Options & Derivatives

The One Period Model

- time t: $t = 0, 1$
- (deterministic) bond B_t: $B_0 = 1,\ B_1 = 1 + R$
- (stochastic) stock S_t: $S_0 = s > 0,\ S_1 = \begin{cases} s \cdot u & \text{with prob. } p_u \\ s \cdot d & \text{with prob. } p_d \end{cases} \equiv s\,Z$:

  $u > d$, $p_u + p_d = 1$.
- The value V_t^h of the portfolio $h = (x, y)$, $x, y \in \mathbb{R}$ at time t:
  $V_t^h = x\,B_t + y\,S_t$ — $V_0^h = x + y\,s,\ V_1^h = x(1 + R) + y\,s\,Z$
- Arbitrage portfolio h: $V_0^h = 0$, $V_1^h > 0$ with prob. 1.

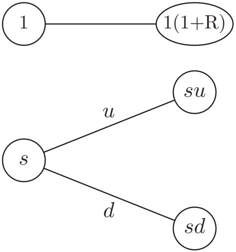


Figure: Asset Dynamics of One Period Model.

Portfolios and Arbitrage I

Theorem

The one period model is arbitrage free $\iff$ $u \geqslant 1 + R \geqslant d$.

Proof

$(\Longrightarrow)$

- Suppose $u \geqslant 1 + R \geqslant d$ does not hold, then $1 + R > u$ or $d > 1 + R$.
- If $1 + R > u$, then $s(1 + R) > s\,u$ and a priori $s(1 + R) > s\,d$.
- Consider $h = (s, -1)$, then $V_0^h = s \cdot 1 + (-1) \cdot s = 0$,
  $V_1^h = s(1 + R) - s \cdot Z > 0$, an arbitrage.
- If $d > 1 + R$, then $s\,d > s(1 + R)$ and a priori $s\,u > s(1 + R)$.
- Consider $h = (-s, 1)$, then $V_0^h = (-s) \cdot 1 + 1 \cdot s = 0$,
  $V_1^h = -s(1 + R) + s \cdot Z > 0$, an arbitrage.

Portfolios and Arbitrage II

Theorem

The one period model is arbitrage free $\iff u \geqslant 1+R \geqslant d$.

Proof

$(\Longleftarrow)$

- Arbitrage $h=(x,y)$: $V_0^h=0$.
- $x+s\cdot y=0 \implies x=-s\cdot y$.
- $V_1^h=\begin{cases} y\,s(u-(1+R)), & Z=u \\ y\,s(d-(1+R)), & Z=d \end{cases}$
- If $y>0$: from $V_1^h>0 \implies u>1+R$ and $d>1+R$; a contradiction.
- If $y<0$: from $V_1^h>0 \implies u<1+R$ and $d<1+R$; a contradiction.

Risk-Neutral / Martingale Measure and Probabilities

- Observation: $u \geqslant 1+R \geqslant d \implies 1+R$ is a convex combination of u and d
- $\exists\, q_u, q_d \geqslant 0,\ q_u + q_d = 1$ s.t. $1+R = q_u \cdot u + q_d \cdot d$
- Define a new probability measure Q and the associated expectation E^Q s.t.

$$
\begin{aligned}
&Q(Z=u) = q_u, \quad Q(Z=d) = q_d \\
&\frac{1}{1+R}\,\mathsf{E}^Q\, S_1 = \frac{1}{1+R}(q_u \cdot s\,u + q_d \cdot s\,d) = \frac{1}{1+R} \cdot s(1+R) = s
\end{aligned}
$$

Definition

- **Risk-Neutral / Martingale Measure**: A measure Q satisfies $S_0 = \dfrac{1}{1+R}\,\mathsf{E}^Q\, S_1.$
- **Martingale Probabilities**: $q_u = \dfrac{(1+R)-d}{u-d},\ q_d = \dfrac{u-(1+R)}{u-d}$

Contingent Claims I

Definition

- A **contingent claim** X is of the form $X = \Phi(Z)$
- Stochastic Z with **contract function** $\Phi(\cdot)$
- **Price** of X at time t: $\Pi(t; X)$

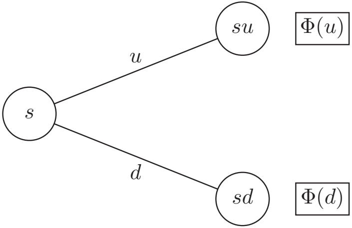


Figure: The Contingent Claim.

Contingent Claims II

Example (European Call Option with Strike K)

Assume $s\,u > K > s\,d$. At $t = 1$,

- Exercise the option if $S_1 > K$.
  - Pay K to get the stock and sell it at $s\,u$, thus making net profit $s\,u - K$.
- Do nothing if $S_1 < K$.

$$X = \begin{cases} s\,u - K, & Z = u \\ 0, & Z = d \end{cases}, \qquad \begin{cases} \Phi(u) = s\,u - K \\ \Phi(d) = 0 \end{cases}$$

Definition

- A contingent claim X is said to be **reachable** if there exists a portfolio h such that $V_1^h = X$ with probability 1; this portfolio h is called a **hedging** or **replicating** portfolio.
- If all claims can be replicated we say the market is **complete**.

Contingent Claims III

Theorem (Pricing Principle)

If a claim X is reachable with replicating portfolio h, then the "reasonable" price of X is given by $\Pi(t;X) = V_t^h,\ t = 0, 1$.

Theorem

An arbitrage free one period model is complete.

Proof

Fixed any $\Phi(\cdot)$, show that $\exists\, h = (x, y)$ s.t.

$$V_1^h = \begin{cases} \Phi(u) & Z = u, \\ \Phi(d) & Z = d. \end{cases} \implies x(1+R) + y\,s\,u = \Phi(u),\ x(1+R) + y\,s\,d = \Phi(d).$$

Solve for x, y: $x = \dfrac{1}{1+R}\,\dfrac{u\Phi(d) - d\,\Phi(u)}{u-d}, \quad y = \dfrac{1}{s}\,\dfrac{\Phi(u) - \Phi(d)}{u-d}.$

Risk Neutral Valuation

- From Pricing Principle ($\Pi(t; X) = V_t^h$, $t = 0, 1$)

$$\begin{aligned}
\Pi(0; X) &= V_0^h = x + s\, y \\
&= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\,\Phi(u)}{u-d} + s \cdot \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d} \\
&= \frac{1}{1+R} \left\{ \frac{(1+R) - d}{u-d}\, \Phi(u) + \frac{u - (1+R)}{u-d}\, \Phi(d) \right\} \\
&= \frac{1}{1+R} \left\{ q_u\, \Phi(u) + q_d\, \Phi(d) \right\} \equiv \frac{1}{1+R}\, \mathsf{E}^Q\, X
\end{aligned}$$

Theorem (The Risk Neutral Valuation Principle)

If the one period binomial model is arbitrage-free, then the price of X is $\Pi(0; X) = \dfrac{1}{1+R}\, \mathsf{E}^Q\, X$.

The Multiperiod Model

- time t: $t = 0, 1, 2, \ldots, T$
- (deterministic) bond B_t with $B_0 = 1,\ B_{n+1} = (1+R)B_n$
- (stochastic) stock S_t with $S_0 = s > 0,\ S_{n+1} = Z_n\, S_n$ where $Z_0, Z_1, Z_2, \ldots, Z_{T-1}$ are iid with $\mathsf{P}(Z_n = u) = p_u,\ \mathsf{P}(Z_n = d) = p_d$

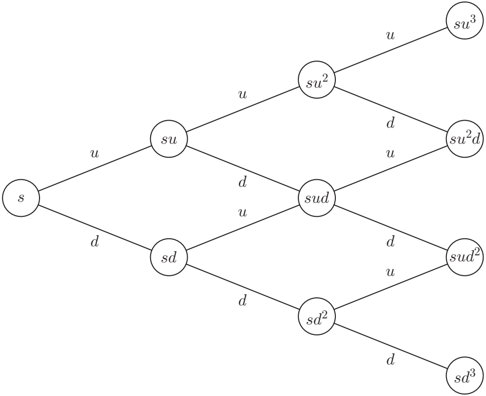


Figure: Asset Dynamics of Multiperiod Model: "Recombining" Tree.

Portfolios and Arbitrage

Definition

The portfolio $h_t \equiv (x_t, y_t)$; The value $V_t^{h_t}$ of portfolio h_t at time t is $V_t^{h_t} = x_t\, B_t + y_t\, S_t$.

- Hereafter we write V_t^h instead of the cumbersome $V_t^{h_t}$.
- x_t is the amount which we invest in the bank at time $t-1$ and keep until t.

Definition

Self-financing portfolio $h_t = (x_t, y_t)$:
$x_t\,(1+R) + y_t\, S_t = x_{t+1} + y_{t+1}\, S_t, \quad \forall t = 0, 1, \ldots, T-1.$

Contingent Claims

Definition

- Arbitrage: there exists a self-financing portfolio h_t with $V_0^h = 0$, $\mathsf{P}(V_T^h \geqslant 0) = 1$, $\mathsf{P}(V_T^h > 0) > 0$.
- A contingent claim X is said to be **reachable** if there exists a self-financing portfolio h such that $V_T^h = X$ with probability 1; this portfolio h is called a **hedging** or **replicating** portfolio.
- If all claims can be replicated we say the market is **complete**.

Theorem (Pricing Principle)

If a claim X is reachable with replicating (and self-financing) portfolio h, then the "reasonable" price process of X is given by $\Pi(t; X) = V_t^h,\ t = 0, 1, 2, \ldots T$.

Theorem

An arbitrage-free multiperiod model is complete.

Theorem (Binomial Algorithms)

- Given a contingent claim $X = \Phi(S_T)$; let $V_t(k)$ denotes the value of the replicating portfolio at node (t, k), then $V_t(k)$ is computed recursively by

$$V_T(k) = \Phi(s\, u^k\, d^{T-k})$$

$$V_t(k) = \frac{1}{1+R}\{q_u\, V_{t+1}(k+1) + q_d\, V_{t+1}(k)\}$$

- The martingale probabilities q_u, q_d are $q_u = \dfrac{(1+R)-d}{u-d}$, $q_d = \dfrac{u-(1+R)}{u-d}$
- The replicating portfolio $h_t = (x_t, y_t)$ is

$$x_t(k) = \frac{1}{1+R}\,\frac{u\,V_t(k) - d\,V_t(k+1)}{u-d}, \quad y_t(k) = \frac{1}{S_{t-1}}\,\frac{V_t(k+1) - V_t(k)}{u-d}$$

- The arbitrage-free price of a contingent claim X at $t = 0$ is

$$\Pi(0; X) = \frac{1}{(1+R)^T}\,\mathsf{E}^Q X = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^{T} \binom{T}{k} q_u^k\, q_d^{T-k} \Phi(s\, u^k\, d^{T-k})$$

Example

Given $T = 3, S_0 = 80, K = 80, u = 1.5, d = 0.5, p_u = 0.6, p_d = 0.4, R = 0,$ compute the European call option price and the replicating portfolio of each node.

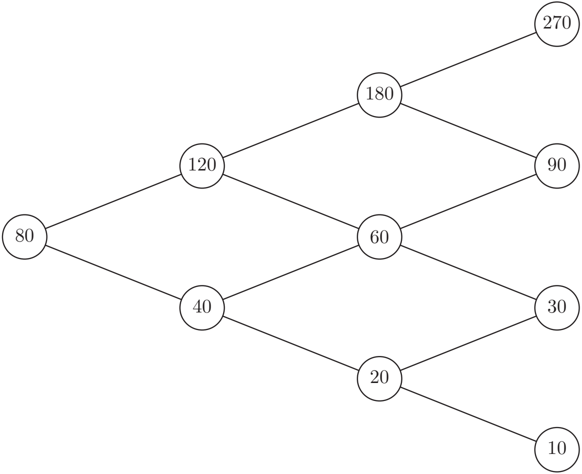


Figure: Asset Dynamics of the Example.

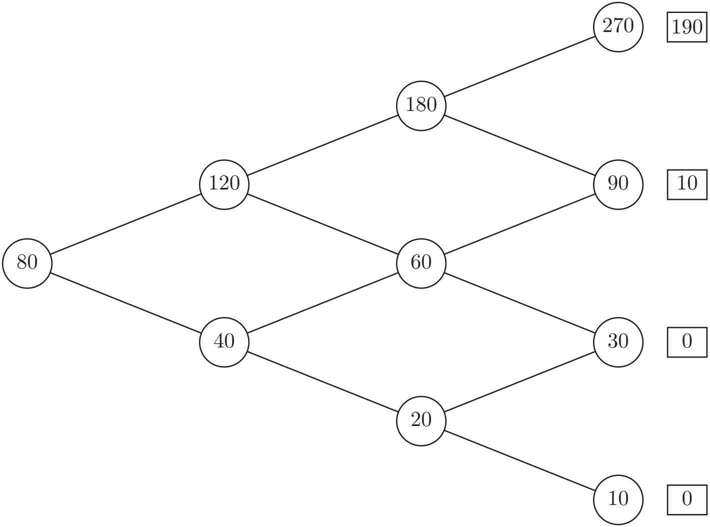


Figure: Payoff at the End of Terms.

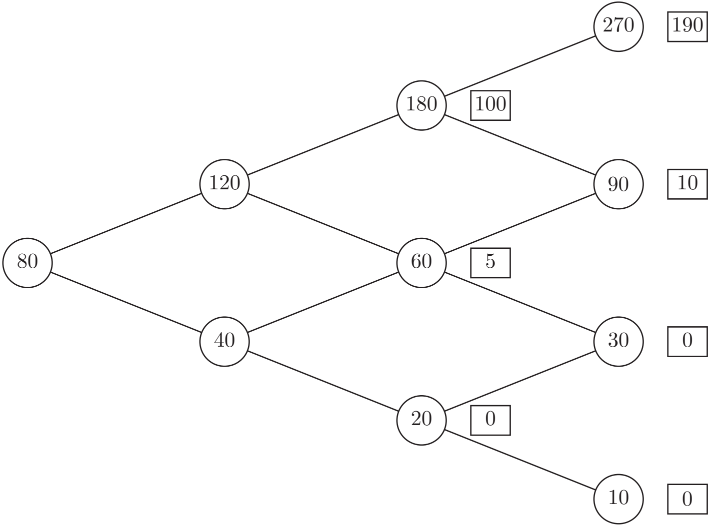


Figure: Iterated Computation of $\Pi(t;X)$: $\Pi(t-1;X) \equiv \dfrac{1}{1+R}\,\mathsf{E}^{Q}\{\Pi(t;X)\}$.

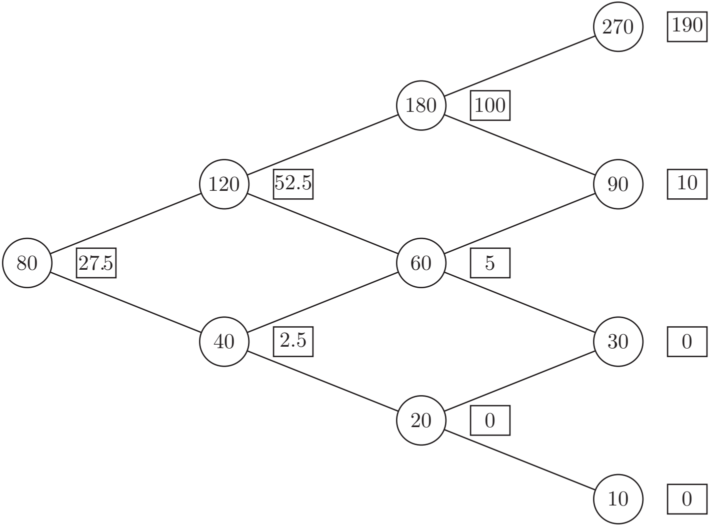


Figure: The Completed $\Pi(t; X)$.

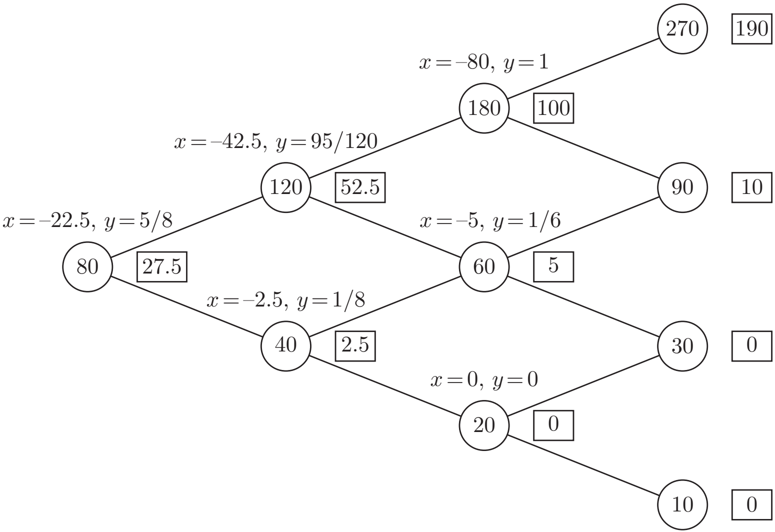


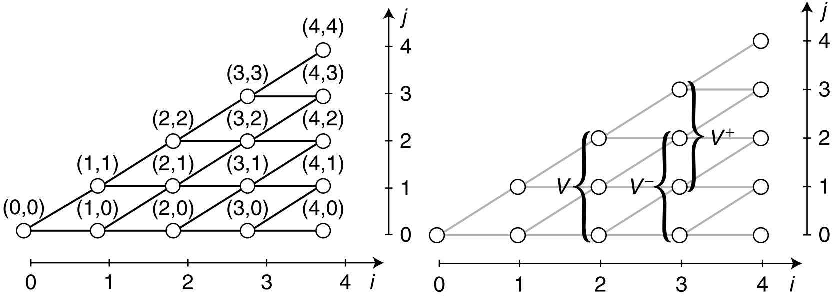
Figure: Replicating $h_t = (x_t, y_t)$: $x_t(k) = \frac{1}{1+R} \frac{u\,V_t(k) - d\,V_t(k+1)}{u-d}$, $y_t(k) = \frac{1}{S_{t-1}} \frac{V_t(k+1) - V_t(k)}{u-d}$

Algorithmic Considerations

$$\Pi(0;X) = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^{T} \binom{T}{k} q_u^k \, q_d^{T-k} \Phi(s\, u^k\, d^{T-k})$$

For big T the formula can't be directly used because of the binomial coefficient

$$V_T(k) = \Phi(s\, u^k\, d^{T-k}), \quad V_t(k) = \frac{1}{1+R} \{q_u\, V_{t+1}(k+1) + q_d\, V_{t+1}(k)\}$$



Python Code Illustration: Common Parts

```python
import numpy as np

S0 = 80; r = 0; K = 80; u = 1.5; d = 0.5;
q = (1 - d) / (u - d); M = 3;
df = 1     # discount factor per time interval
# exhibit stock paths
S = np.zeros((M + 1, M + 1), dtype=np.float)
S[0, 0] = S0
for j in range(1, M + 1, 1):
    for i in range(j + 1):
        S[i, j] = S[0, 0] * (u ** (j - i)) * (d ** i)
```

Python Codes: Traditional Loops

```python
iv = np.zeros((M + 1, M + 1), dtype=np.float); z = 0  # inner values
for j in range(0, M + 1, 1):
    for i in range(z + 1):
        iv[i, j] = round(max(S[i, j] - K, 0), 8)
    z += 1

pv = np.zeros((M + 1, M + 1), dtype=np.float)         # present values
pv[:, M] = iv[:, M]
z = M + 1
for j in range(M - 1, -1, -1):
    z -= 1
    for i in range(z):
        pv[i, j] = (q * pv[i, j + 1] + (1 - q) * pv[i + 1, j + 1]) * df
```

Python Codes: Vectorized Loops

```
import numpy as np
from params import *
import time

mu = np.arange(M + 1)
mu = np.resize(mu, (M + 1, M + 1))
md = np.transpose(mu)
mu = u ** (mu - md)
md = d ** md
S = S0 * mu * md

start_time = time.time()

# present value array initialized with inner values
pv = np.maximum(S - K, 0)
z = 0
for i in range(M - 1, -1, -1): # backwards induction
    pv[0:M-z, i] = (q * pv[0:M-z, i+1] + (1 - q) * pv[1:M-z+1, i+1]) * df
    z += 1

print(pv)
print('Value of European call option is %8.3f' % pv[0, 0])
print('vector elapsed: %f seconds.' % (time.time() - start_time,))
```

Option Pricing in Continuous Time

- Option pricing in discrete time: for contract X

$$\Pi(0;\, X) = \frac{1}{(1+R)^T}\,\mathsf{E}^Q\, X_T$$

- Discretize each interval further into m sections, then the compounding factor $(1+R)^T$ becomes $(1+\frac{R}{m})^{mT}$
- Let $m \to \infty$ (continuous time), $(1+\frac{R}{m})^{mT} \to e^{RT}$
- So option pricing in continuous time: for contract X

$$\Pi(0;\, X) = e^{-RT}\,\mathsf{E}^Q\, X_T$$

- Hereafter r, instead of R, is the underlying interest rate

Option Pricing: The Black-Scholes Formula I

- Under the risk-neutral probability measure Q, the stock S evolves as $S(t) = S(0)\,\exp\left\{(r-\delta-\frac{\sigma^2}{2})t + \sigma\sqrt{t}Z\right\}$, where $Z \sim N(0,1)$.
- For the European call option with strike K, the contract is $X(t) = \max\{S(t)-K,\,0\} \equiv (S(t)-K)_+$.
- So the price of the call option at $t=0$ is

$$
\begin{aligned}
\Pi_c(0;\,X) &= e^{-rT}\,\mathsf{E}^Q\{X(T)\} = e^{-rT}\,\mathsf{E}^Q\{(S(T)-K)_+\} \\
&= e^{-rT}\,\mathsf{E}^Q\{(S(T)-K)_+\,|\,S(T)>K\}\,\mathsf{P}^Q\{S(T)>K\} \\
&\qquad + e^{-rT}\,\underbrace{\mathsf{E}^Q\{(S(T)-K)_+\,|\,S(T)<K\}}_{=0}\,\mathsf{P}^Q\{S(T)<K\} \\
&= e^{-rT}\,\mathsf{E}^Q\{(S(T)-K)_+\,|\,S(T)>K\}\,\mathsf{P}^Q\{S(T)>K\} \\
&= e^{-rT}\,\mathsf{E}^Q\{S(T)-K\,|\,S(T)>K\}\,\mathsf{P}^Q\{S(T)>K\} \\
&= e^{-rT}\left(\mathsf{E}^Q\{S(T)\,|\,S(T)>K\}-K\right)\mathsf{P}^Q\{S(T)>K\}
\end{aligned}
$$

Option Pricing: The Black-Scholes Formula II

- As $S(T) = S(0)\,\exp\{(r-\delta-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\}$, evaluate $\mathsf{P}^Q\{S(T) > K\}$ and $\mathsf{E}^Q\{S(T)\,|\,S(T) > K\}$
- Let $\Phi(\cdot)$ be the CDF of $N(0,1)$, then

$$\begin{aligned}
\mathsf{P}^Q\{S(T) > K\} &= \mathsf{P}^Q\left\{S(0)\,\exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right\} > K\right\} \\
&= \mathsf{P}^Q\left\{\exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right\} > \frac{K}{S(0)}\right\} \\
&= \mathsf{P}^Q\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z > \ln\frac{K}{S(0)}\right\} \\
&= \mathsf{P}^Q\left\{Z > \frac{\ln\frac{K}{S(0)} - (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right\} \\
&= 1 - \Phi\left(\frac{\ln\frac{K}{S(0)} - (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\
&= \Phi\left(\frac{\ln\frac{S(0)}{K} + (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \equiv \Phi(d_2)
\end{aligned}$$

Option Pricing: The Black-Scholes Formula III

- Define $d_2 = \dfrac{\ln \frac{S(0)}{K} + (r - \delta - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_1 = \dfrac{\ln \frac{S(0)}{K} + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_2 + \sigma\sqrt{T}$; $\mathsf{E}^Q\{S(T) \,|\, S(T) > K\} = \dfrac{\mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{S(T)>K\}}\right\}}{\mathsf{P}^Q\{S(T) > K\}}$ and

$$
\begin{aligned}
\mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{S(T)>K\}}\right\} &= \mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{Z>-d_2\}}\right\} \\
&= \int_{-d_2}^{\infty} S(0)\, e^{\left(r-\delta-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2+\sigma\sqrt{T}z-\frac{1}{2}\sigma^2 T}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_2-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,\mathrm{d}z = S(0)\, e^{(r-\delta)T}\,\Phi(d_1)
\end{aligned}
$$

Option Pricing: The Black-Scholes Formula IV

- The price of the call option with strike K at $t = 0$ is

$$\begin{aligned}
\Pi_c(0;\, X) &= e^{-rT}\big(\,\mathsf{E}^Q\{S(T)\,|\,S(T) > K\} - K\big)\,\mathsf{P}^Q\{S(T) > K\} \\
&= e^{-rT}\,\mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{S(T)>K\}}\right\} - Ke^{-rT}\,\mathsf{P}^Q\{S(T) > K\} \\
&= e^{-rT}S(0)\,e^{(r-\delta)T}\Phi(d_1) - K\,e^{-rT}\Phi(d_2) \\
&= S(0)\,e^{-\delta T}\Phi(d_1) - K\,e^{-rT}\Phi(d_2)
\end{aligned}$$

- Note that

$$\begin{aligned}
(S(T) - K)_+ - (K - S(T))_+ &= \max\{S(T) - K, 0\} - \max\{K - S(T), 0\} \\
&= \max\{S(T) - K, 0\} + \min\{S(T) - K, 0\} \\
&= S(T) - K
\end{aligned}$$

- Let the price of the put option with strike K at $t = 0$ be $\Pi_p(0;\, X)$, then

$$\Pi_c(0;\, X) - \Pi_p(0;\, X) = e^{-rT}\,\mathsf{E}^Q\{S(T) - K\}$$

Option Pricing: The Black-Scholes Formula V

- Note that $\mathsf{E}^Q\{e^{kz}\}$ for $z \sim N(0,1)$ is $e^{\frac{1}{2}k^2}$, then

$$
\begin{aligned}
e^{-rT}\,\mathsf{E}^Q\{S(T)-K\} &= e^{-rT}S(0)\,e^{(r-\delta-\frac{1}{2}\sigma^2)T}\,\mathsf{E}^Q\left\{e^{\sigma\sqrt{T}Z}\right\} - Ke^{-rT} \\
&= S(0)\,e^{(-\delta-\frac{1}{2}\sigma^2)T}\underbrace{\mathsf{E}^Q\left\{e^{\sigma\sqrt{T}Z}\right\}}_{=e^{\frac{1}{2}\sigma^2 T}} - Ke^{-rT} \\
&= S(0)\,e^{-\delta T} - Ke^{-rT}
\end{aligned}
$$

- By $\Phi(x)+\Phi(-x)=1$,

$$
\begin{aligned}
\Pi_p(0;\,X) &= \Pi_c(0;\,X) - S(0)\,e^{-\delta T} + Ke^{-rT} \\
&= S(0)\,e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2) - S(0)\,e^{-\delta T} + Ke^{-rT} \\
&= -S(0)\,e^{-\delta T}(1-\Phi(d_1)) + Ke^{-rT}(1-\Phi(d_2)) \\
&= -S(0)\,e^{-\delta T}\Phi(-d_1) + Ke^{-rT}\Phi(-d_2)
\end{aligned}
$$

Example

You are asked to determine the price of a European put option on a stock. Assuming the Black-Scholes model, you are given

- The stock price now is 100.
- The option expires in 6 months.
- The strike price is 98.
- The interest rate $r = 0.055$.
- $\delta = 0.01$.
- $\sigma = 0.5$.

What is the price?

Solution

Note that $S(0) = 100$, $T = 0.5$, $K = 98$, $d_1 = \dfrac{\ln \frac{100}{98} + (0.055 - 0.01 + \frac{0.5^2}{2})\, 0.5}{0.5\sqrt{0.5}}$
$= 0.29756$, $d_2 = d_1 - 0.5\sqrt{0.5} = -0.056$, $\Phi(-d_1) = 0.38302$, $\Phi(-d_2) = 0.52233$.
The price of the put is

$$
\begin{aligned}
&K e^{-rT}\Phi(-d_2) - S(0)\, e^{-\delta T}\Phi(-d_1) \\
&\qquad = 98\, e^{-0.055\cdot 0.5} \cdot 0.52233 - 100\, e^{-0.01\cdot 0.5} \cdot 0.38302 = 11.6889.
\end{aligned}
$$