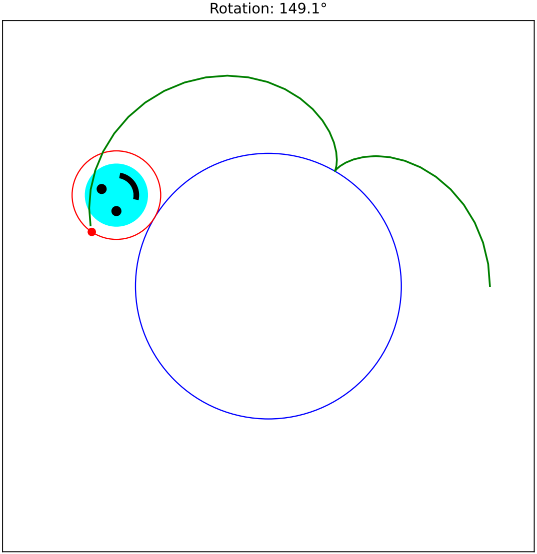
Animation of Coin Rotation Paradox



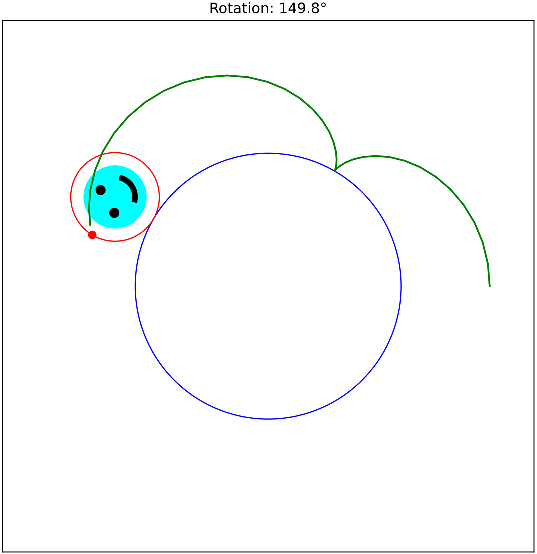
Animation of Coin Rotation Paradox



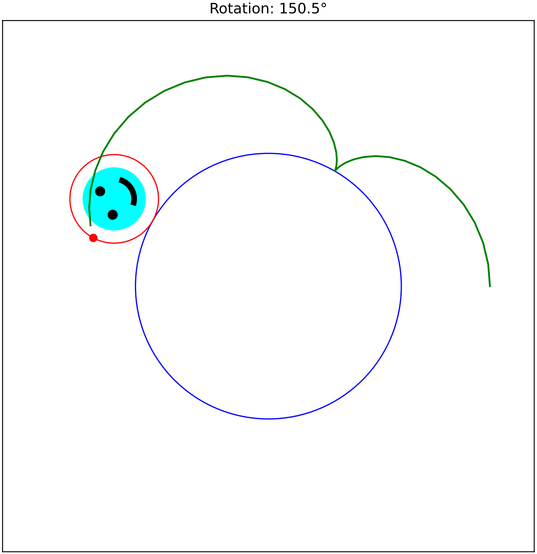
Animation of Coin Rotation Paradox



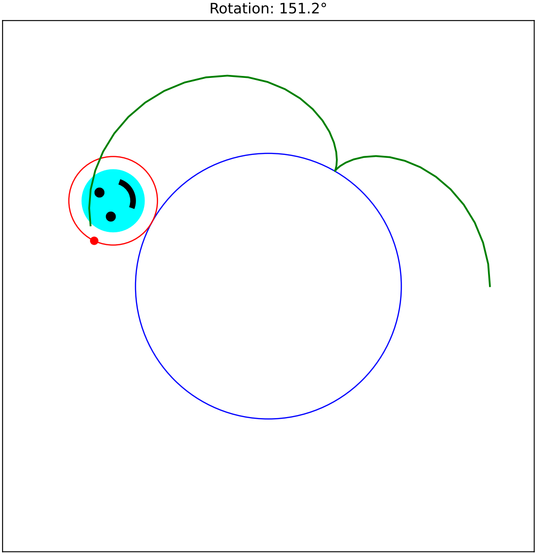
Animation of Coin Rotation Paradox



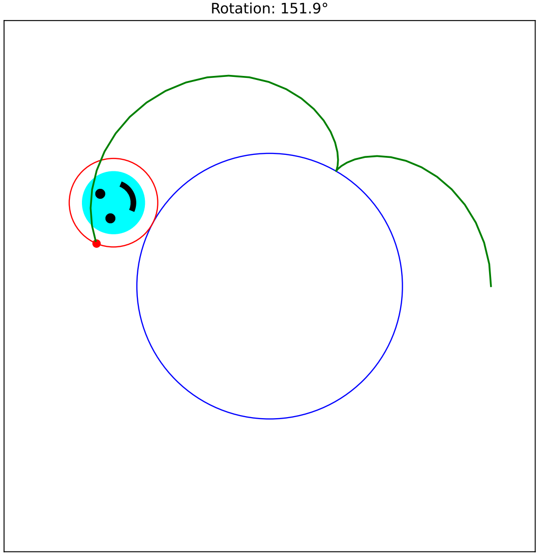
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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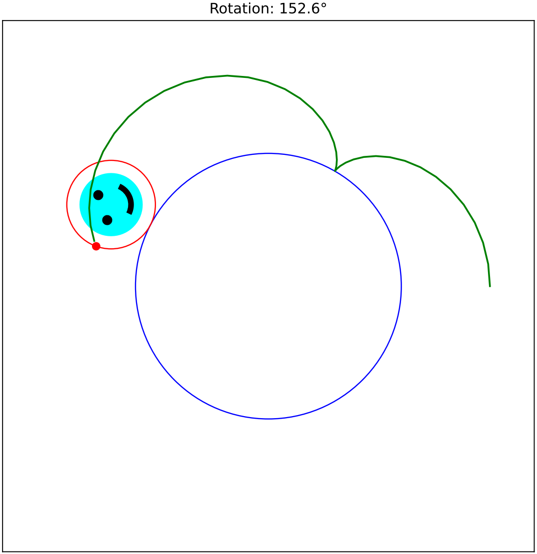
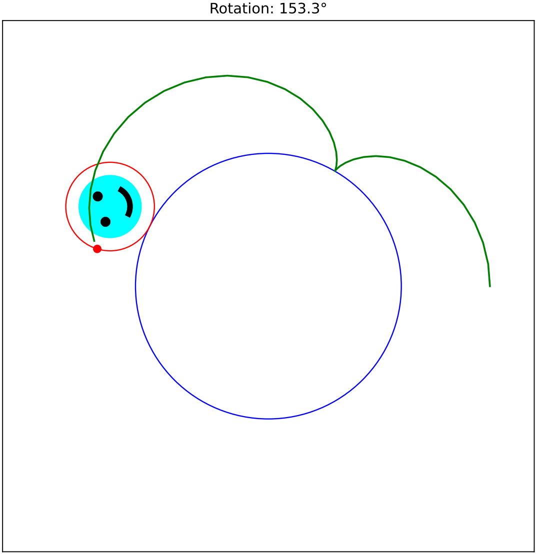
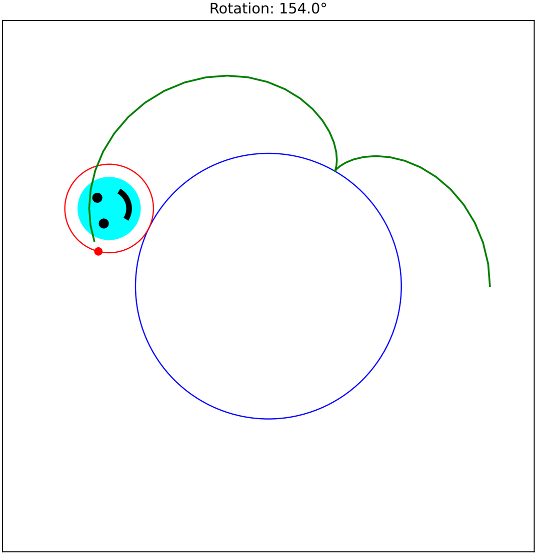


Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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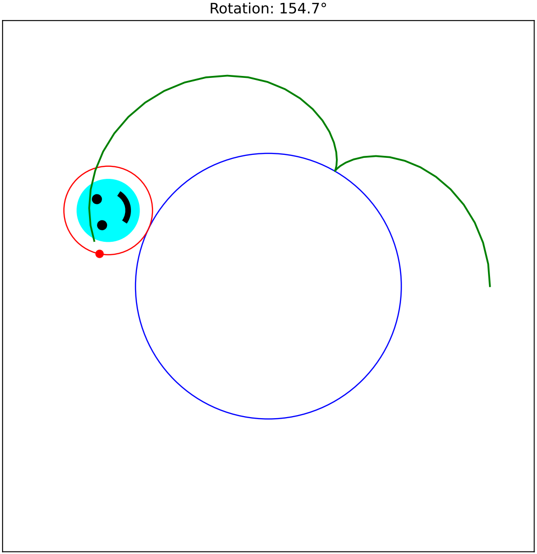

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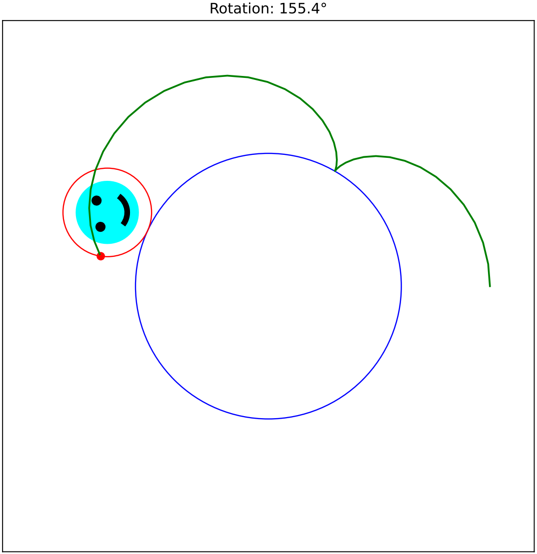
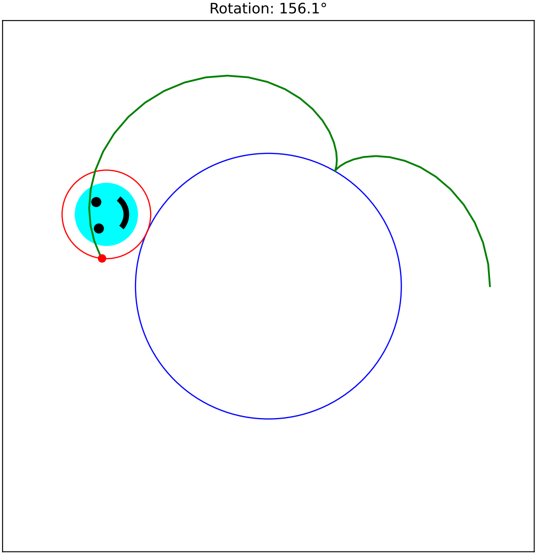
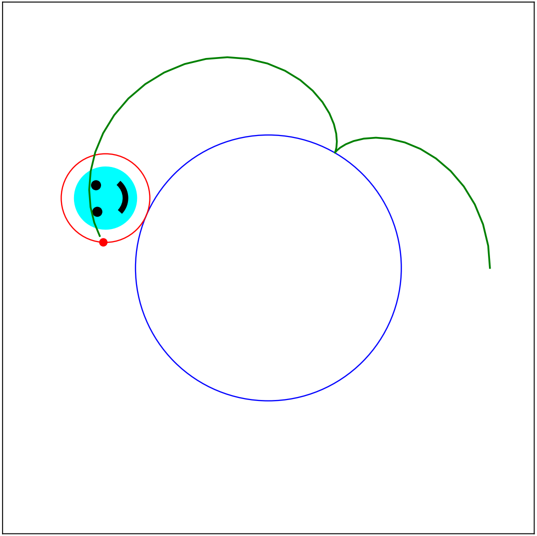


Animation of Coin Rotation Paradox

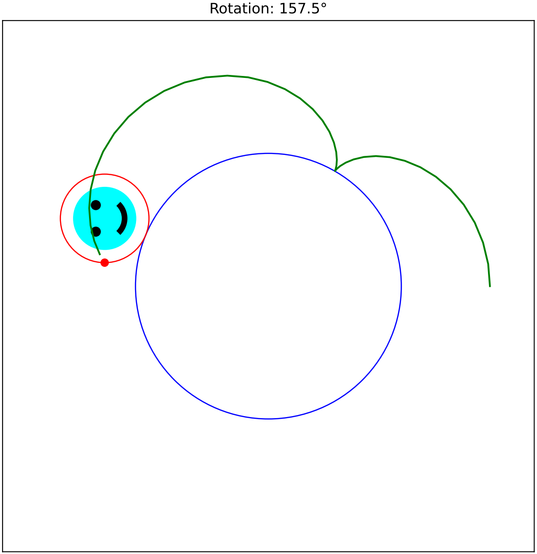


Animation of Coin Rotation Paradox

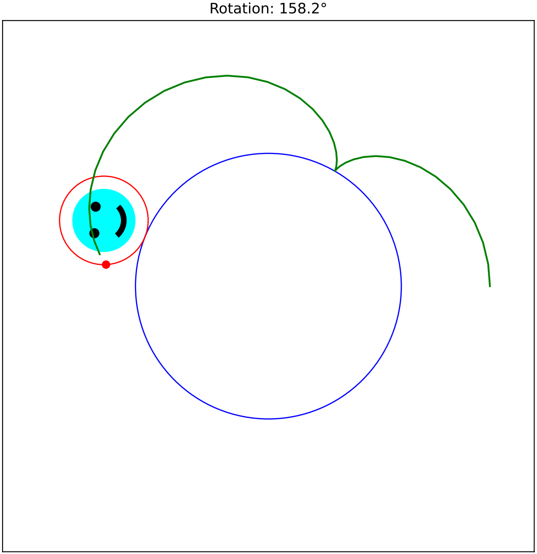
Rotation: 156.8°



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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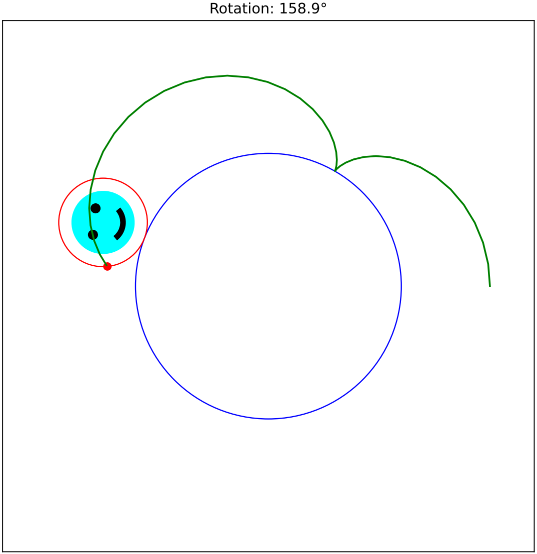

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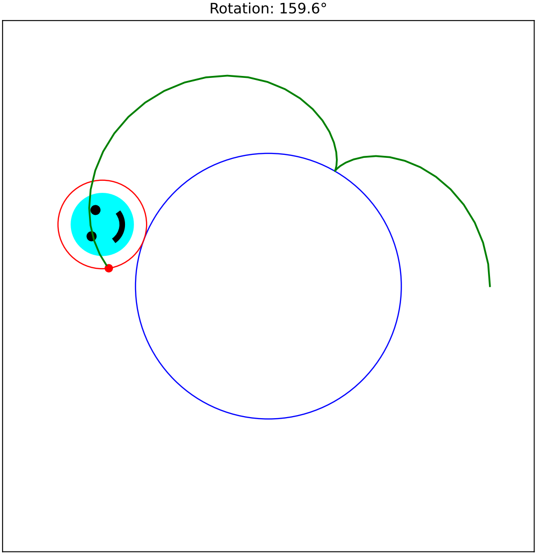
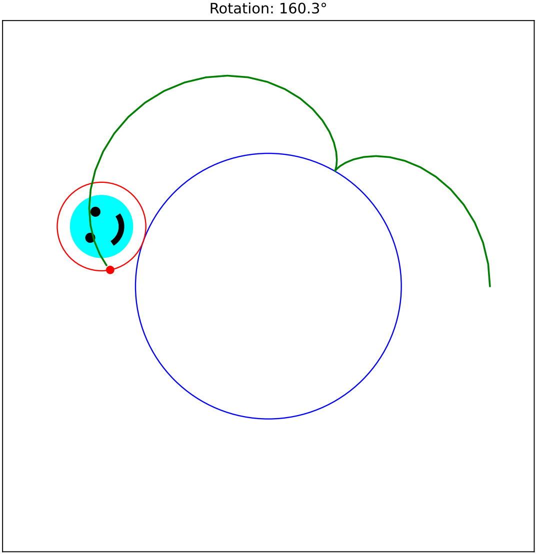
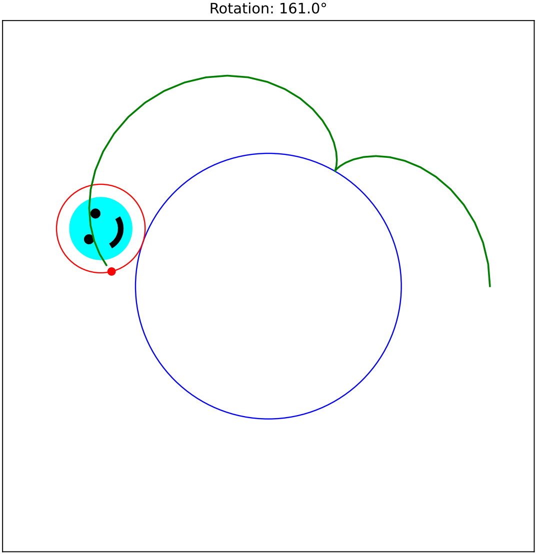


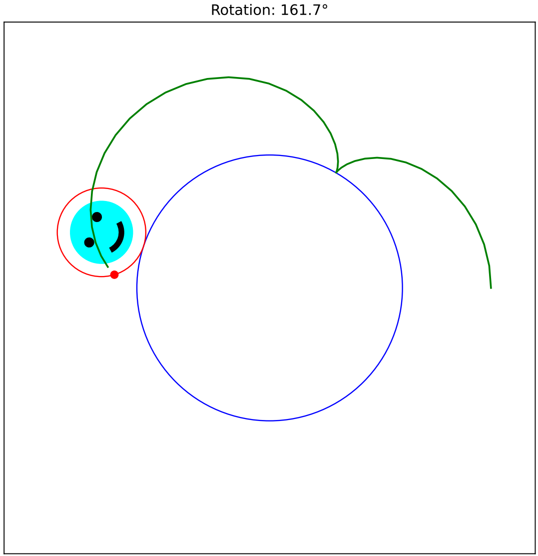
Animation of Coin Rotation Paradox



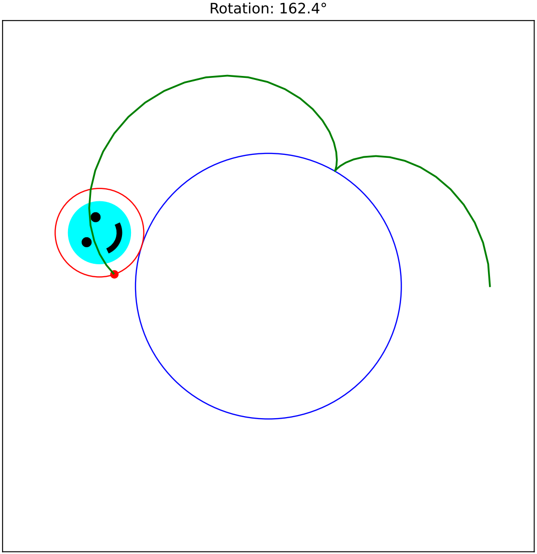
Animation of Coin Rotation Paradox



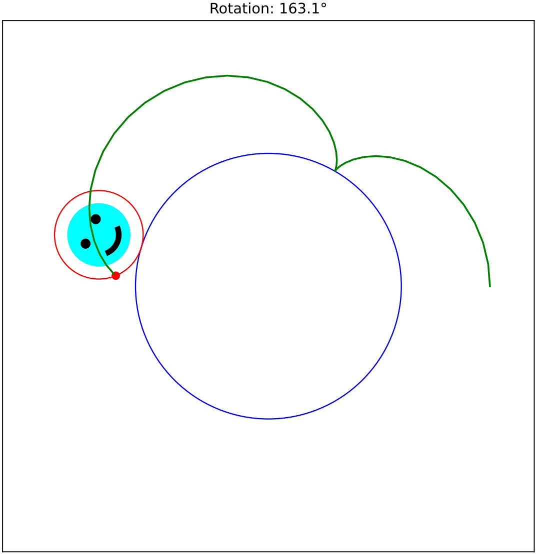
Animation of Coin Rotation Paradox



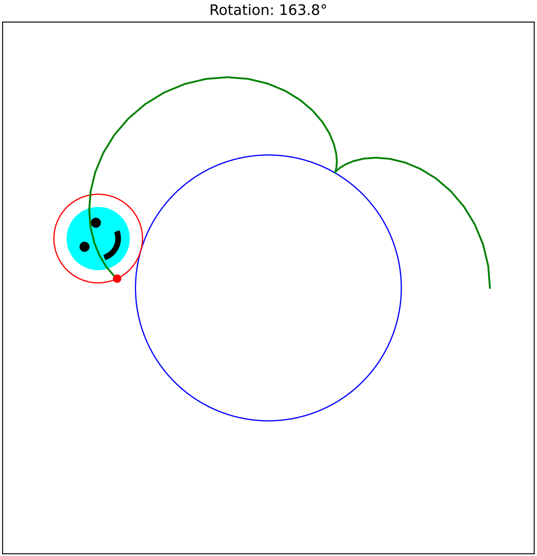
Animation of Coin Rotation Paradox



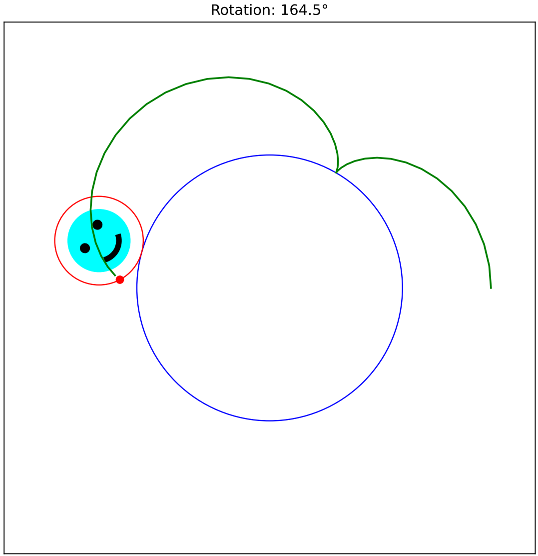
Animation of Coin Rotation Paradox



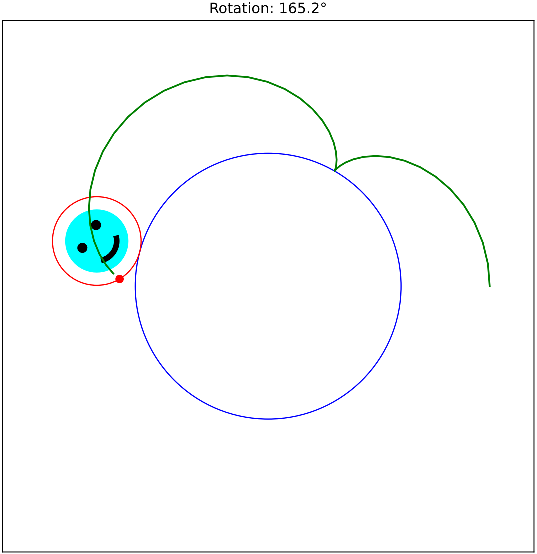
Animation of Coin Rotation Paradox



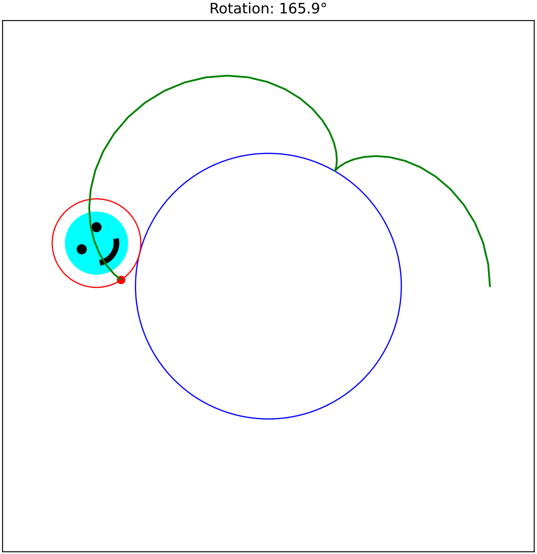
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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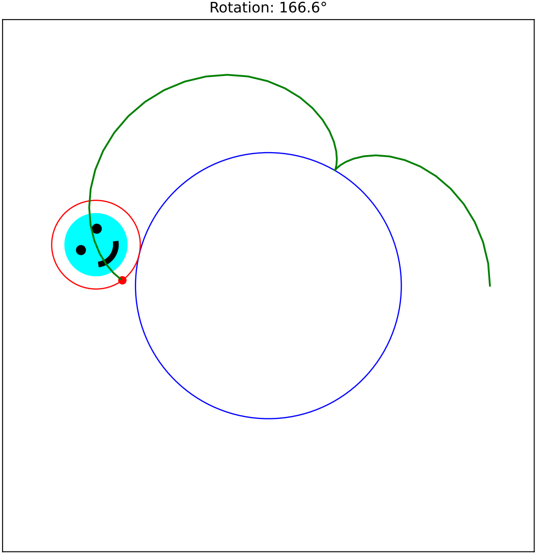

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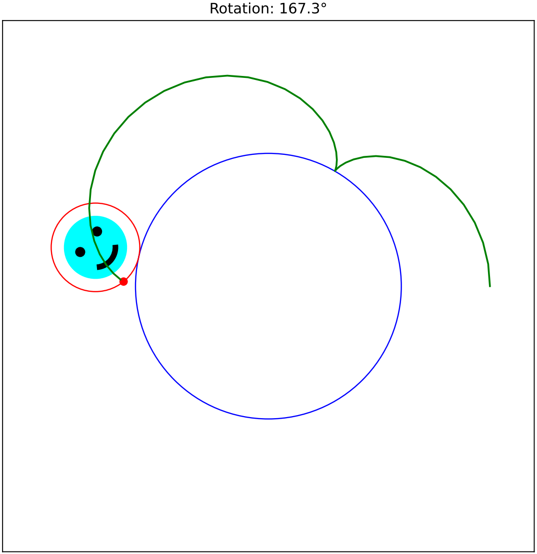

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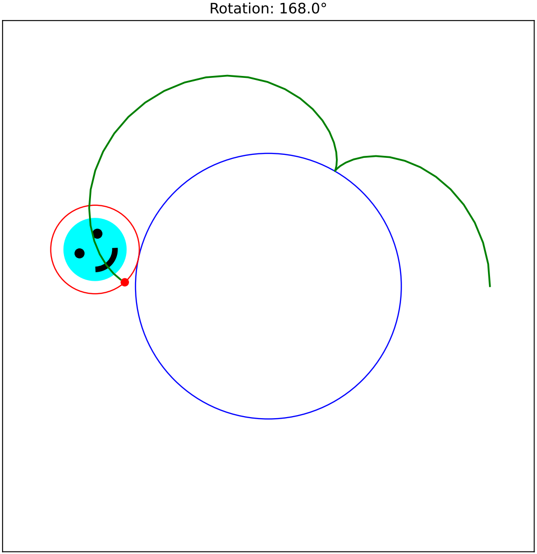
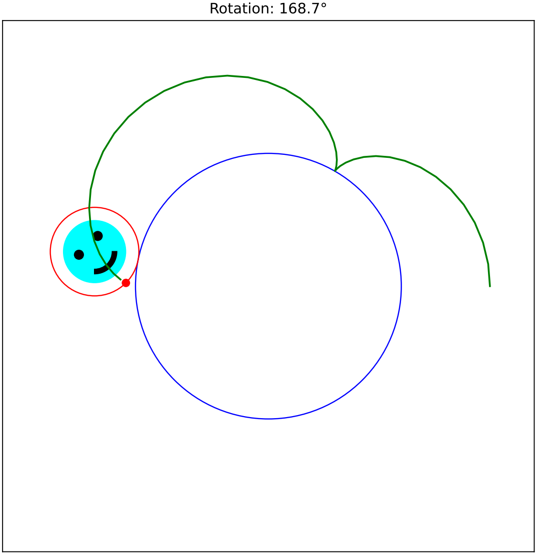
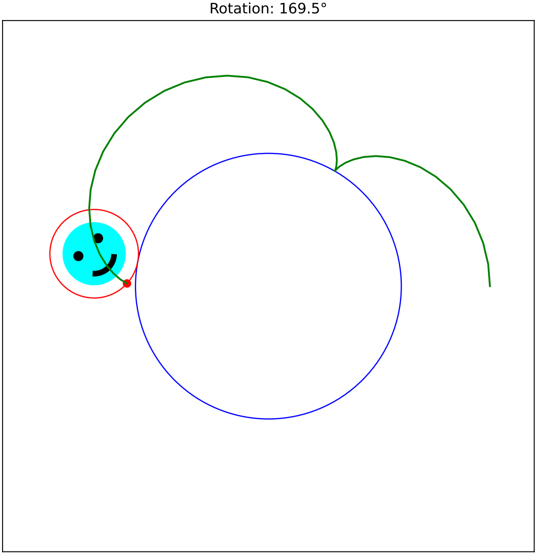


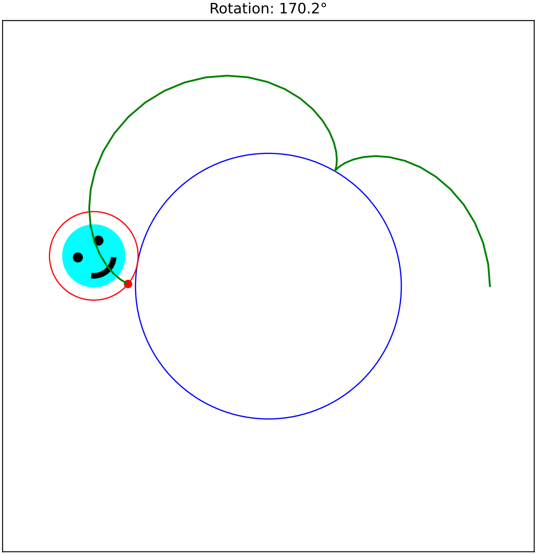
Animation of Coin Rotation Paradox



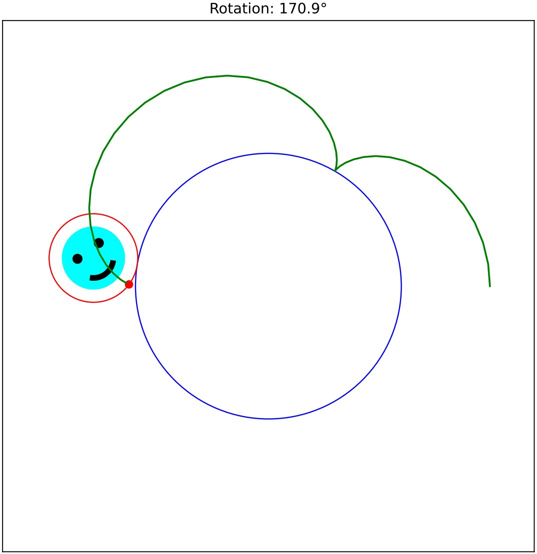
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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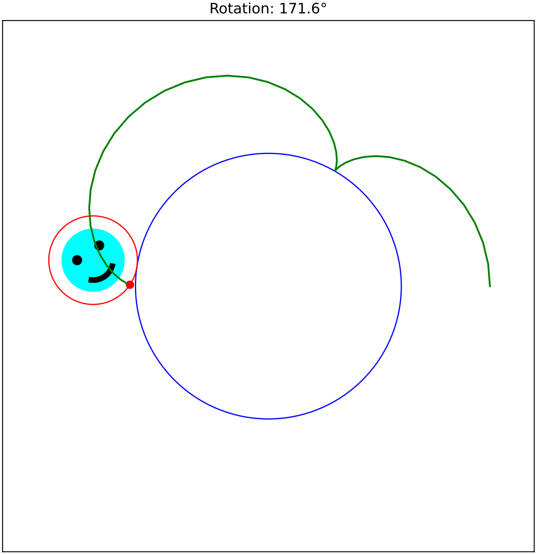
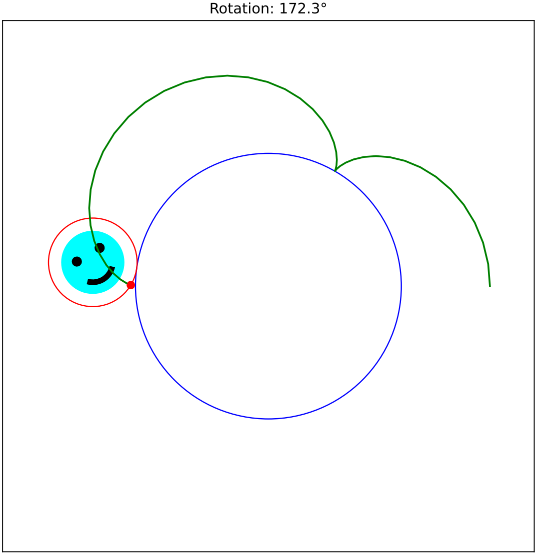


Animation of Coin Rotation Paradox



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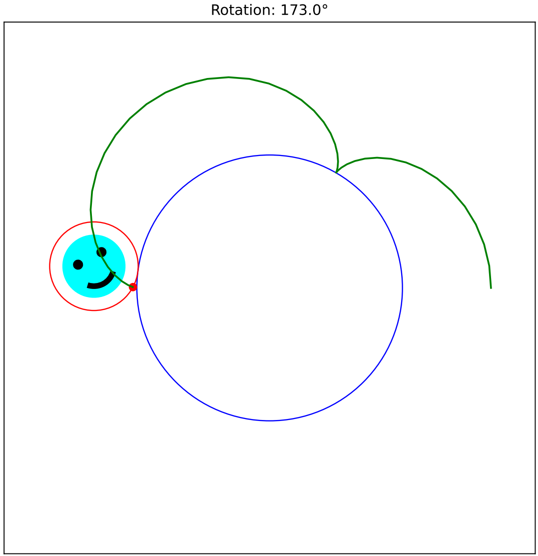

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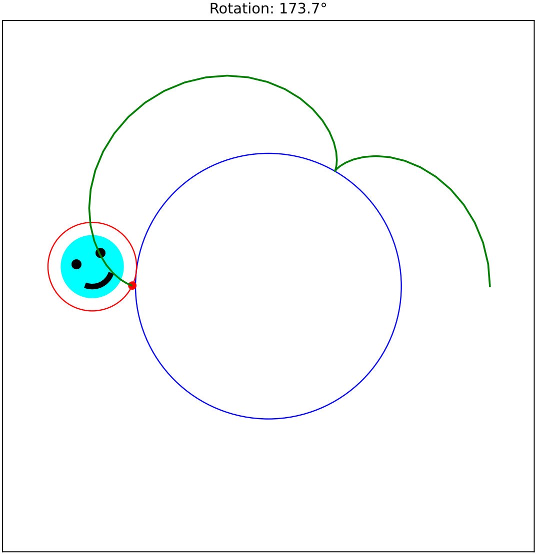

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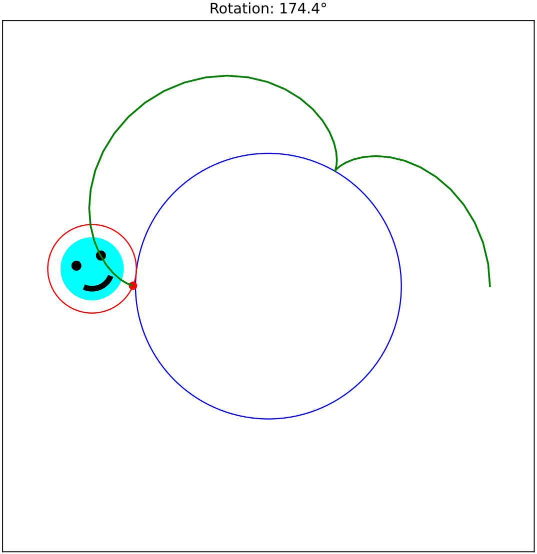

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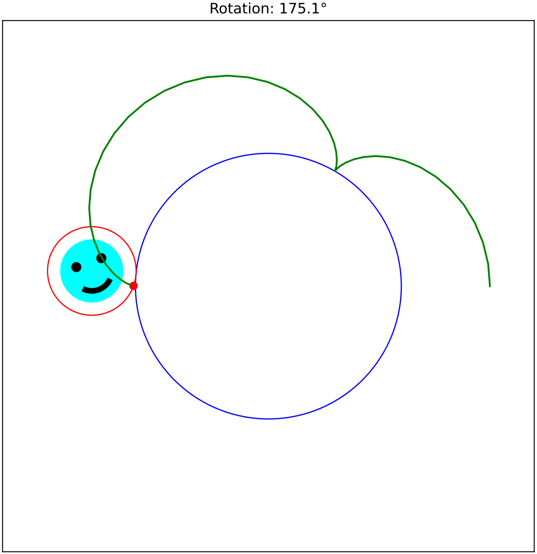

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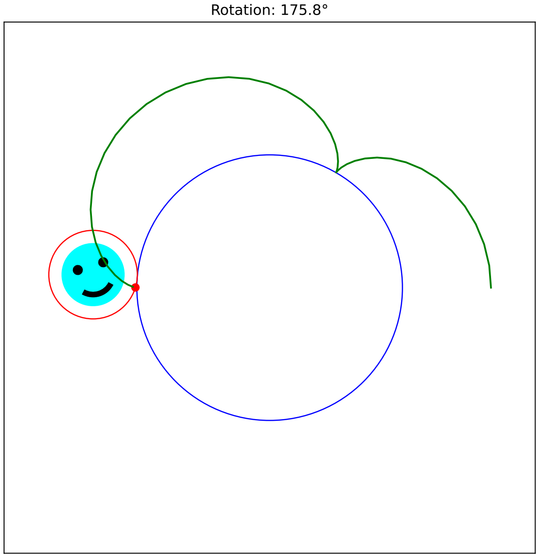
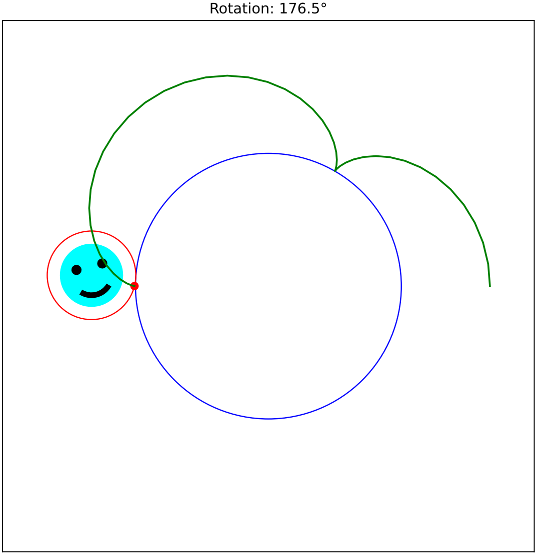
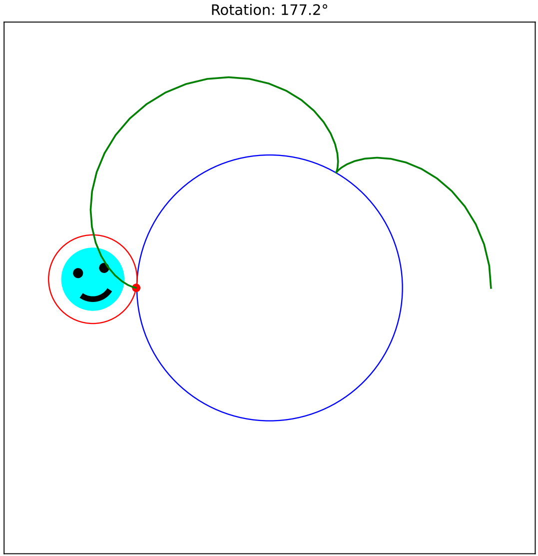


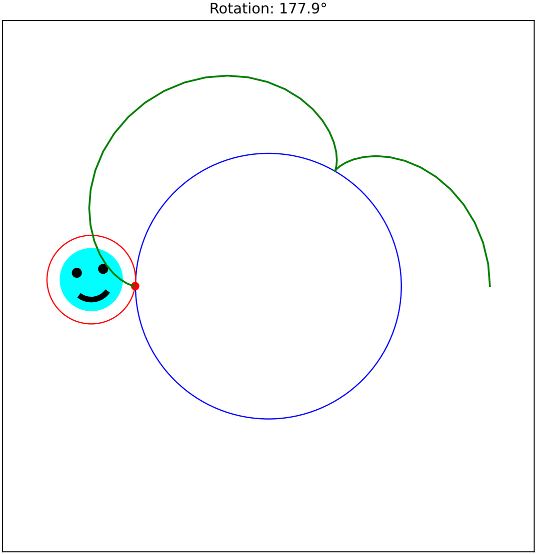
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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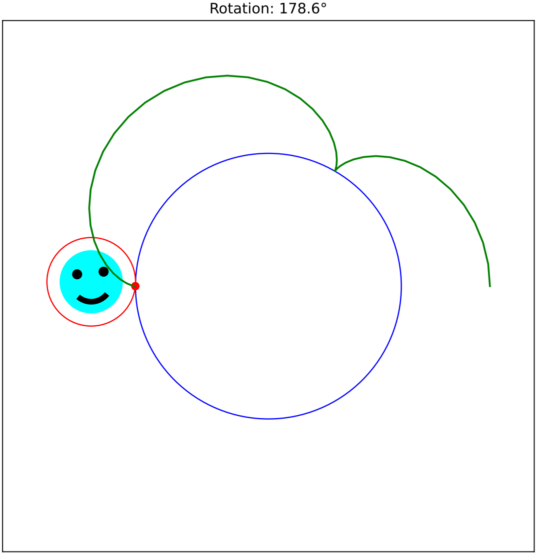
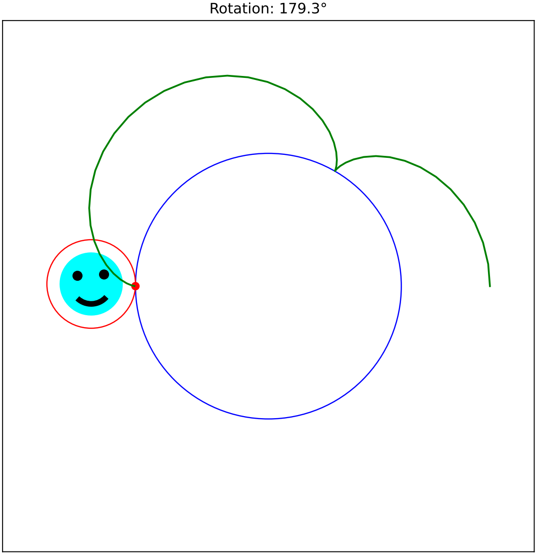
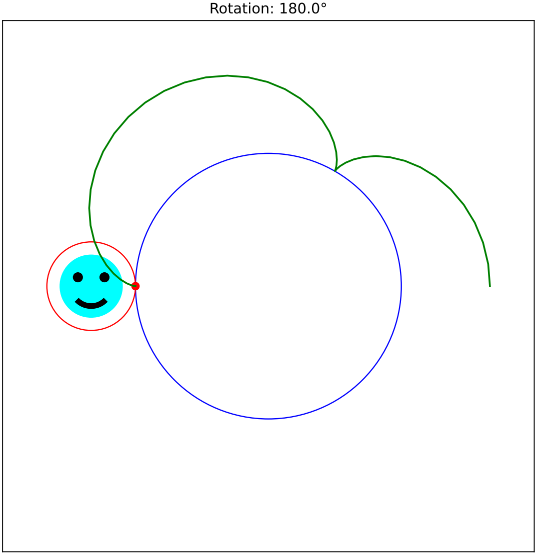


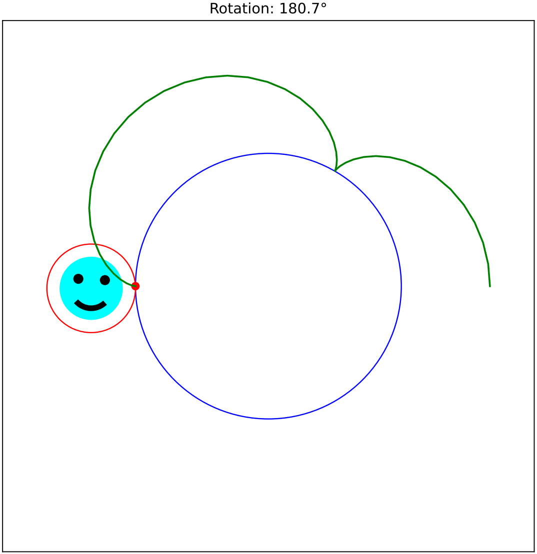
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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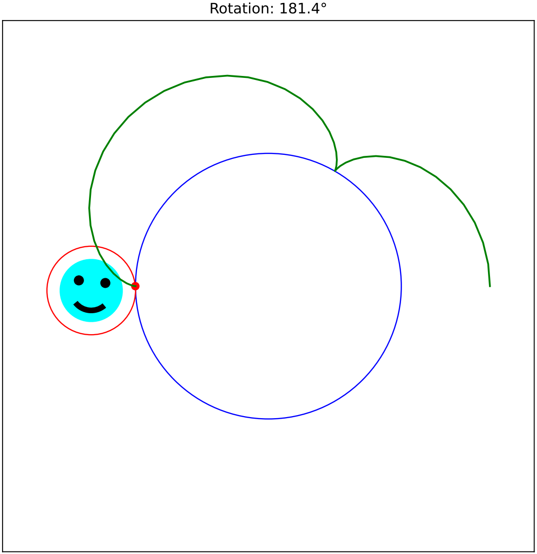
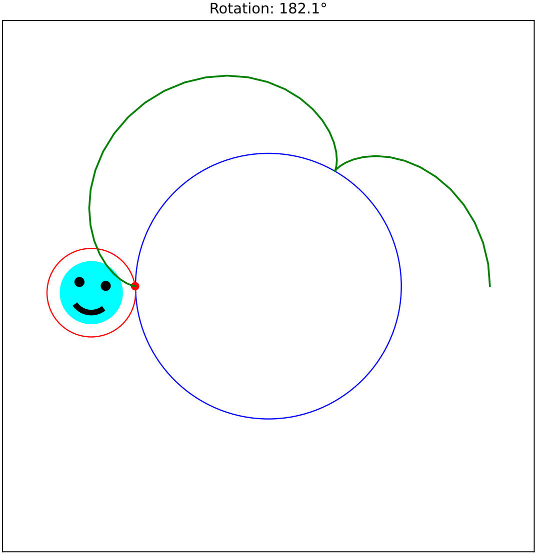
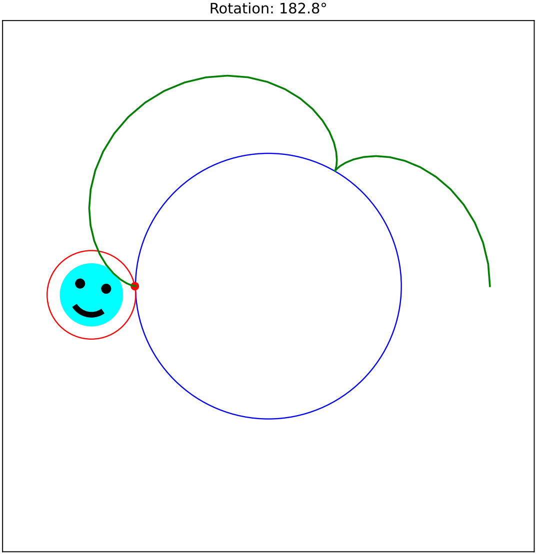


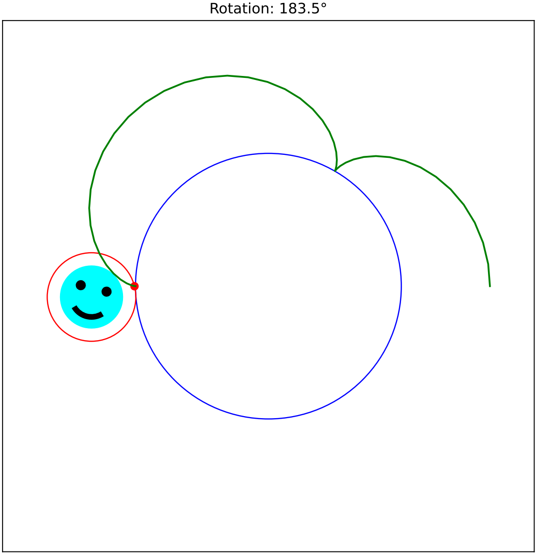
Animation of Coin Rotation Paradox



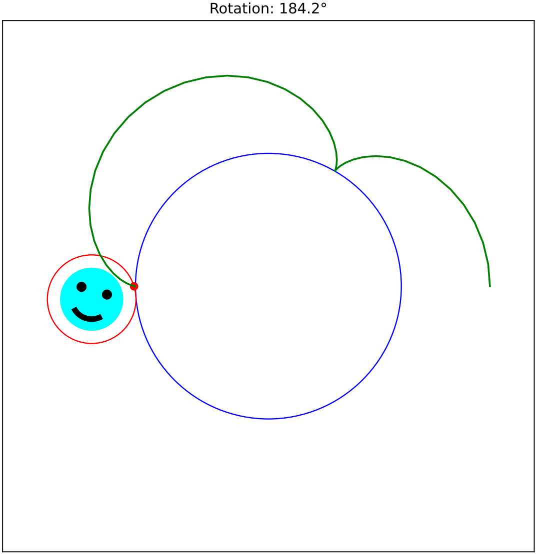
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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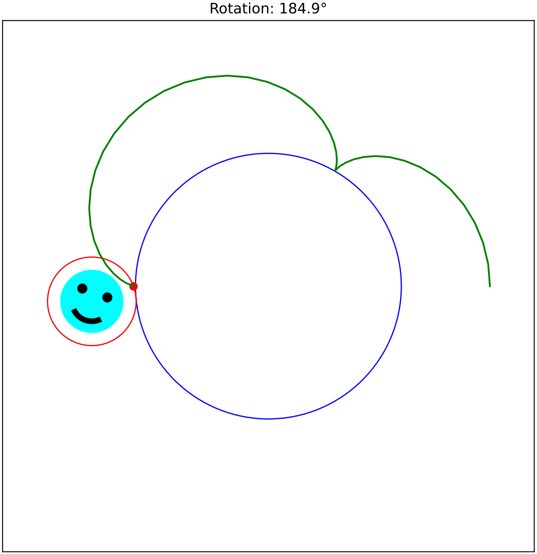

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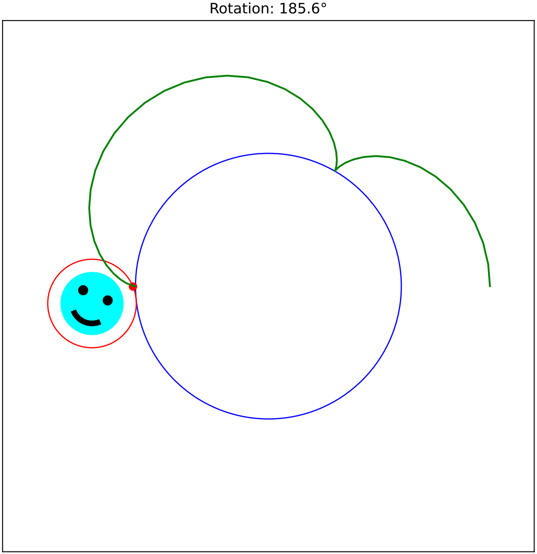

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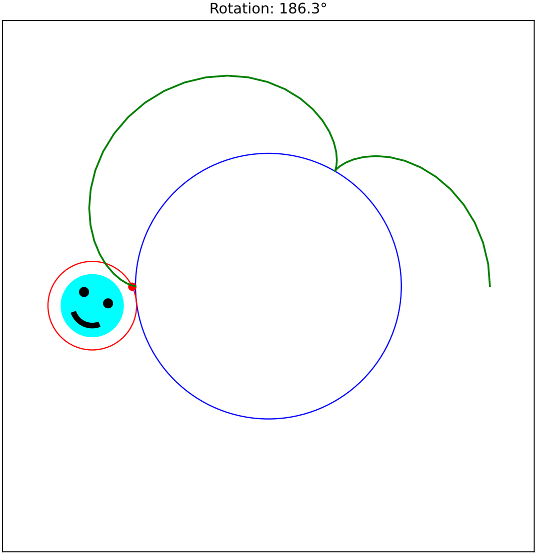

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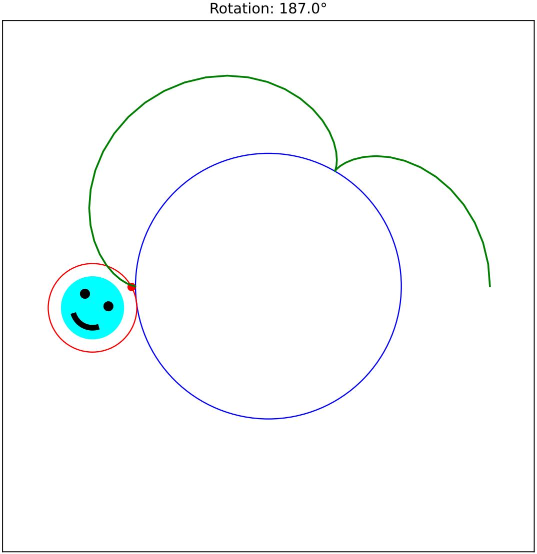

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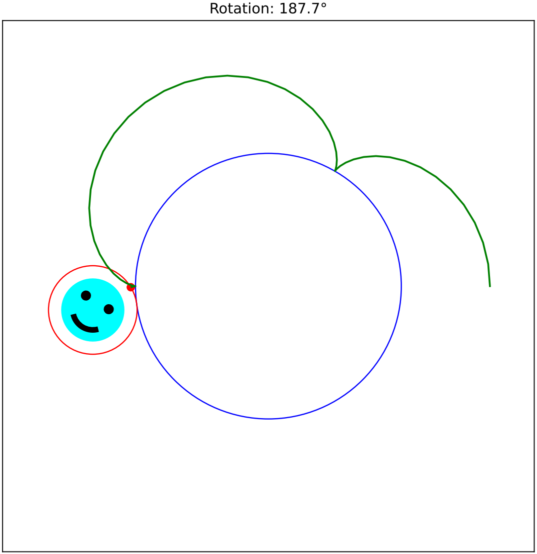

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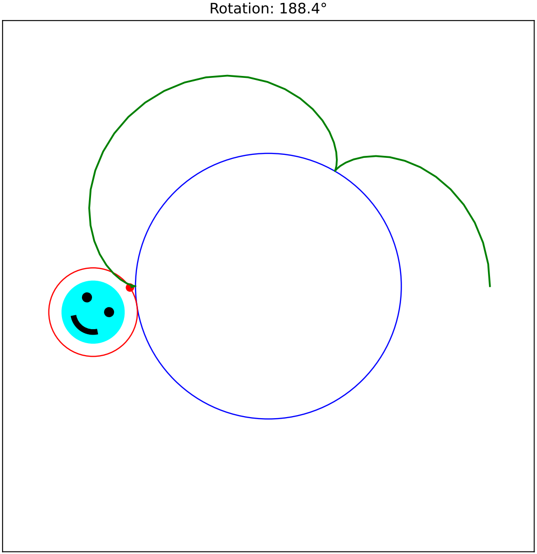

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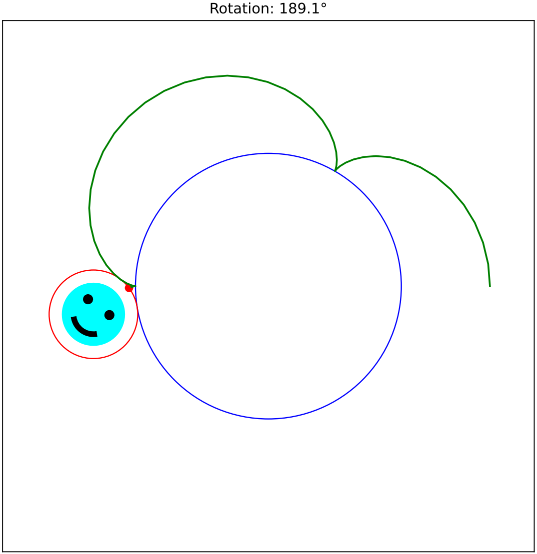

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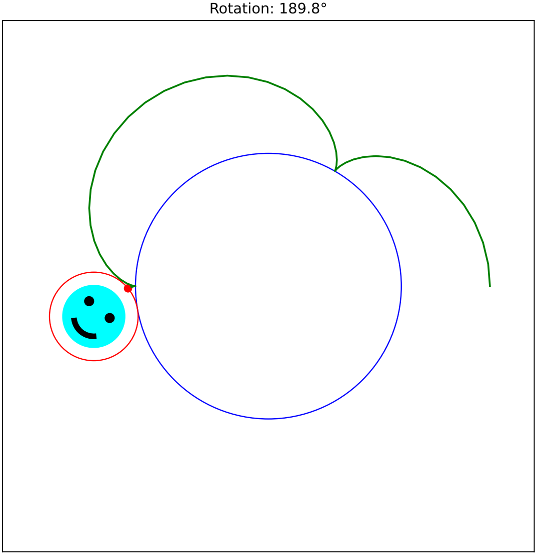
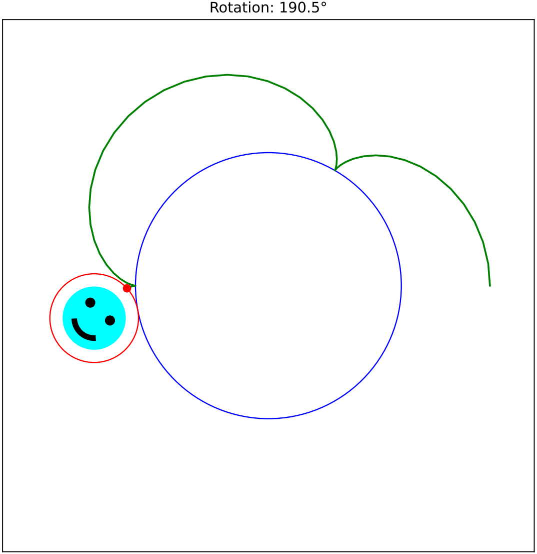
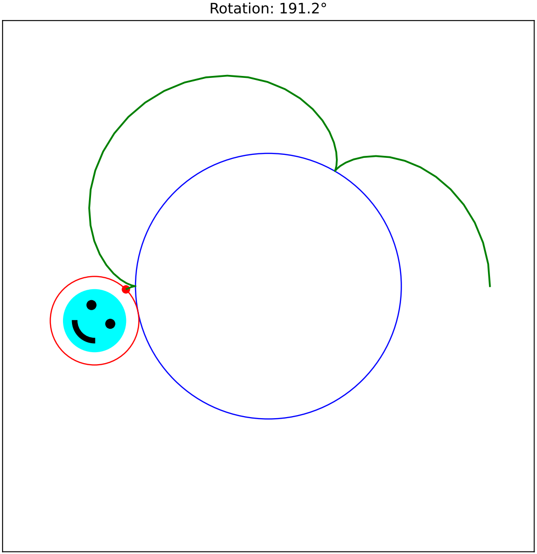


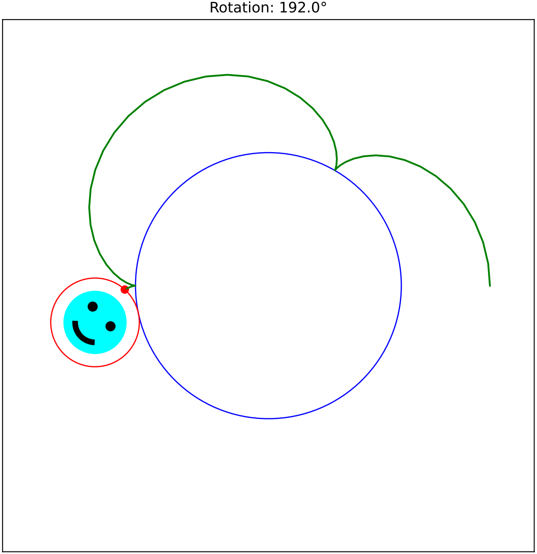
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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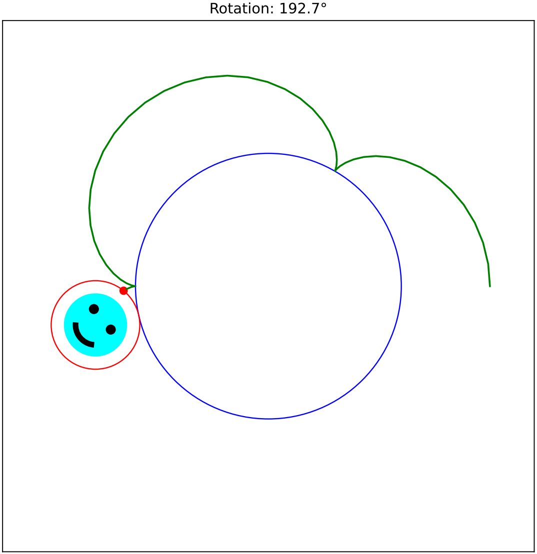
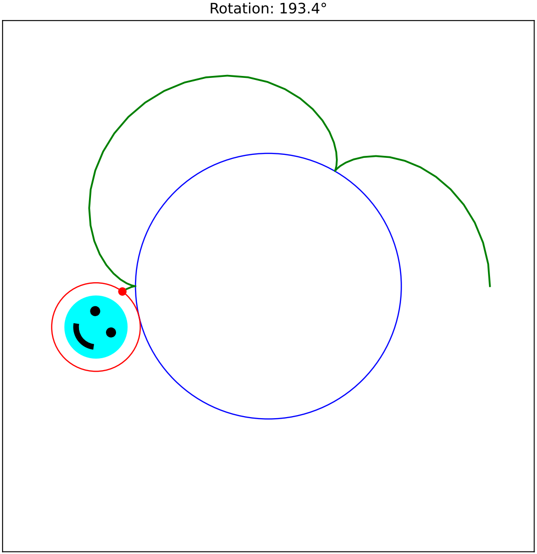
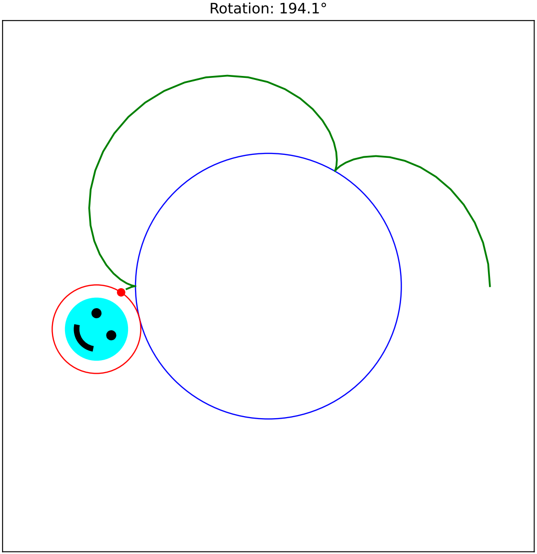


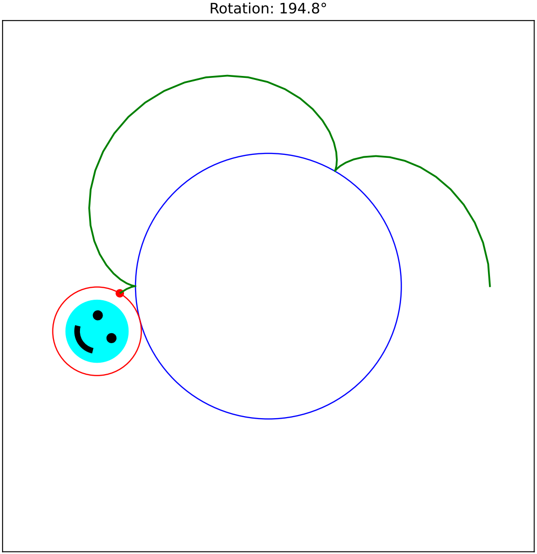
Animation of Coin Rotation Paradox



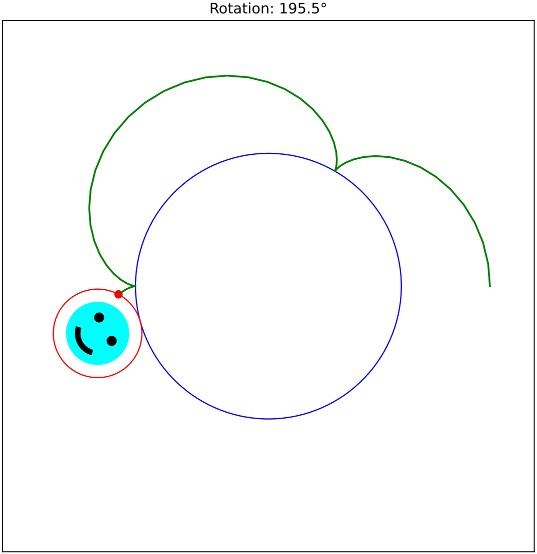
Animation of Coin Rotation Paradox



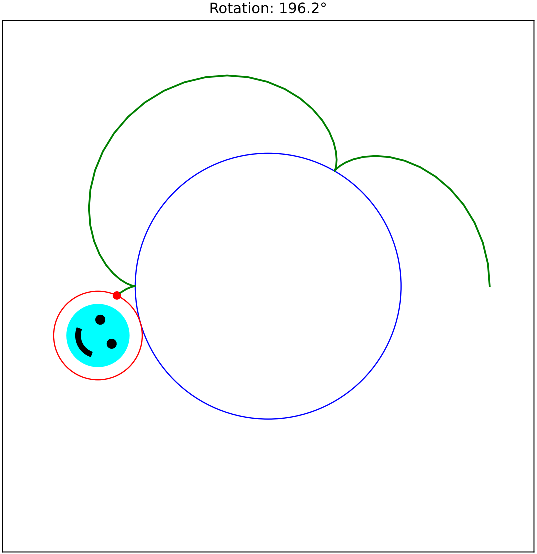
Animation of Coin Rotation Paradox



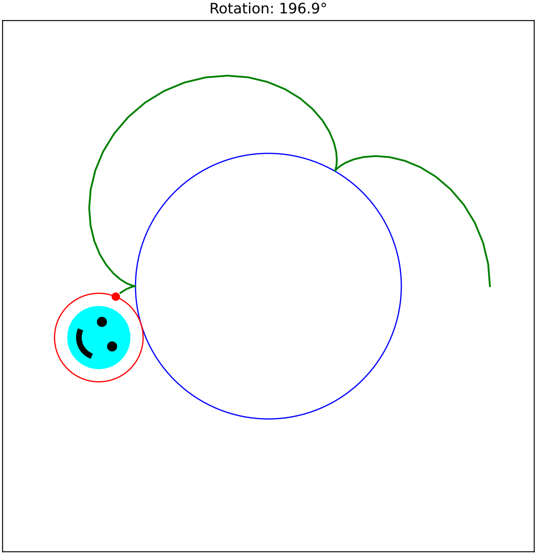
Animation of Coin Rotation Paradox



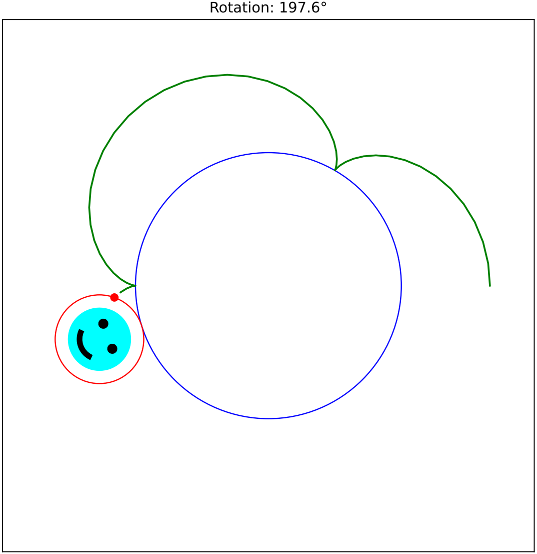
Animation of Coin Rotation Paradox



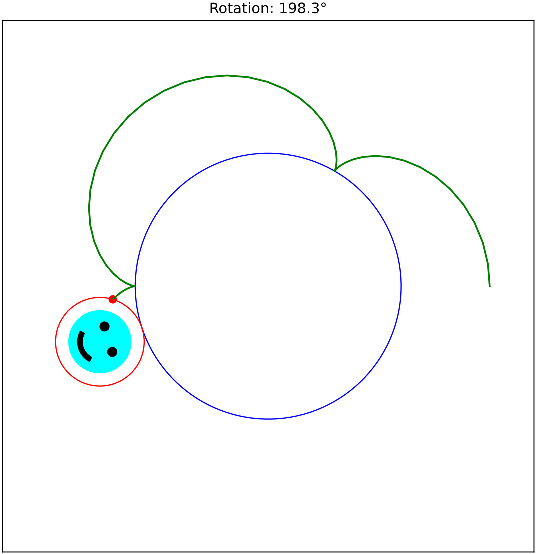
Animation of Coin Rotation Paradox



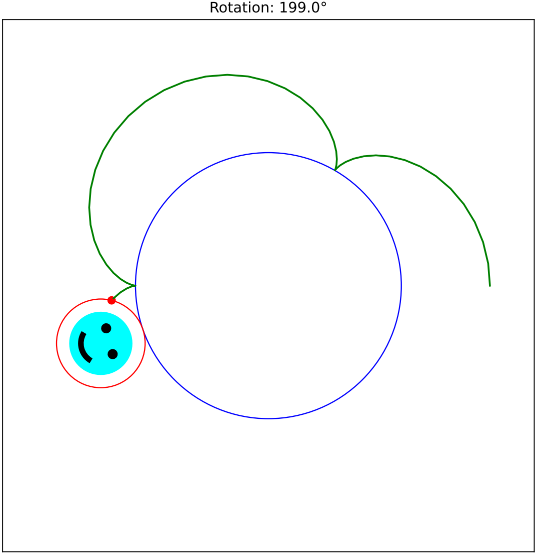
Animation of Coin Rotation Paradox



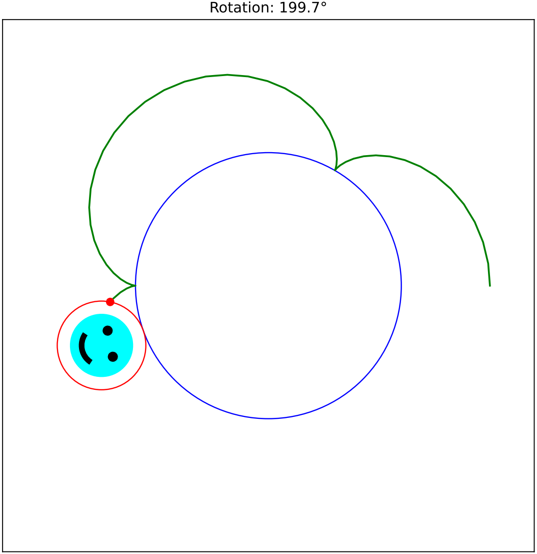
Animation of Coin Rotation Paradox



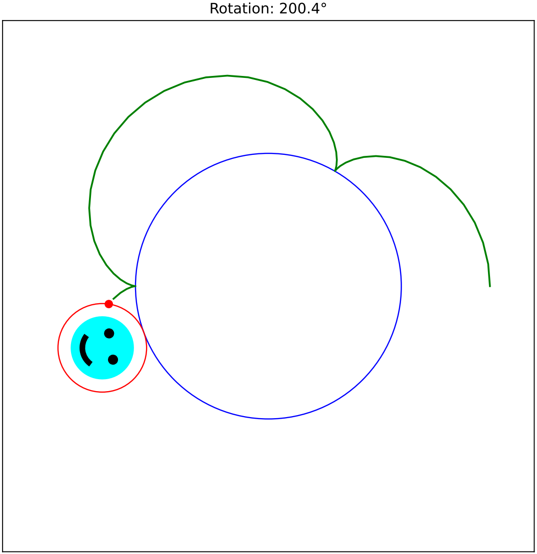
Animation of Coin Rotation Paradox



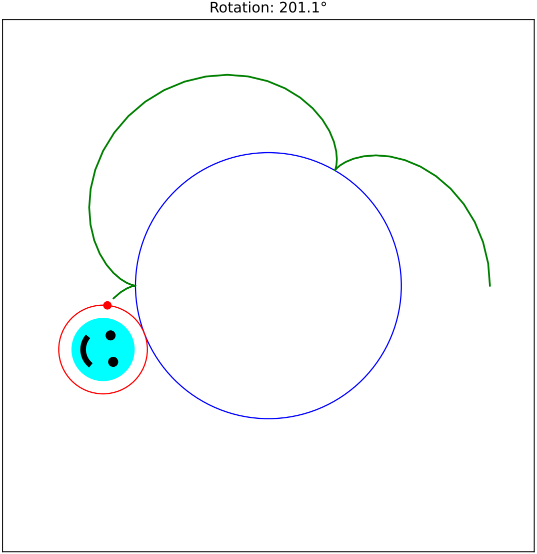
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

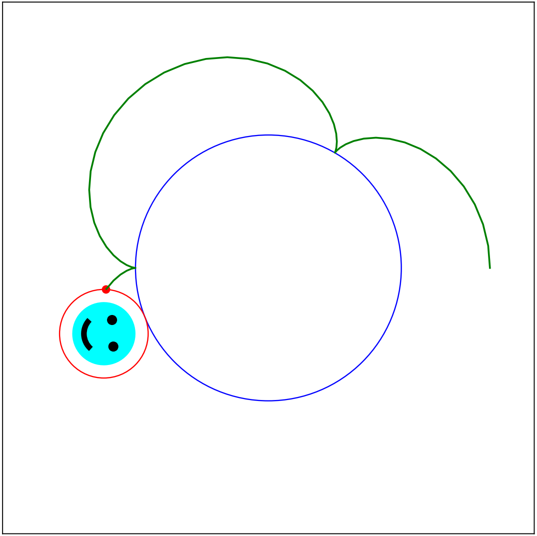


Animation of Coin Rotation Paradox

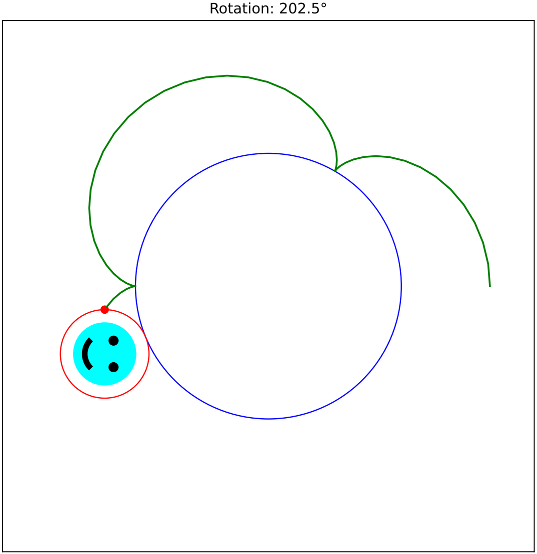


Animation of Coin Rotation Paradox

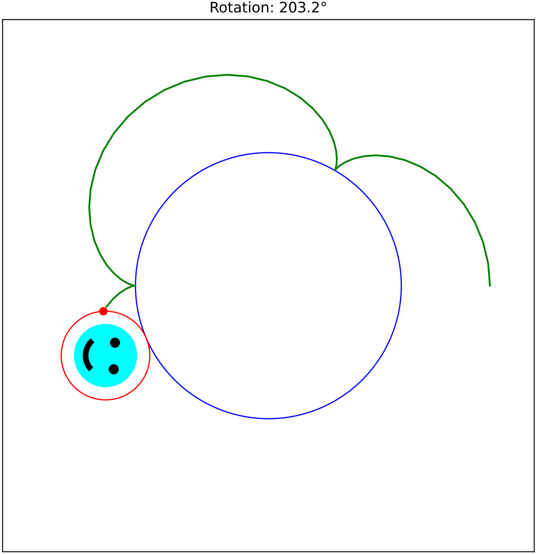
Rotation: 201.8°



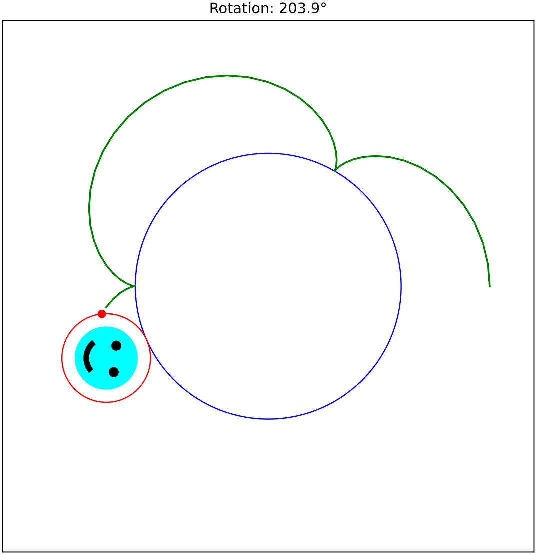
Animation of Coin Rotation Paradox



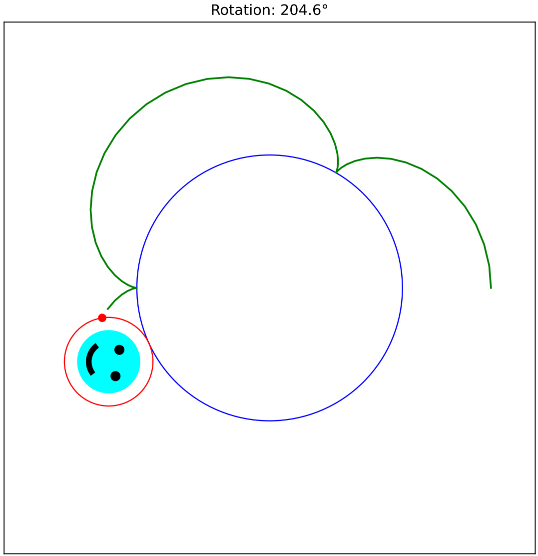
Animation of Coin Rotation Paradox



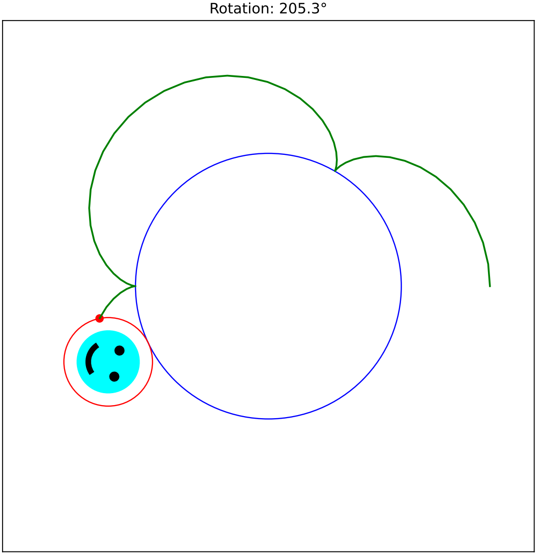
Animation of Coin Rotation Paradox



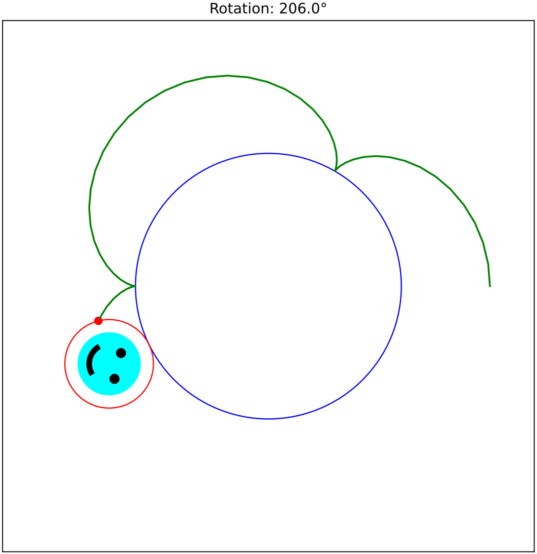
Animation of Coin Rotation Paradox



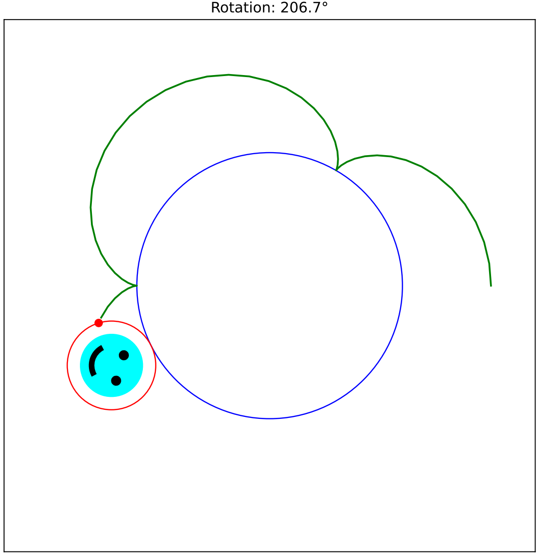
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

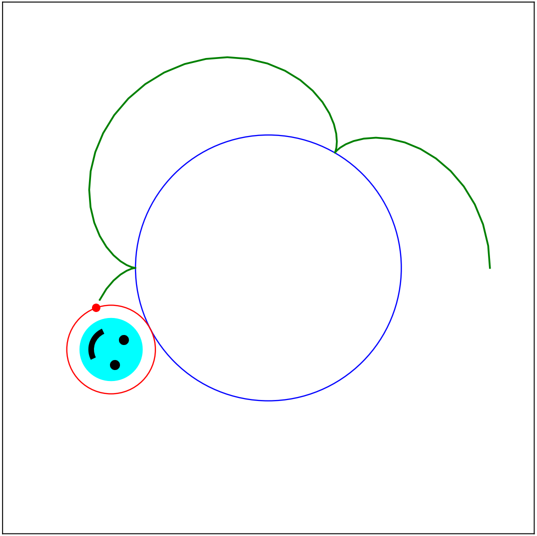


Animation of Coin Rotation Paradox

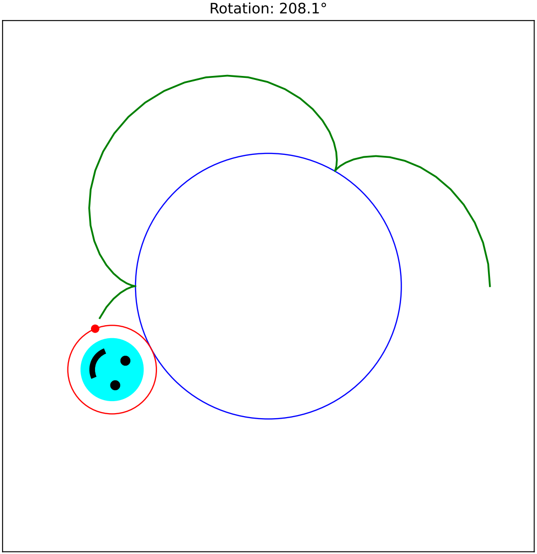


Animation of Coin Rotation Paradox

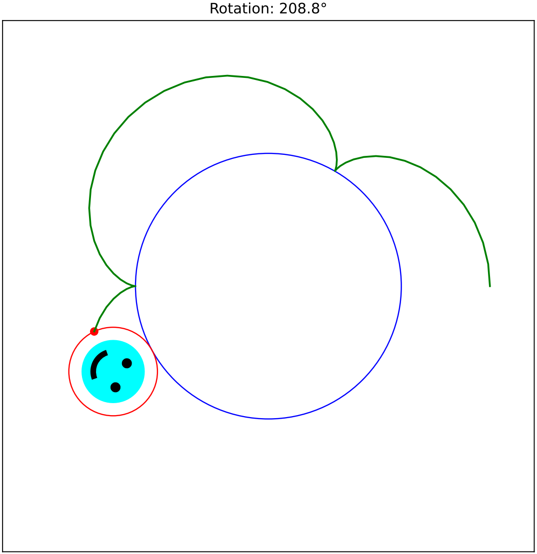
Rotation: 207.4°



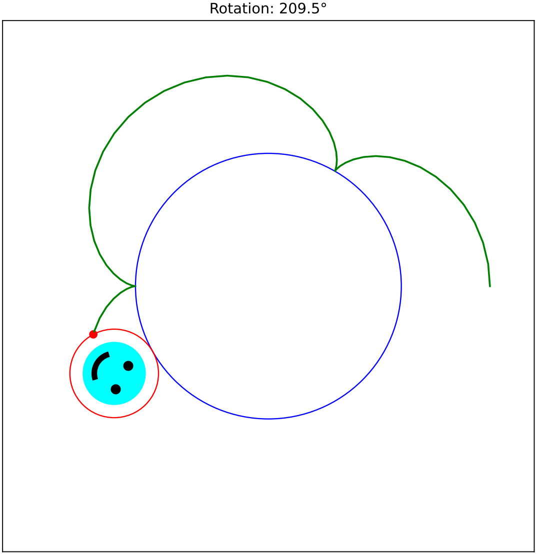
Animation of Coin Rotation Paradox



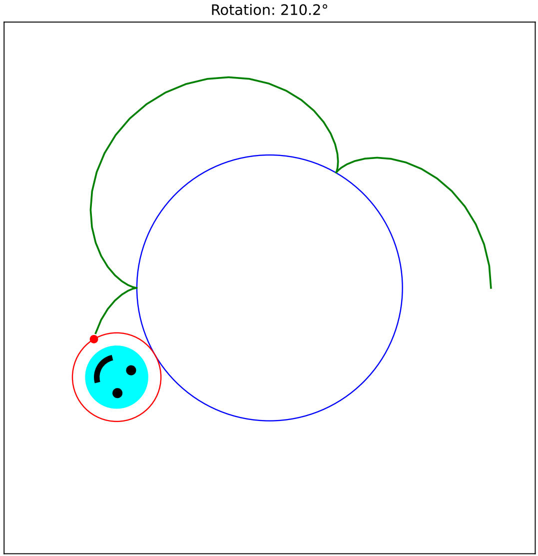
Animation of Coin Rotation Paradox



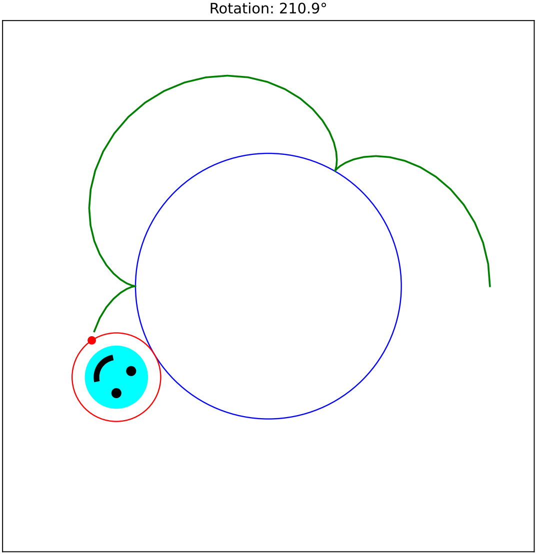
Animation of Coin Rotation Paradox



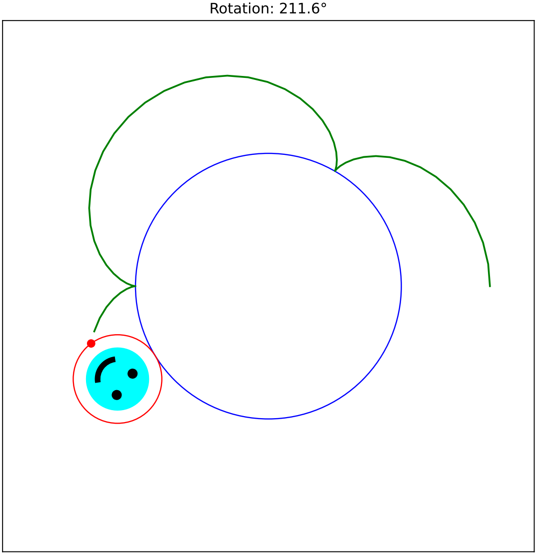
Animation of Coin Rotation Paradox



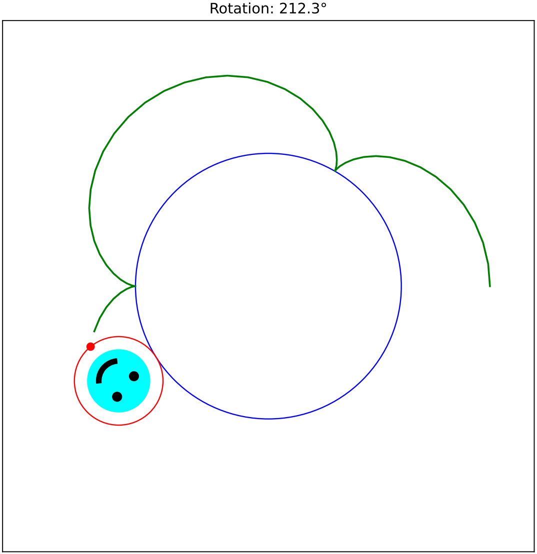
Animation of Coin Rotation Paradox



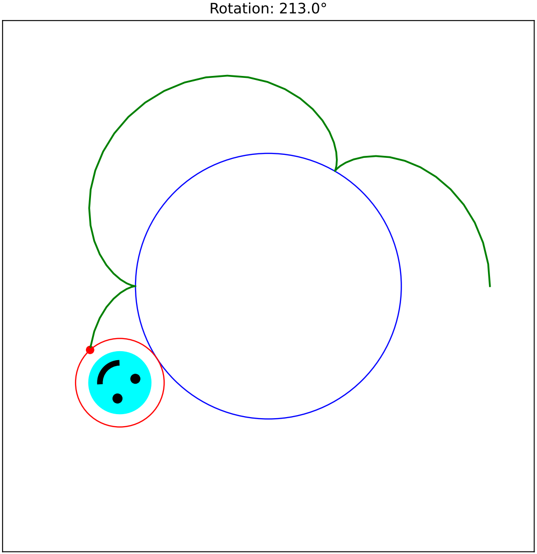
Animation of Coin Rotation Paradox



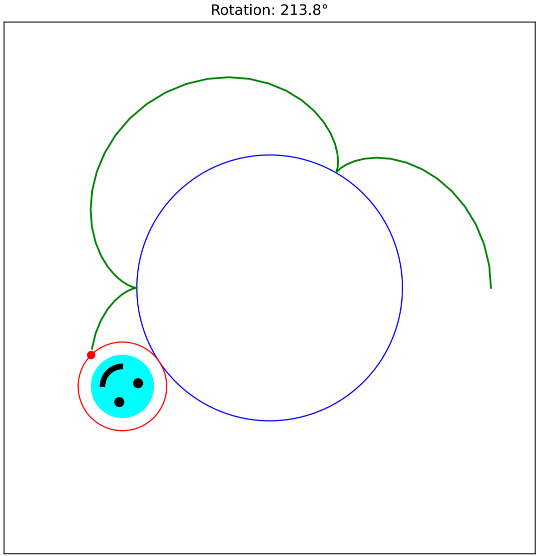
Animation of Coin Rotation Paradox



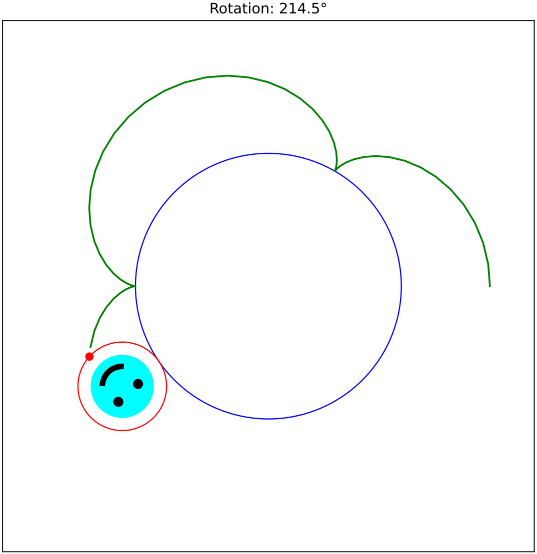
Animation of Coin Rotation Paradox



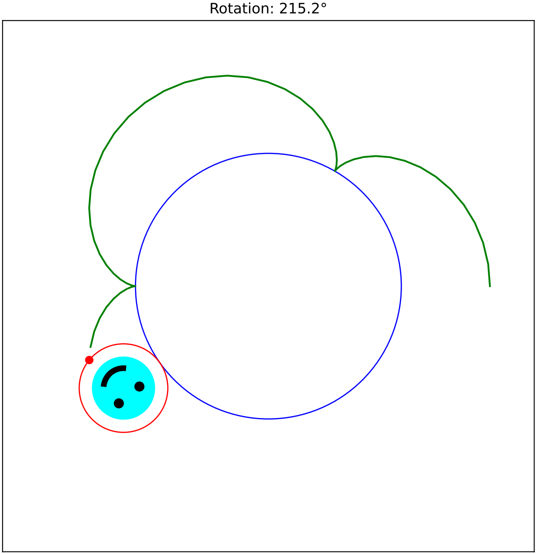
Animation of Coin Rotation Paradox



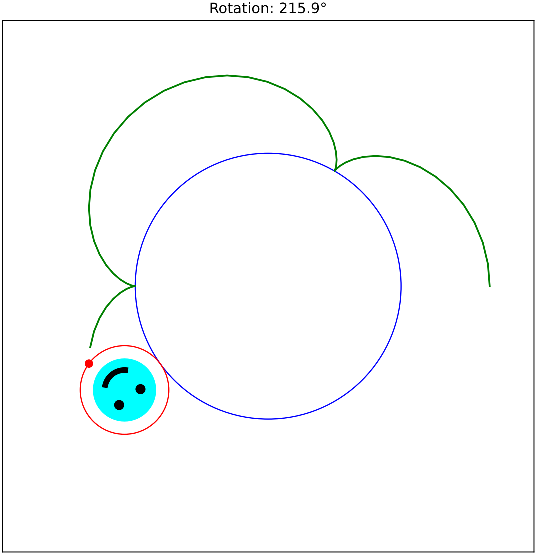
Animation of Coin Rotation Paradox



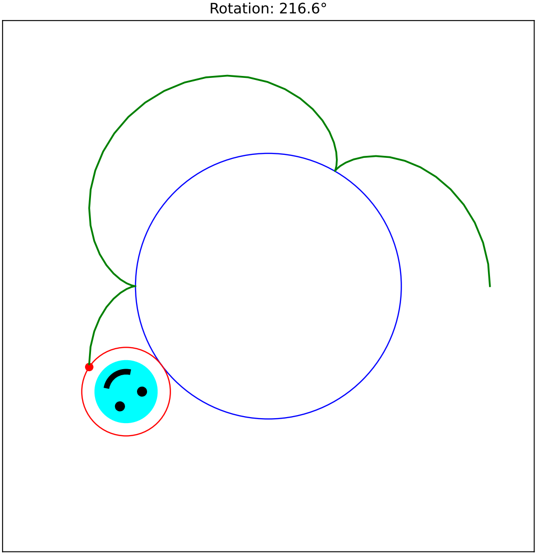
Animation of Coin Rotation Paradox



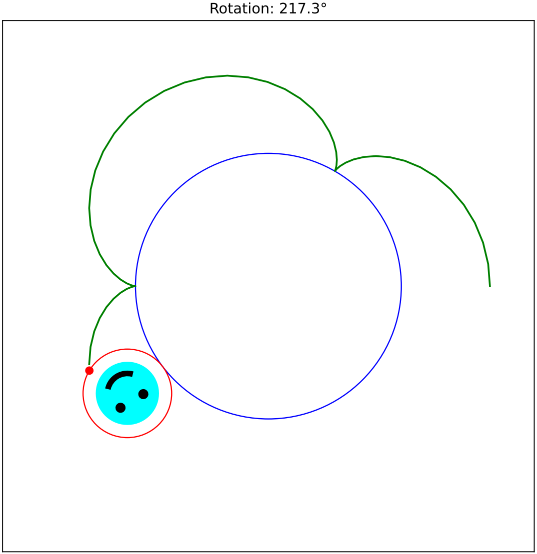
Animation of Coin Rotation Paradox



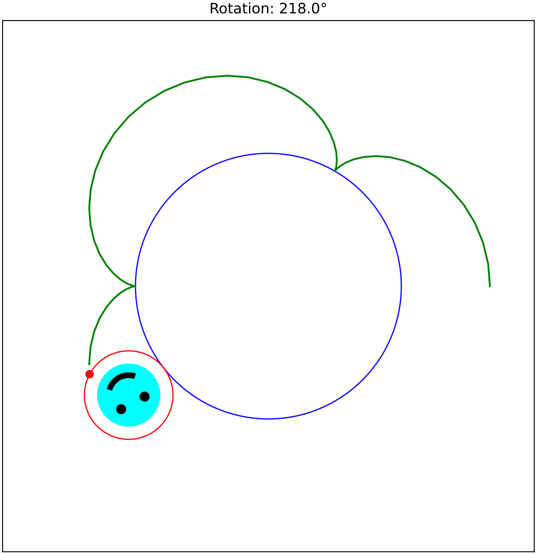
Animation of Coin Rotation Paradox



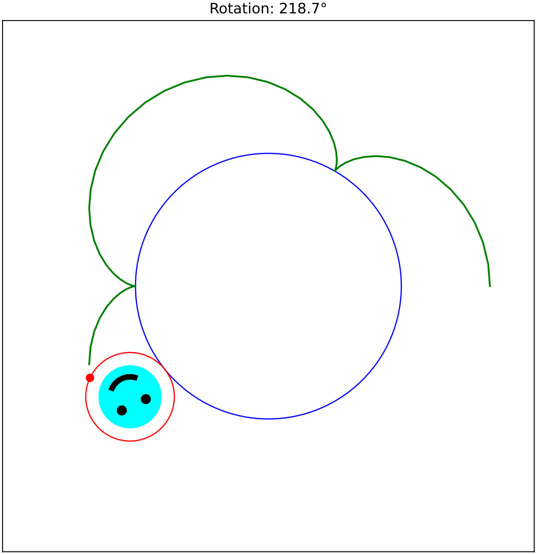
Animation of Coin Rotation Paradox



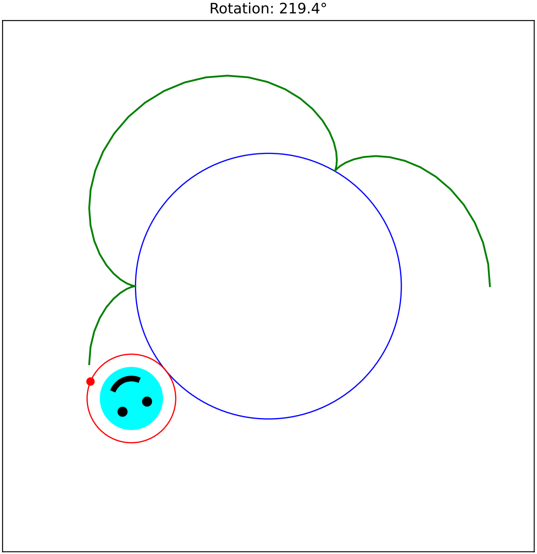
Animation of Coin Rotation Paradox



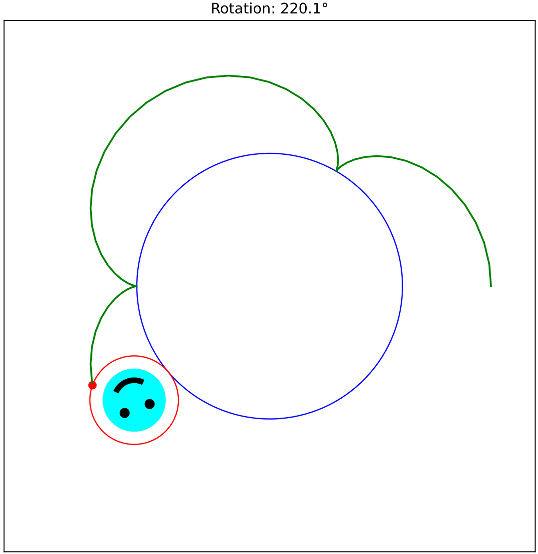
Animation of Coin Rotation Paradox



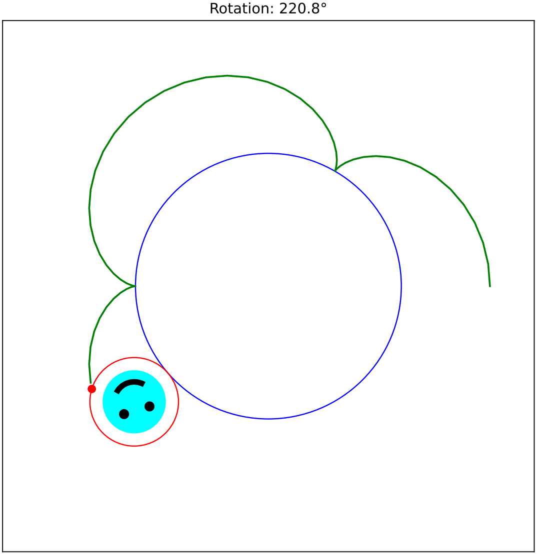
Animation of Coin Rotation Paradox



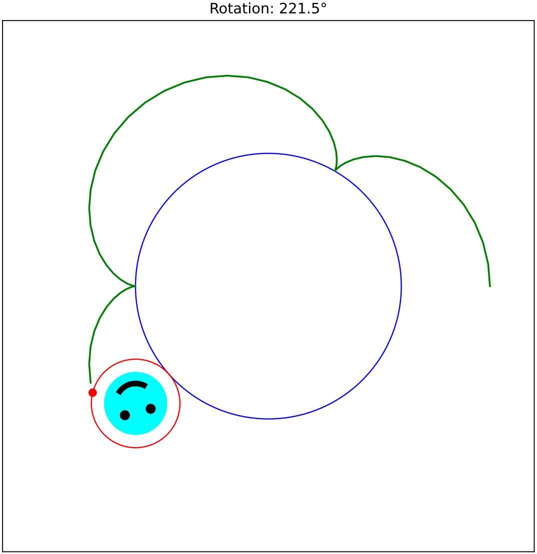
Animation of Coin Rotation Paradox



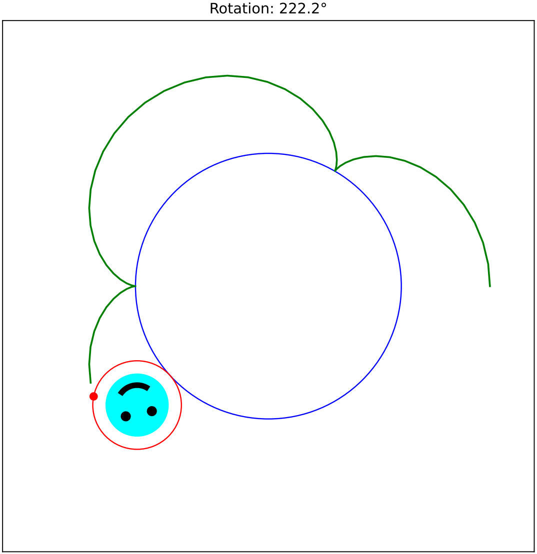
Animation of Coin Rotation Paradox



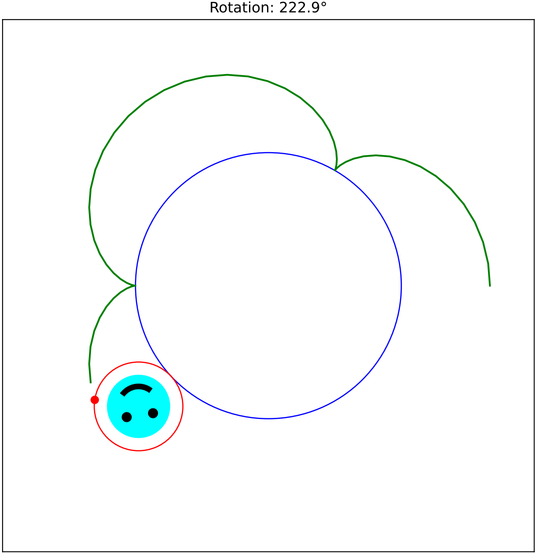
Animation of Coin Rotation Paradox



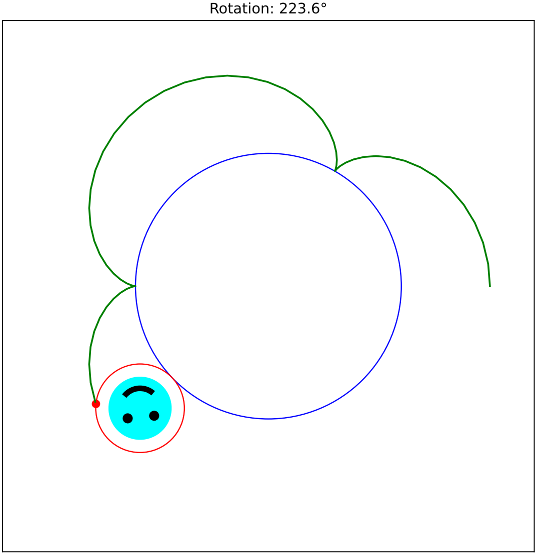
Animation of Coin Rotation Paradox



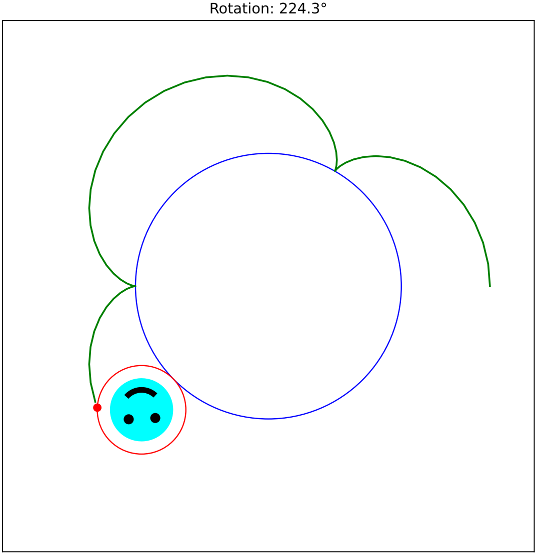
Animation of Coin Rotation Paradox



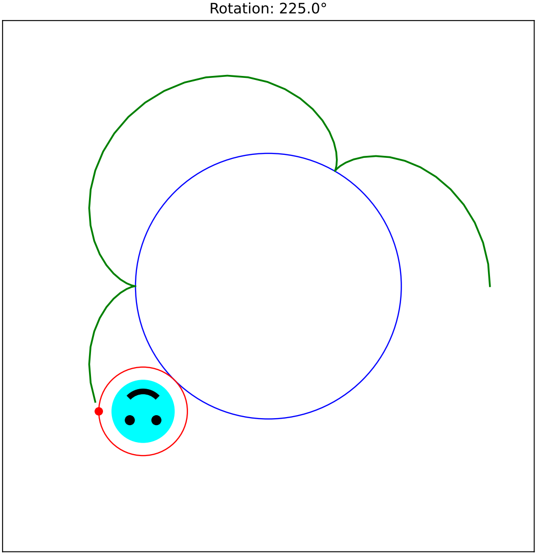
Animation of Coin Rotation Paradox



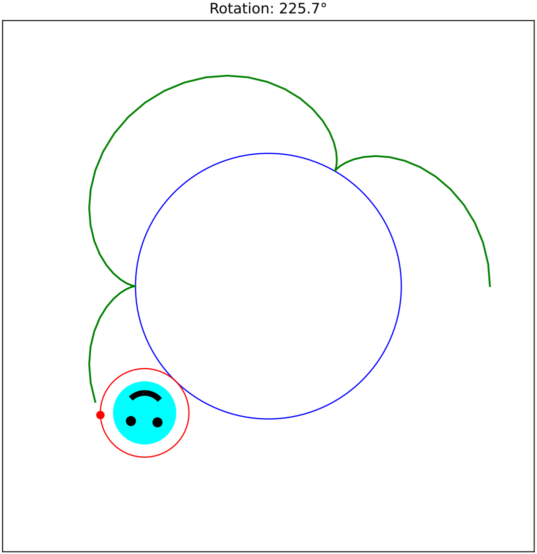
Animation of Coin Rotation Paradox



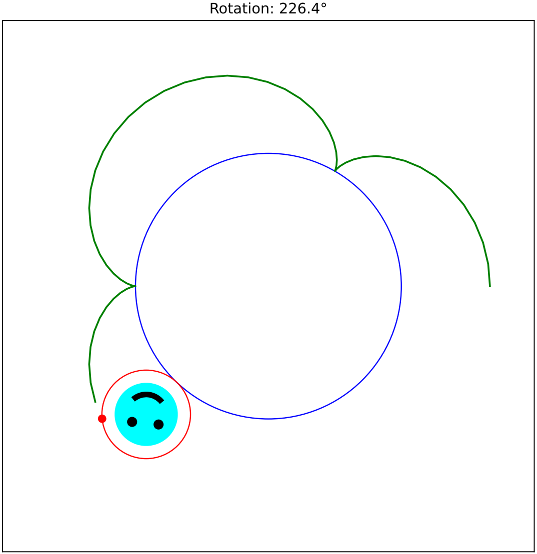
Animation of Coin Rotation Paradox



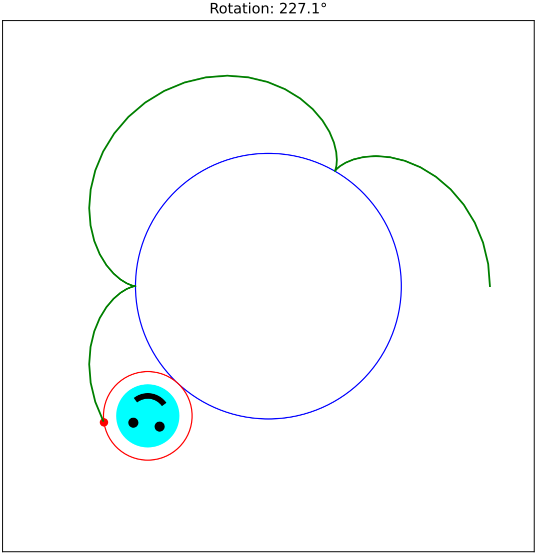
Animation of Coin Rotation Paradox



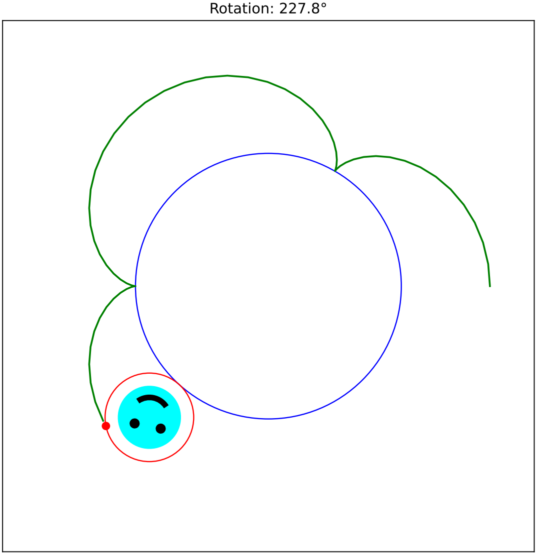
Animation of Coin Rotation Paradox



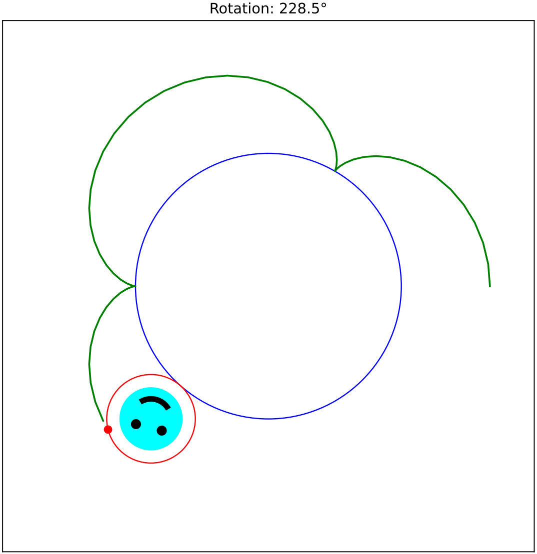
Animation of Coin Rotation Paradox



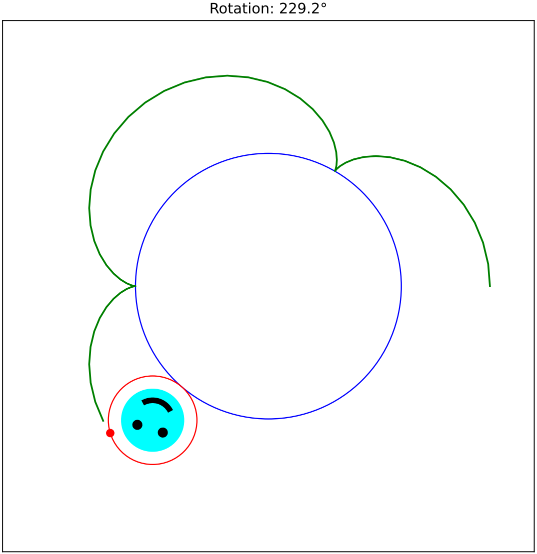
Animation of Coin Rotation Paradox



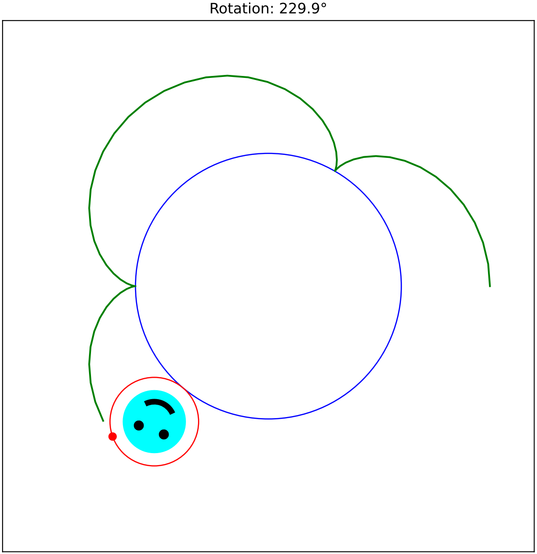
Animation of Coin Rotation Paradox



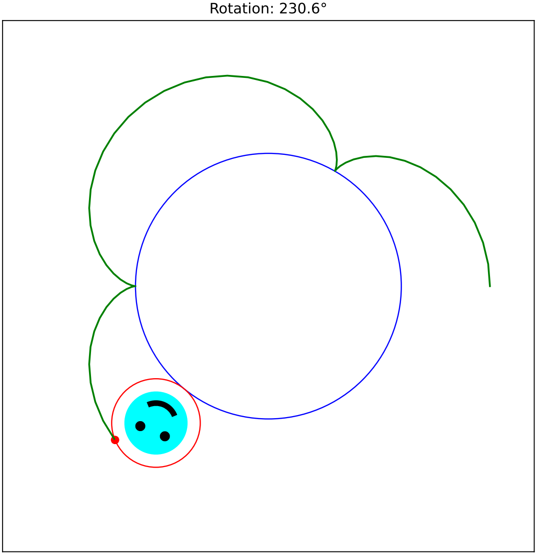
Animation of Coin Rotation Paradox



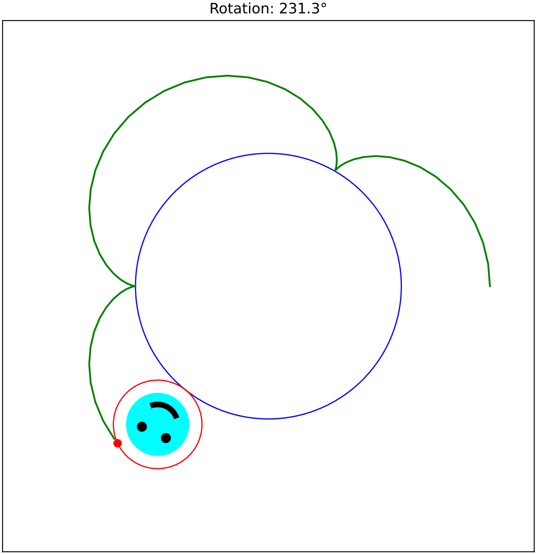
Animation of Coin Rotation Paradox



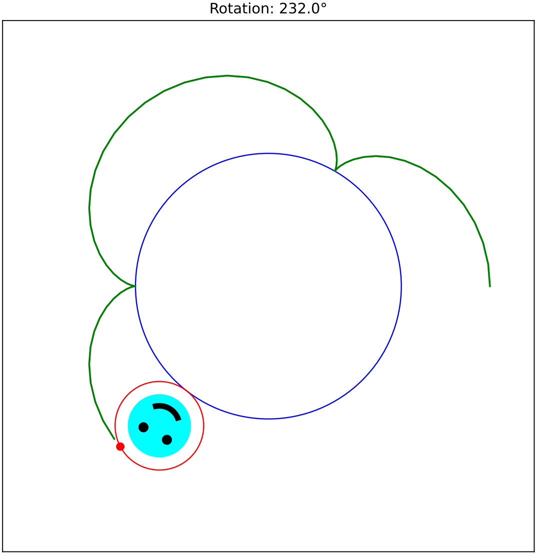
Animation of Coin Rotation Paradox



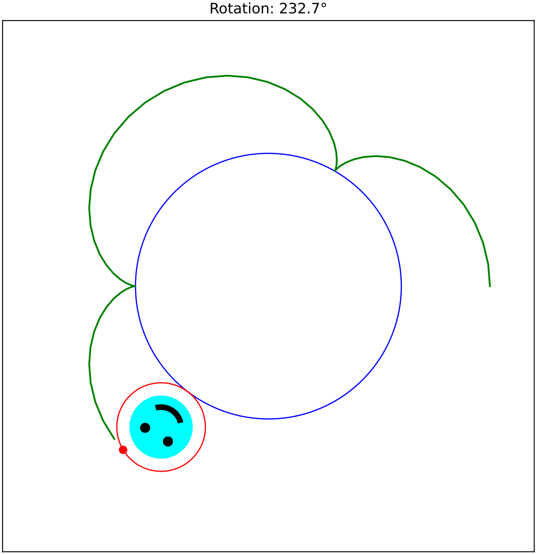
Animation of Coin Rotation Paradox



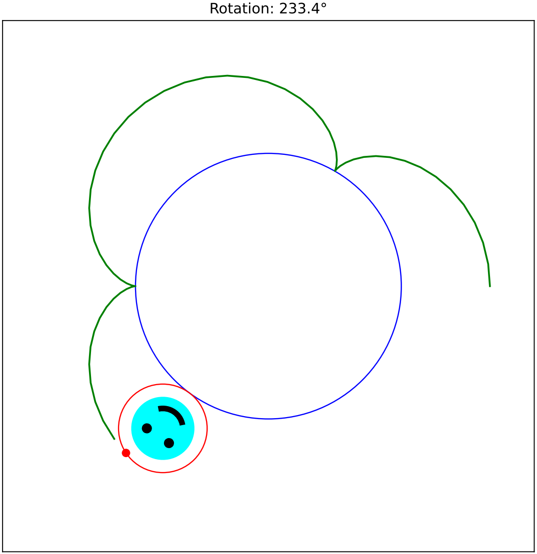
Animation of Coin Rotation Paradox



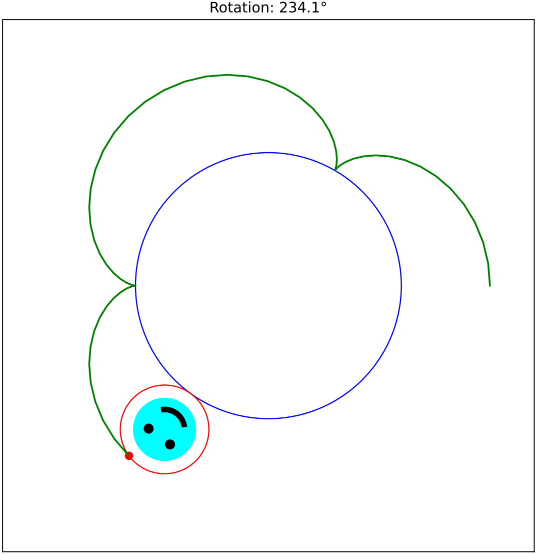
Animation of Coin Rotation Paradox



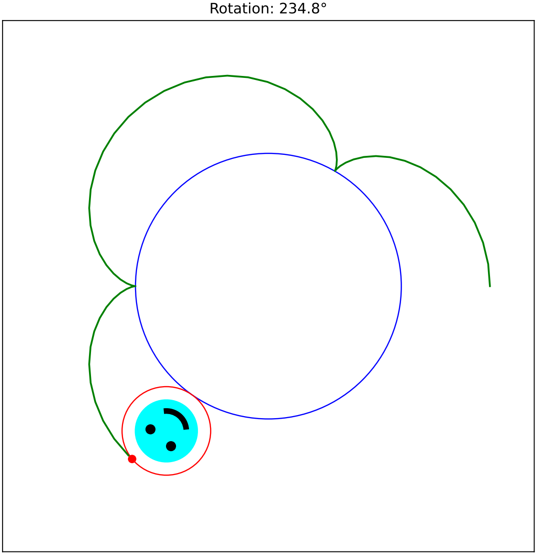
Animation of Coin Rotation Paradox



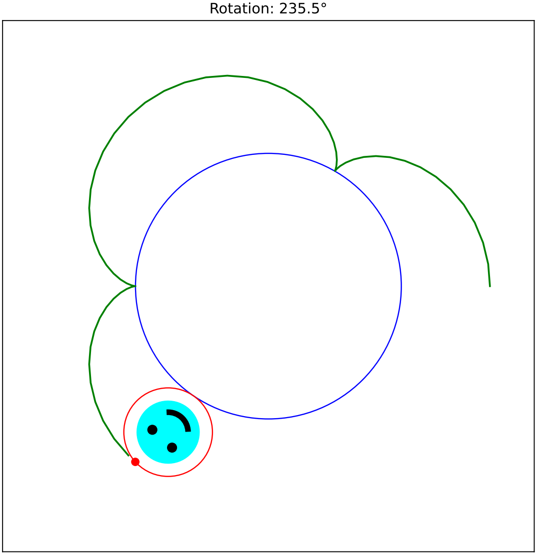
Animation of Coin Rotation Paradox



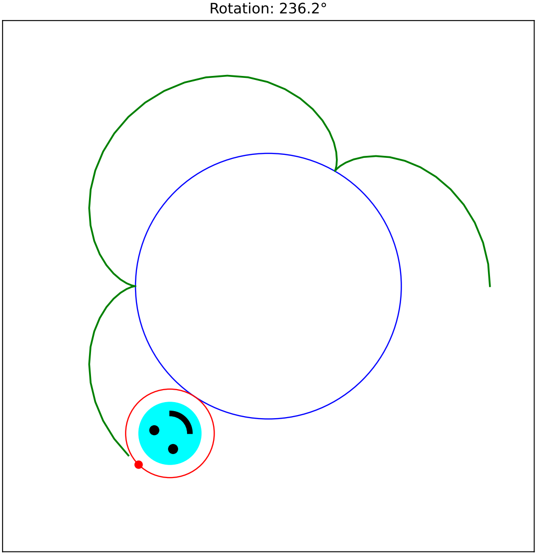
Animation of Coin Rotation Paradox



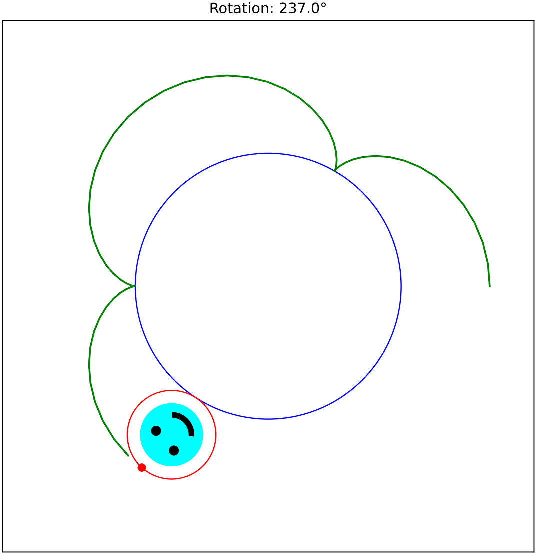
Animation of Coin Rotation Paradox



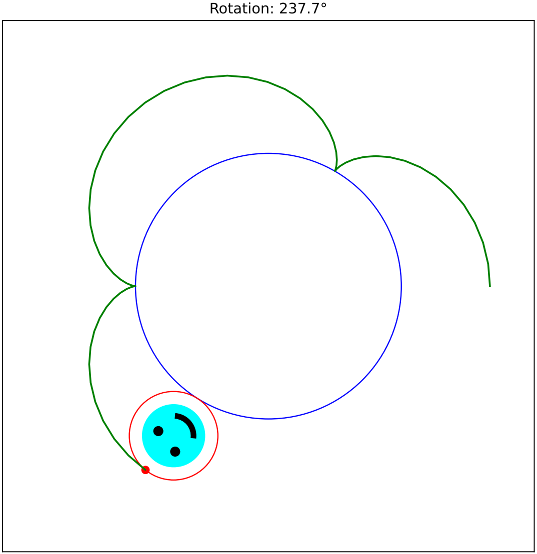
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

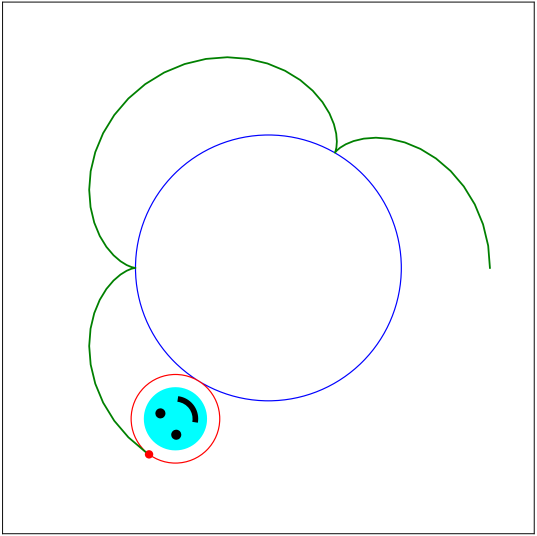


Animation of Coin Rotation Paradox

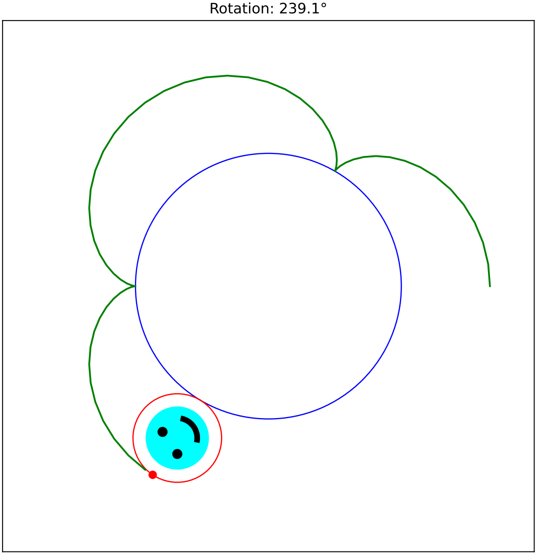


Animation of Coin Rotation Paradox

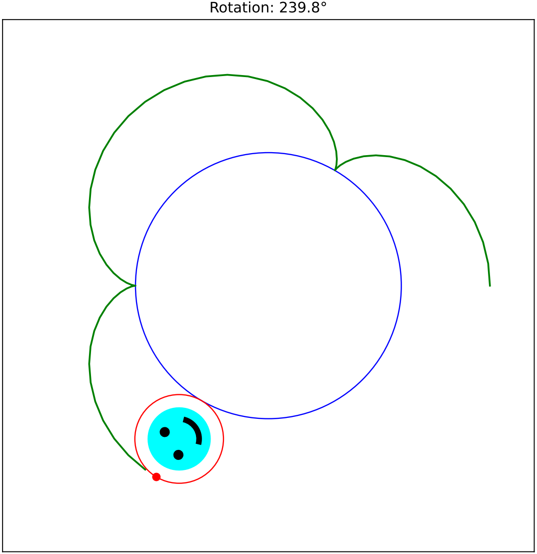
Rotation: 238.4°



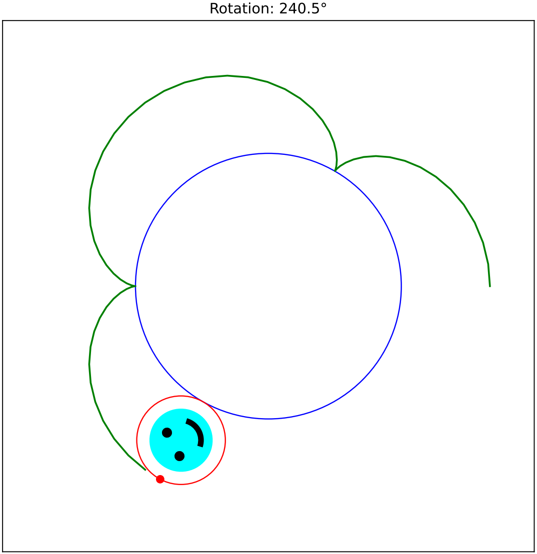
Animation of Coin Rotation Paradox



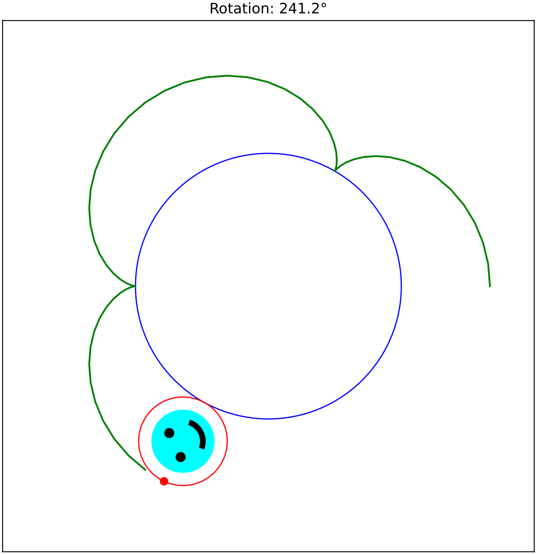
Animation of Coin Rotation Paradox



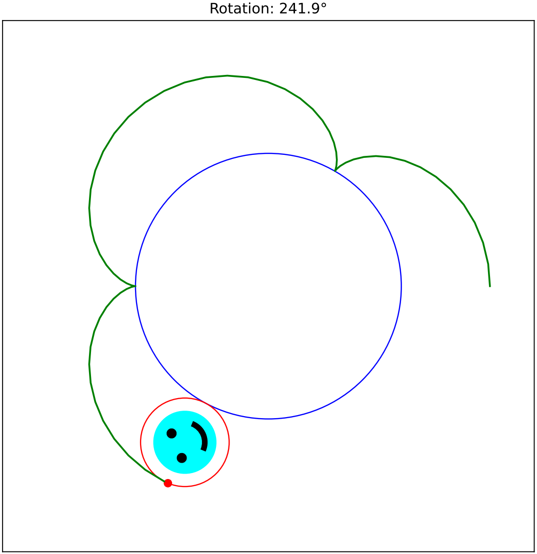
Animation of Coin Rotation Paradox



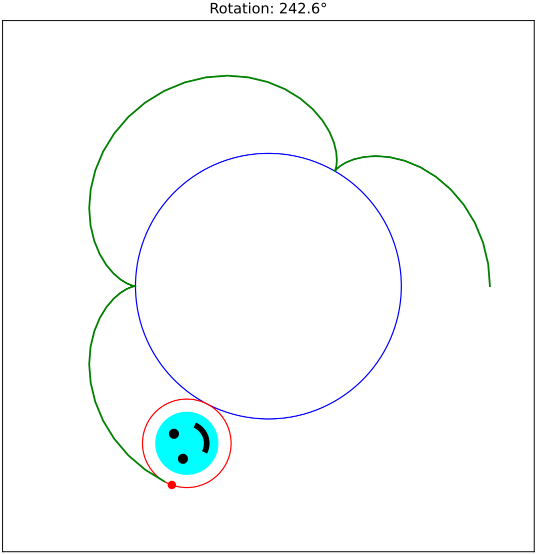
Animation of Coin Rotation Paradox



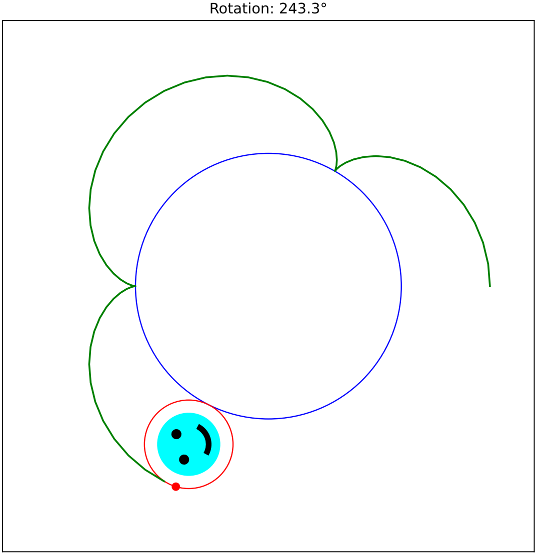
Animation of Coin Rotation Paradox



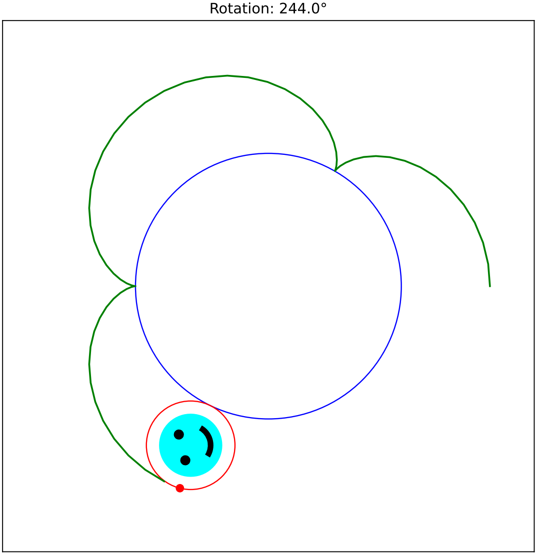
Animation of Coin Rotation Paradox



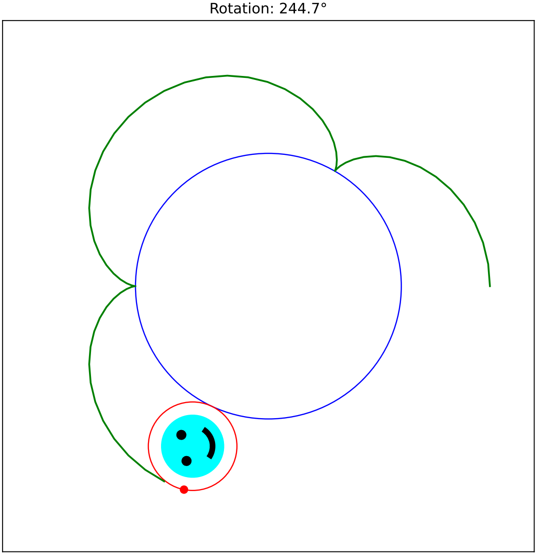
Animation of Coin Rotation Paradox



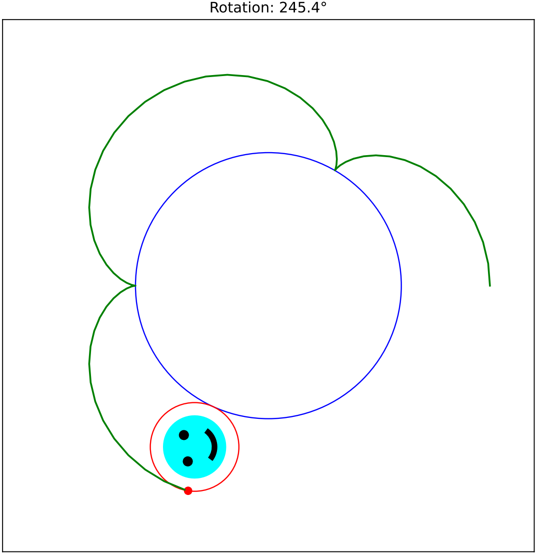
Animation of Coin Rotation Paradox



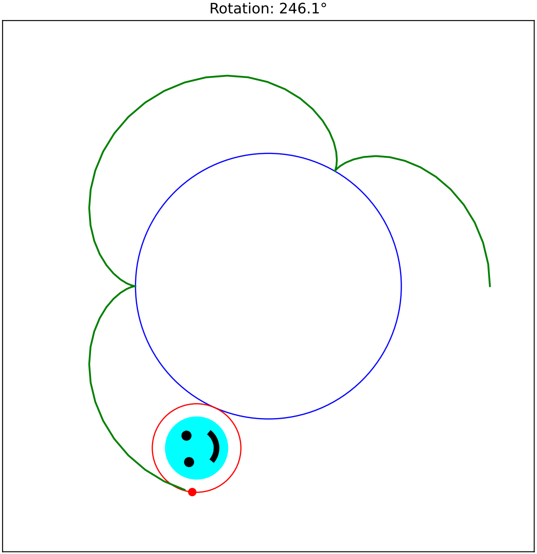
Animation of Coin Rotation Paradox



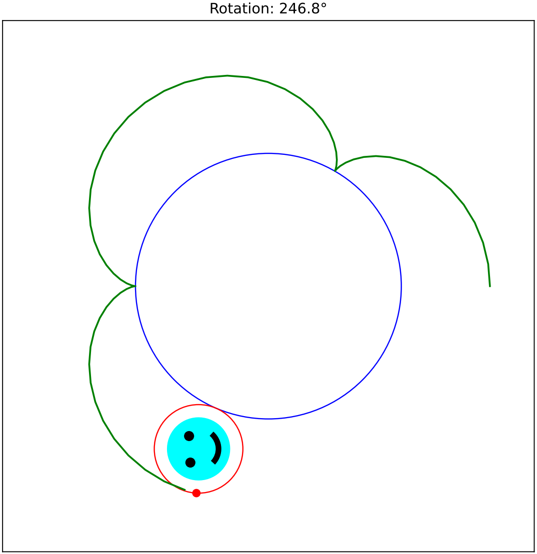
Animation of Coin Rotation Paradox



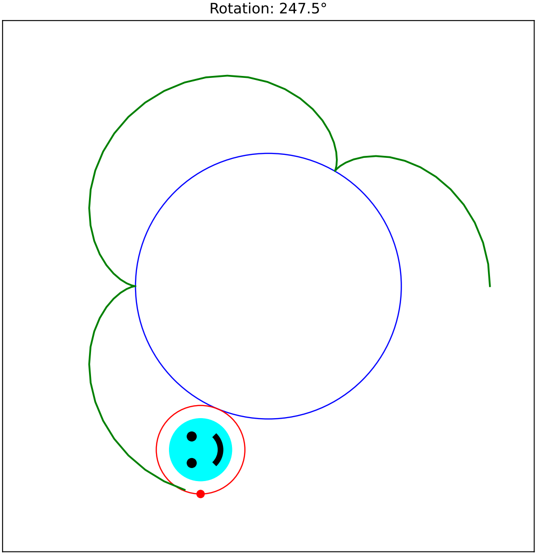
Animation of Coin Rotation Paradox



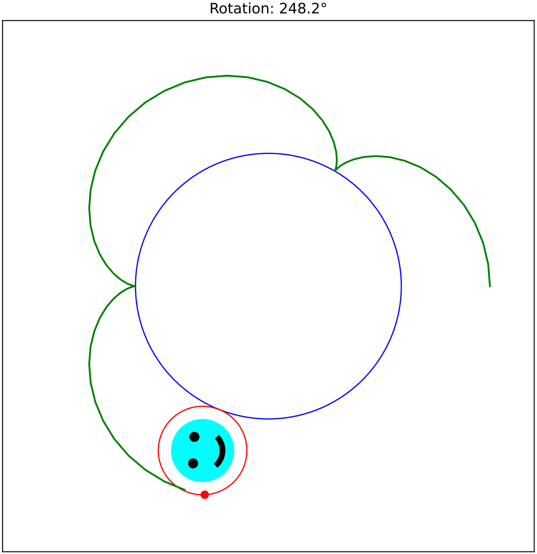
Animation of Coin Rotation Paradox



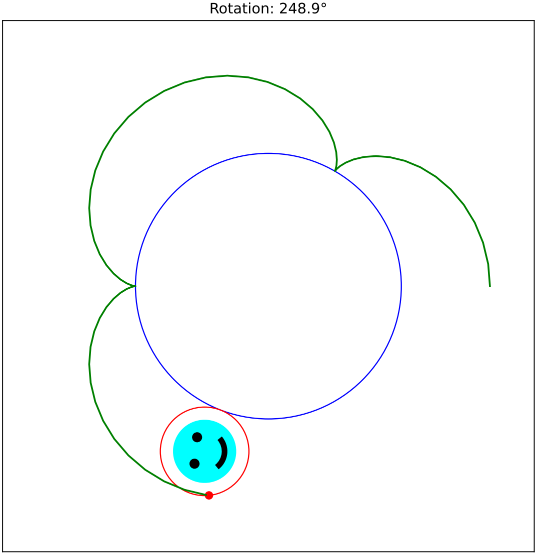
Animation of Coin Rotation Paradox



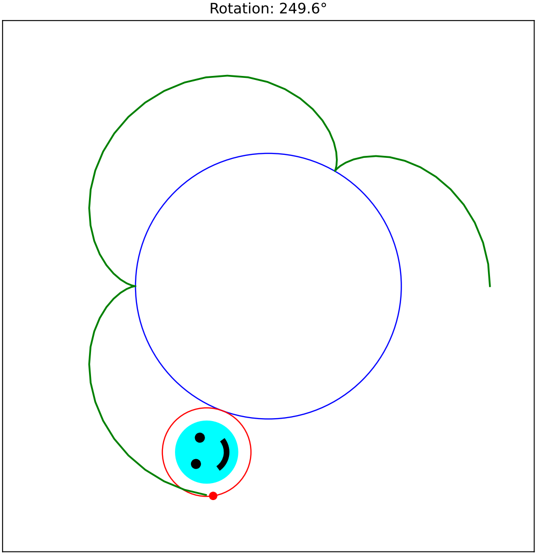
Animation of Coin Rotation Paradox



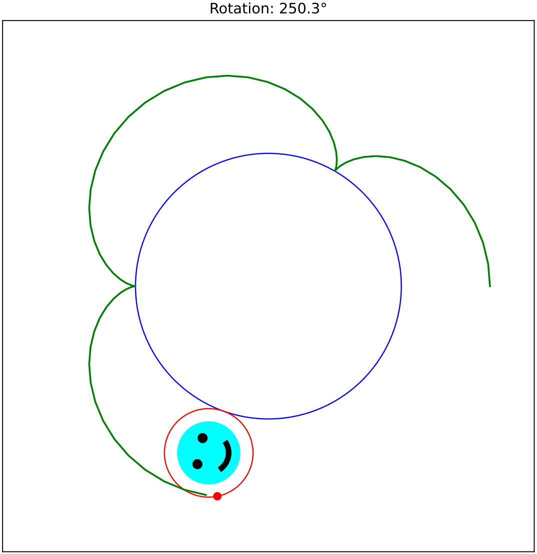
Animation of Coin Rotation Paradox



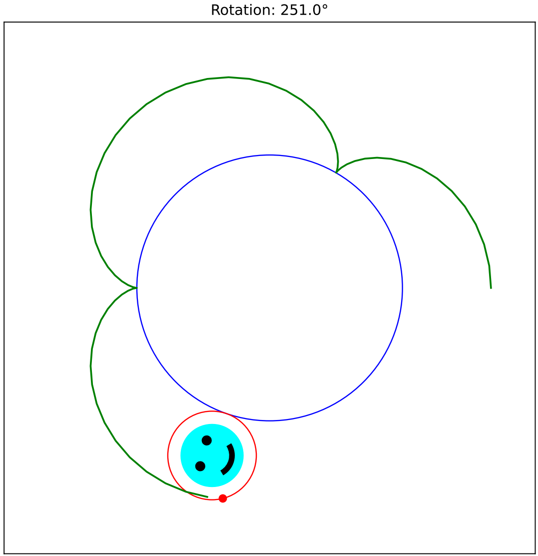
Animation of Coin Rotation Paradox



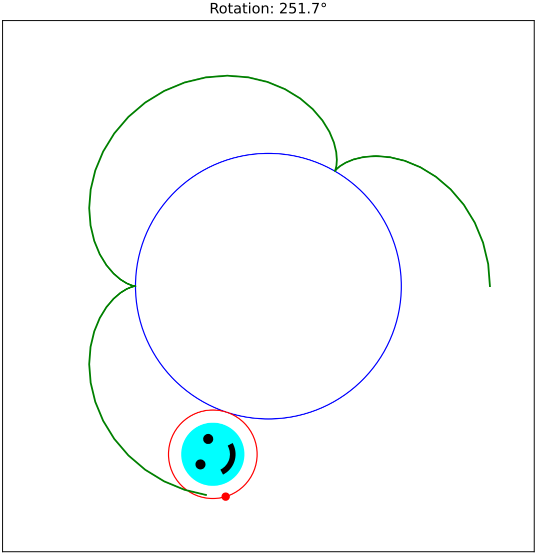
Animation of Coin Rotation Paradox



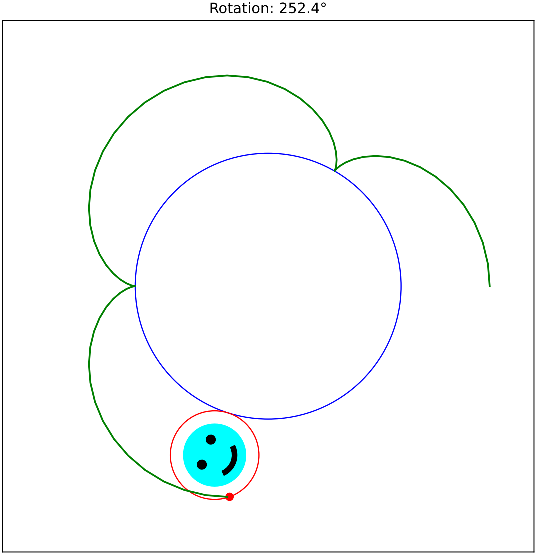
Animation of Coin Rotation Paradox



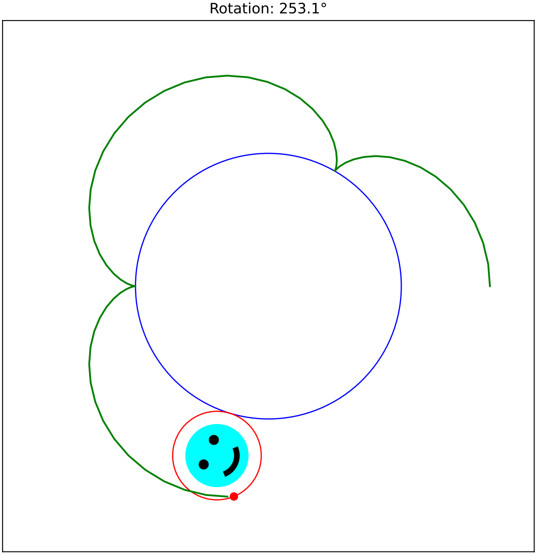
Animation of Coin Rotation Paradox



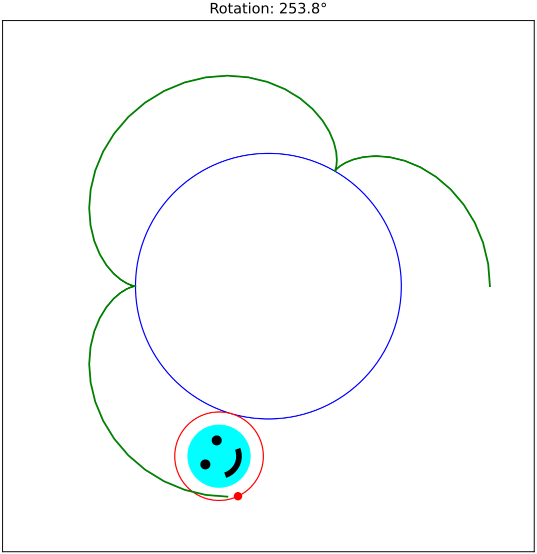
Animation of Coin Rotation Paradox



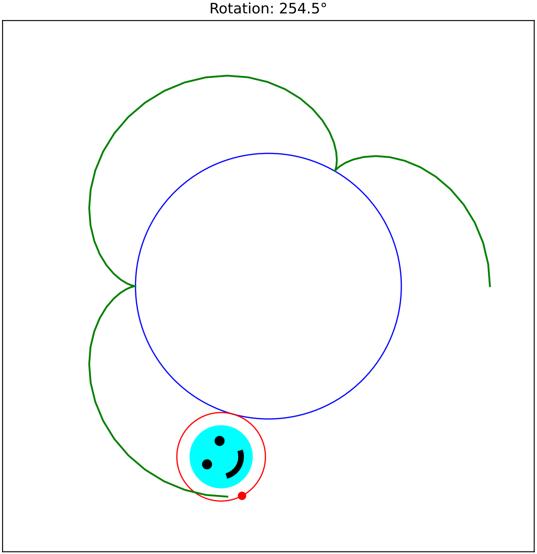
Animation of Coin Rotation Paradox



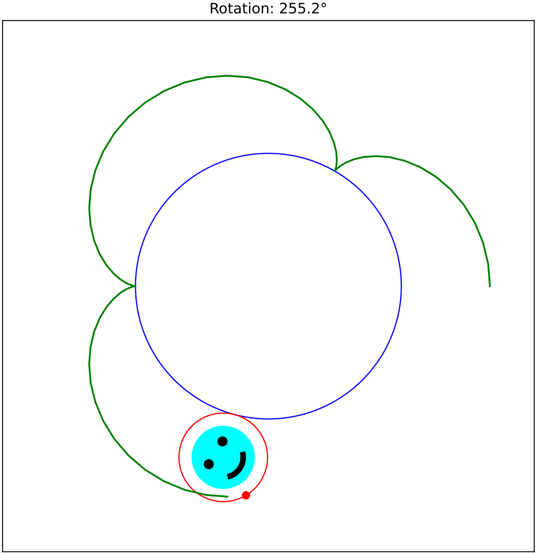
Animation of Coin Rotation Paradox



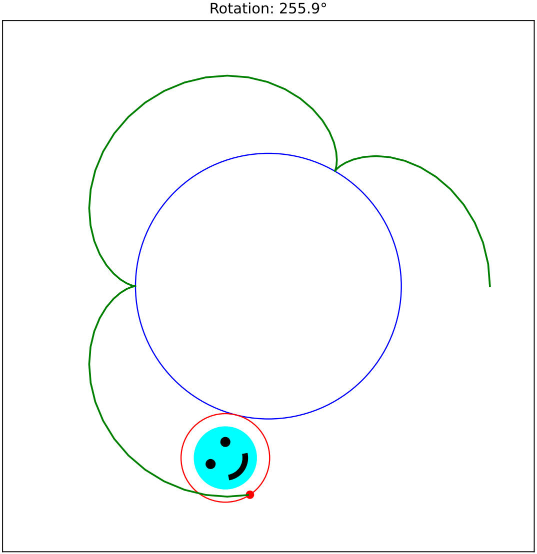
Animation of Coin Rotation Paradox



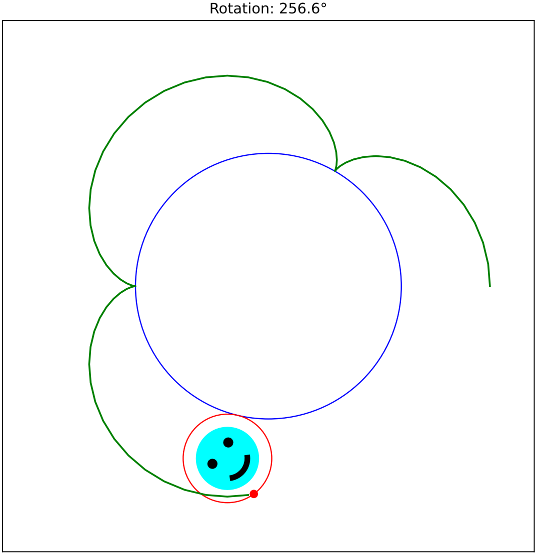
Animation of Coin Rotation Paradox



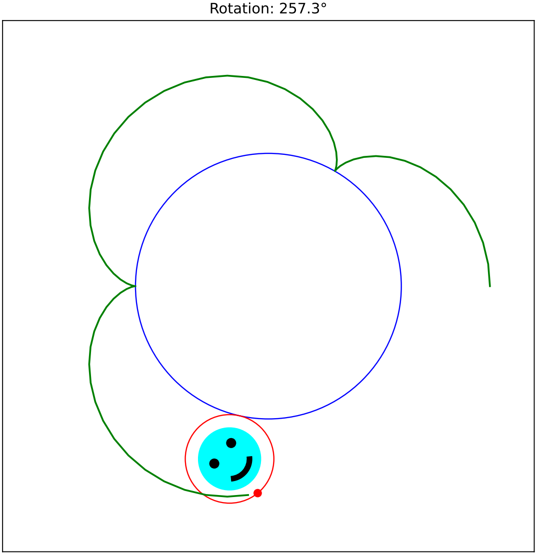
Animation of Coin Rotation Paradox



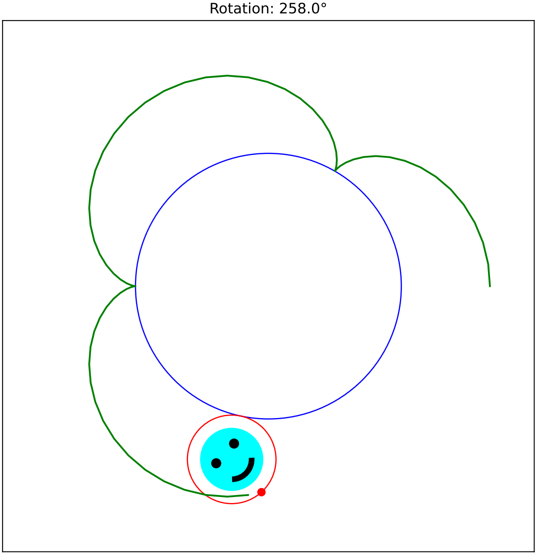
Animation of Coin Rotation Paradox



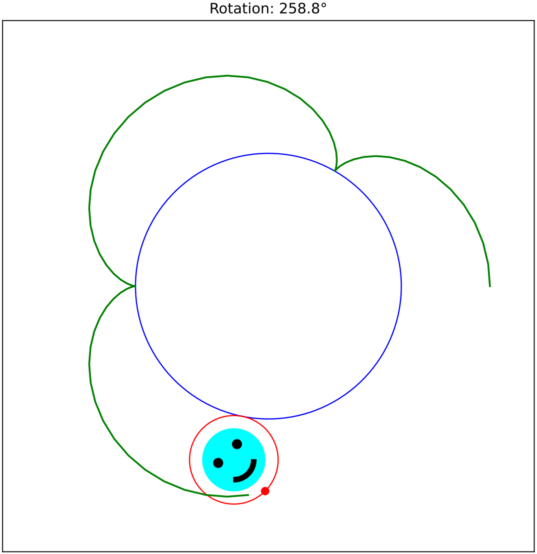
Animation of Coin Rotation Paradox



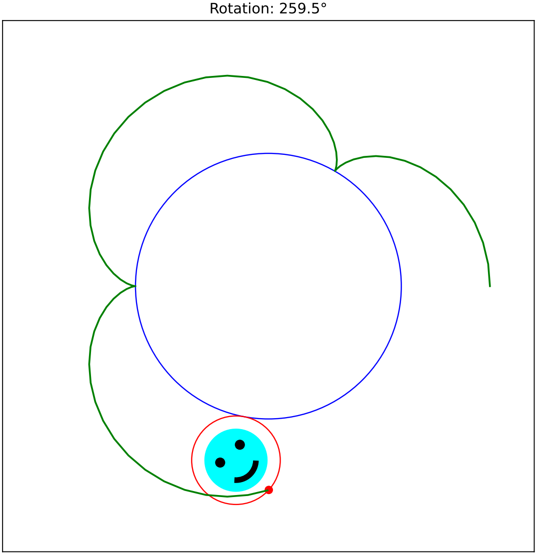
Animation of Coin Rotation Paradox



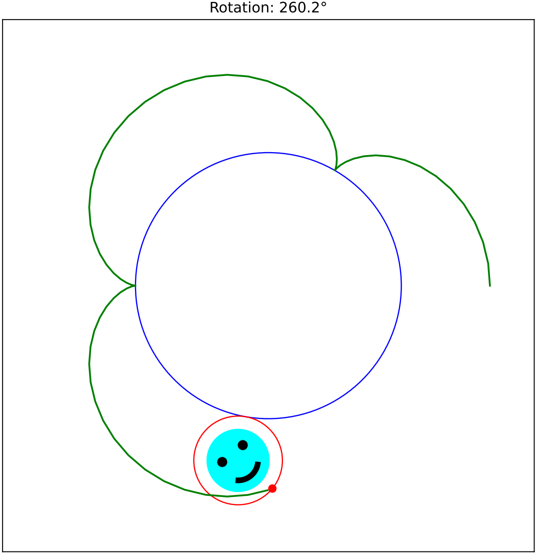
Animation of Coin Rotation Paradox



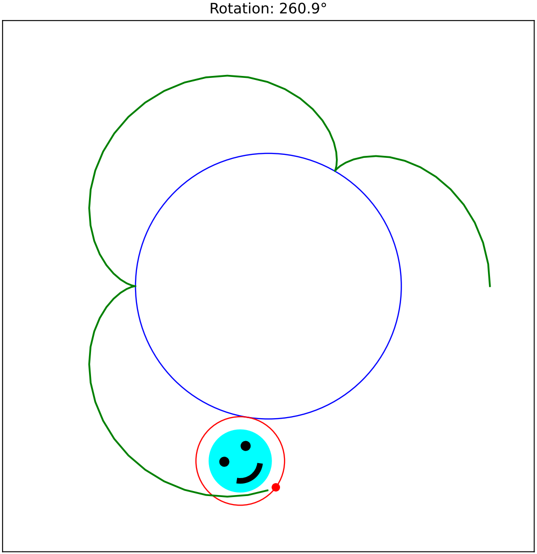
Animation of Coin Rotation Paradox



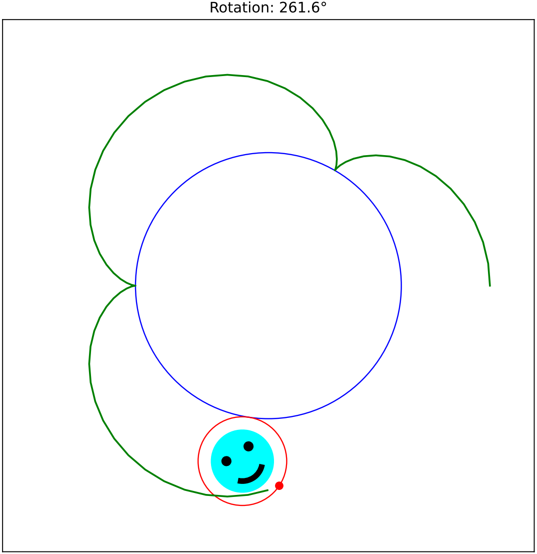
Animation of Coin Rotation Paradox



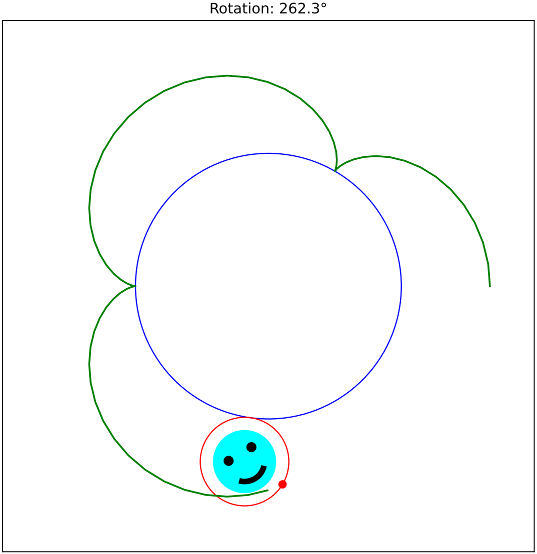
Animation of Coin Rotation Paradox



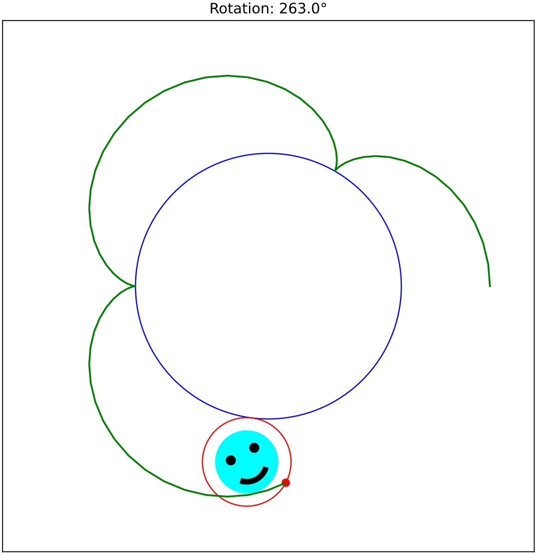
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

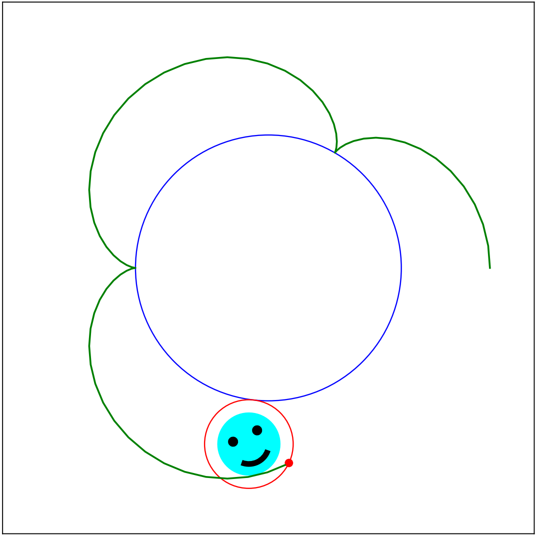


Animation of Coin Rotation Paradox

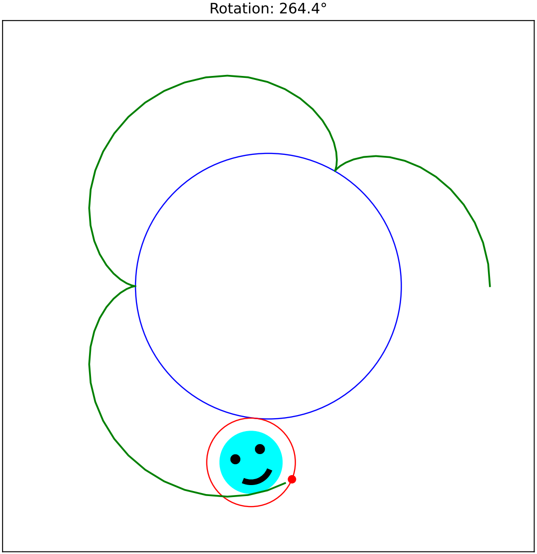


Animation of Coin Rotation Paradox

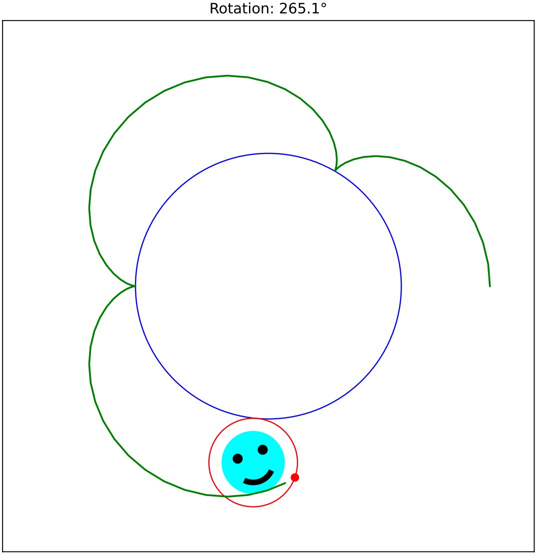
Rotation: 263.7°



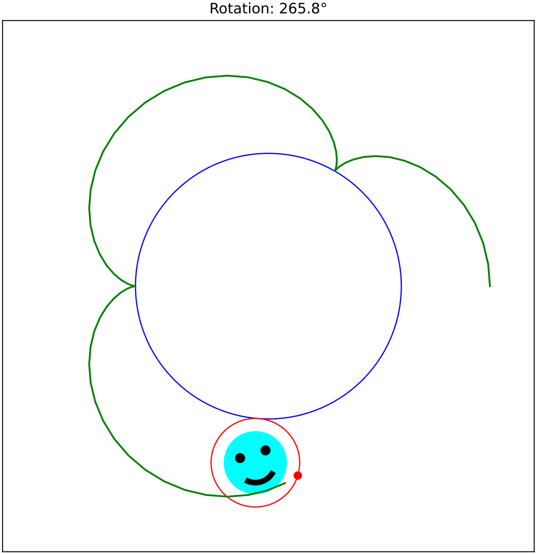
Animation of Coin Rotation Paradox



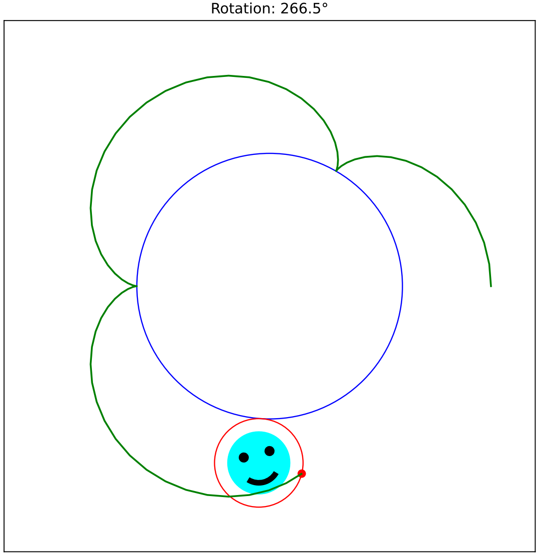
Animation of Coin Rotation Paradox



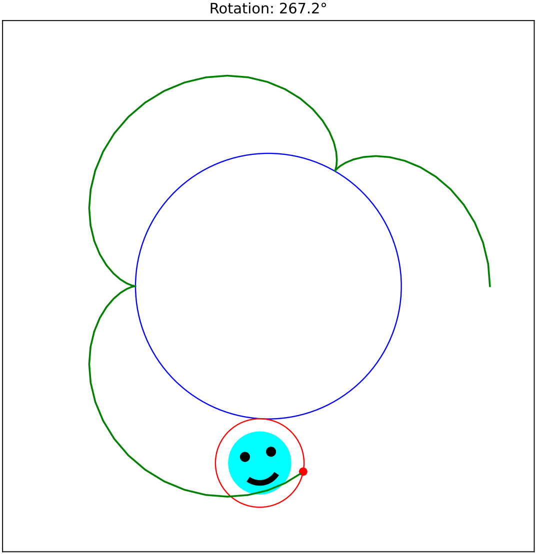
Animation of Coin Rotation Paradox



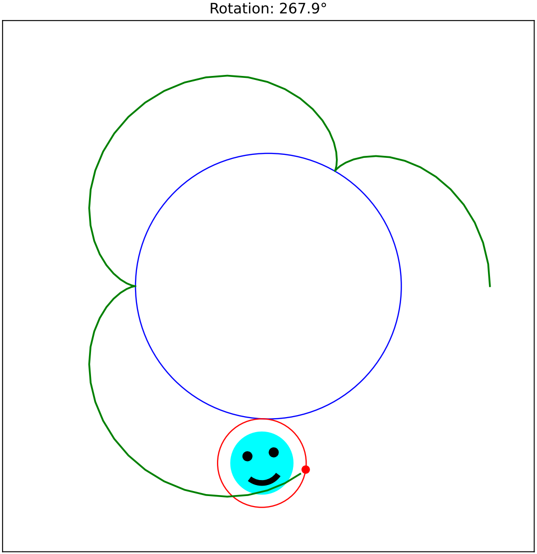
Animation of Coin Rotation Paradox



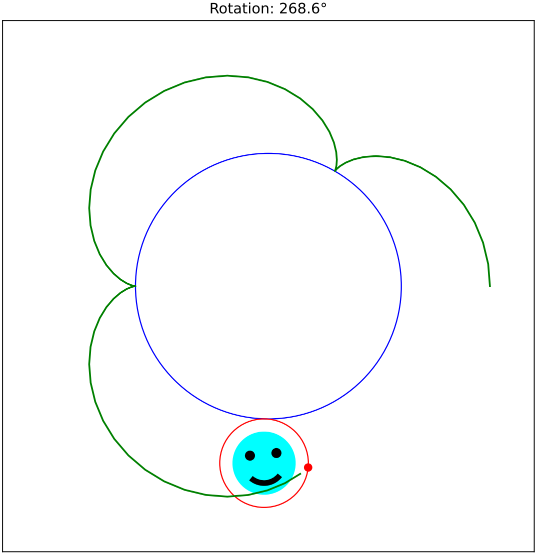
Animation of Coin Rotation Paradox



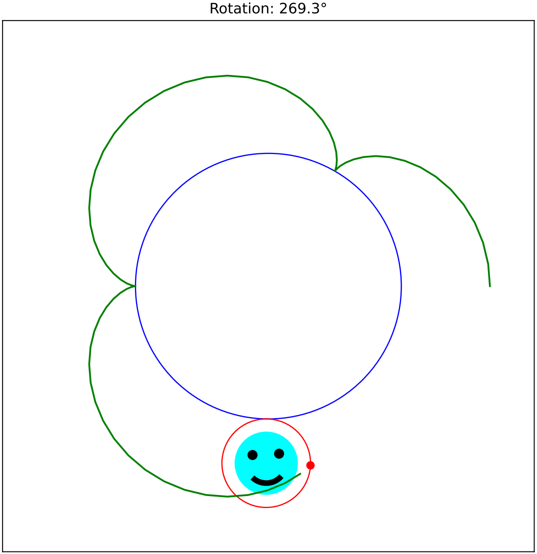
Animation of Coin Rotation Paradox



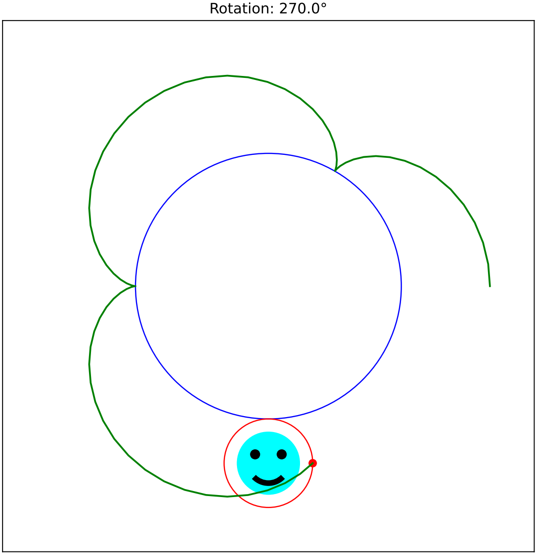
Animation of Coin Rotation Paradox



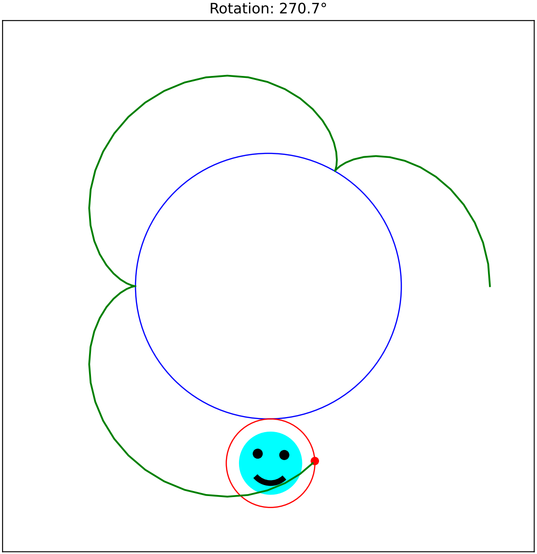
Animation of Coin Rotation Paradox



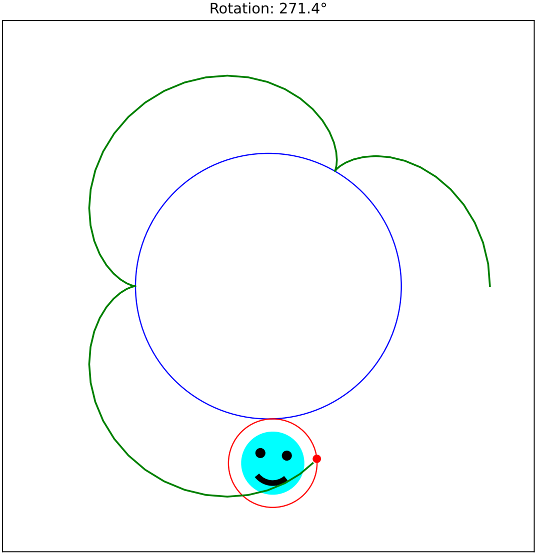
Animation of Coin Rotation Paradox



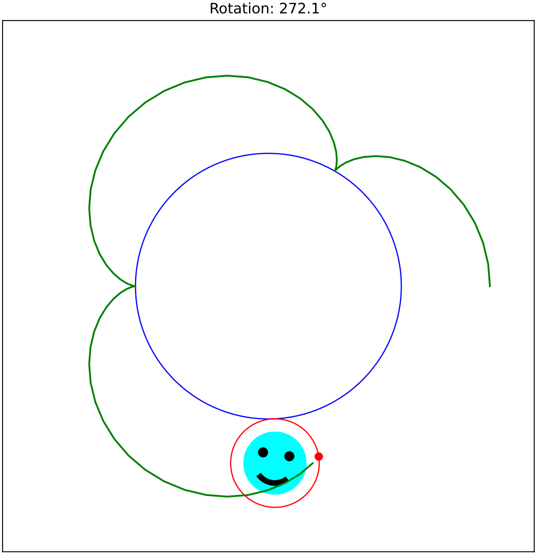
Animation of Coin Rotation Paradox



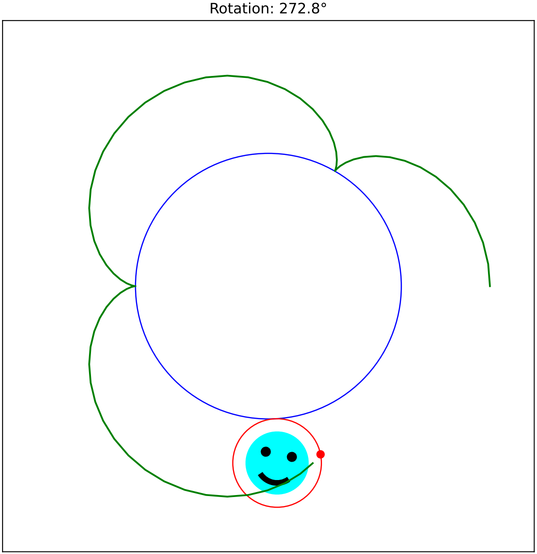
Animation of Coin Rotation Paradox



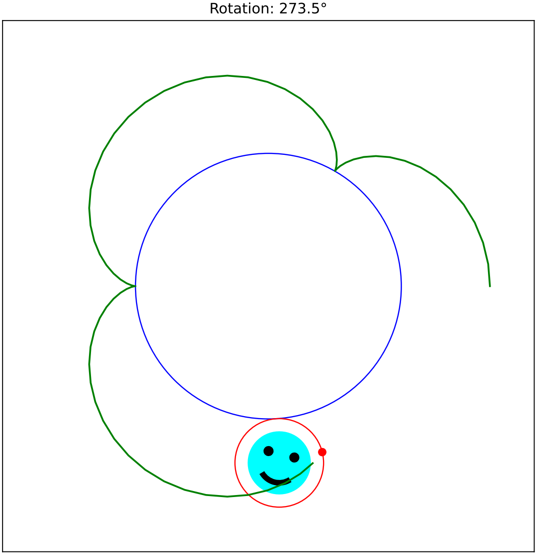
Animation of Coin Rotation Paradox



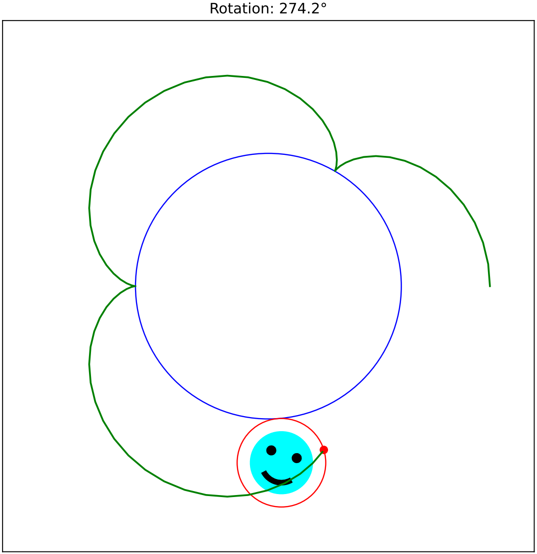
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

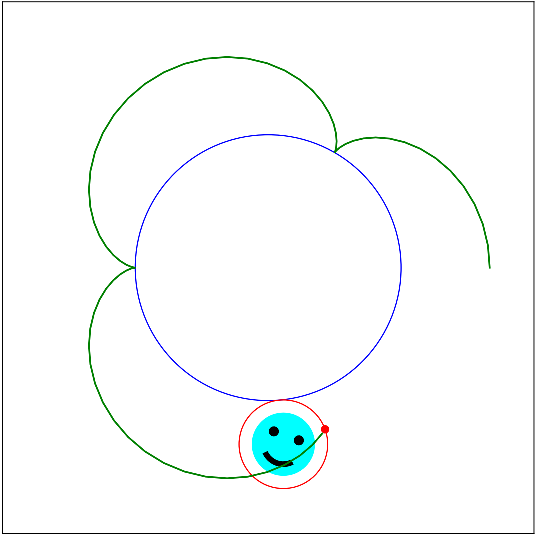


Animation of Coin Rotation Paradox

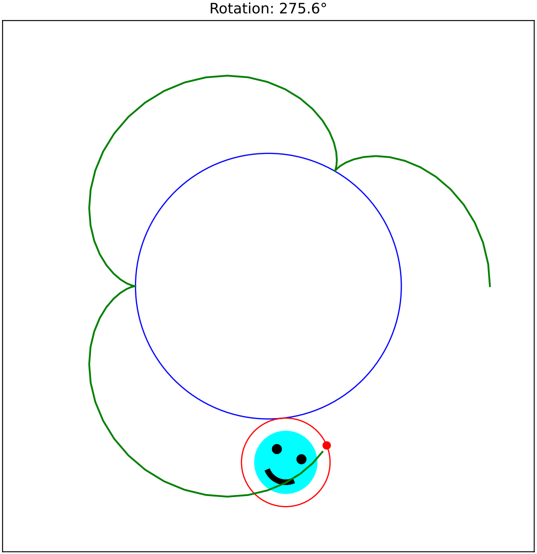


Animation of Coin Rotation Paradox

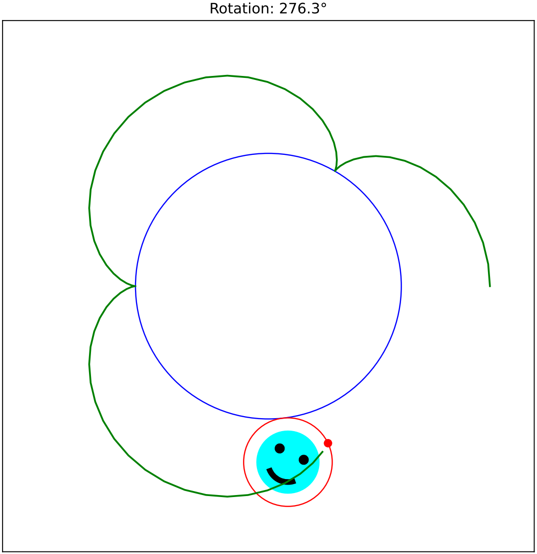
Rotation: 274.9°



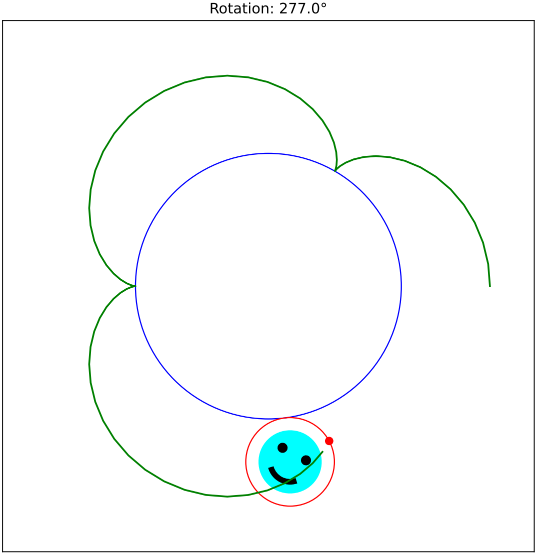
Animation of Coin Rotation Paradox



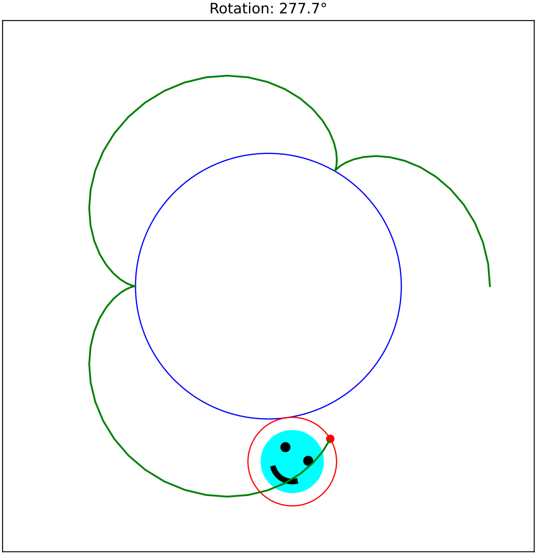
Animation of Coin Rotation Paradox



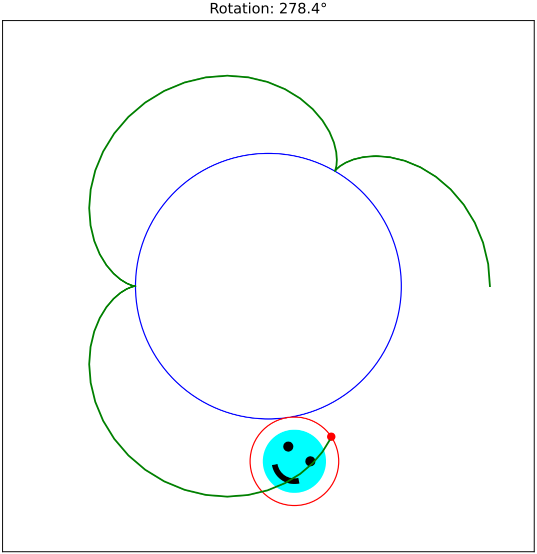
Animation of Coin Rotation Paradox



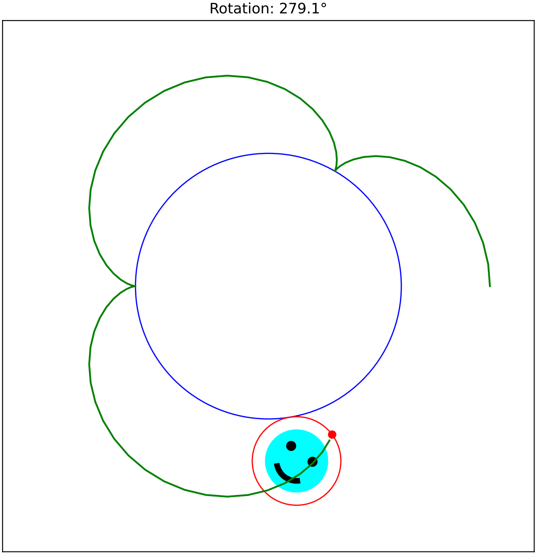
Animation of Coin Rotation Paradox



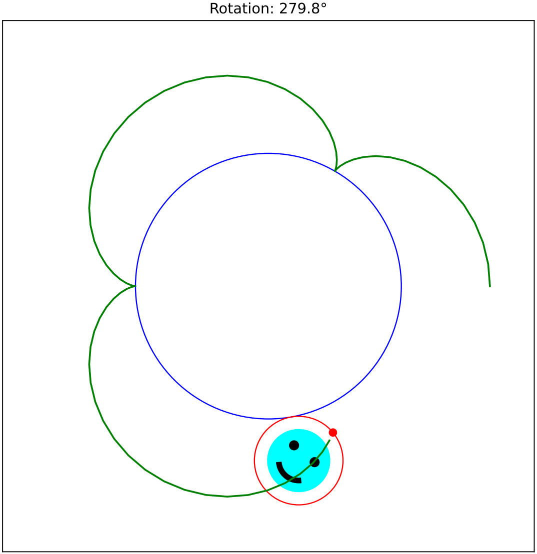
Animation of Coin Rotation Paradox



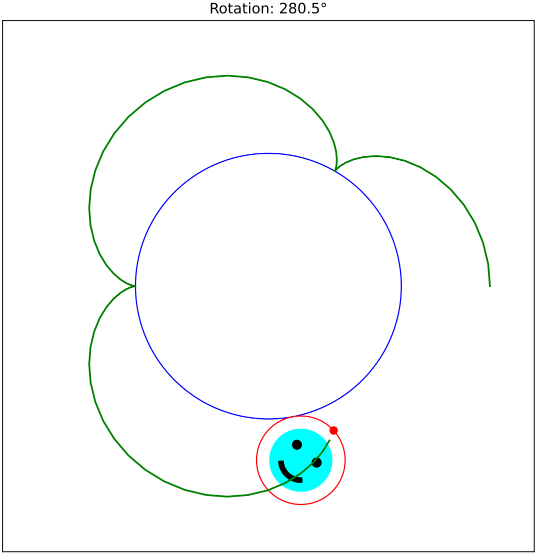
Animation of Coin Rotation Paradox



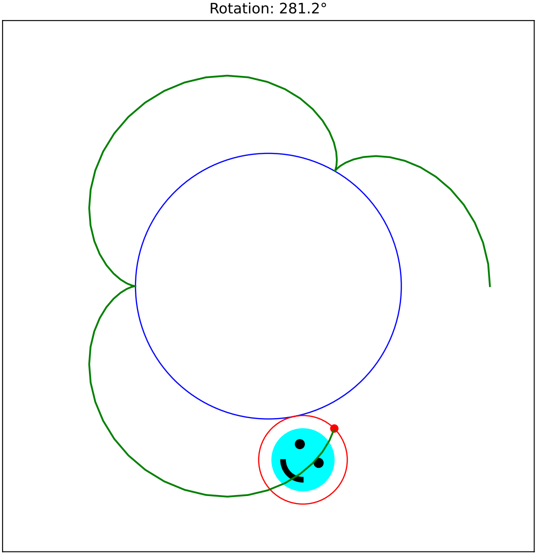
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

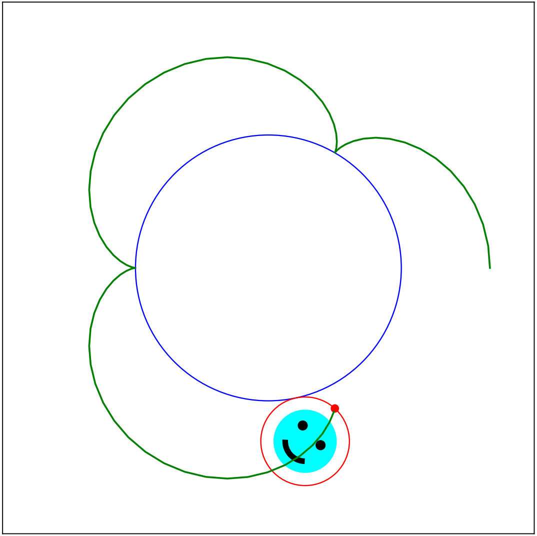


Animation of Coin Rotation Paradox

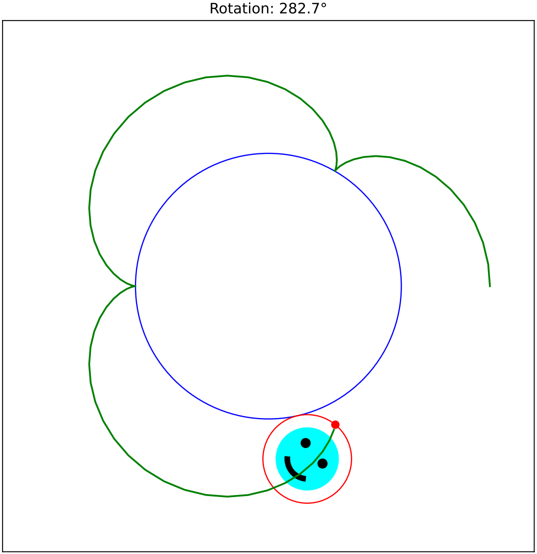


Animation of Coin Rotation Paradox

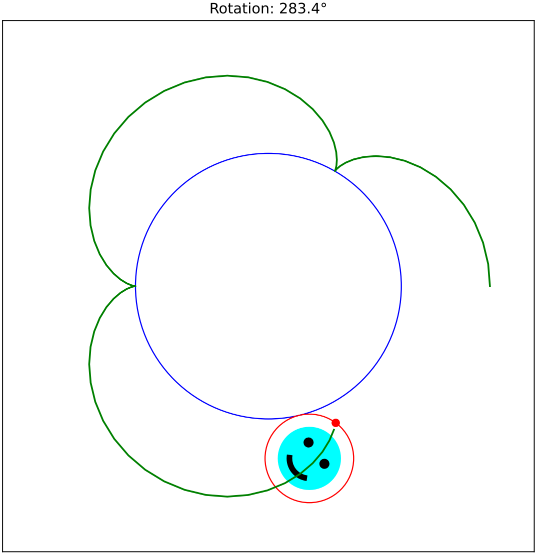
Rotation: 282.0°



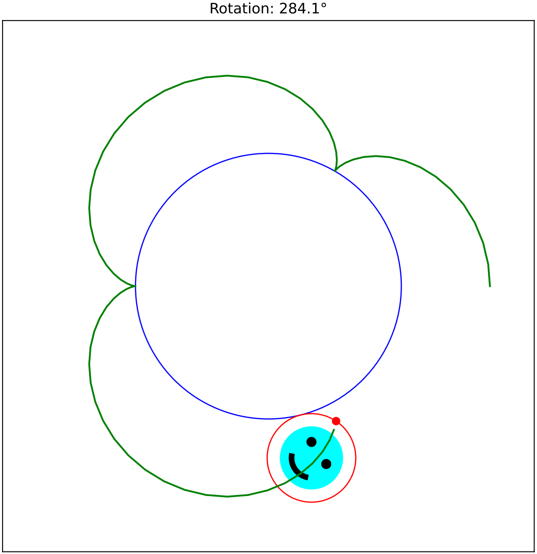
Animation of Coin Rotation Paradox



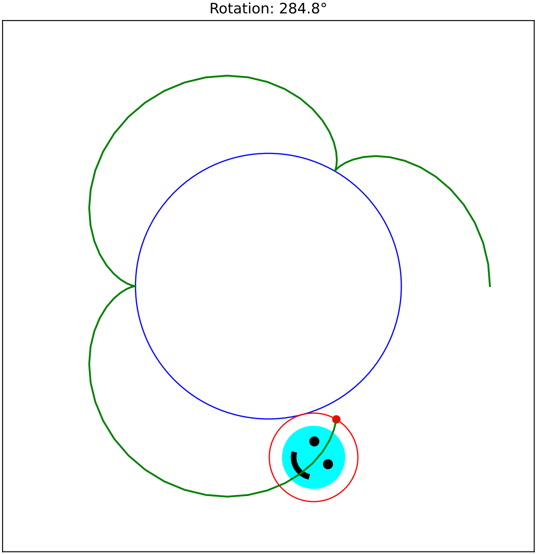
Animation of Coin Rotation Paradox



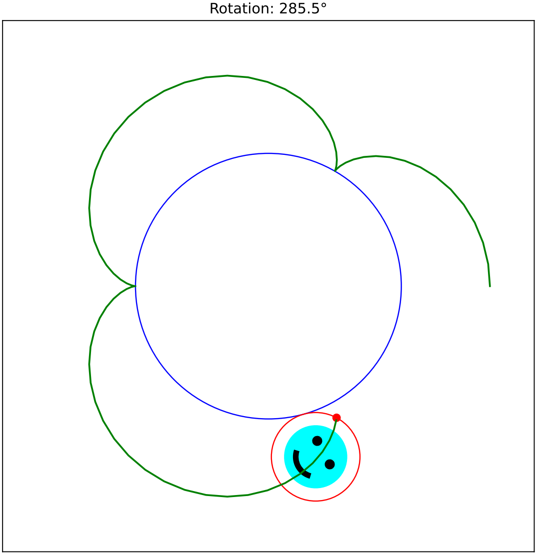
Animation of Coin Rotation Paradox



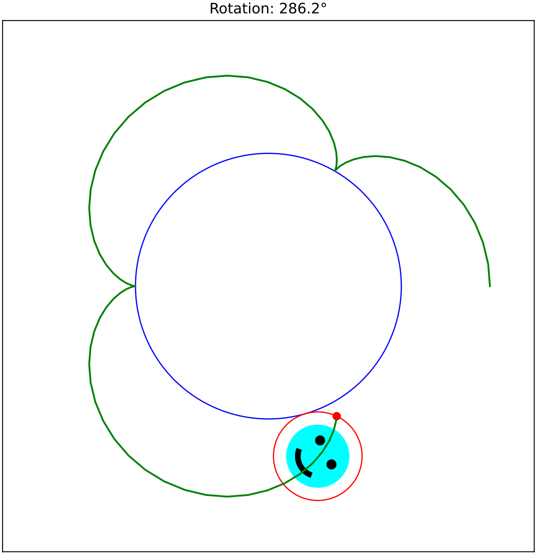
Animation of Coin Rotation Paradox



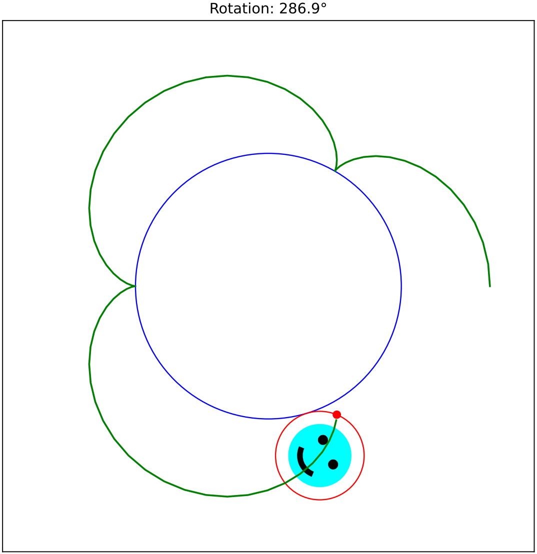
Animation of Coin Rotation Paradox



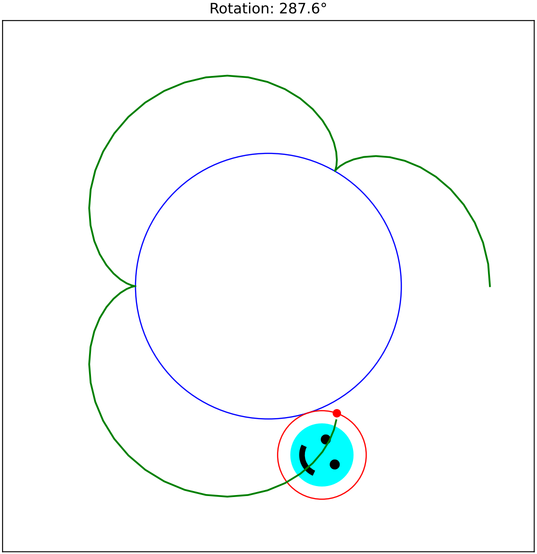
Animation of Coin Rotation Paradox



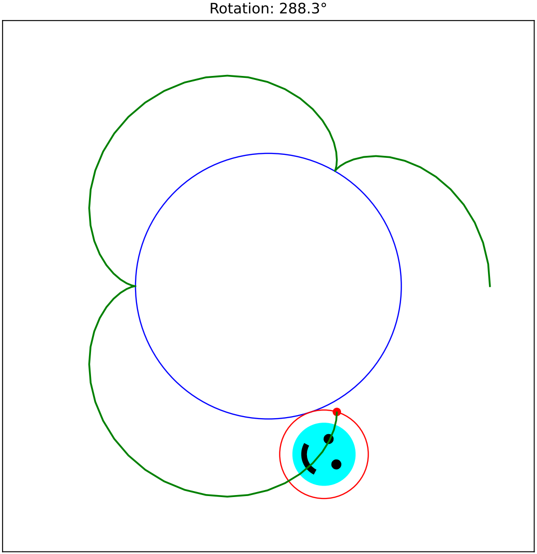
Animation of Coin Rotation Paradox



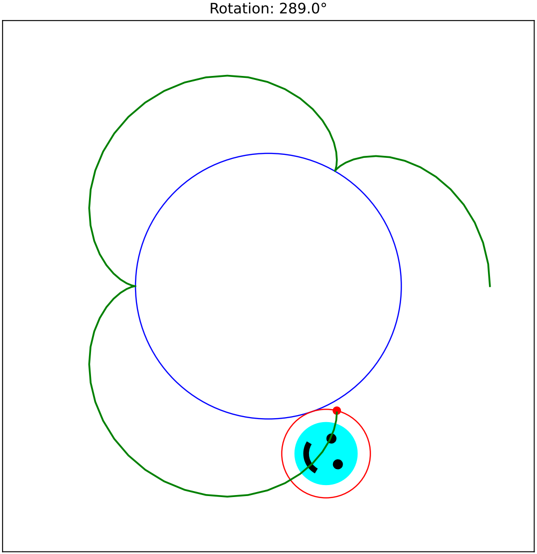
Animation of Coin Rotation Paradox



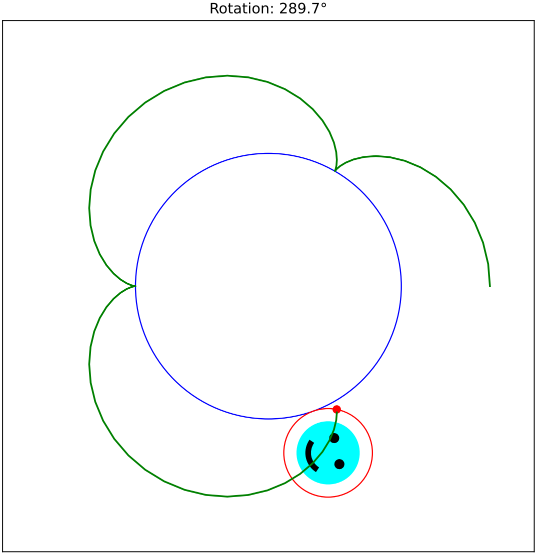
Animation of Coin Rotation Paradox



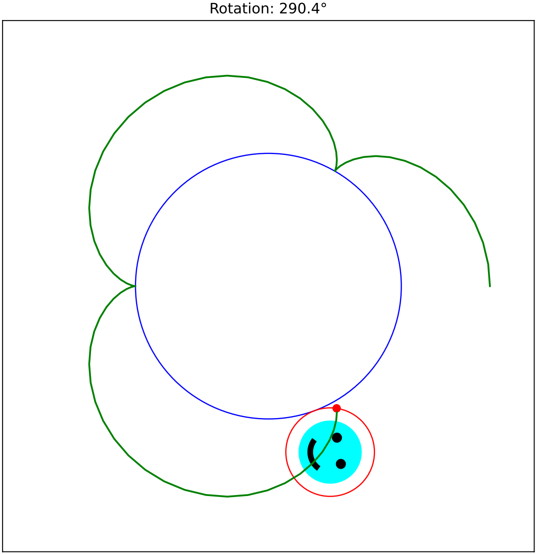
Animation of Coin Rotation Paradox



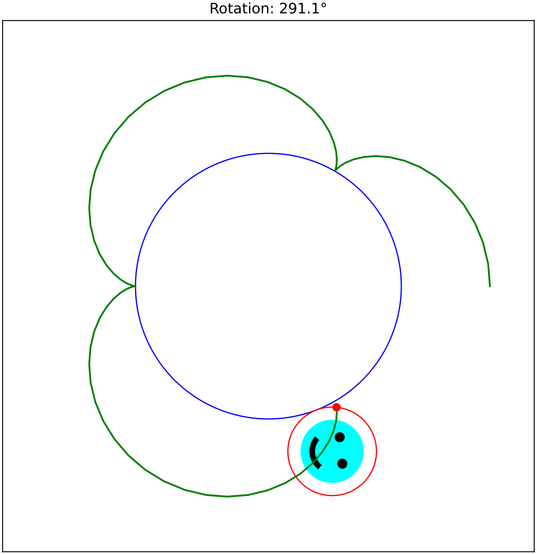
Animation of Coin Rotation Paradox



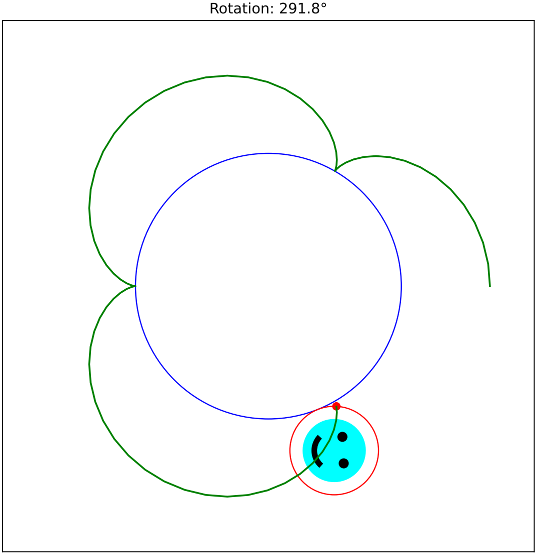
Animation of Coin Rotation Paradox



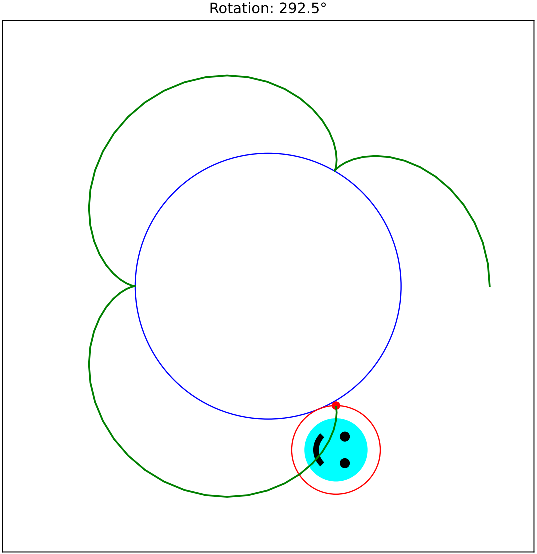
Animation of Coin Rotation Paradox



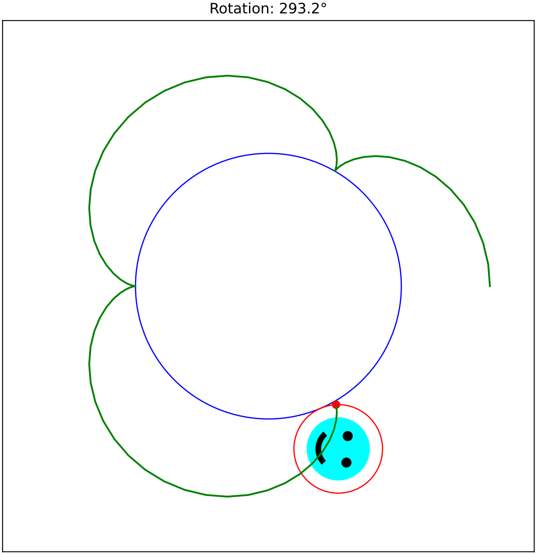
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

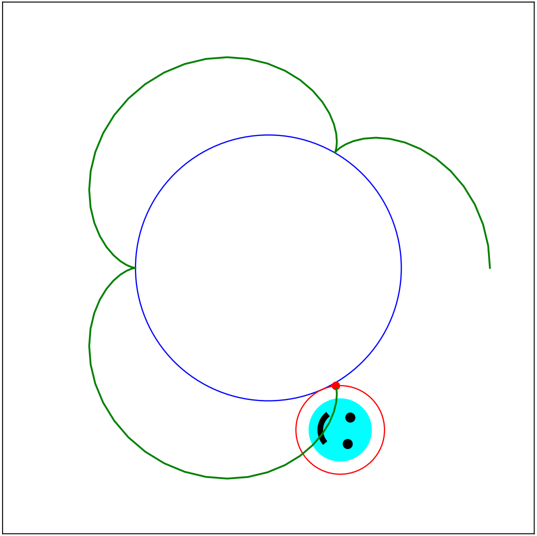


Animation of Coin Rotation Paradox

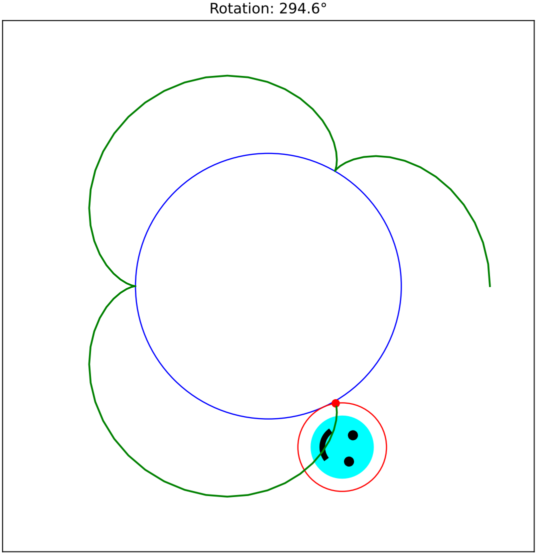


Animation of Coin Rotation Paradox

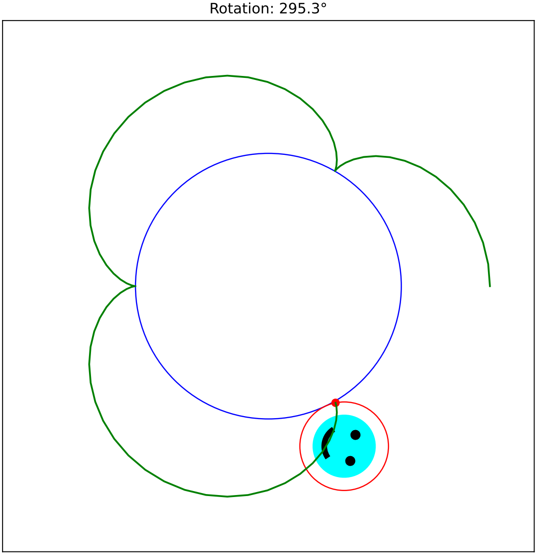
Rotation: 293.9°



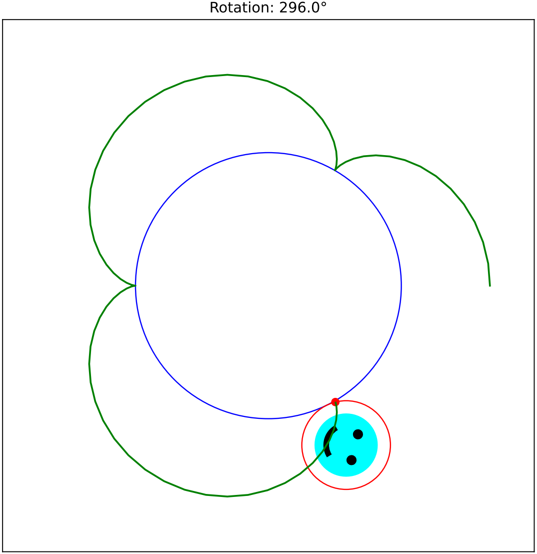
Animation of Coin Rotation Paradox



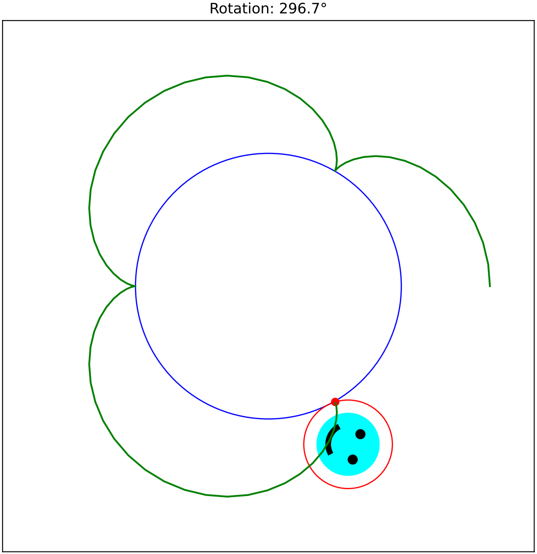
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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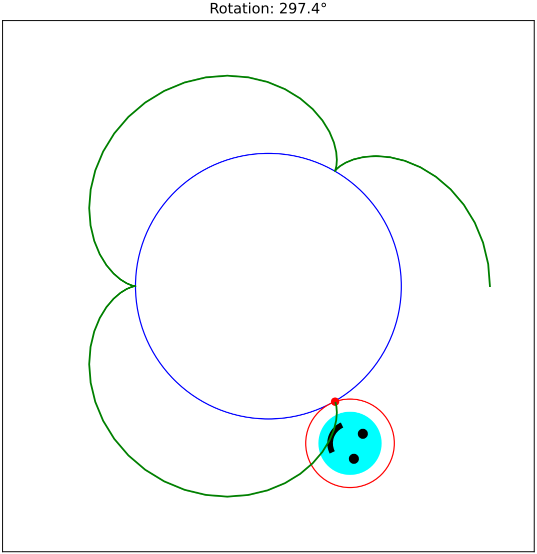

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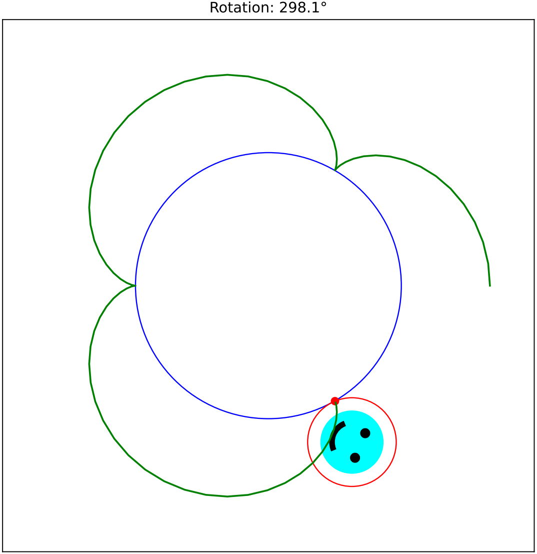

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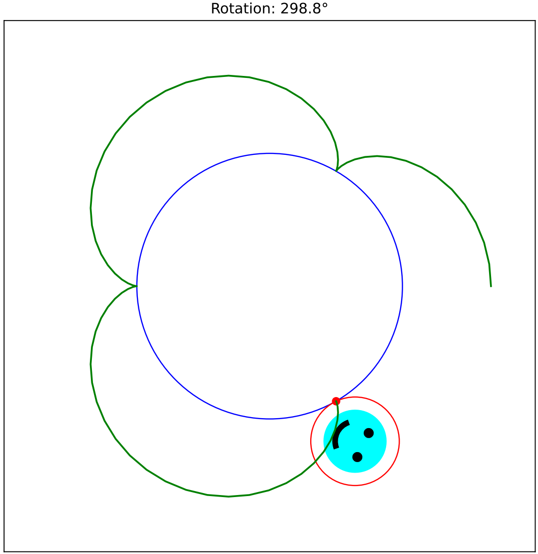

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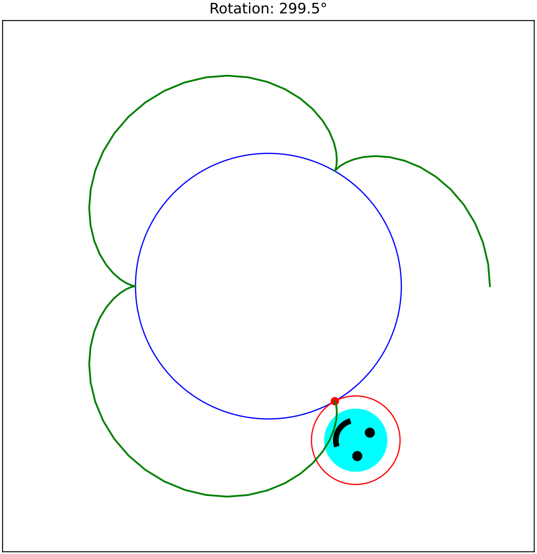

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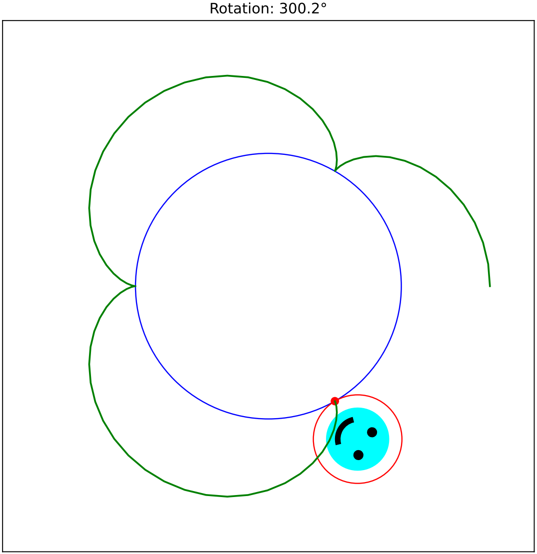

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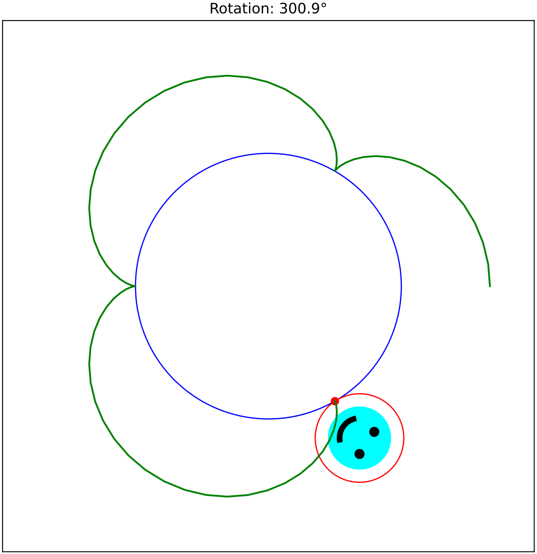

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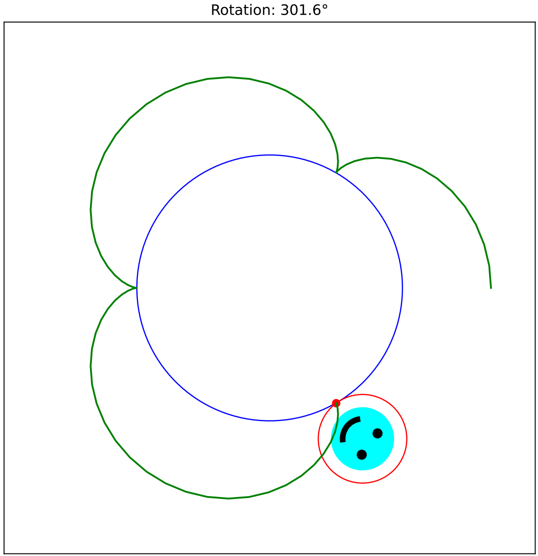

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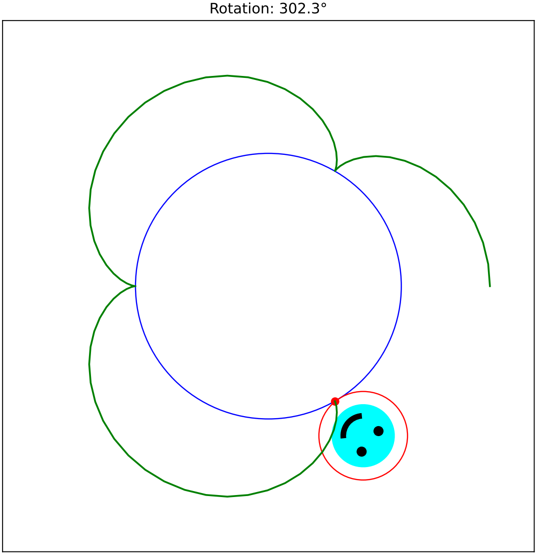

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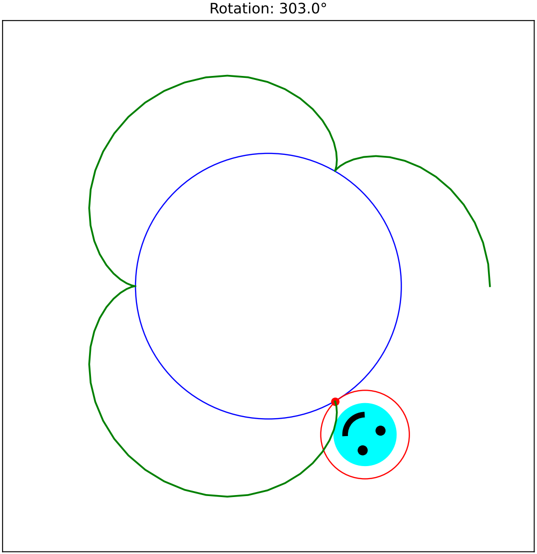

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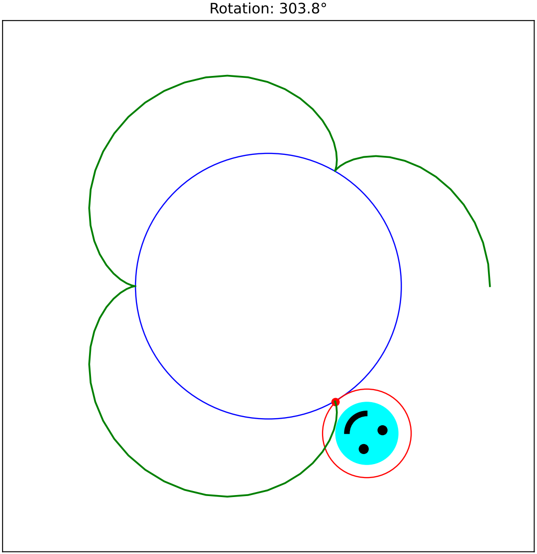

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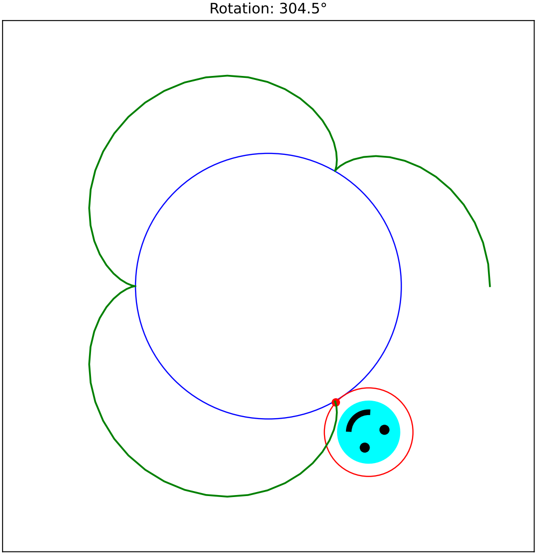

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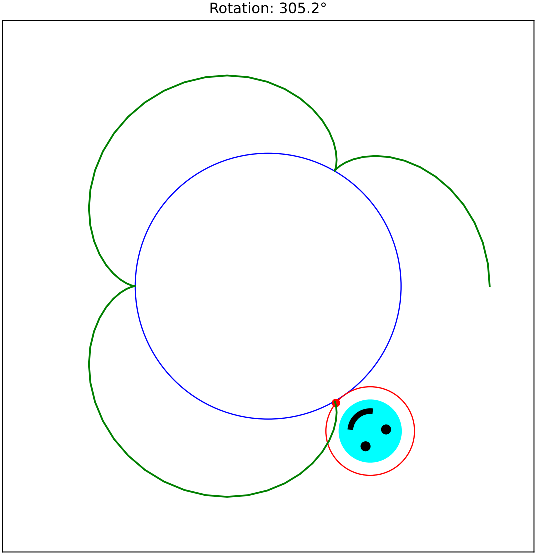

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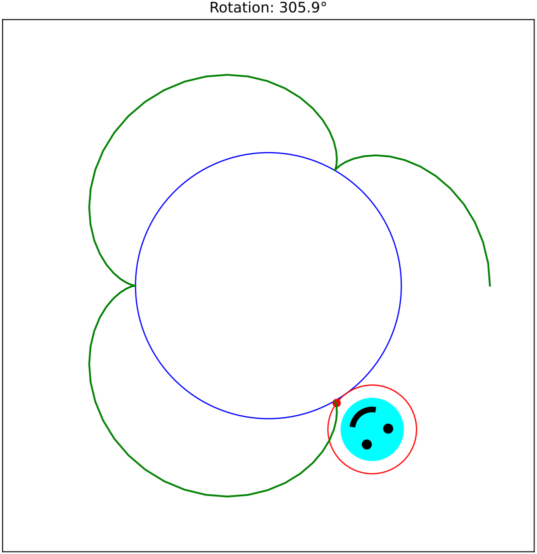

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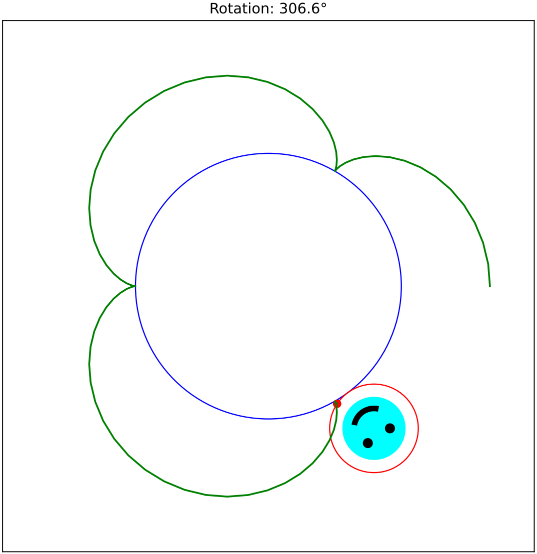

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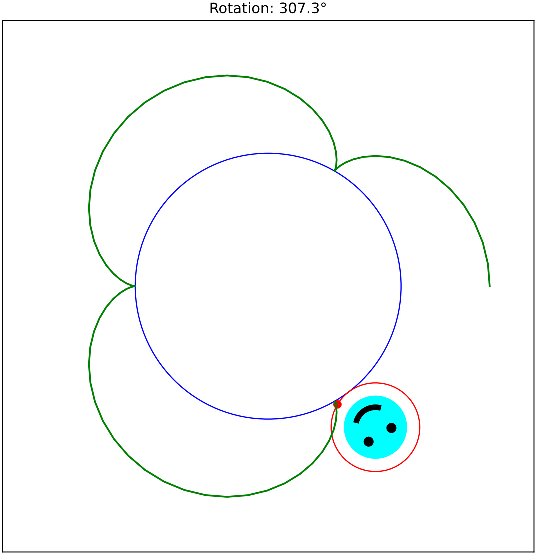

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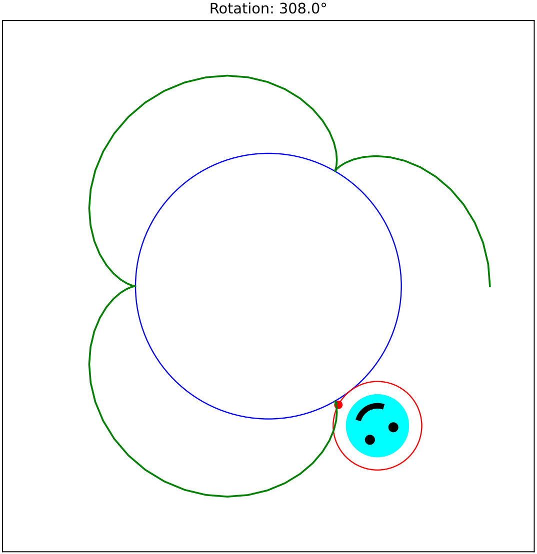

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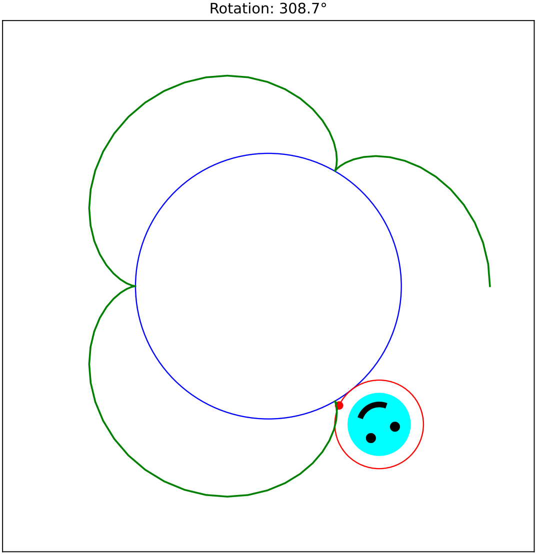

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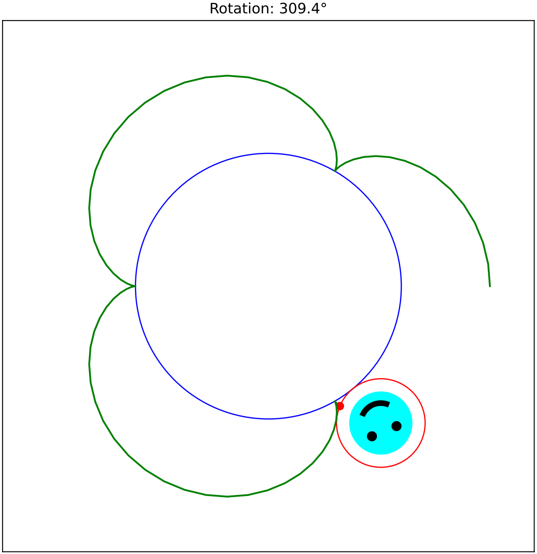

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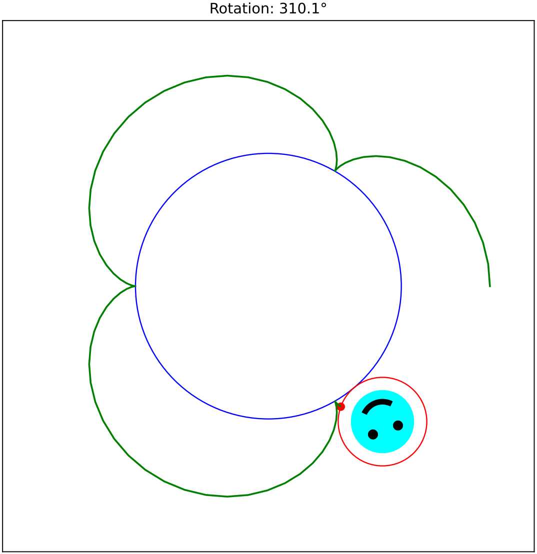

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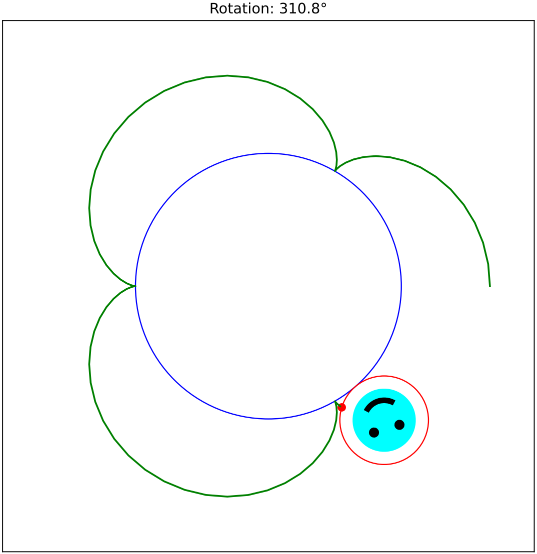

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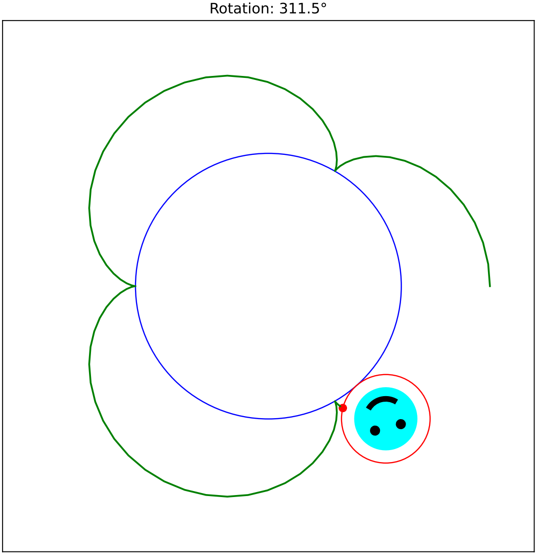

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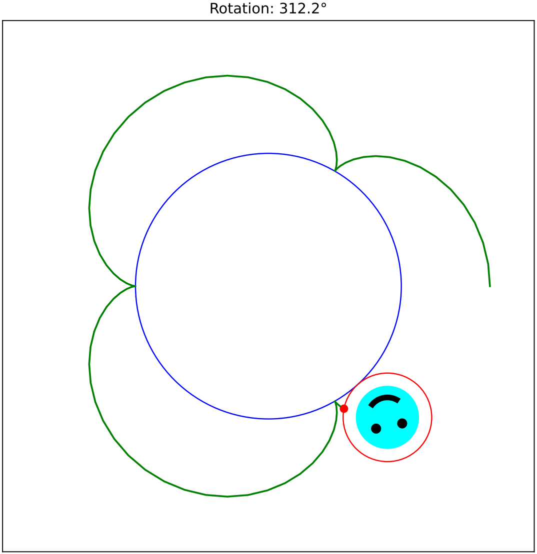

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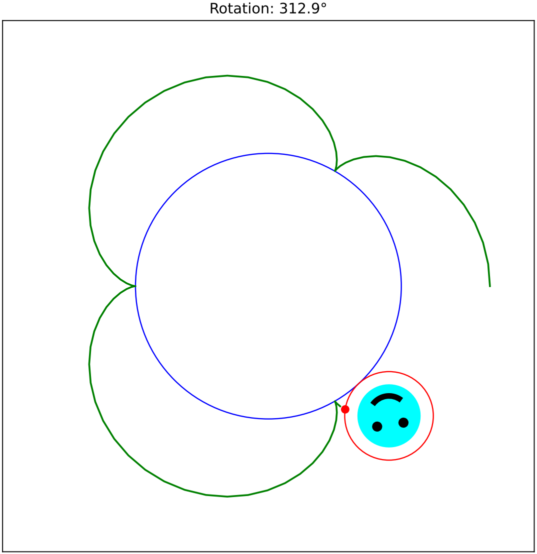

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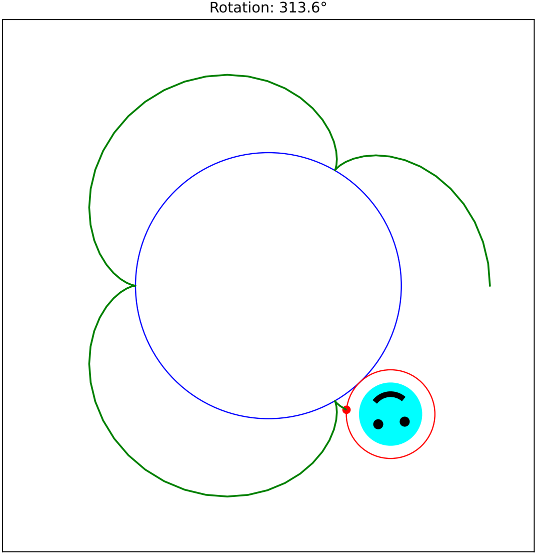

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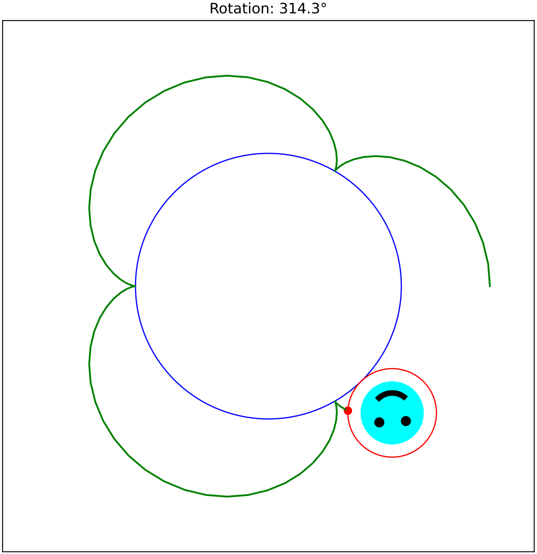

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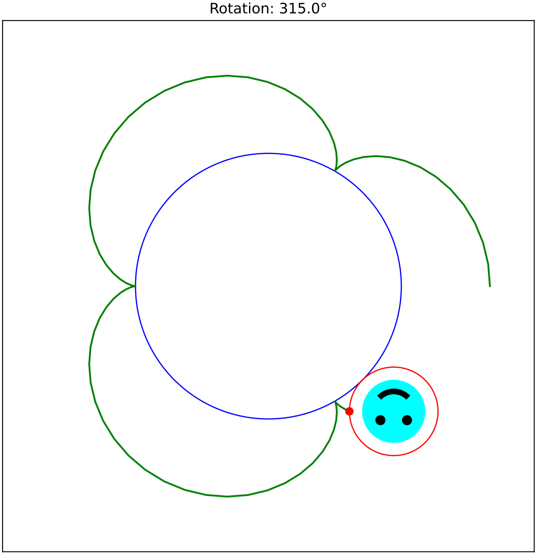

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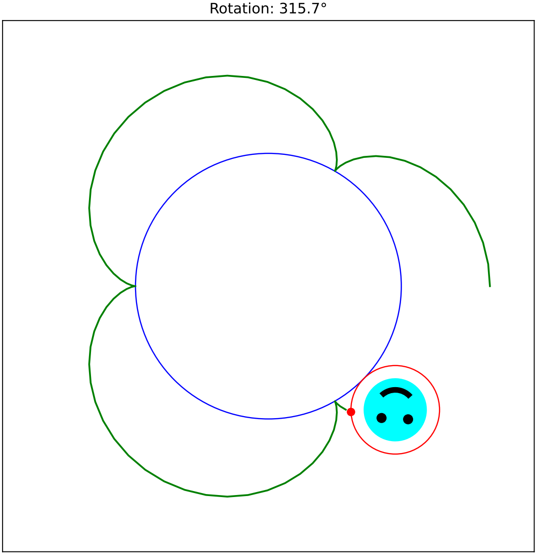

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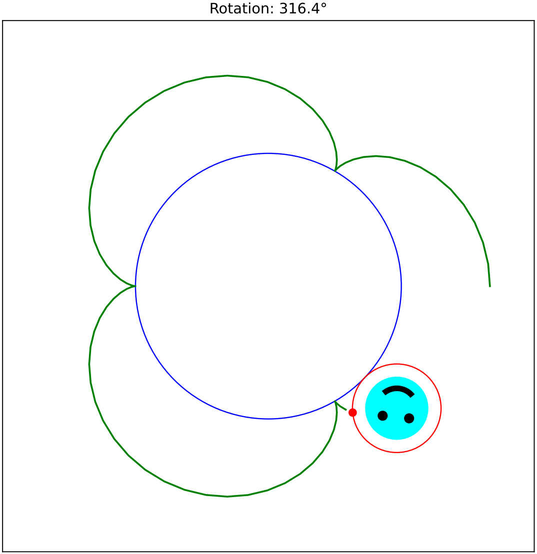

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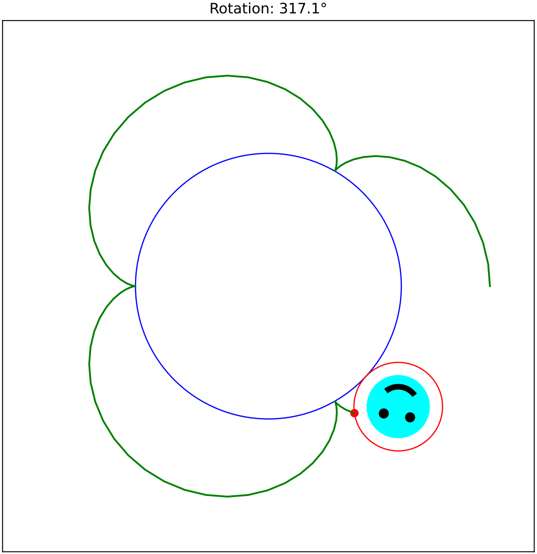

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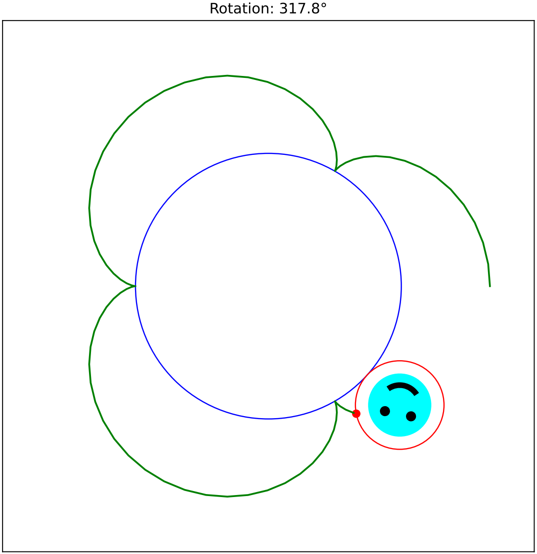

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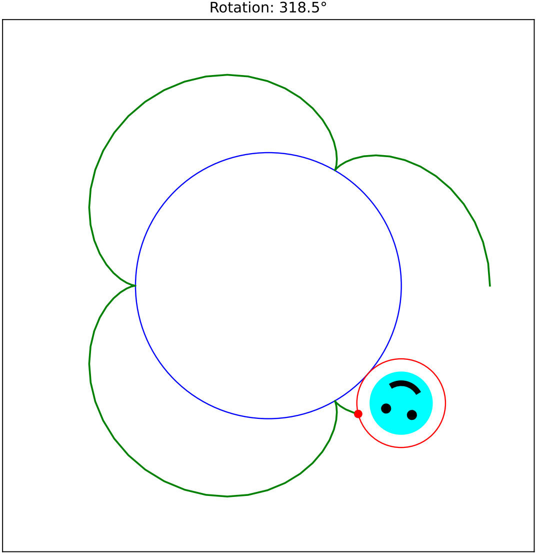

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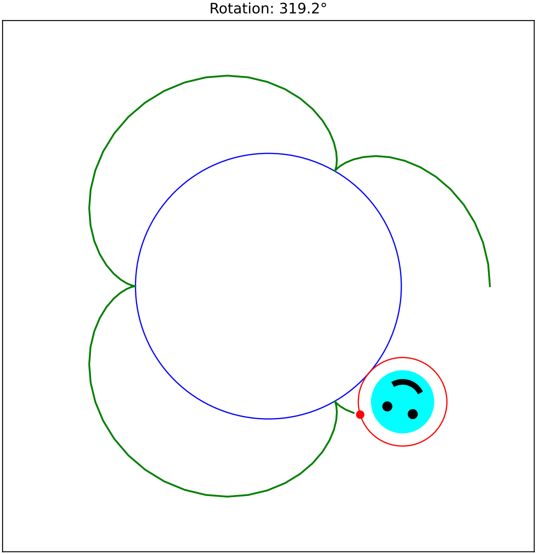

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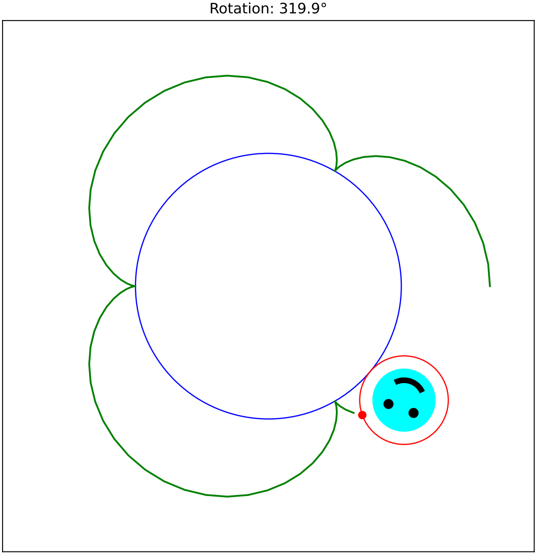

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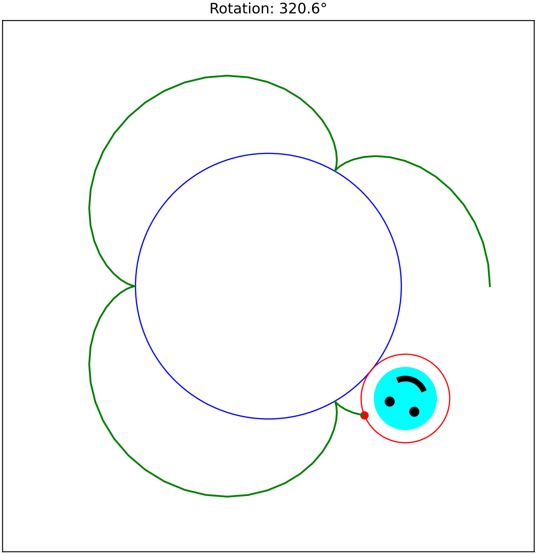

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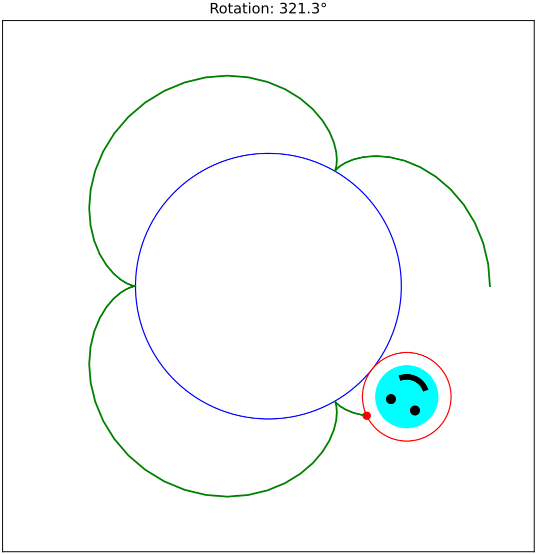

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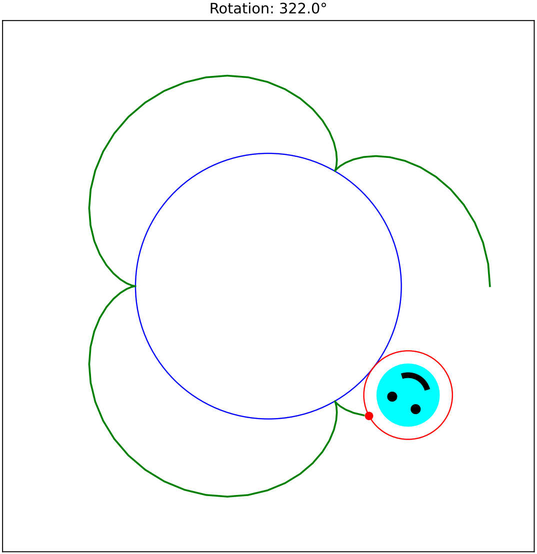

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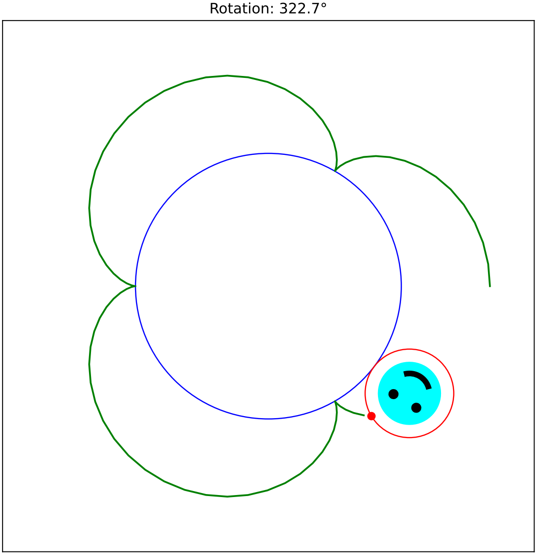

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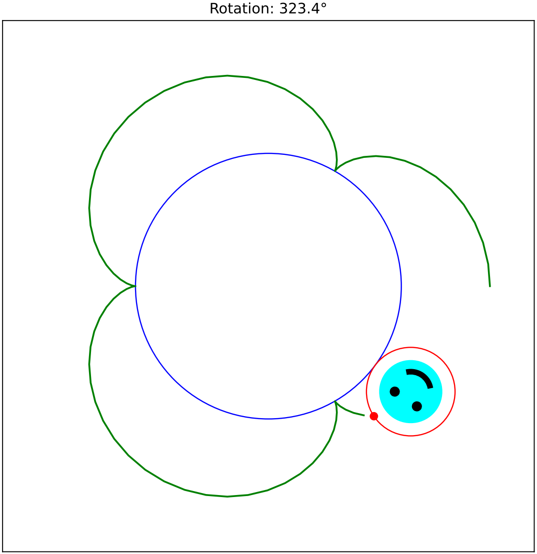

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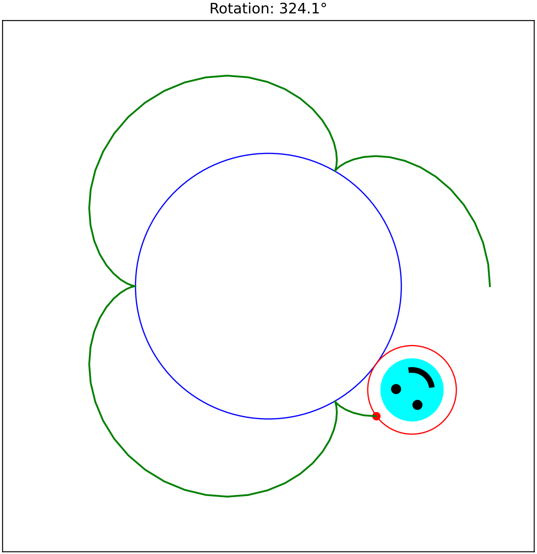

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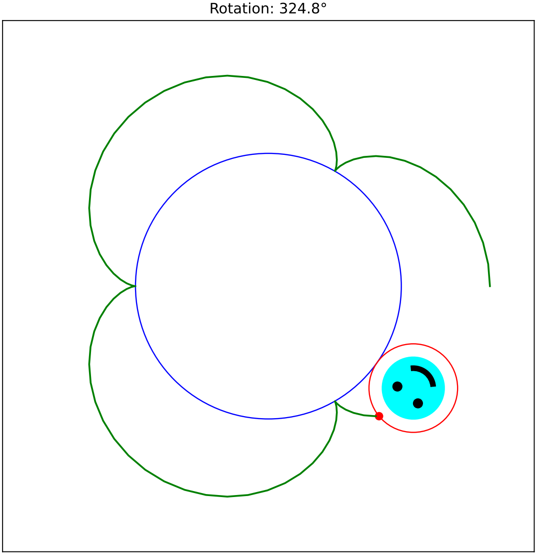

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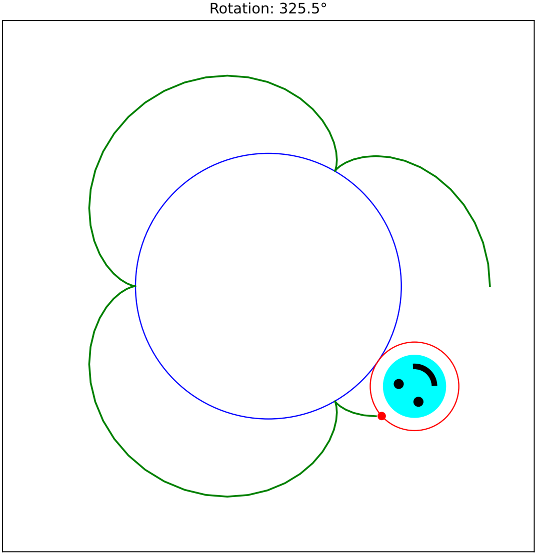

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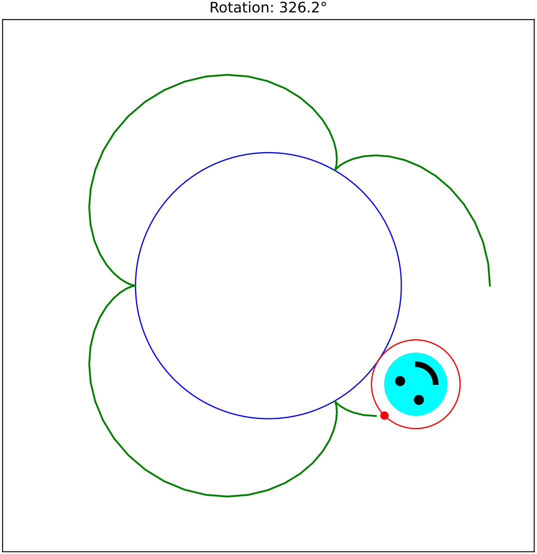

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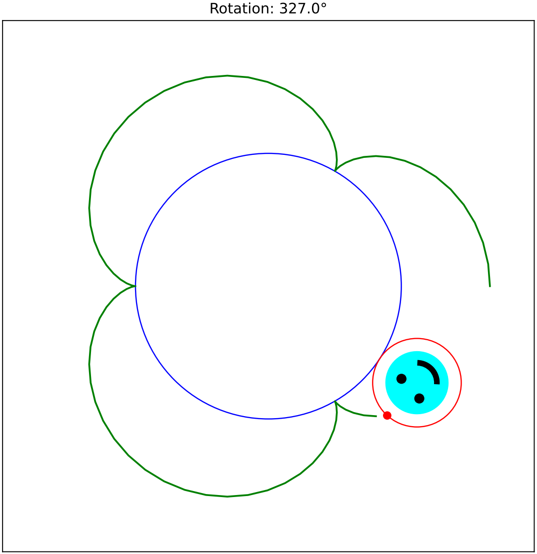

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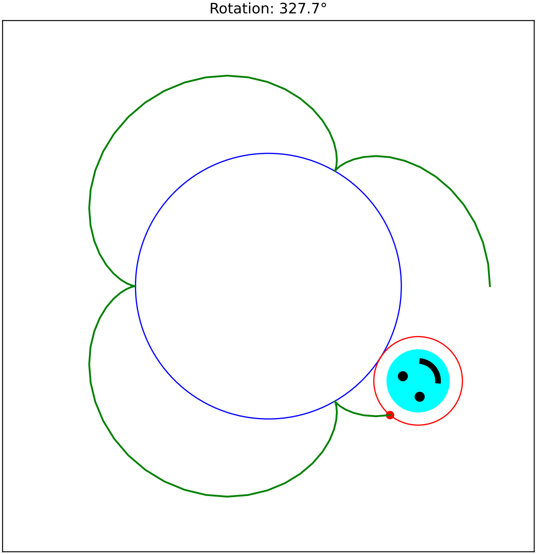

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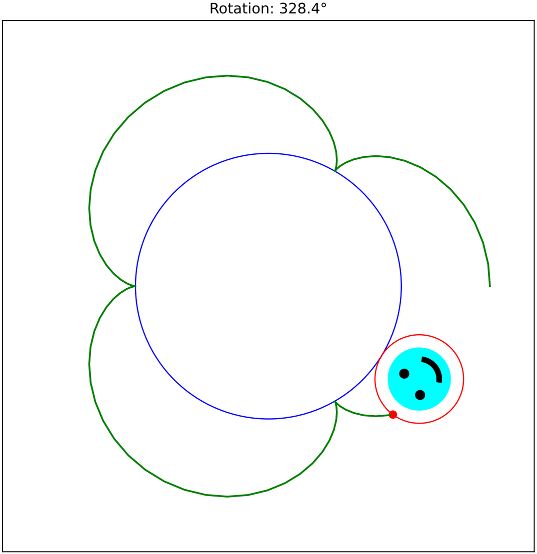

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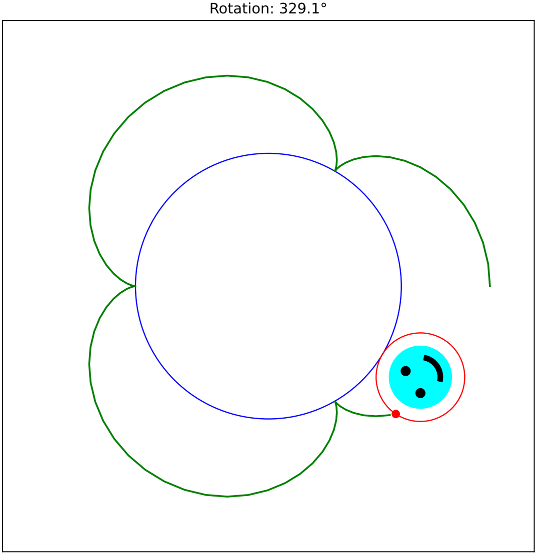

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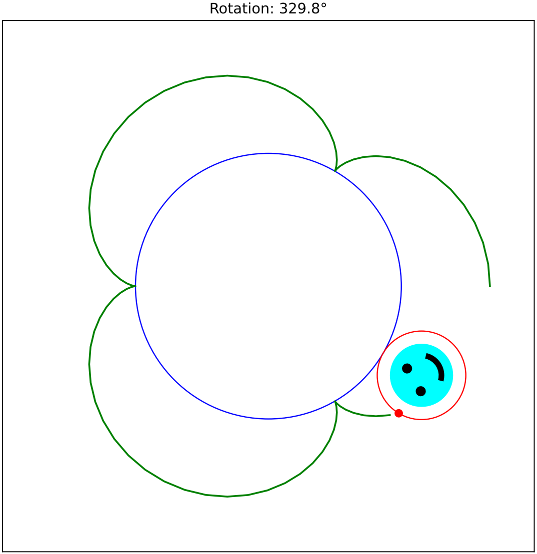

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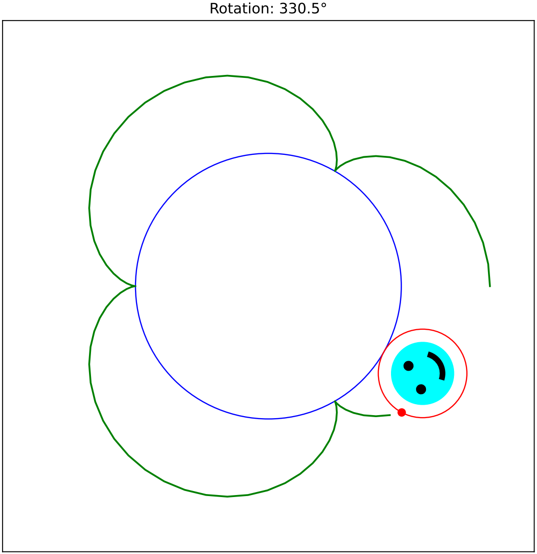

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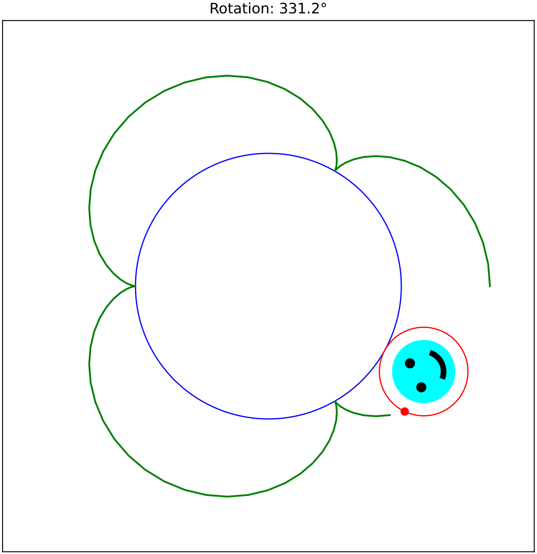

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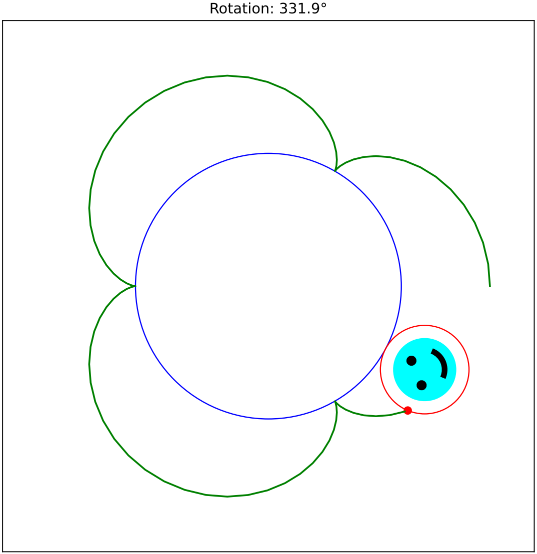

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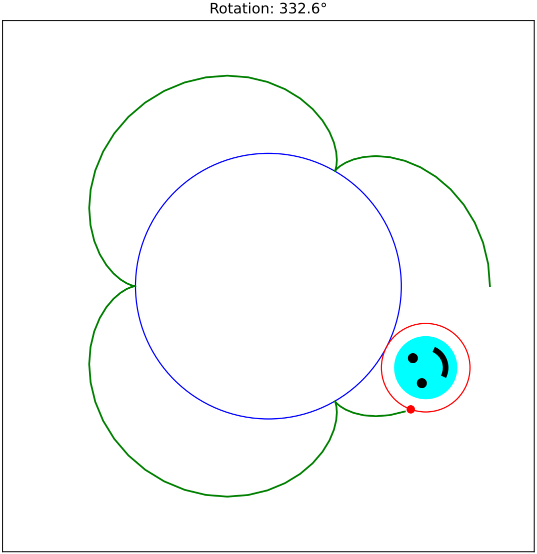

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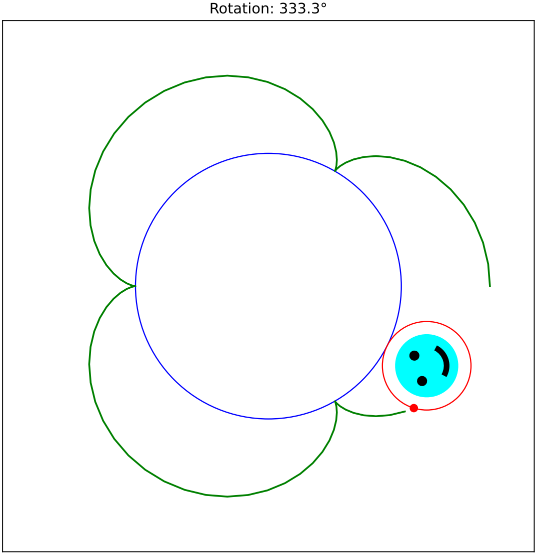

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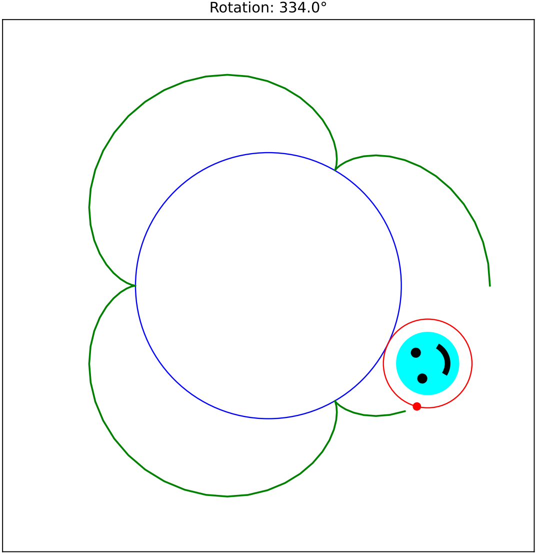

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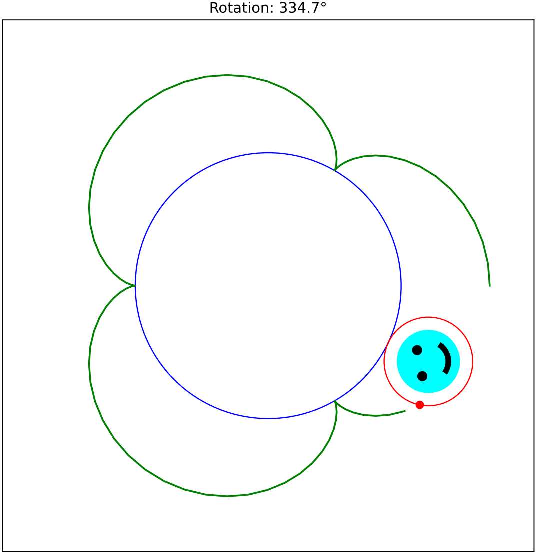

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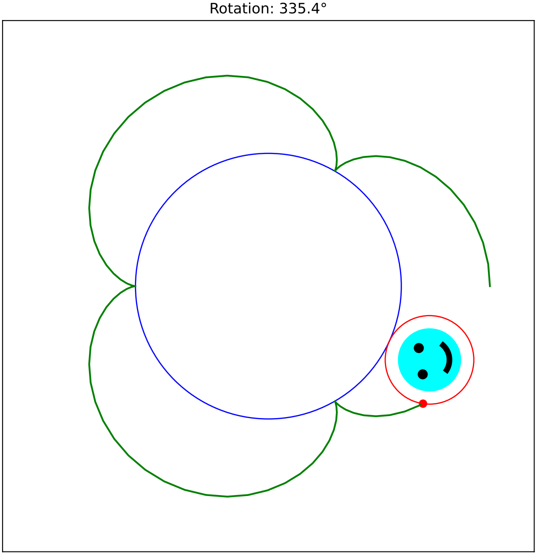

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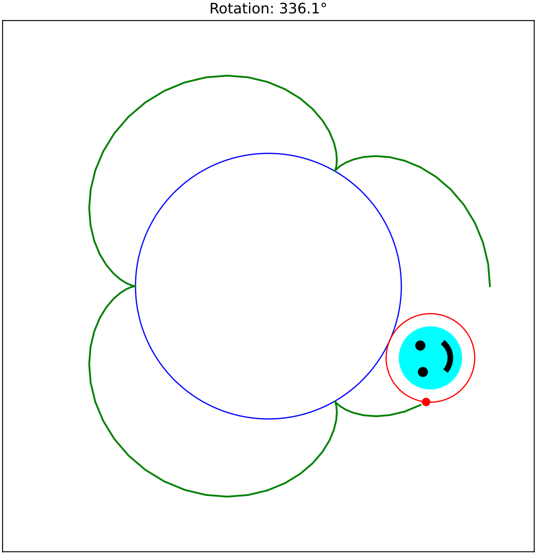

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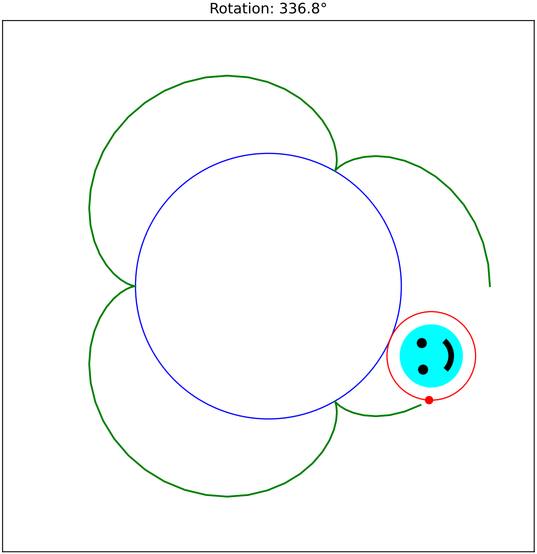

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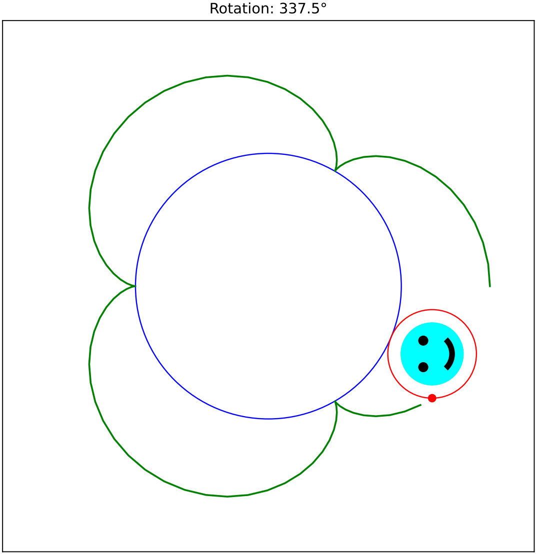

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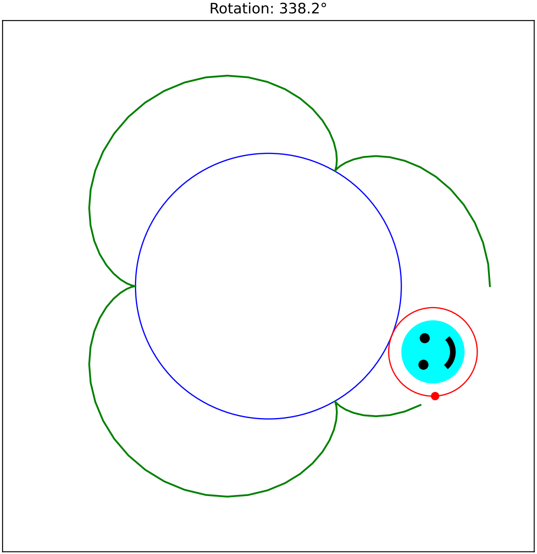

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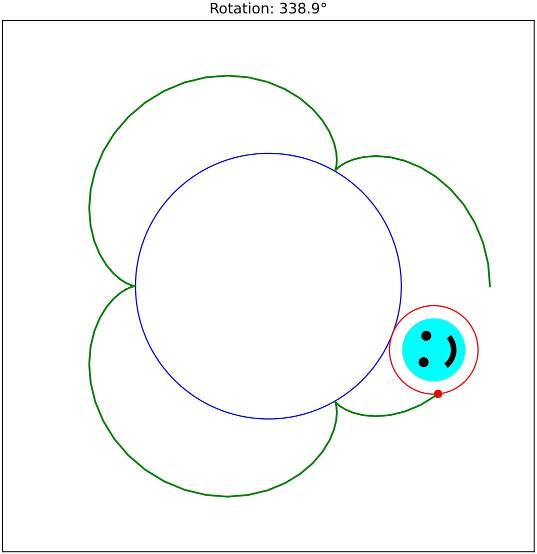

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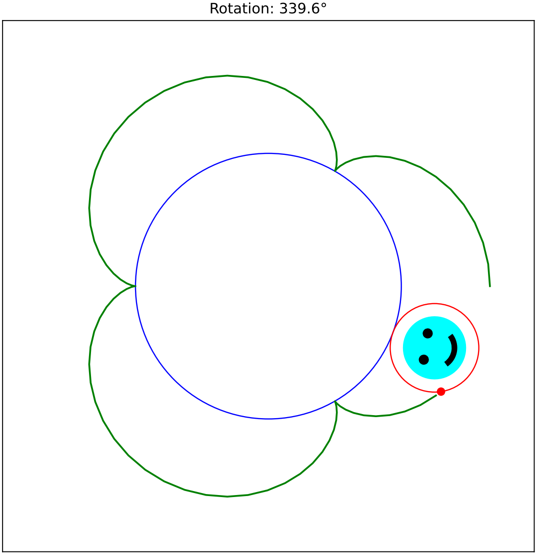

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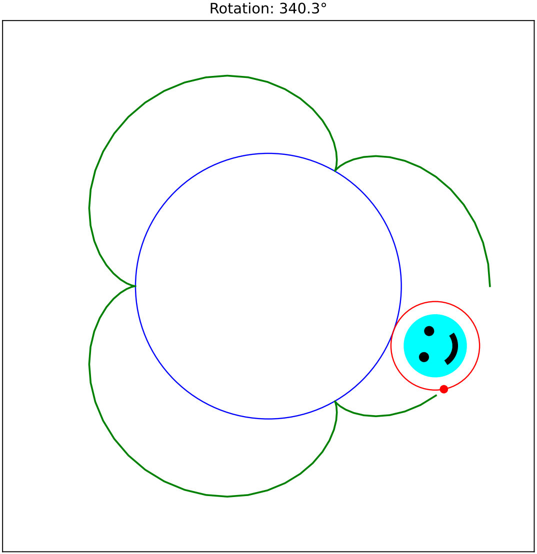

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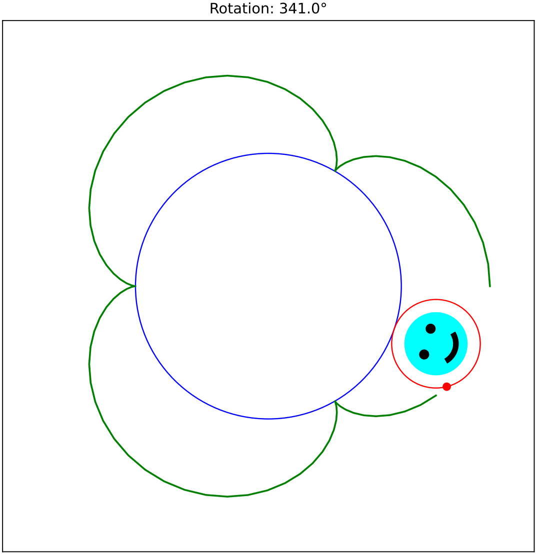

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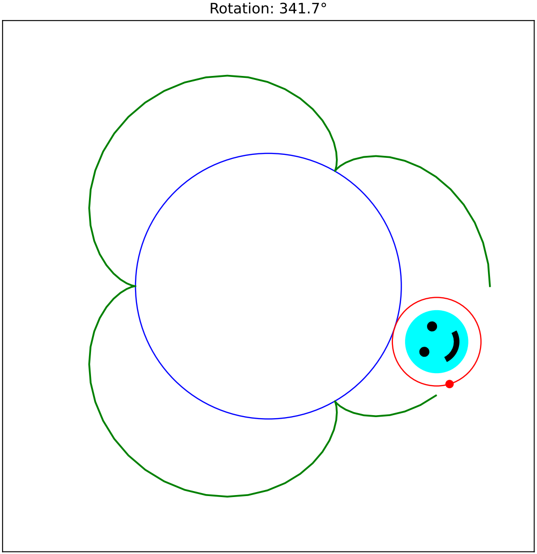

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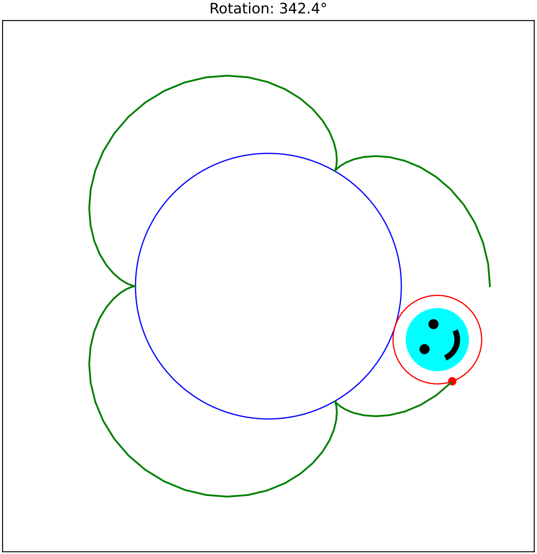

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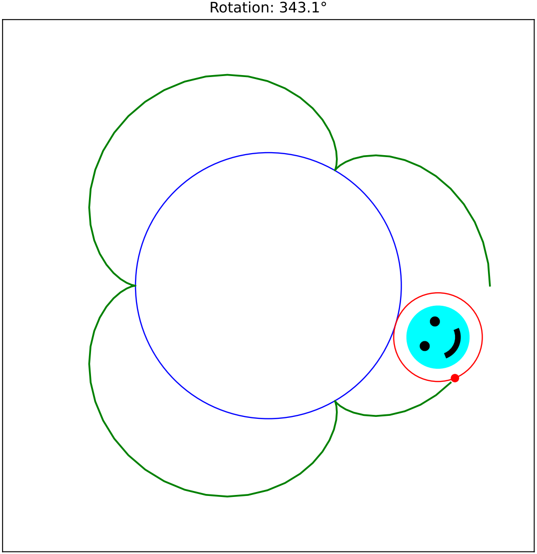

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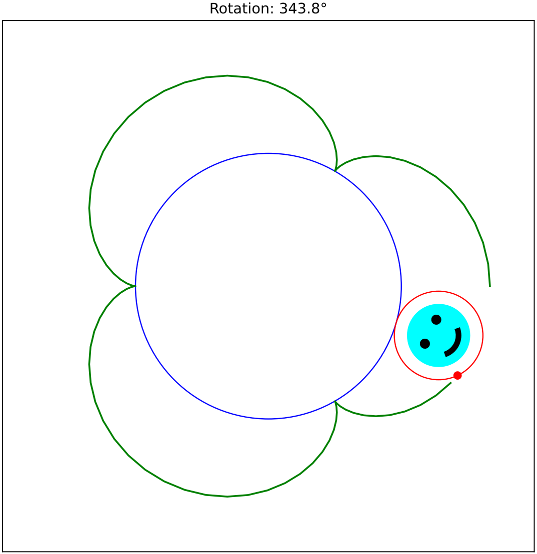
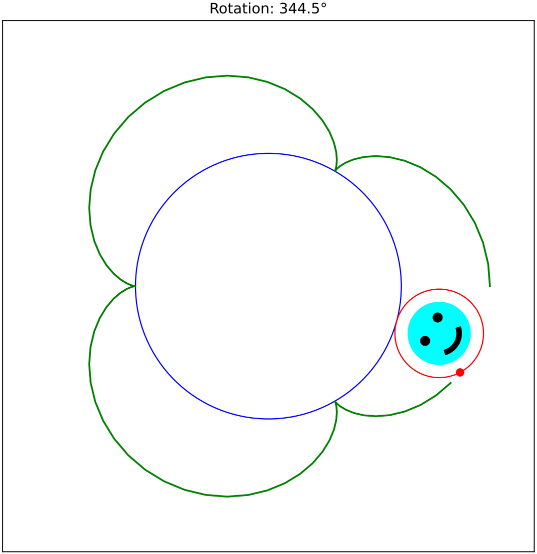


Animation of Coin Rotation Paradox



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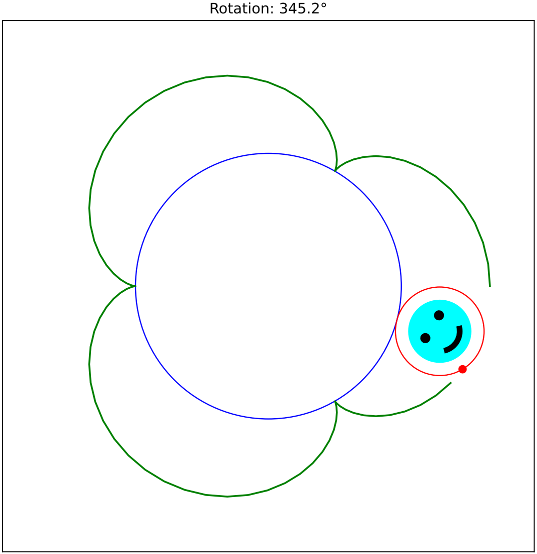

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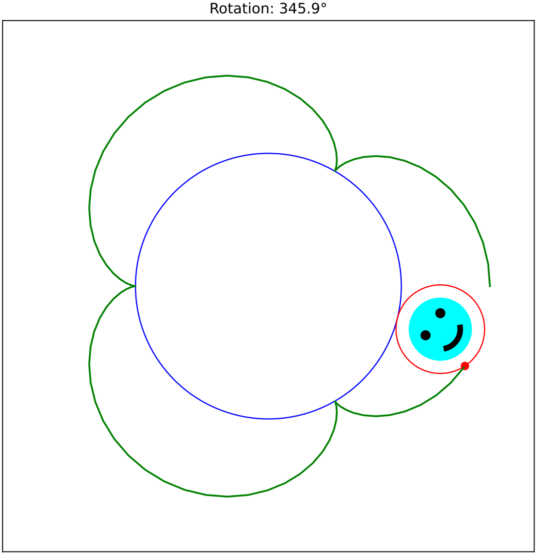

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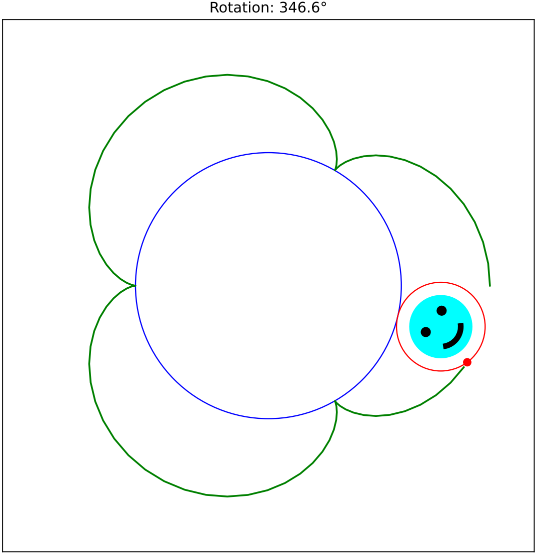
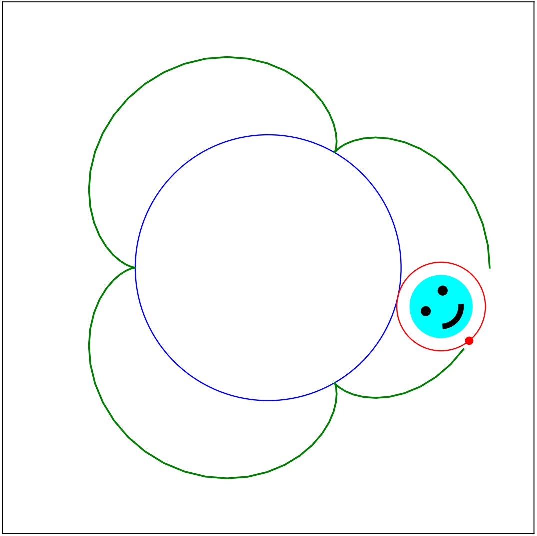


Animation of Coin Rotation Paradox

Rotation: 347.3°



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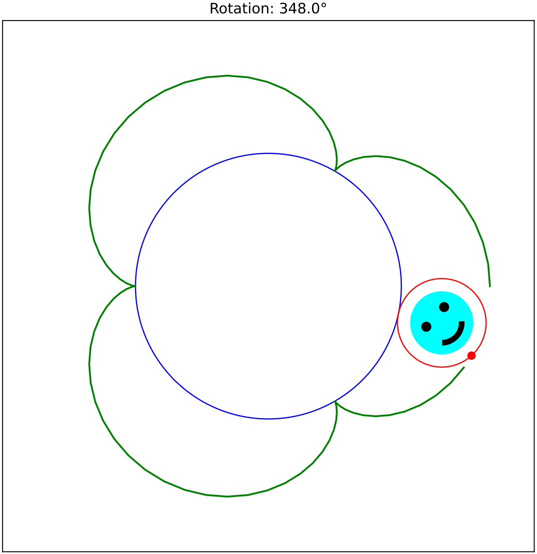

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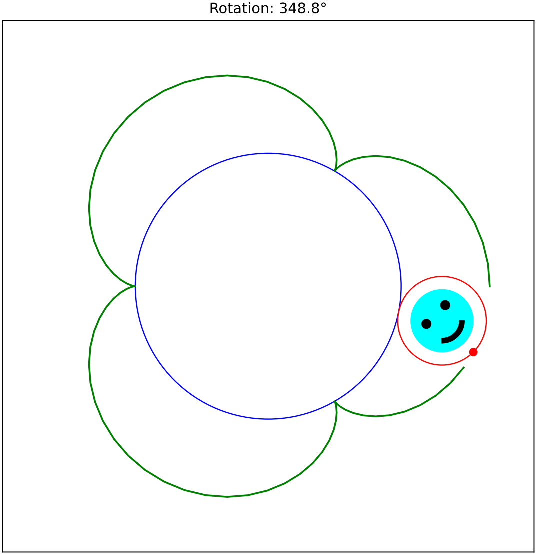

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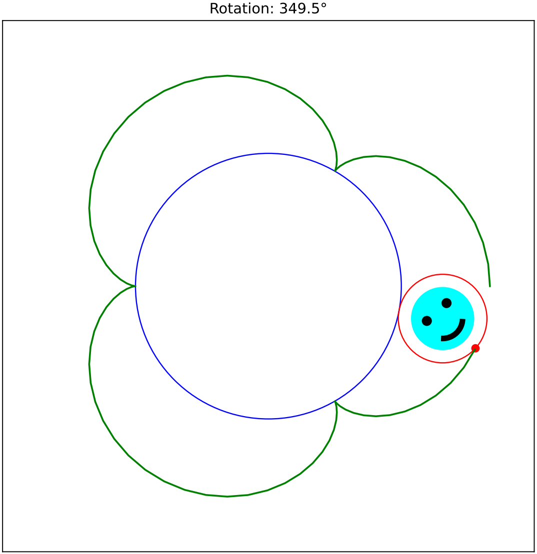

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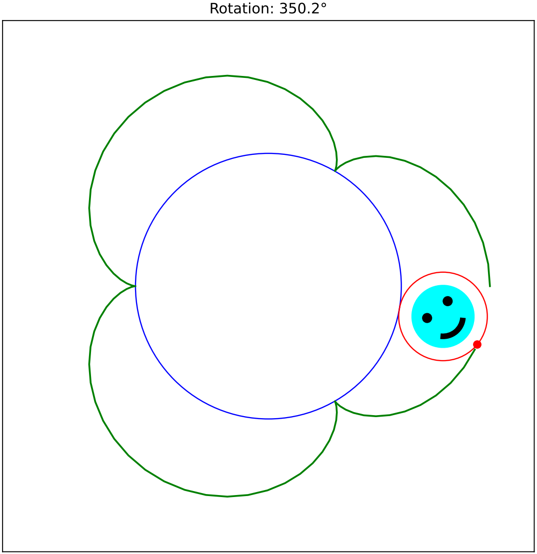

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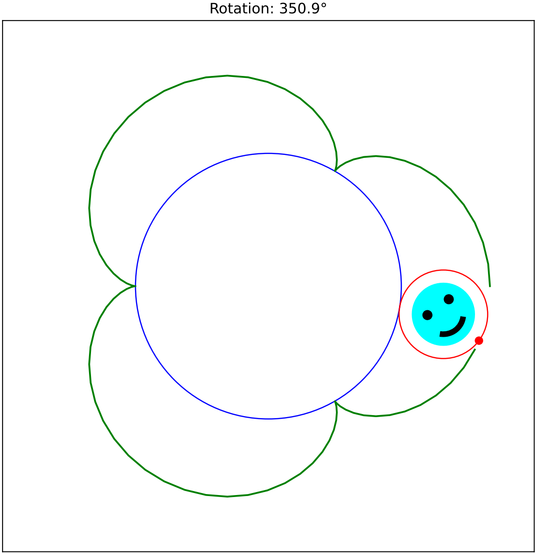

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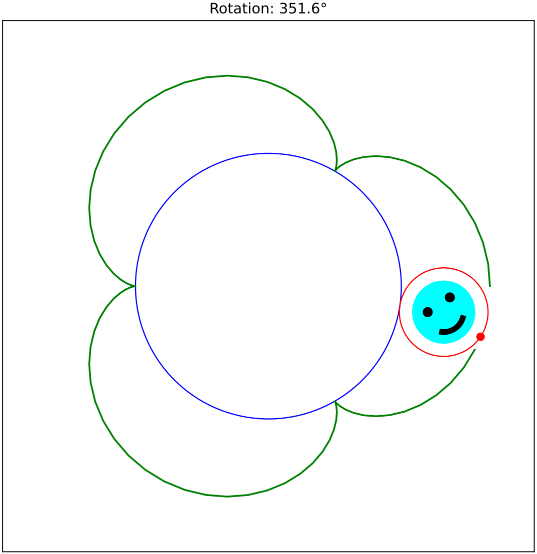

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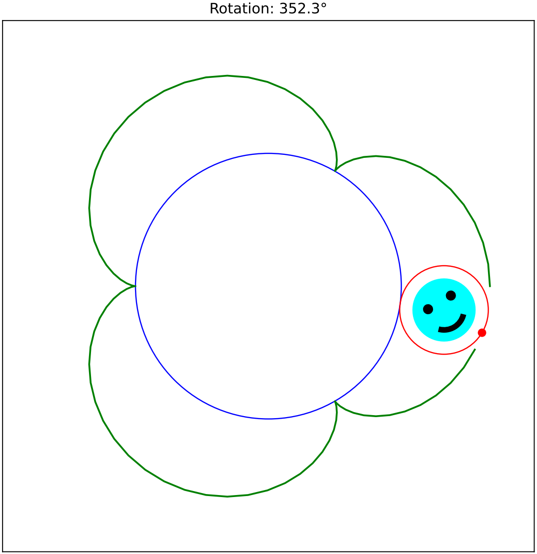

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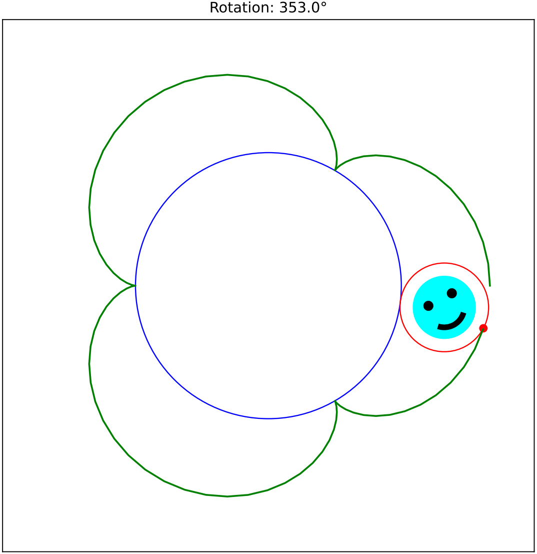

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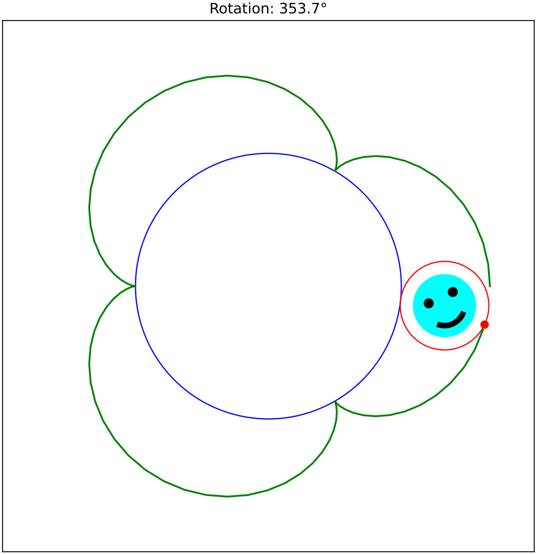

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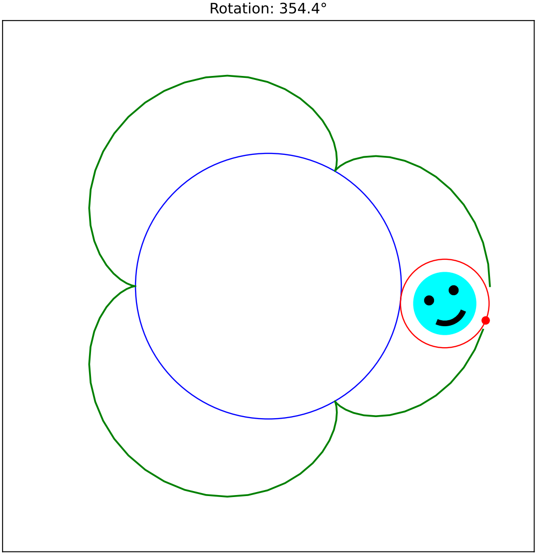

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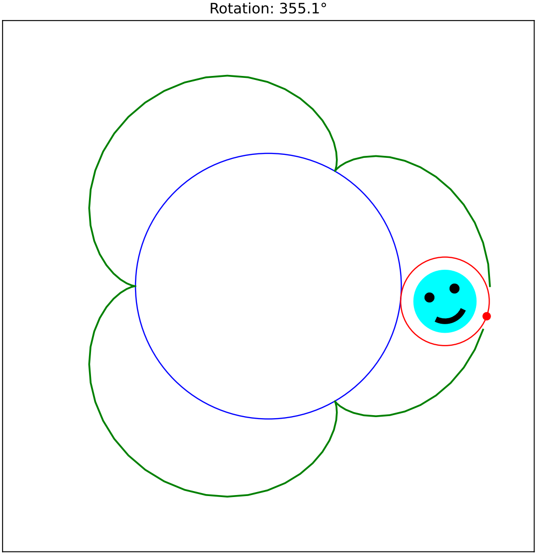

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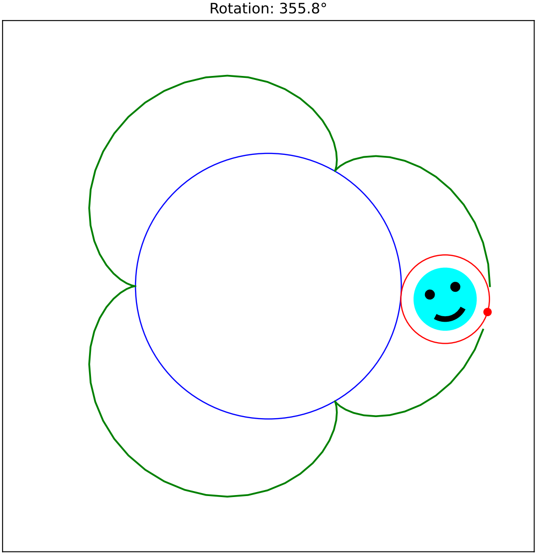

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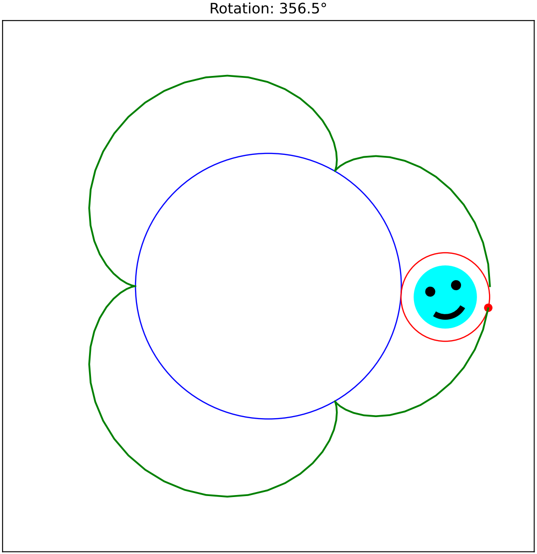

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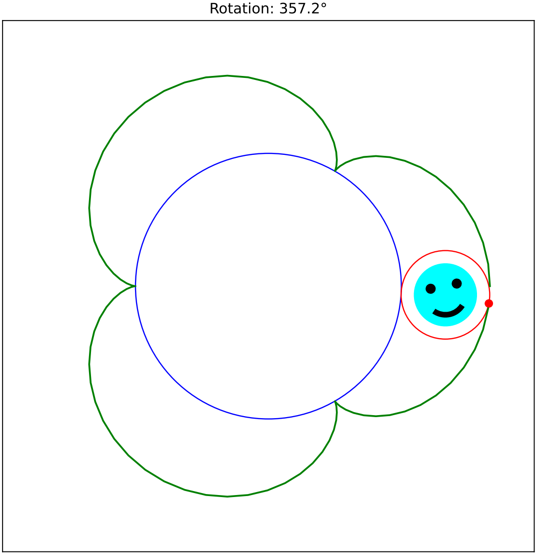

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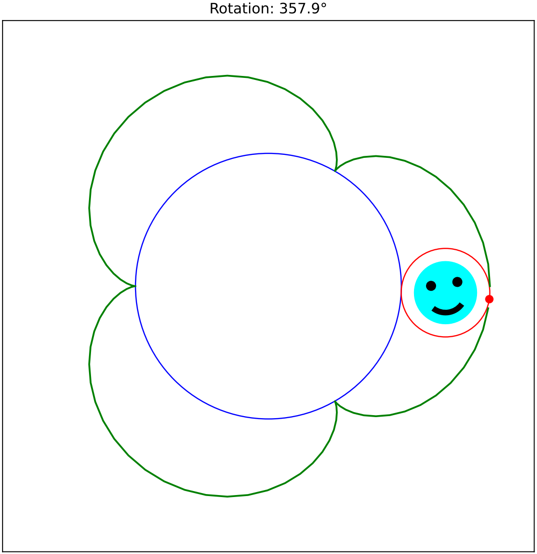

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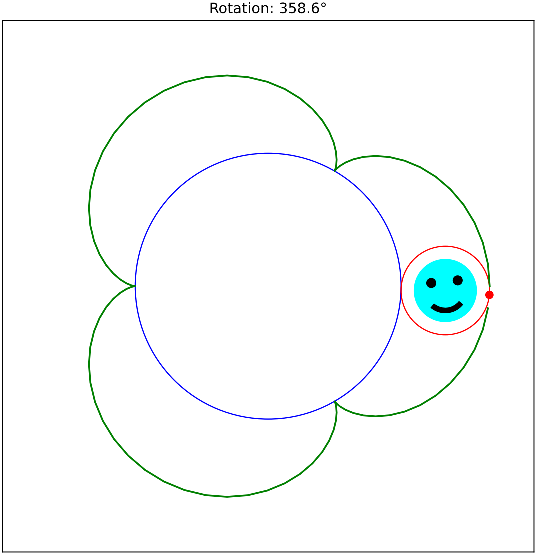

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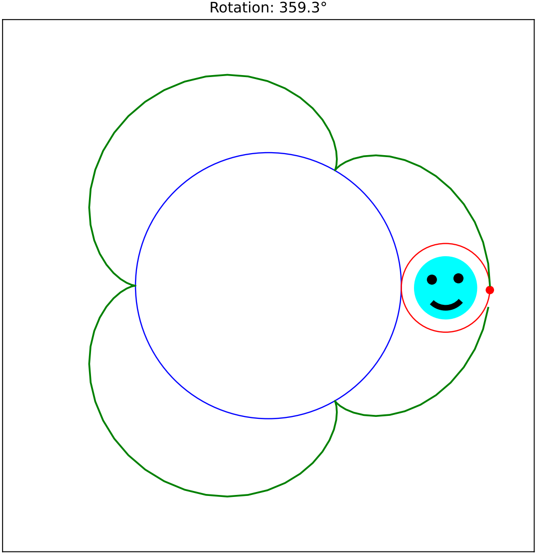

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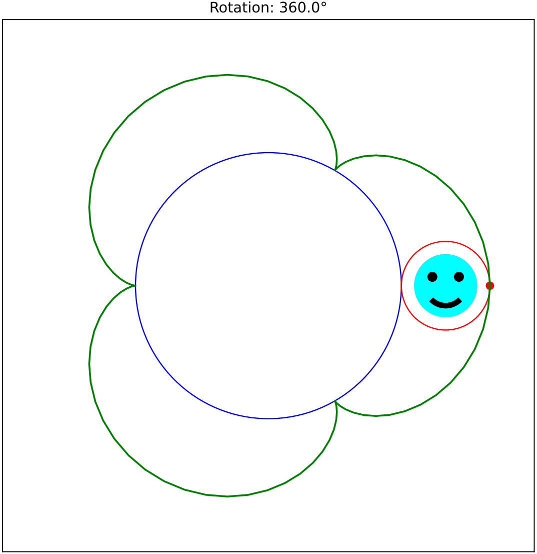
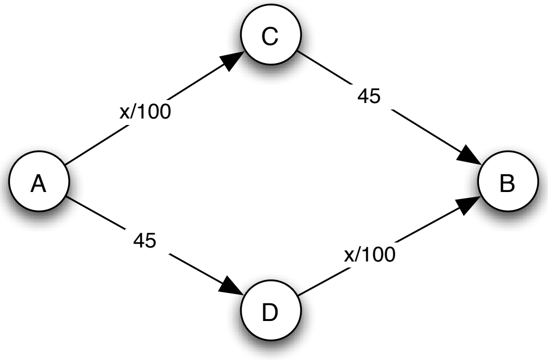


Braess Paradox

A Typical Case



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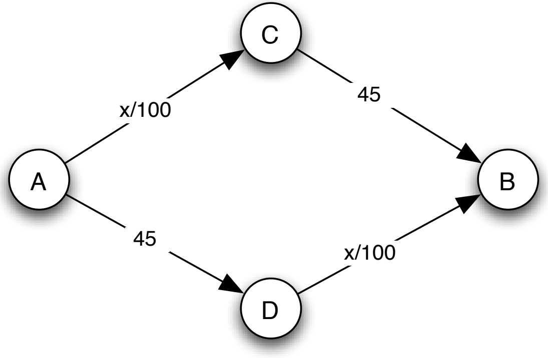

- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it

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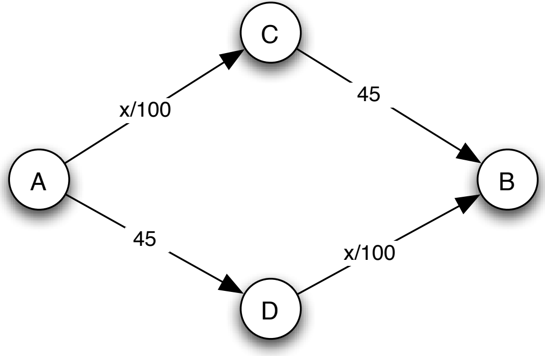

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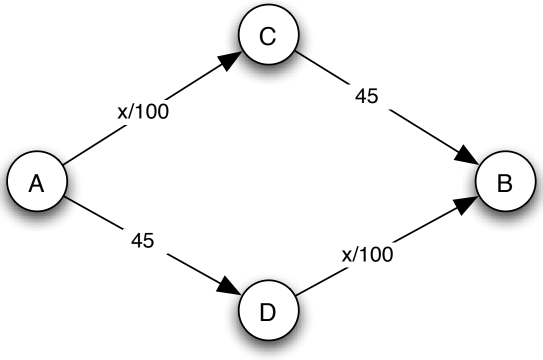
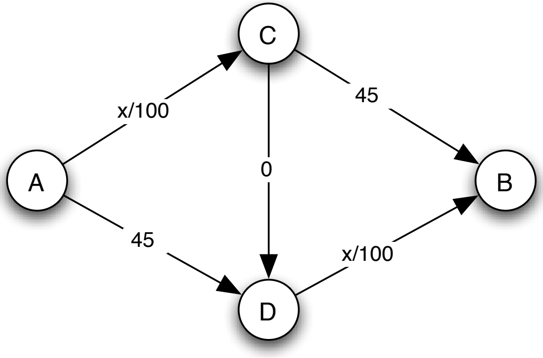


- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it
- Suppose there are 4000 cars need to get from A to B
- They divide evenly over the two routes at equilibrium; the travel time is $45 + 2000/100 = 65$ mins

A Typical Case: Cont'd



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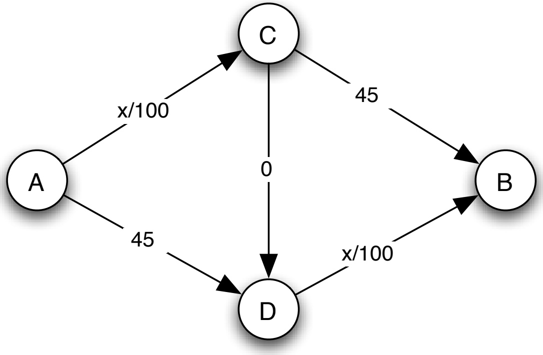

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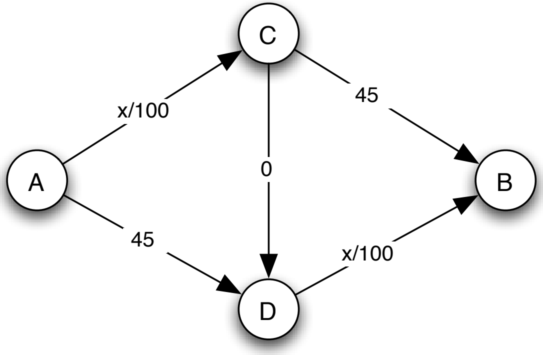

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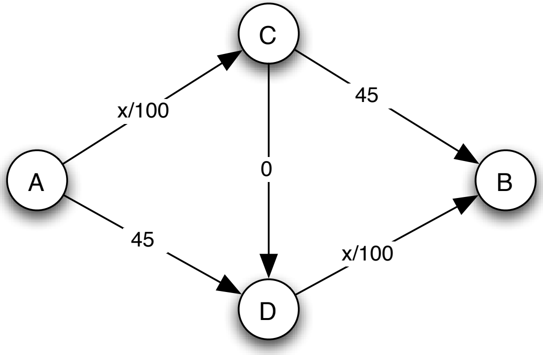
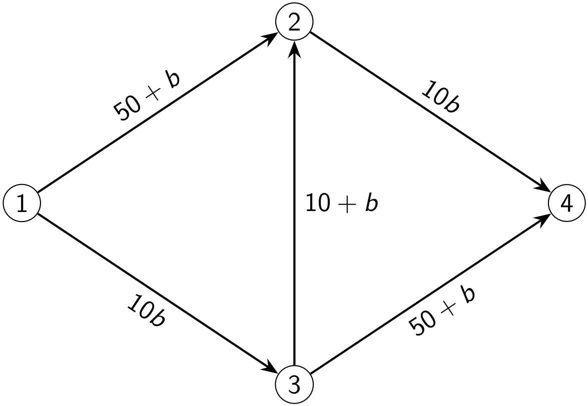


- Now a very fast edge is added from C to D to the previous highway network
- At equilibrium, every user uses the route through C and D
- As a result, the travel time is $4000/10 + 0 + 4000/100 = 80$ mins!

[Braess et al., 2005] Example



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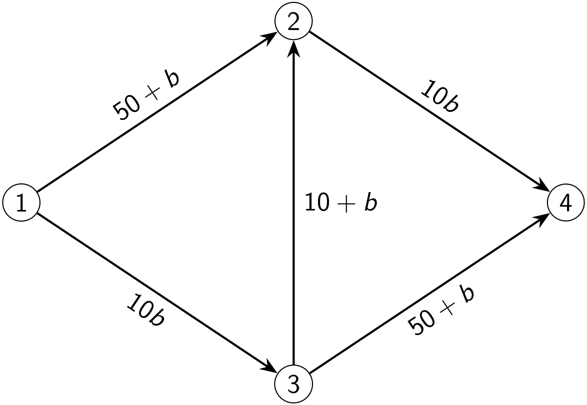

- Assume a flow of 6 units (e.g., 6000 vehicles) must travel from 1 to 4

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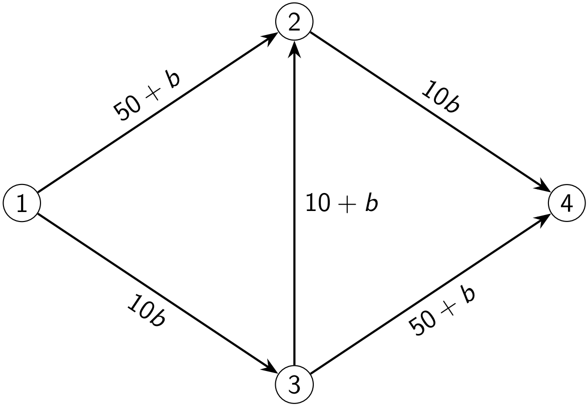

- Assume a flow of 6 units (e.g., 6000 vehicles) must travel from 1 to 4
- Three paths exist: $B_1 = 124$, $B_2 = 1324$, $B_3 = 134$

[Braess et al., 2005] Example: Cont'd

- Case 1: All Paths Open. Split the flow equally (2 units each):

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$$b_{12} = 2,\ b_{24} = 4,\ b_{34} = 2,\ b_{13} = 4,\ b_{32} = 2$$
$$d_{12} = 52,\ d_{24} = 40,\ d_{34} = 52,\ d_{13} = 40,\ d_{32} = 12$$
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[Braess et al., 2005] Example: Cont'd

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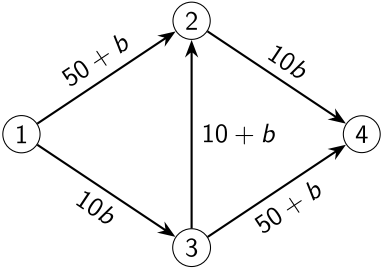
  Despite losing a connection, all paths are shorter!

[Braess et al., 2005] Example: Selfish Deviation



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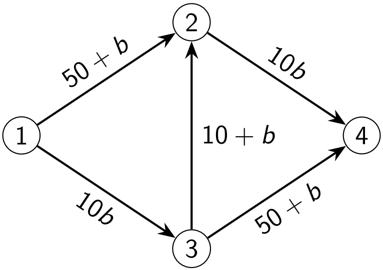

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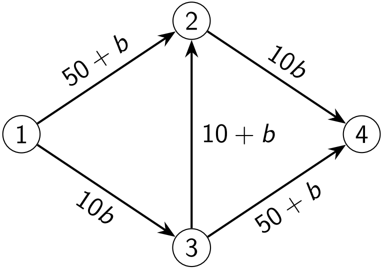

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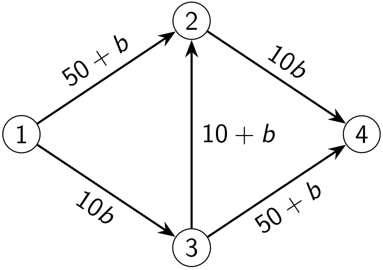

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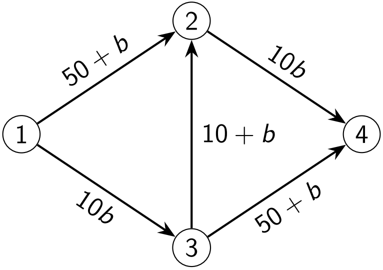

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- Consider an intermediate step: 5 units on B_1 and B_3, 1 on B_2, then $L(B_1) = L(B_3) = 87.5$, $L(B_2) = 82.5$.

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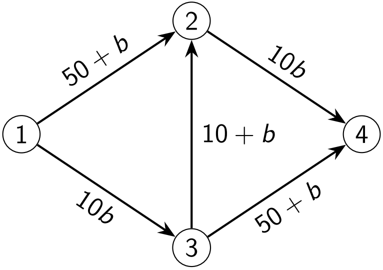

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- Consider an intermediate step: 5 units on B_1 and B_3, 1 on B_2, then $L(B_1) = L(B_3) = 87.5$, $L(B_2) = 82.5$.
- B_2 users clog k_{13} and k_{24} (factor 10 in load-time relation), worsening times for B_1 and B_3 beyond 83; Yet B_2 remains shortest, attracting more traffic and degrading the system for all.

Real-Life Examples

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- The mechanical analogy: the spring paradox

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  - Replaced a six lane highway with a five mile long park, traffic flow improved

The Social Cost of Traffic at Equilibrium

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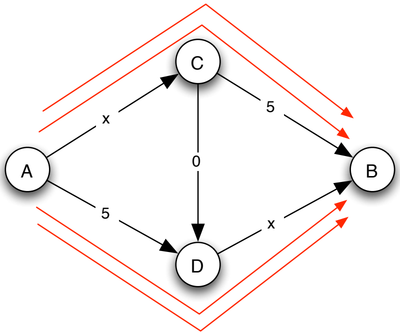
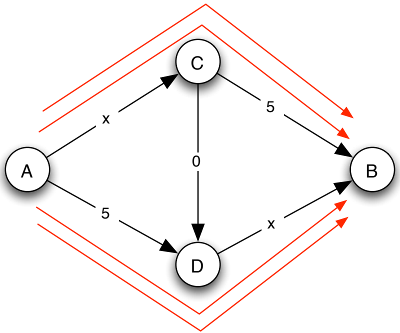
- **Nash Equilibrium**: No driver can reduce their travel time by switching paths, given others' choices.



(a) The social optimum.



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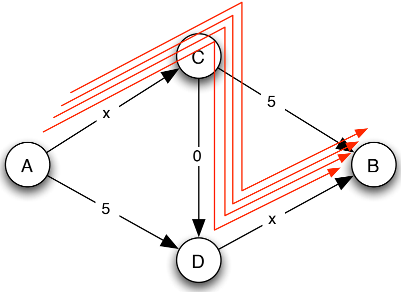
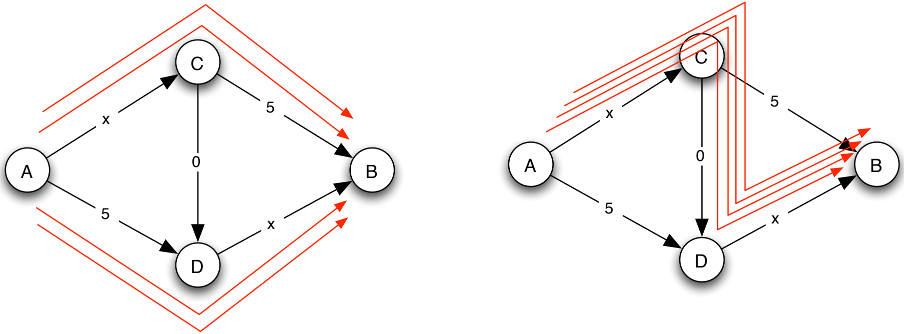


(b) The Nash equilibrium.



(a) The social optimum.

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Figure: A version of Braess's Paradox: In the socially optimal traffic pattern, the social cost is 28, while in the unique Nash equilibrium, the social cost is 32.

Best-Response Dynamics

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Does the *best-response dynamics* process always converge?

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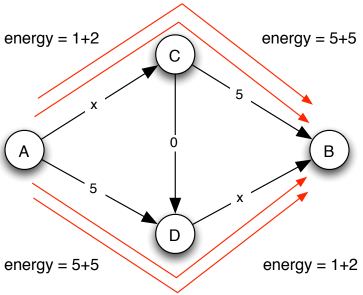
- Social cost can increase or decrease with best-response steps (e.g., from 28 to 32 in the Braess example), but potential energy strictly decreases, serving as a progress measure.

Analyzing Dynamics



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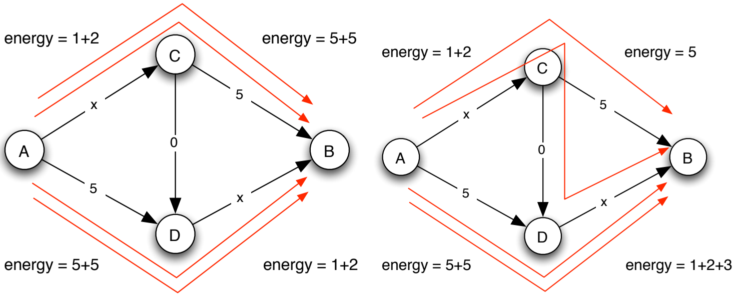

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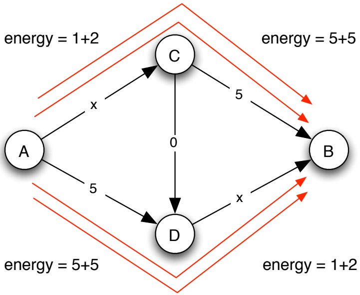

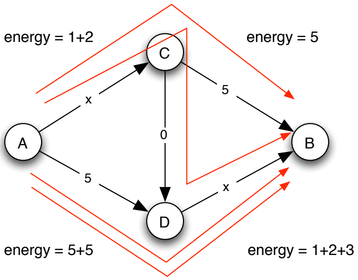

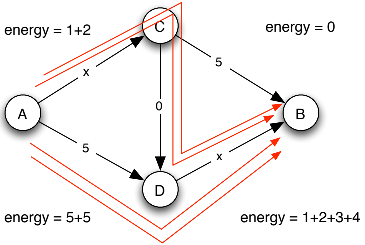

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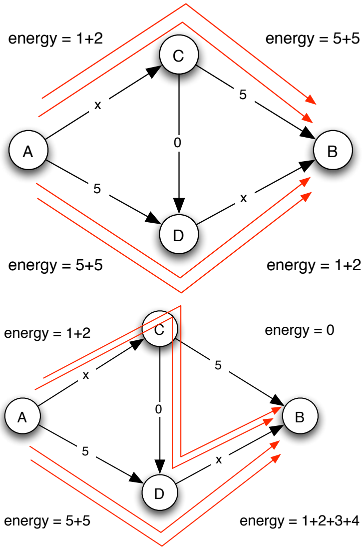

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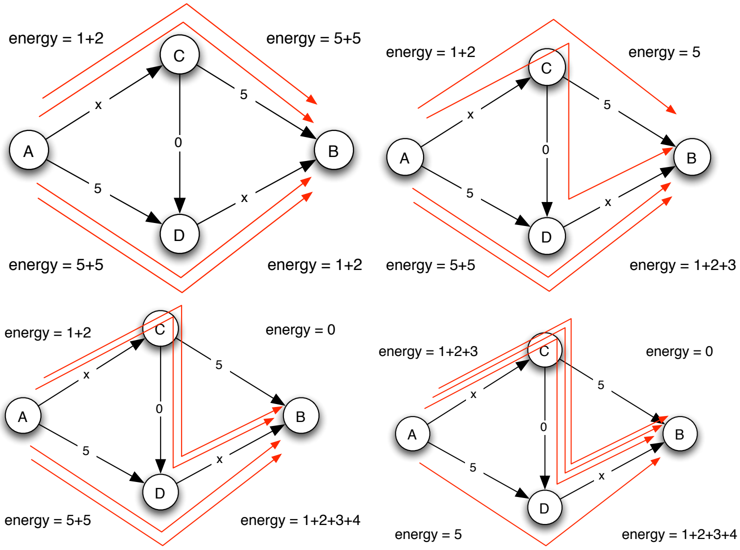

Figure: Steps of best-response dynamics with potential energy changes.

The Best-Response Dynamics Process Converges: Proof

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- With finite patterns, dynamics must stop at an equilibrium.

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- Implications: Network design and tolls can mitigate inefficiencies.

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