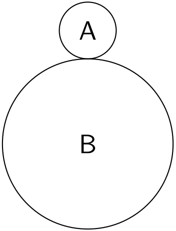
Introduction to Financial Models
Lecture 02: Surprises & Paradoxes II

Coin Rotation Paradox

The 1982 SAT Question Everyone Got Wrong



The radius of circle A is $\frac{1}{3}$ of the radius of circle B. Circle A rolls around circle B one trip back to its starting point. How many times will circle A revolve in total?

(a) $\frac{3}{2}$ (b) 3 (c) 6 (d) $\frac{9}{2}$ (e) 9

Animation of Coin Rotation Paradox: Python Code I

```
import math
import matplotlib.pyplot as plt
from matplotlib.patches import Circle, Wedge
from matplotlib.transforms import Affine2D
from matplotlib.backends.backend_pdf import PdfPages
import os; os.chdir(os.path.dirname(__file__))

# Set the radii
R = 1.5  # Radius of the outer (stationary) coin
r = 0.5  # Radius of the inner (rolling) coin

def create_logo(ax, x, y, radius, angle):
    # Create a simple smiley face logo
    face = Circle((0, 0), radius * 0.7, fill=True, color="cyan")
    left_eye = Circle((-radius * 0.3, radius * 0.2), radius *
    ↪ 0.1, fill=True, color="black")
    right_eye = Circle((radius * 0.3, radius * 0.2), radius *
    ↪ 0.1, fill=True, color="black")
```

Animation of Coin Rotation Paradox: Python Code II

```python
    mouth = Wedge((0, 0), radius * 0.5, 225, 315, width=radius *
     ↪  0.1, color="black")

    # Create a transformation
    t = Affine2D().rotate(-angle).translate(x, y) + ax.transData

    # Apply the transformation to all elements
    face.set_transform(t)
    left_eye.set_transform(t)
    right_eye.set_transform(t)
    mouth.set_transform(t)

    ax.add_artist(face)
    ax.add_artist(left_eye)
    ax.add_artist(right_eye)
    ax.add_artist(mouth)

def create_frame(angle):
```

Animation of Coin Rotation Paradox: Python Code III

```
# Create figure and axis
fig, ax = plt.subplots(figsize=(8, 8))

# Set limits and aspect ratio
ax.set_xlim(-(R + r + 1), R + r + 1)
ax.set_ylim(-(R + r + 1), R + r + 1)
ax.set_aspect("equal")

# Remove axis ticks
ax.set_xticks([])
ax.set_yticks([])

# Calculate position of the rolling coin's center
x = (R + r) * math.cos(angle)
y = (R + r) * math.sin(angle)

# Calculate position of the point on the edge of the moving
↪   coin
```

Animation of Coin Rotation Paradox: Python Code IV

```
point_x = x + r * math.cos((R + r) / r * angle)
point_y = y + r * math.sin((R + r) / r * angle)

# Draw stationary coin
stationary_coin = Circle((0, 0), R, fill=False, color="blue")
ax.add_artist(stationary_coin)

# Draw rotating coin
rotating_coin = Circle((x, y), r, fill=False, color="red")
ax.add_artist(rotating_coin)

# Add logo to the rotating coin
create_logo(ax, x, y, r, (R + r) / r * angle)

# Draw point on the edge of the moving coin
ax.plot(point_x, point_y, "ro")

# Draw cardioid path
```

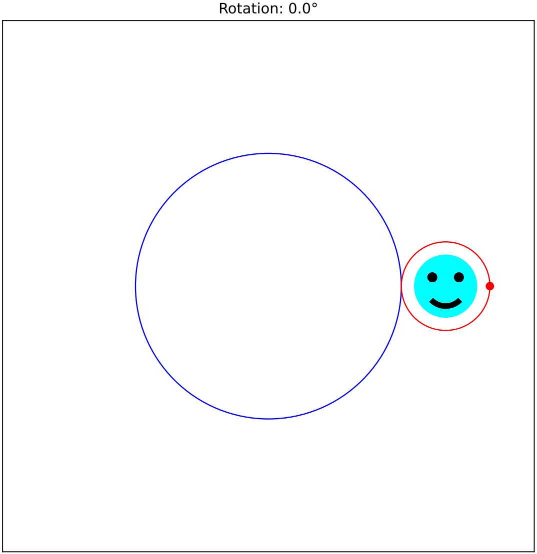
Animation of Coin Rotation Paradox: Python Code V

```
    theta = [i * 2 * math.pi / 100 for i in range(int(100 * angle
    ↪  / (2 * math.pi)) + 1)]
    cardioid_x = [(R + r) * math.cos(t) + r * math.cos((R + r) /
    ↪  r * t) for t in theta]
    cardioid_y = [(R + r) * math.sin(t) + r * math.sin((R + r) /
    ↪  r * t) for t in theta]
    ax.plot(cardioid_x, cardioid_y, "g-")

    # Add title
    ax.set_title(f"Rotation: {angle/(2*math.pi)*360:.1f}°")

    return fig

# Generate 512 frames and save them into a single PDF
num_frames = 512
pdf_filename = 'coin_rotation_paradox.pdf'
tex = []
```

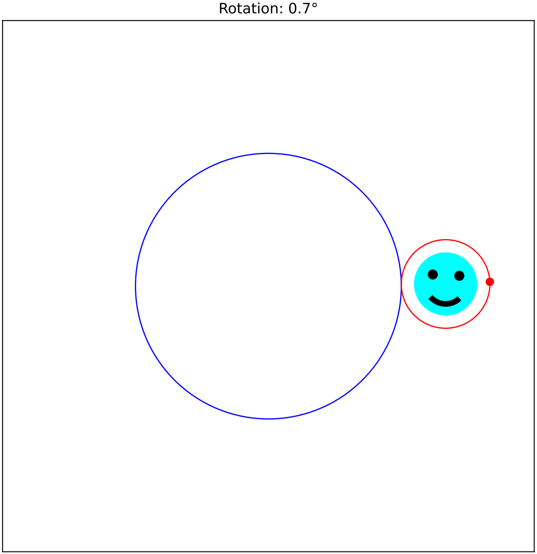
Animation of Coin Rotation Paradox: Python Code VI

```
with PdfPages(pdf_filename) as pdf:
    for i in range(num_frames + 1):
        angle = i * 2 * math.pi / num_frames
        fig = create_frame(angle)
        pdf.savefig(fig, bbox_inches='tight', pad_inches=0.1,
         ↪  dpi=300)
        plt.close(fig)
        # Generate LaTeX commands referencing pages (1-based
         ↪  indexing)
        tex.append(f'\\only<{i +
         ↪  1}>{{\\includegraphics[width=.6\\textwidth, page={i
         ↪  + 1}]{{fig/note02/coin_rotation_paradox.pdf}}\\noind
         ↪  ent}}')

# Write the LaTeX overlayarea content to a .tex file
open('../../coin.tex',
 ↪  'w').write(f"\\hspace*{{\\dimexpr(\\textwidth - .6\\textwidt
 ↪  h)/2\\relax}}\n\\begin{{overlayarea}}{{\\textwidth}}{{0.7\\t
 ↪  extheight}}\n{'\n'.join(tex)}\n\\end{{overlayarea}}")
```

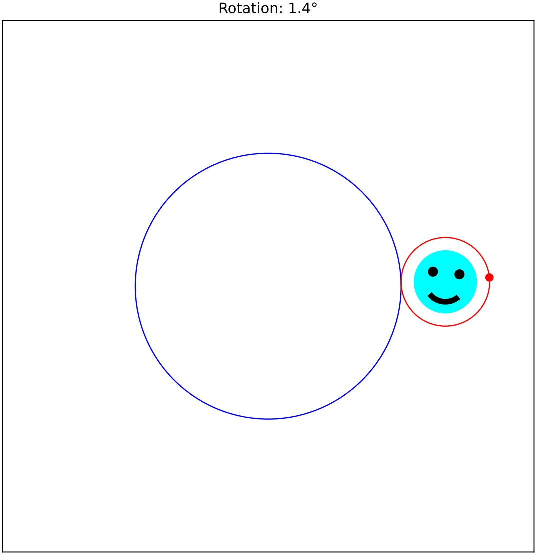
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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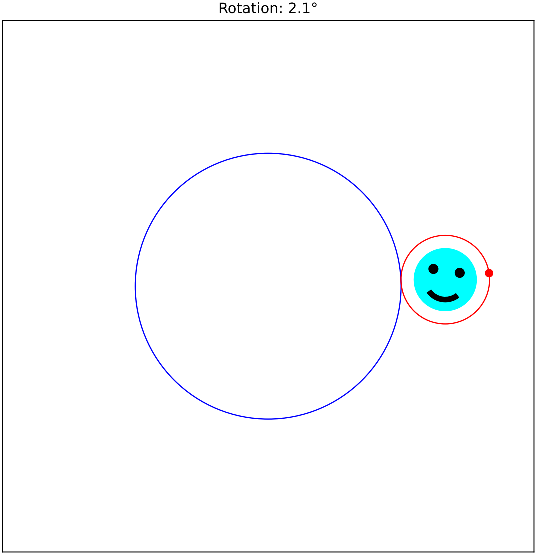

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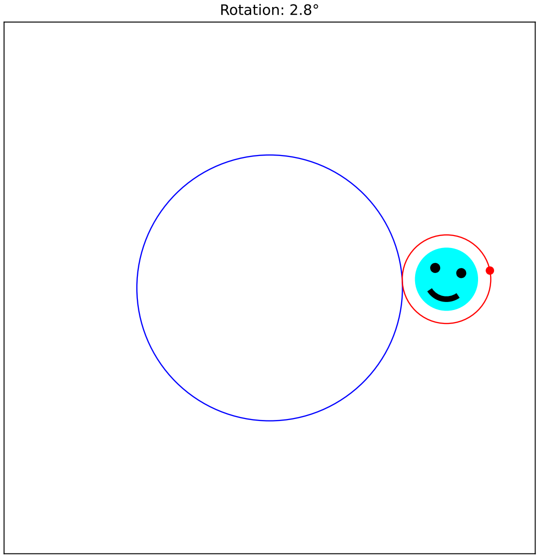

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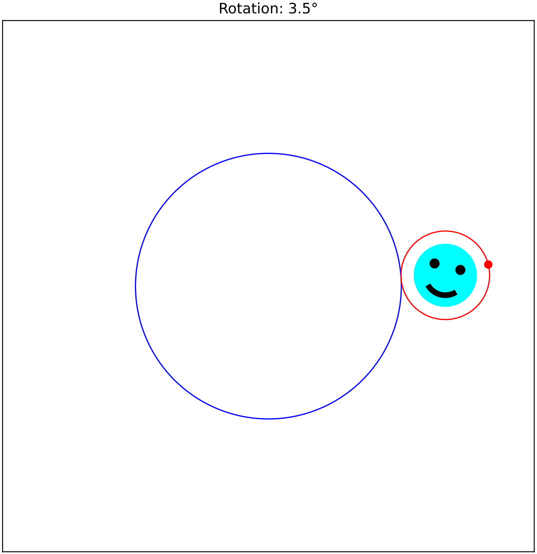

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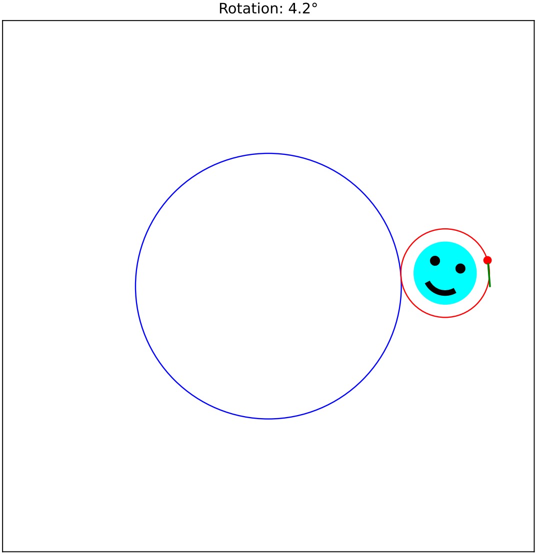

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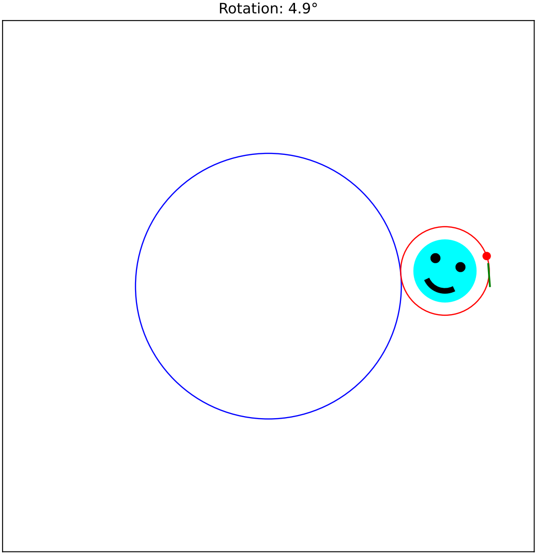

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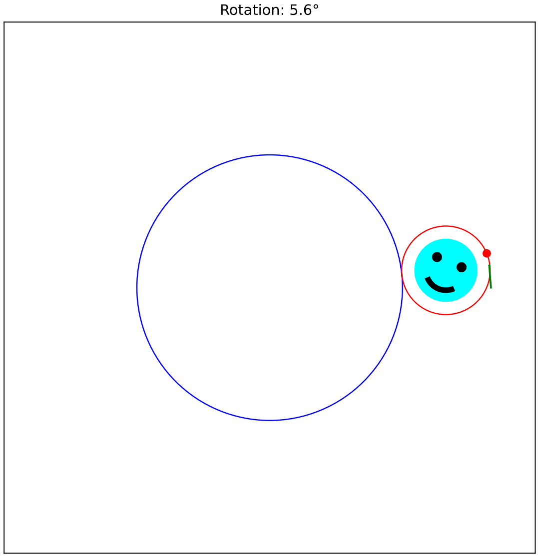

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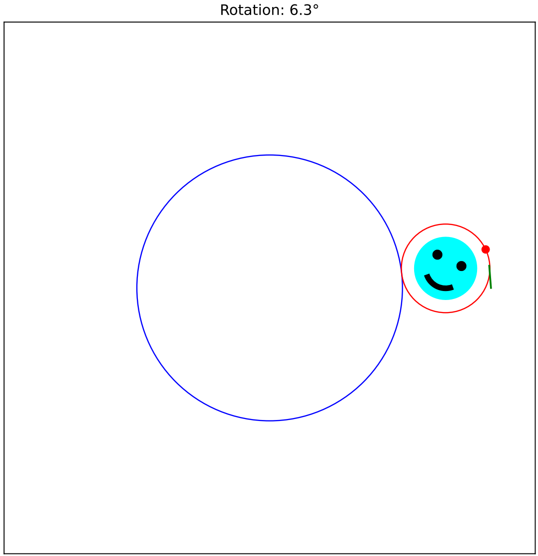

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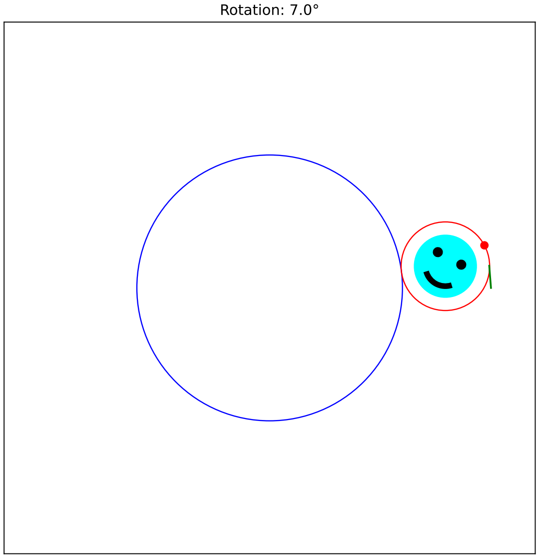

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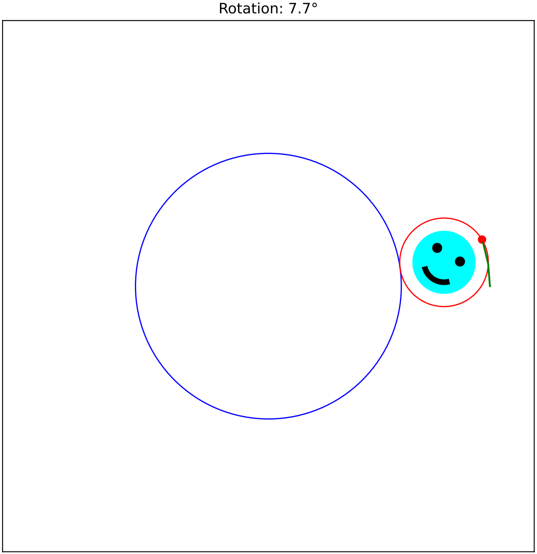

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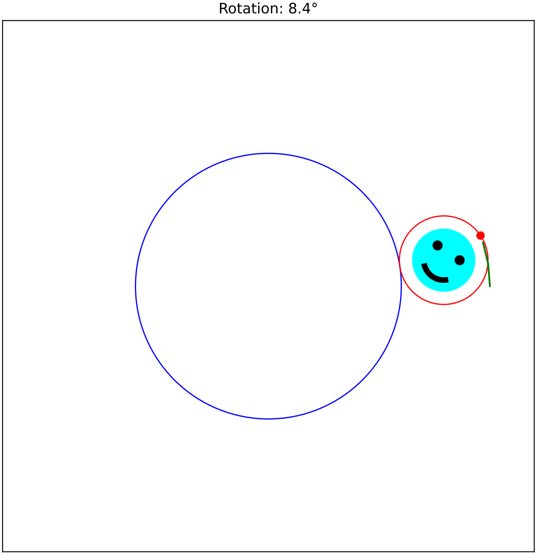
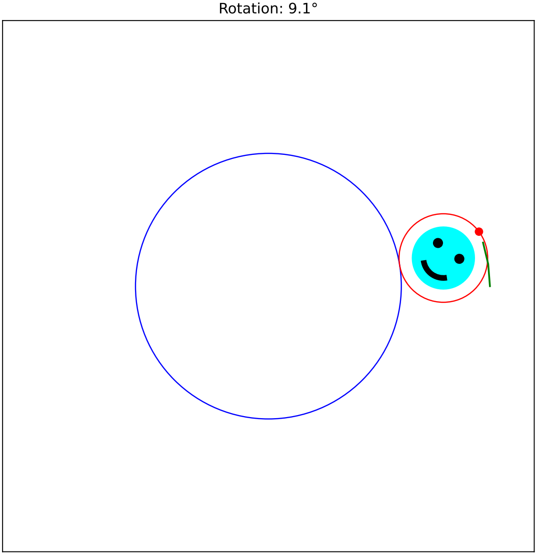


Animation of Coin Rotation Paradox



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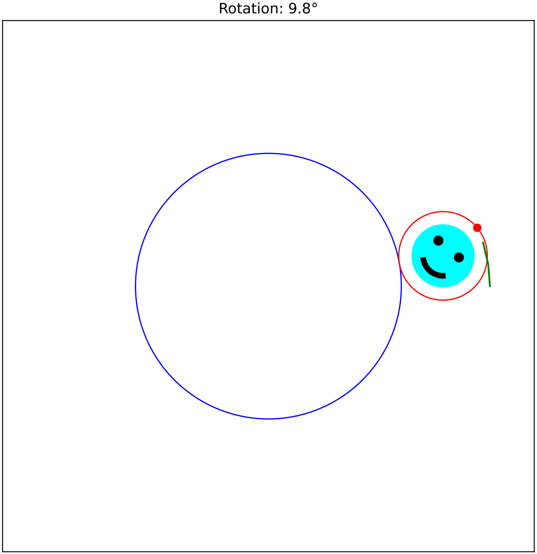

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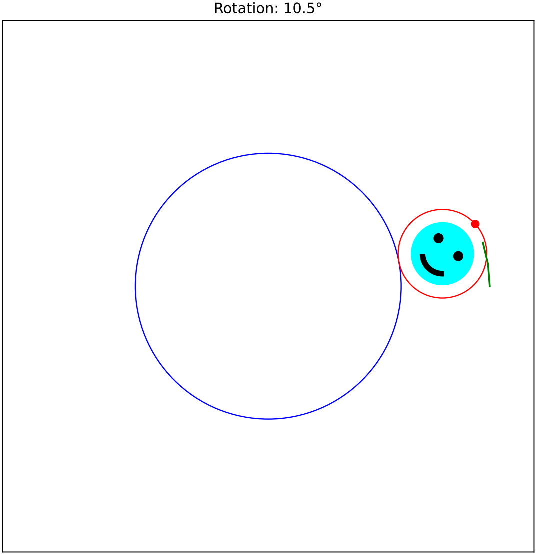

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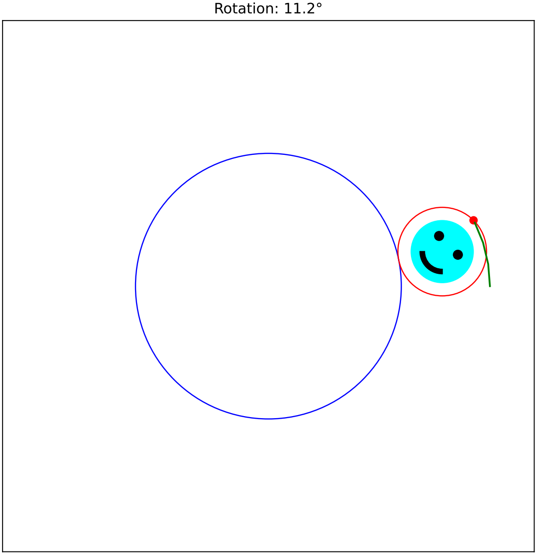

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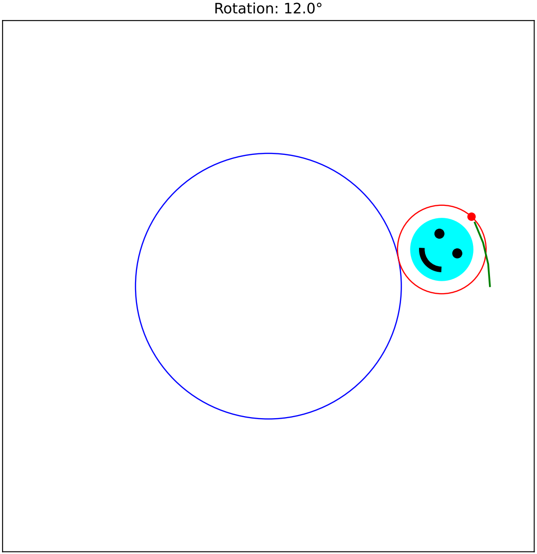

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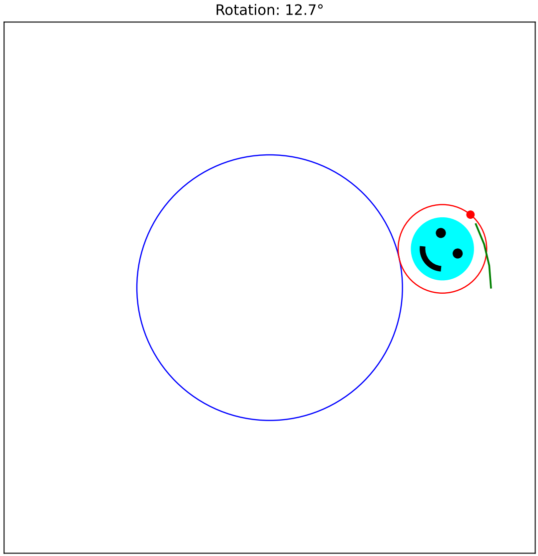

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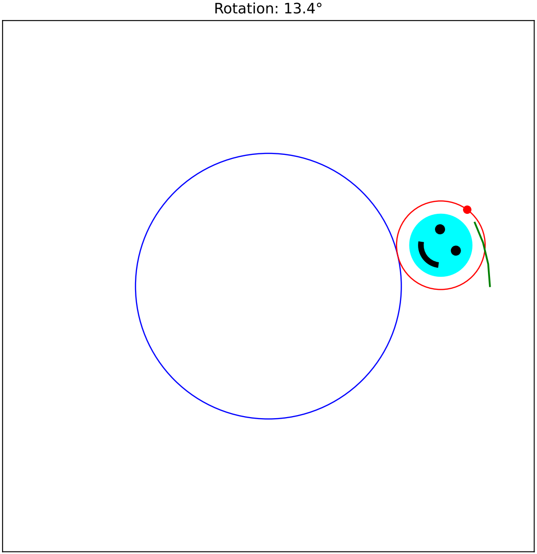

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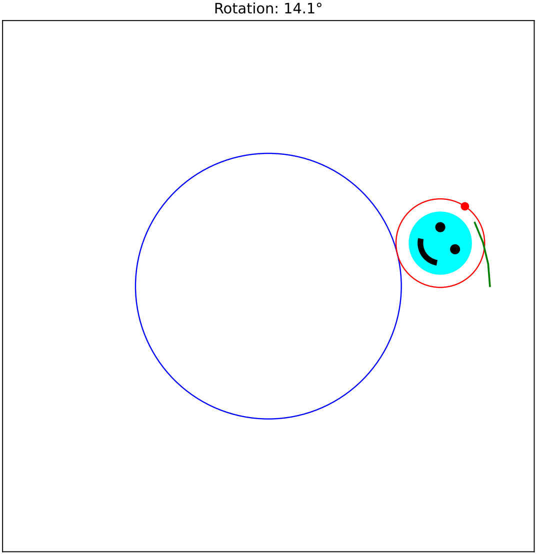

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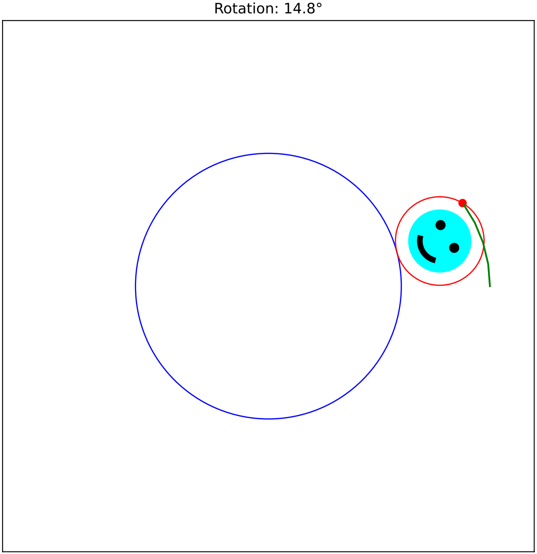

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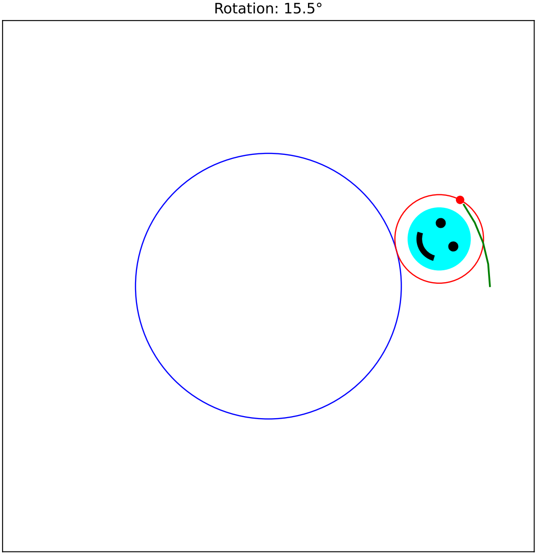

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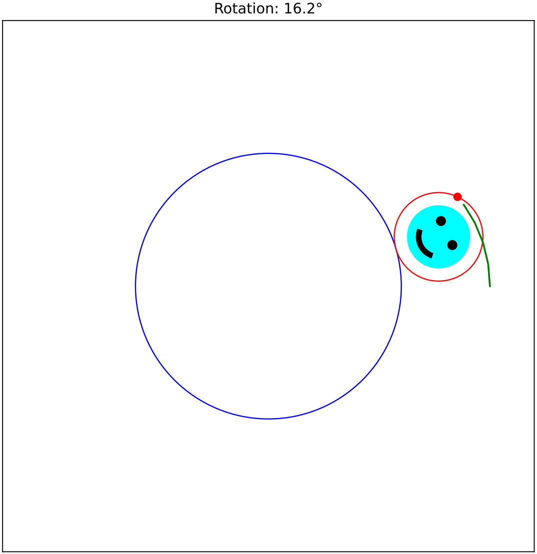

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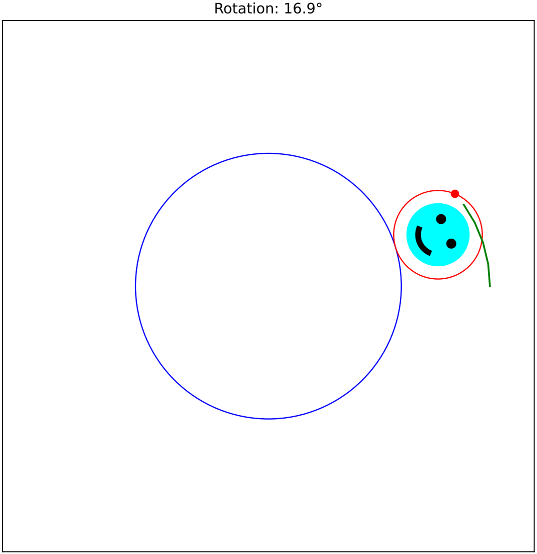

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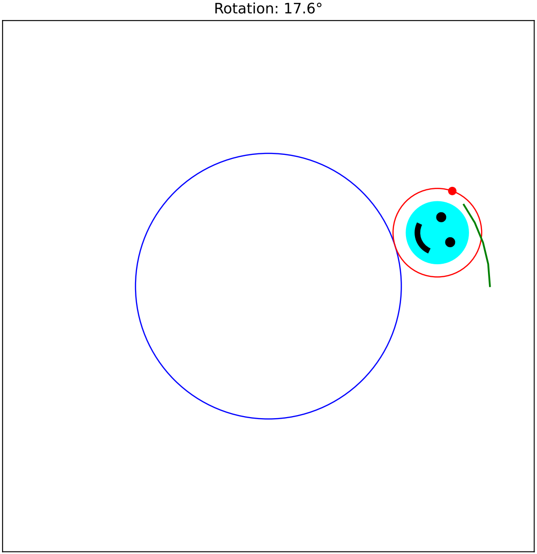

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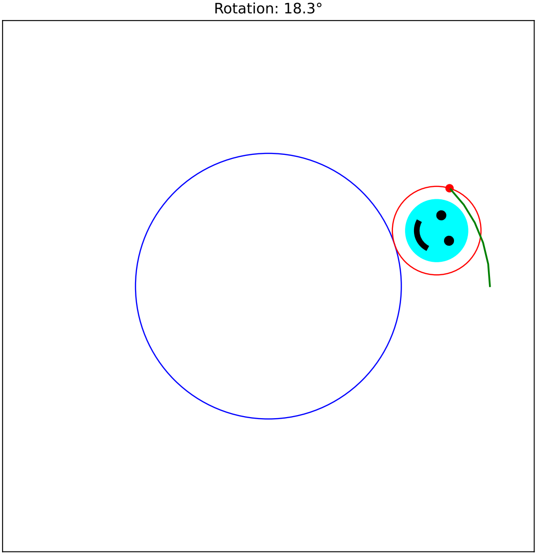

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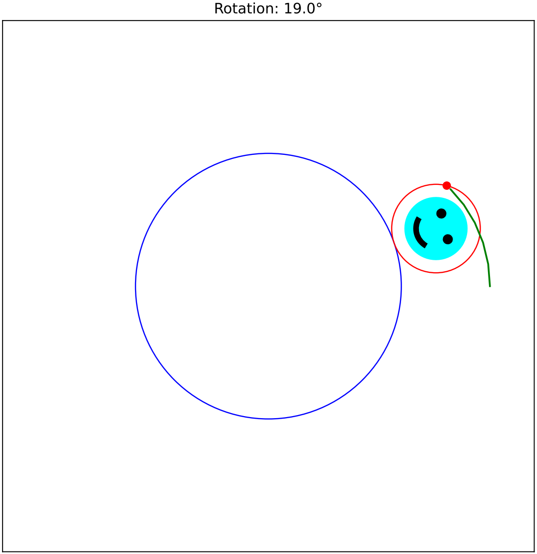

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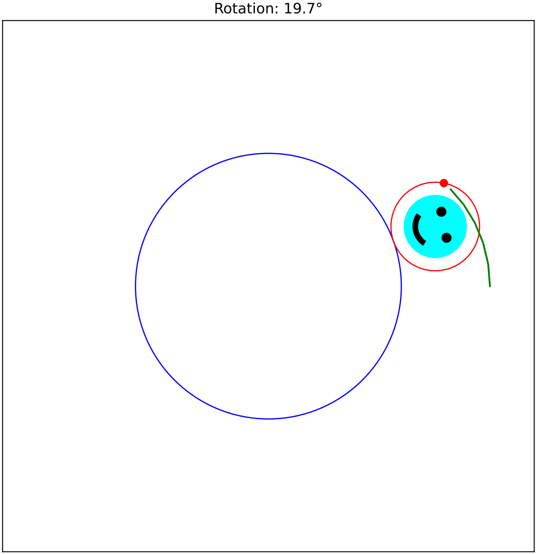

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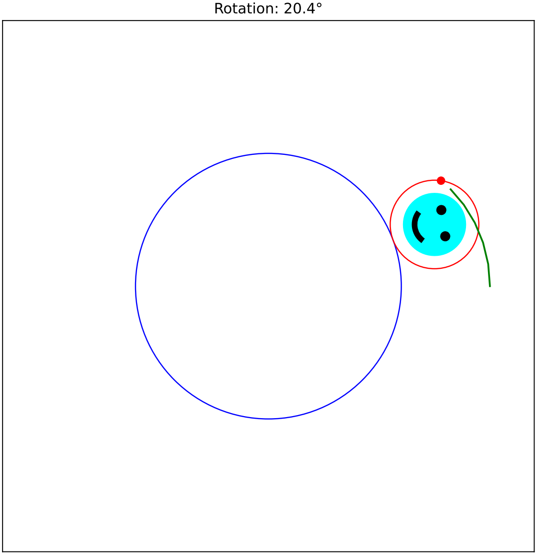
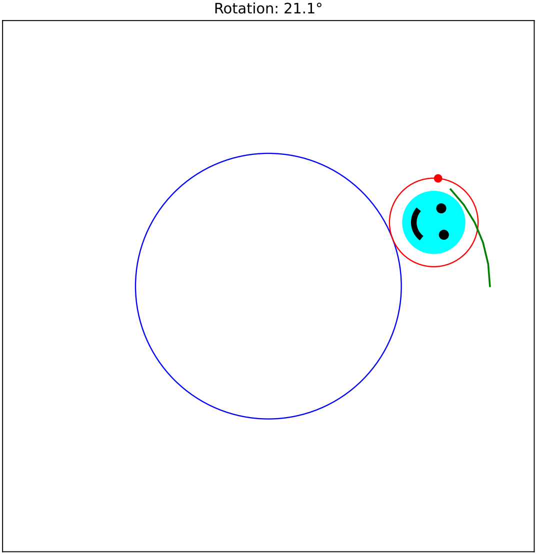


Animation of Coin Rotation Paradox



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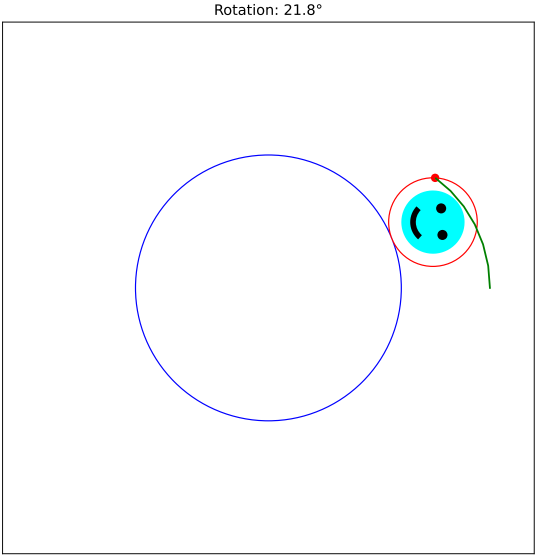

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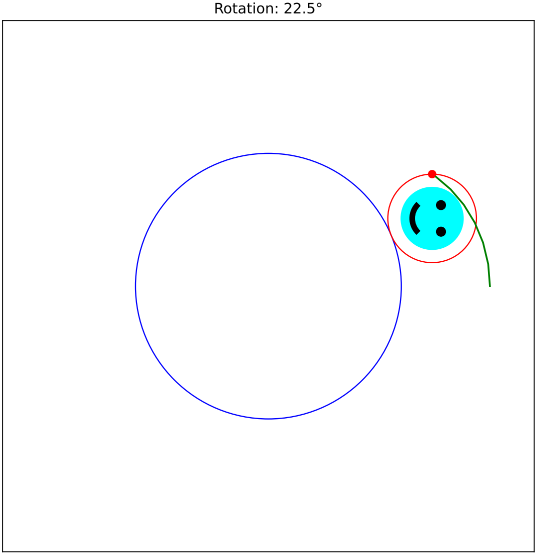

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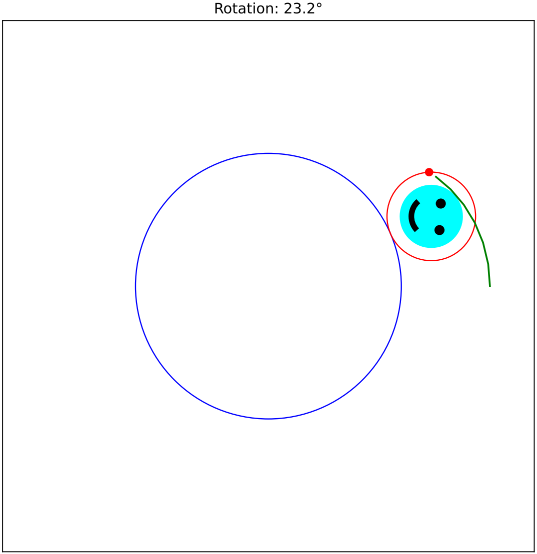

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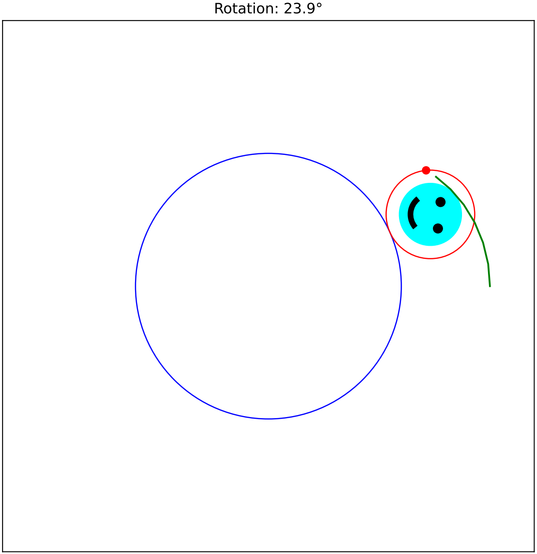
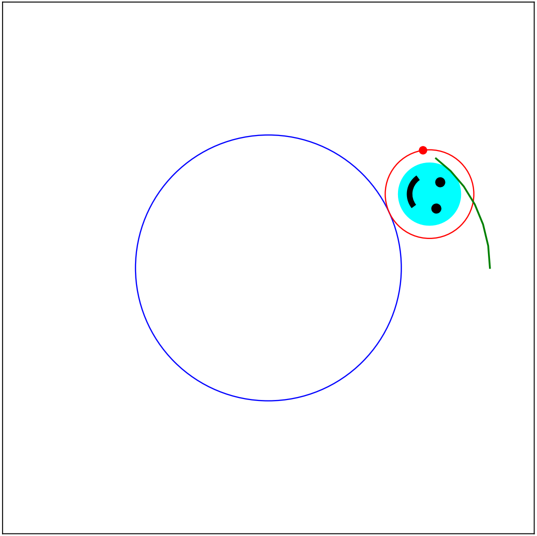
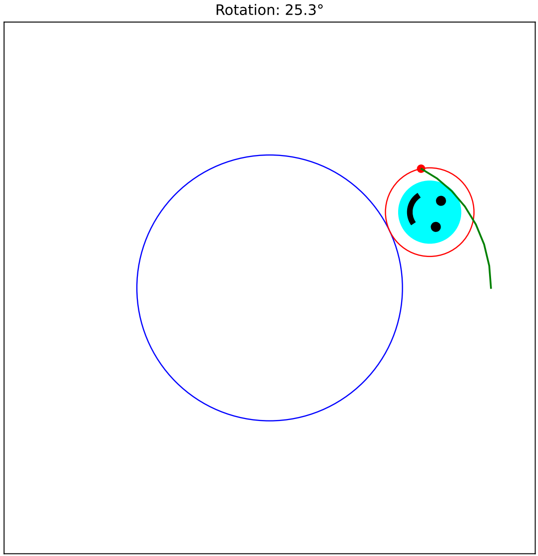


Animation of Coin Rotation Paradox

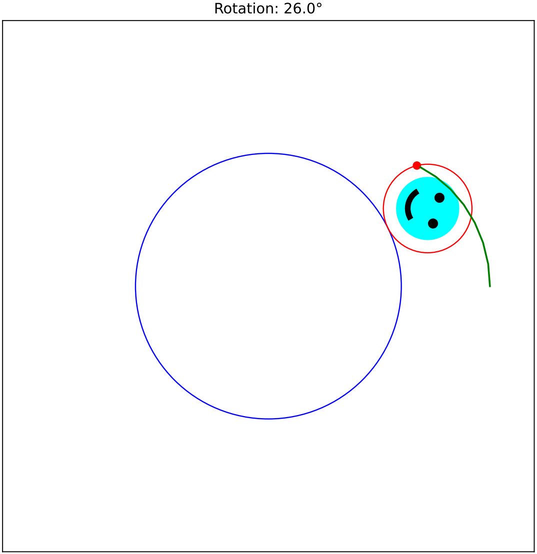
Rotation: 24.6°



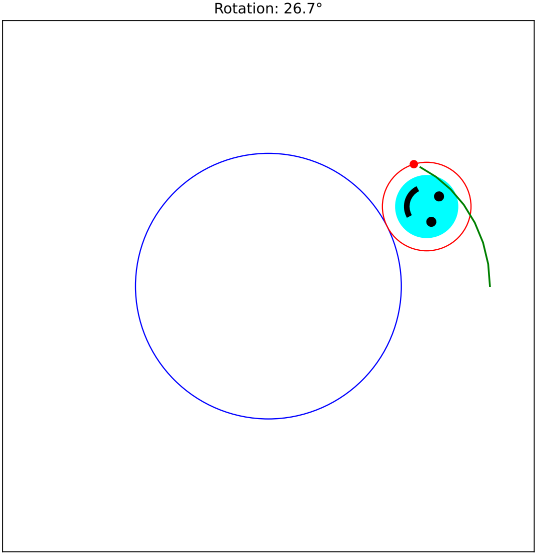
Animation of Coin Rotation Paradox



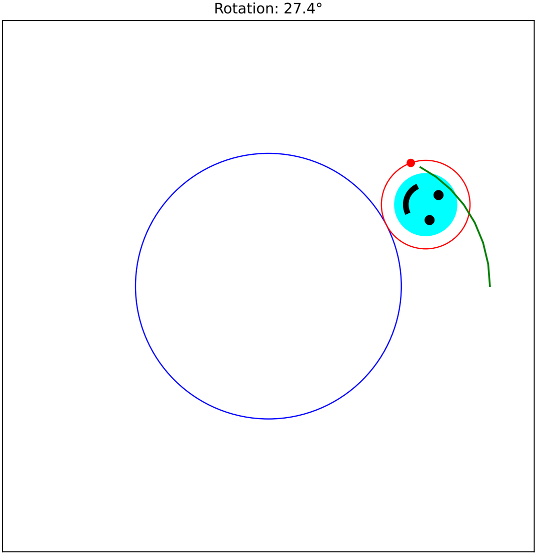
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

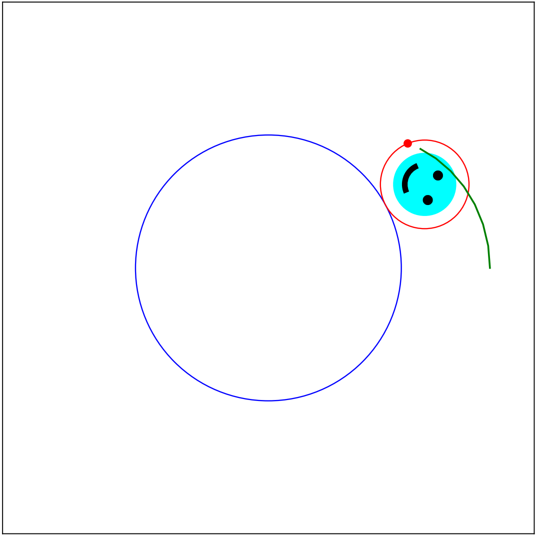


Animation of Coin Rotation Paradox

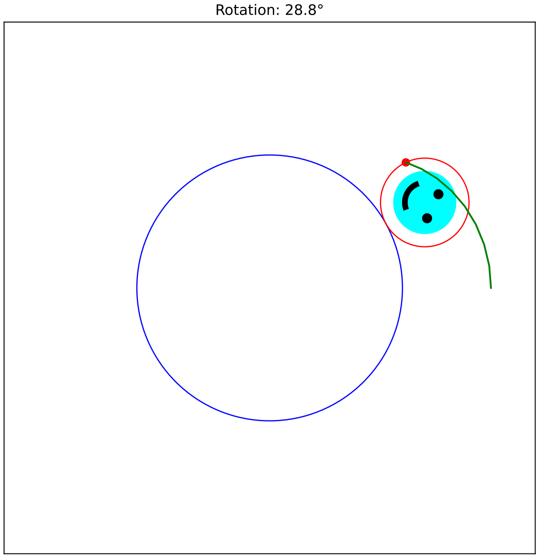


Animation of Coin Rotation Paradox

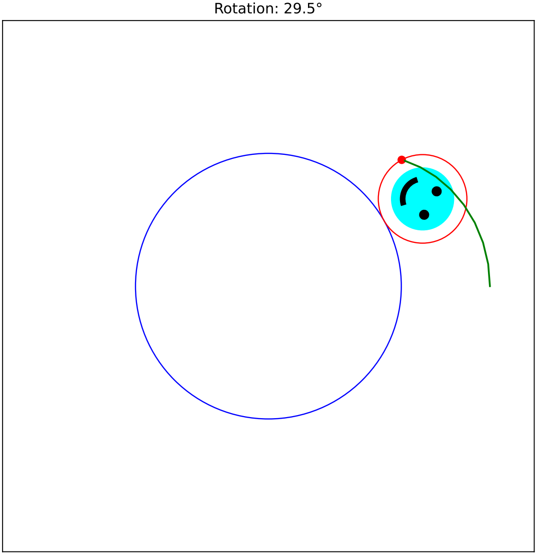
Rotation: 28.1°



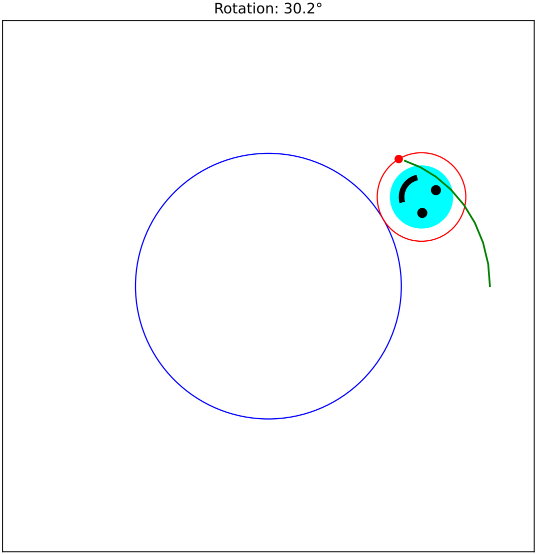
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

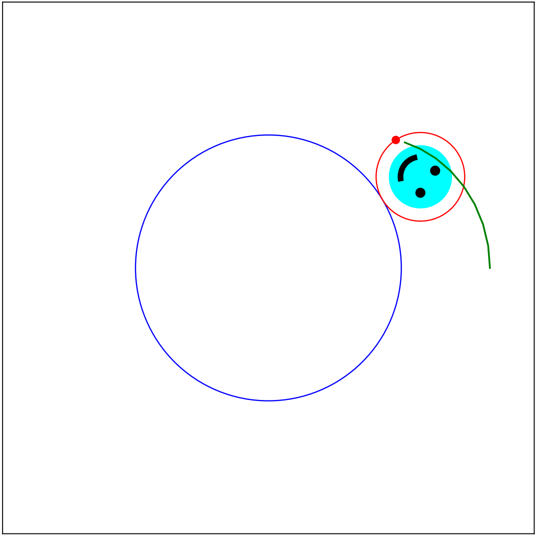


Animation of Coin Rotation Paradox

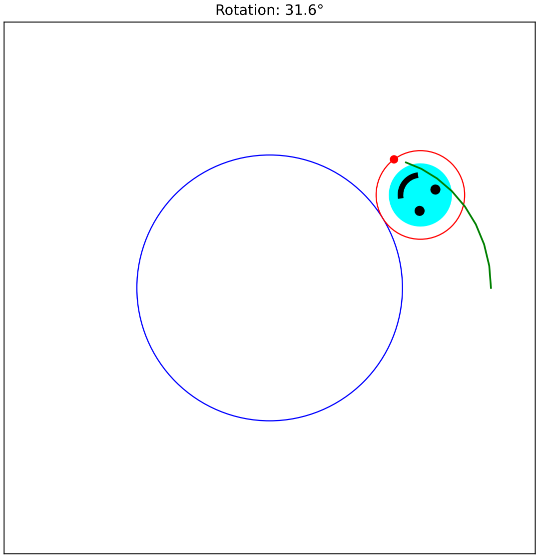


Animation of Coin Rotation Paradox

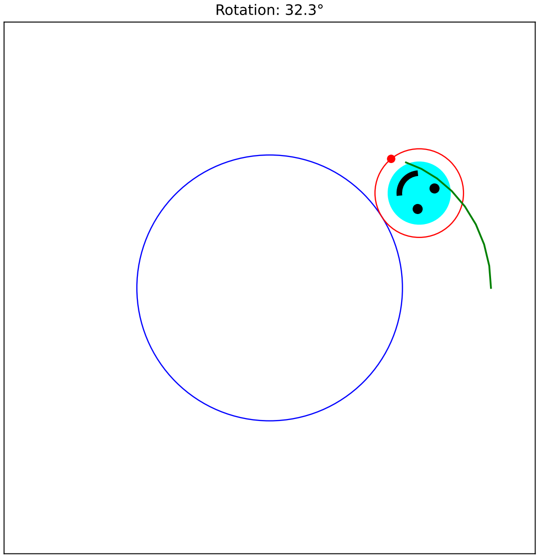
Rotation: 30.9°



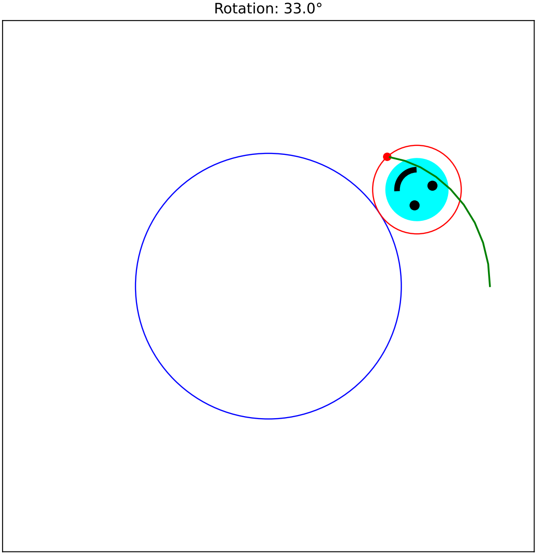
Animation of Coin Rotation Paradox



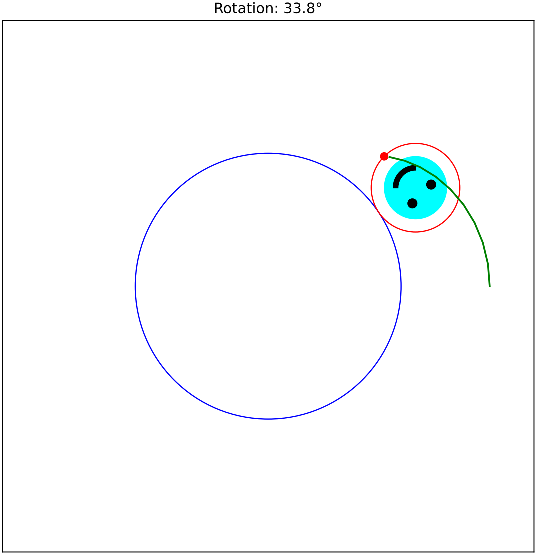
Animation of Coin Rotation Paradox



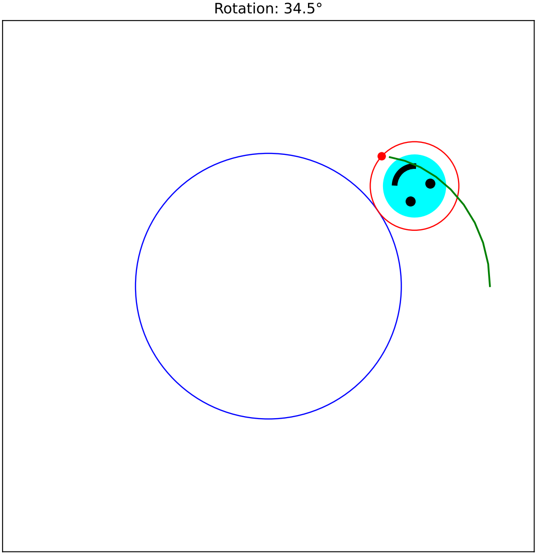
Animation of Coin Rotation Paradox



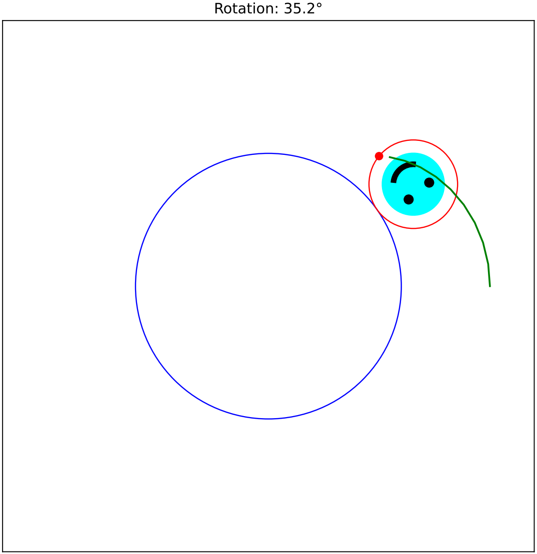
Animation of Coin Rotation Paradox



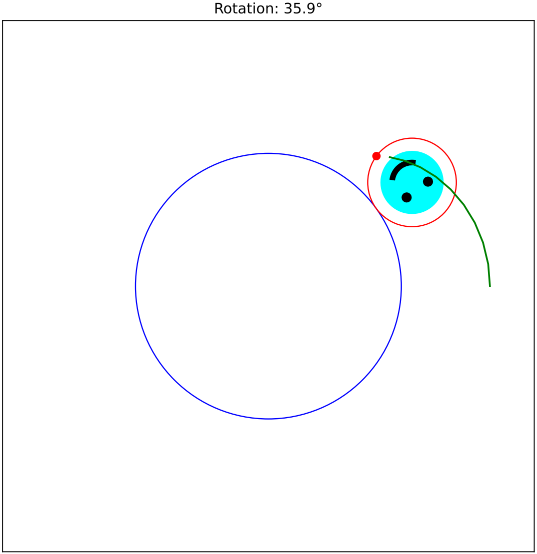
Animation of Coin Rotation Paradox



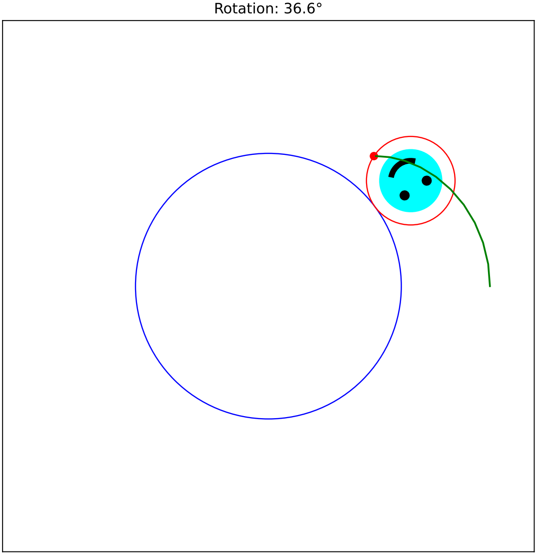
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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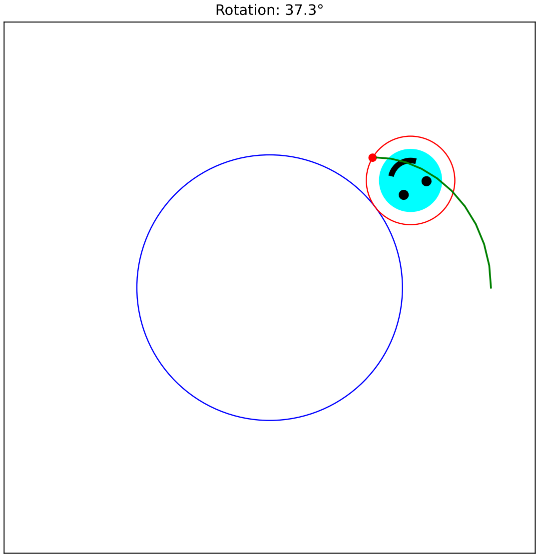

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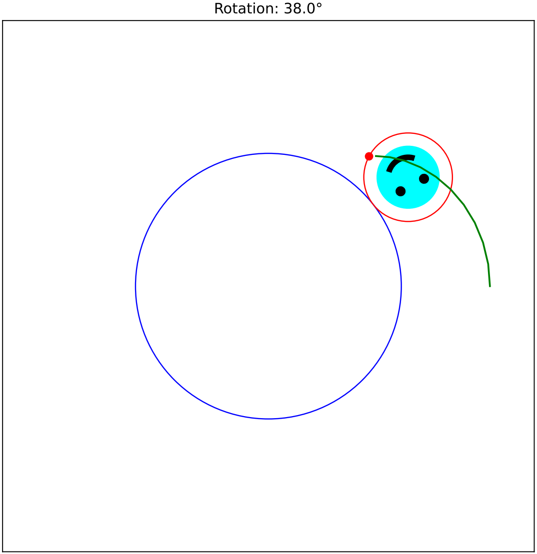

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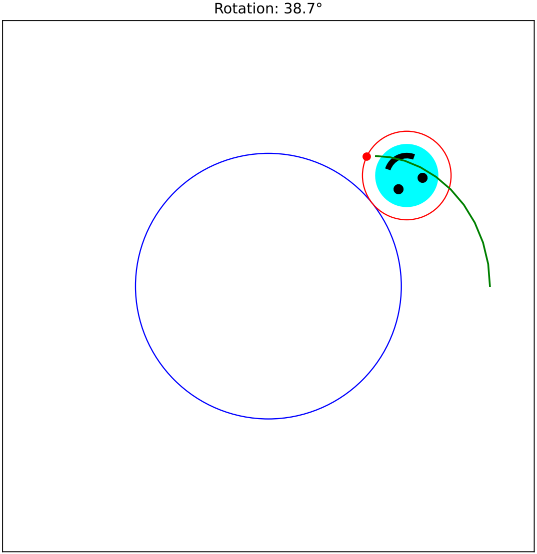
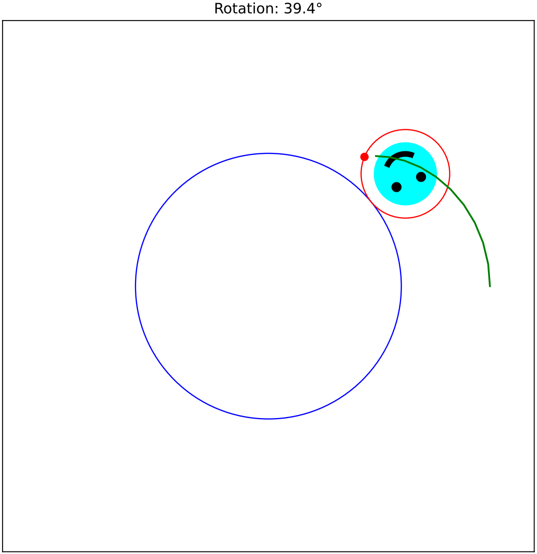
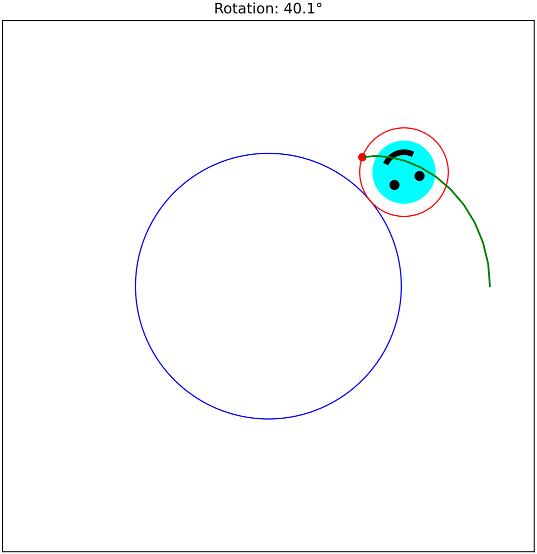


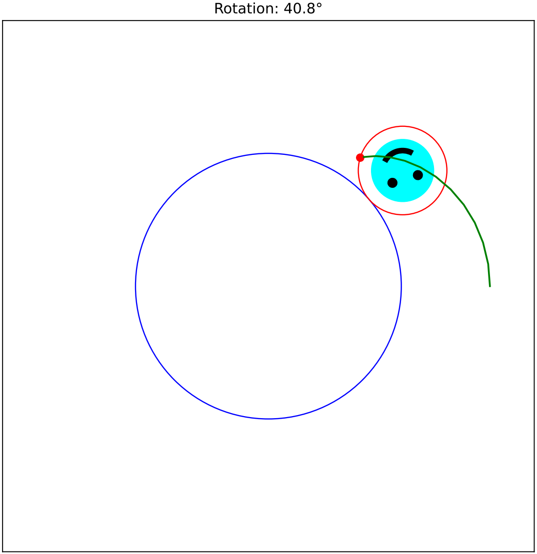
Animation of Coin Rotation Paradox



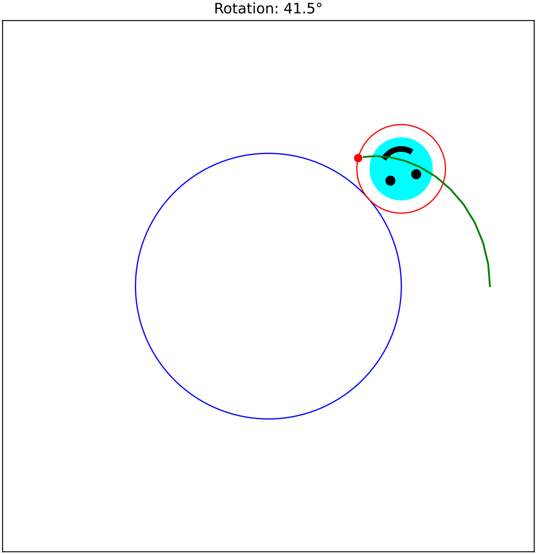
Animation of Coin Rotation Paradox



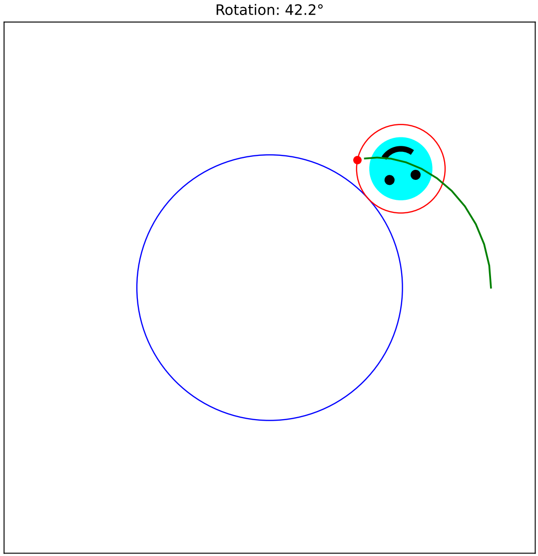
Animation of Coin Rotation Paradox



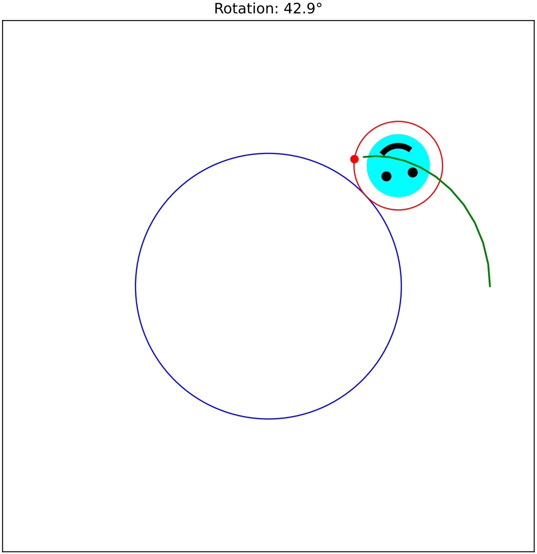
Animation of Coin Rotation Paradox



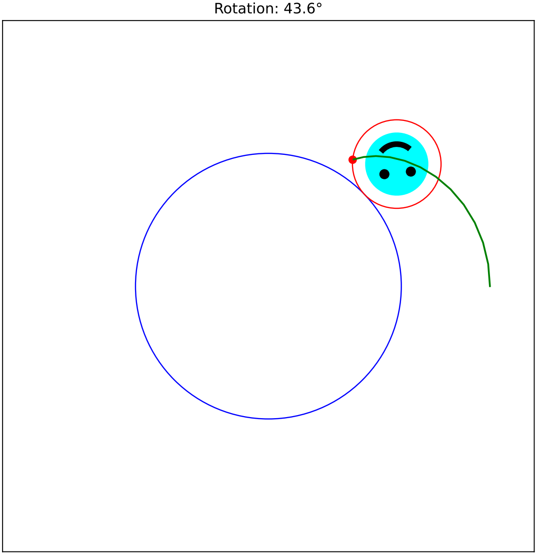
Animation of Coin Rotation Paradox



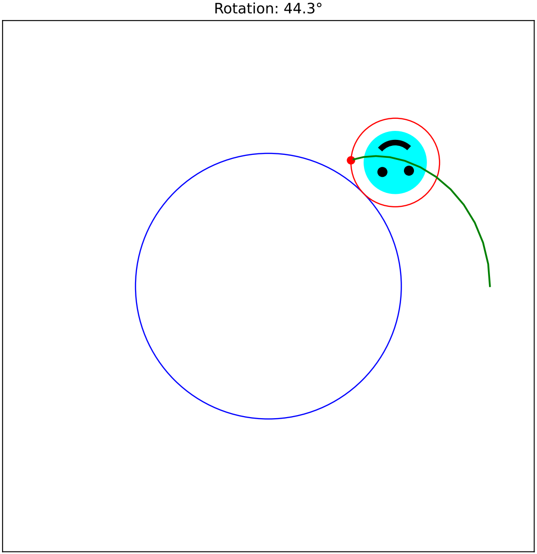
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

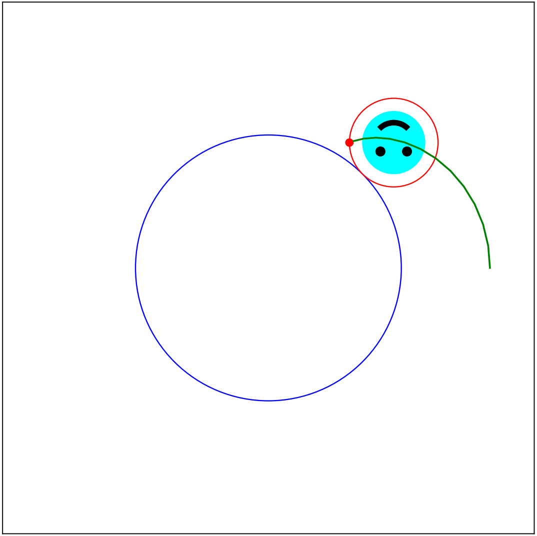


Animation of Coin Rotation Paradox

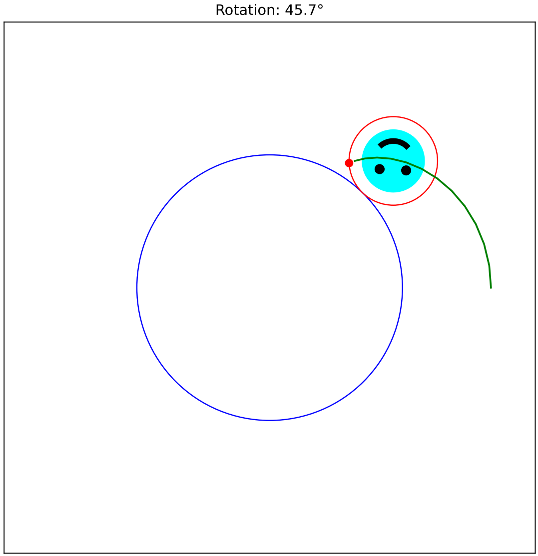


Animation of Coin Rotation Paradox

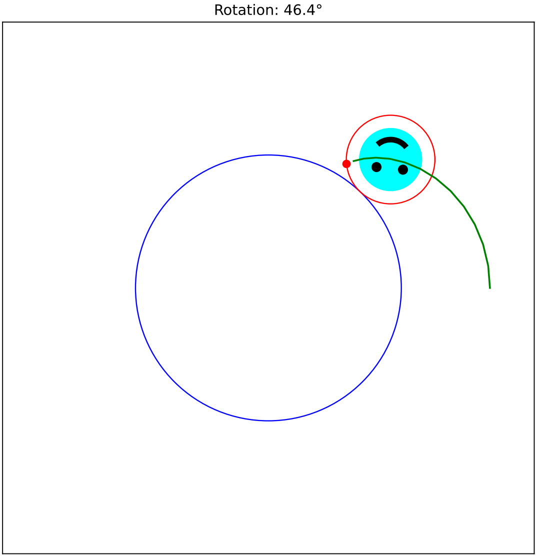
Rotation: 45.0°



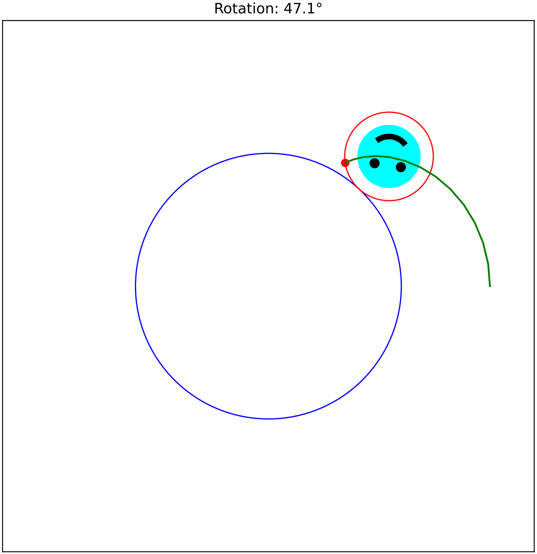
Animation of Coin Rotation Paradox



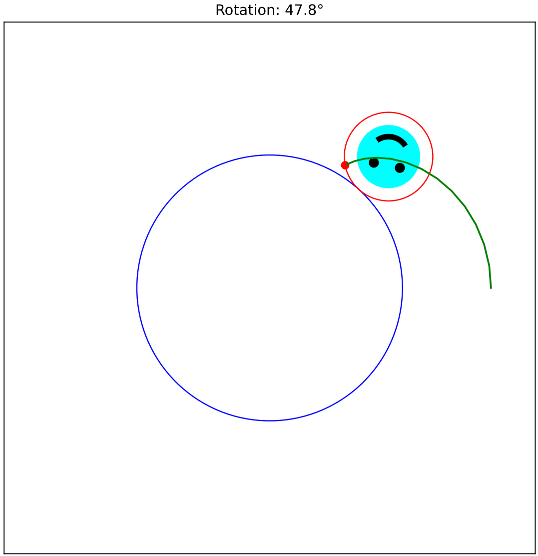
Animation of Coin Rotation Paradox



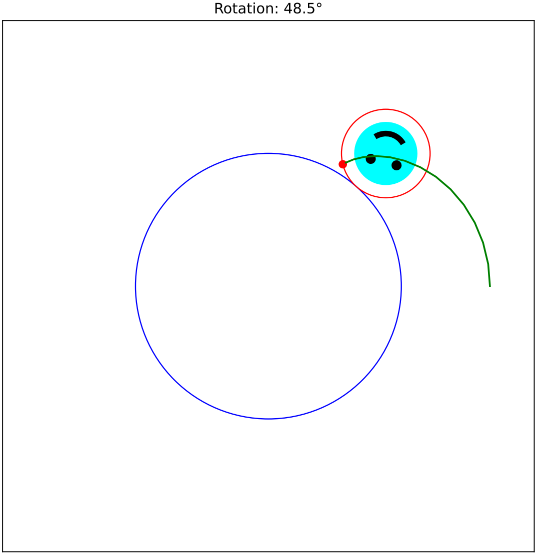
Animation of Coin Rotation Paradox



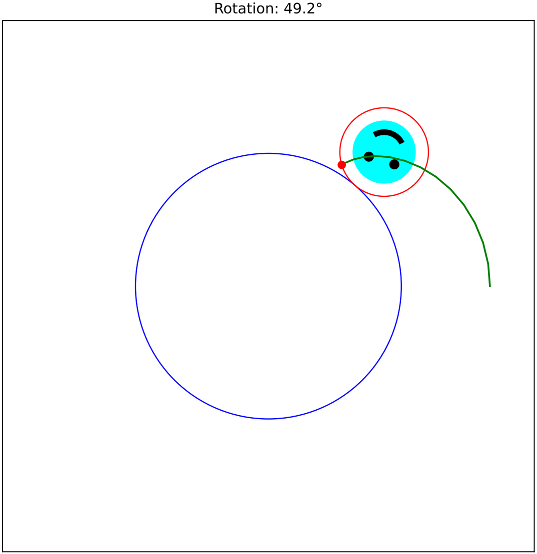
Animation of Coin Rotation Paradox



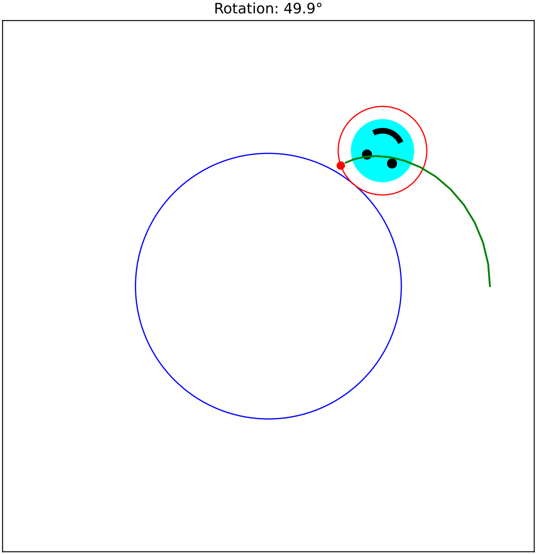
Animation of Coin Rotation Paradox



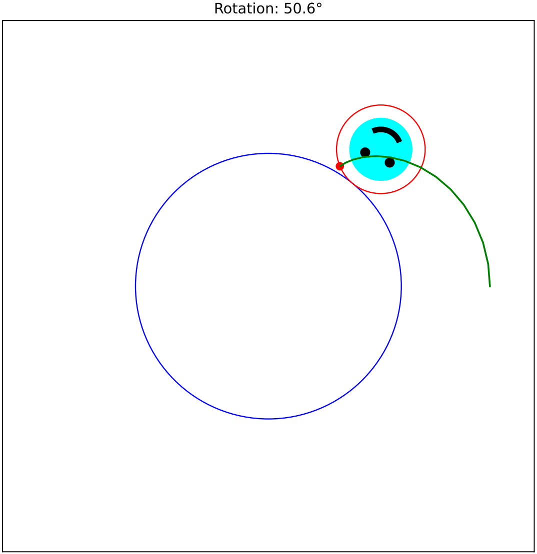
Animation of Coin Rotation Paradox



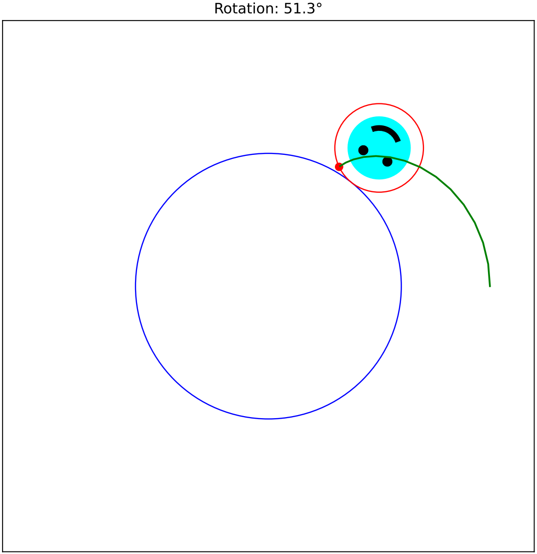
Animation of Coin Rotation Paradox



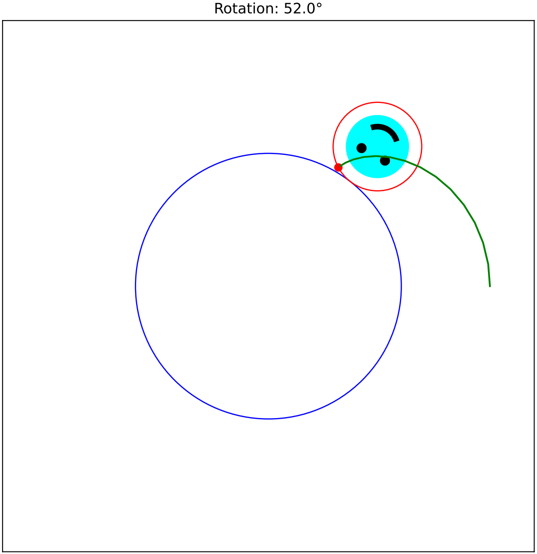
Animation of Coin Rotation Paradox



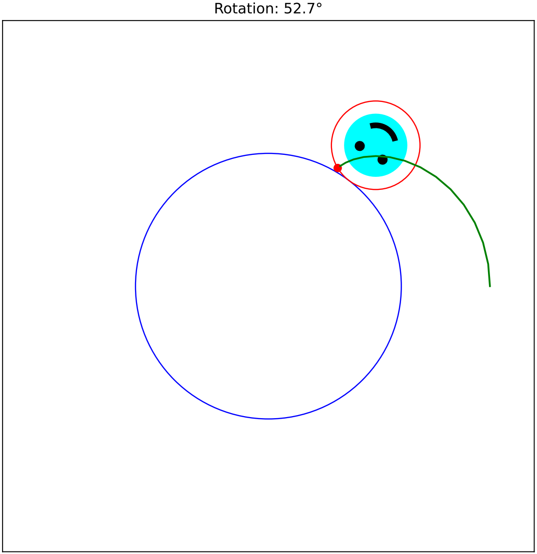
Animation of Coin Rotation Paradox



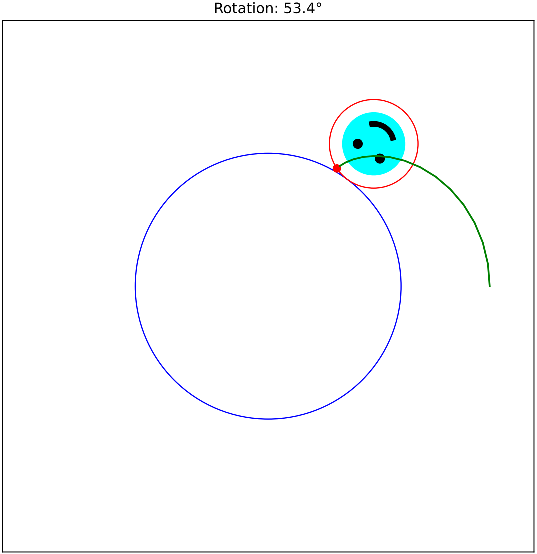
Animation of Coin Rotation Paradox



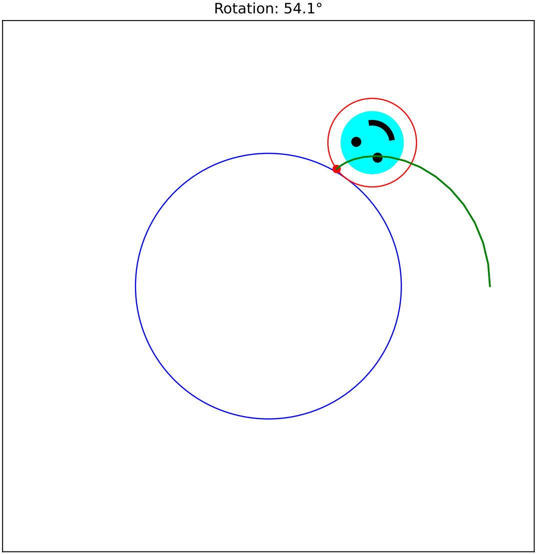
Animation of Coin Rotation Paradox



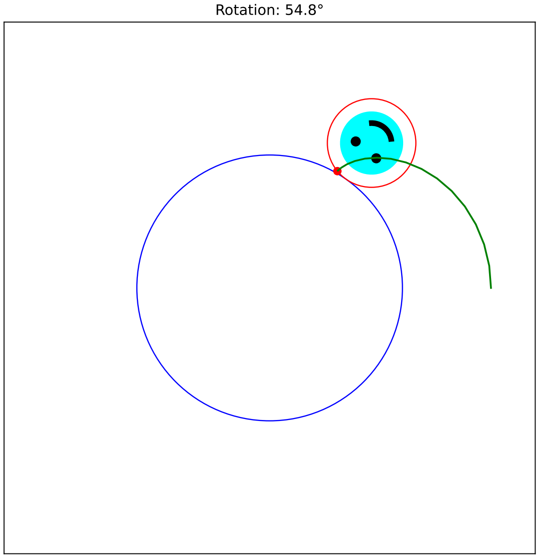
Animation of Coin Rotation Paradox



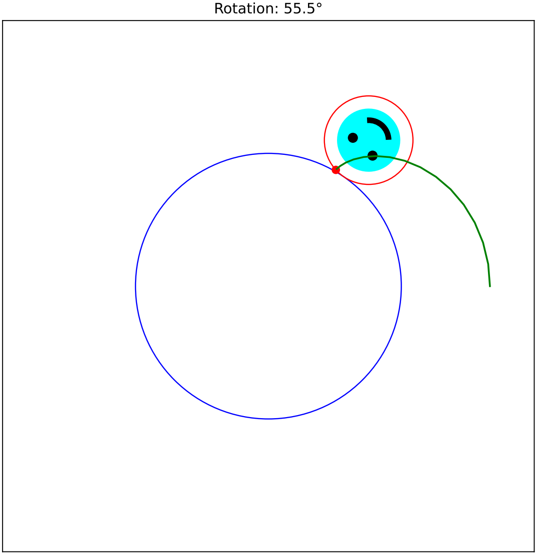
Animation of Coin Rotation Paradox



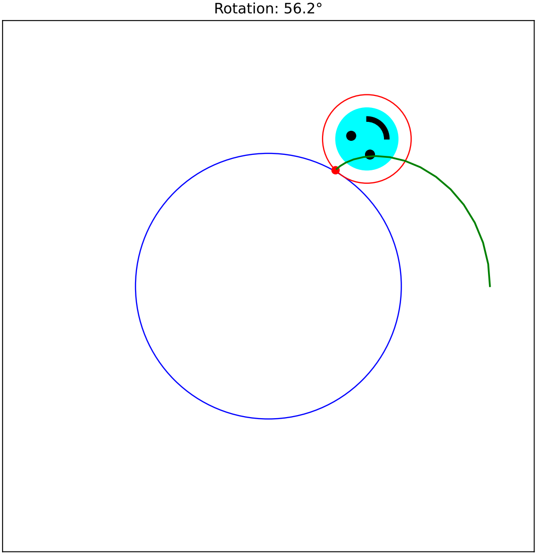
Animation of Coin Rotation Paradox



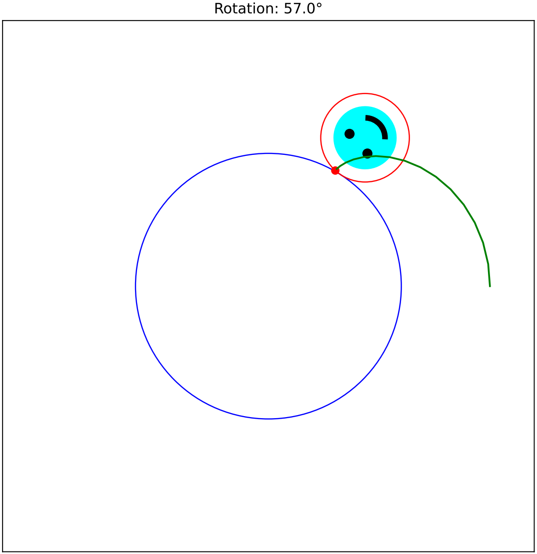
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

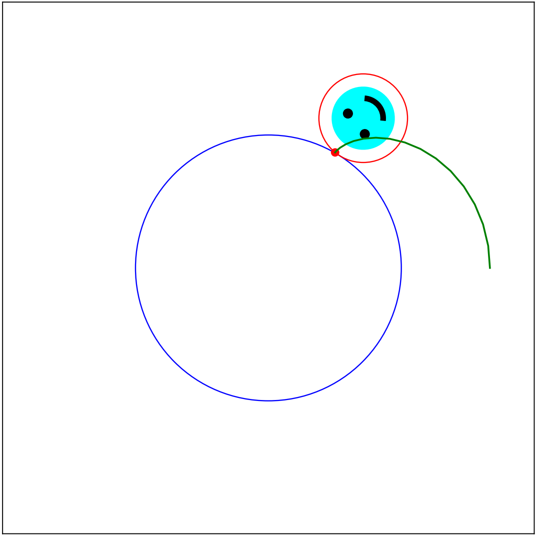


Animation of Coin Rotation Paradox

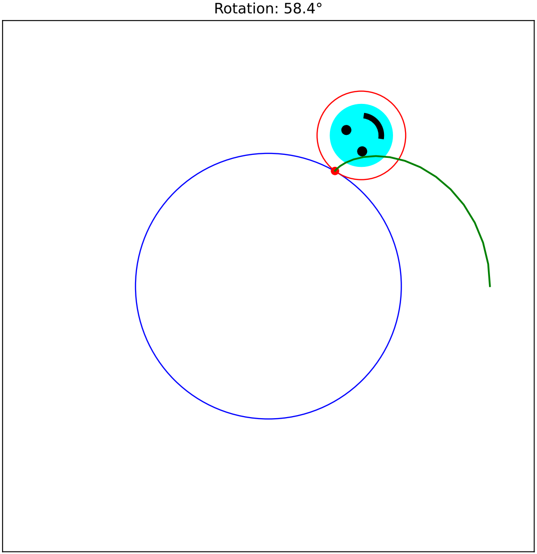


Animation of Coin Rotation Paradox

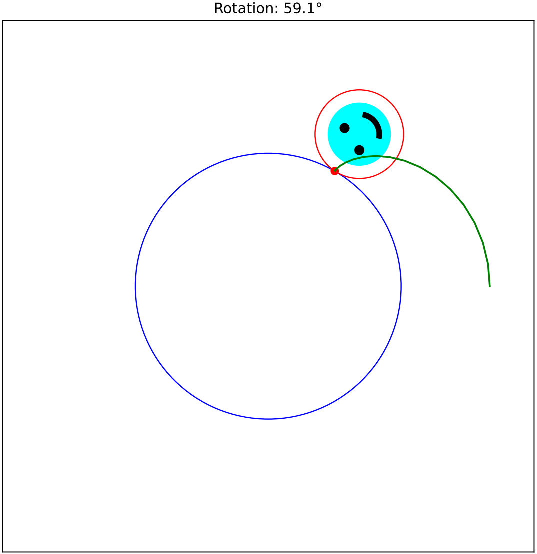
Rotation: 57.7°



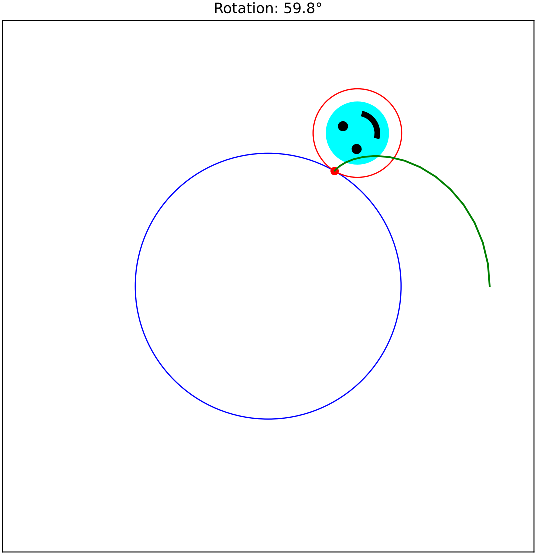
Animation of Coin Rotation Paradox



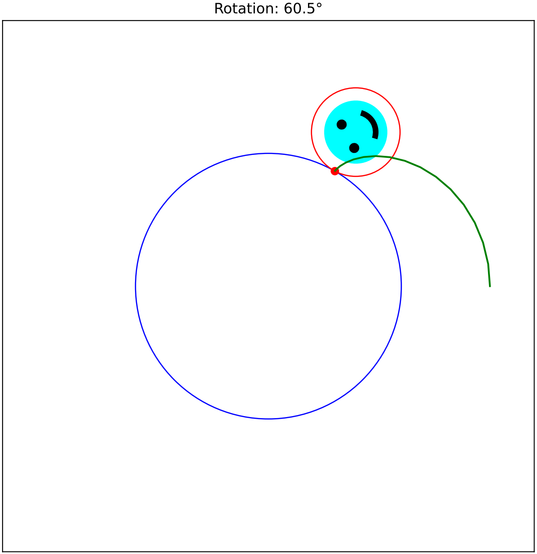
Animation of Coin Rotation Paradox



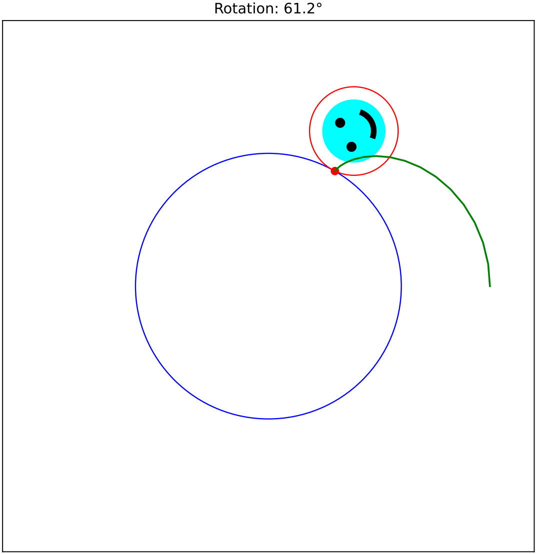
Animation of Coin Rotation Paradox



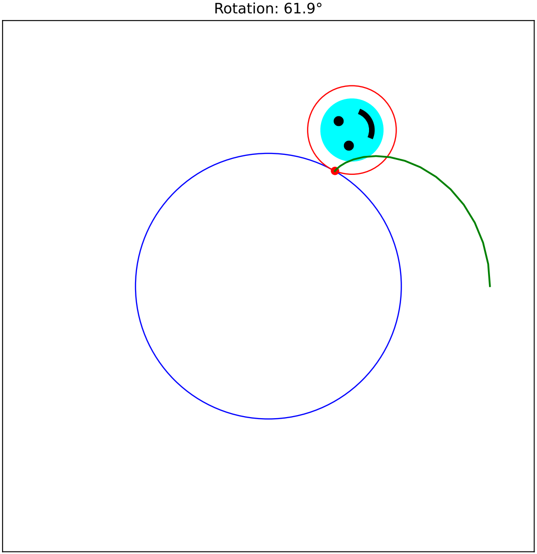
Animation of Coin Rotation Paradox



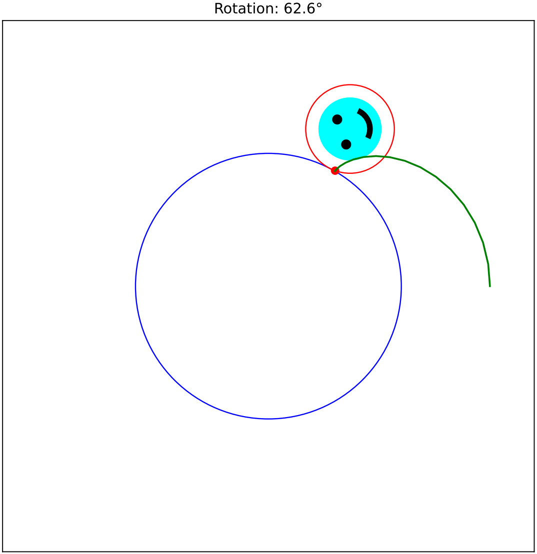
Animation of Coin Rotation Paradox



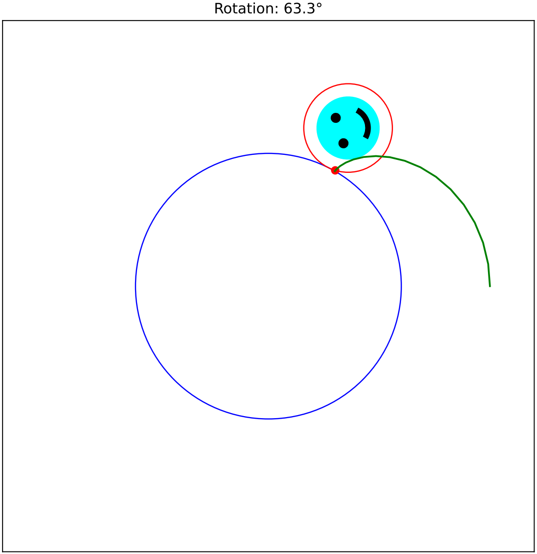
Animation of Coin Rotation Paradox



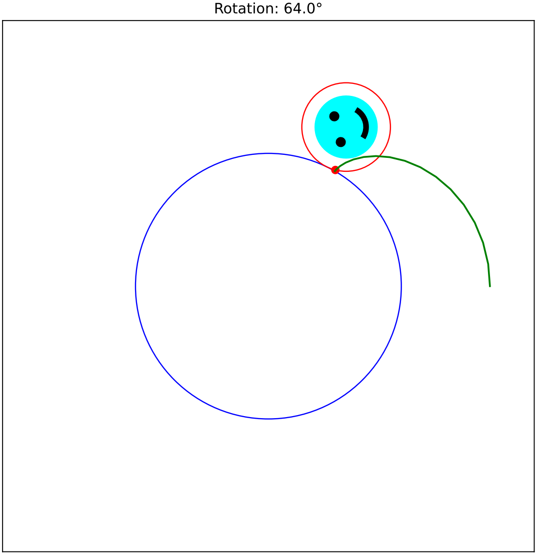
Animation of Coin Rotation Paradox



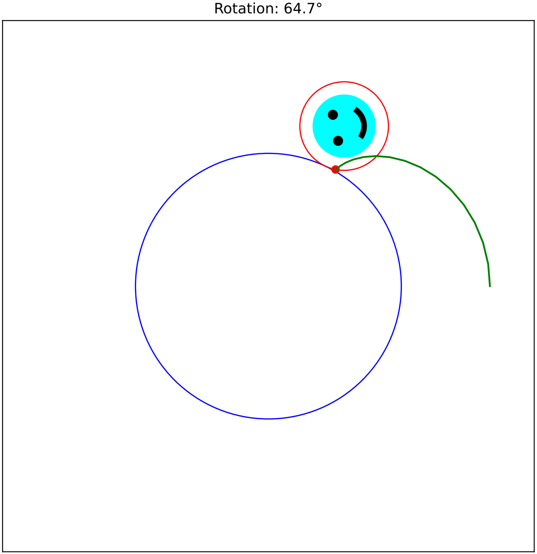
Animation of Coin Rotation Paradox



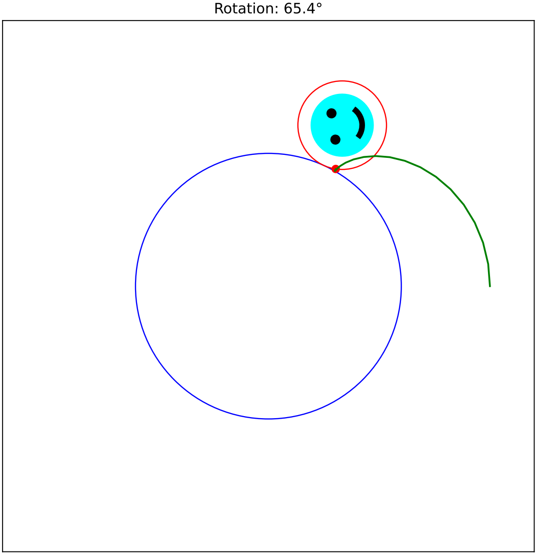
Animation of Coin Rotation Paradox



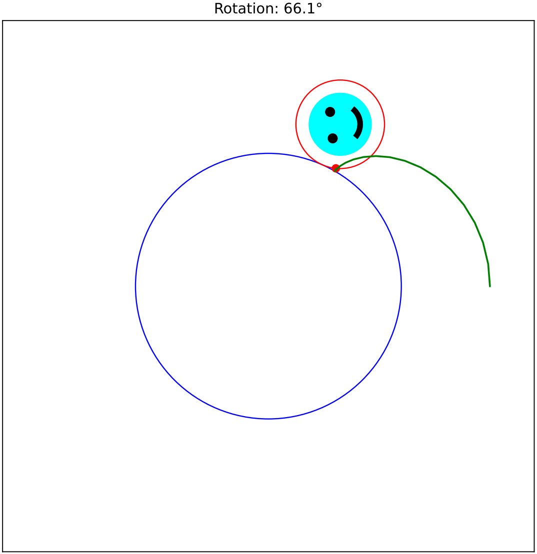
Animation of Coin Rotation Paradox



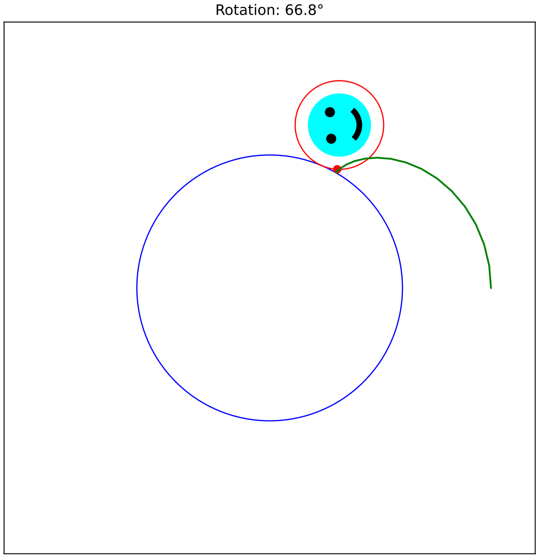
Animation of Coin Rotation Paradox



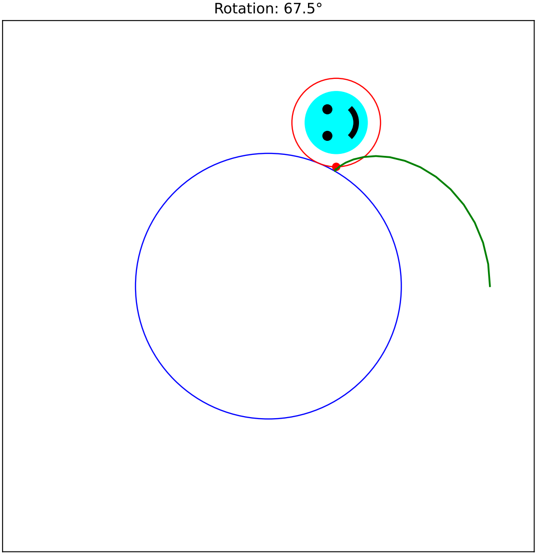
Animation of Coin Rotation Paradox



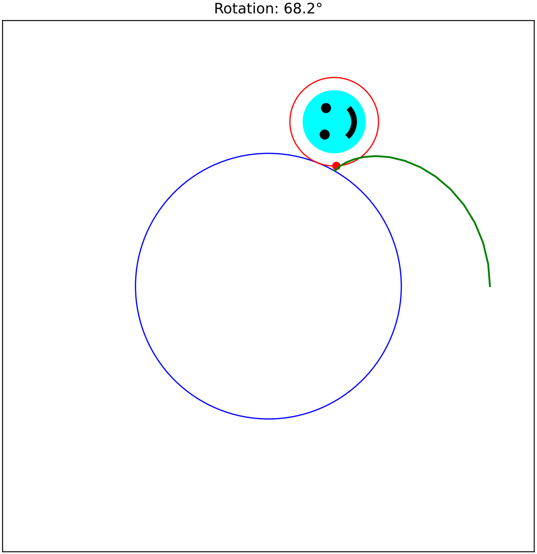
Animation of Coin Rotation Paradox



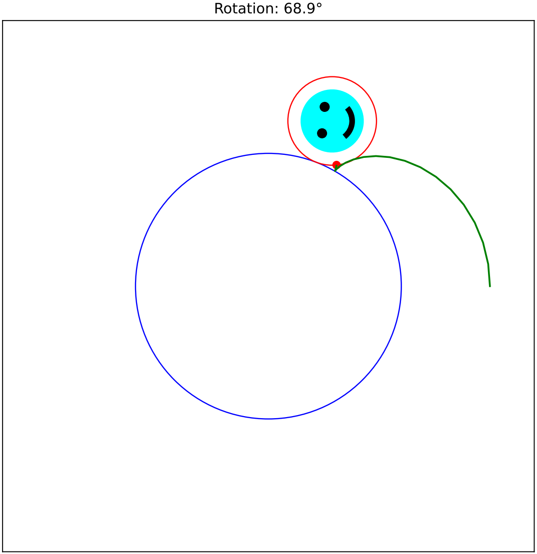
Animation of Coin Rotation Paradox



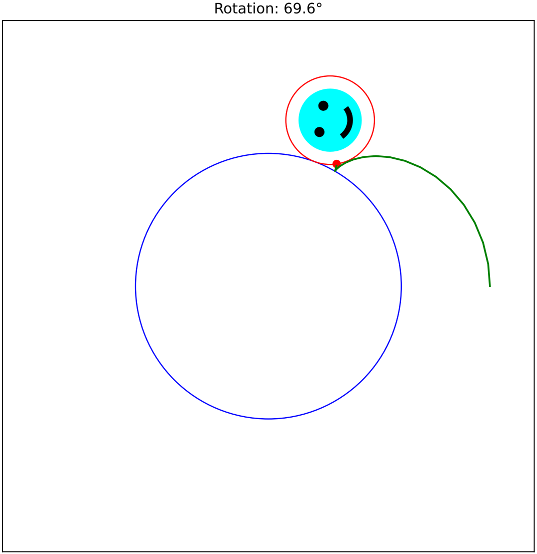
Animation of Coin Rotation Paradox



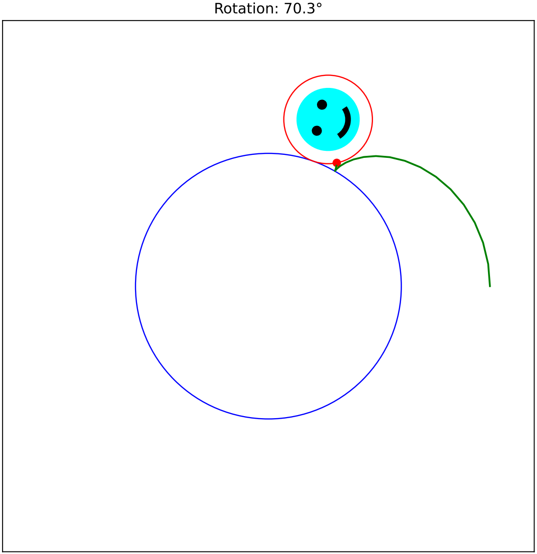
Animation of Coin Rotation Paradox



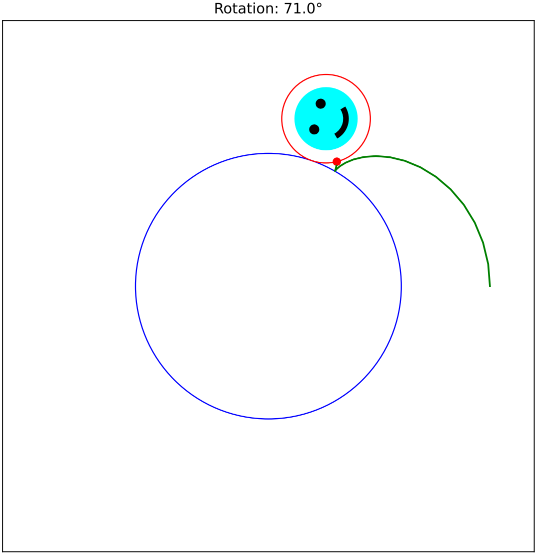
Animation of Coin Rotation Paradox



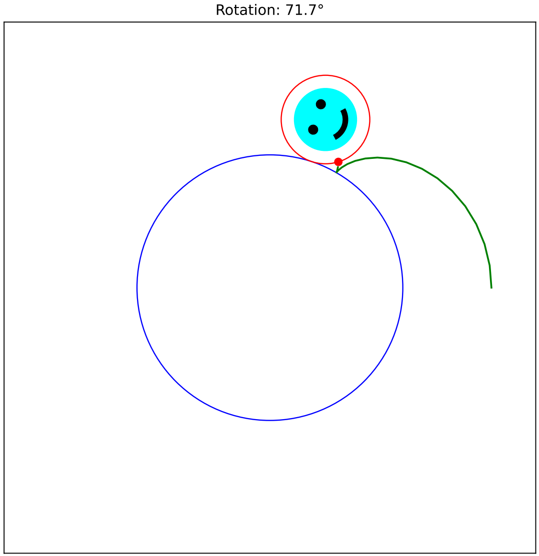
Animation of Coin Rotation Paradox



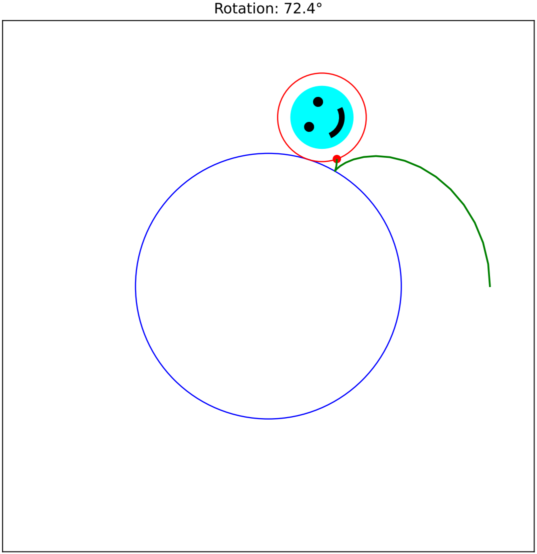
Animation of Coin Rotation Paradox



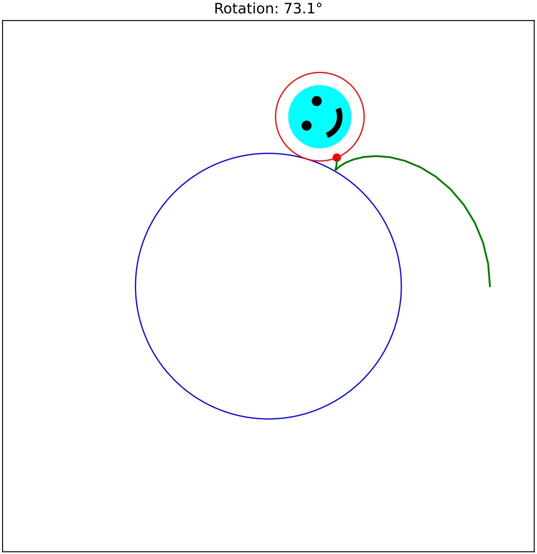
Animation of Coin Rotation Paradox



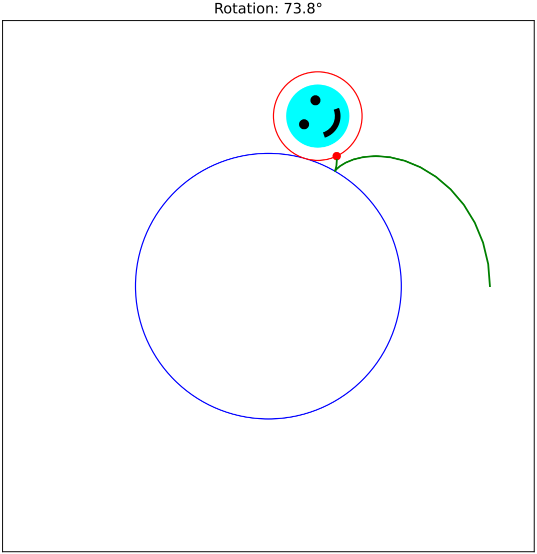
Animation of Coin Rotation Paradox



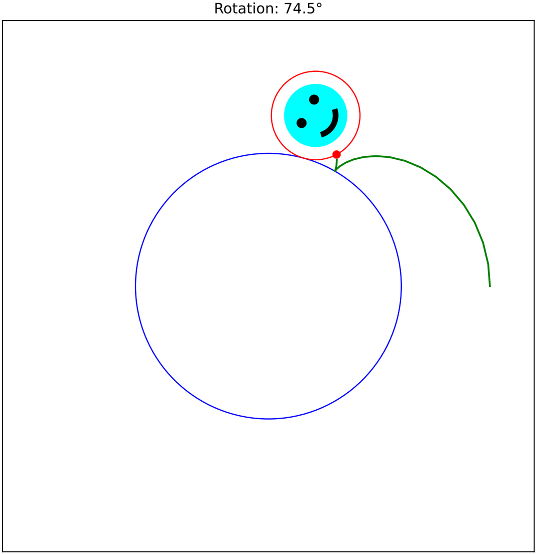
Animation of Coin Rotation Paradox



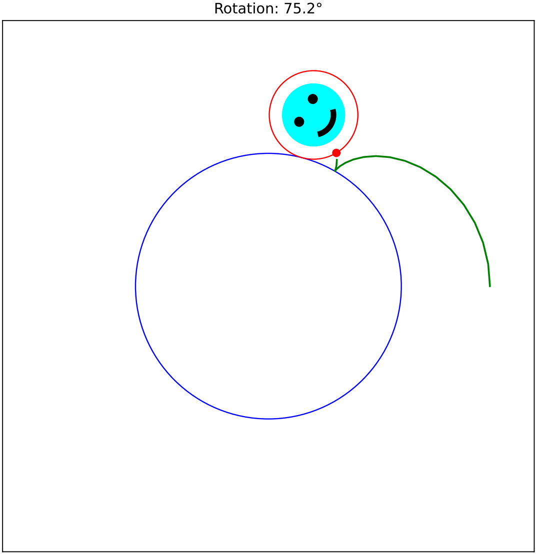
Animation of Coin Rotation Paradox



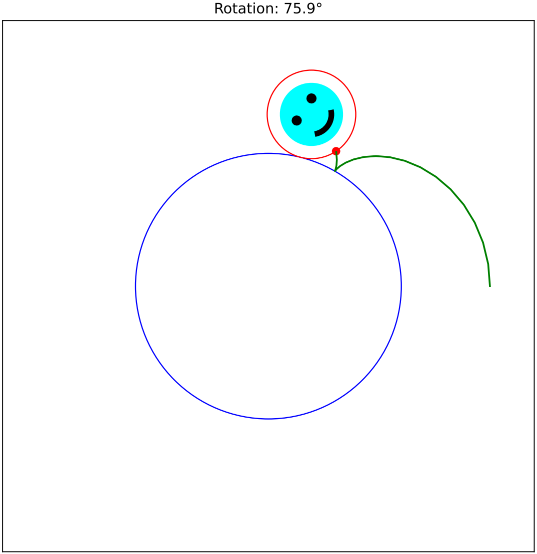
Animation of Coin Rotation Paradox



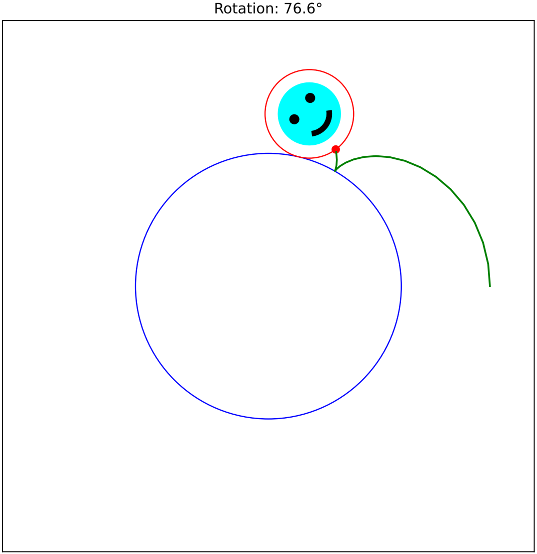
Animation of Coin Rotation Paradox



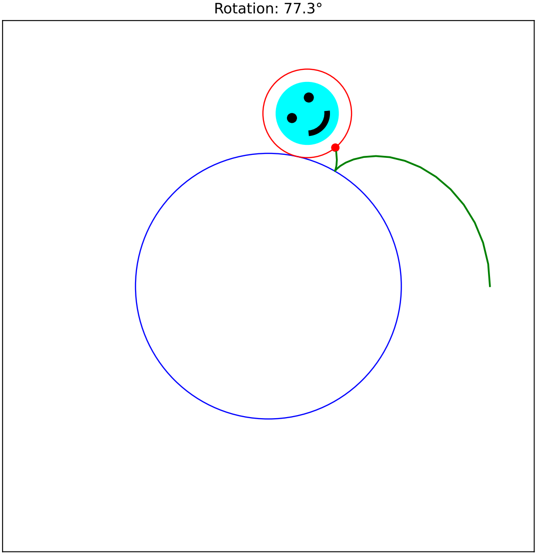
Animation of Coin Rotation Paradox



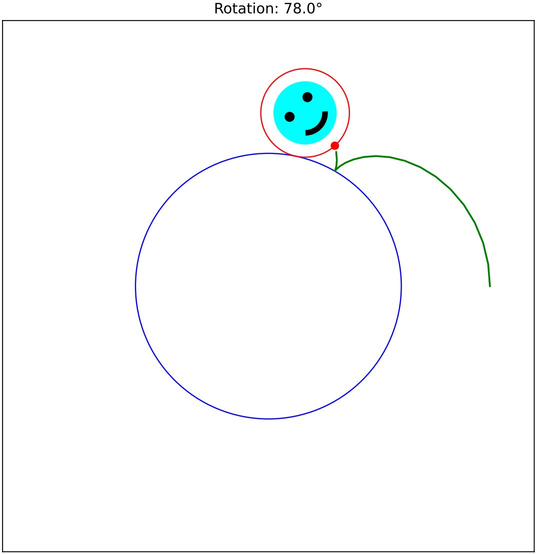
Animation of Coin Rotation Paradox



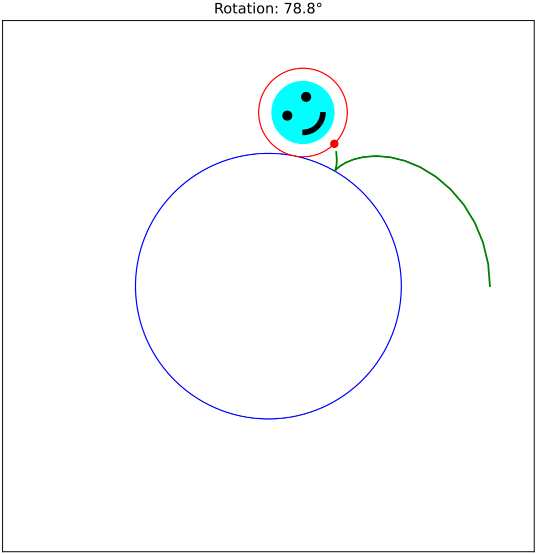
Animation of Coin Rotation Paradox



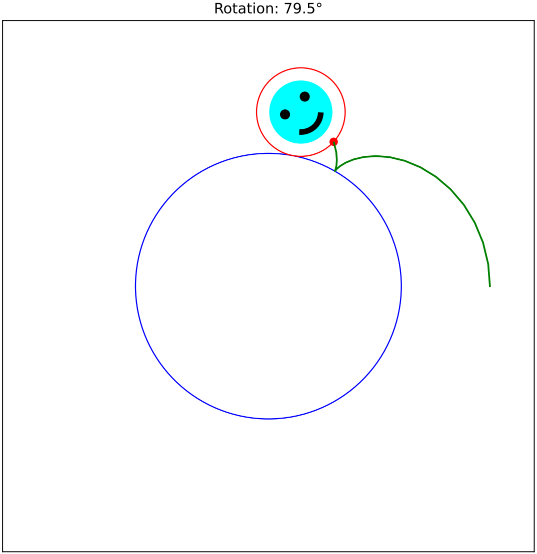
Animation of Coin Rotation Paradox



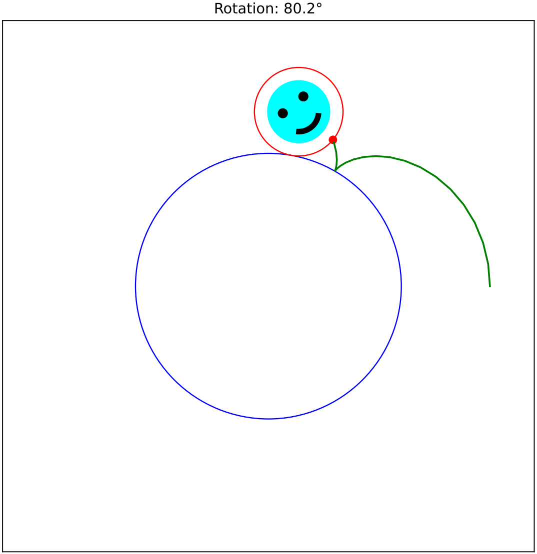
Animation of Coin Rotation Paradox



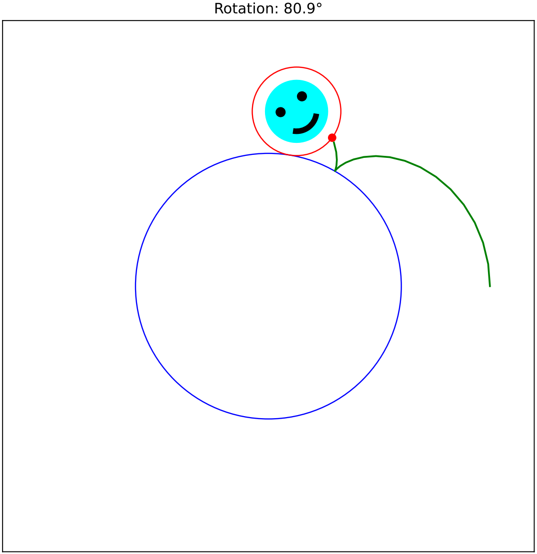
Animation of Coin Rotation Paradox



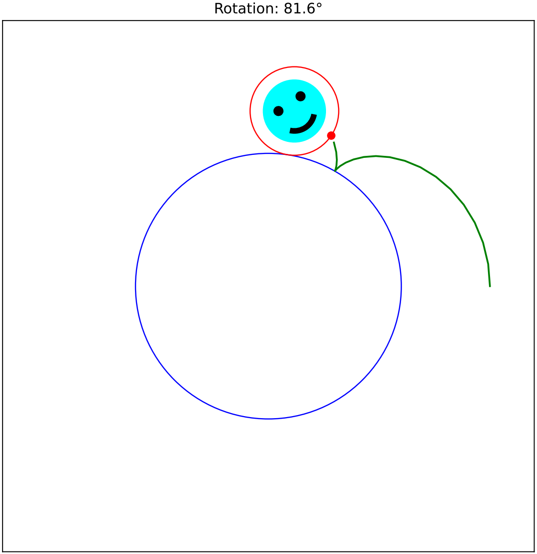
Animation of Coin Rotation Paradox



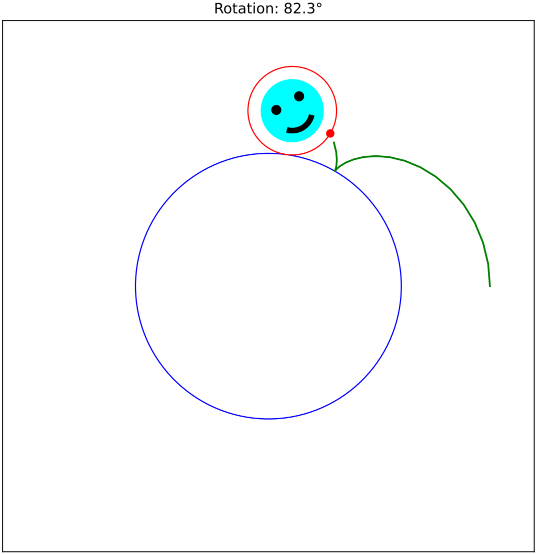
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

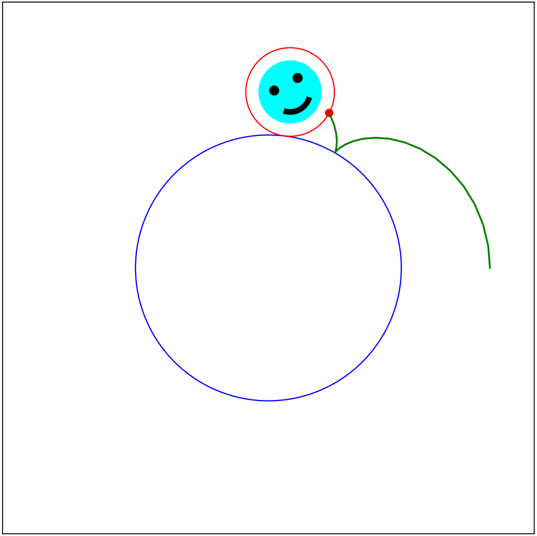


Animation of Coin Rotation Paradox

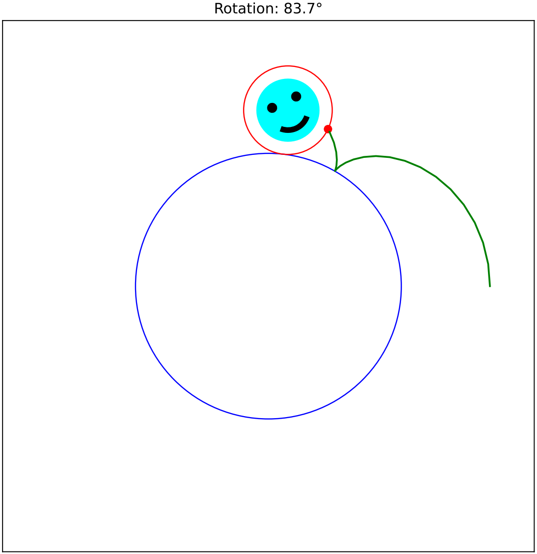


Animation of Coin Rotation Paradox

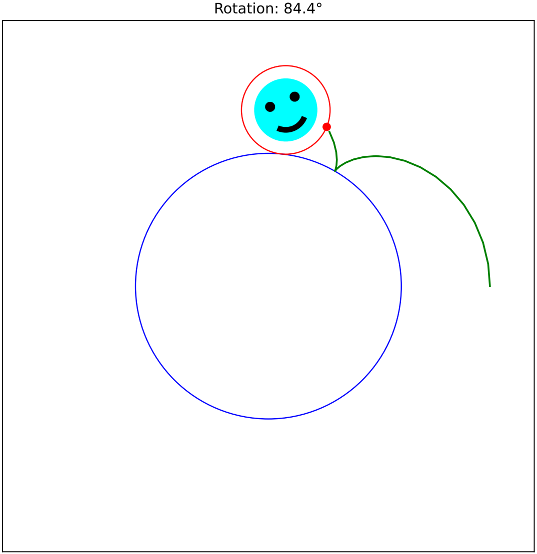
Rotation: 83.0°



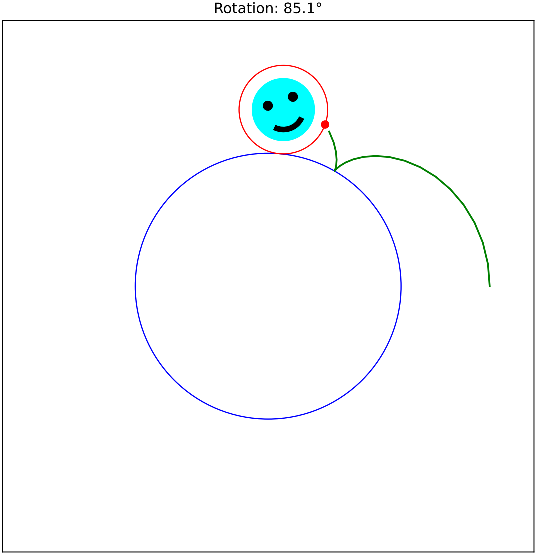
Animation of Coin Rotation Paradox



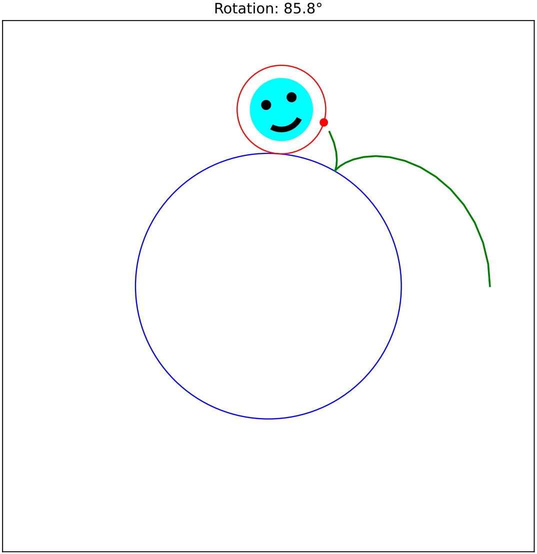
Animation of Coin Rotation Paradox



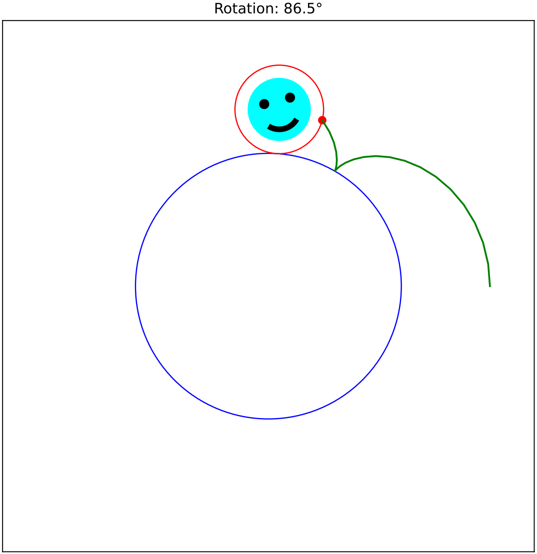
Animation of Coin Rotation Paradox



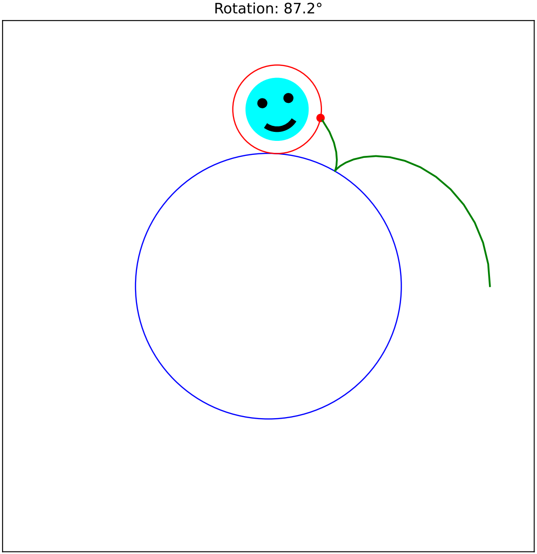
Animation of Coin Rotation Paradox



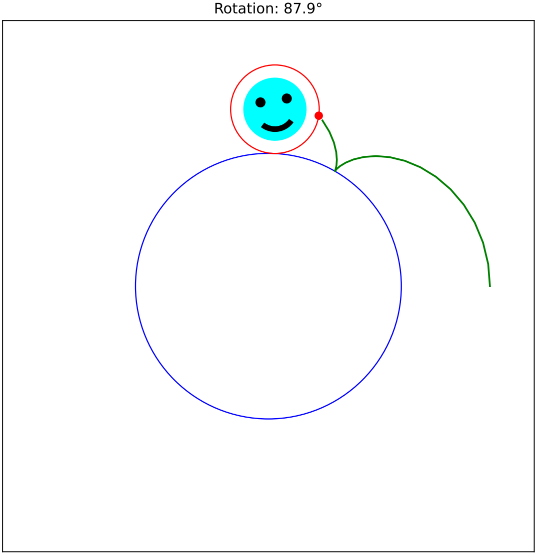
Animation of Coin Rotation Paradox



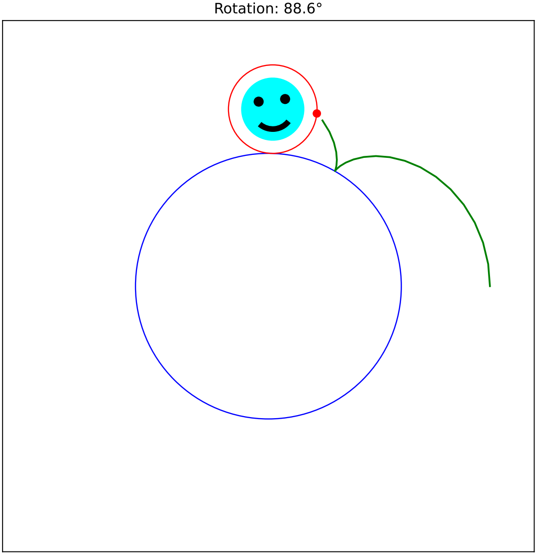
Animation of Coin Rotation Paradox



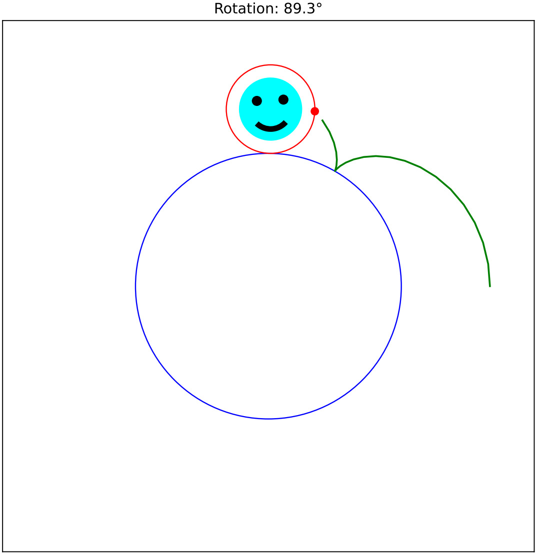
Animation of Coin Rotation Paradox



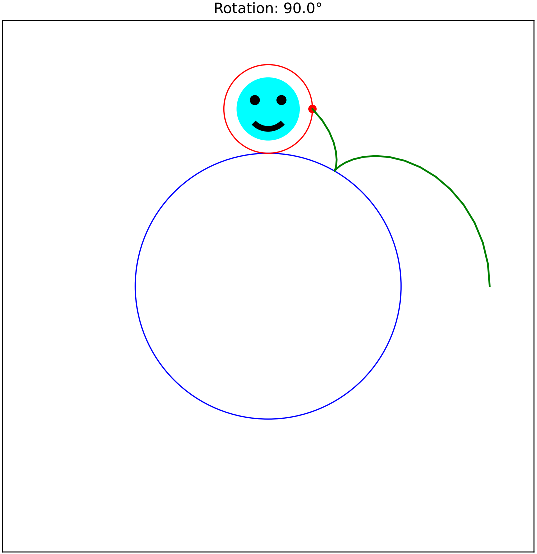
Animation of Coin Rotation Paradox



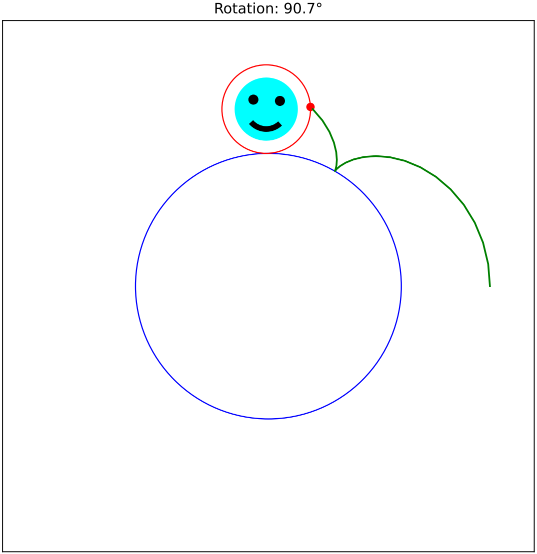
Animation of Coin Rotation Paradox



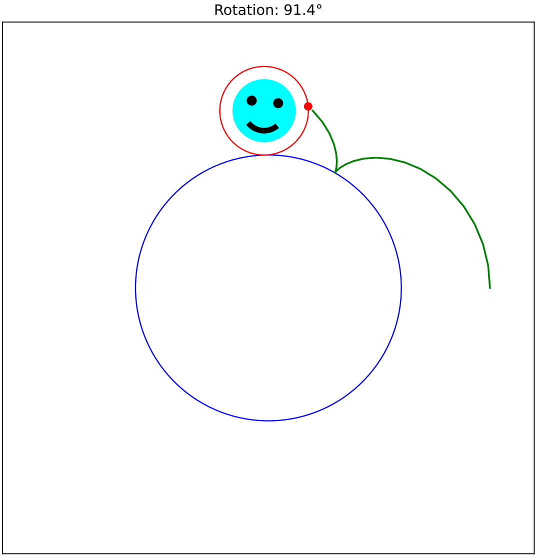
Animation of Coin Rotation Paradox



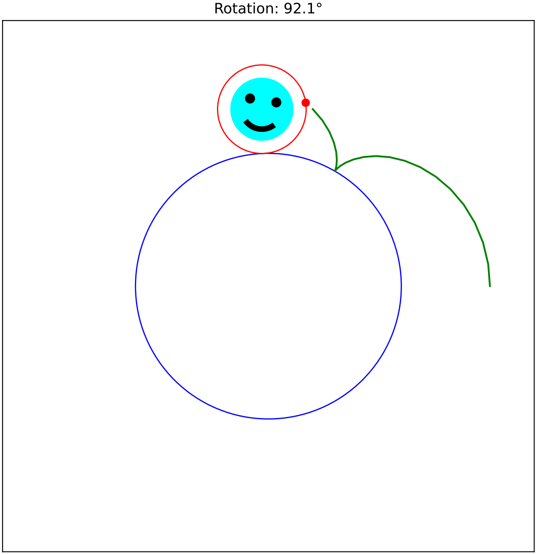
Animation of Coin Rotation Paradox



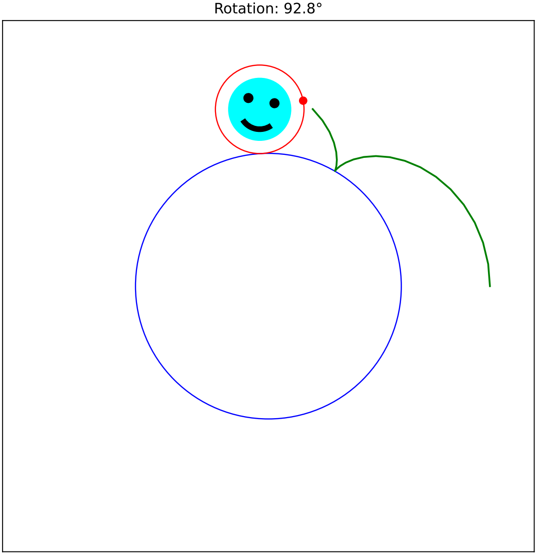
Animation of Coin Rotation Paradox



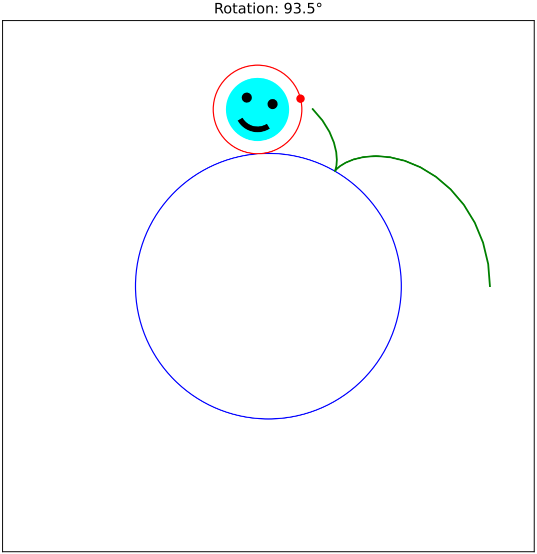
Animation of Coin Rotation Paradox



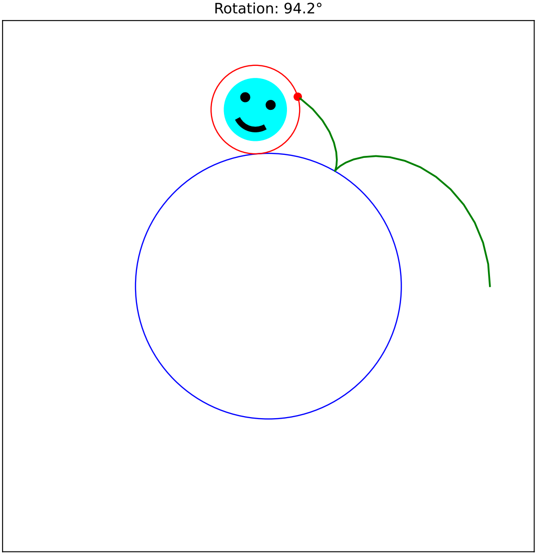
Animation of Coin Rotation Paradox



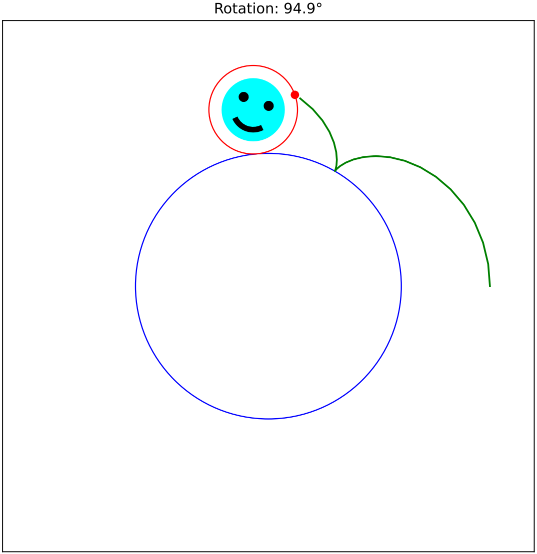
Animation of Coin Rotation Paradox



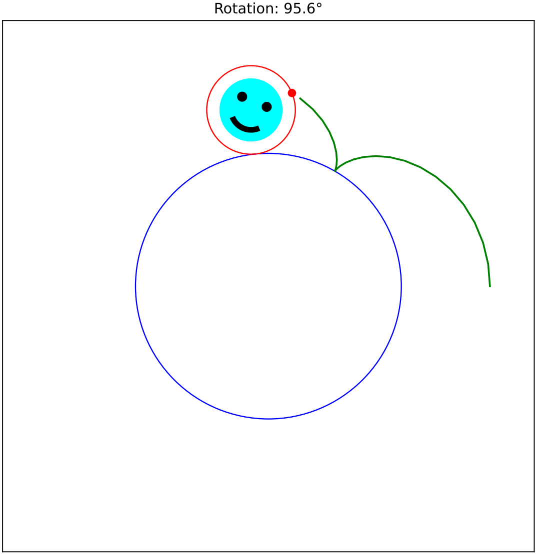
Animation of Coin Rotation Paradox



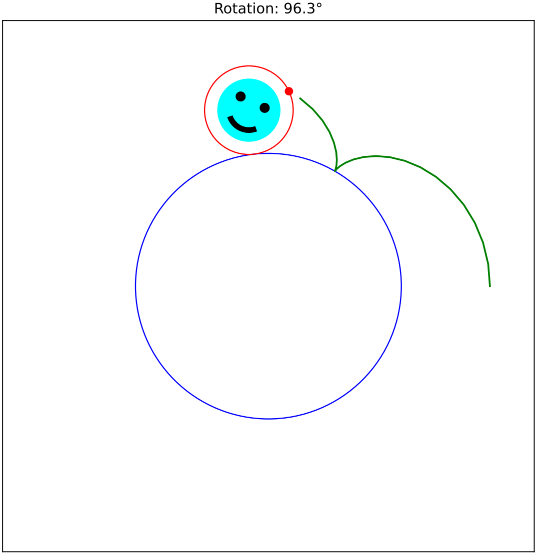
Animation of Coin Rotation Paradox



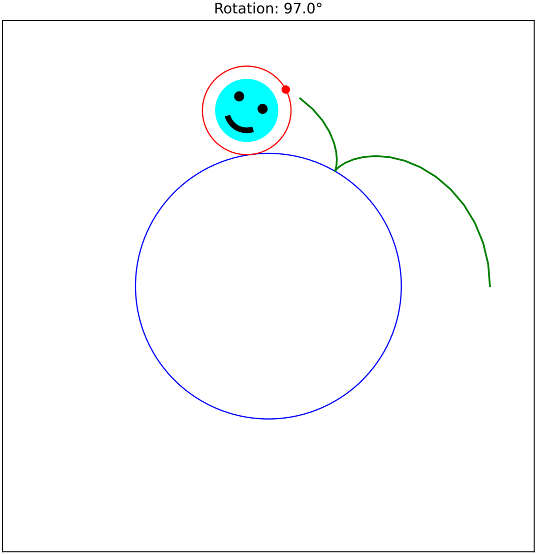
Animation of Coin Rotation Paradox



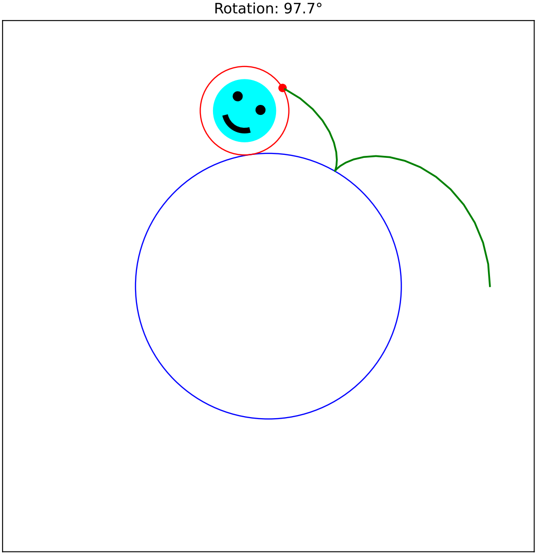
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

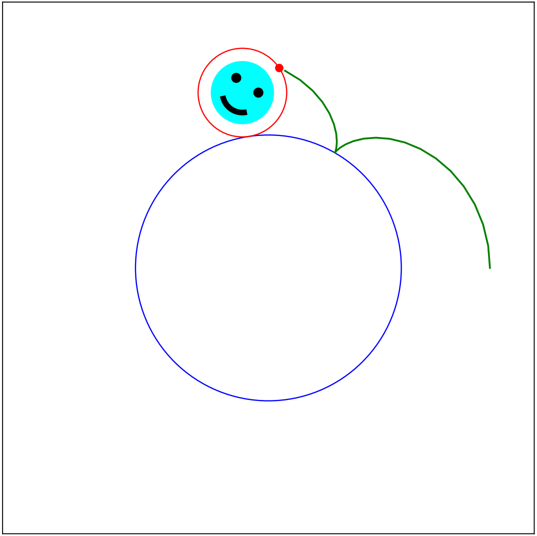


Animation of Coin Rotation Paradox

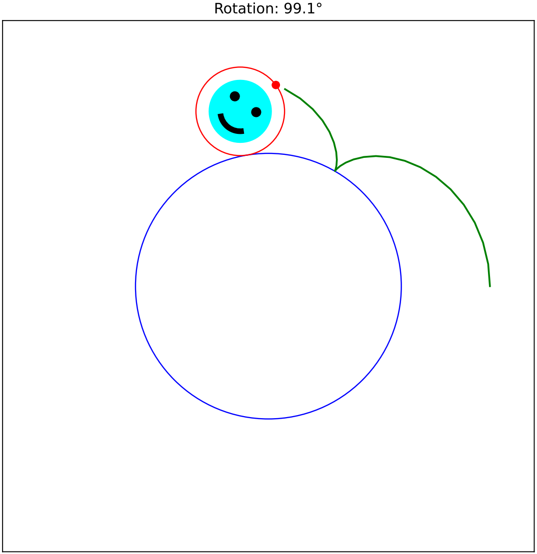


Animation of Coin Rotation Paradox

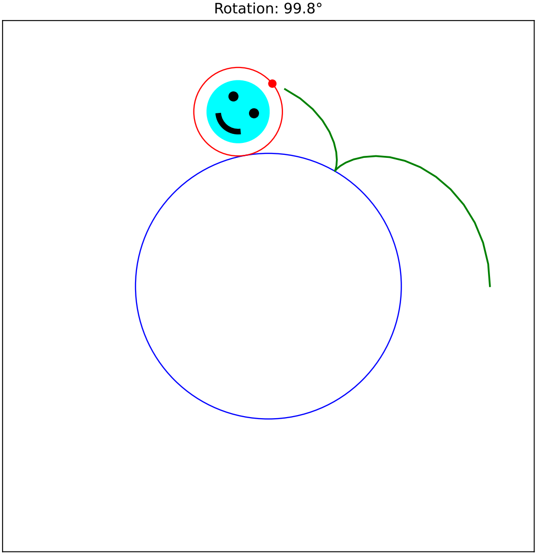
Rotation: 98.4°



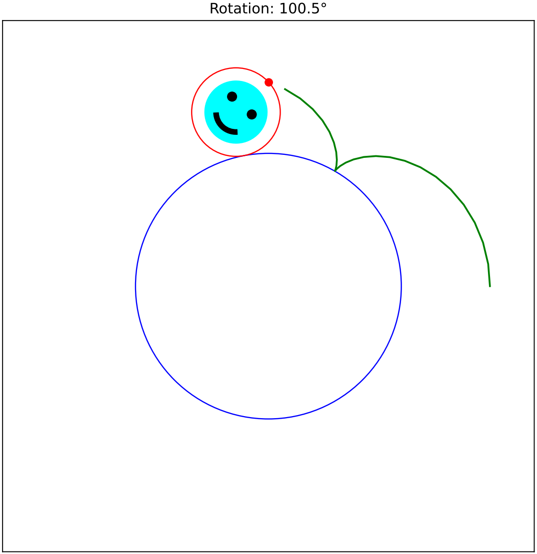
Animation of Coin Rotation Paradox



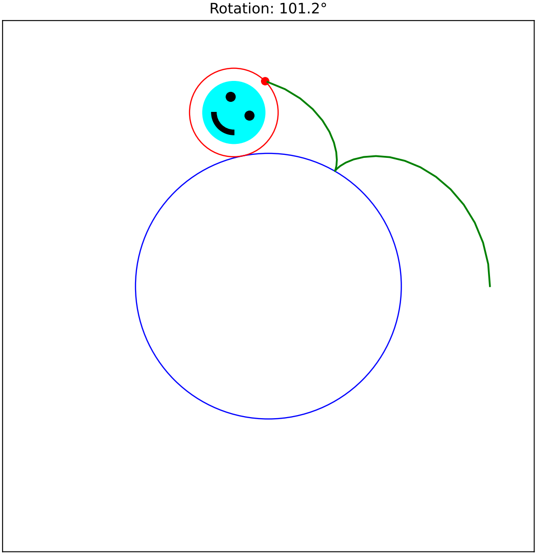
Animation of Coin Rotation Paradox



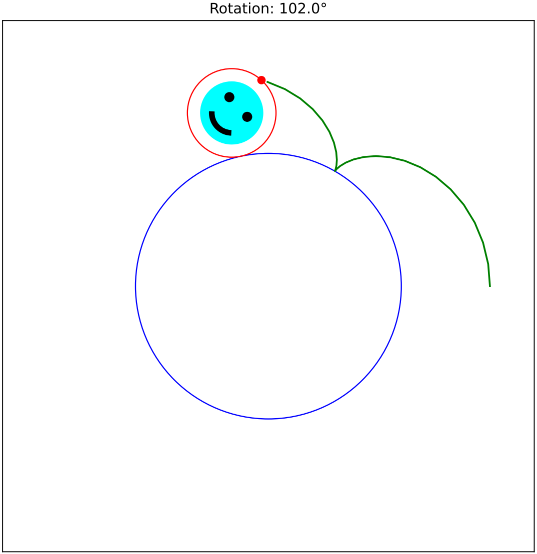
Animation of Coin Rotation Paradox



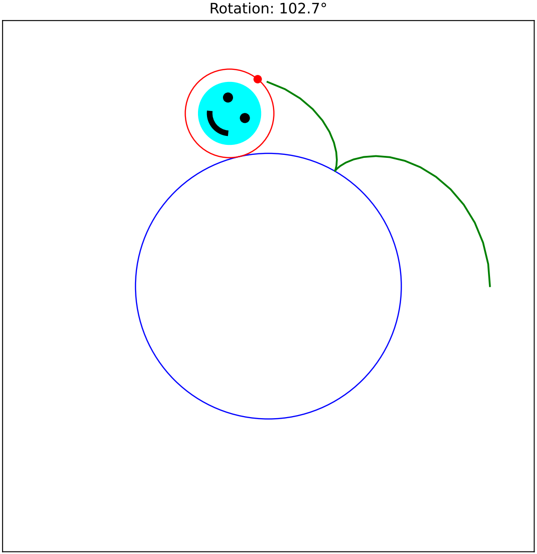
Animation of Coin Rotation Paradox



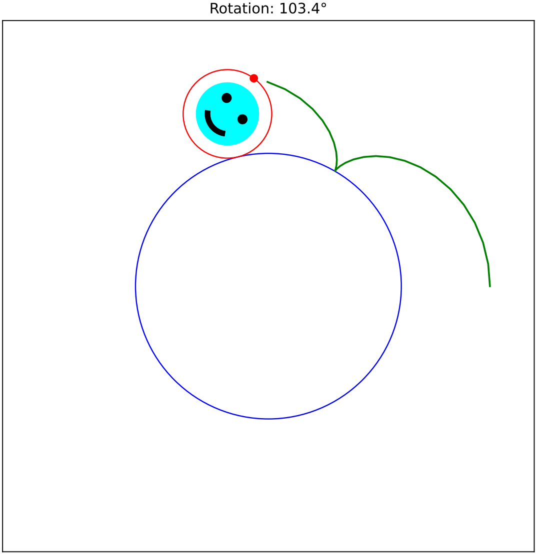
Animation of Coin Rotation Paradox



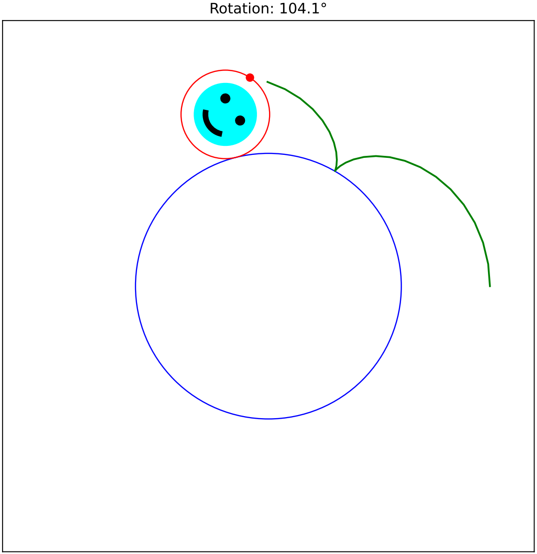
Animation of Coin Rotation Paradox



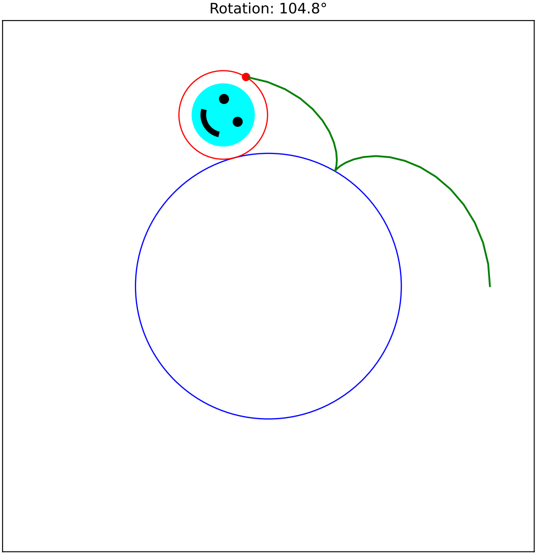
Animation of Coin Rotation Paradox



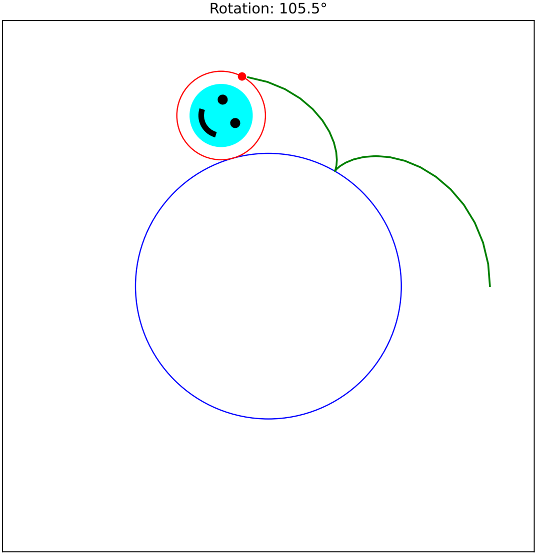
Animation of Coin Rotation Paradox



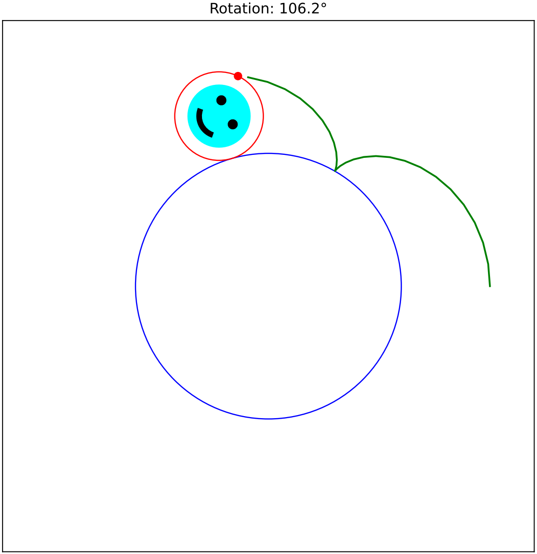
Animation of Coin Rotation Paradox



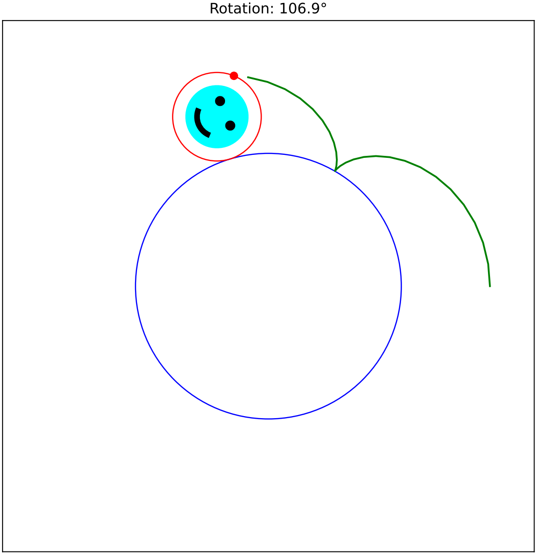
Animation of Coin Rotation Paradox



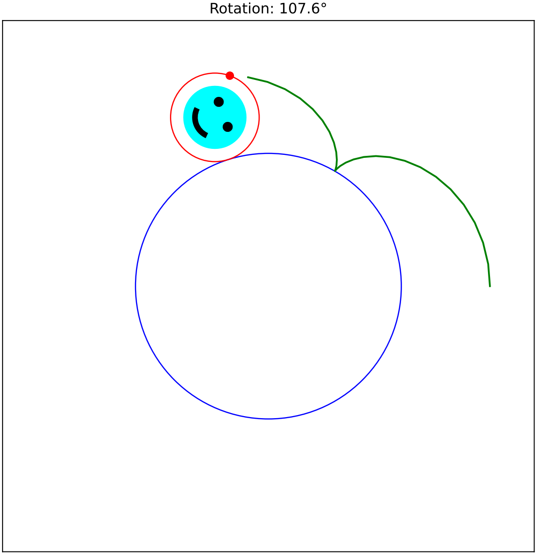
Animation of Coin Rotation Paradox



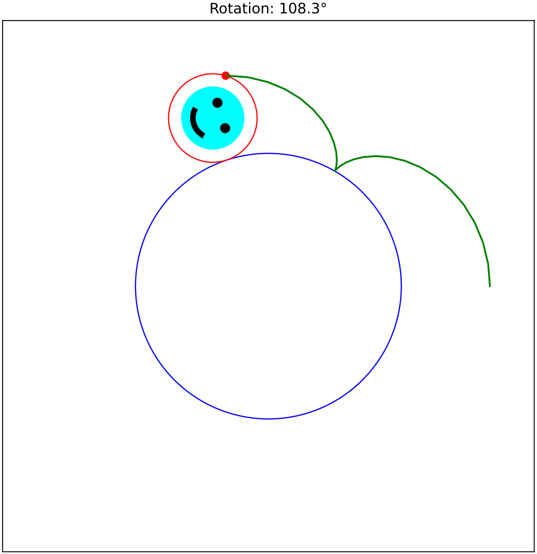
Animation of Coin Rotation Paradox



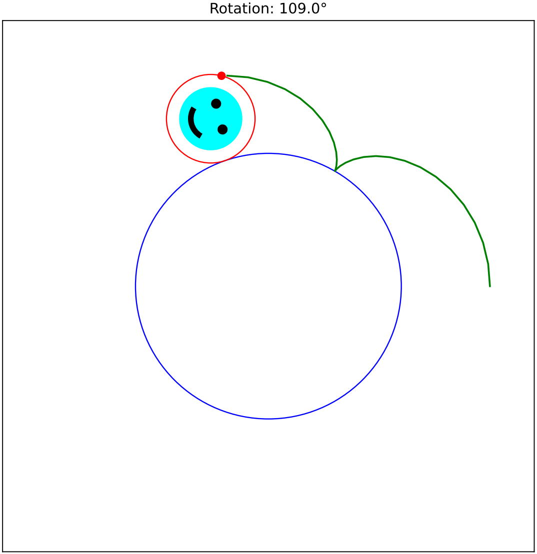
Animation of Coin Rotation Paradox



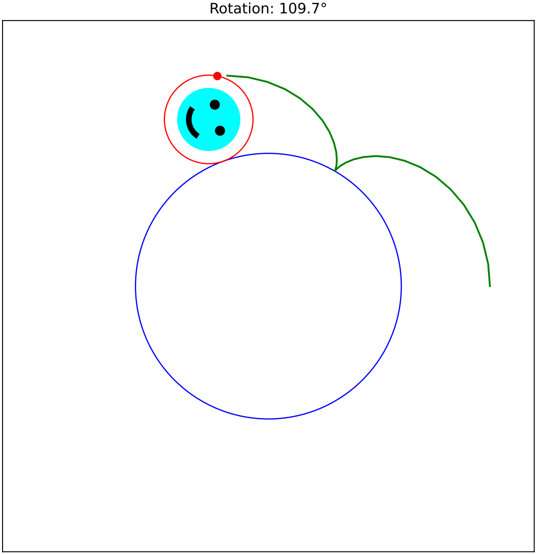
Animation of Coin Rotation Paradox



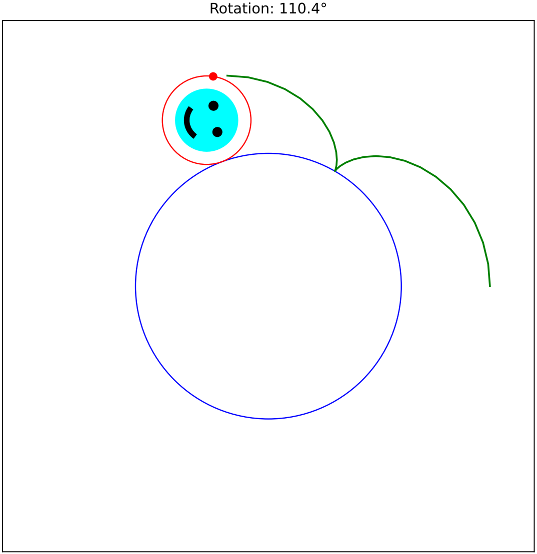
Animation of Coin Rotation Paradox



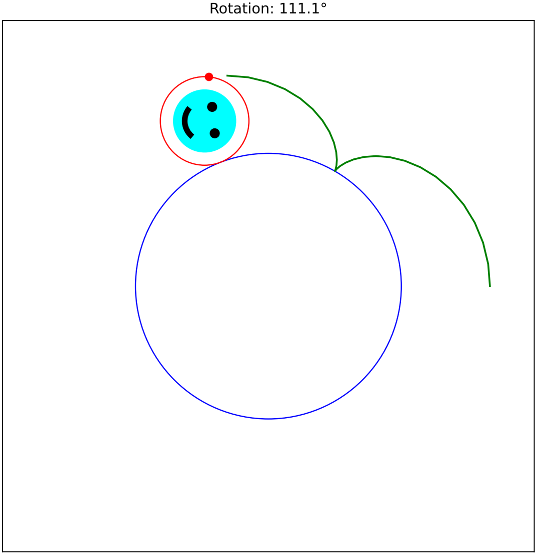
Animation of Coin Rotation Paradox



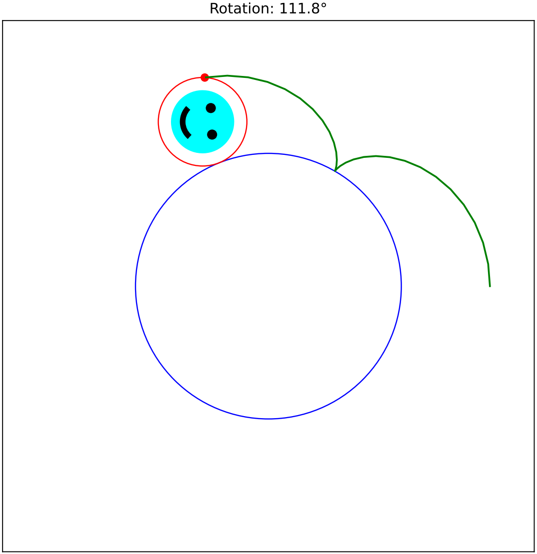
Animation of Coin Rotation Paradox



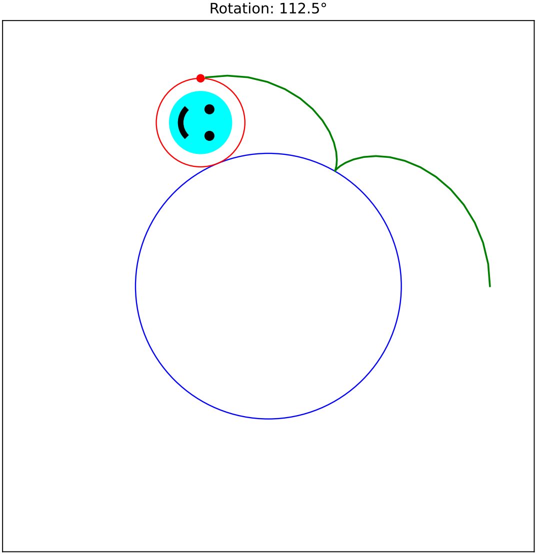
Animation of Coin Rotation Paradox



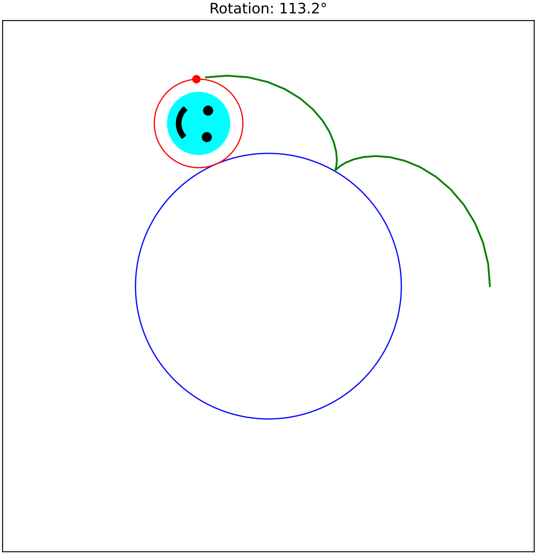
Animation of Coin Rotation Paradox



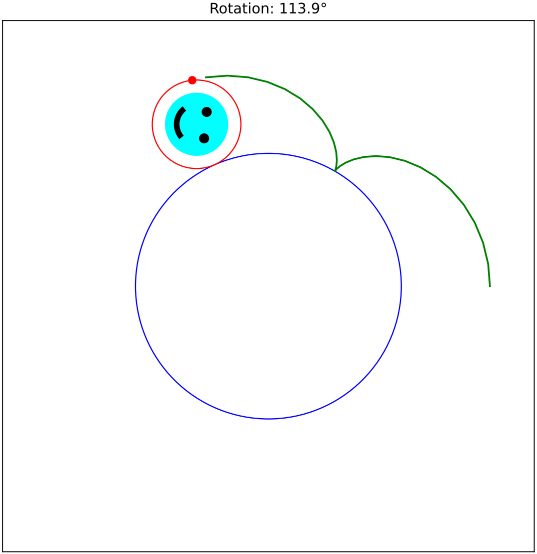
Animation of Coin Rotation Paradox



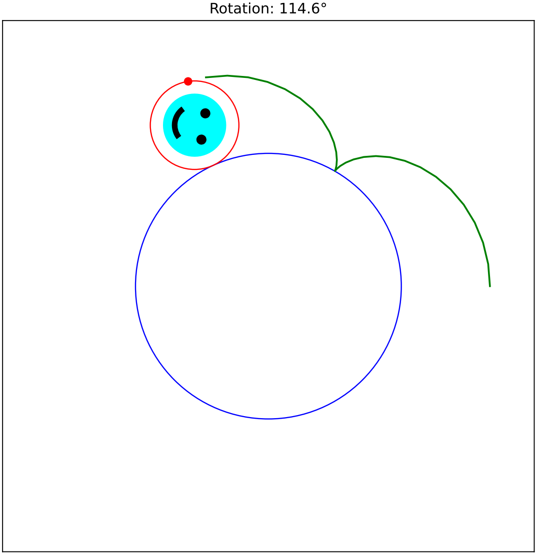
Animation of Coin Rotation Paradox



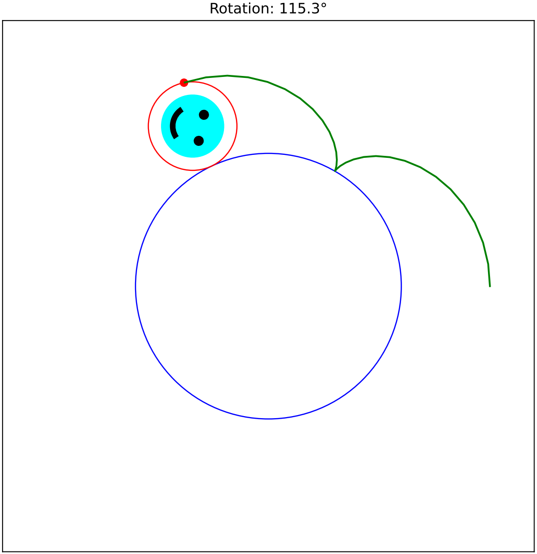
Animation of Coin Rotation Paradox



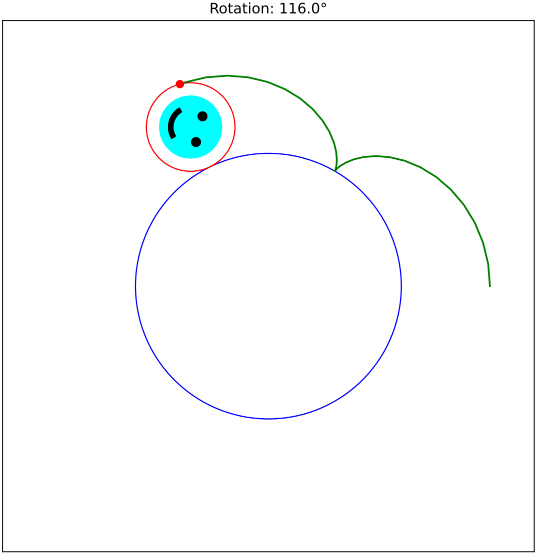
Animation of Coin Rotation Paradox



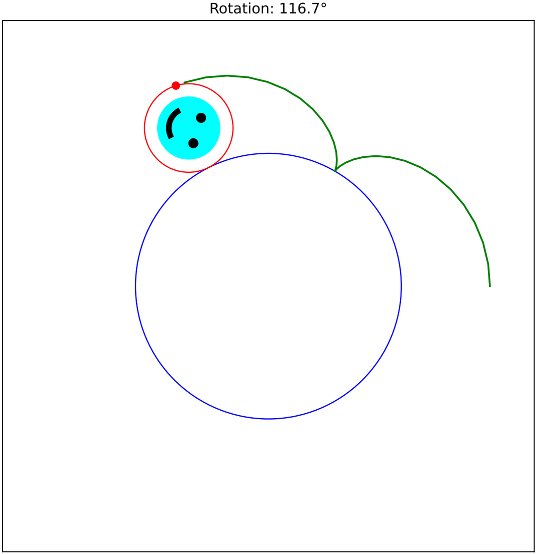
Animation of Coin Rotation Paradox



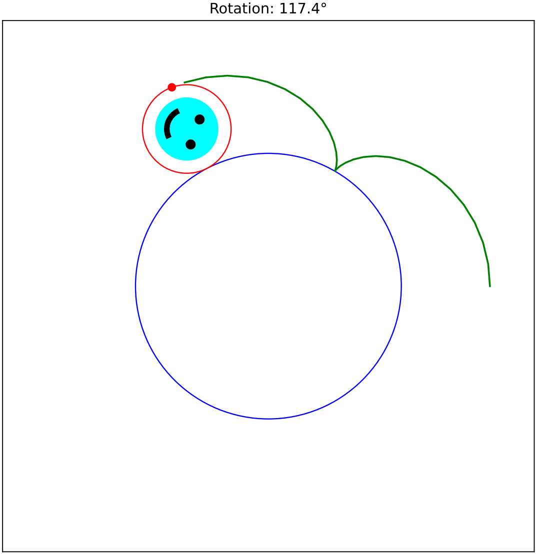
Animation of Coin Rotation Paradox



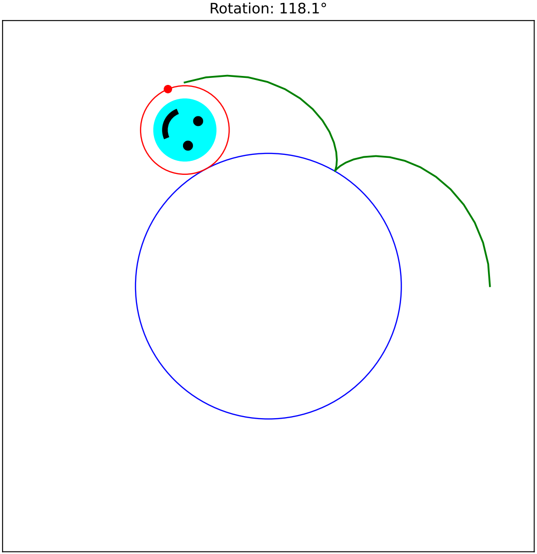
Animation of Coin Rotation Paradox



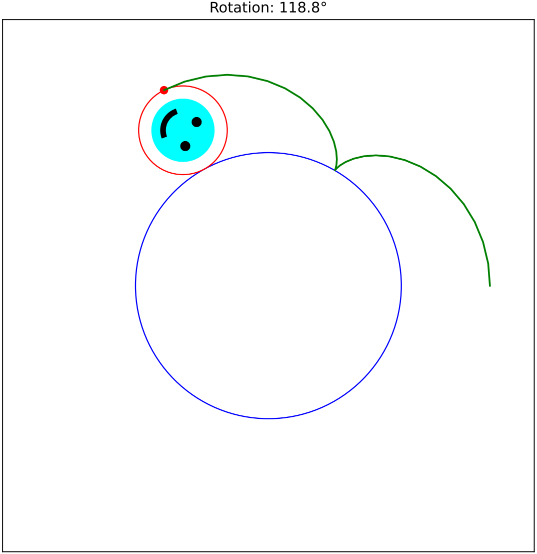
Animation of Coin Rotation Paradox



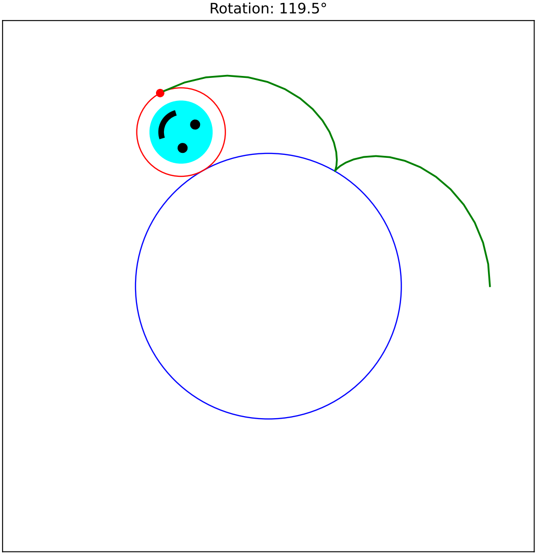
Animation of Coin Rotation Paradox



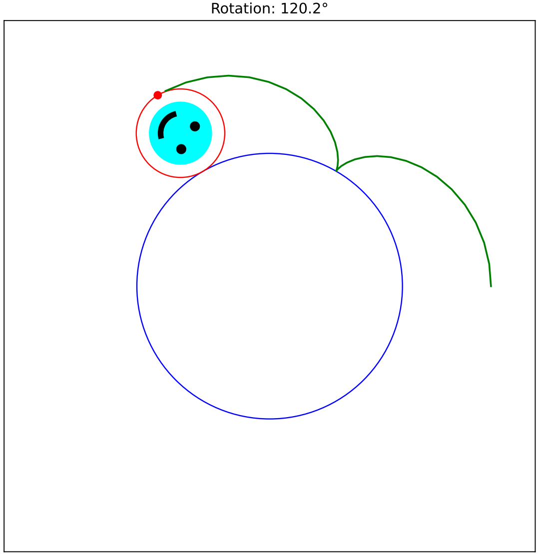
Animation of Coin Rotation Paradox



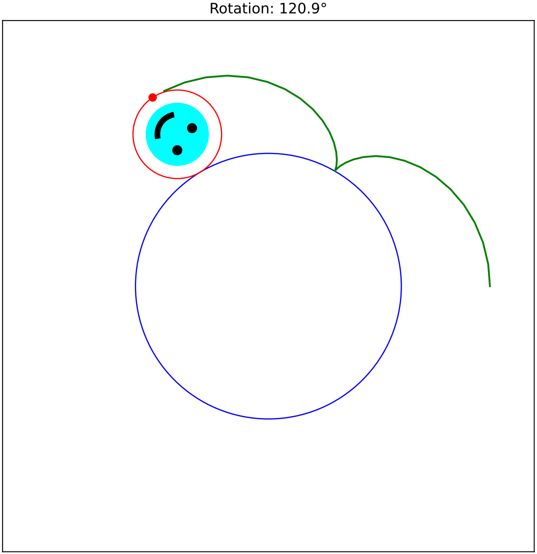
Animation of Coin Rotation Paradox



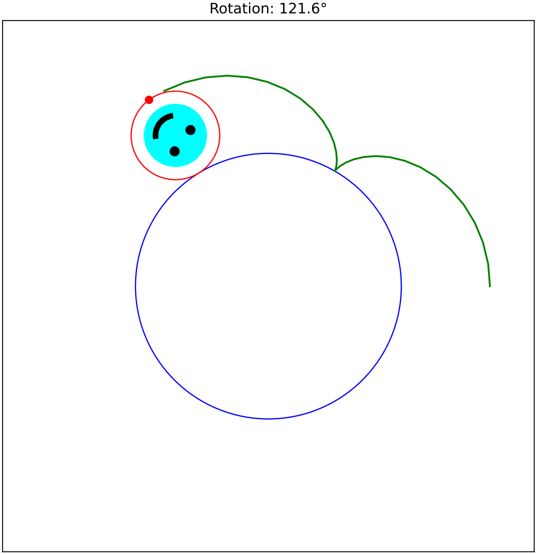
Animation of Coin Rotation Paradox



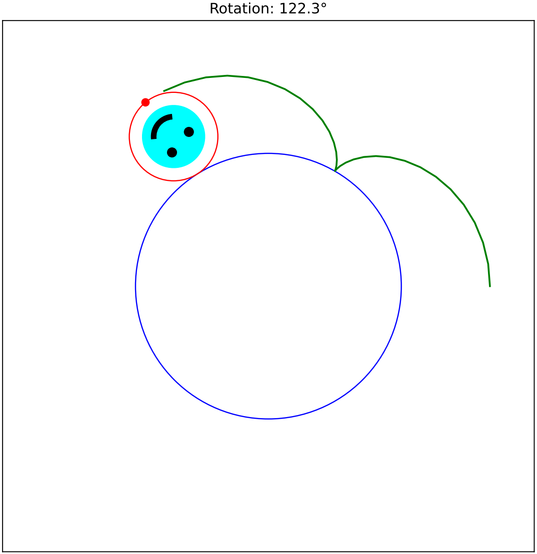
Animation of Coin Rotation Paradox



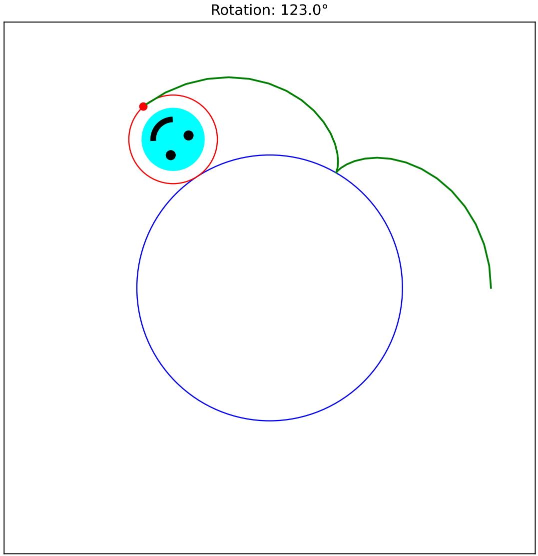
Animation of Coin Rotation Paradox



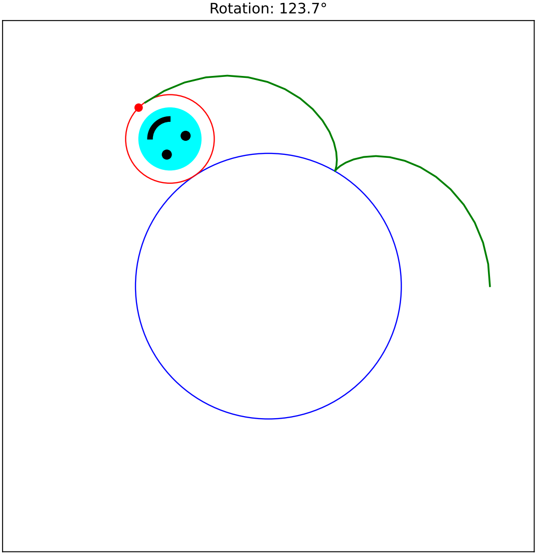
Animation of Coin Rotation Paradox



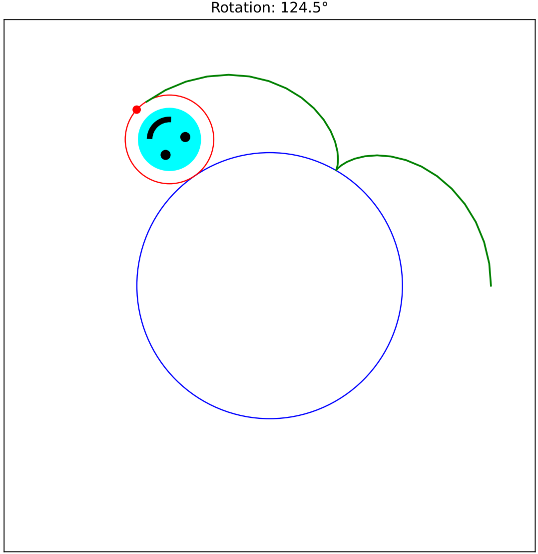
Animation of Coin Rotation Paradox



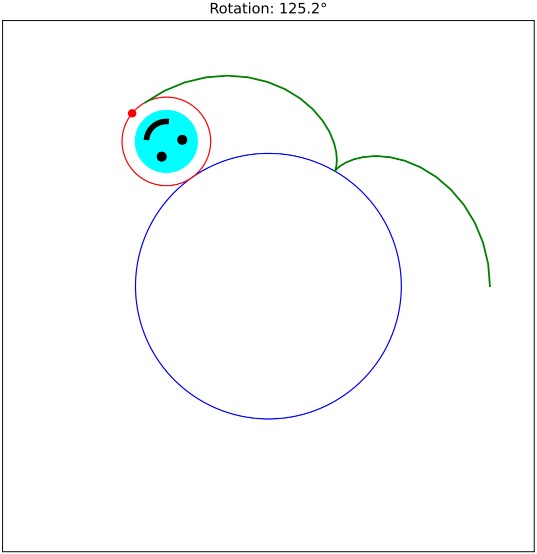
Animation of Coin Rotation Paradox



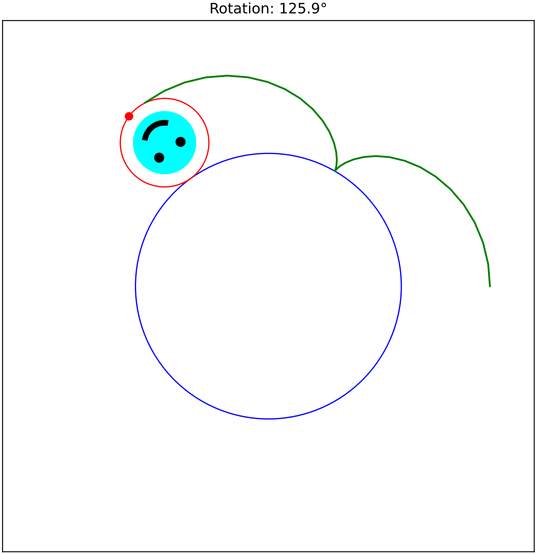
Animation of Coin Rotation Paradox



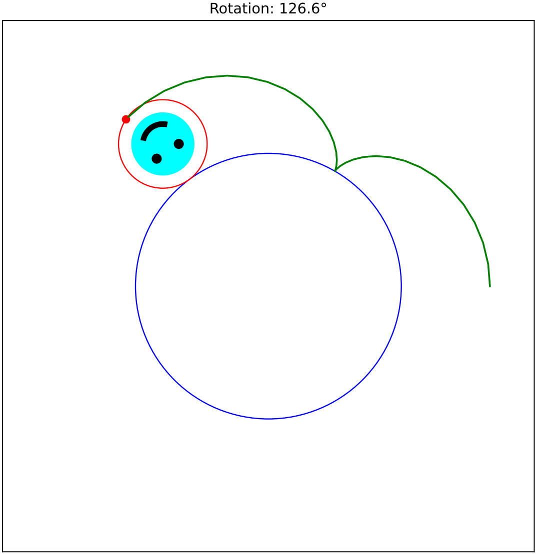
Animation of Coin Rotation Paradox



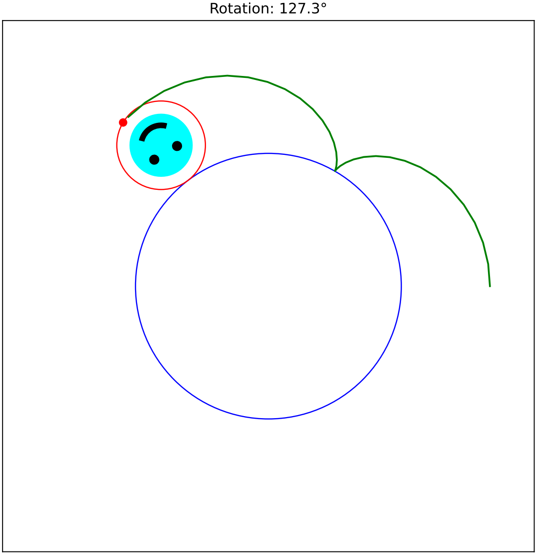
Animation of Coin Rotation Paradox



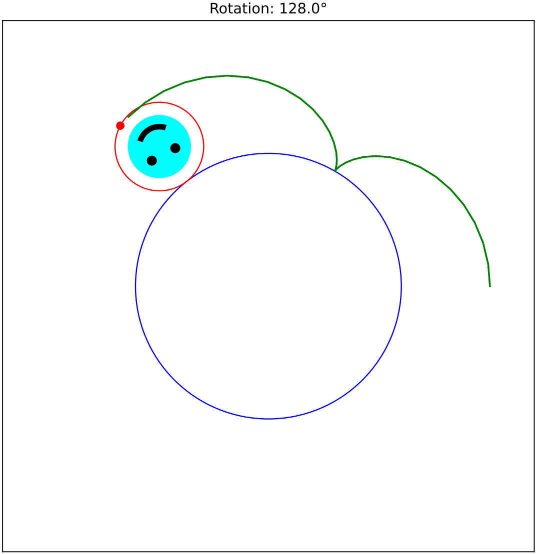
Animation of Coin Rotation Paradox



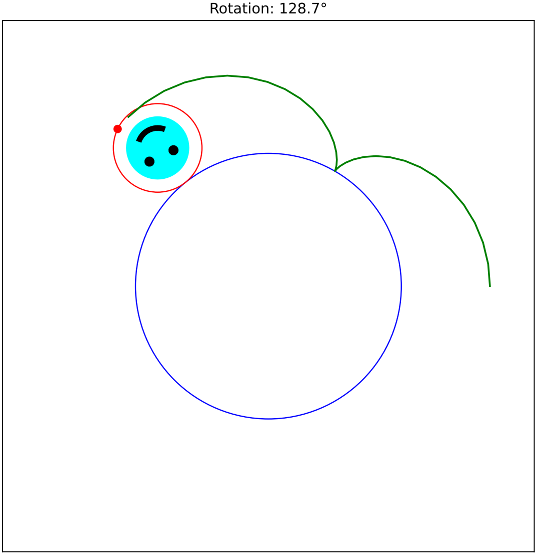
Animation of Coin Rotation Paradox



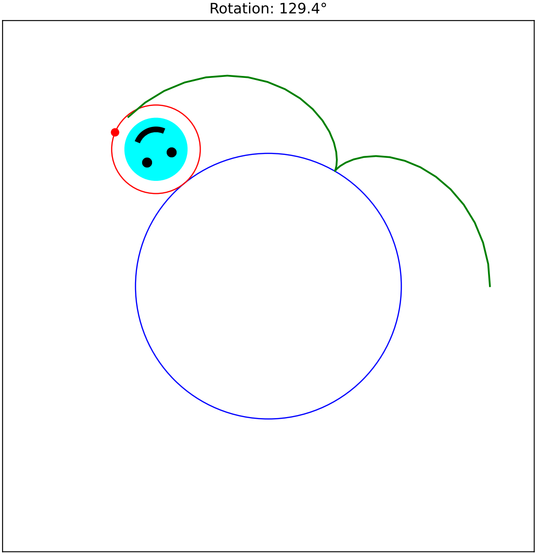
Animation of Coin Rotation Paradox



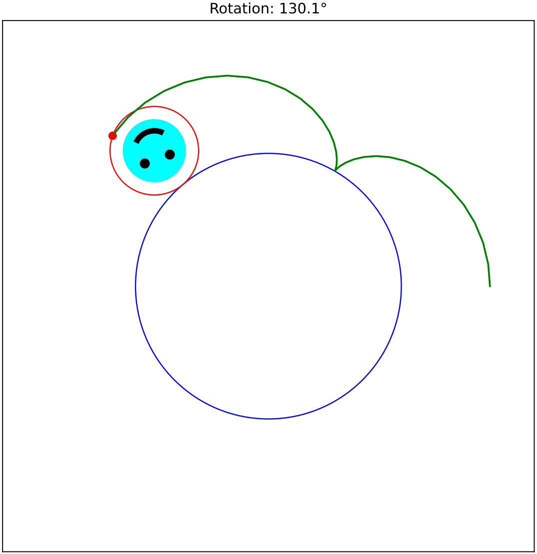
Animation of Coin Rotation Paradox



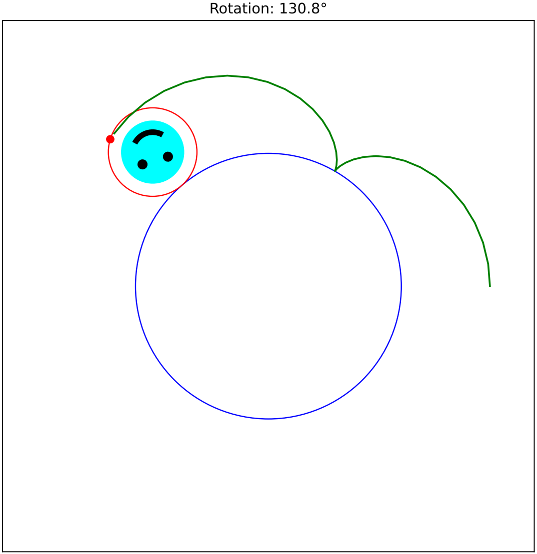
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

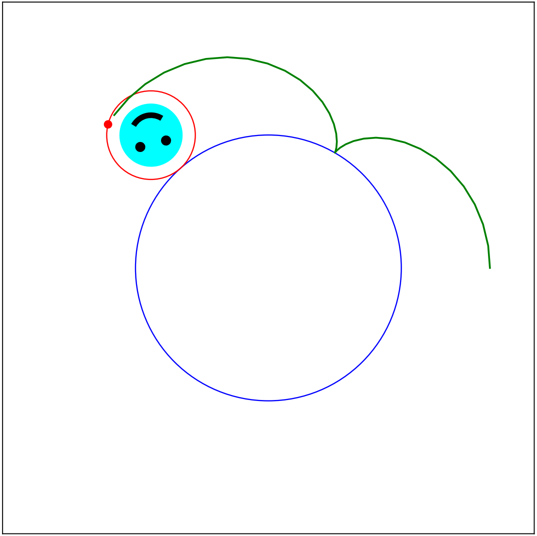


Animation of Coin Rotation Paradox

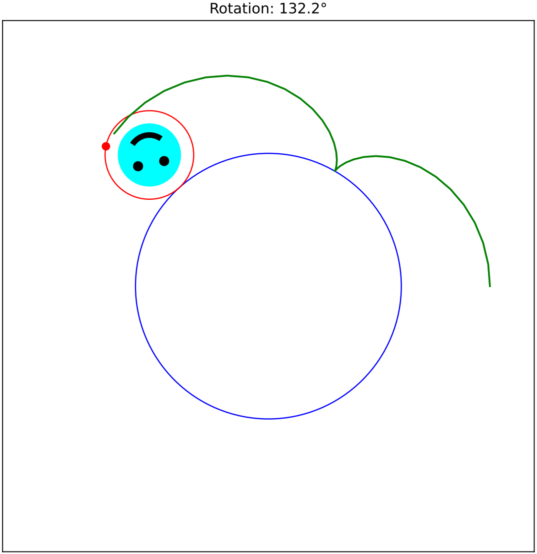


Animation of Coin Rotation Paradox

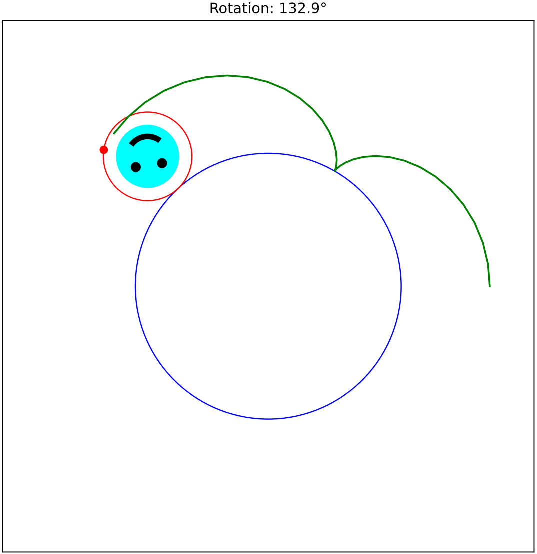
Rotation: 131.5°



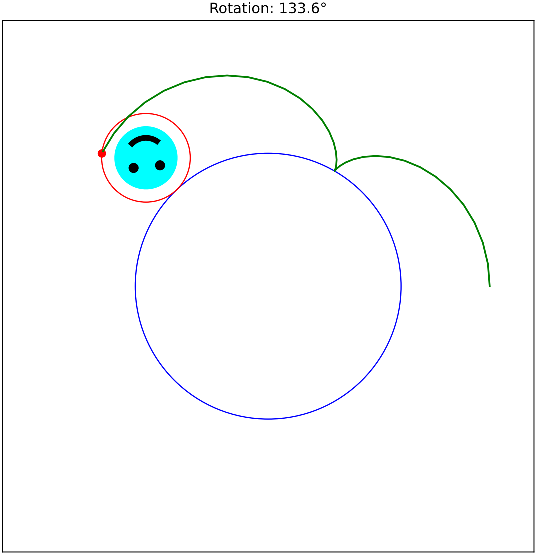
Animation of Coin Rotation Paradox



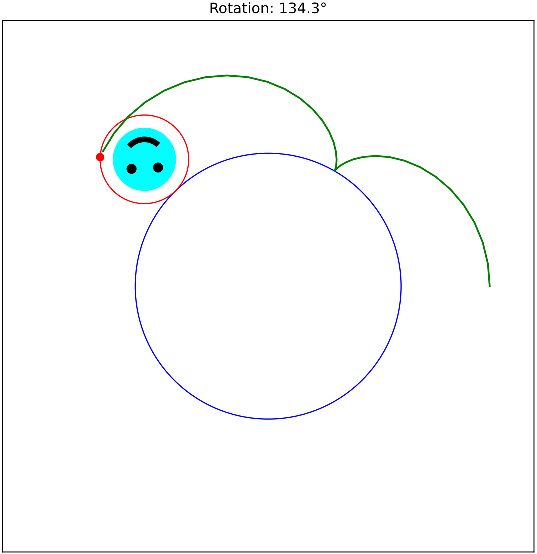
Animation of Coin Rotation Paradox



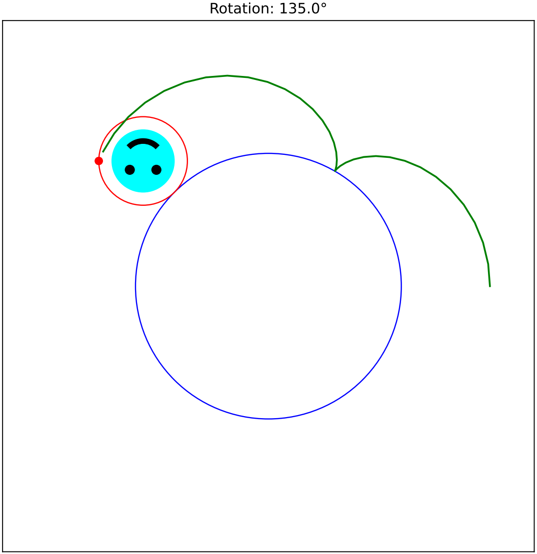
Animation of Coin Rotation Paradox



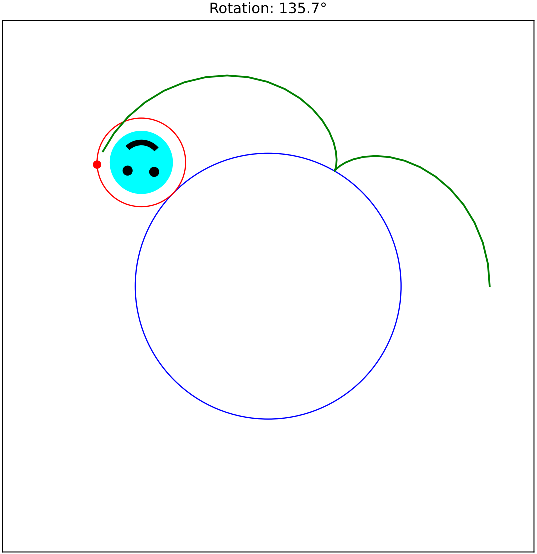
Animation of Coin Rotation Paradox



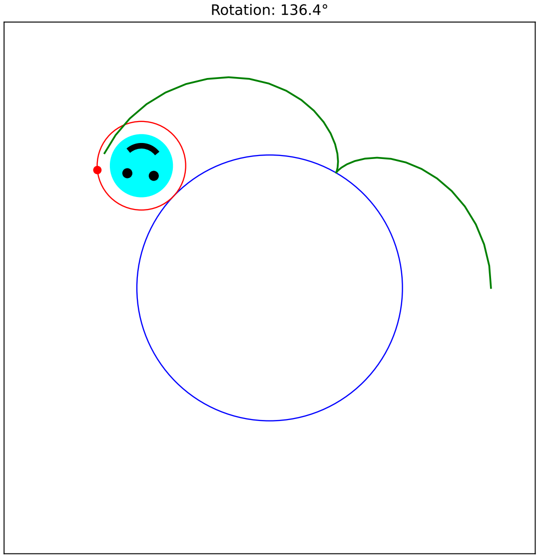
Animation of Coin Rotation Paradox



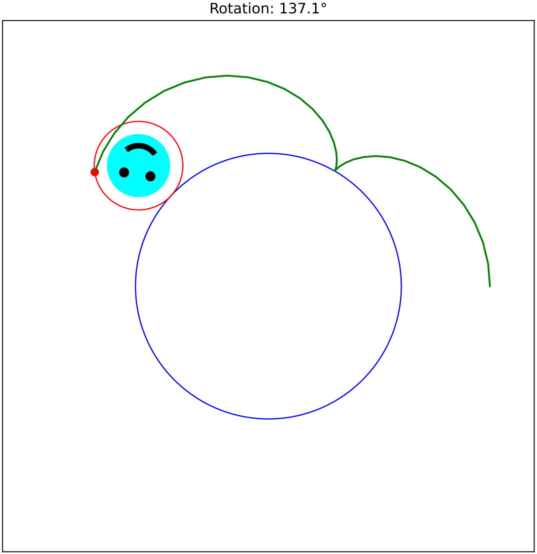
Animation of Coin Rotation Paradox



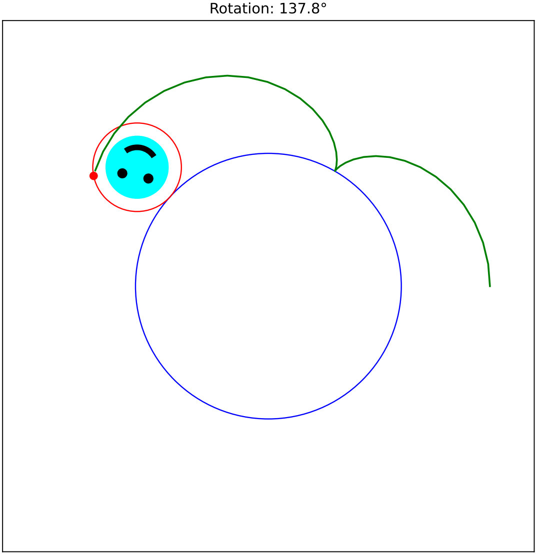
Animation of Coin Rotation Paradox



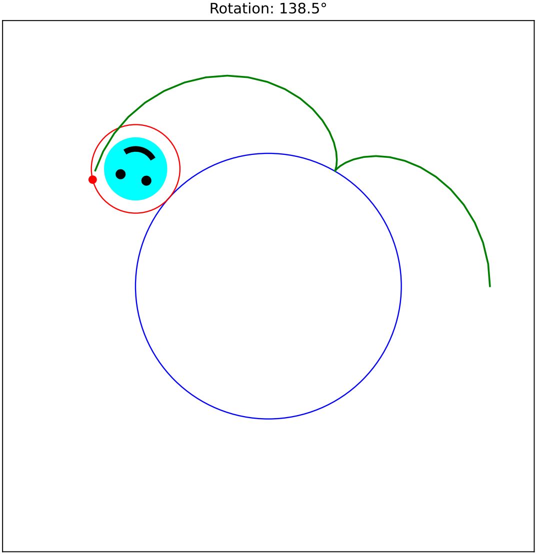
Animation of Coin Rotation Paradox



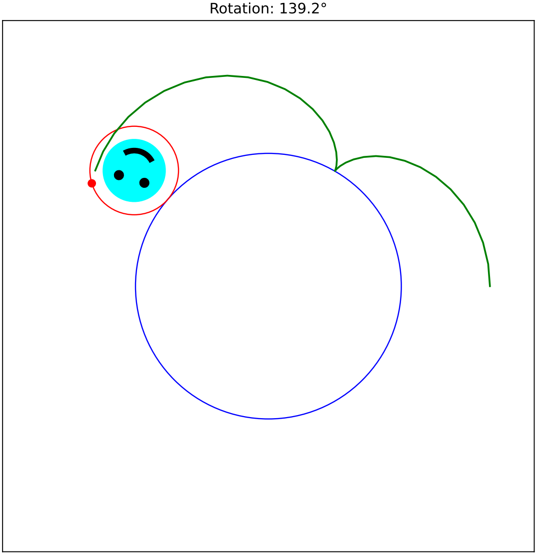
Animation of Coin Rotation Paradox



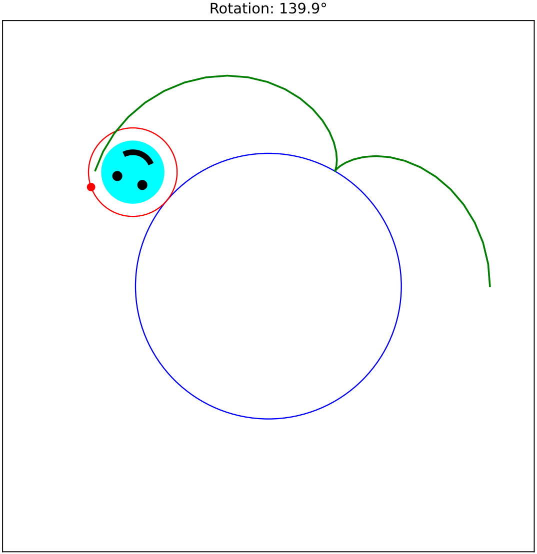
Animation of Coin Rotation Paradox



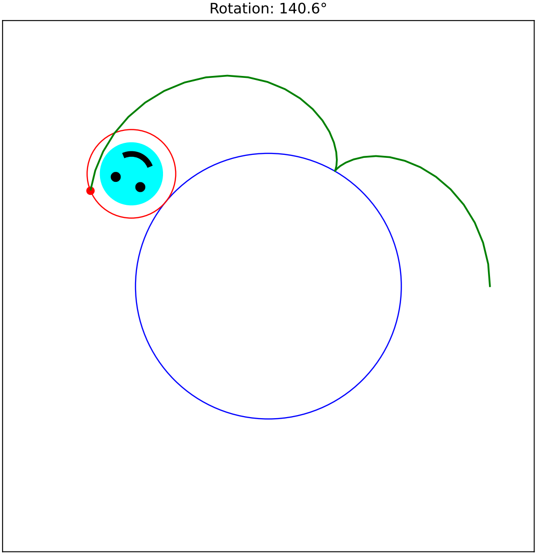
Animation of Coin Rotation Paradox



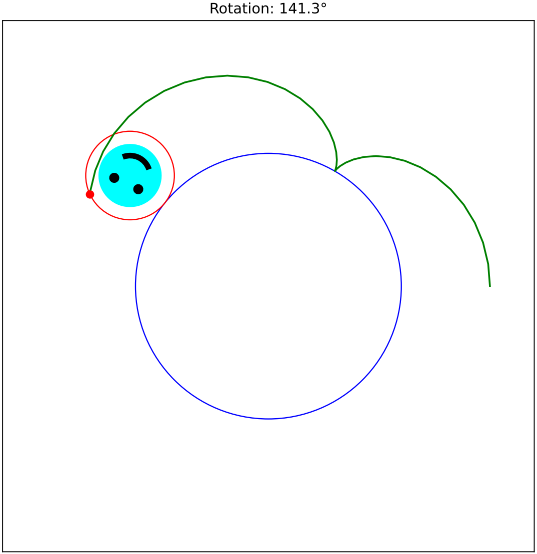
Animation of Coin Rotation Paradox



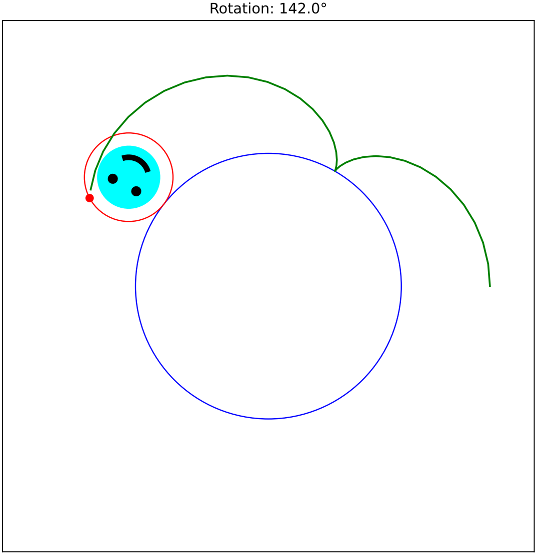
Animation of Coin Rotation Paradox



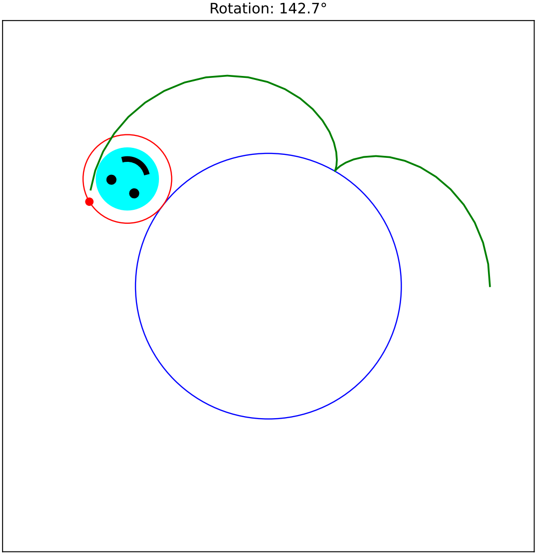
Animation of Coin Rotation Paradox



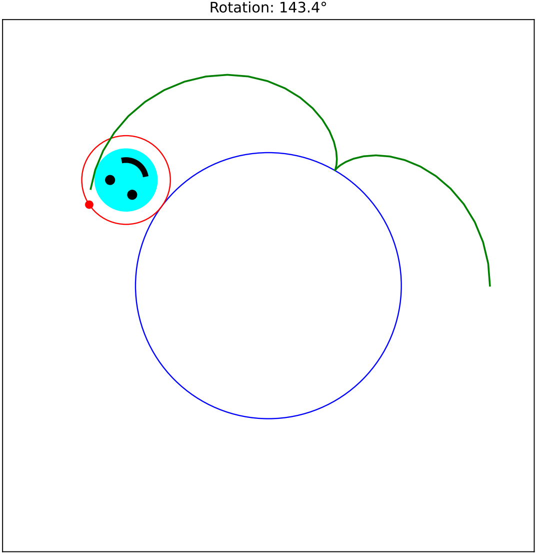
Animation of Coin Rotation Paradox



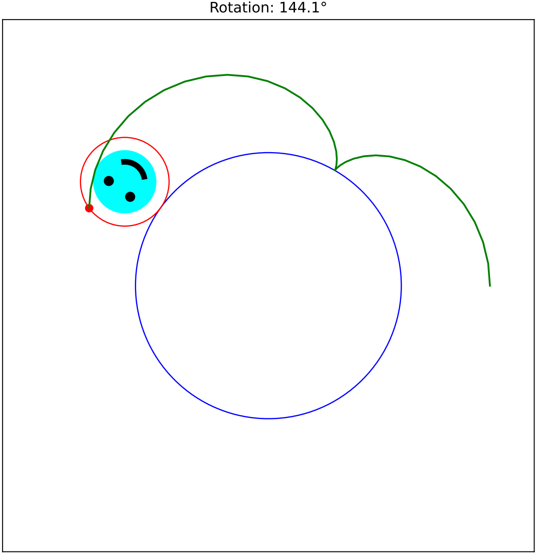
Animation of Coin Rotation Paradox



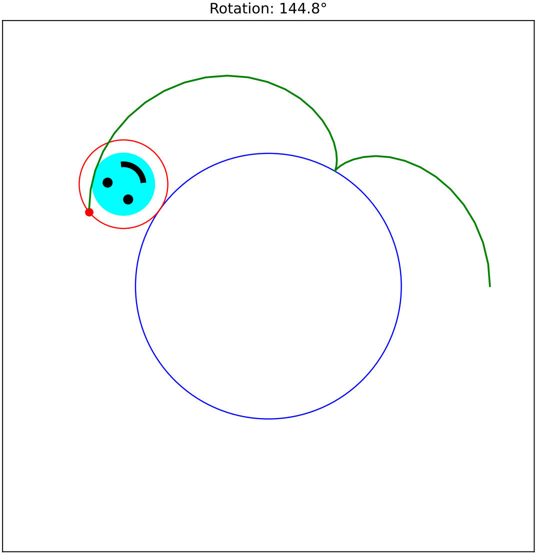
Animation of Coin Rotation Paradox



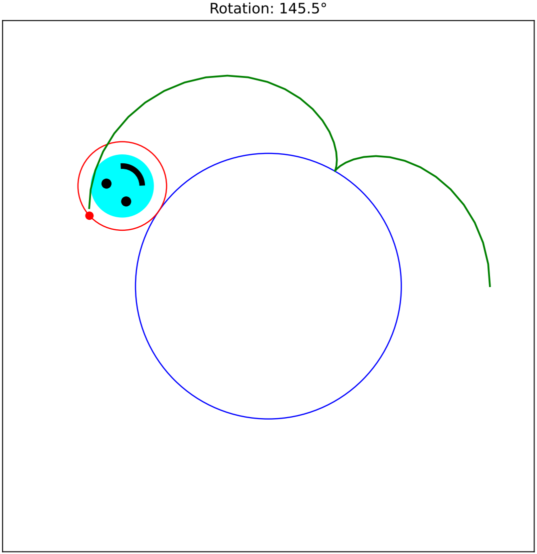
Animation of Coin Rotation Paradox



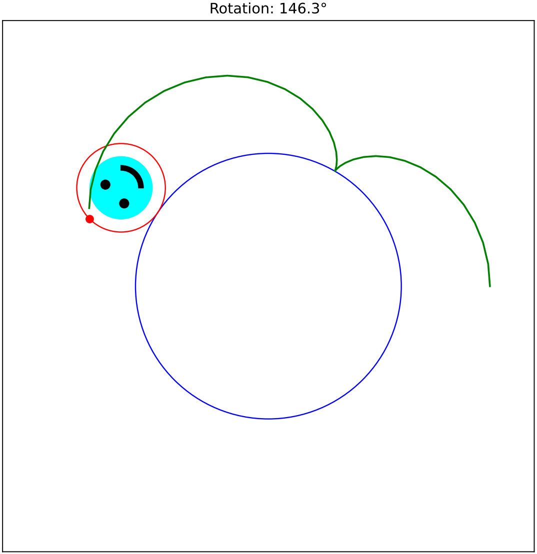
Animation of Coin Rotation Paradox



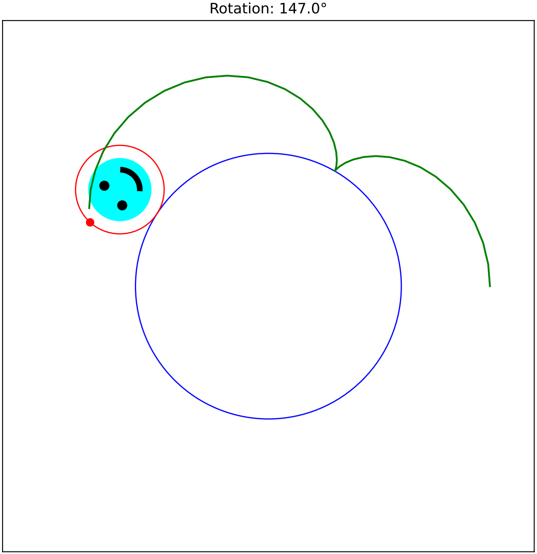
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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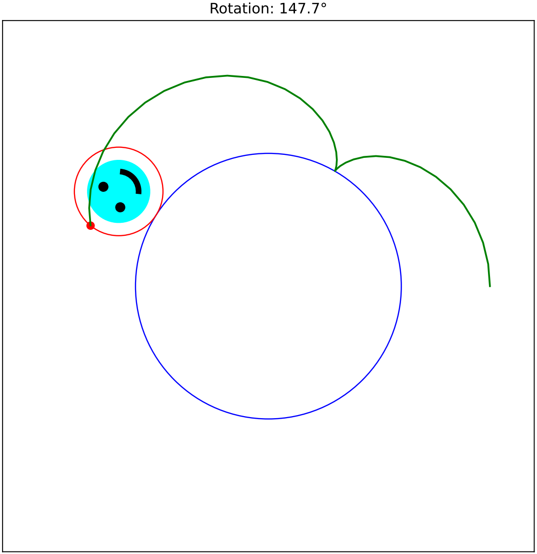
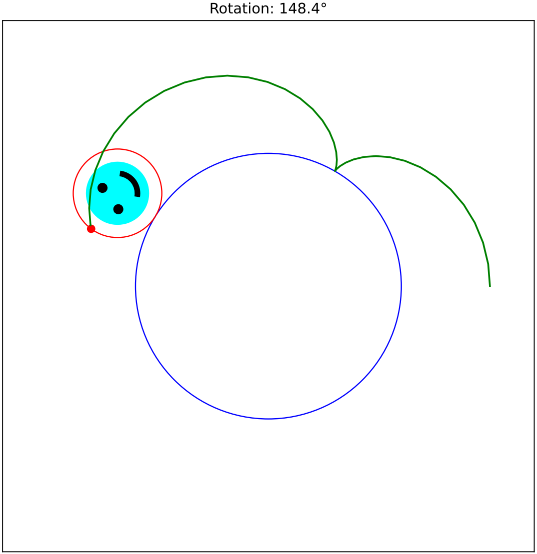
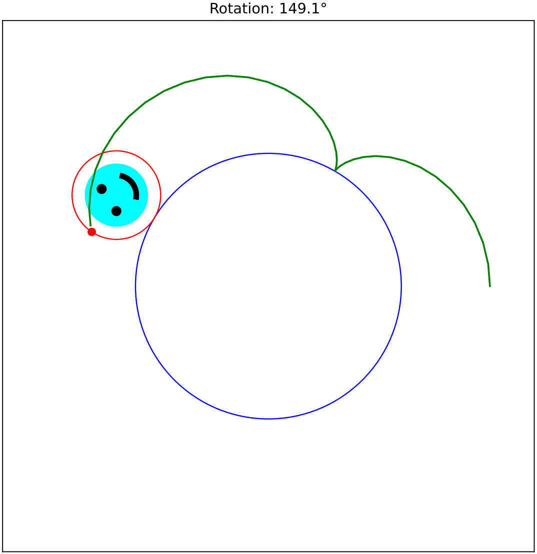


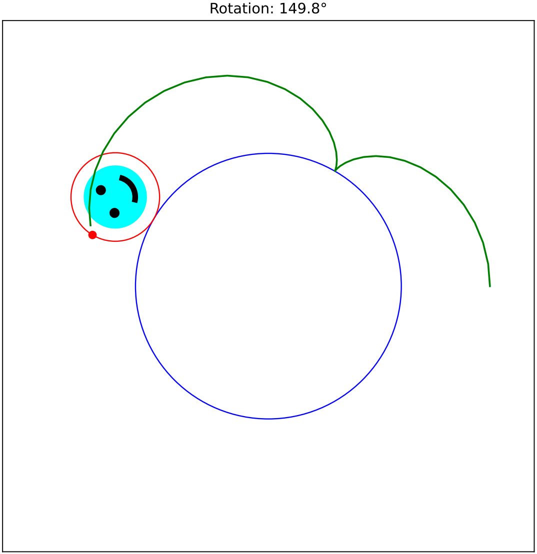
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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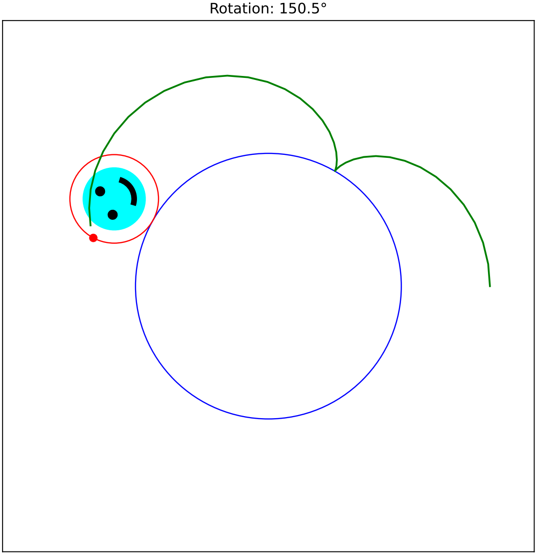

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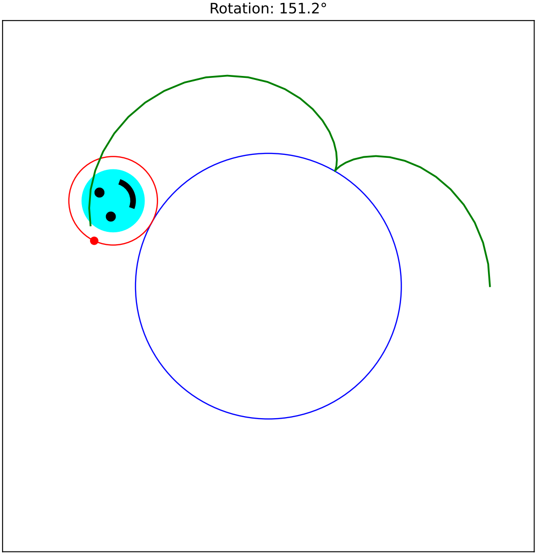
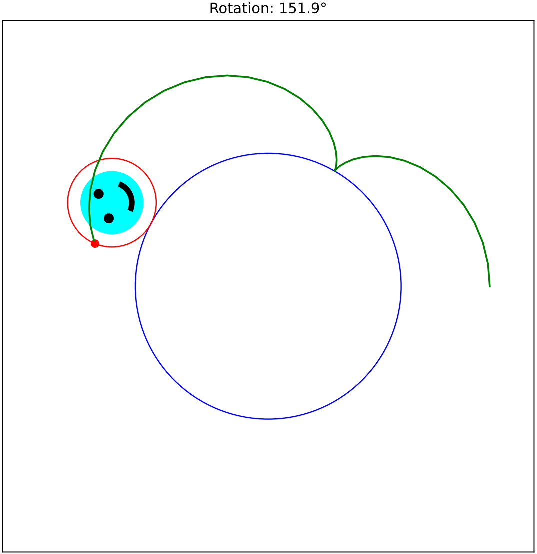


Animation of Coin Rotation Paradox



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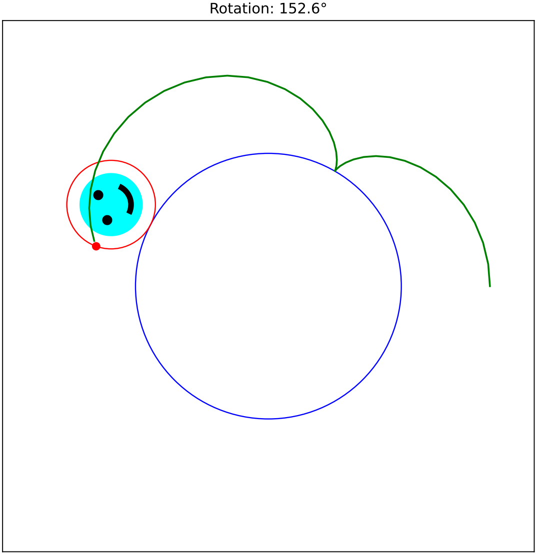
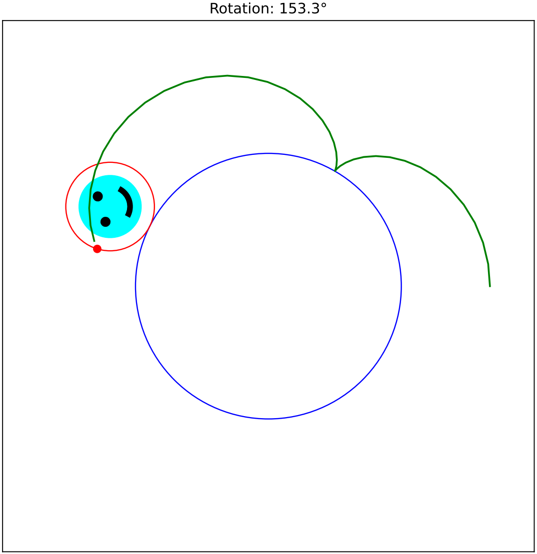
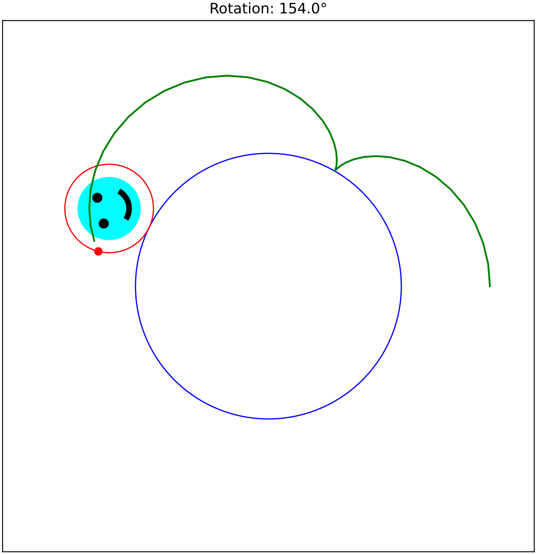


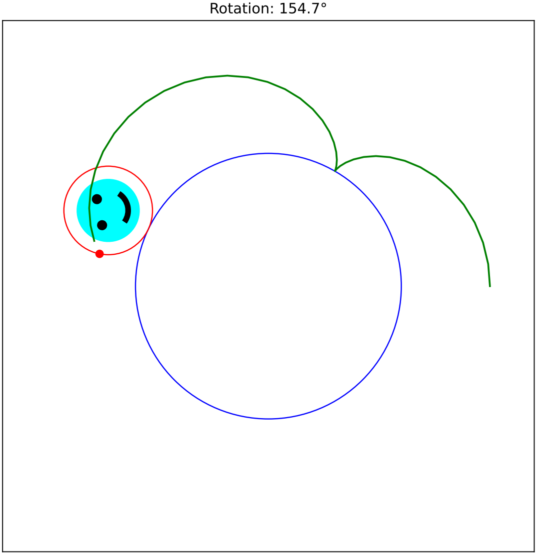
Animation of Coin Rotation Paradox



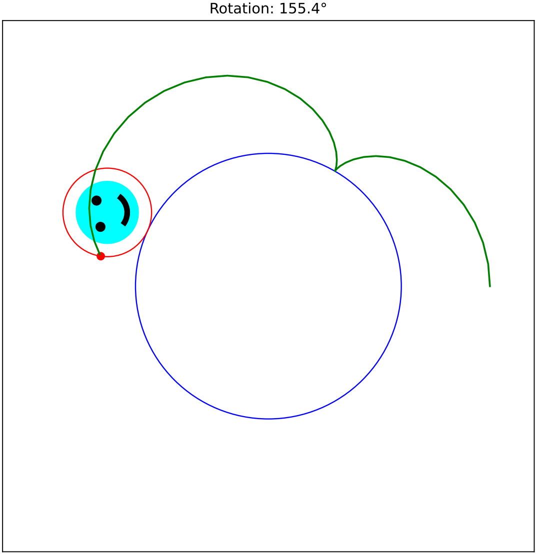
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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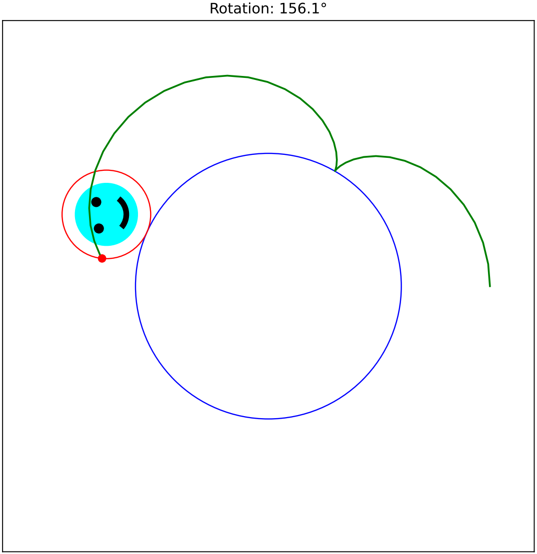

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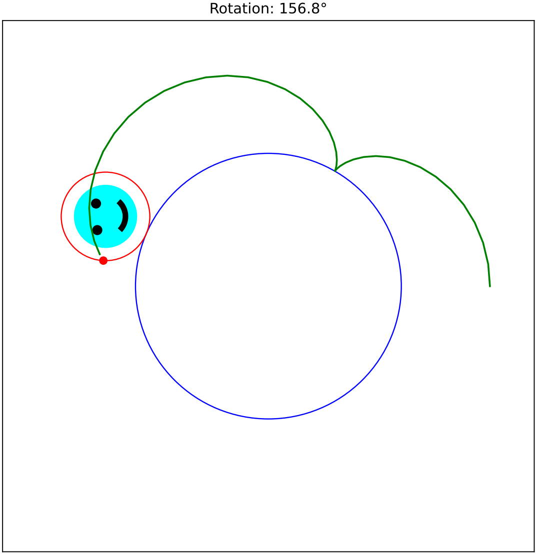

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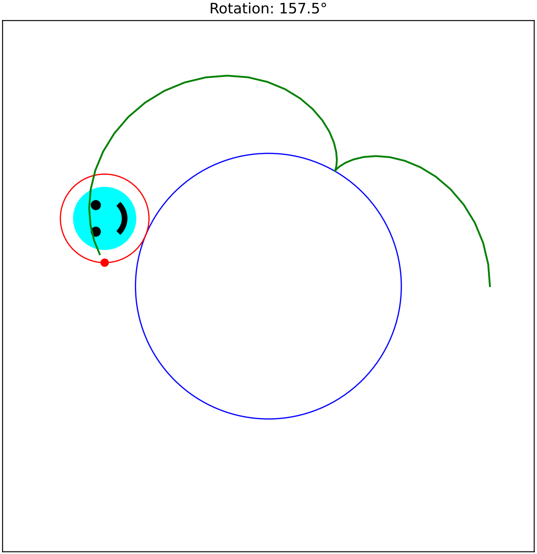
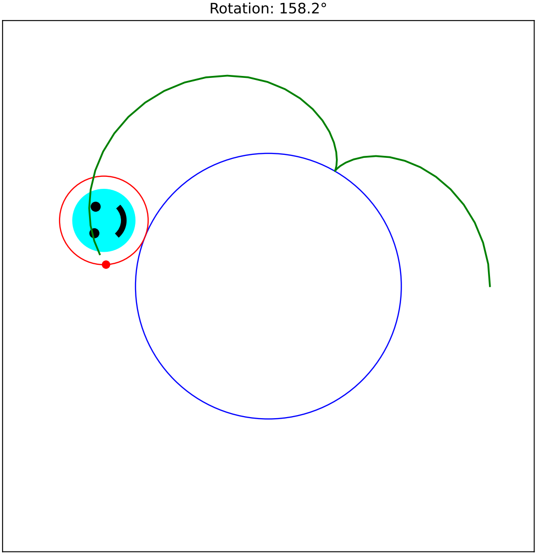
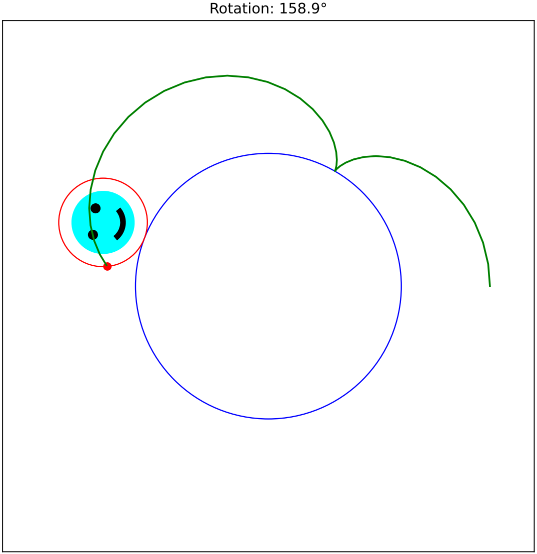


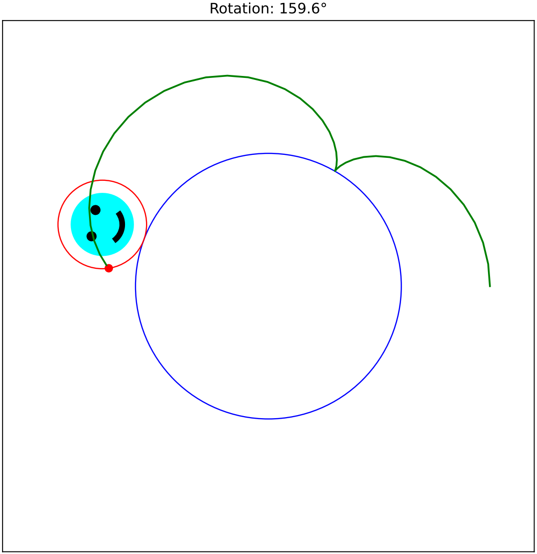
Animation of Coin Rotation Paradox



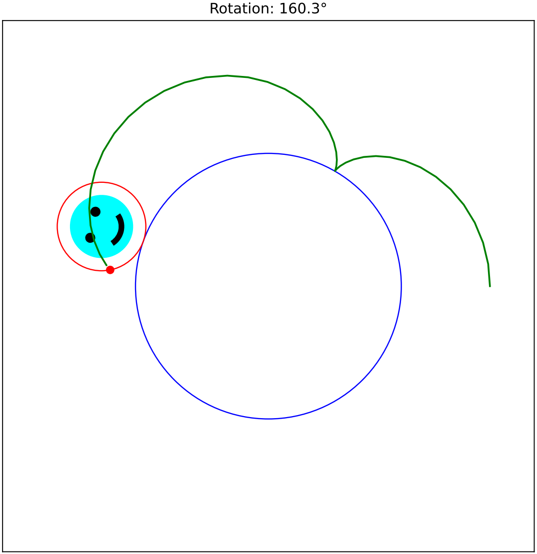
Animation of Coin Rotation Paradox



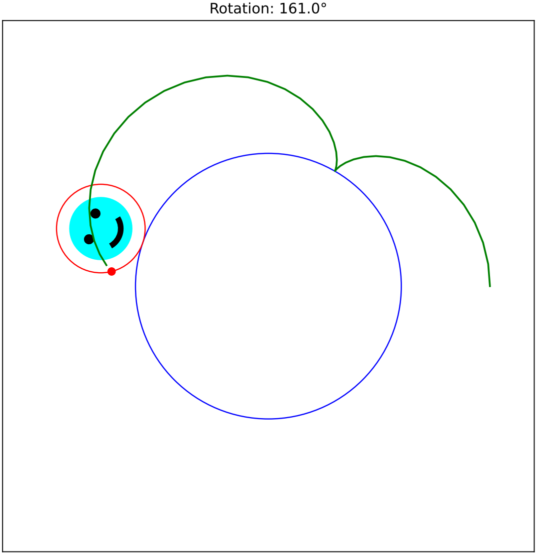
Animation of Coin Rotation Paradox



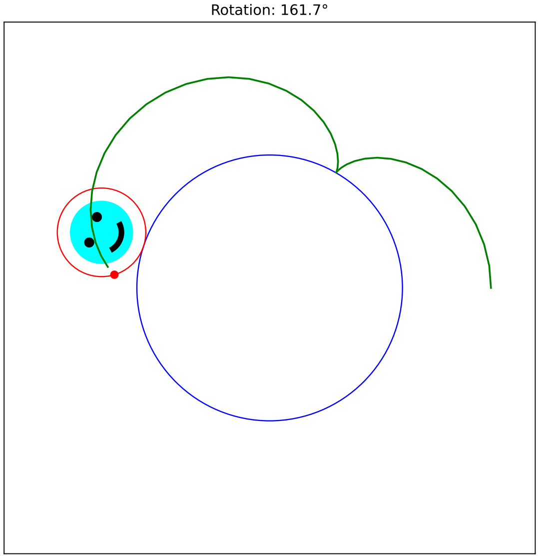
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

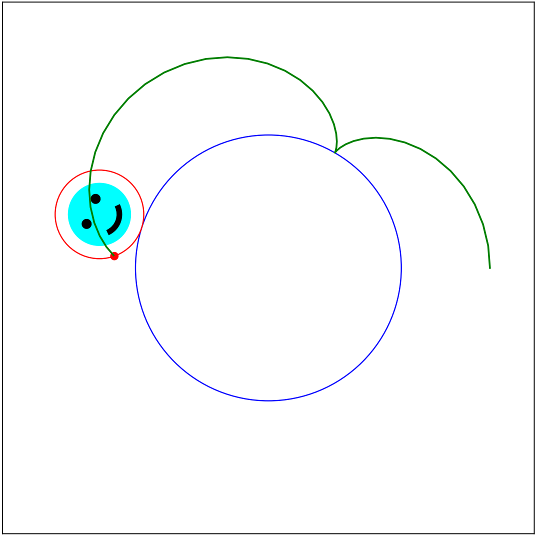


Animation of Coin Rotation Paradox

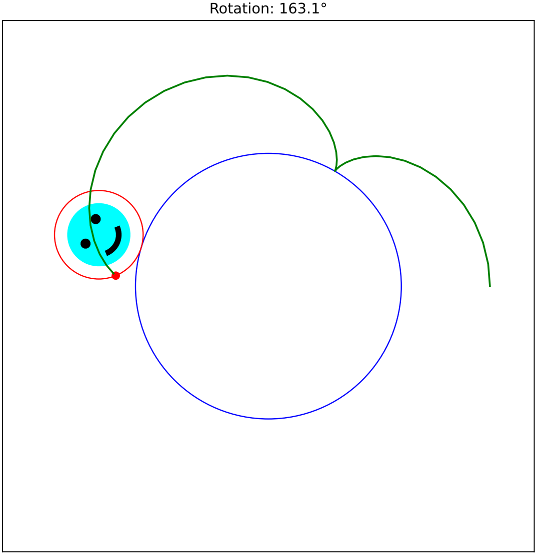


Animation of Coin Rotation Paradox

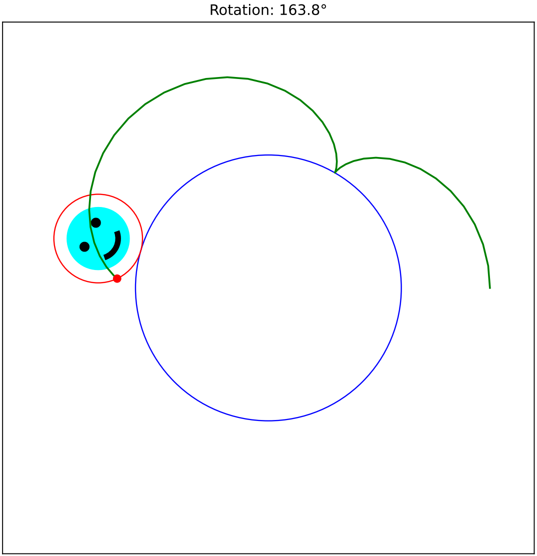
Rotation: 162.4°



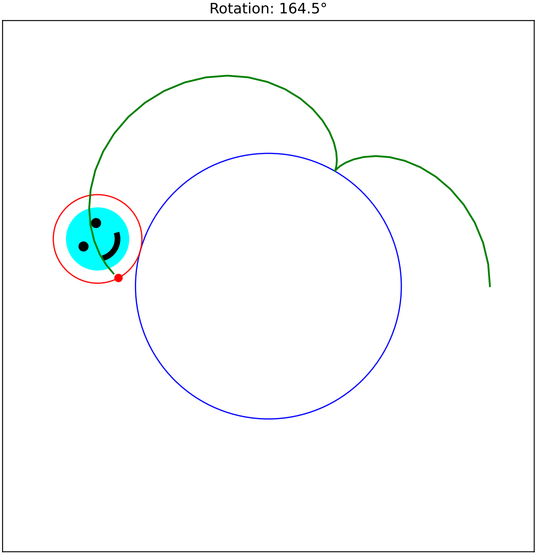
Animation of Coin Rotation Paradox



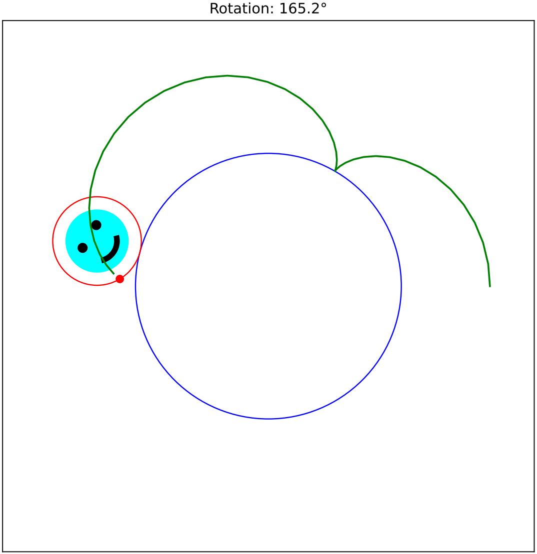
Animation of Coin Rotation Paradox



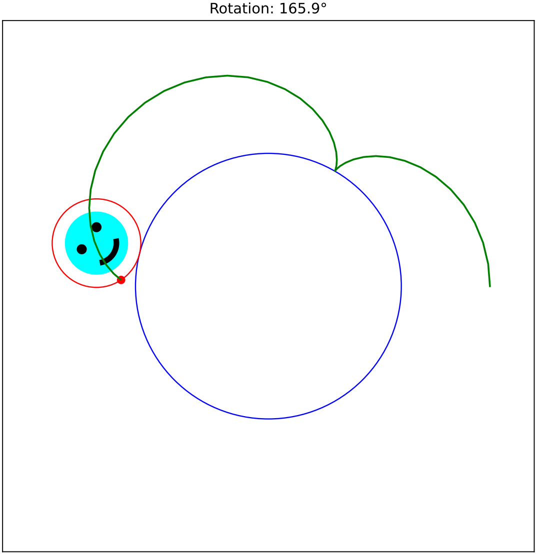
Animation of Coin Rotation Paradox



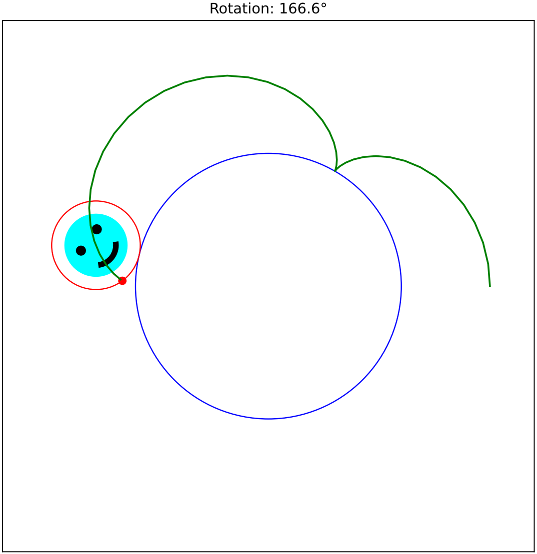
Animation of Coin Rotation Paradox



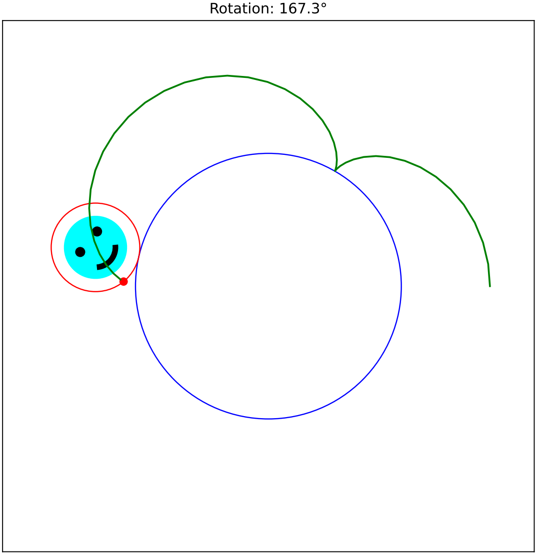
Animation of Coin Rotation Paradox



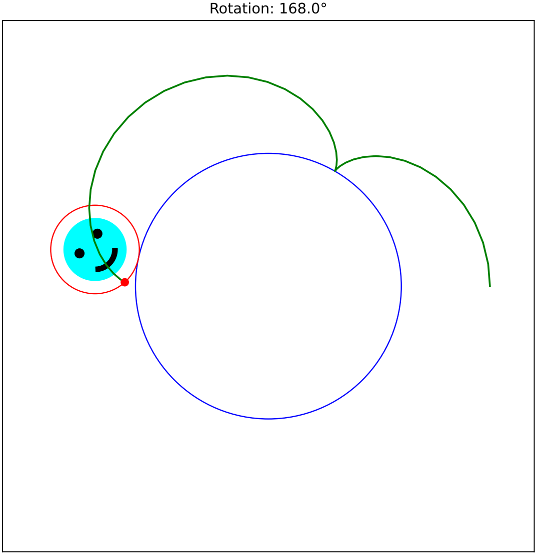
Animation of Coin Rotation Paradox



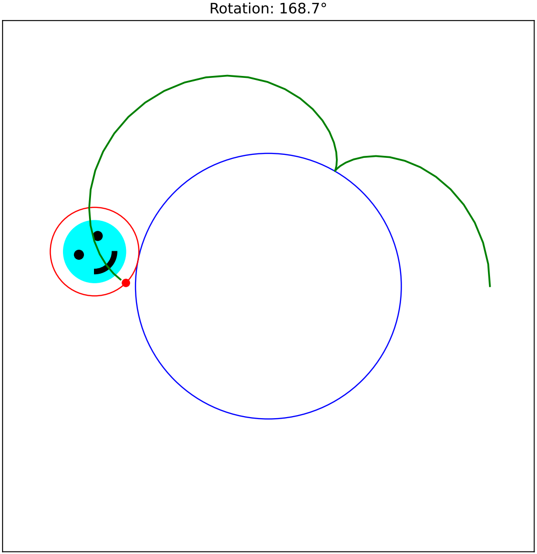
Animation of Coin Rotation Paradox



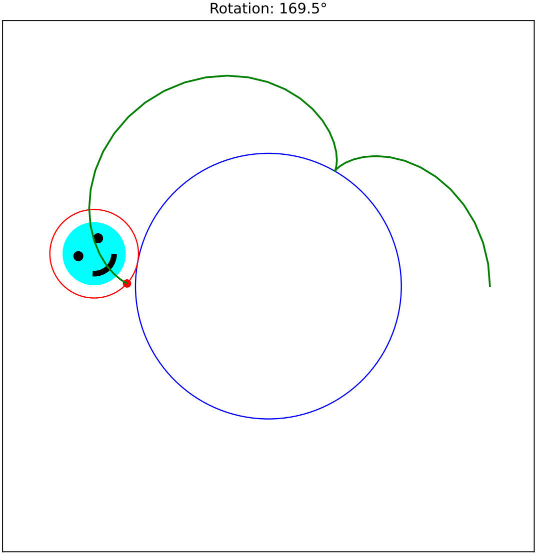
Animation of Coin Rotation Paradox



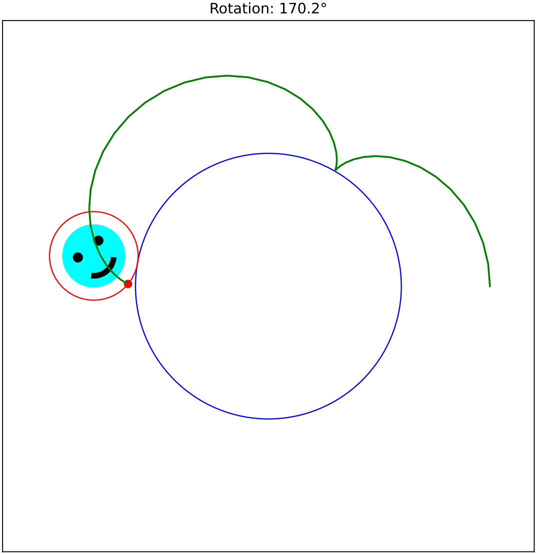
Animation of Coin Rotation Paradox



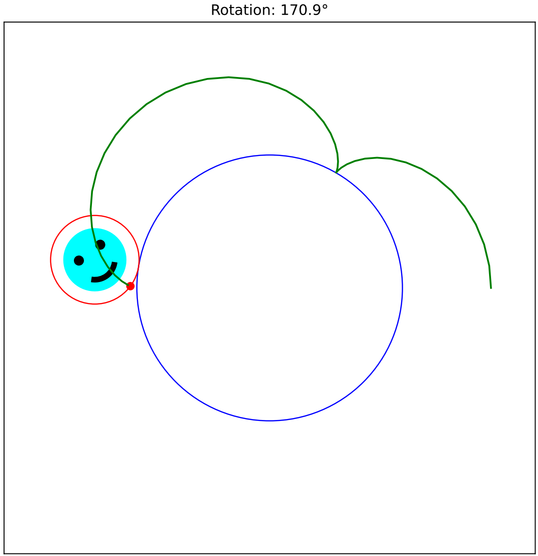
Animation of Coin Rotation Paradox



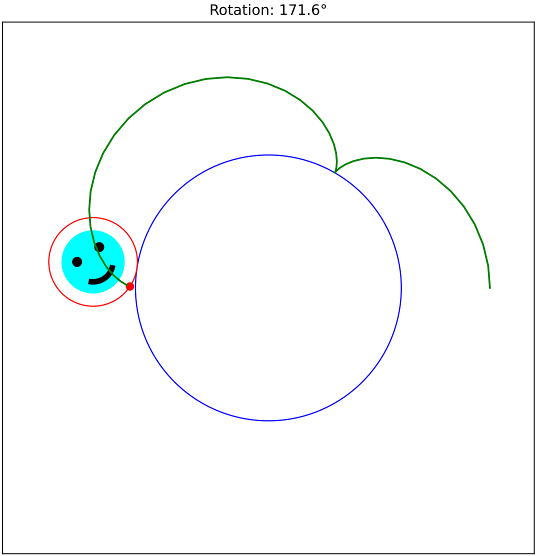
Animation of Coin Rotation Paradox



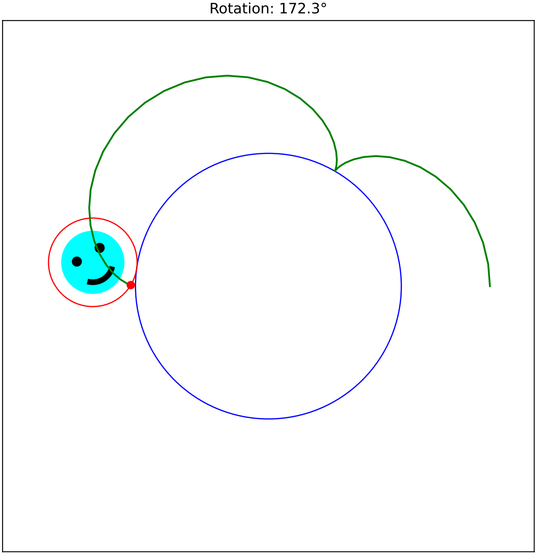
Animation of Coin Rotation Paradox



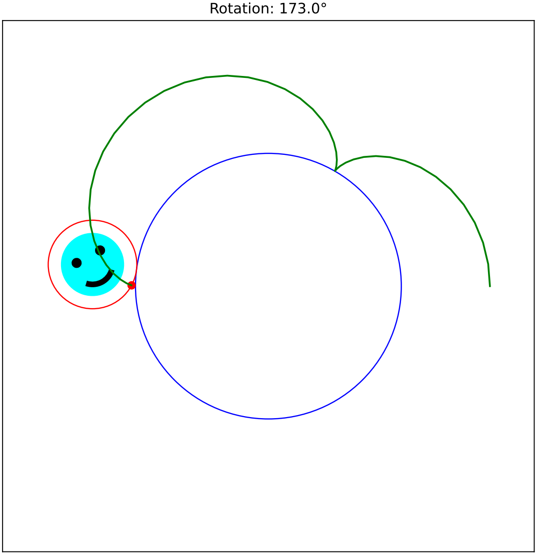
Animation of Coin Rotation Paradox



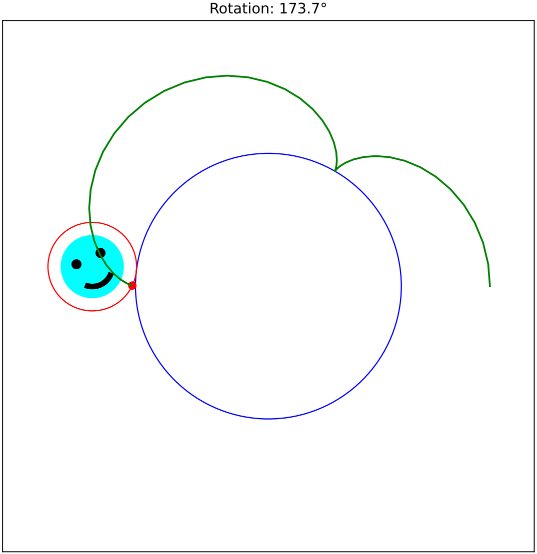
Animation of Coin Rotation Paradox



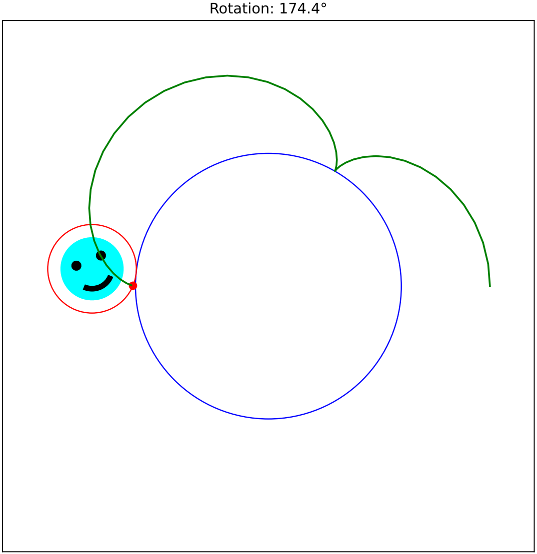
Animation of Coin Rotation Paradox



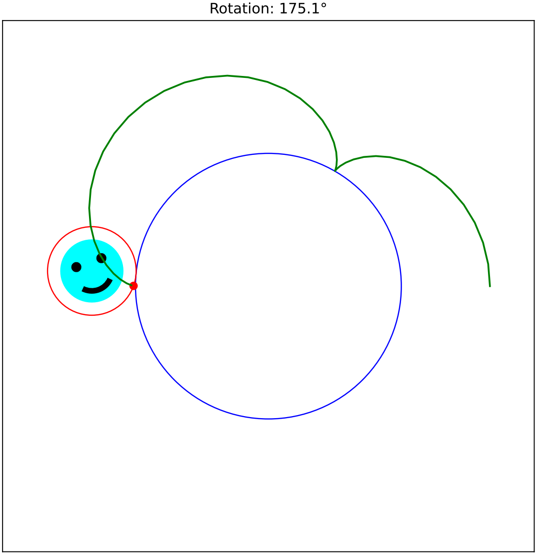
Animation of Coin Rotation Paradox



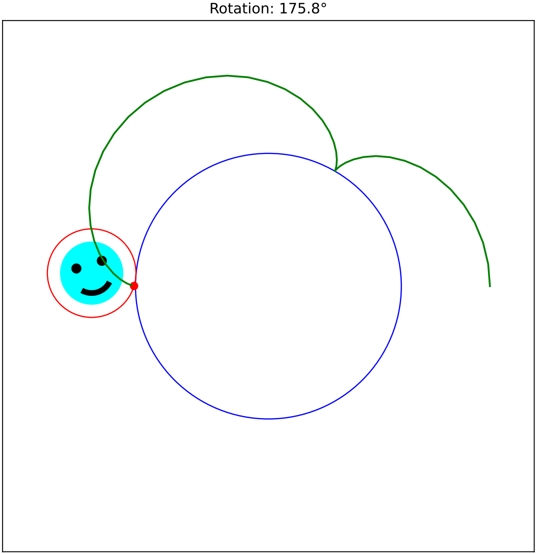
Animation of Coin Rotation Paradox



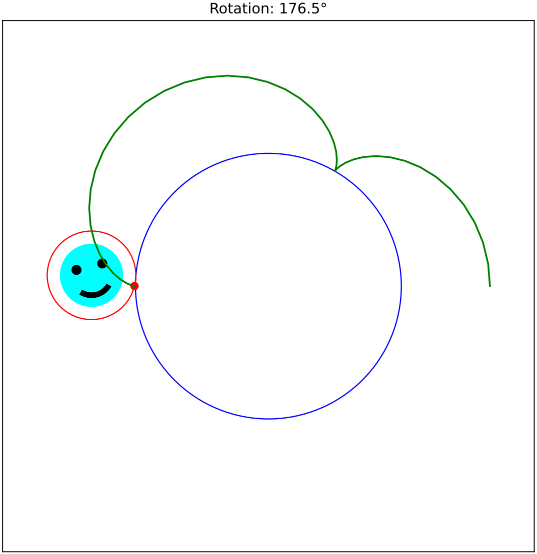
Animation of Coin Rotation Paradox



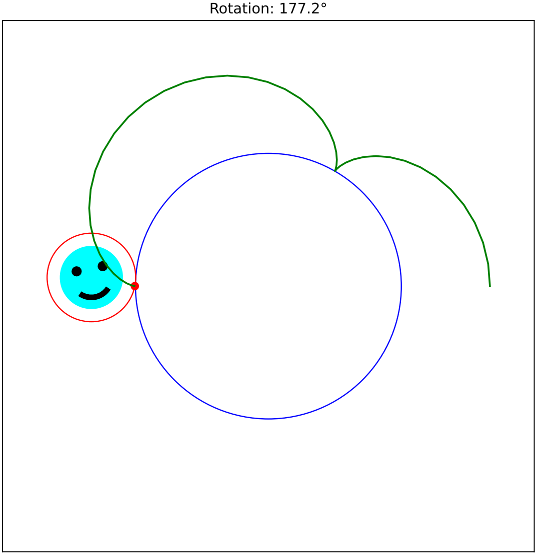
Animation of Coin Rotation Paradox



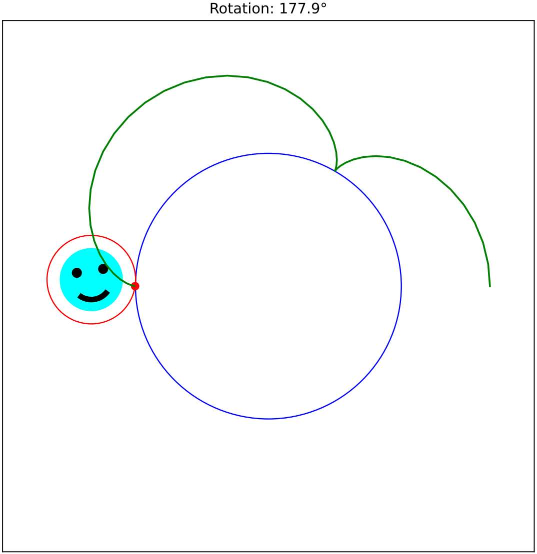
Animation of Coin Rotation Paradox



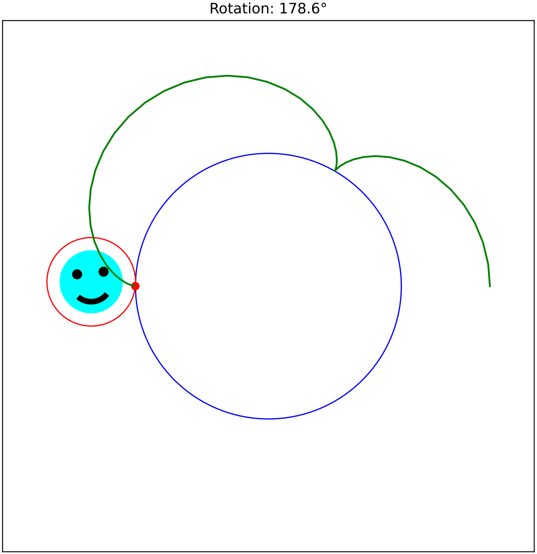
Animation of Coin Rotation Paradox



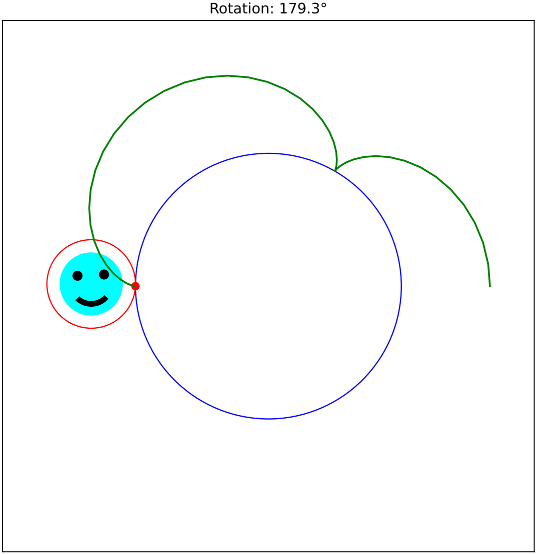
Animation of Coin Rotation Paradox



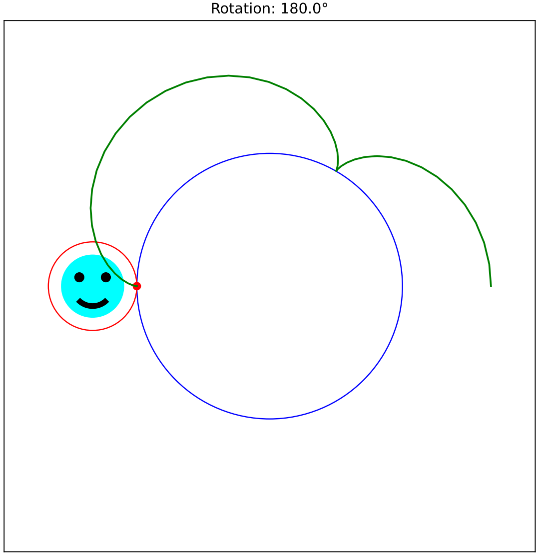
Animation of Coin Rotation Paradox



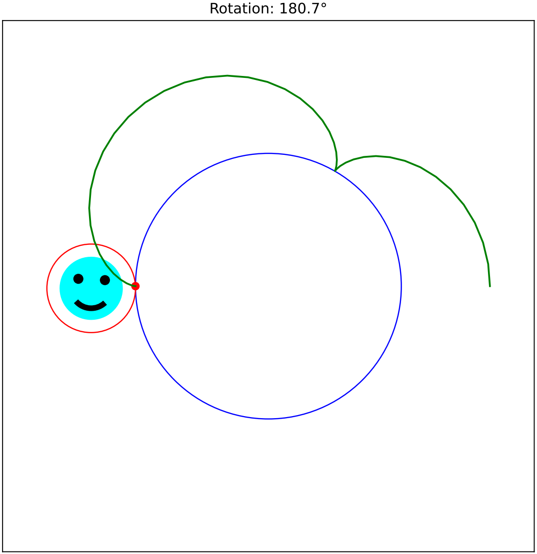
Animation of Coin Rotation Paradox



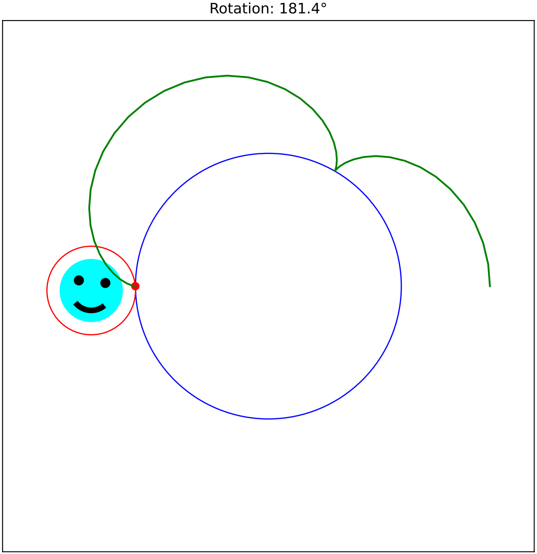
Animation of Coin Rotation Paradox



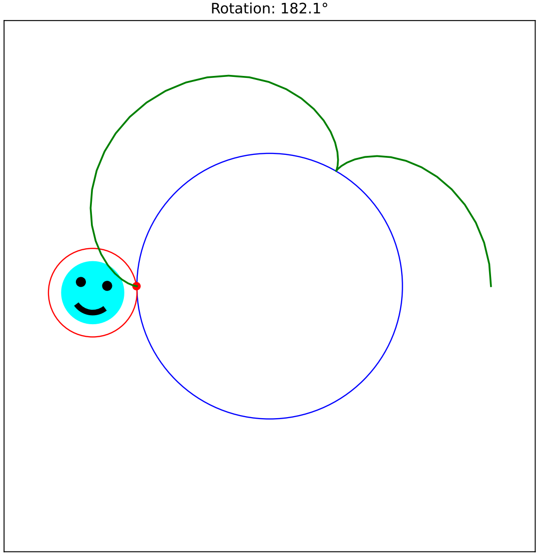
Animation of Coin Rotation Paradox



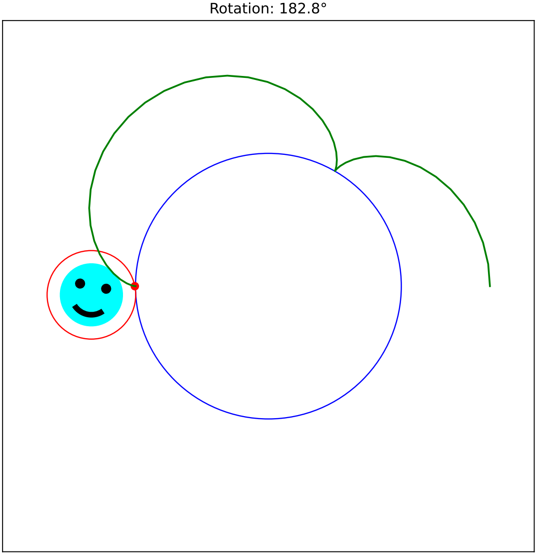
Animation of Coin Rotation Paradox



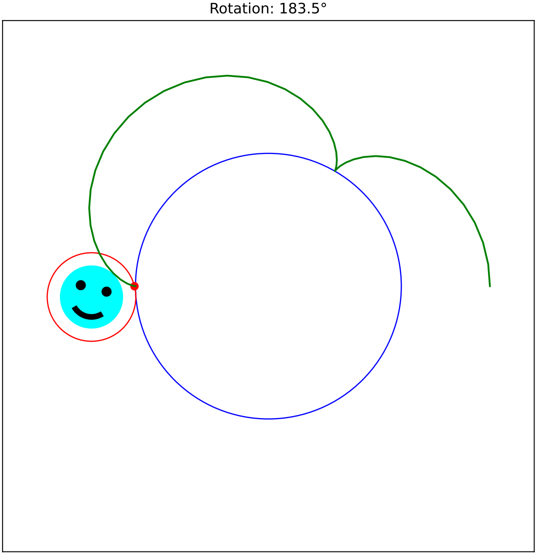
Animation of Coin Rotation Paradox



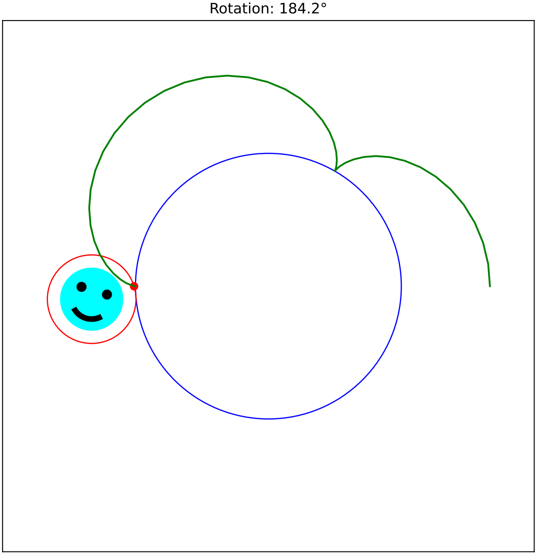
Animation of Coin Rotation Paradox



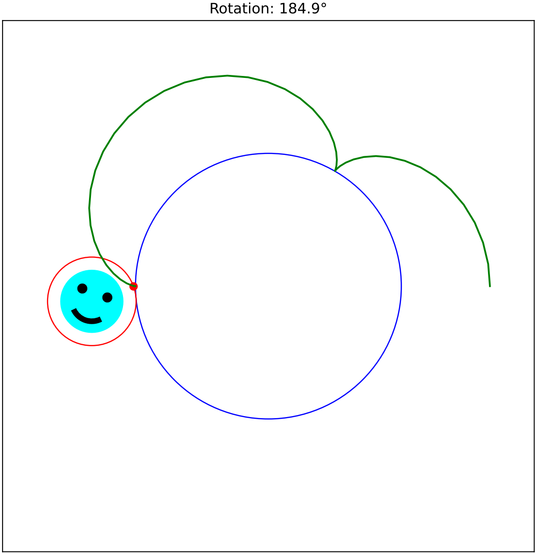
Animation of Coin Rotation Paradox



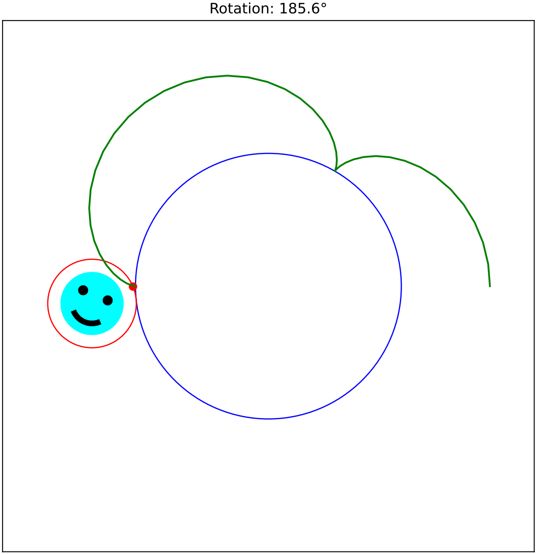
Animation of Coin Rotation Paradox



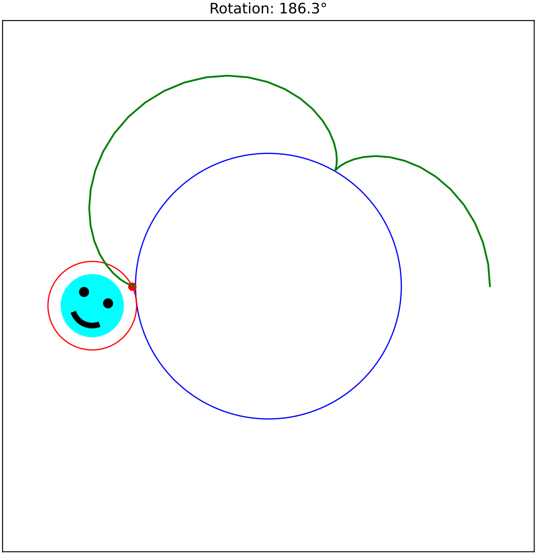
Animation of Coin Rotation Paradox



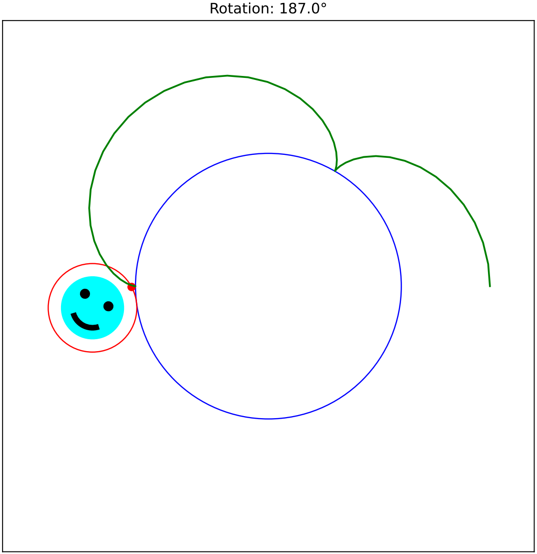
Animation of Coin Rotation Paradox



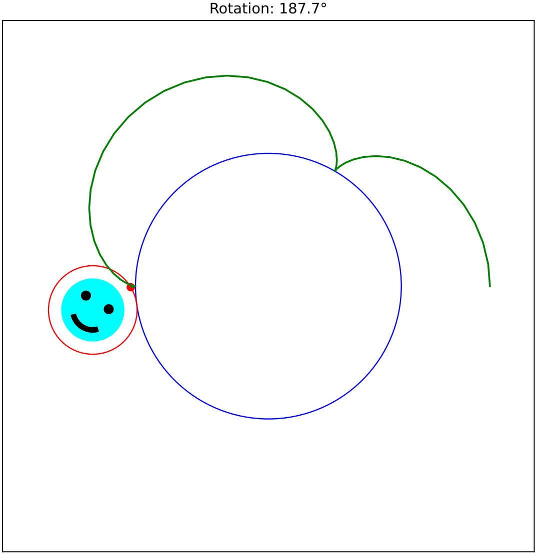
Animation of Coin Rotation Paradox



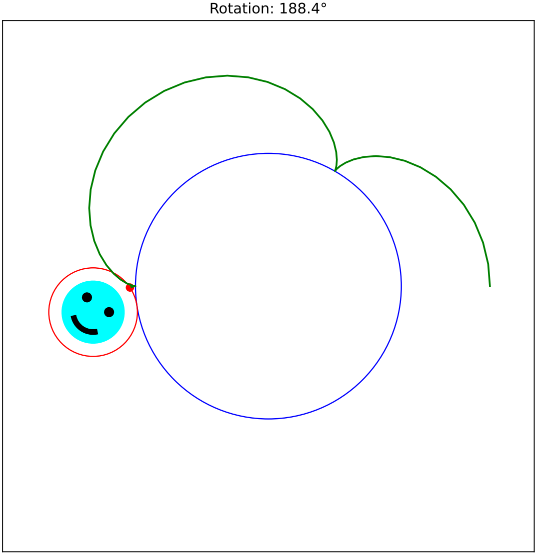
Animation of Coin Rotation Paradox



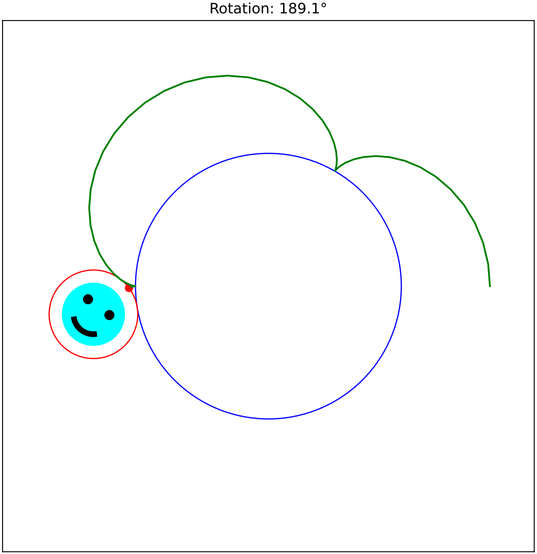
Animation of Coin Rotation Paradox



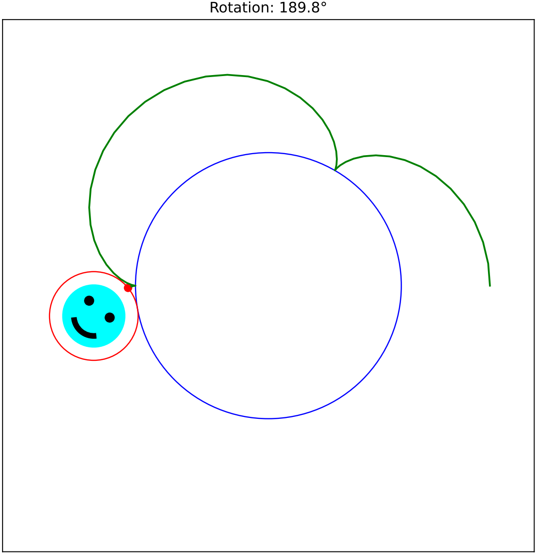
Animation of Coin Rotation Paradox



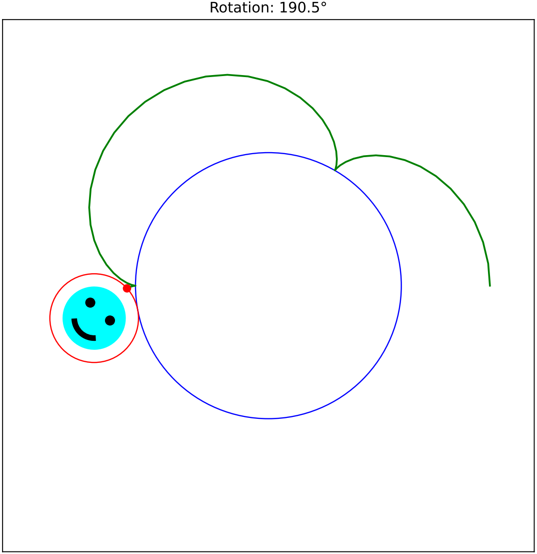
Animation of Coin Rotation Paradox



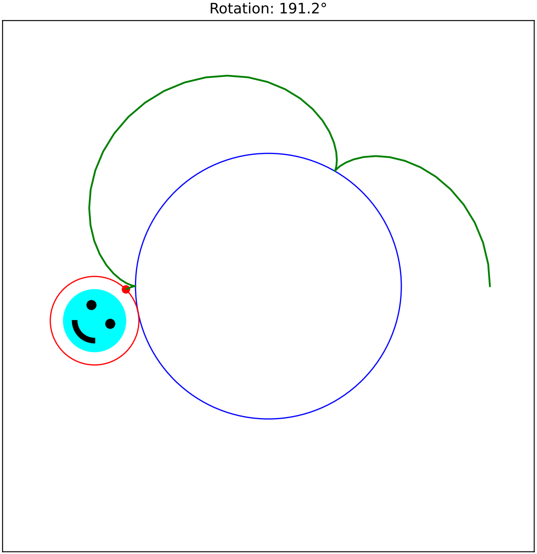
Animation of Coin Rotation Paradox



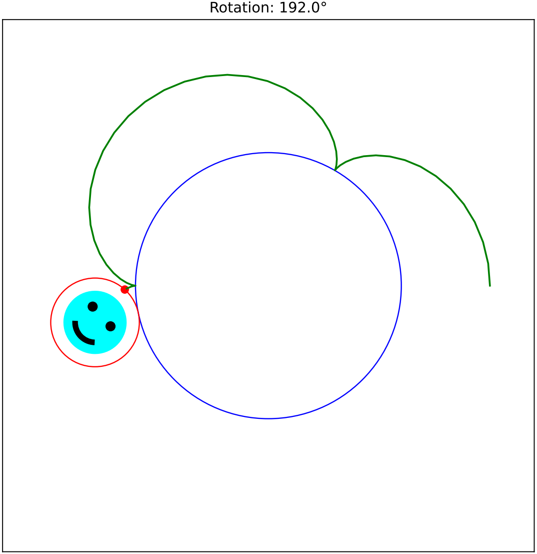
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

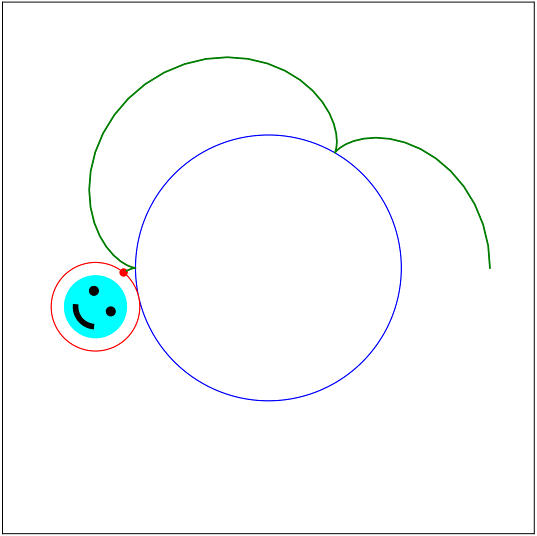


Animation of Coin Rotation Paradox

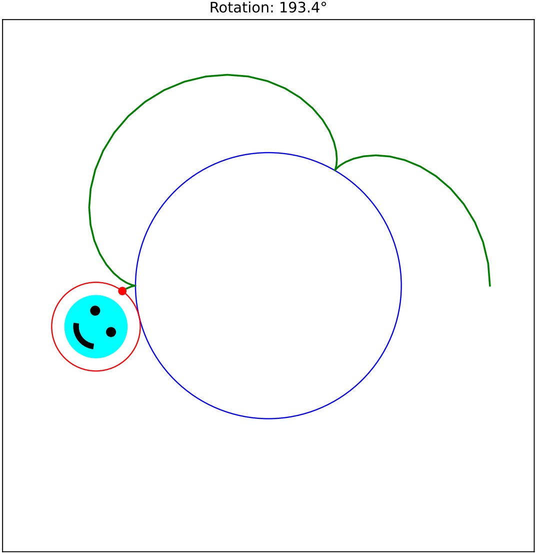


Animation of Coin Rotation Paradox

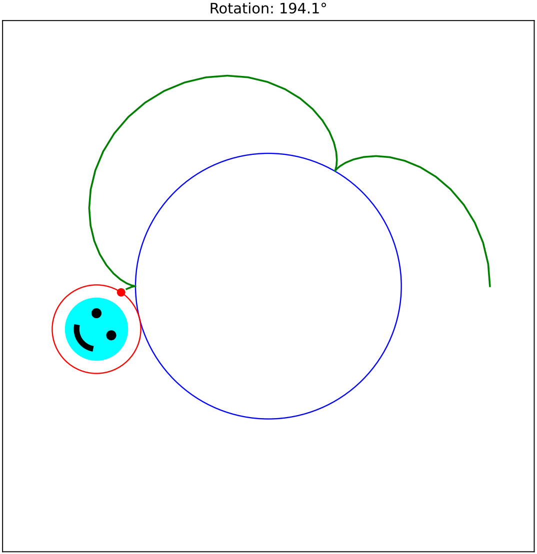
Rotation: 192.7°



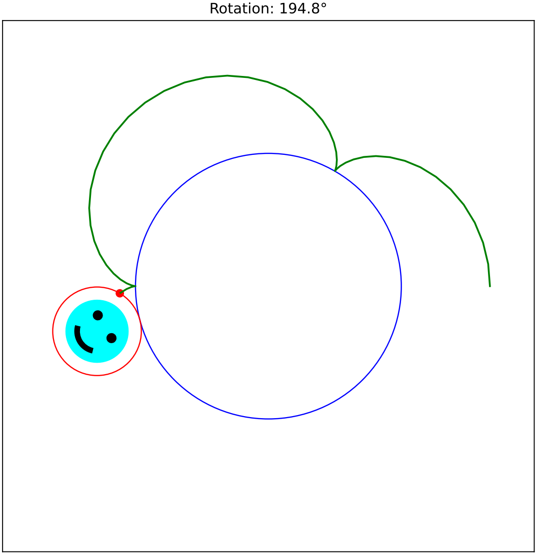
Animation of Coin Rotation Paradox



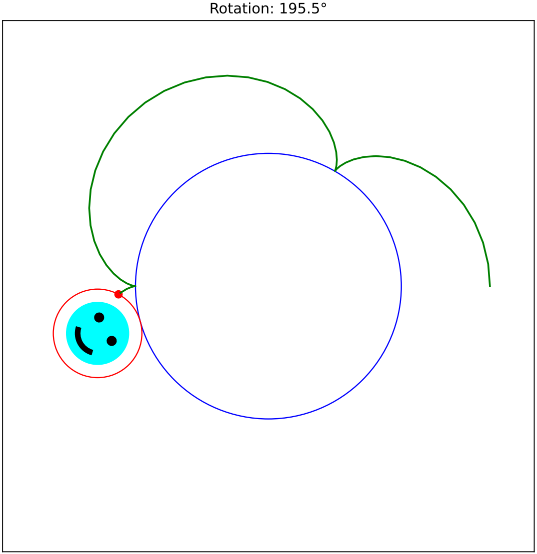
Animation of Coin Rotation Paradox



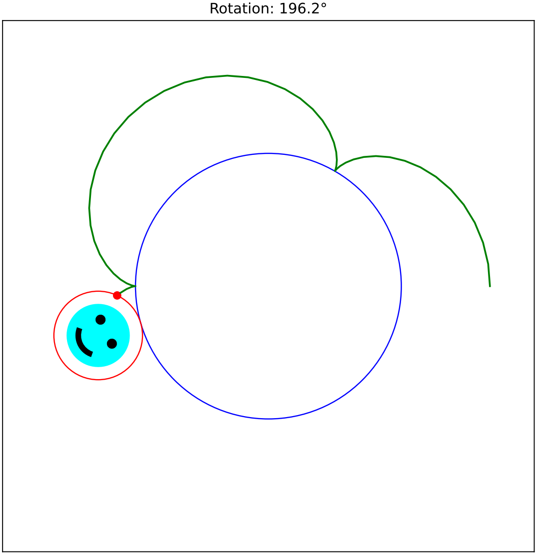
Animation of Coin Rotation Paradox



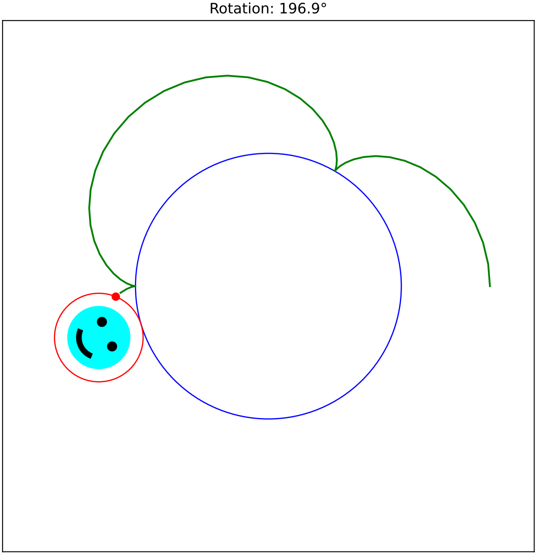
Animation of Coin Rotation Paradox



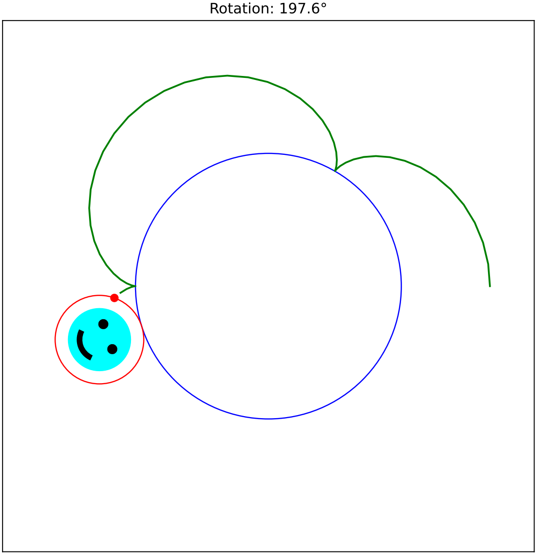
Animation of Coin Rotation Paradox



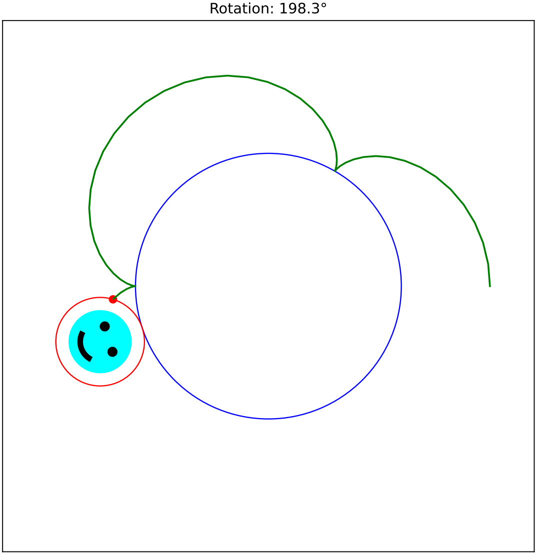
Animation of Coin Rotation Paradox



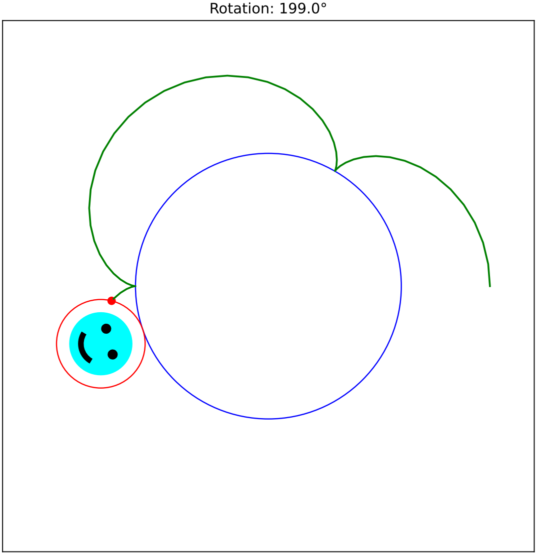
Animation of Coin Rotation Paradox



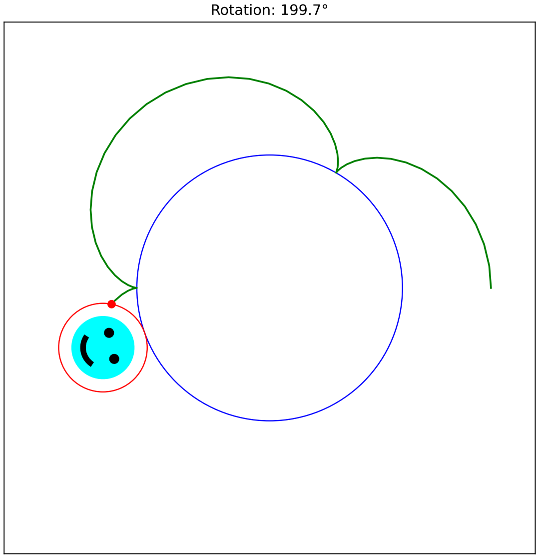
Animation of Coin Rotation Paradox



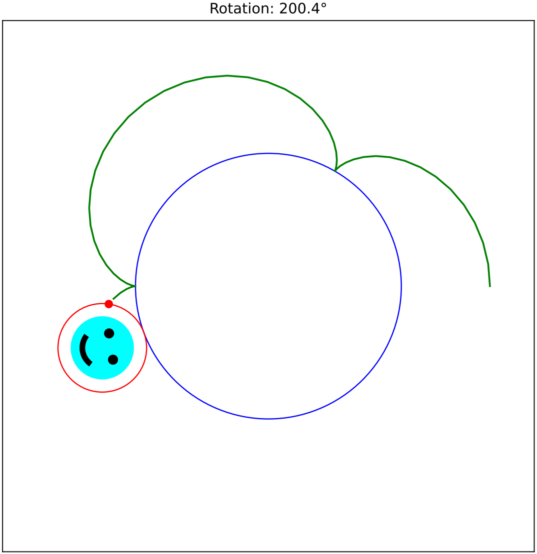
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

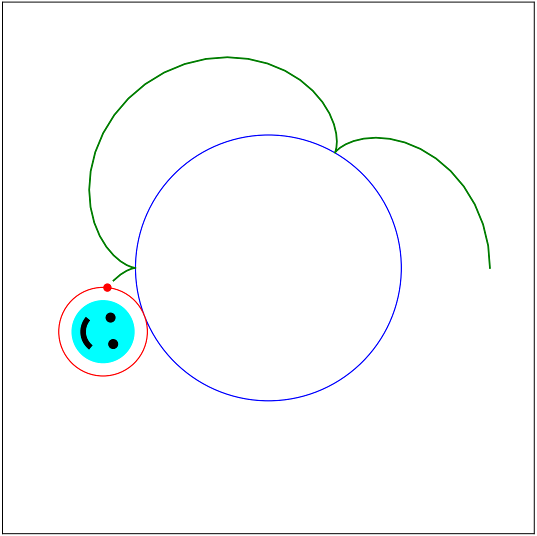


Animation of Coin Rotation Paradox

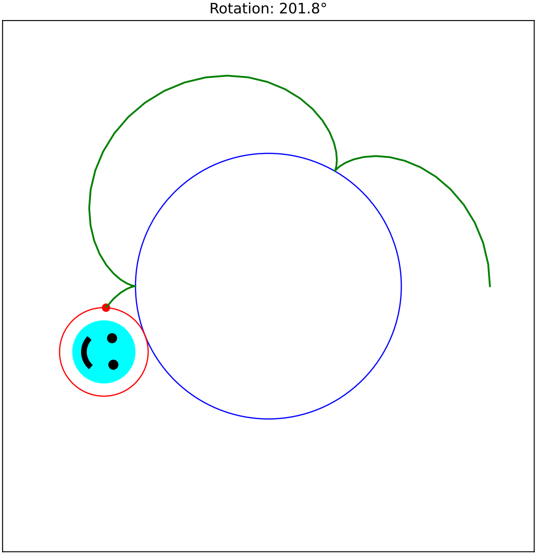


Animation of Coin Rotation Paradox

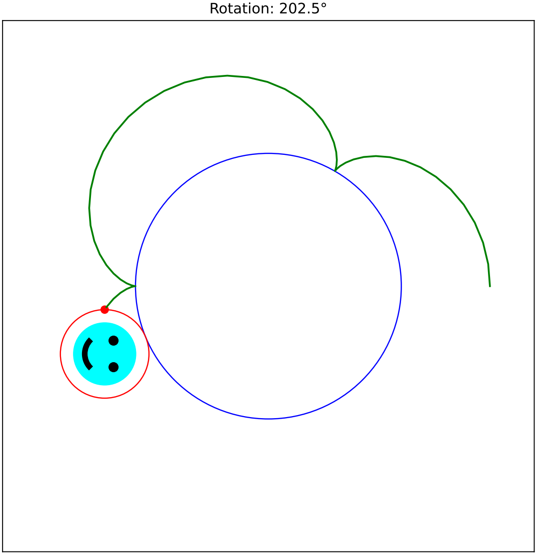
Rotation: 201.1°



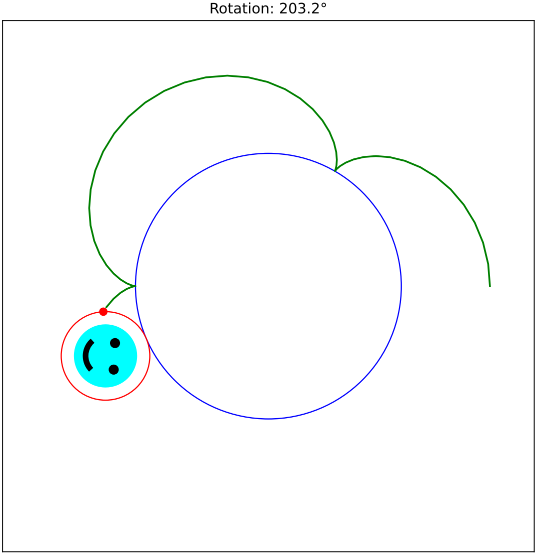
Animation of Coin Rotation Paradox



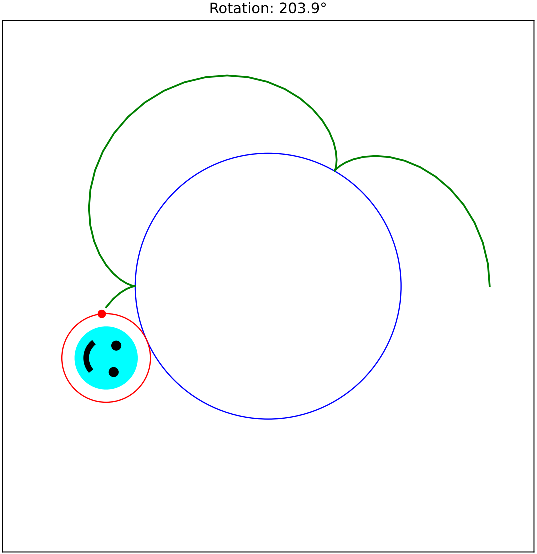
Animation of Coin Rotation Paradox



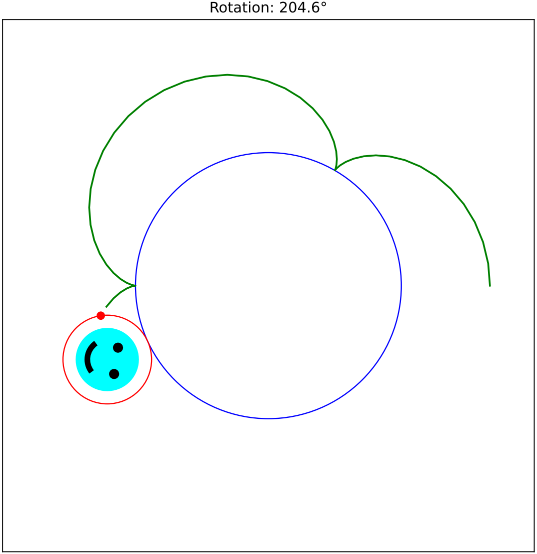
Animation of Coin Rotation Paradox



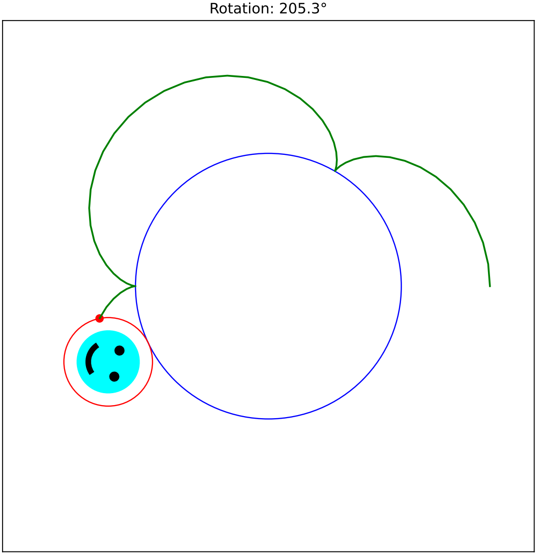
Animation of Coin Rotation Paradox



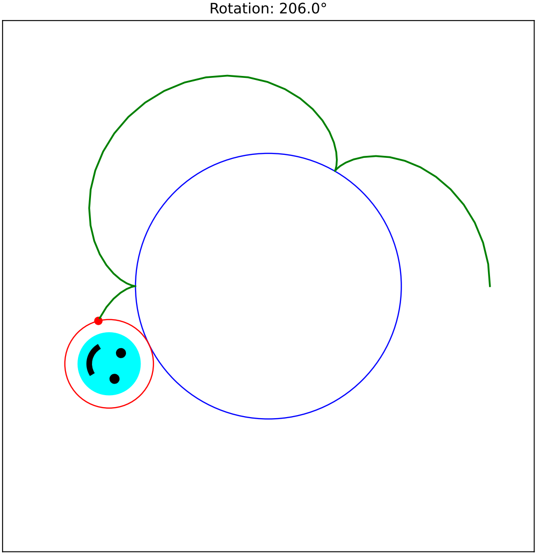
Animation of Coin Rotation Paradox



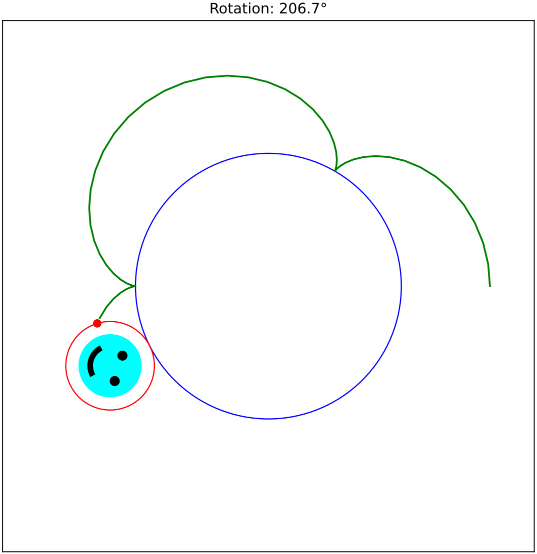
Animation of Coin Rotation Paradox



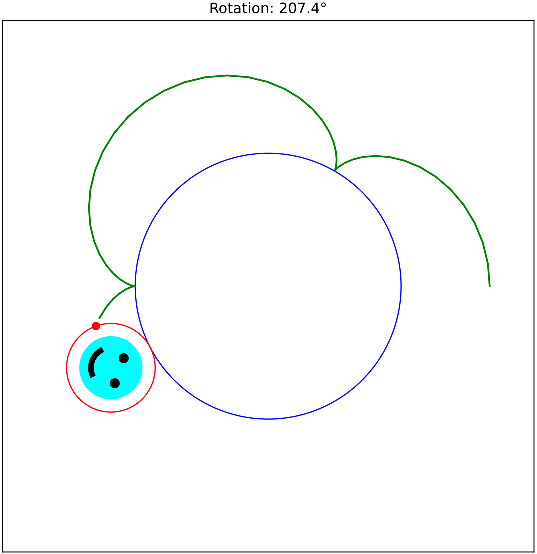
Animation of Coin Rotation Paradox



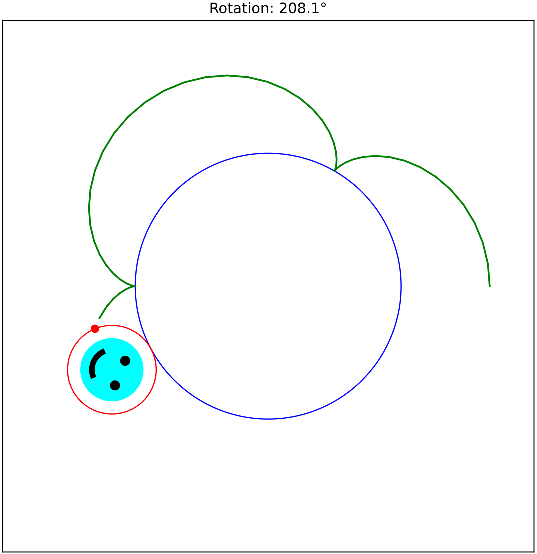
Animation of Coin Rotation Paradox



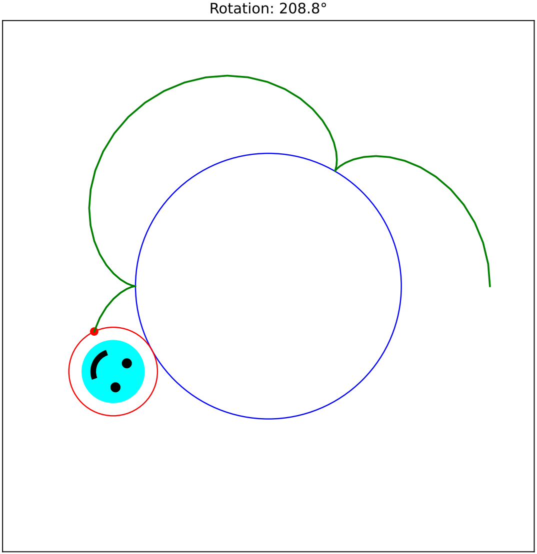
Animation of Coin Rotation Paradox



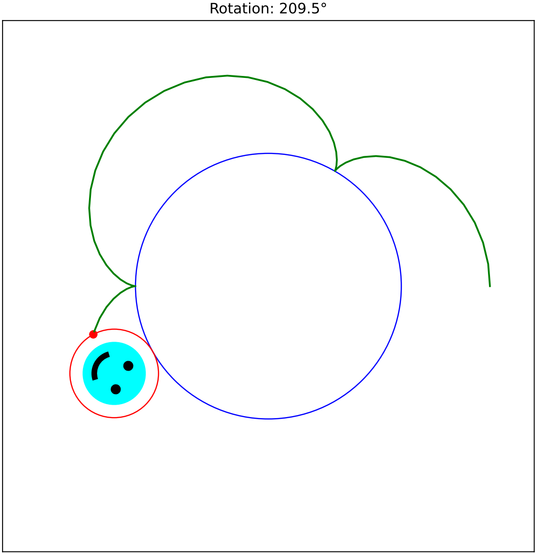
Animation of Coin Rotation Paradox



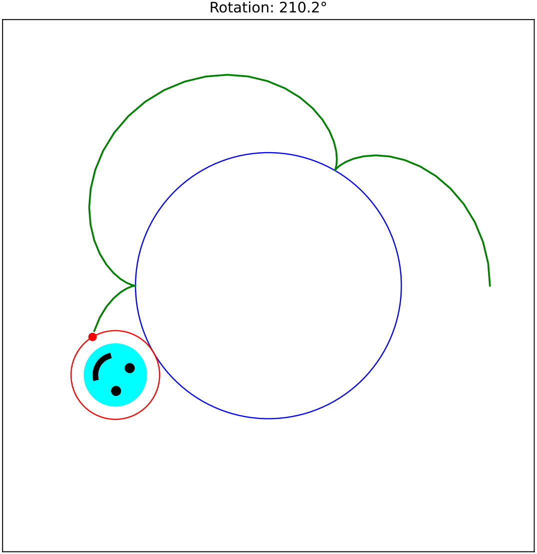
Animation of Coin Rotation Paradox



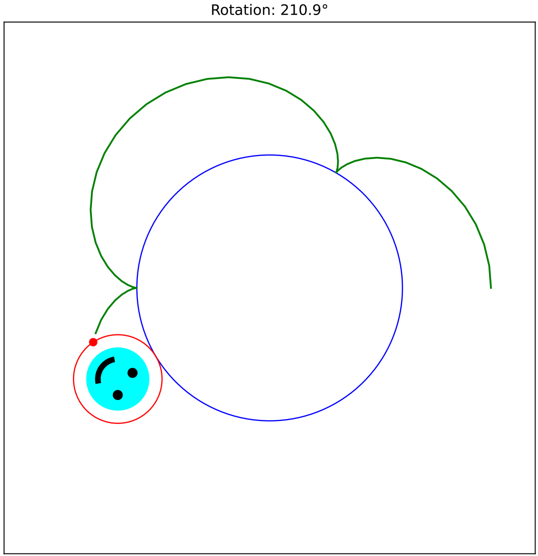
Animation of Coin Rotation Paradox



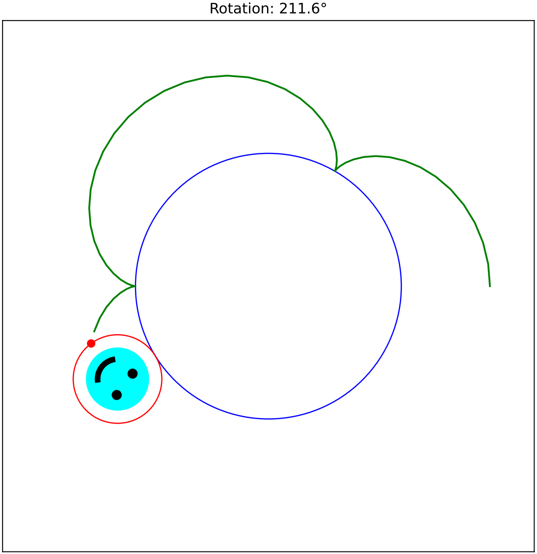
Animation of Coin Rotation Paradox



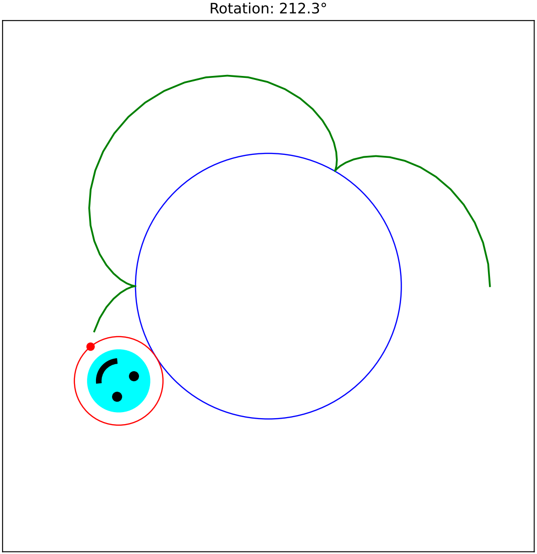
Animation of Coin Rotation Paradox



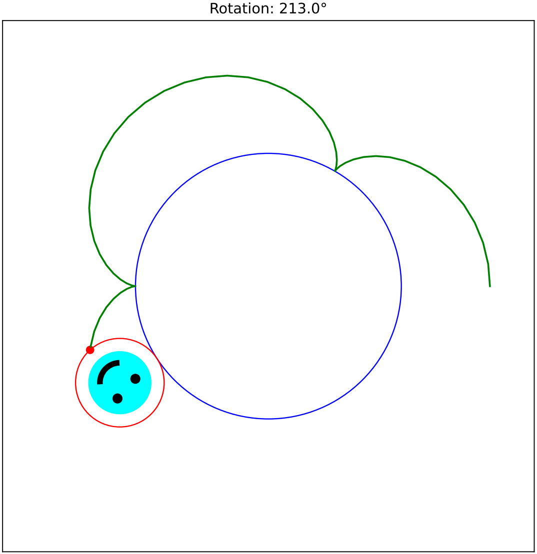
Animation of Coin Rotation Paradox



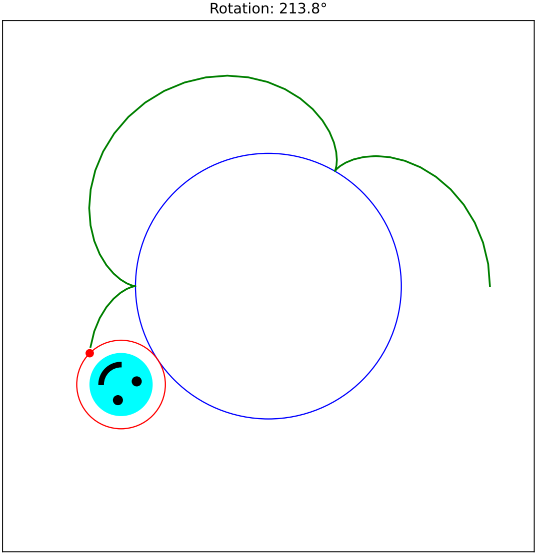
Animation of Coin Rotation Paradox



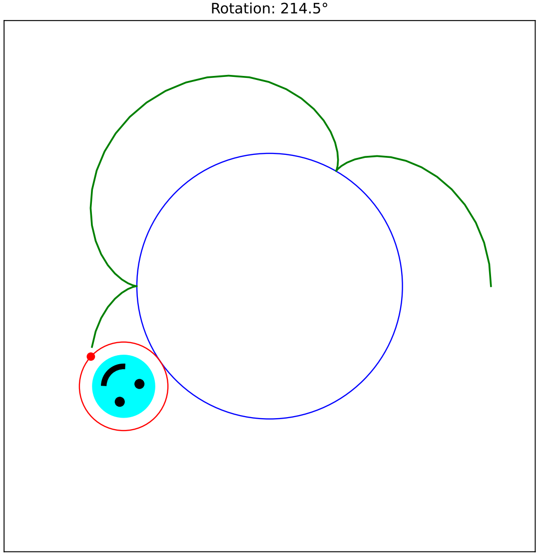
Animation of Coin Rotation Paradox



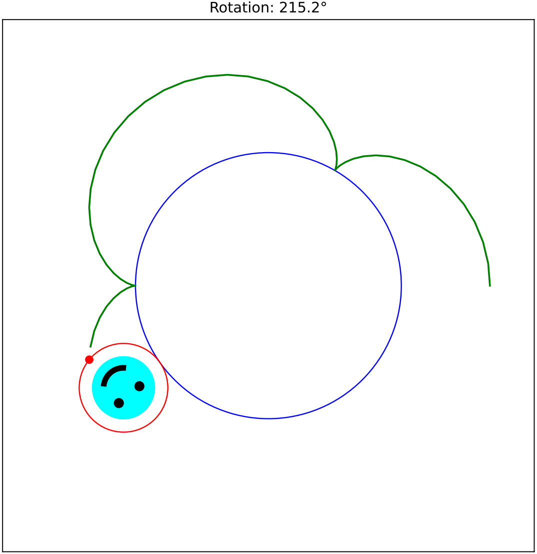
Animation of Coin Rotation Paradox



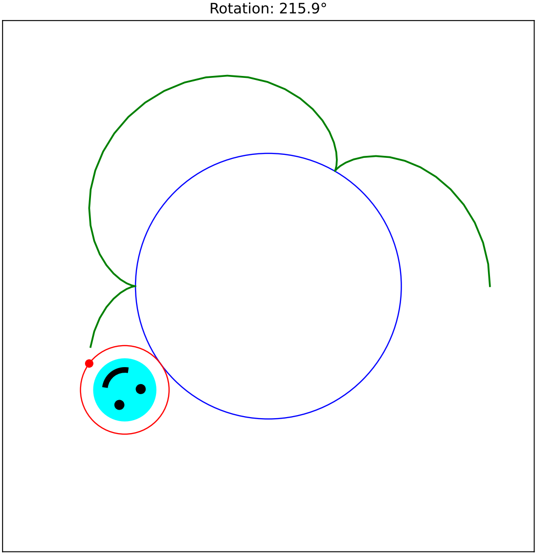
Animation of Coin Rotation Paradox



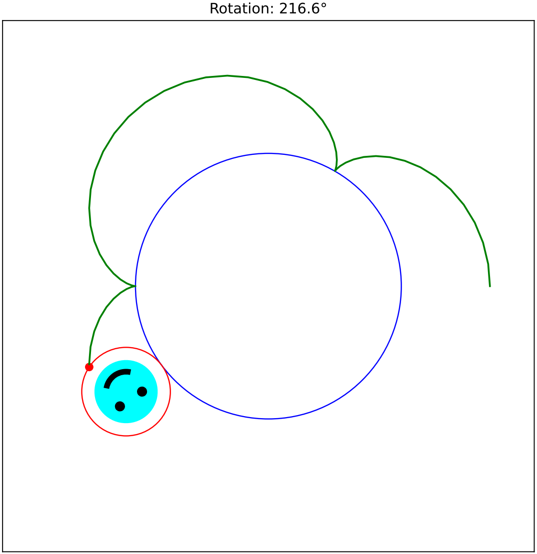
Animation of Coin Rotation Paradox



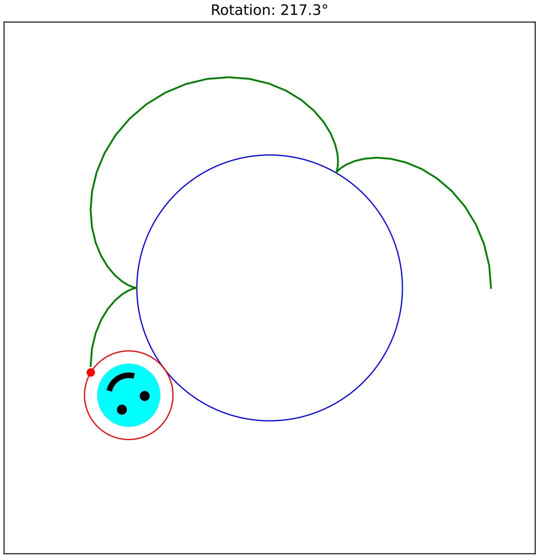
Animation of Coin Rotation Paradox



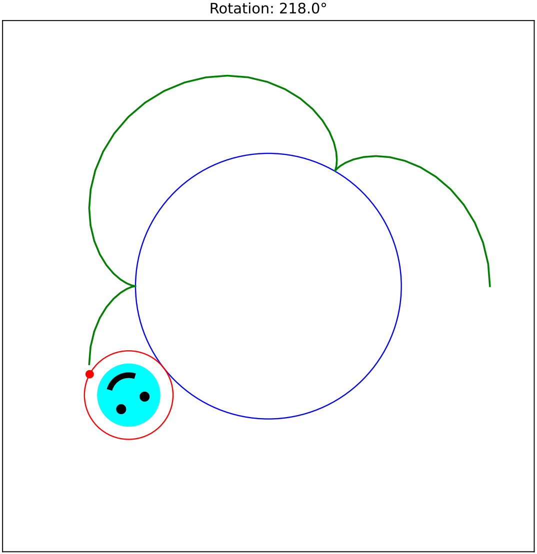
Animation of Coin Rotation Paradox



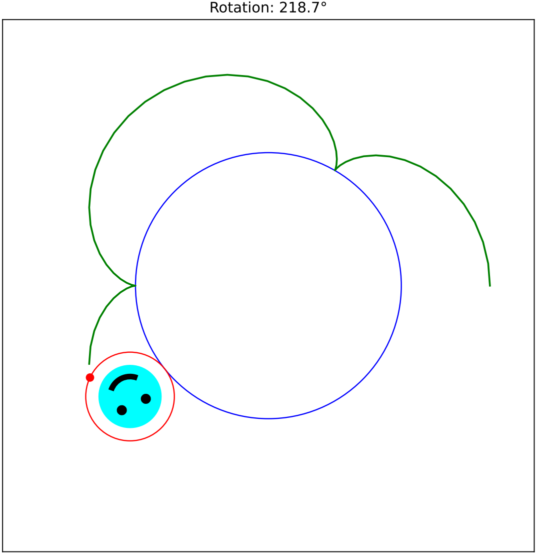
Animation of Coin Rotation Paradox



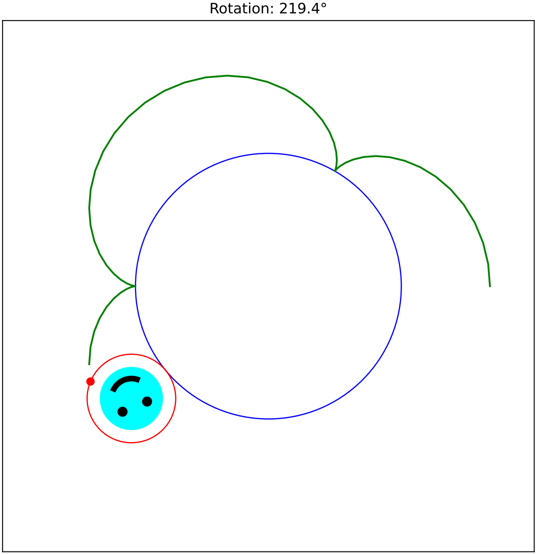
Animation of Coin Rotation Paradox



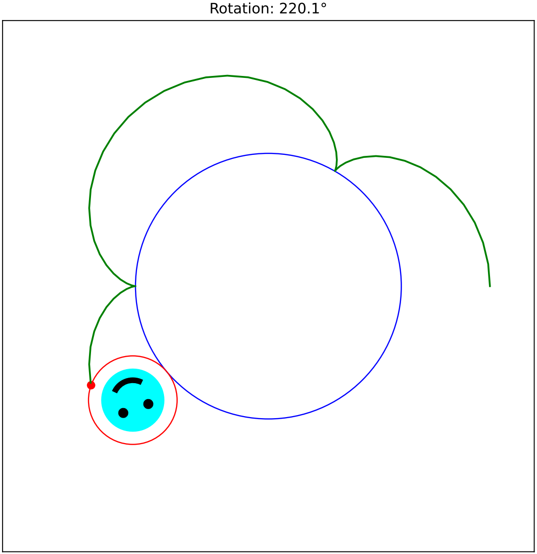
Animation of Coin Rotation Paradox



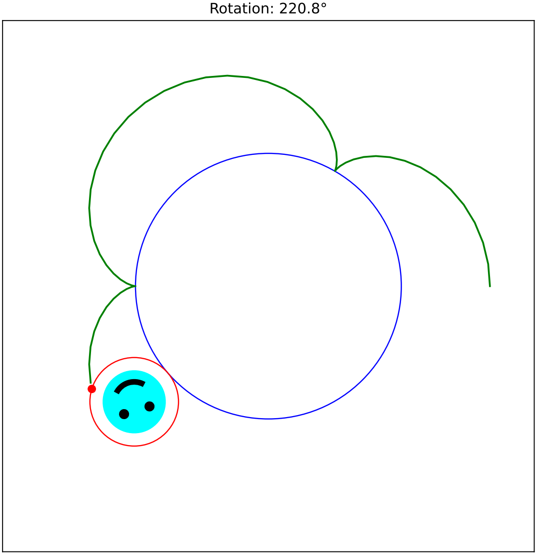
Animation of Coin Rotation Paradox



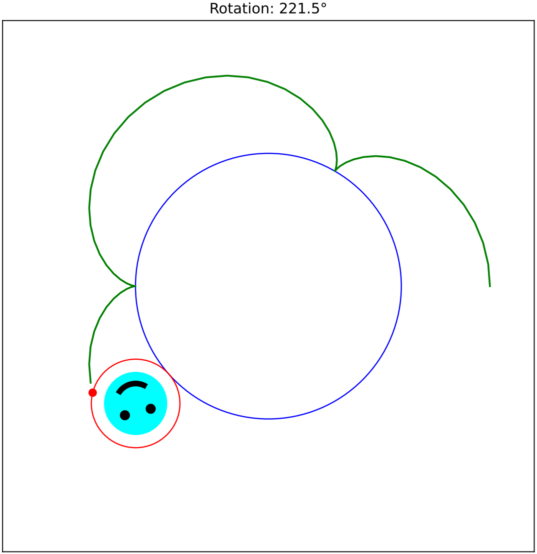
Animation of Coin Rotation Paradox



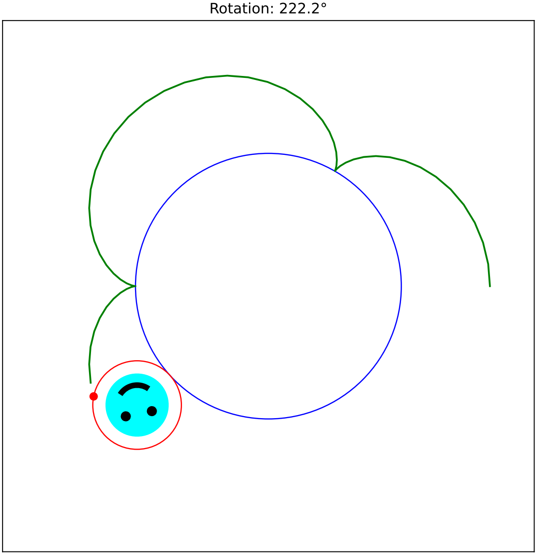
Animation of Coin Rotation Paradox



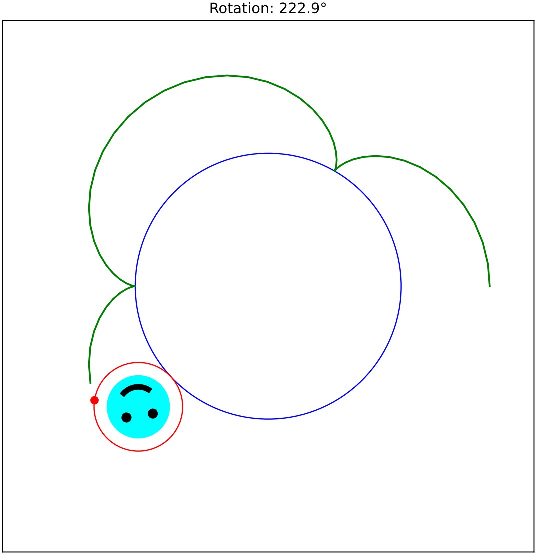
Animation of Coin Rotation Paradox



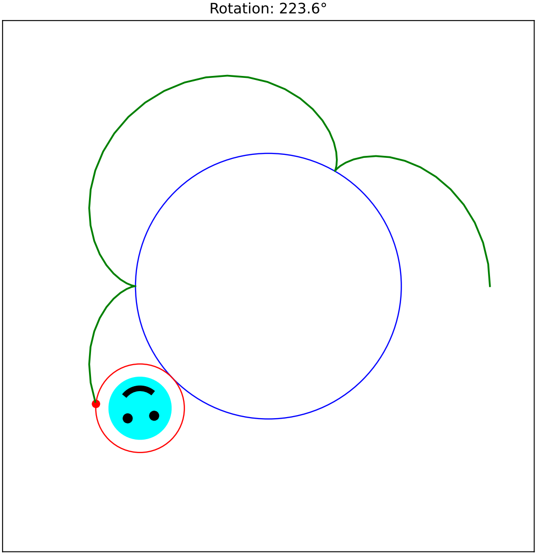
Animation of Coin Rotation Paradox



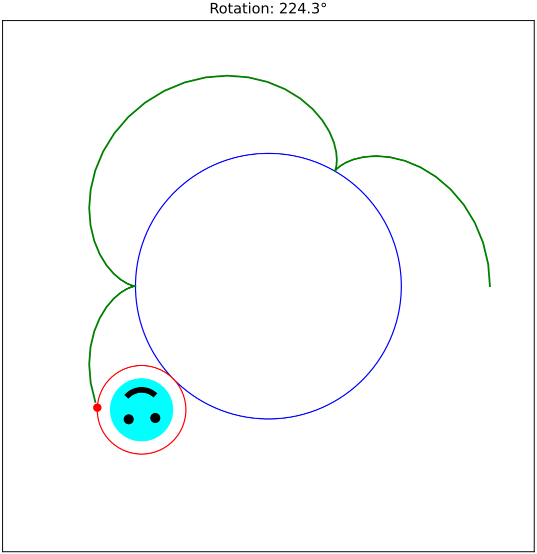
Animation of Coin Rotation Paradox



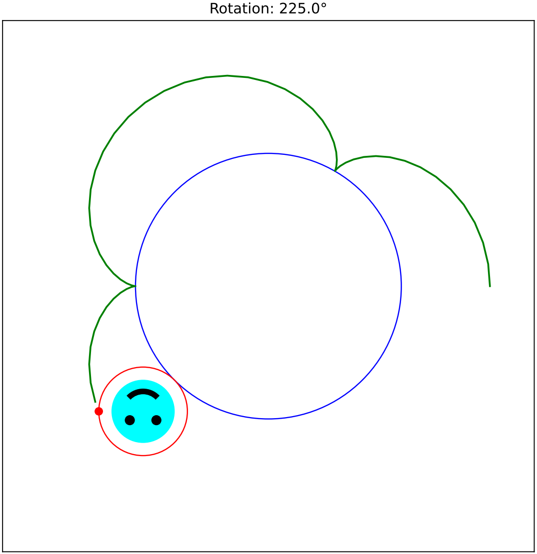
Animation of Coin Rotation Paradox



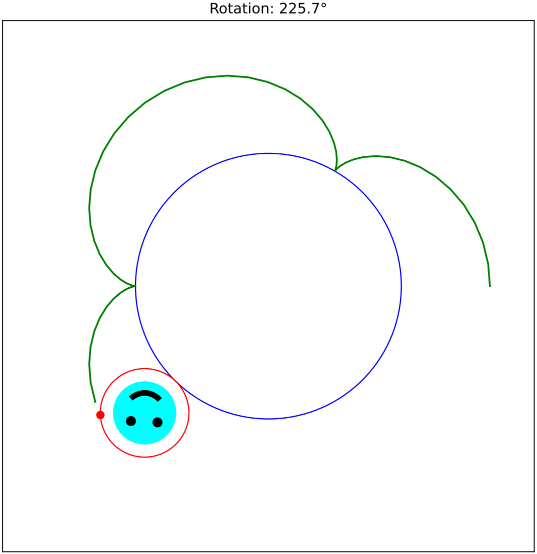
Animation of Coin Rotation Paradox



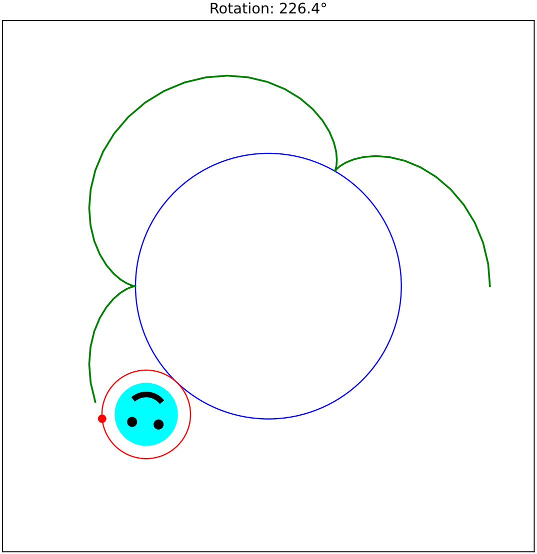
Animation of Coin Rotation Paradox



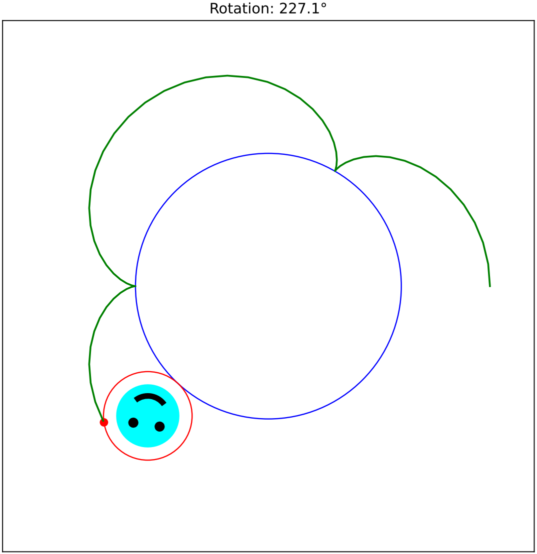
Animation of Coin Rotation Paradox



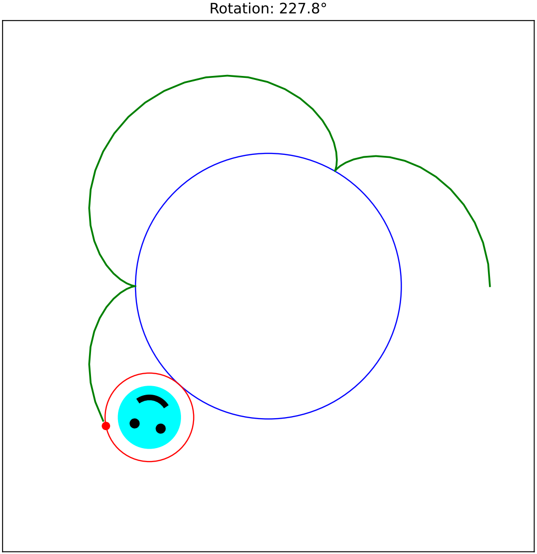
Animation of Coin Rotation Paradox



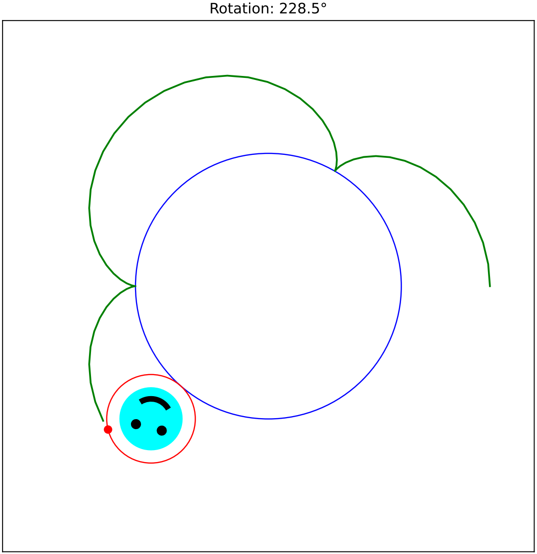
Animation of Coin Rotation Paradox



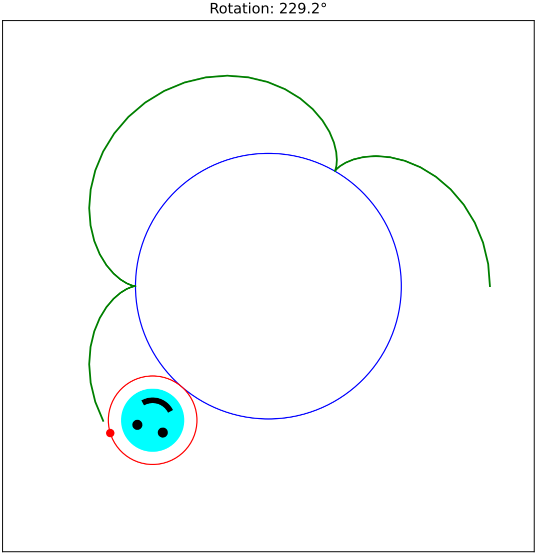
Animation of Coin Rotation Paradox



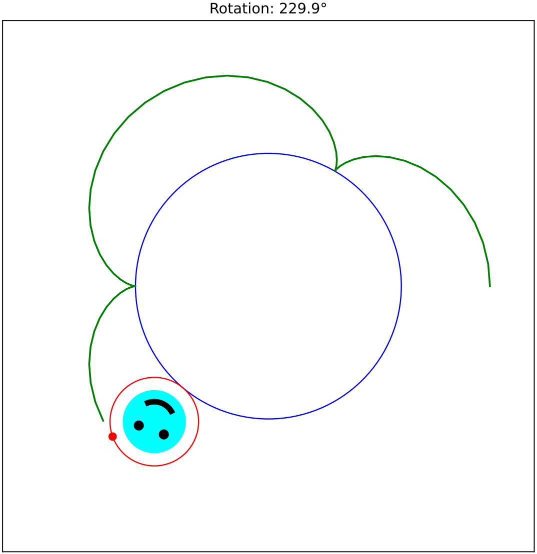
Animation of Coin Rotation Paradox



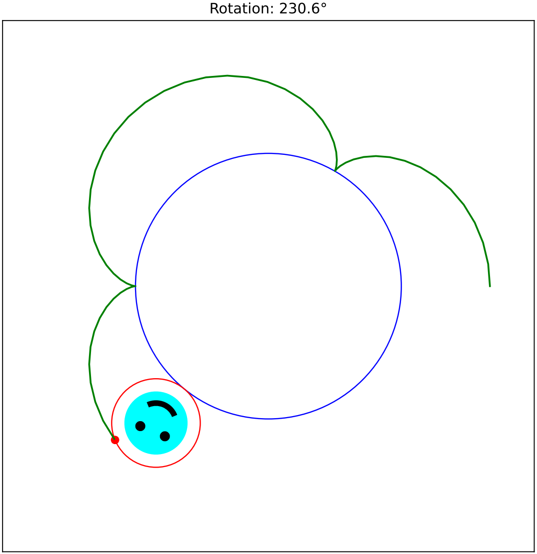
Animation of Coin Rotation Paradox



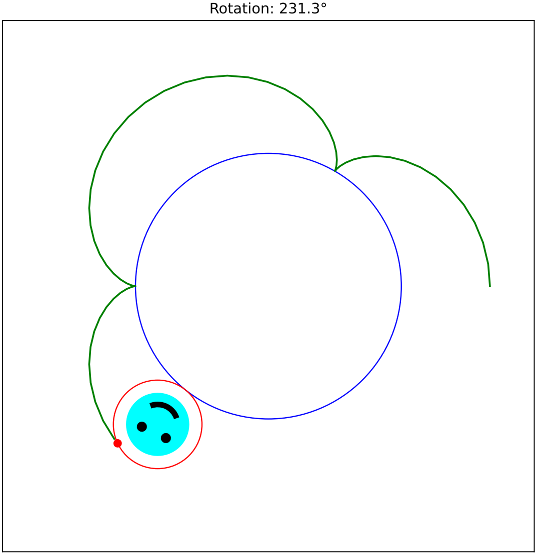
Animation of Coin Rotation Paradox



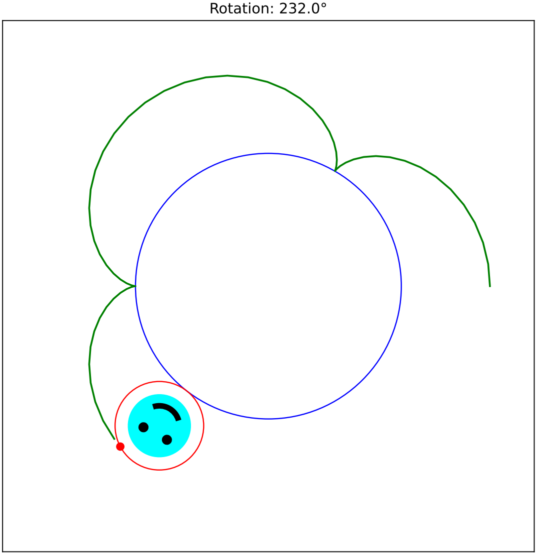
Animation of Coin Rotation Paradox



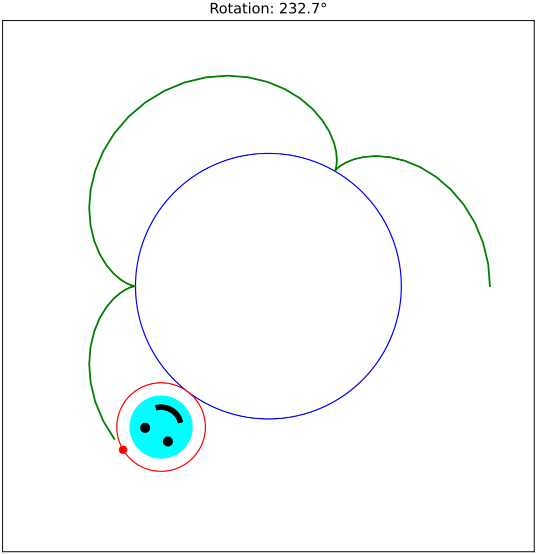
Animation of Coin Rotation Paradox



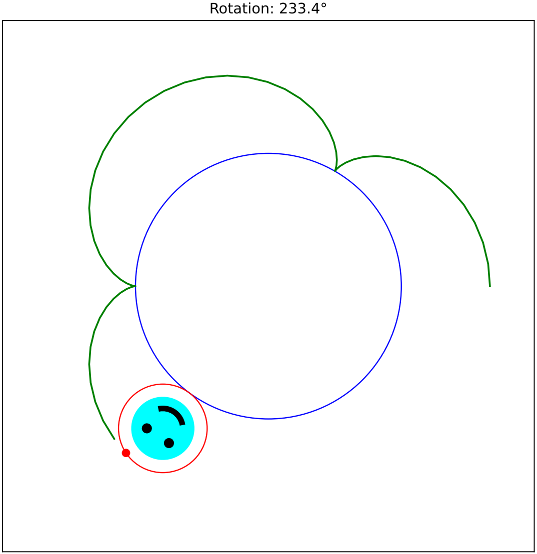
Animation of Coin Rotation Paradox



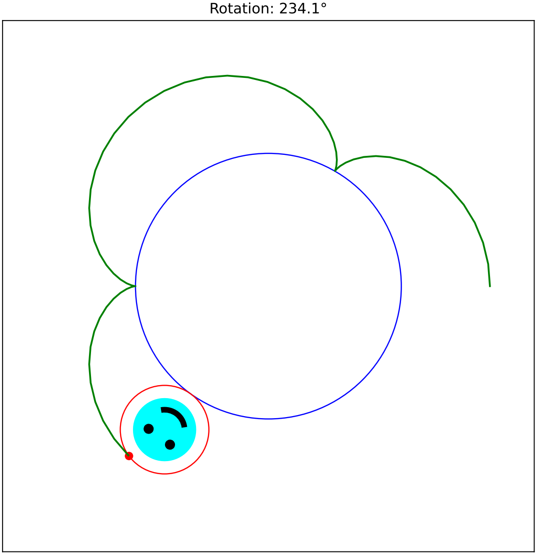
Animation of Coin Rotation Paradox



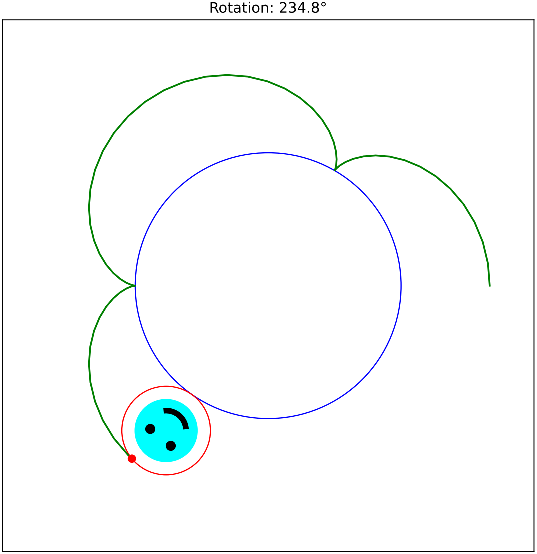
Animation of Coin Rotation Paradox



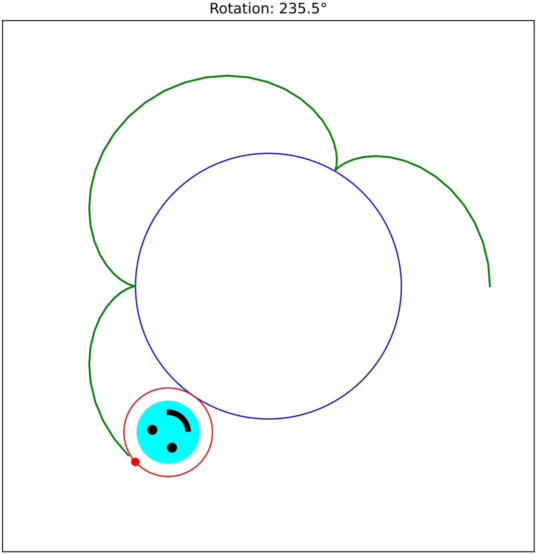
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

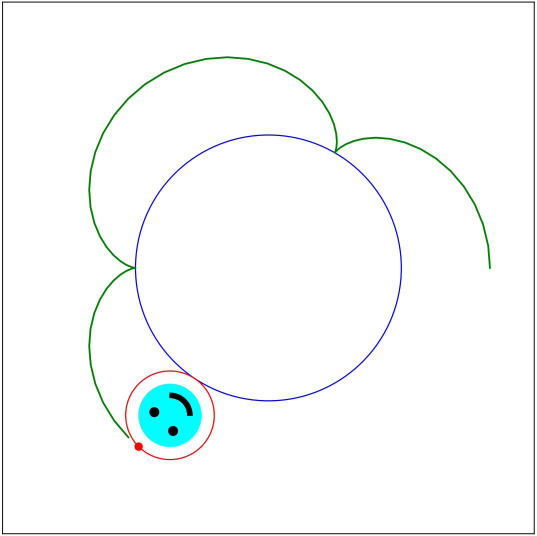


Animation of Coin Rotation Paradox

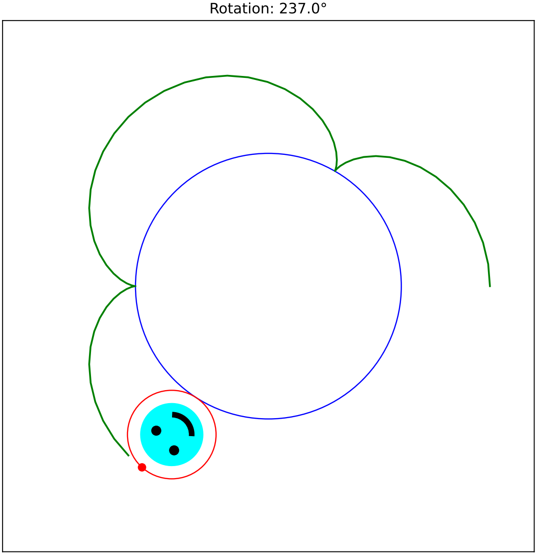


Animation of Coin Rotation Paradox

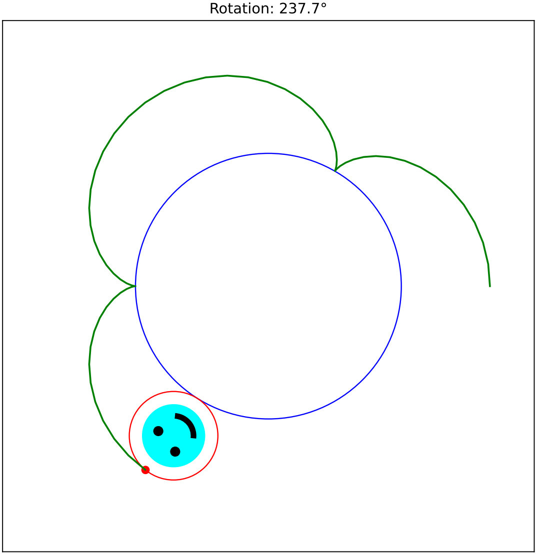
Rotation: 236.2°



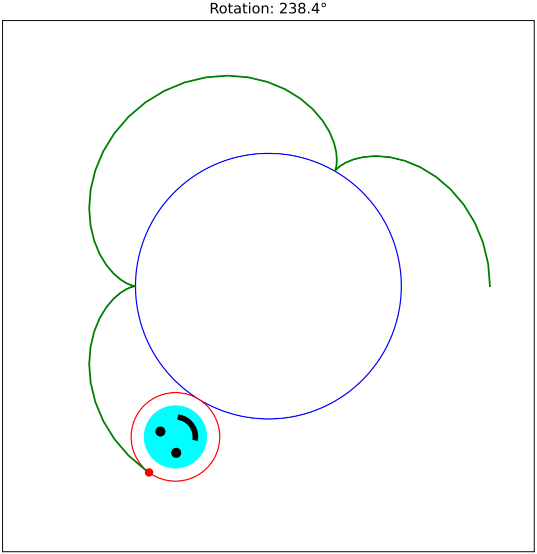
Animation of Coin Rotation Paradox



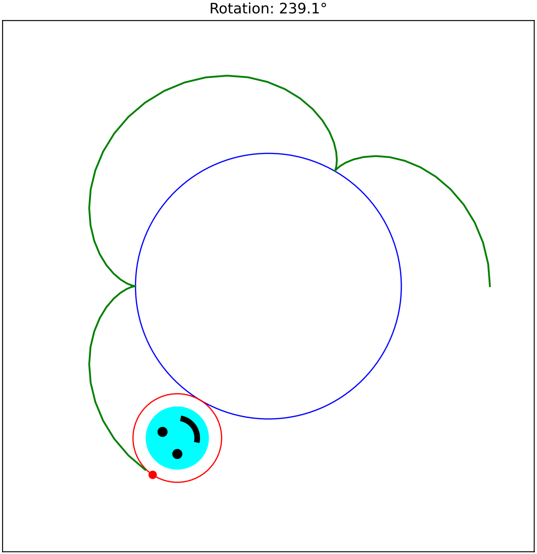
Animation of Coin Rotation Paradox



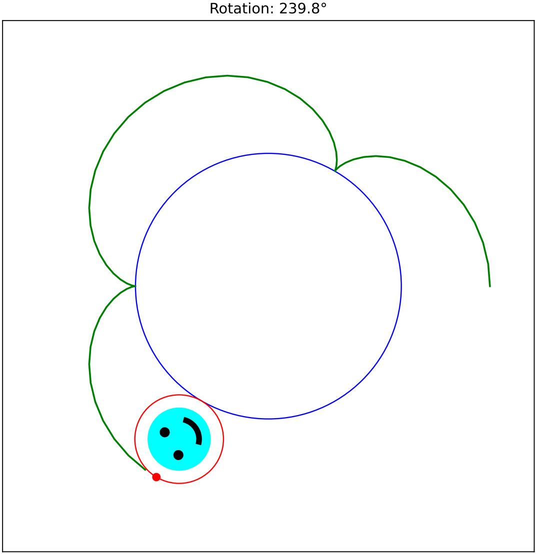
Animation of Coin Rotation Paradox



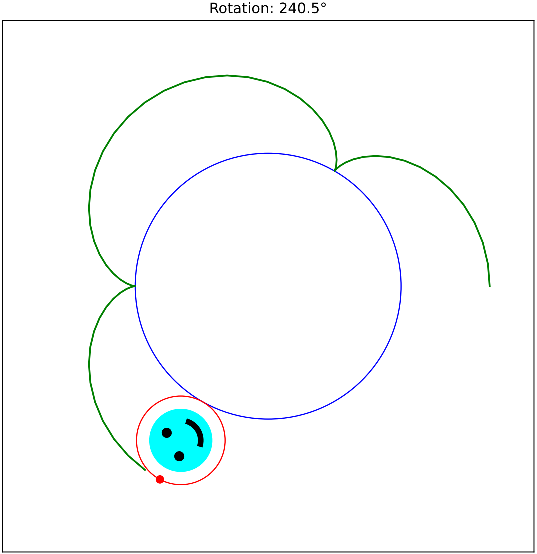
Animation of Coin Rotation Paradox



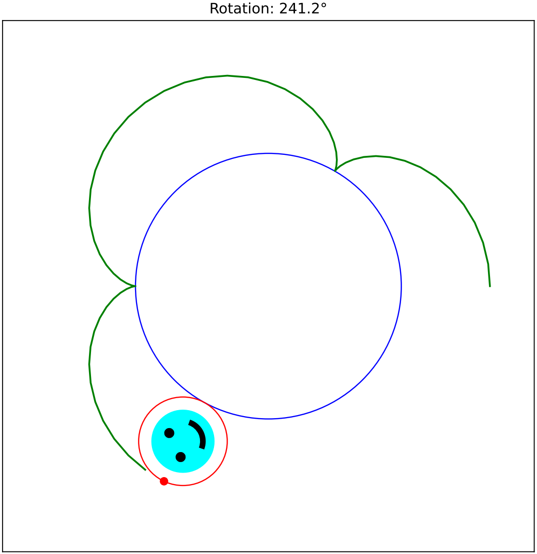
Animation of Coin Rotation Paradox



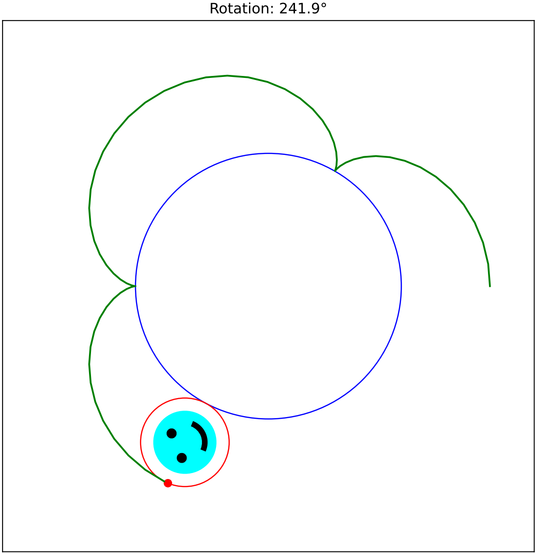
Animation of Coin Rotation Paradox



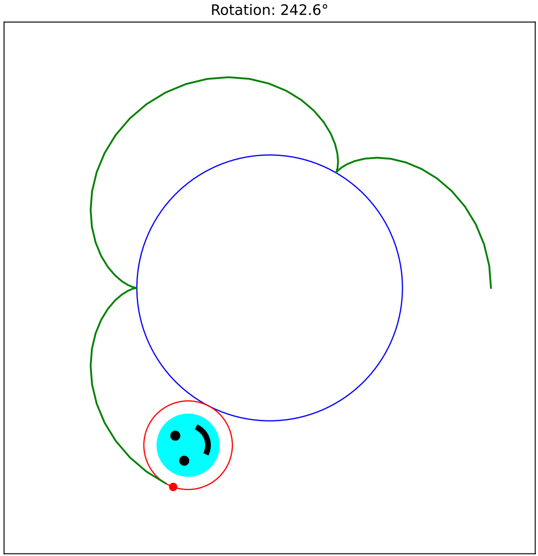
Animation of Coin Rotation Paradox



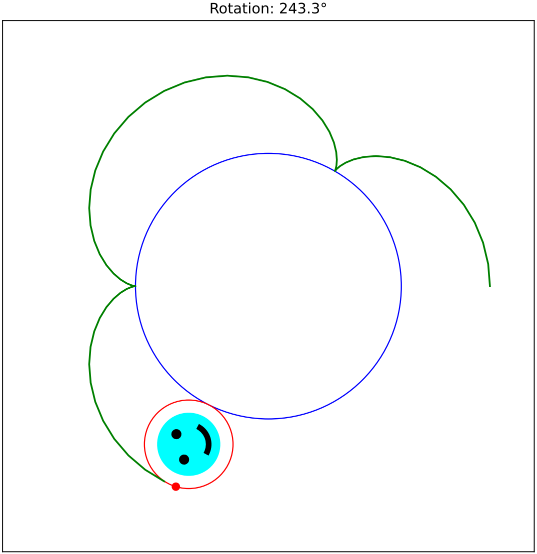
Animation of Coin Rotation Paradox



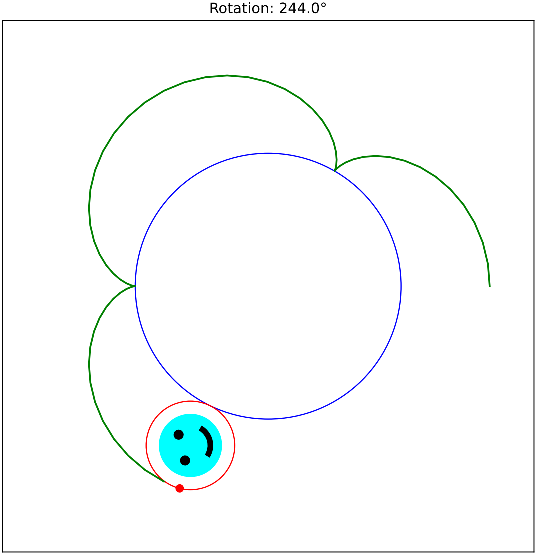
Animation of Coin Rotation Paradox



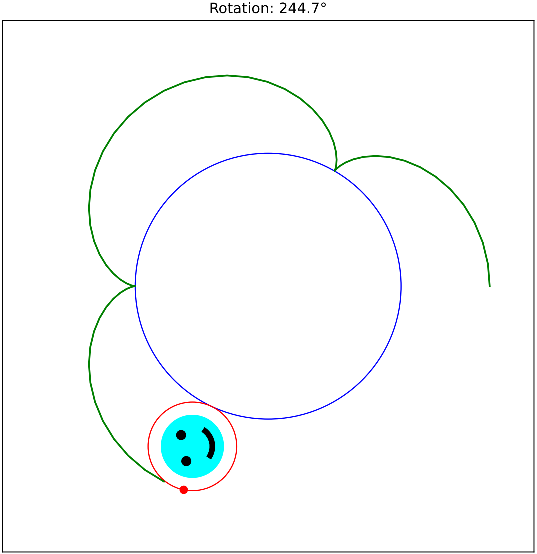
Animation of Coin Rotation Paradox



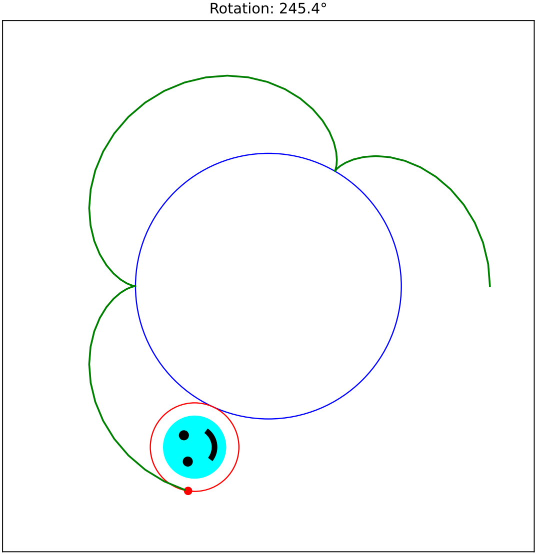
Animation of Coin Rotation Paradox



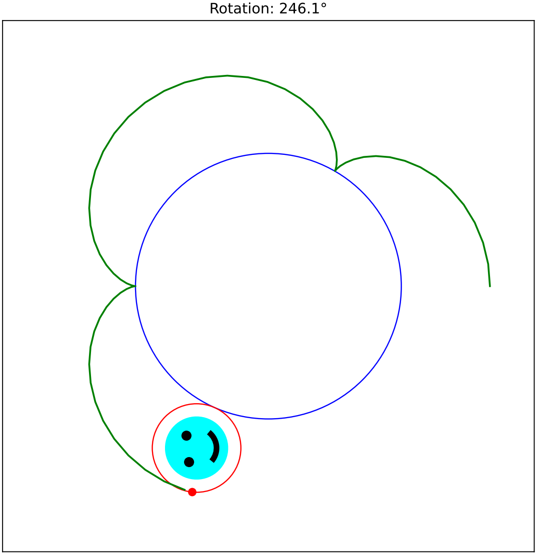
Animation of Coin Rotation Paradox



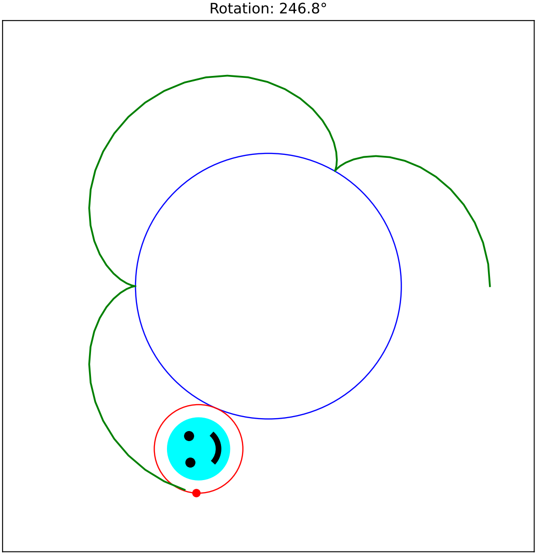
Animation of Coin Rotation Paradox



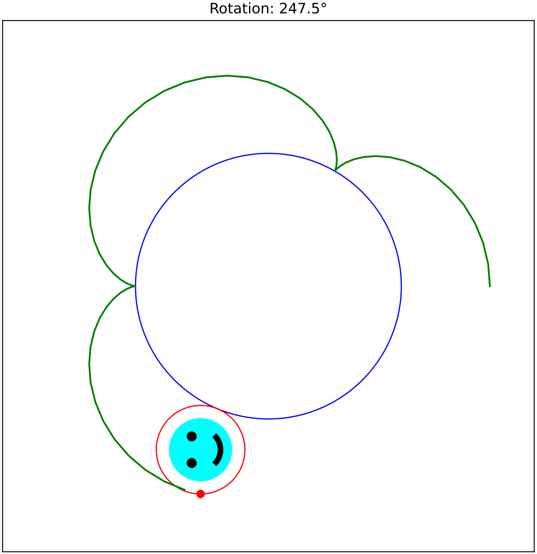
Animation of Coin Rotation Paradox



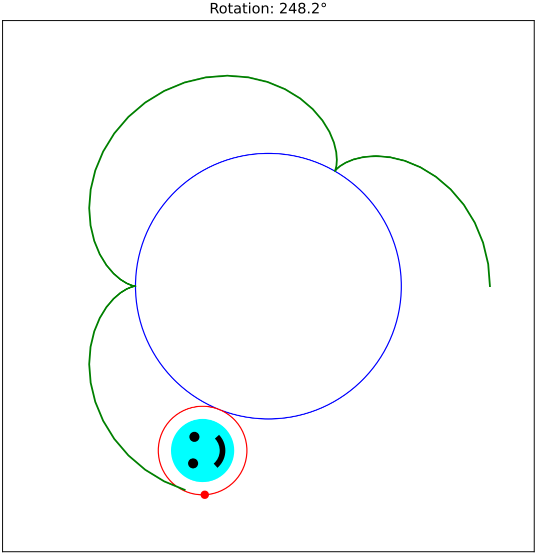
Animation of Coin Rotation Paradox



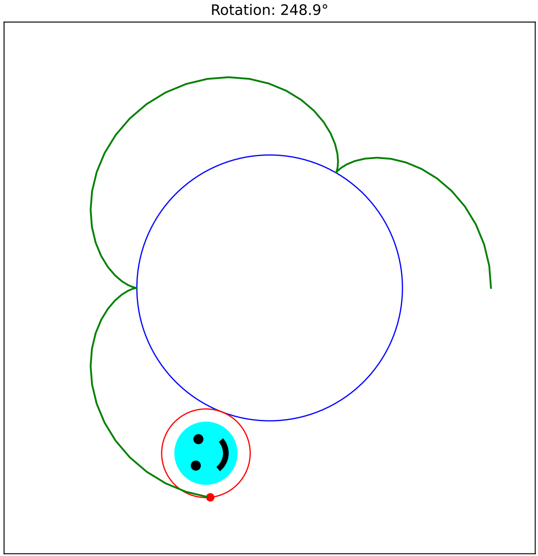
Animation of Coin Rotation Paradox



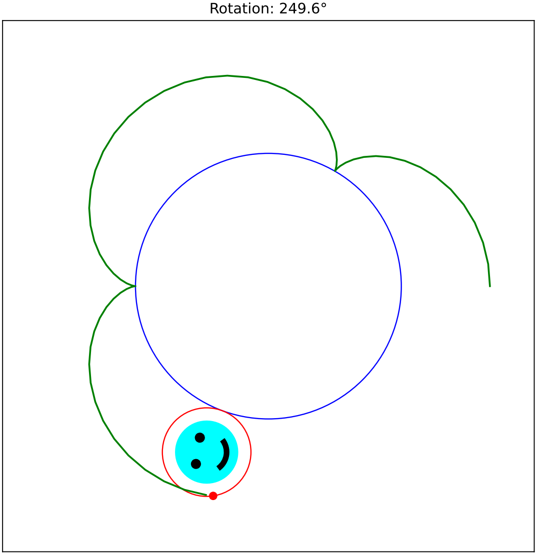
Animation of Coin Rotation Paradox



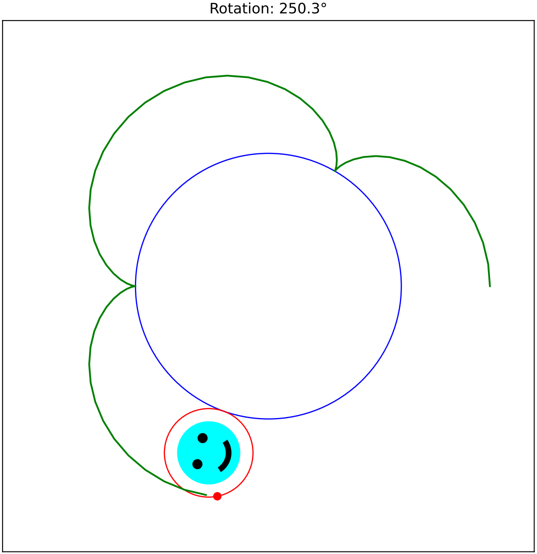
Animation of Coin Rotation Paradox



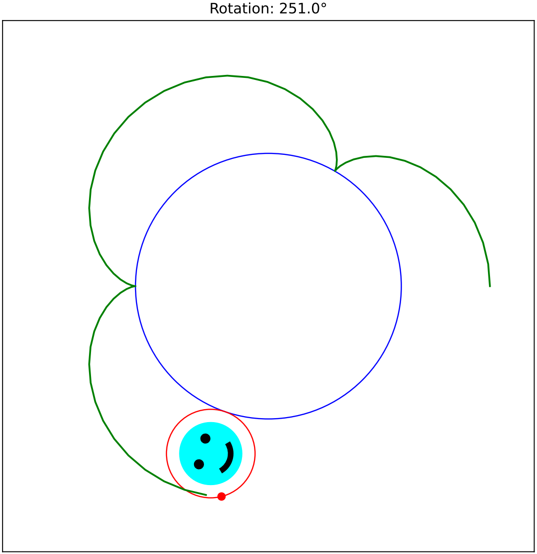
Animation of Coin Rotation Paradox



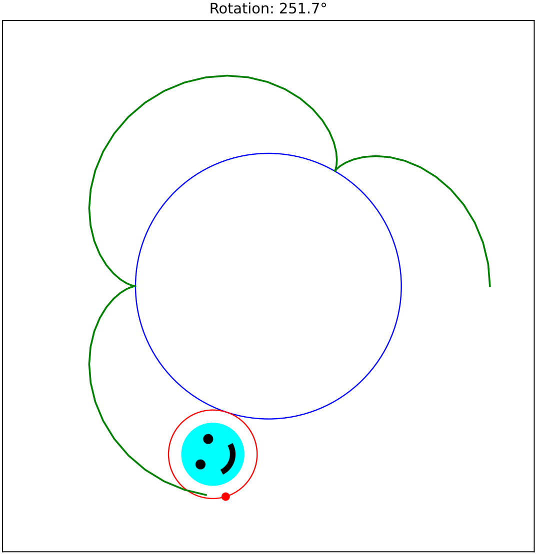
Animation of Coin Rotation Paradox



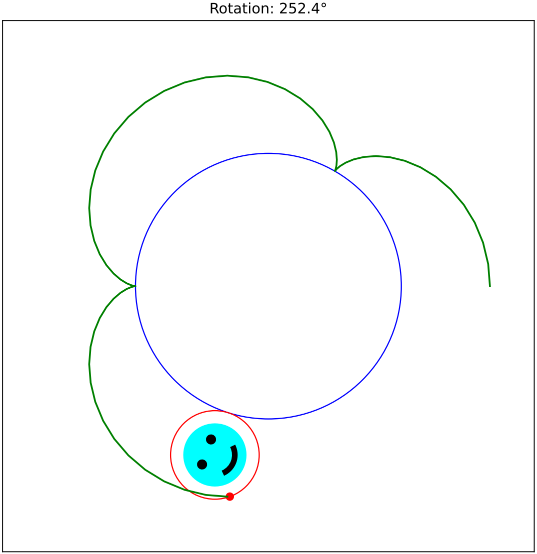
Animation of Coin Rotation Paradox



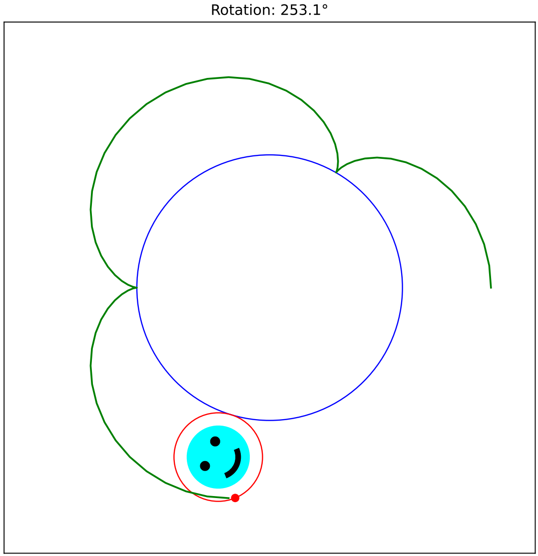
Animation of Coin Rotation Paradox



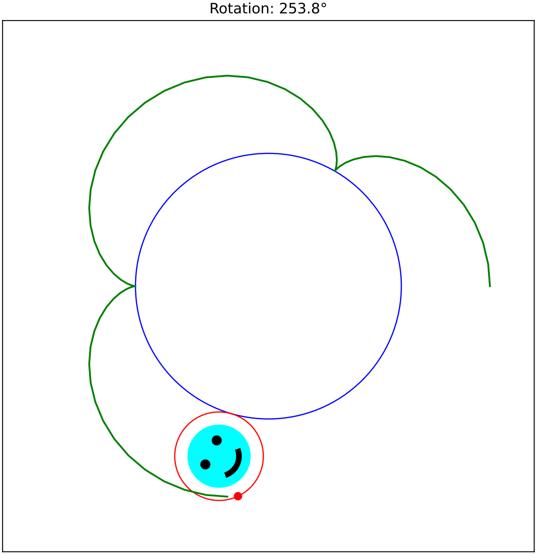
Animation of Coin Rotation Paradox



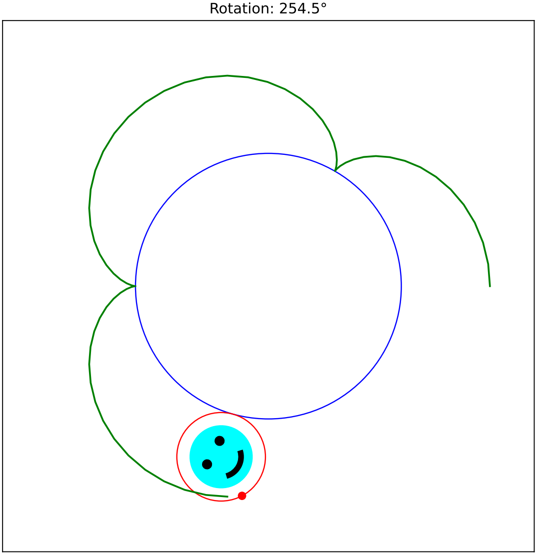
Animation of Coin Rotation Paradox



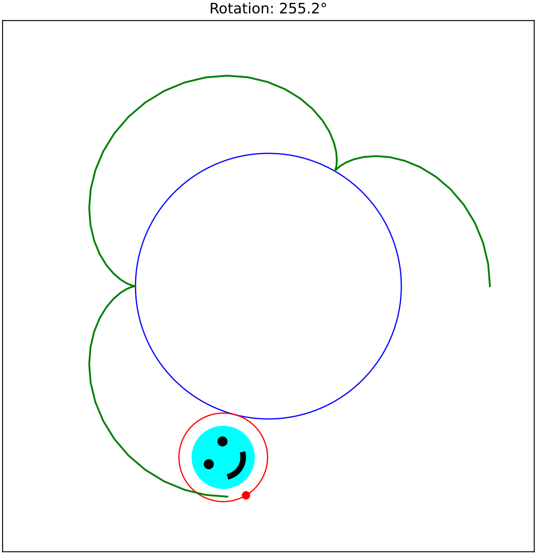
Animation of Coin Rotation Paradox



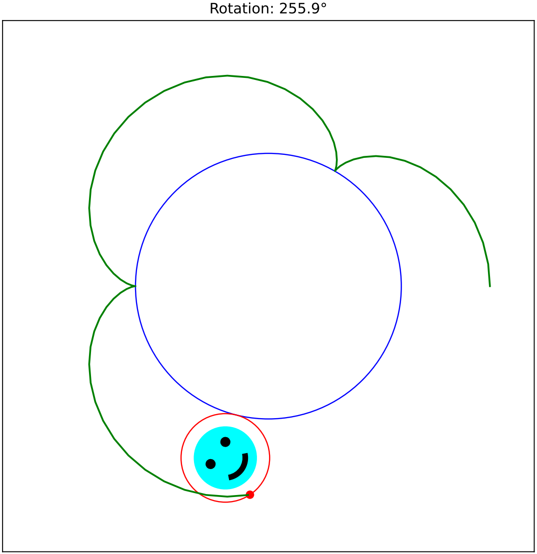
Animation of Coin Rotation Paradox



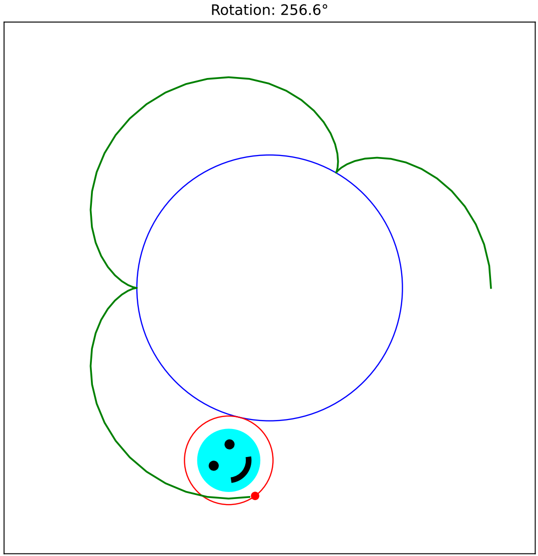
Animation of Coin Rotation Paradox



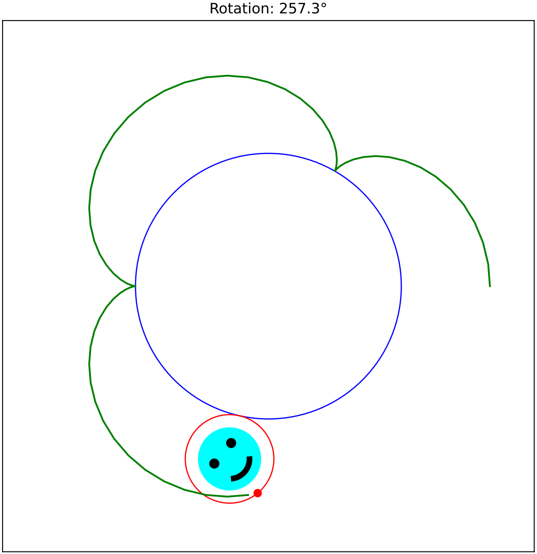
Animation of Coin Rotation Paradox



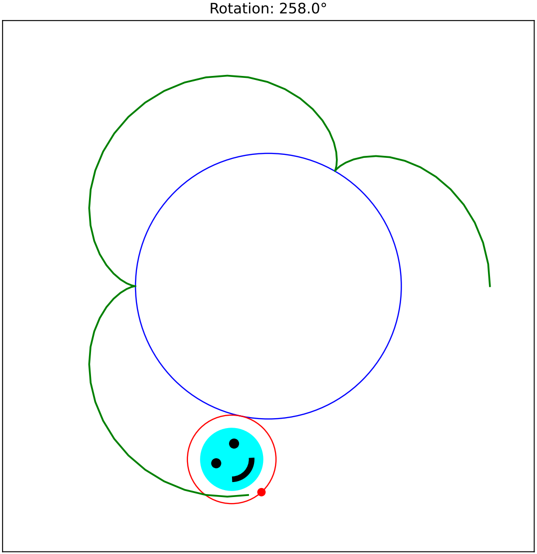
Animation of Coin Rotation Paradox



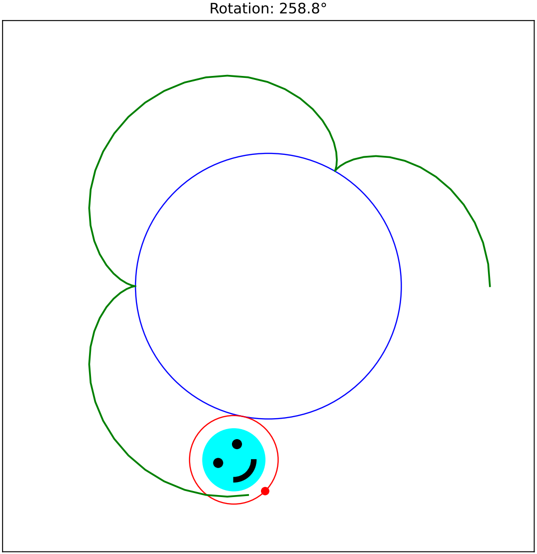
Animation of Coin Rotation Paradox



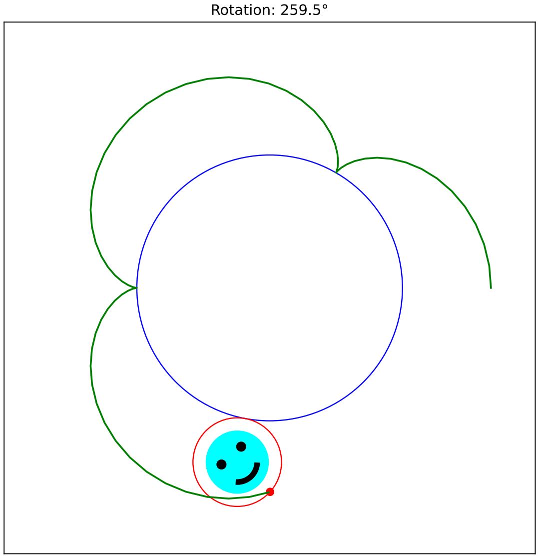
Animation of Coin Rotation Paradox



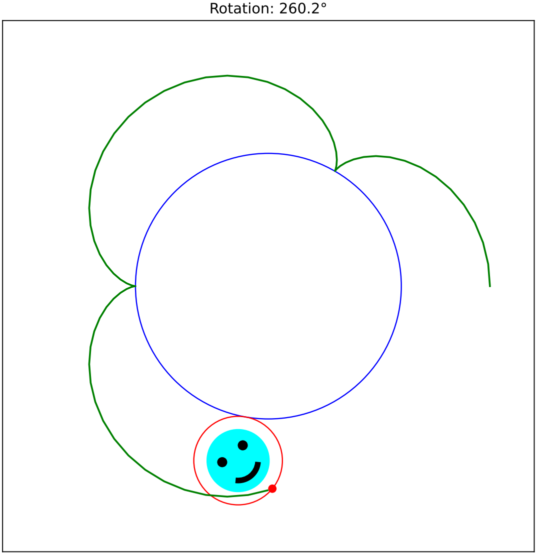
Animation of Coin Rotation Paradox



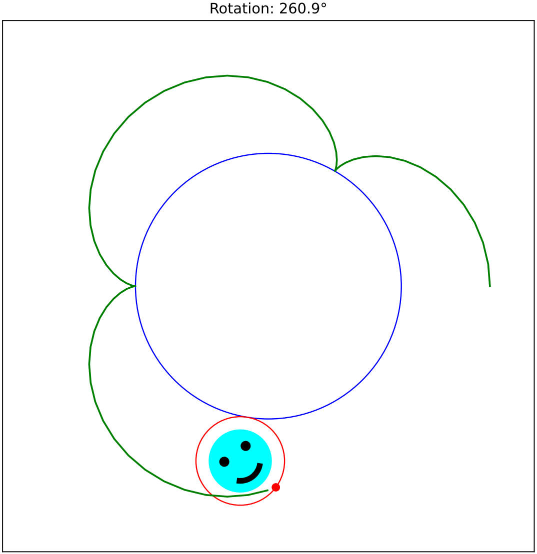
Animation of Coin Rotation Paradox



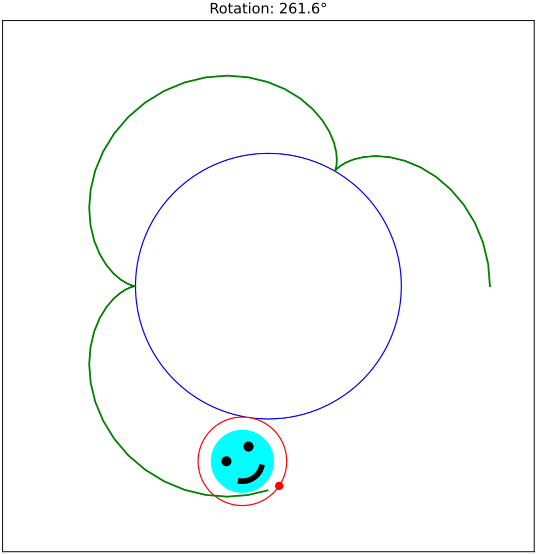
Animation of Coin Rotation Paradox



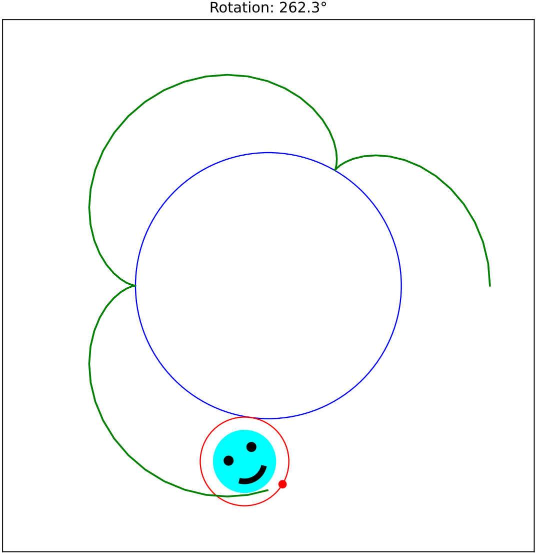
Animation of Coin Rotation Paradox



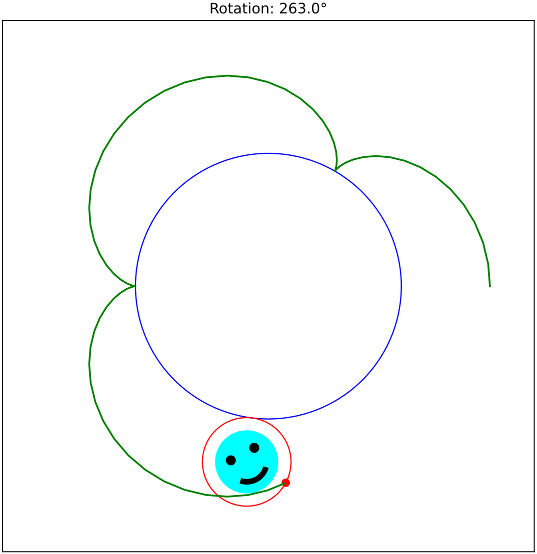
Animation of Coin Rotation Paradox



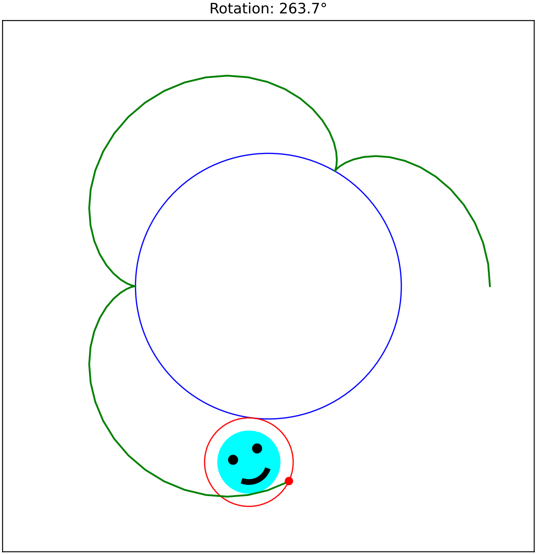
Animation of Coin Rotation Paradox



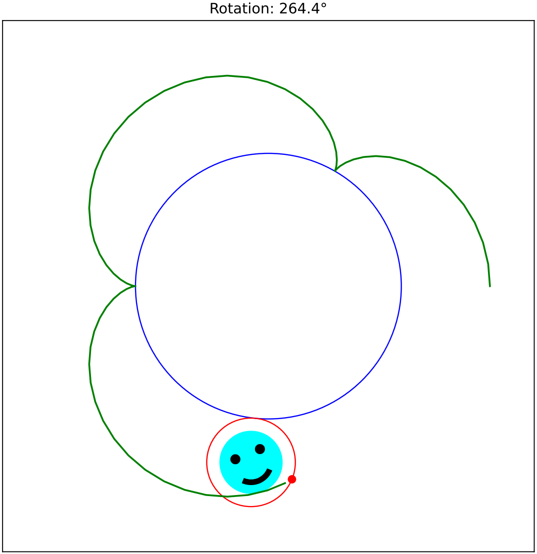
Animation of Coin Rotation Paradox



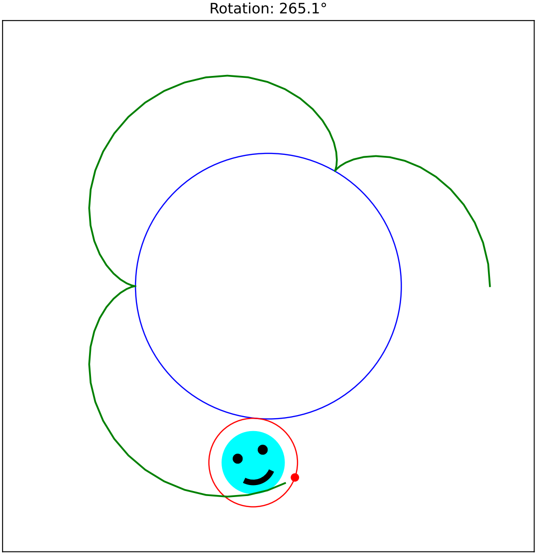
Animation of Coin Rotation Paradox



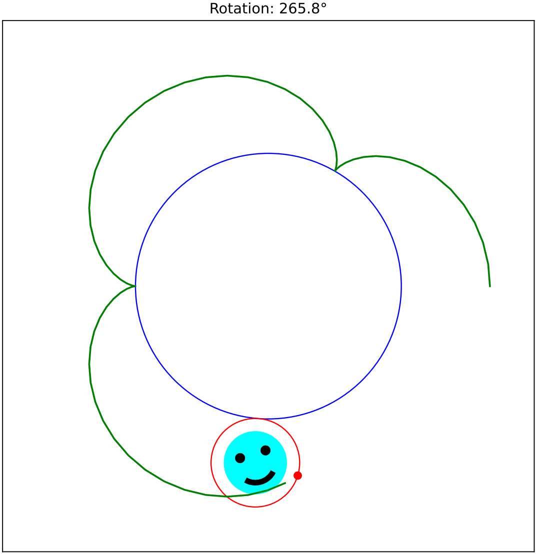
Animation of Coin Rotation Paradox



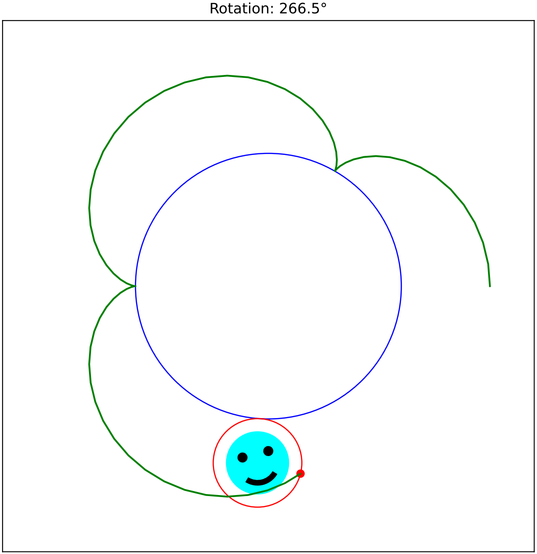
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

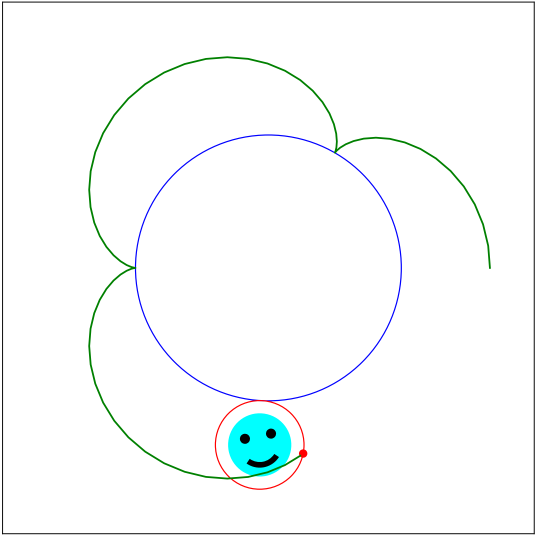


Animation of Coin Rotation Paradox

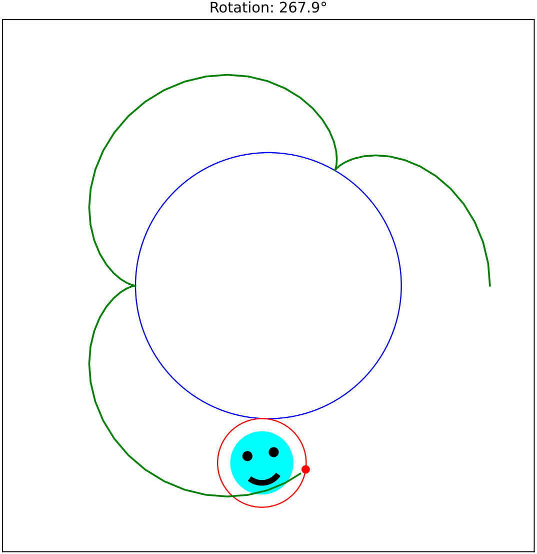


Animation of Coin Rotation Paradox

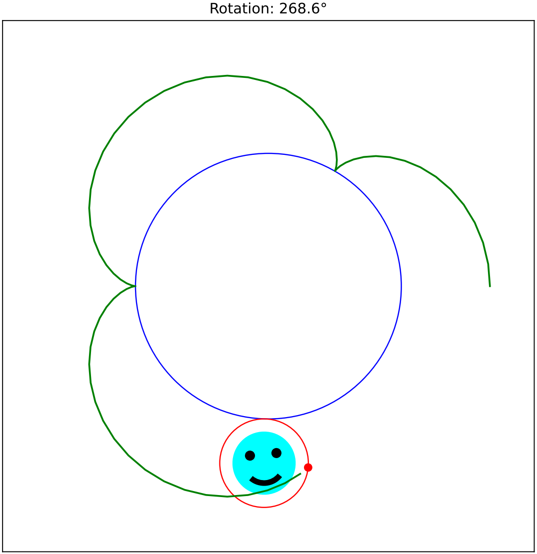
Rotation: 267.2°



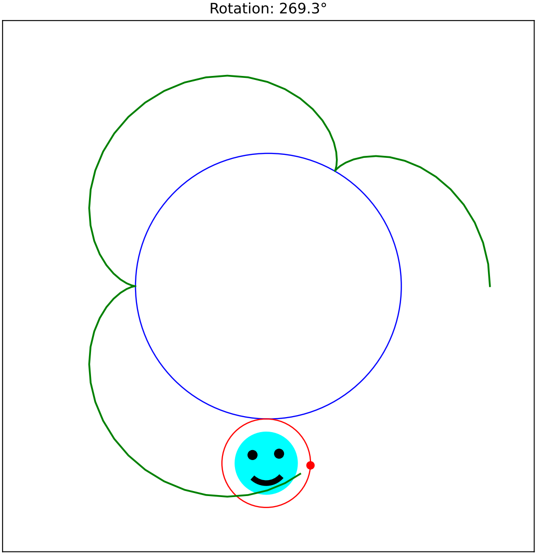
Animation of Coin Rotation Paradox



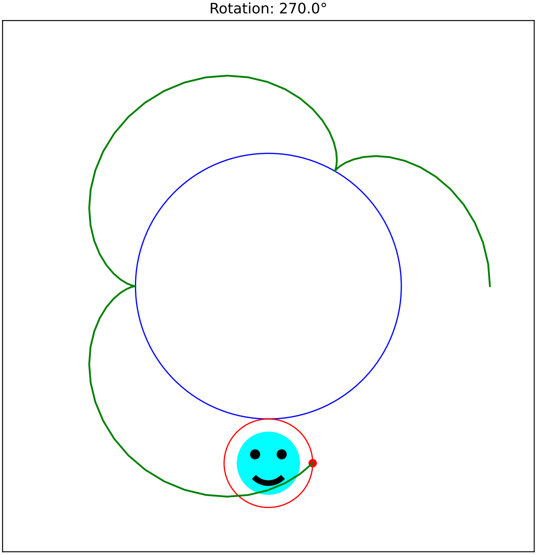
Animation of Coin Rotation Paradox



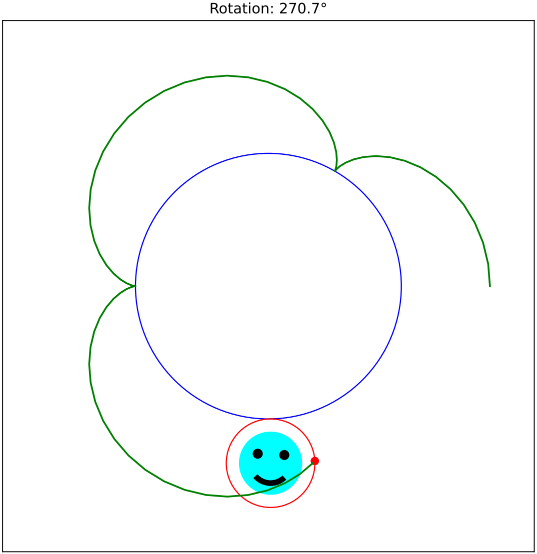
Animation of Coin Rotation Paradox



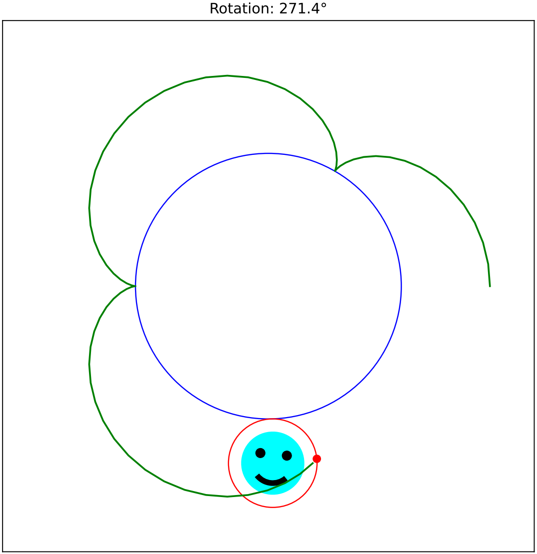
Animation of Coin Rotation Paradox



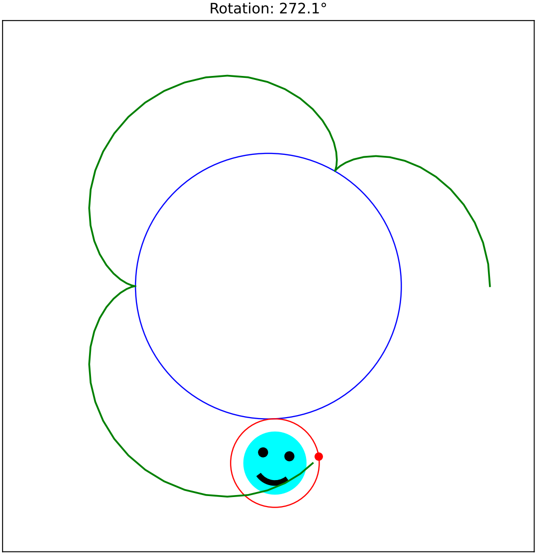
Animation of Coin Rotation Paradox



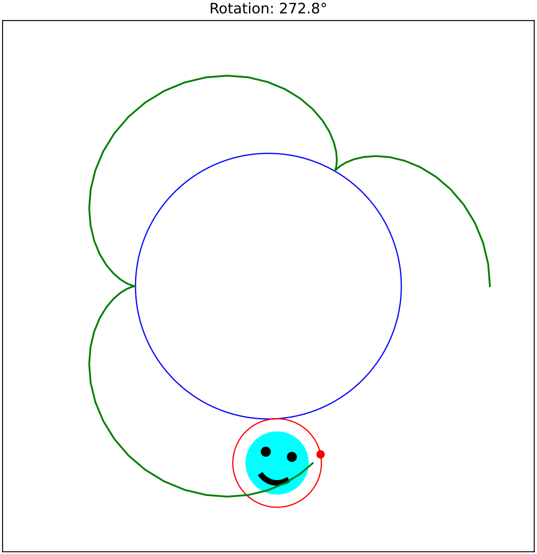
Animation of Coin Rotation Paradox



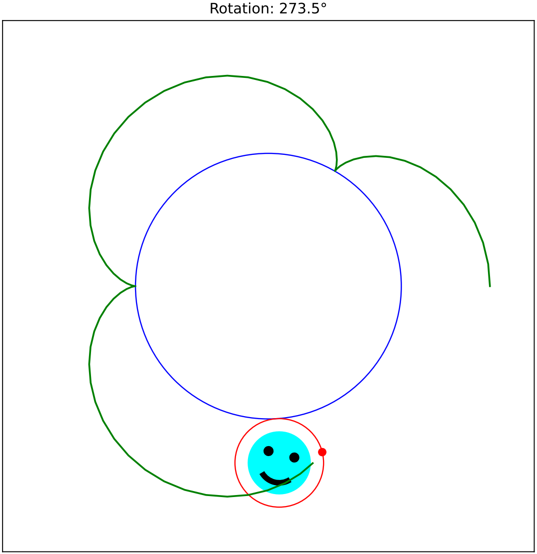
Animation of Coin Rotation Paradox



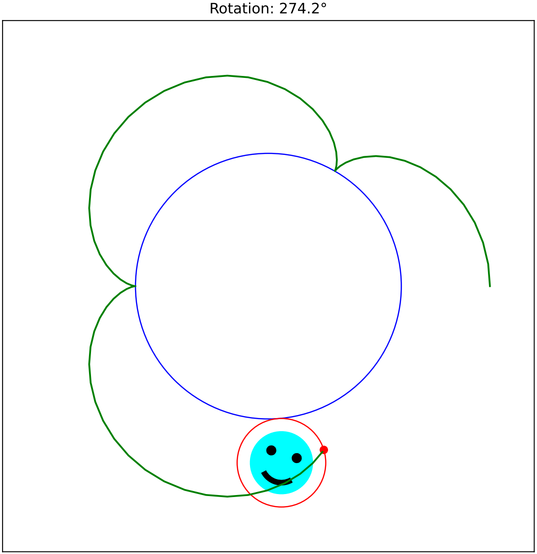
Animation of Coin Rotation Paradox



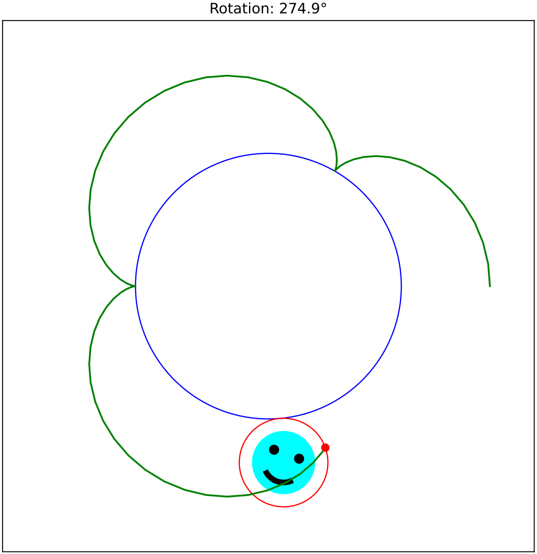
Animation of Coin Rotation Paradox



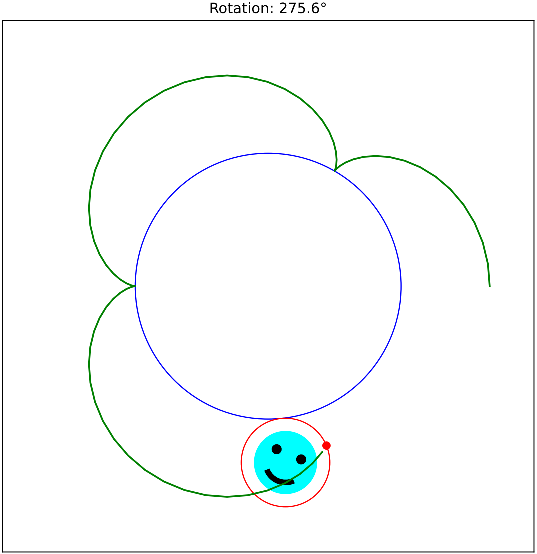
Animation of Coin Rotation Paradox



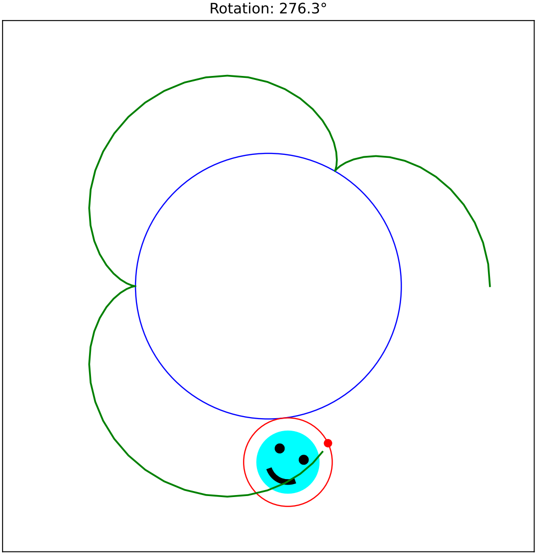
Animation of Coin Rotation Paradox



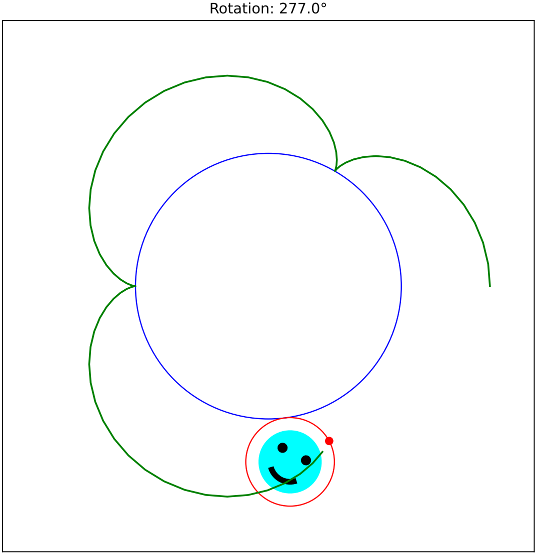
Animation of Coin Rotation Paradox



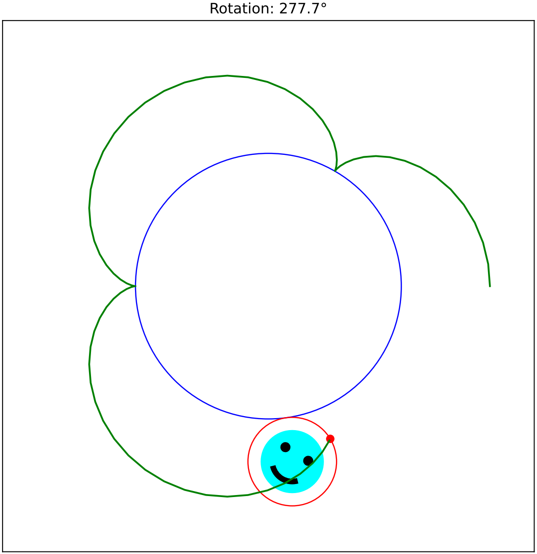
Animation of Coin Rotation Paradox



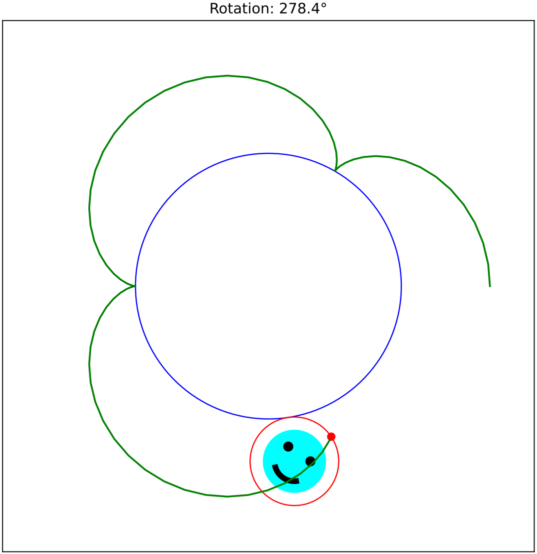
Animation of Coin Rotation Paradox



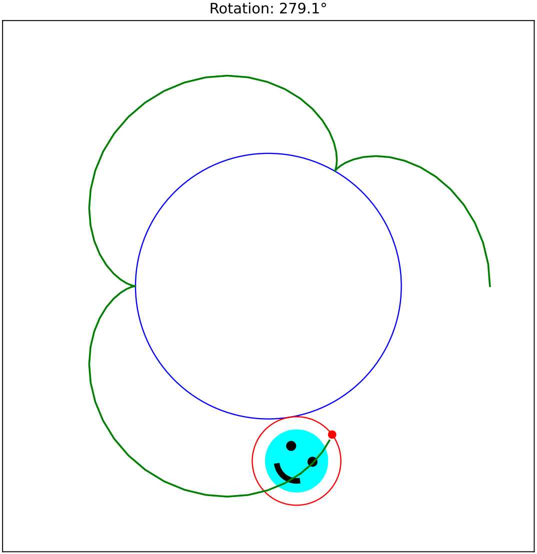
Animation of Coin Rotation Paradox



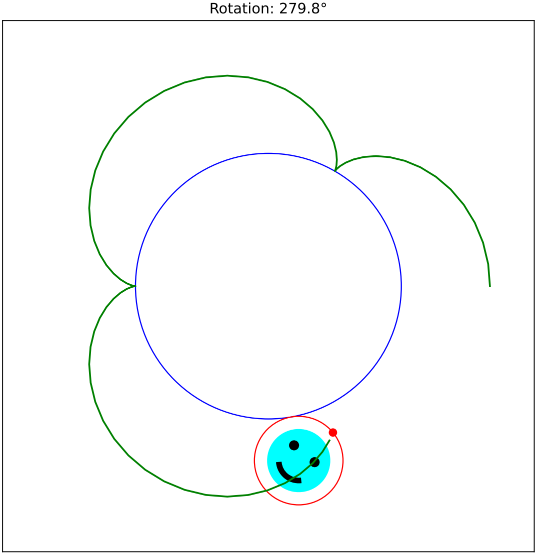
Animation of Coin Rotation Paradox



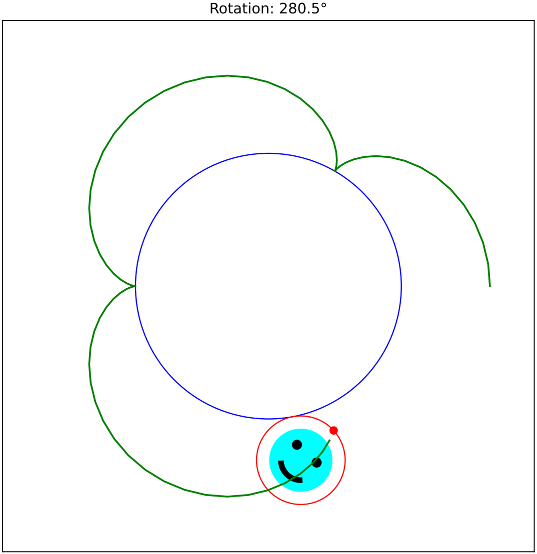
Animation of Coin Rotation Paradox



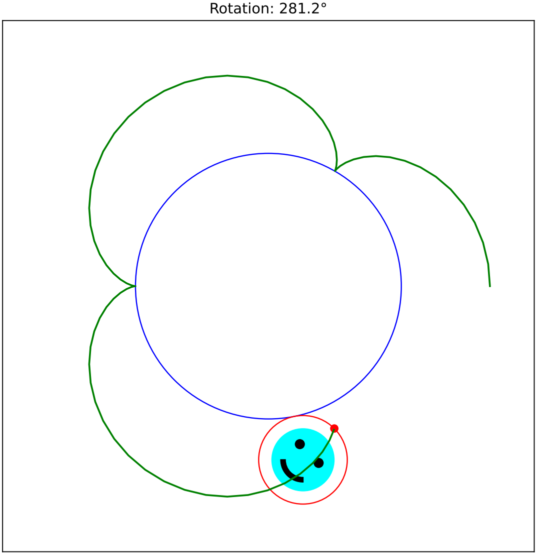
Animation of Coin Rotation Paradox



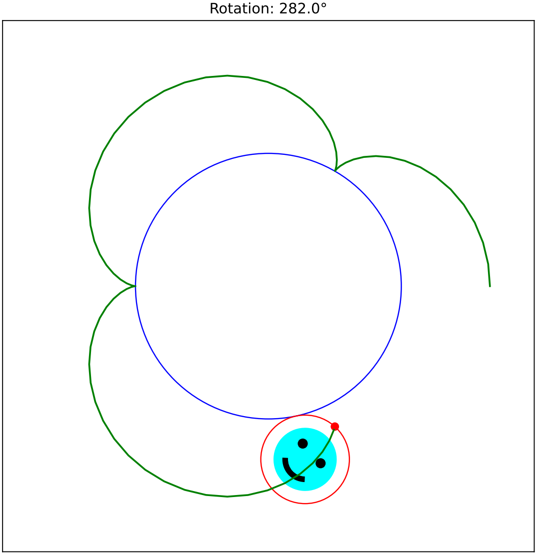
Animation of Coin Rotation Paradox



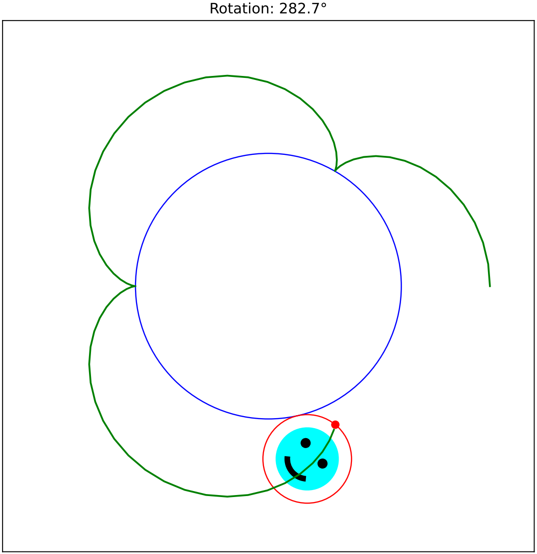
Animation of Coin Rotation Paradox



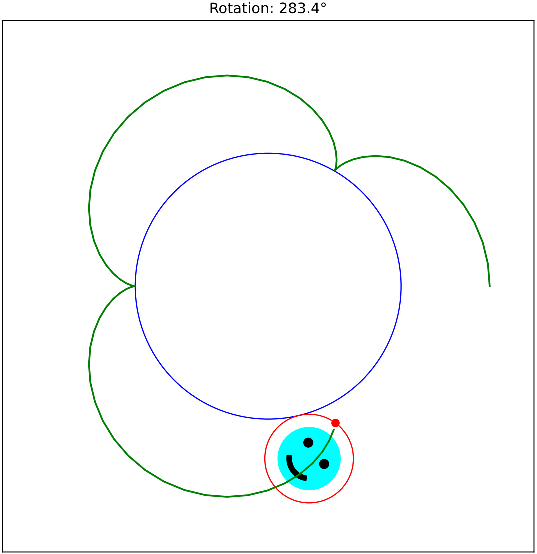
Animation of Coin Rotation Paradox



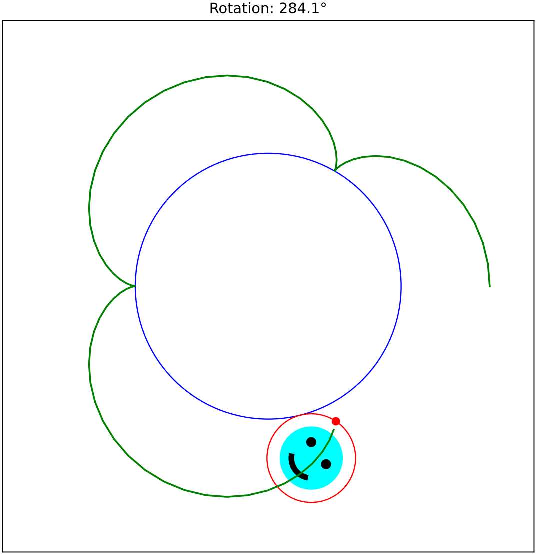
Animation of Coin Rotation Paradox



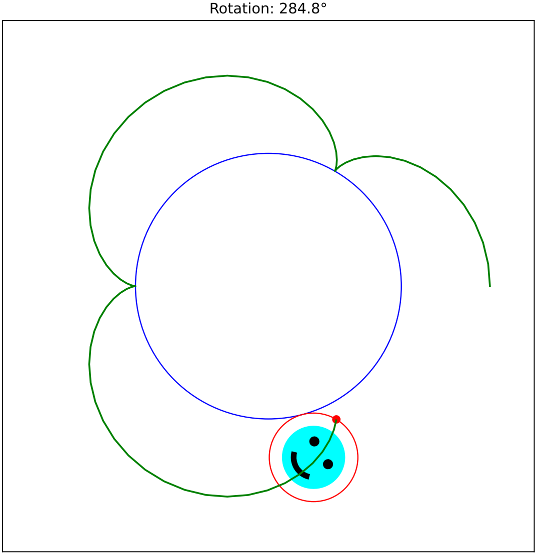
Animation of Coin Rotation Paradox



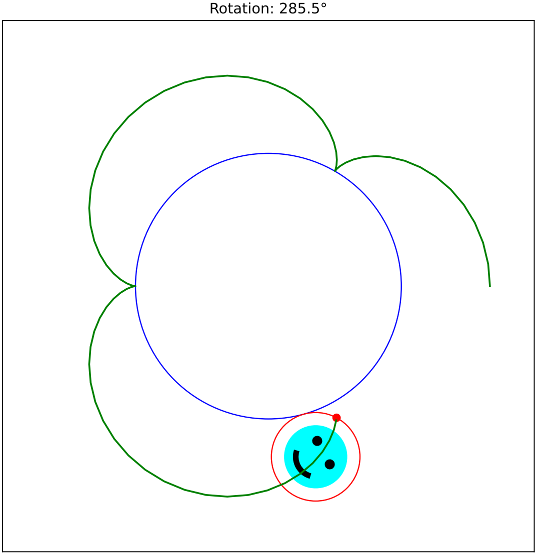
Animation of Coin Rotation Paradox



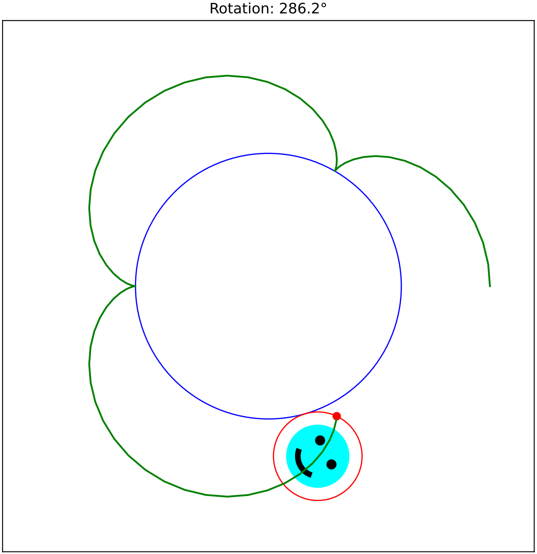
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

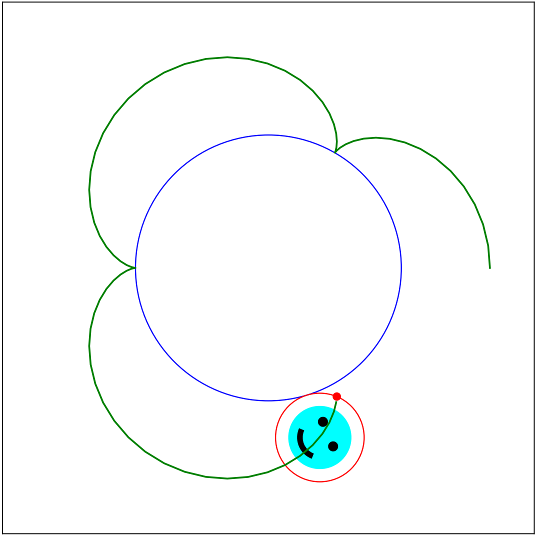


Animation of Coin Rotation Paradox

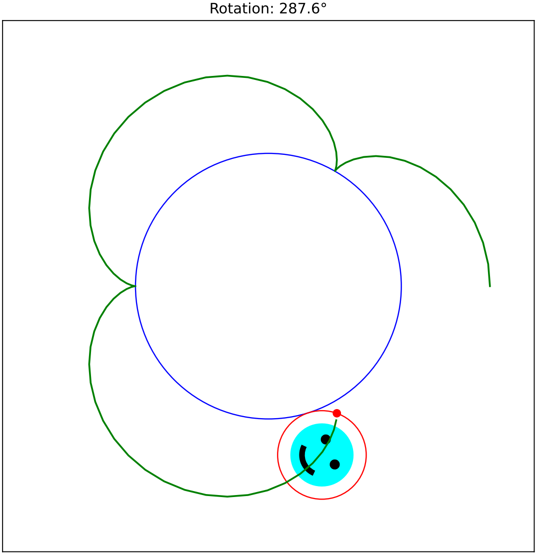


Animation of Coin Rotation Paradox

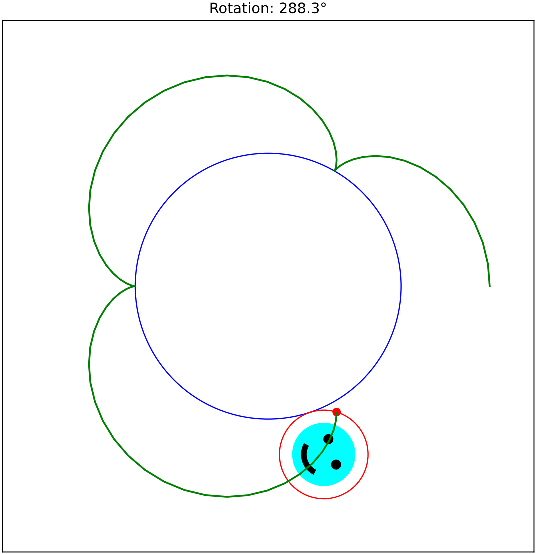
Rotation: 286.9°



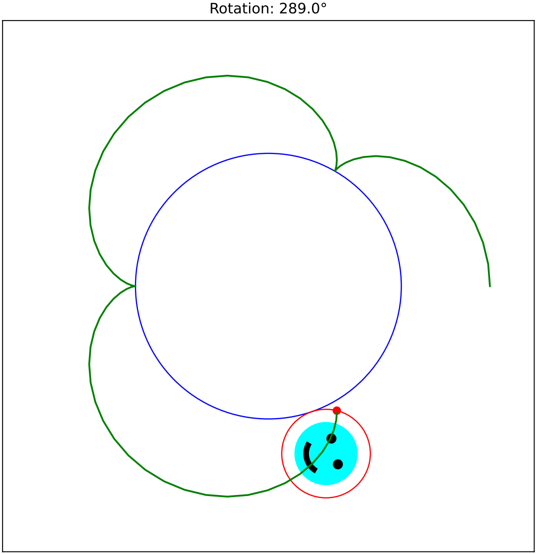
Animation of Coin Rotation Paradox



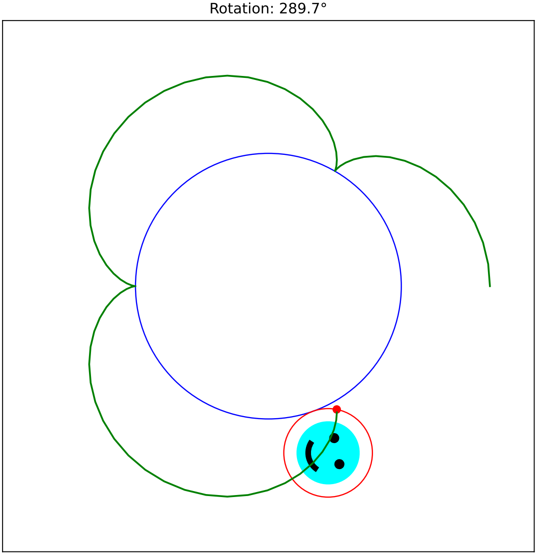
Animation of Coin Rotation Paradox



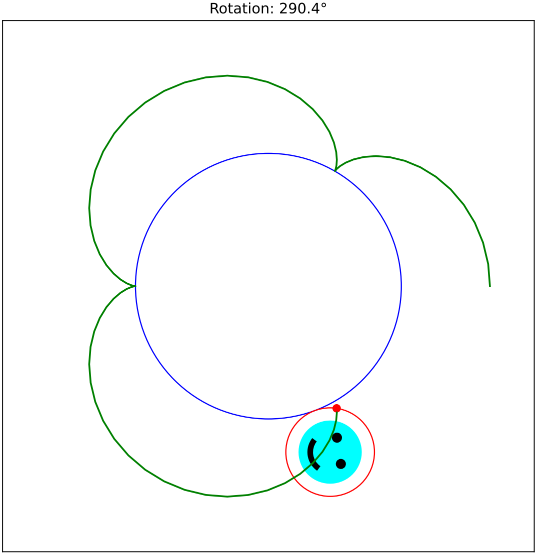
Animation of Coin Rotation Paradox



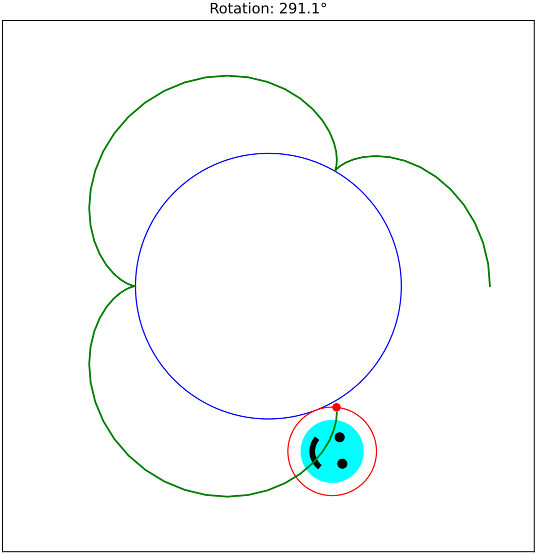
Animation of Coin Rotation Paradox



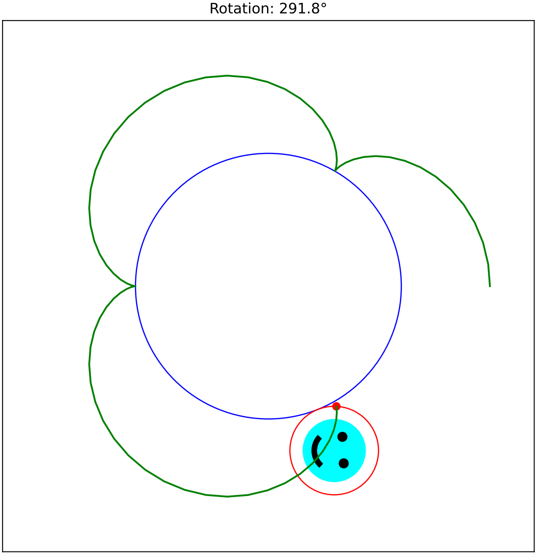
Animation of Coin Rotation Paradox



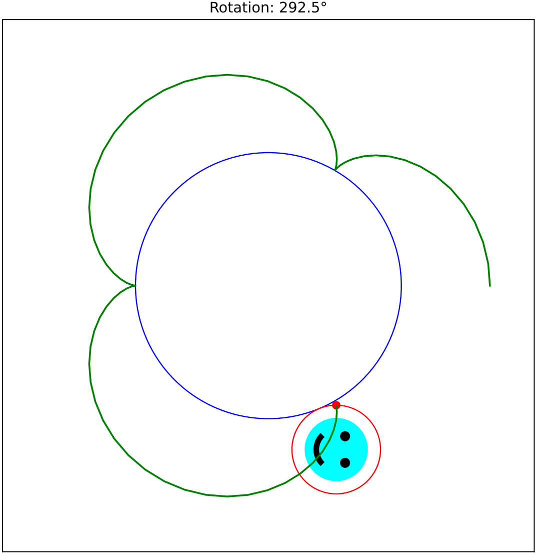
Animation of Coin Rotation Paradox



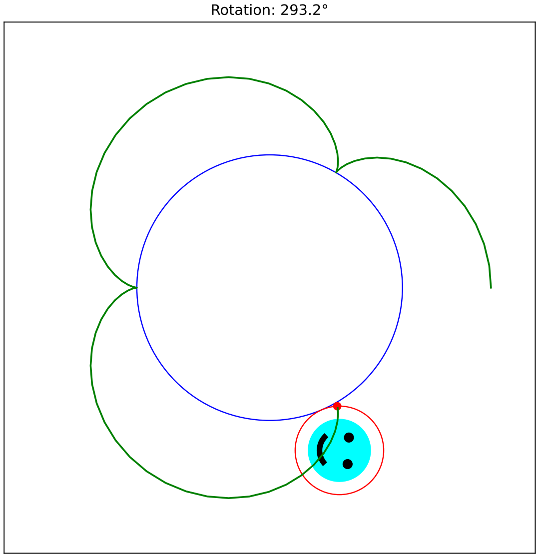
Animation of Coin Rotation Paradox



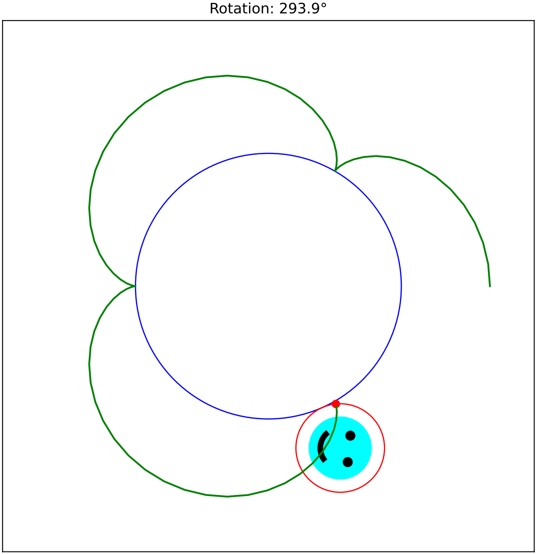
Animation of Coin Rotation Paradox



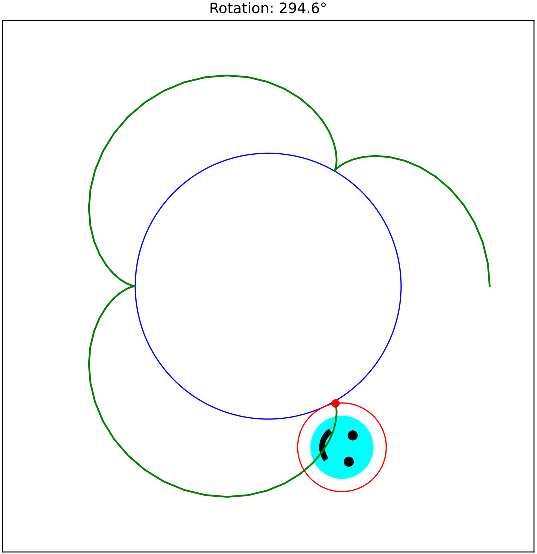
Animation of Coin Rotation Paradox



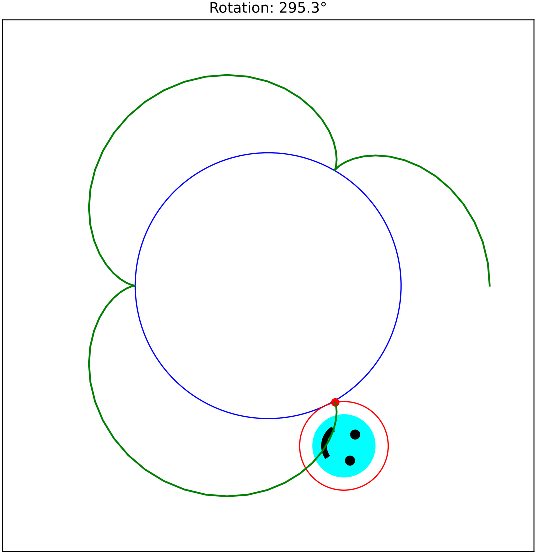
Animation of Coin Rotation Paradox



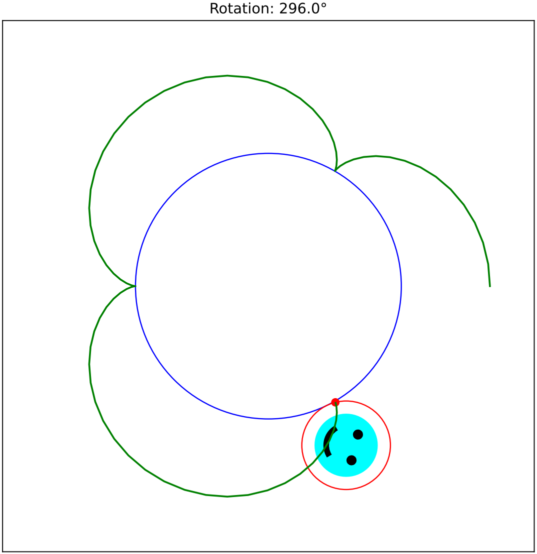
Animation of Coin Rotation Paradox



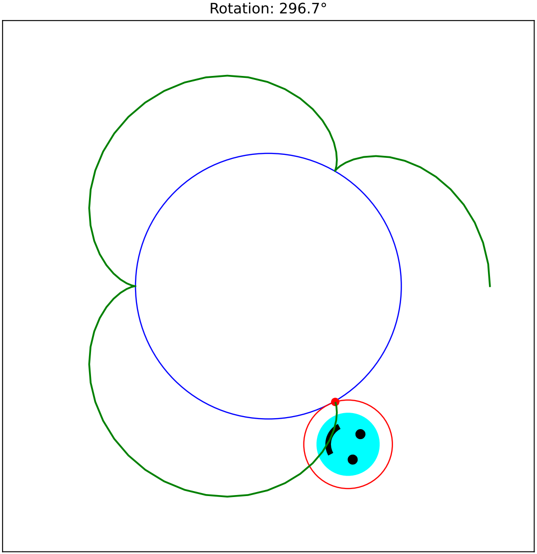
Animation of Coin Rotation Paradox



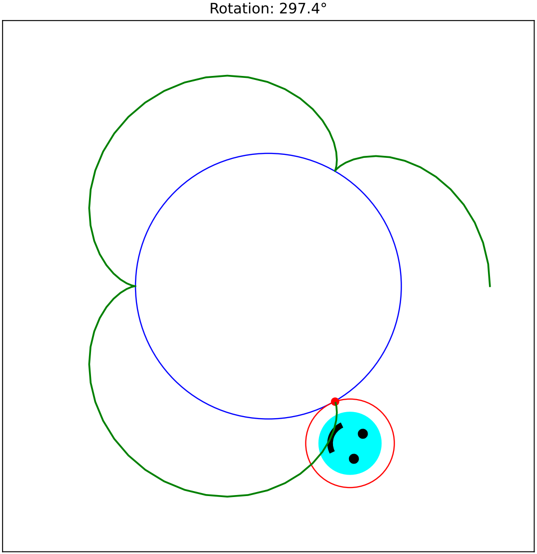
Animation of Coin Rotation Paradox



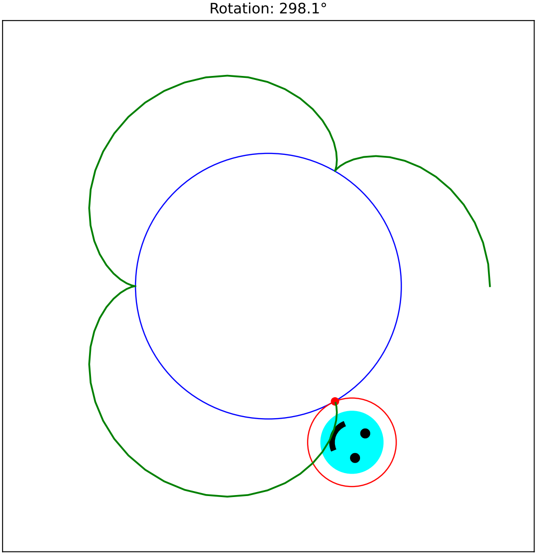
Animation of Coin Rotation Paradox



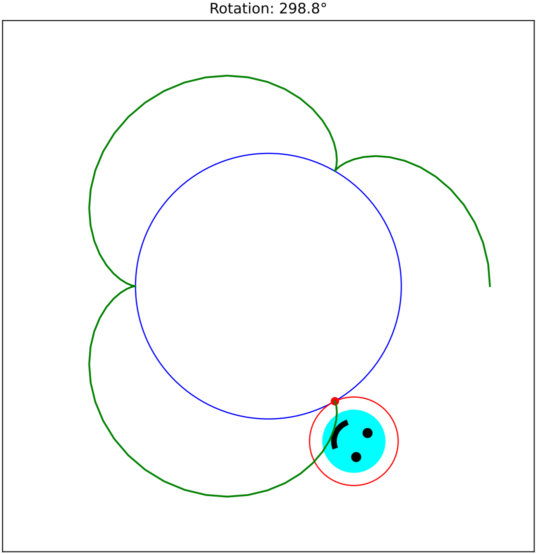
Animation of Coin Rotation Paradox



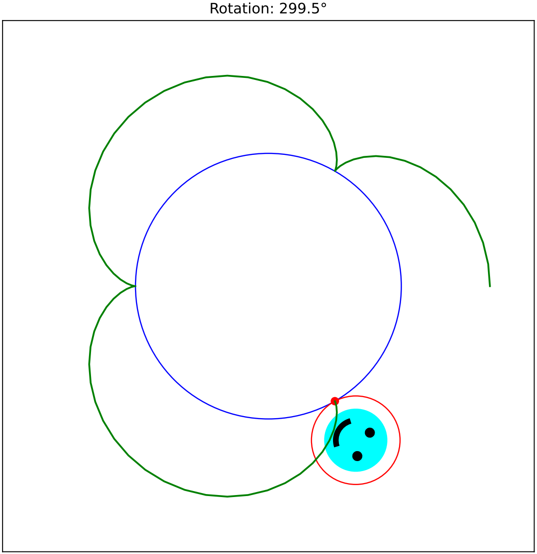
Animation of Coin Rotation Paradox



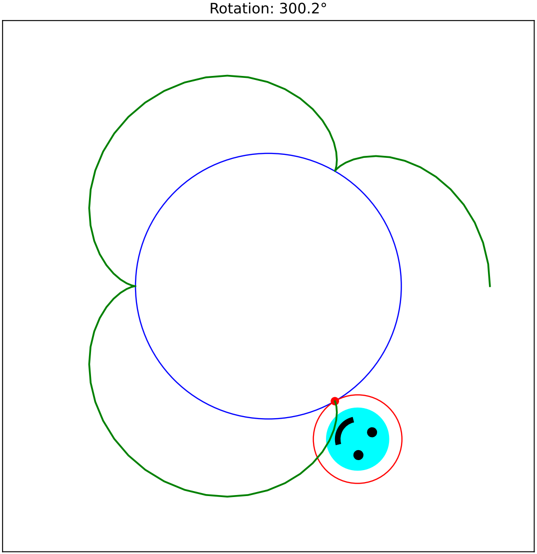
Animation of Coin Rotation Paradox



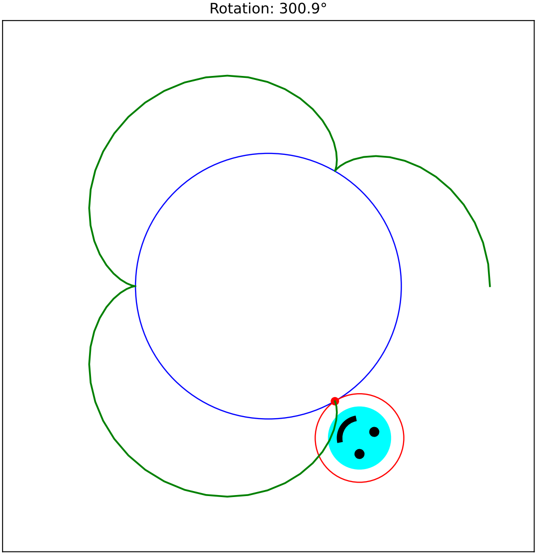
Animation of Coin Rotation Paradox



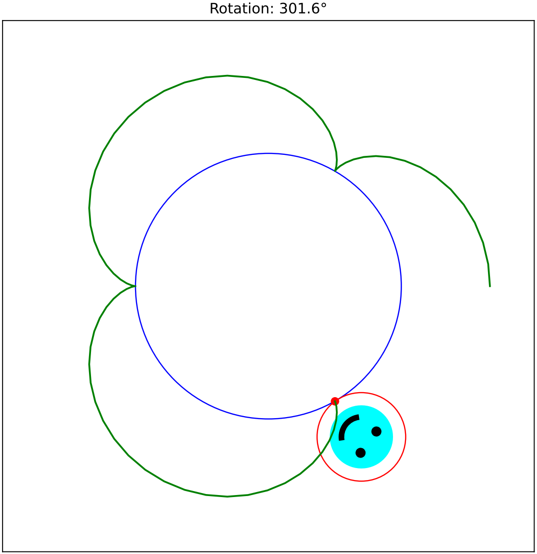
Animation of Coin Rotation Paradox



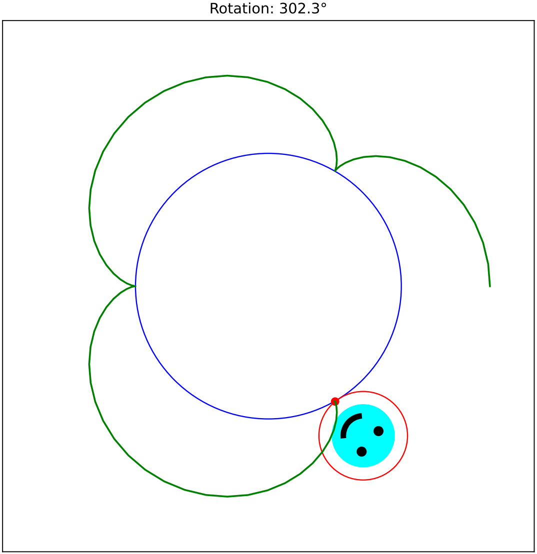
Animation of Coin Rotation Paradox



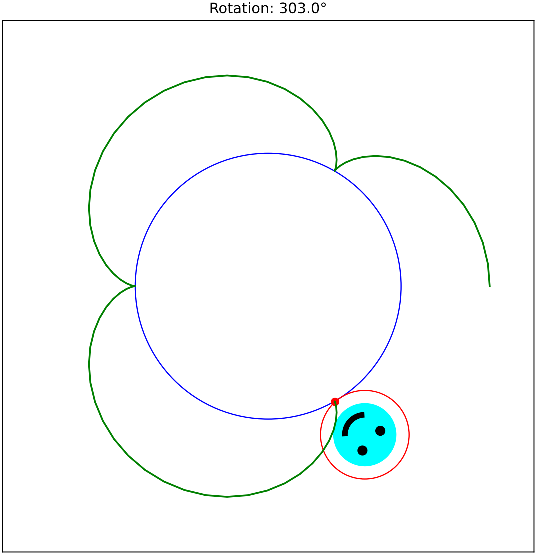
Animation of Coin Rotation Paradox



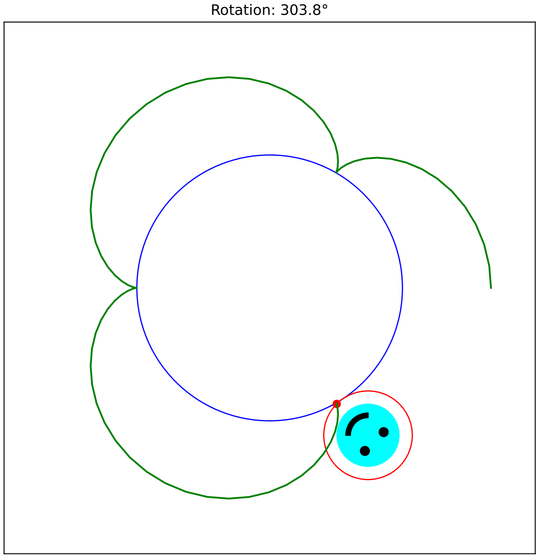
Animation of Coin Rotation Paradox



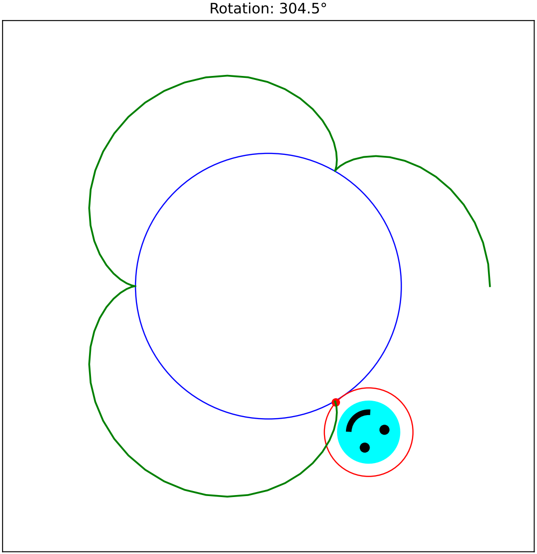
Animation of Coin Rotation Paradox



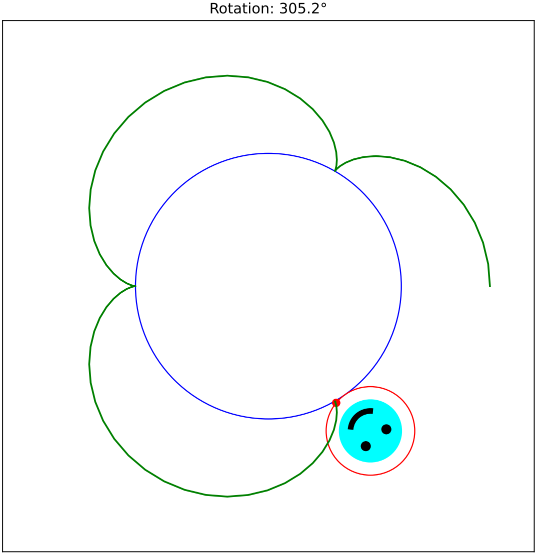
Animation of Coin Rotation Paradox



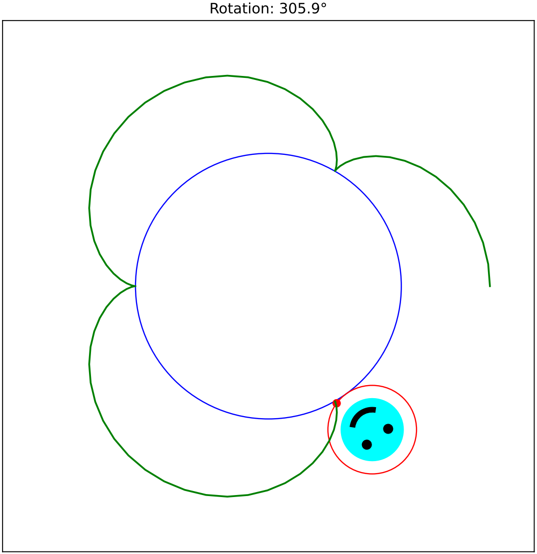
Animation of Coin Rotation Paradox



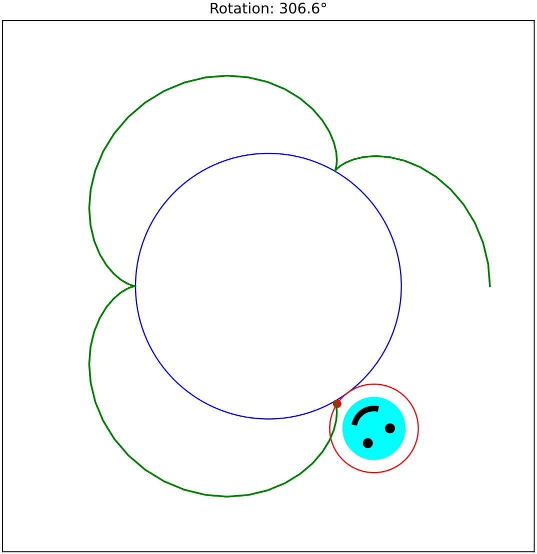
Animation of Coin Rotation Paradox



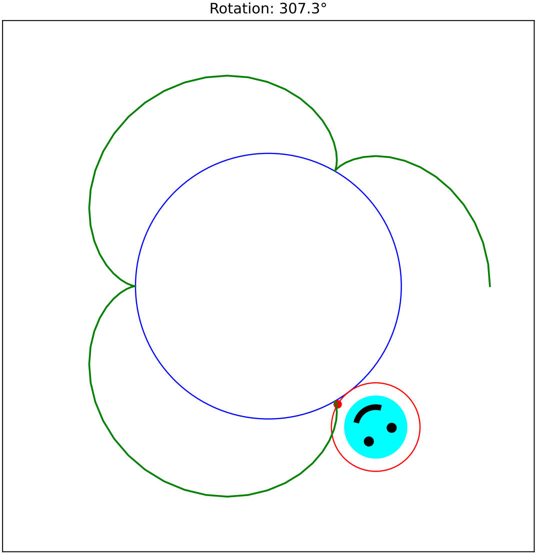
Animation of Coin Rotation Paradox



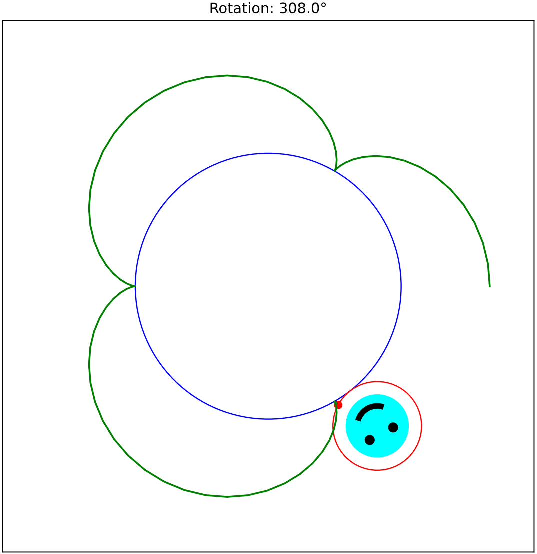
Animation of Coin Rotation Paradox



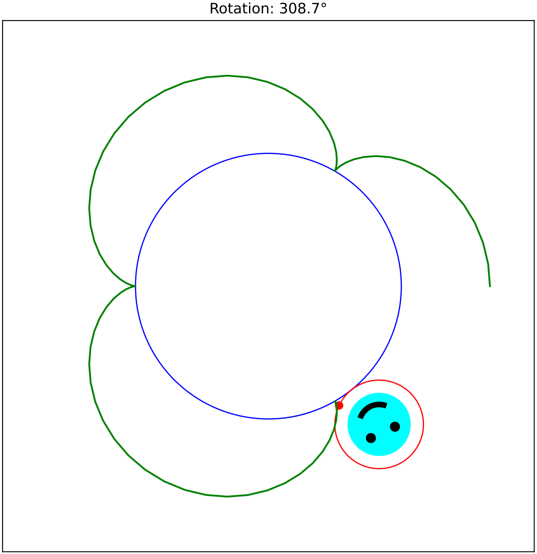
Animation of Coin Rotation Paradox



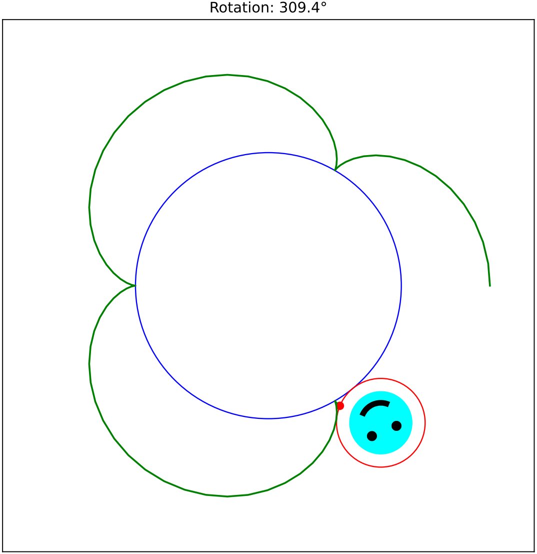
Animation of Coin Rotation Paradox



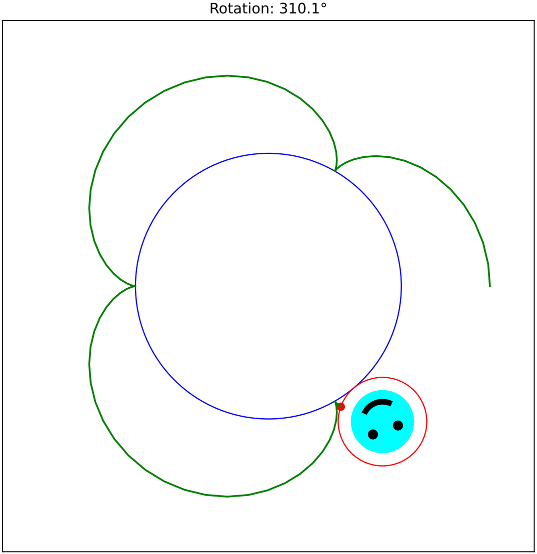
Animation of Coin Rotation Paradox



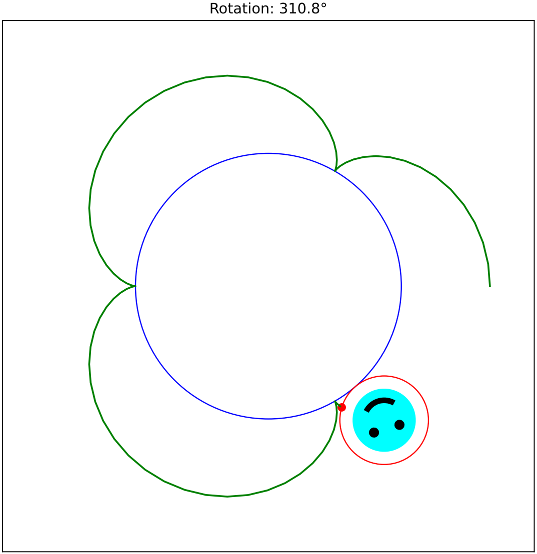
Animation of Coin Rotation Paradox



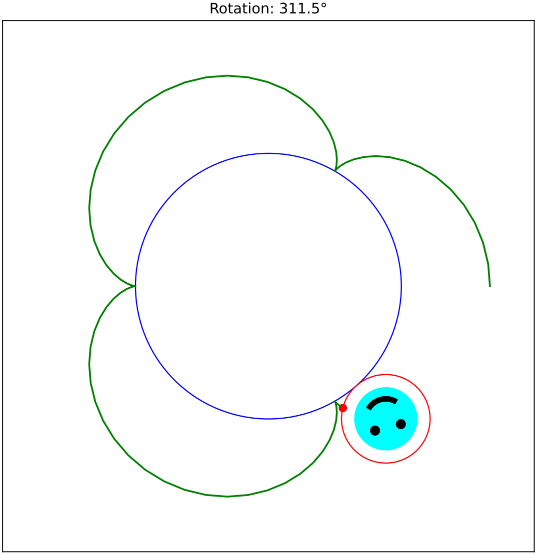
Animation of Coin Rotation Paradox



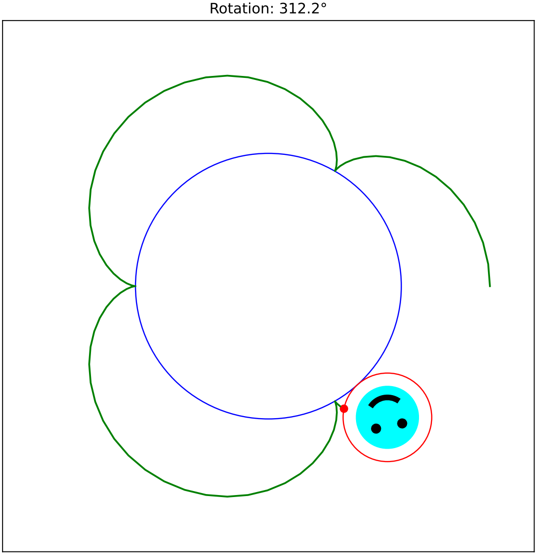
Animation of Coin Rotation Paradox



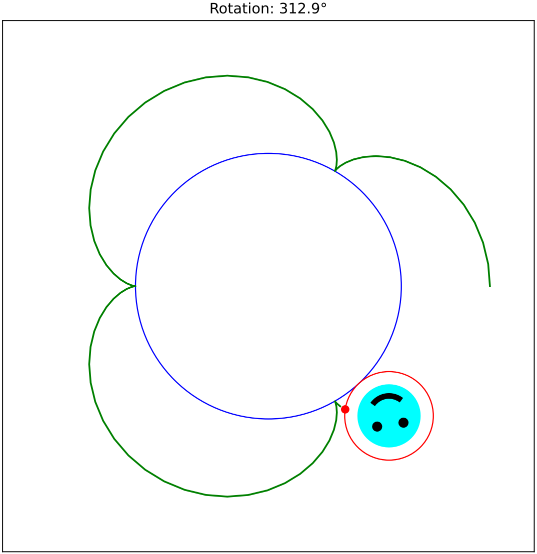
Animation of Coin Rotation Paradox



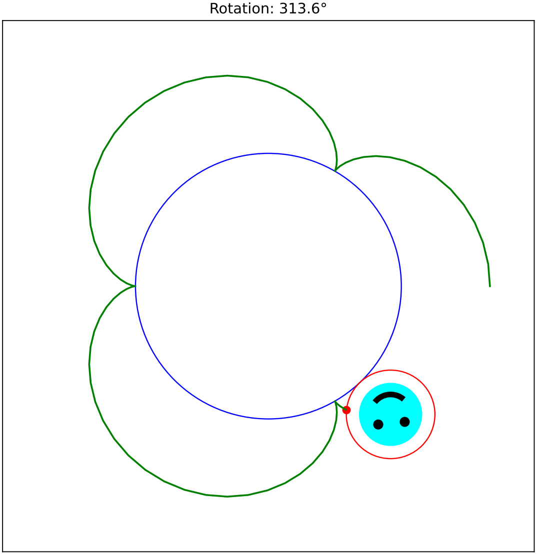
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

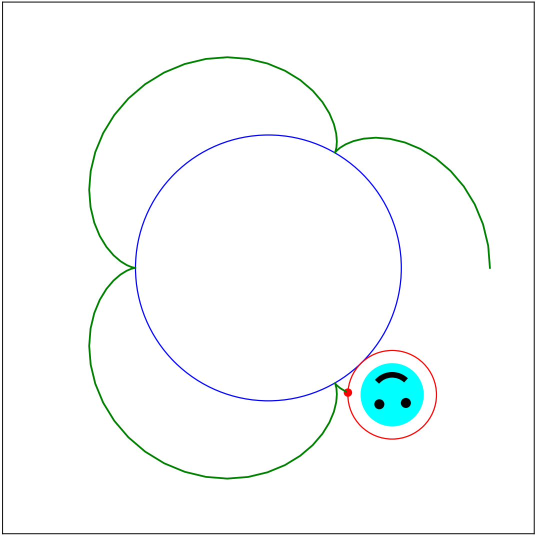


Animation of Coin Rotation Paradox

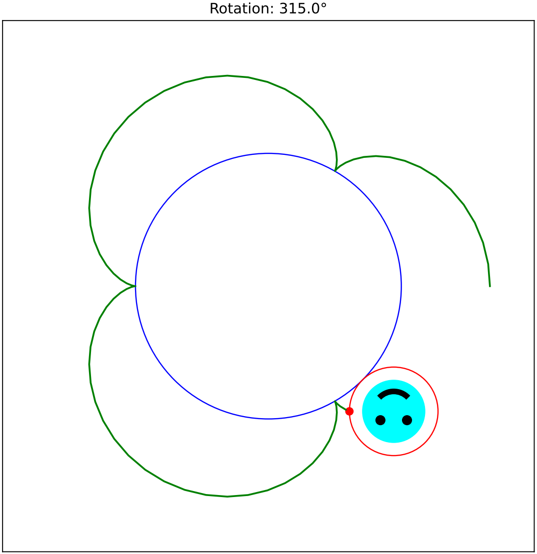


Animation of Coin Rotation Paradox

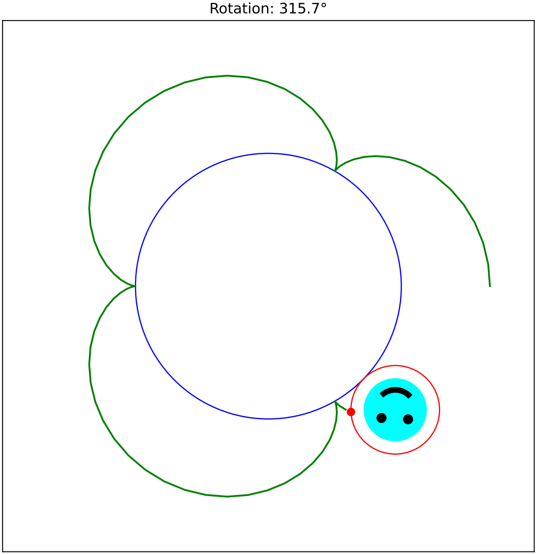
Rotation: 314.3°



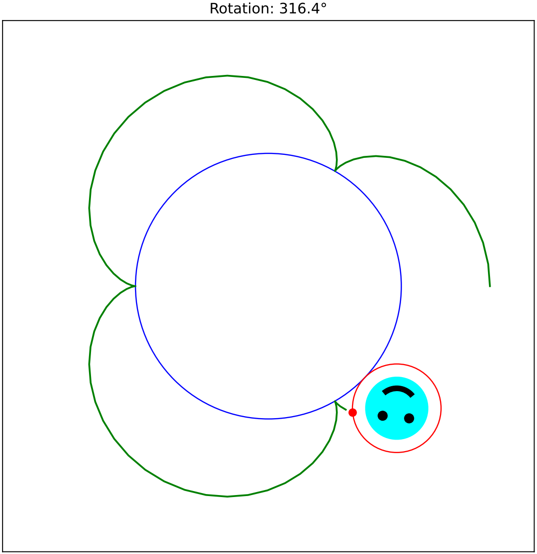
Animation of Coin Rotation Paradox



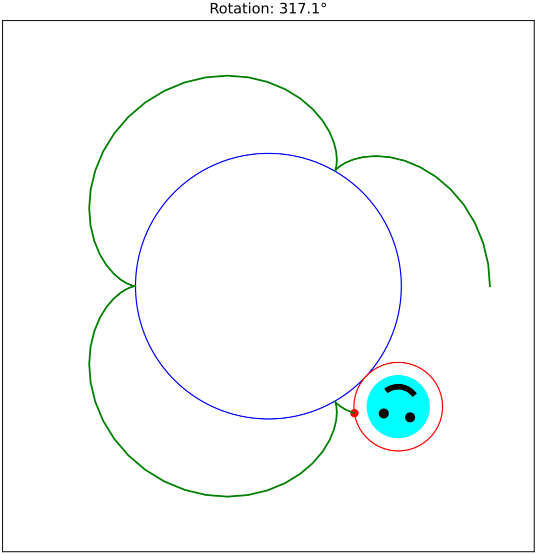
Animation of Coin Rotation Paradox



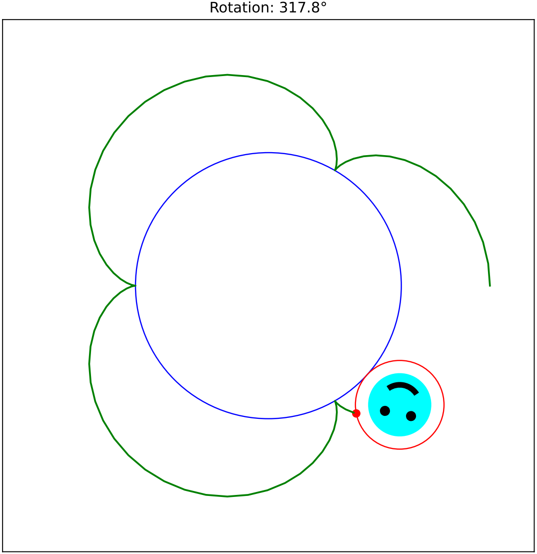
Animation of Coin Rotation Paradox



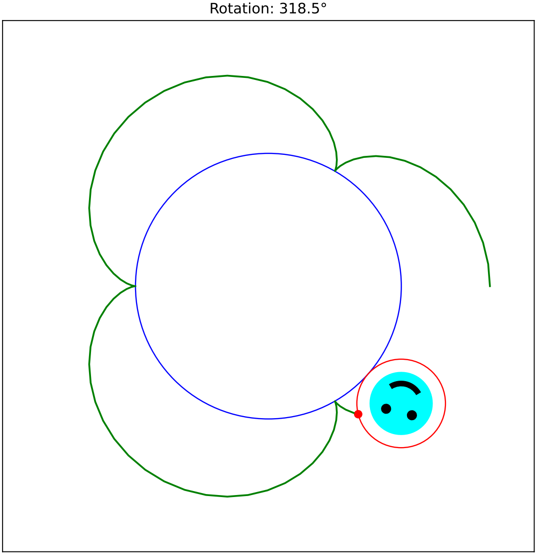
Animation of Coin Rotation Paradox



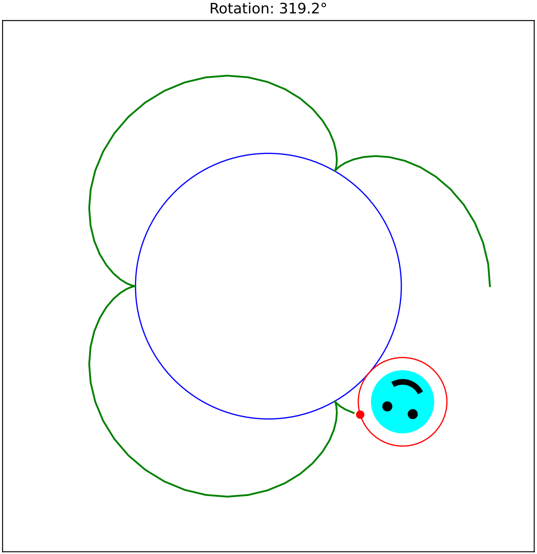
Animation of Coin Rotation Paradox



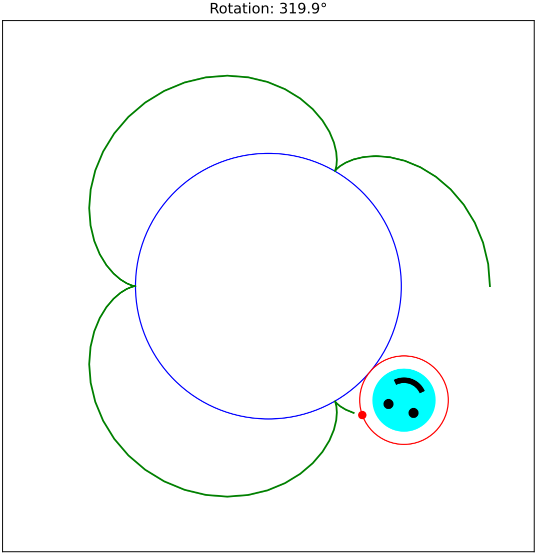
Animation of Coin Rotation Paradox



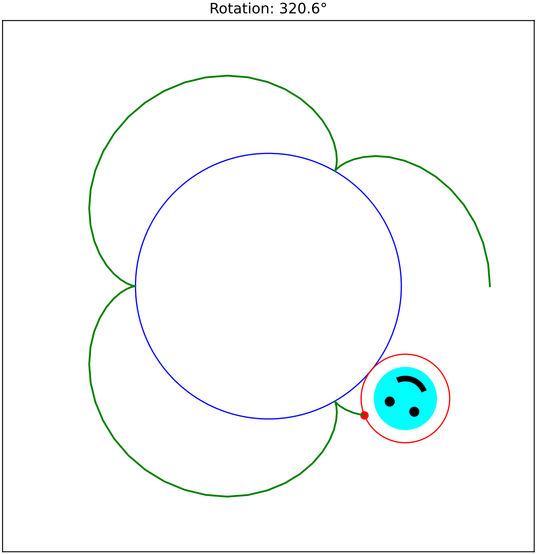
Animation of Coin Rotation Paradox



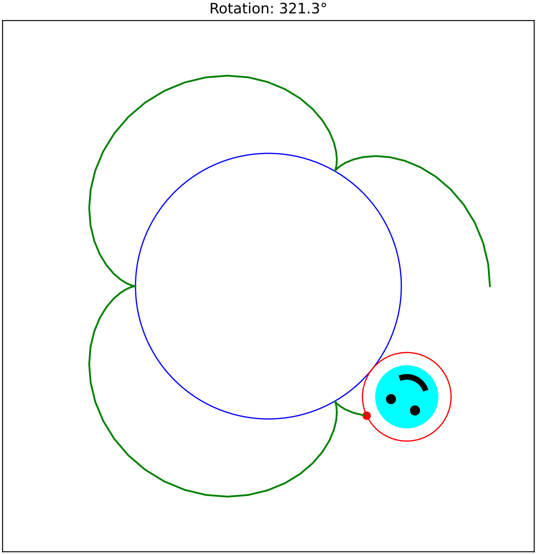
Animation of Coin Rotation Paradox



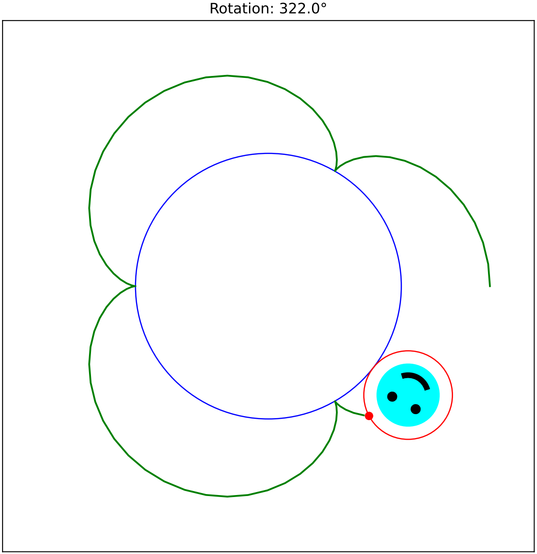
Animation of Coin Rotation Paradox



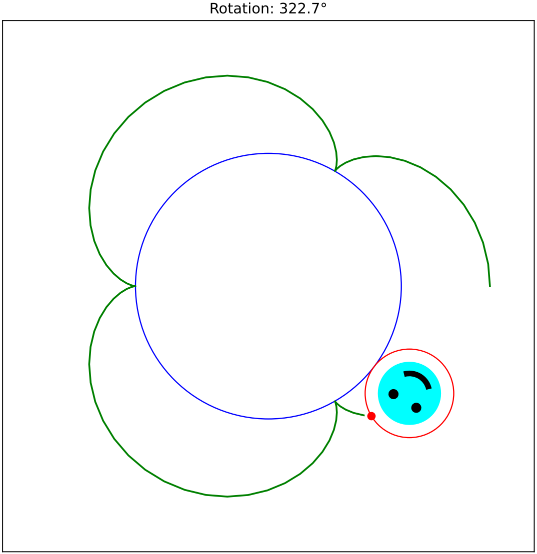
Animation of Coin Rotation Paradox



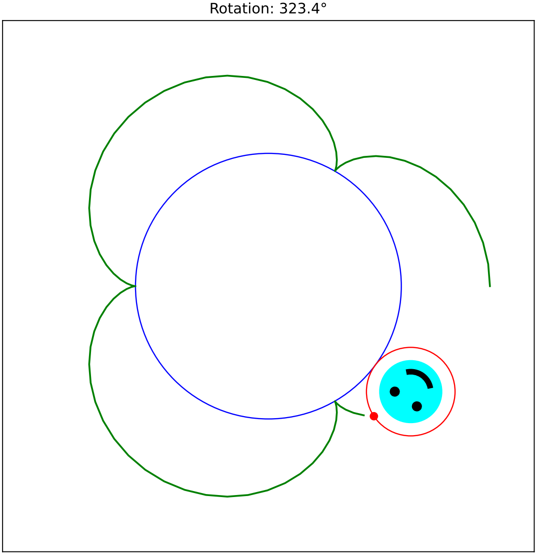
Animation of Coin Rotation Paradox



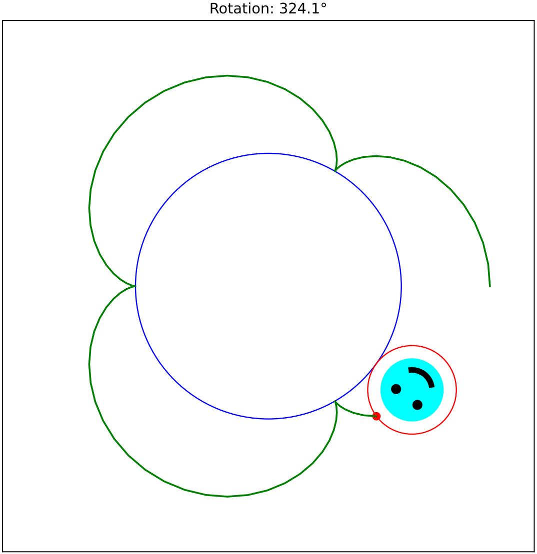
Animation of Coin Rotation Paradox



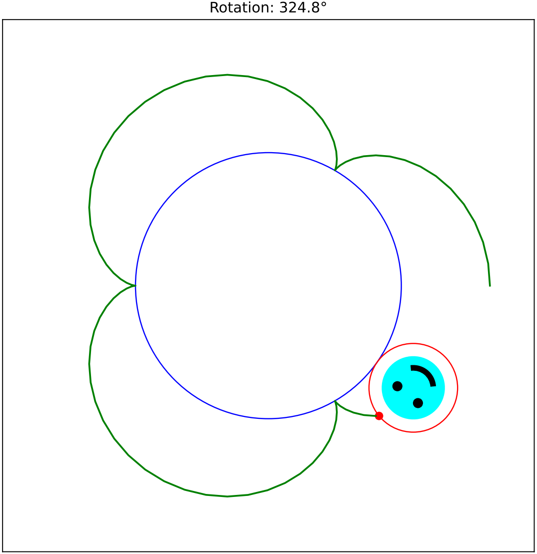
Animation of Coin Rotation Paradox



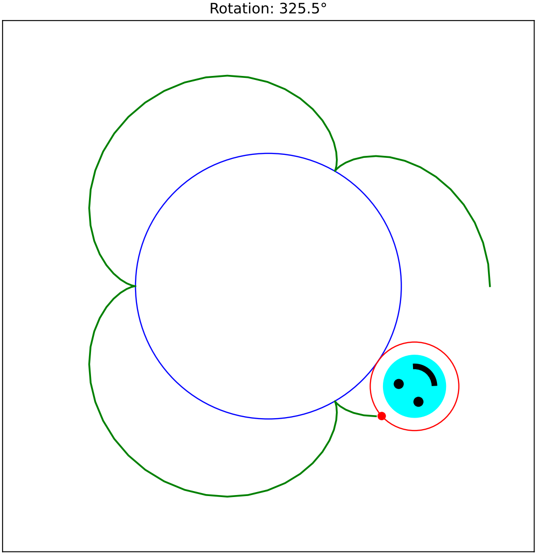
Animation of Coin Rotation Paradox



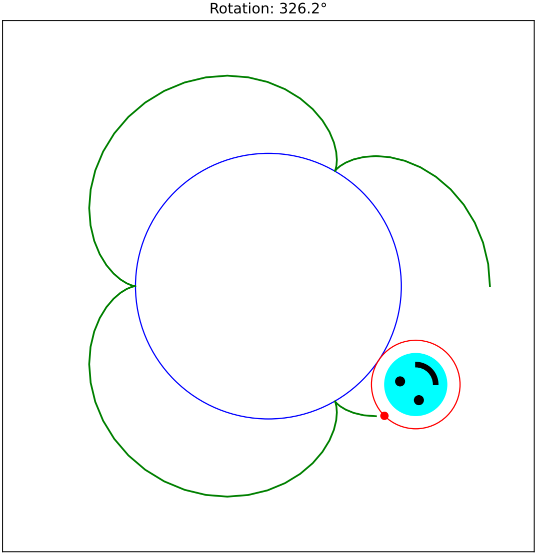
Animation of Coin Rotation Paradox



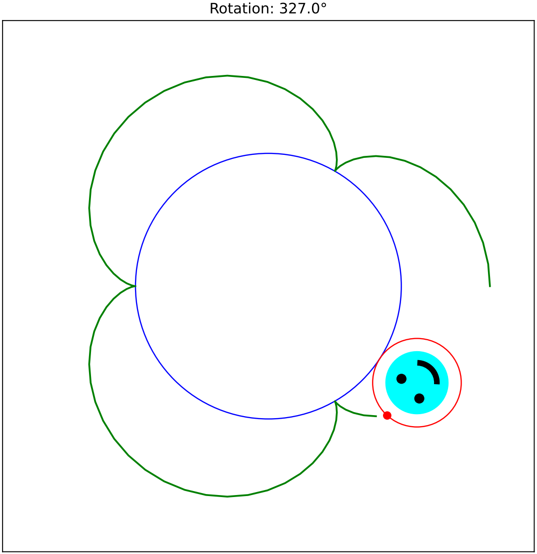
Animation of Coin Rotation Paradox



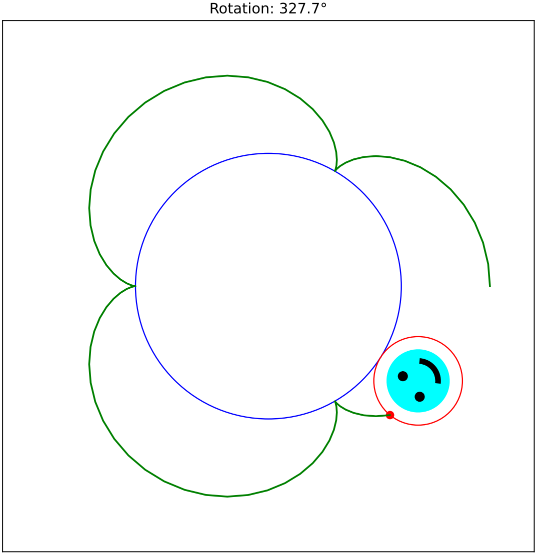
Animation of Coin Rotation Paradox



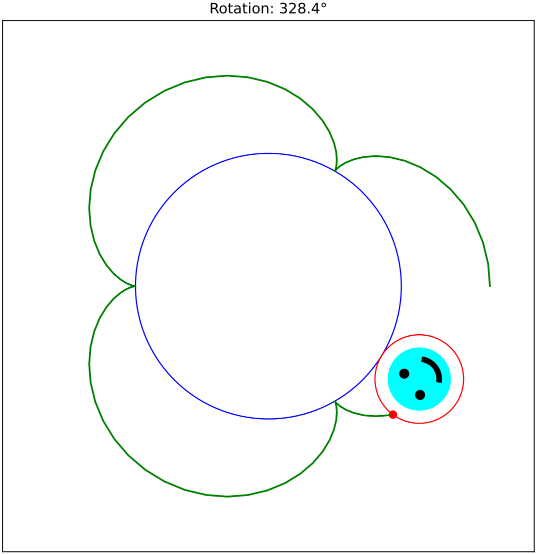
Animation of Coin Rotation Paradox



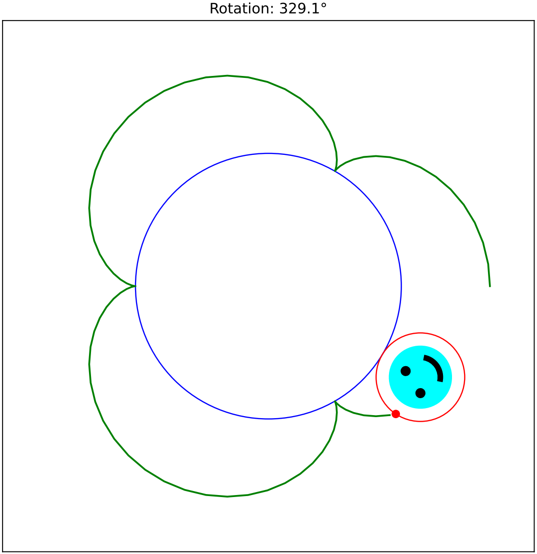
Animation of Coin Rotation Paradox



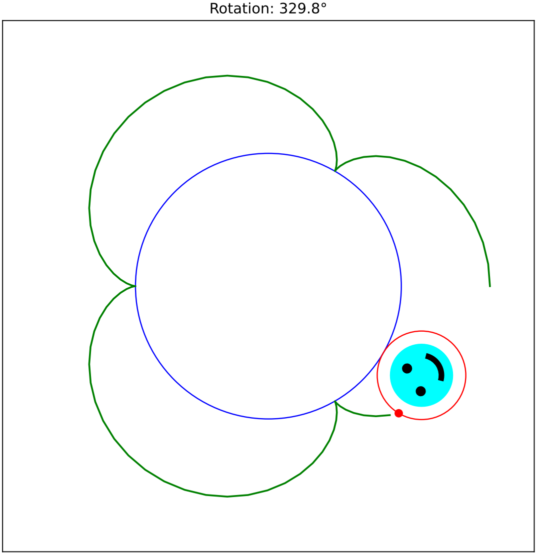
Animation of Coin Rotation Paradox



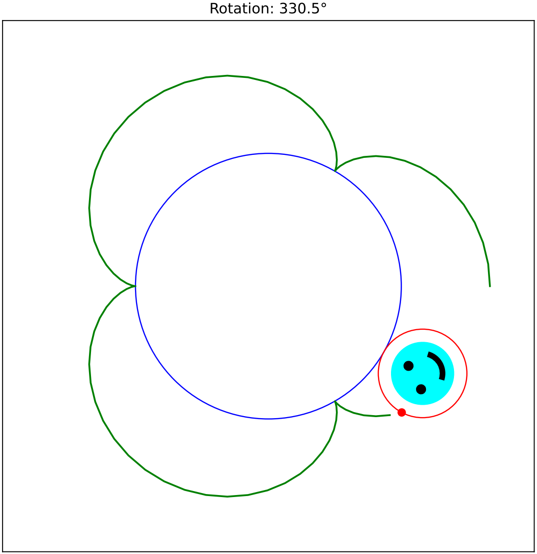
Animation of Coin Rotation Paradox



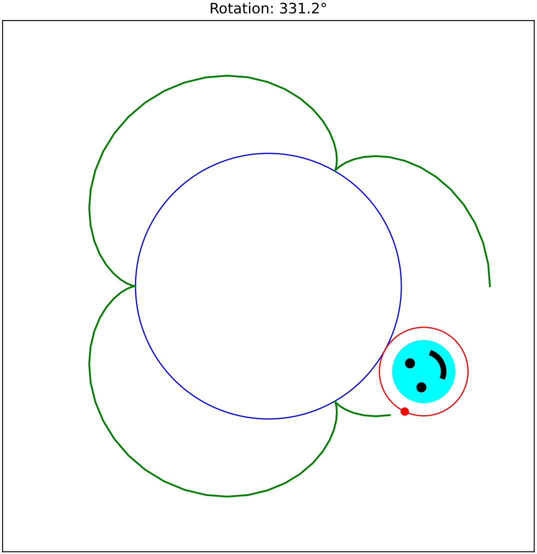
Animation of Coin Rotation Paradox



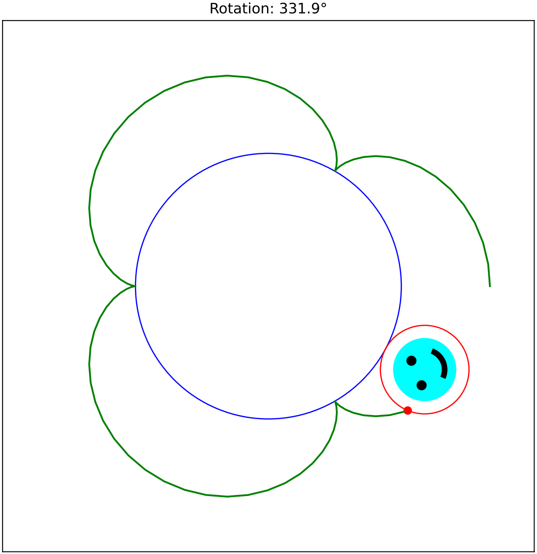
Animation of Coin Rotation Paradox



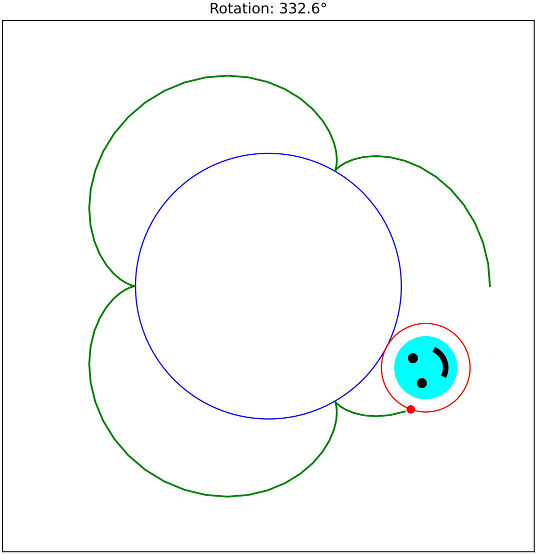
Animation of Coin Rotation Paradox



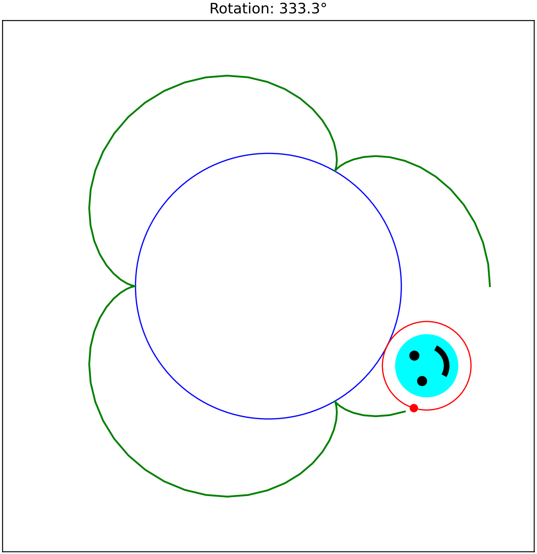
Animation of Coin Rotation Paradox



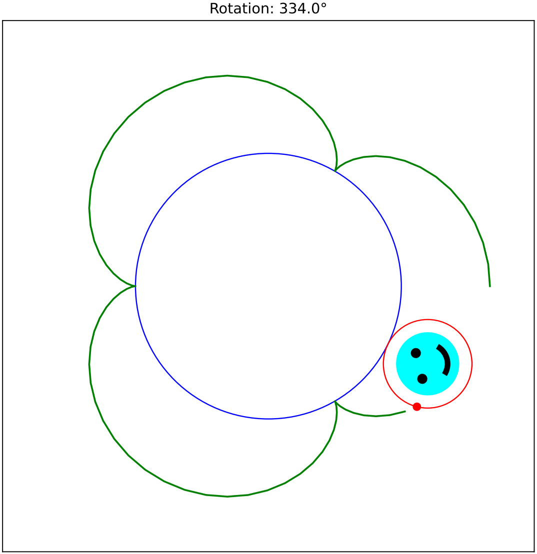
Animation of Coin Rotation Paradox



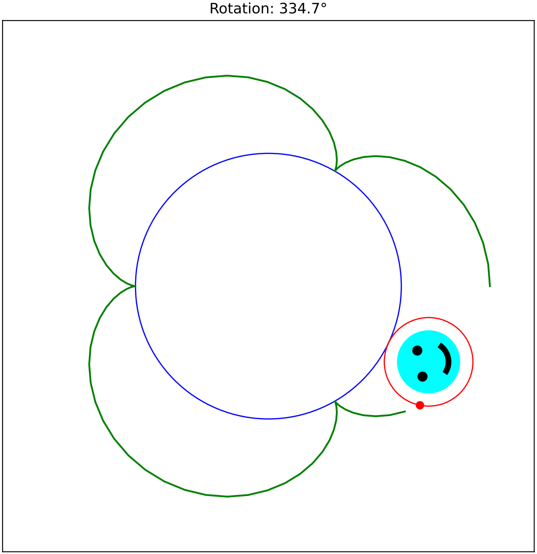
Animation of Coin Rotation Paradox



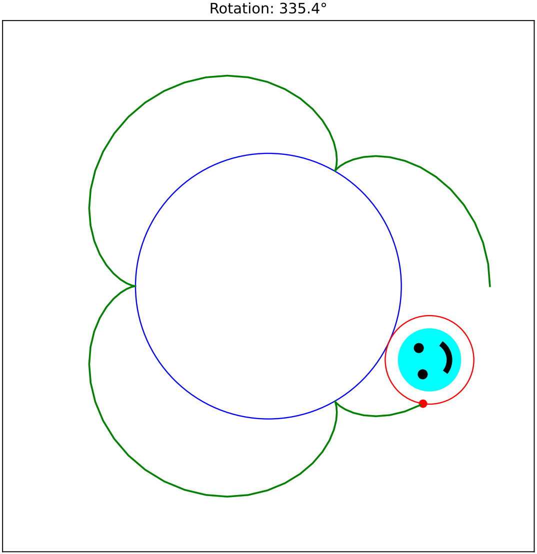
Animation of Coin Rotation Paradox



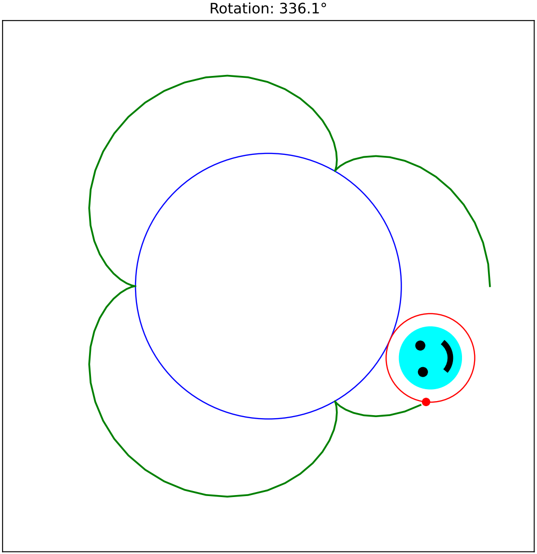
Animation of Coin Rotation Paradox



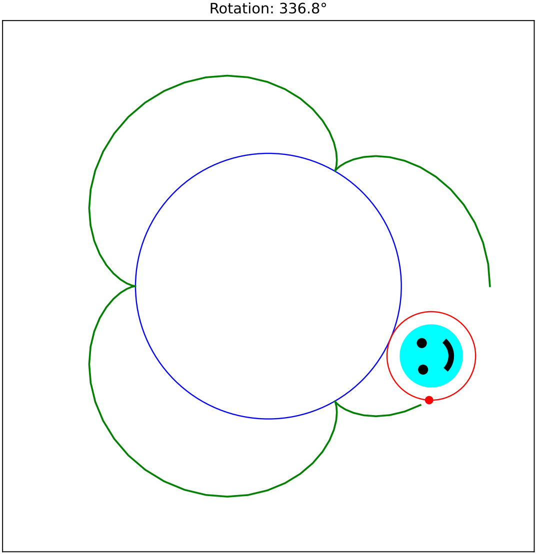
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox



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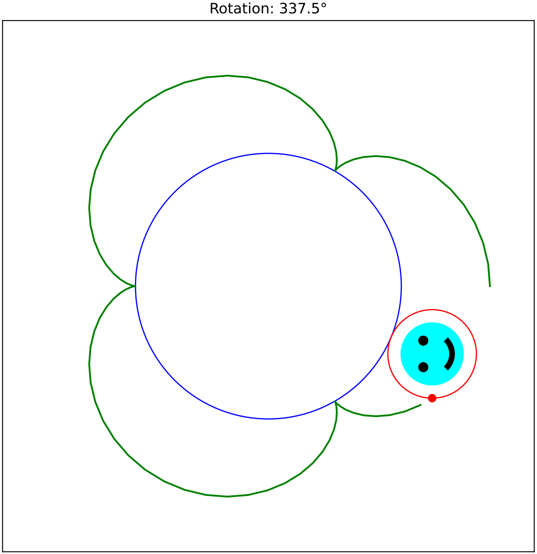

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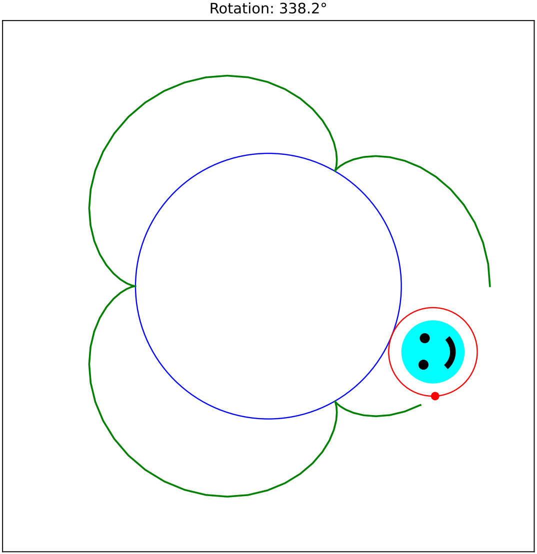

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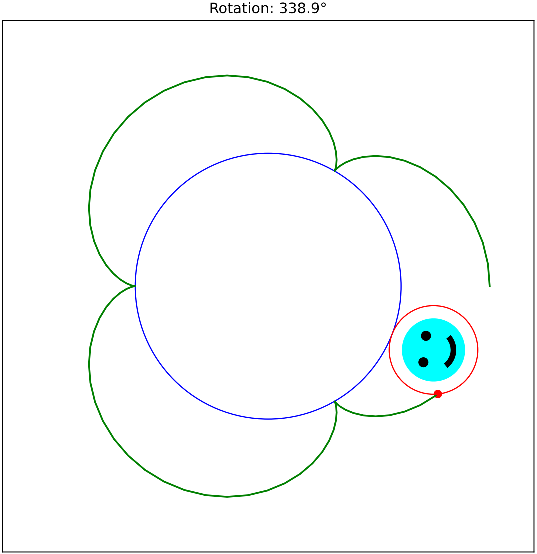

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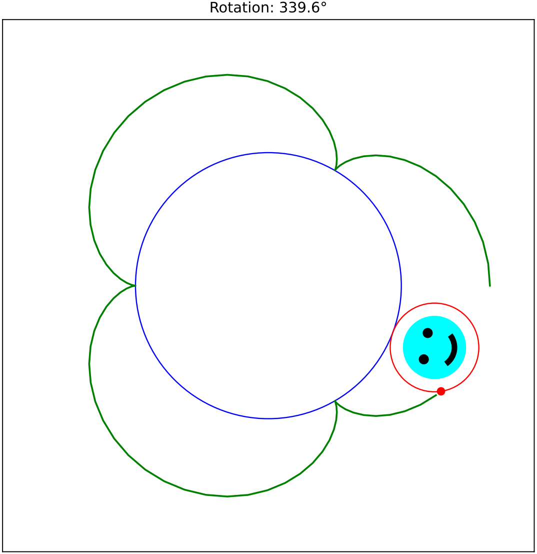

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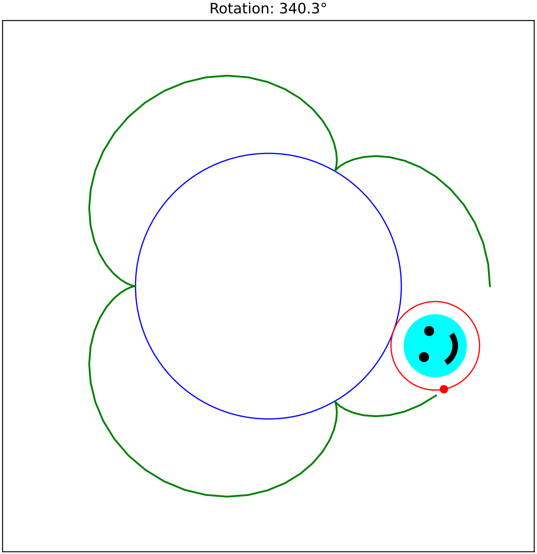

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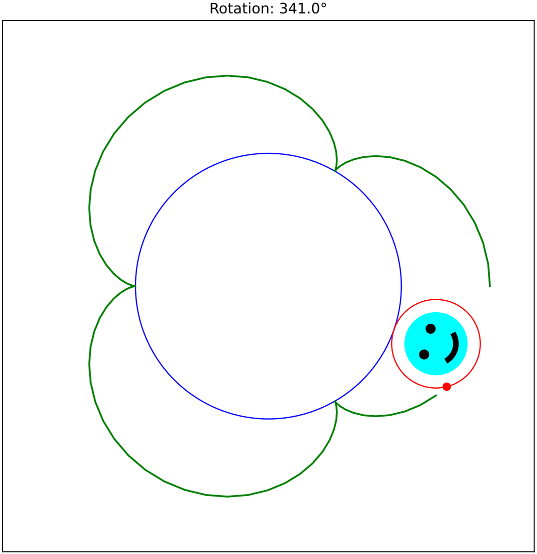

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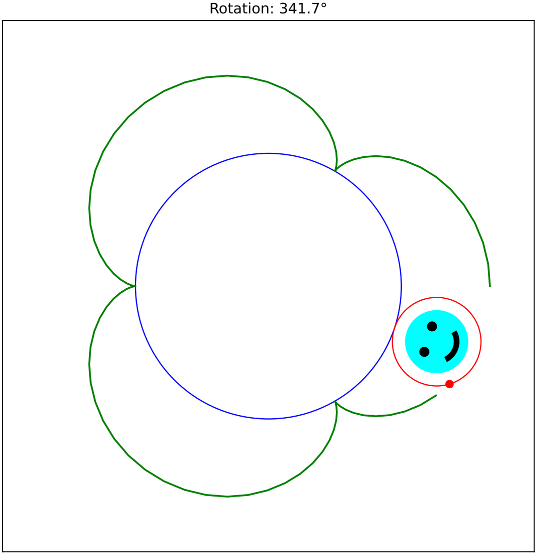

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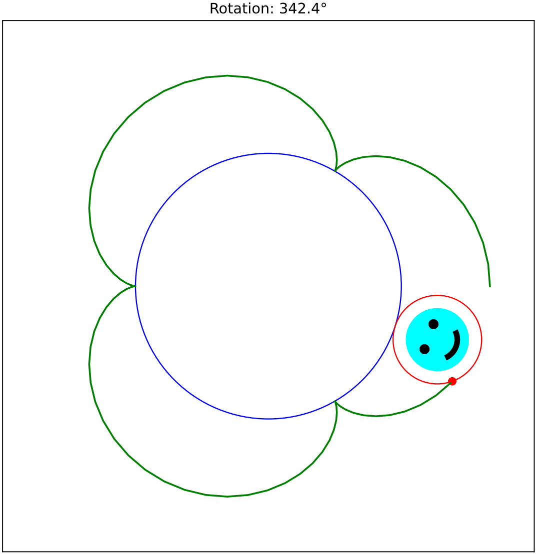

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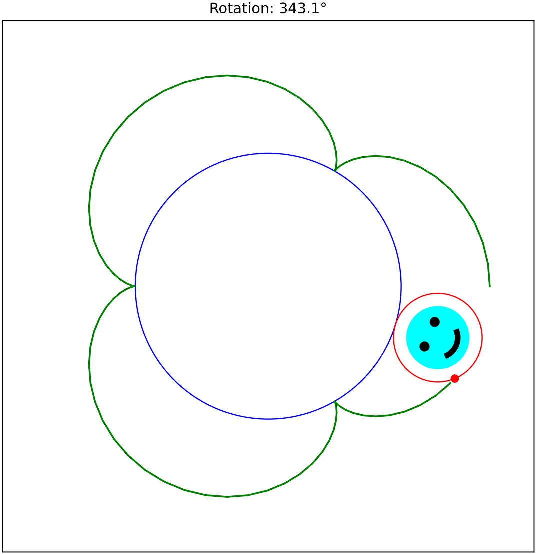

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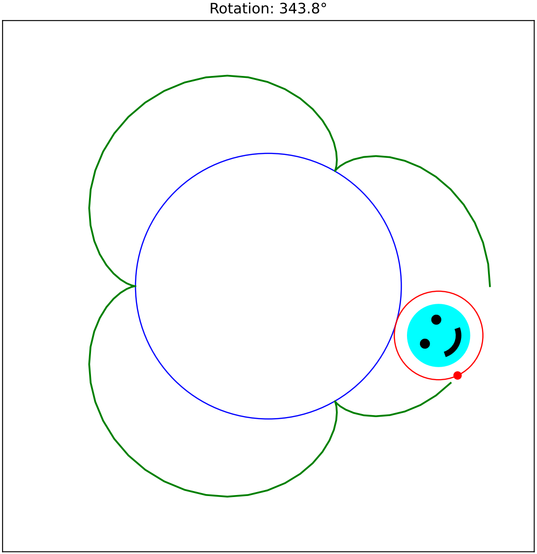

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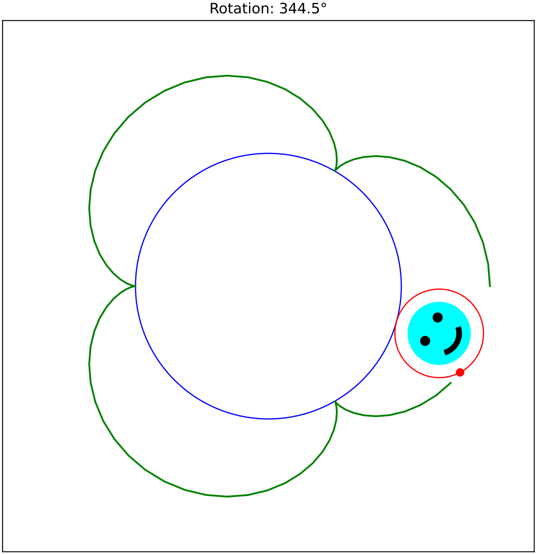

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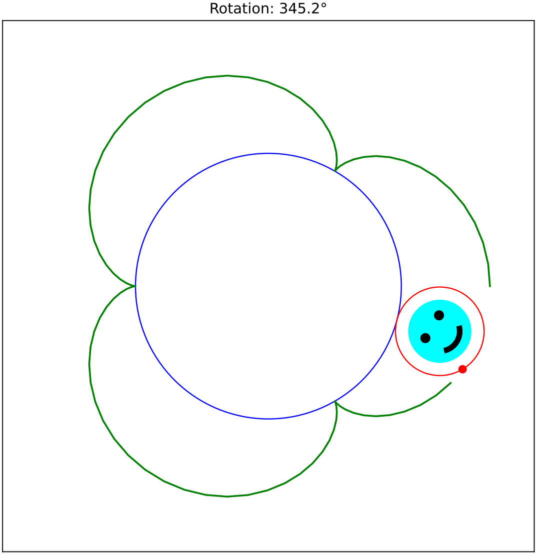

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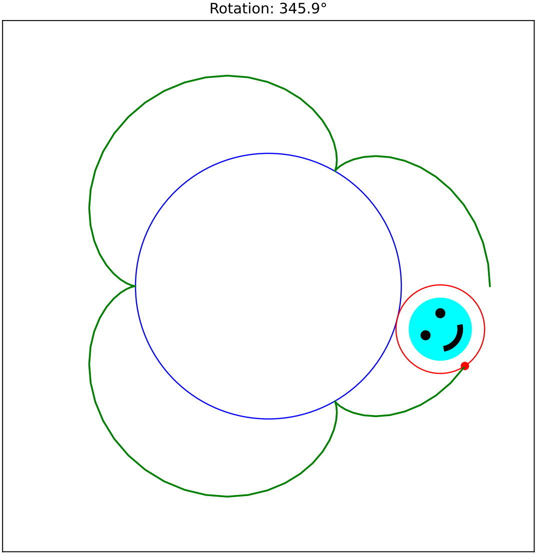

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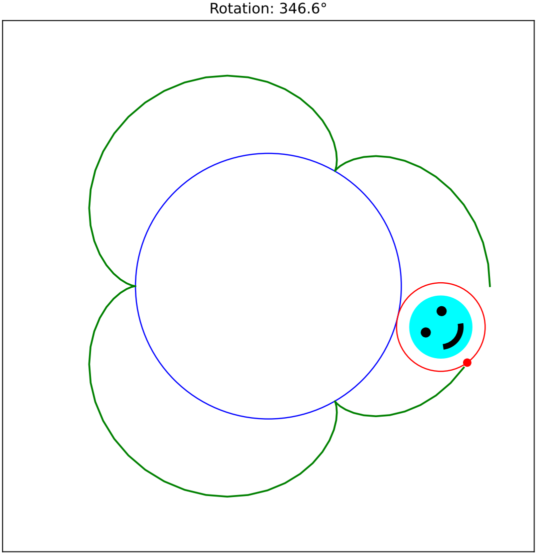

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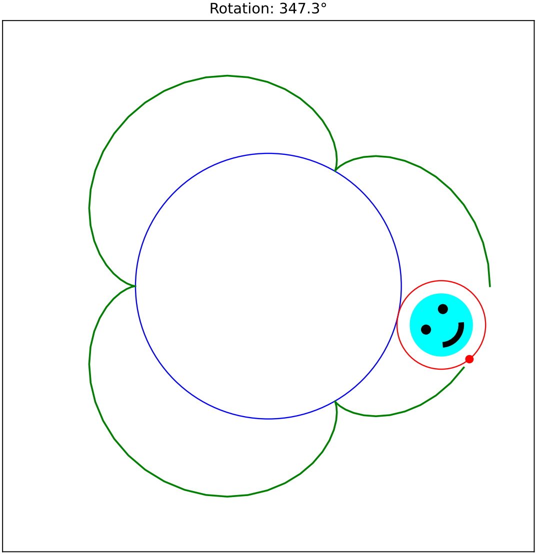

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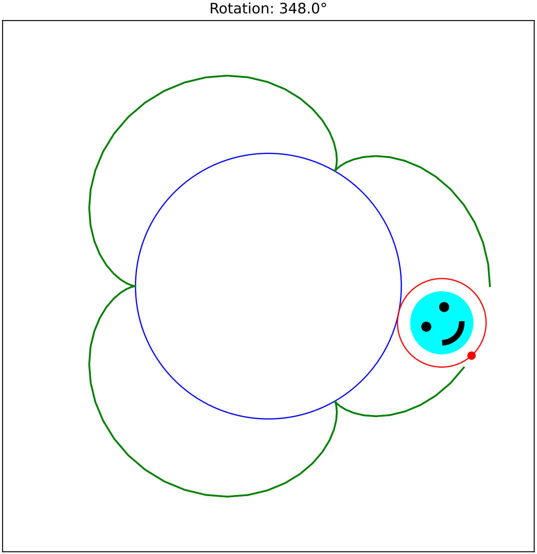

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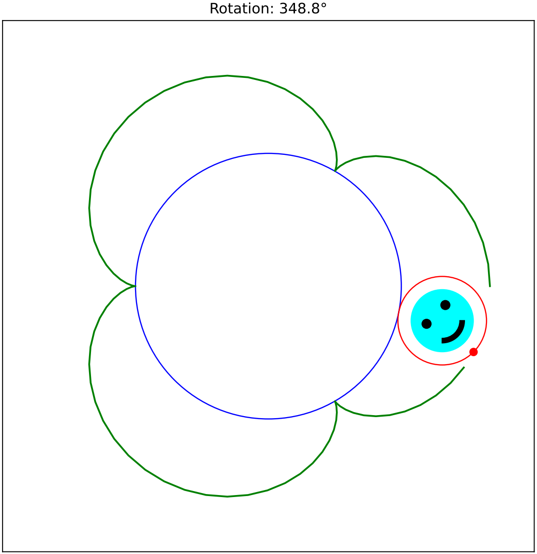

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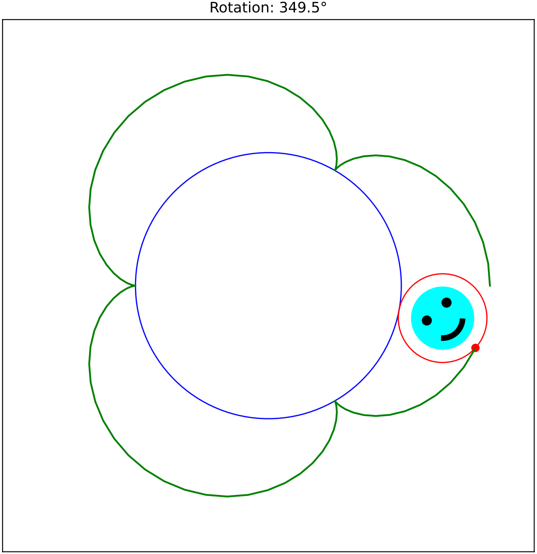

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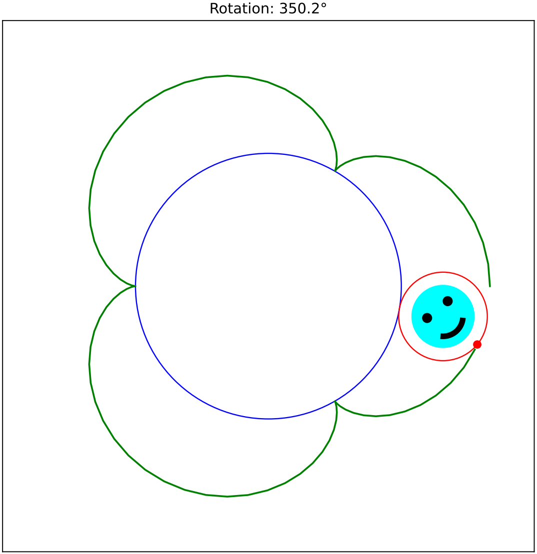

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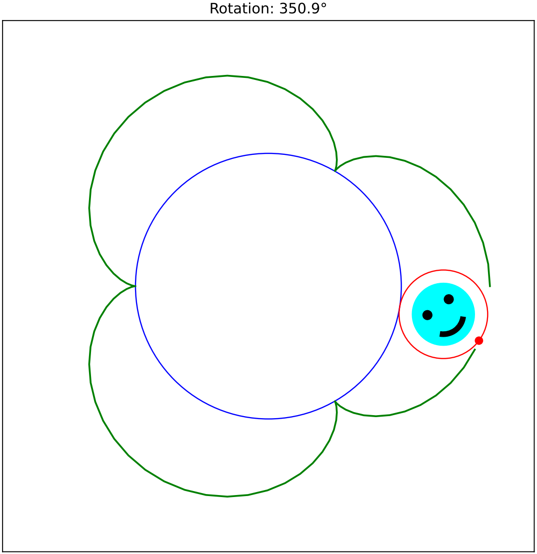

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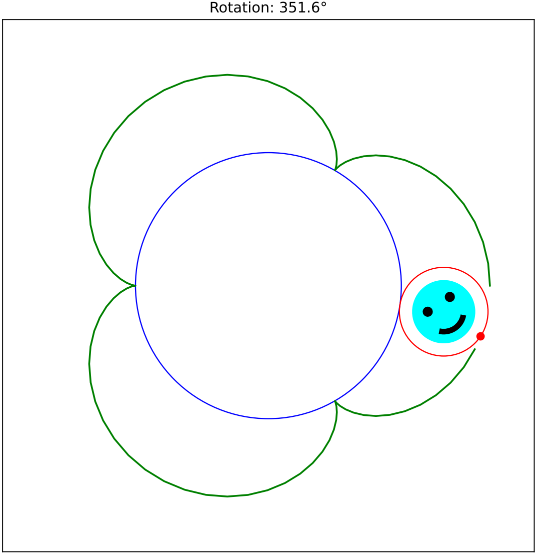

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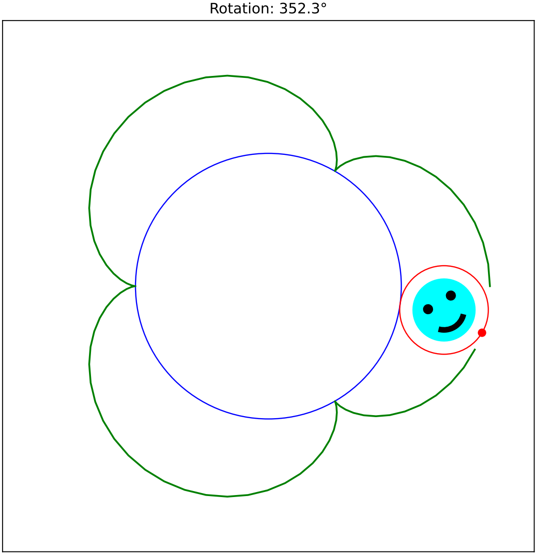

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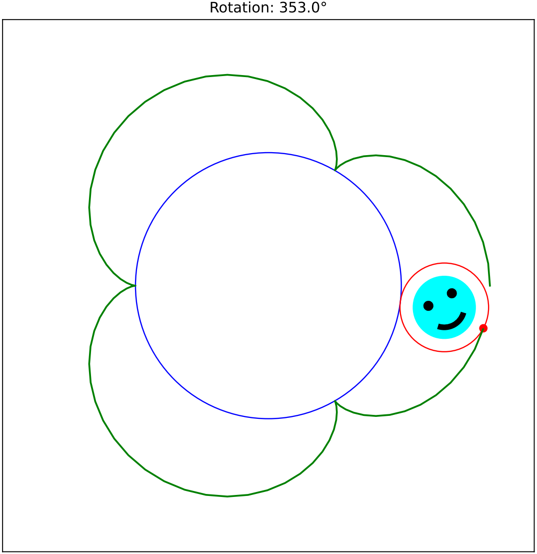

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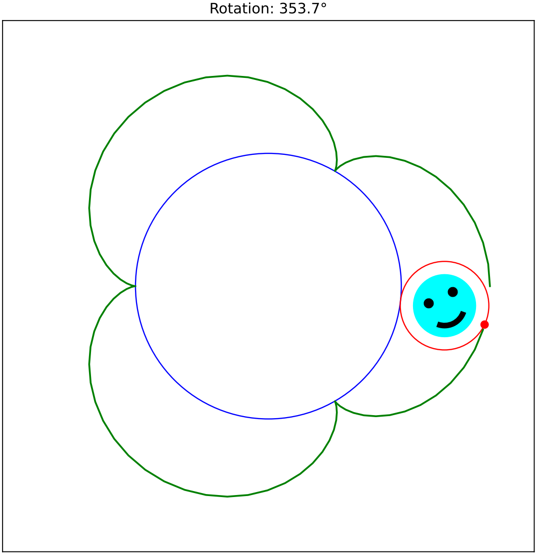

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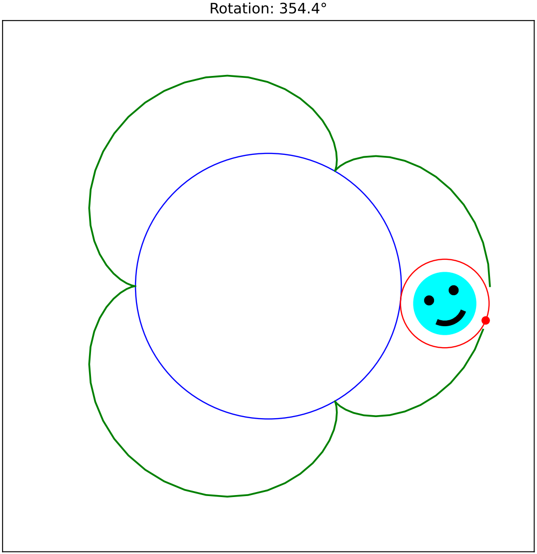

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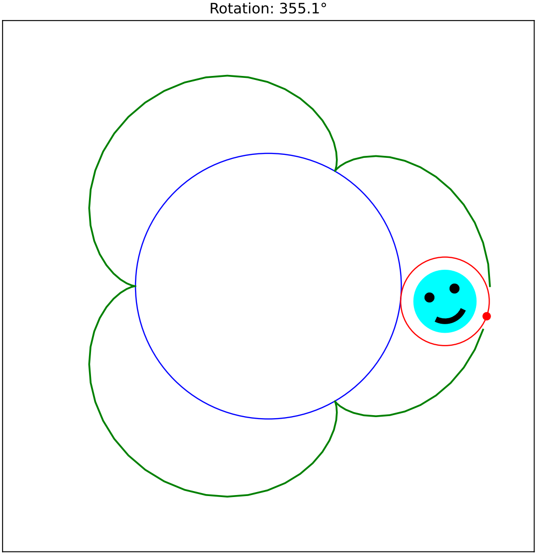

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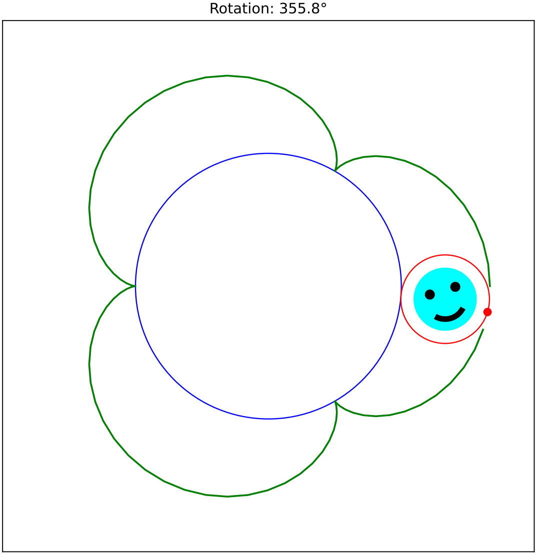

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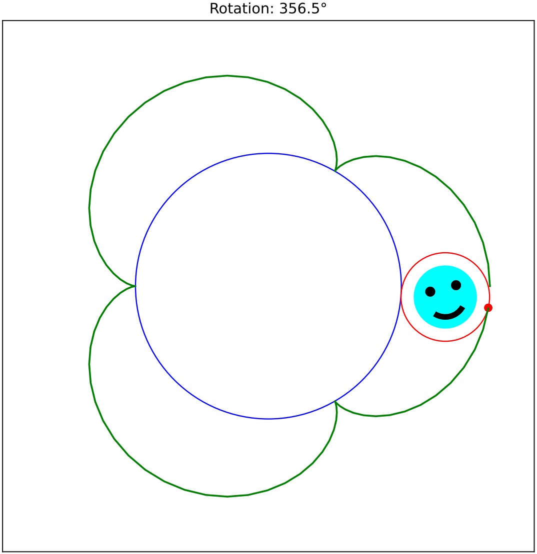

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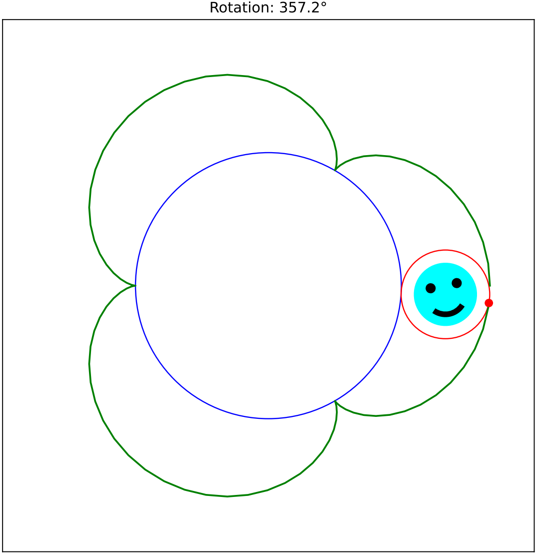

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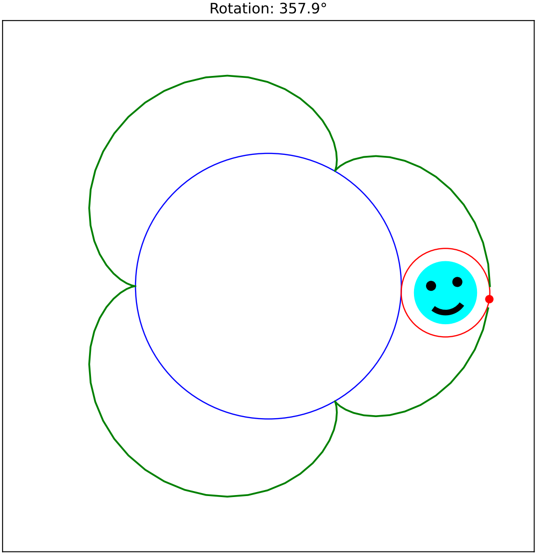

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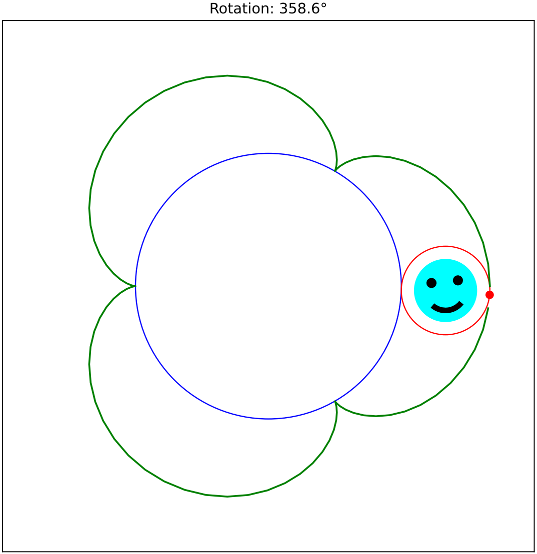

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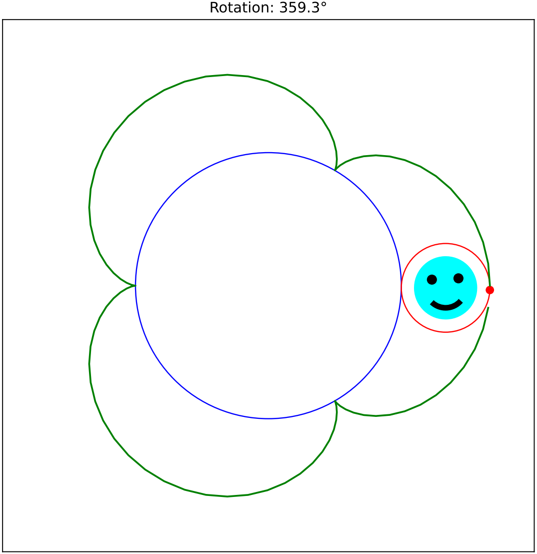

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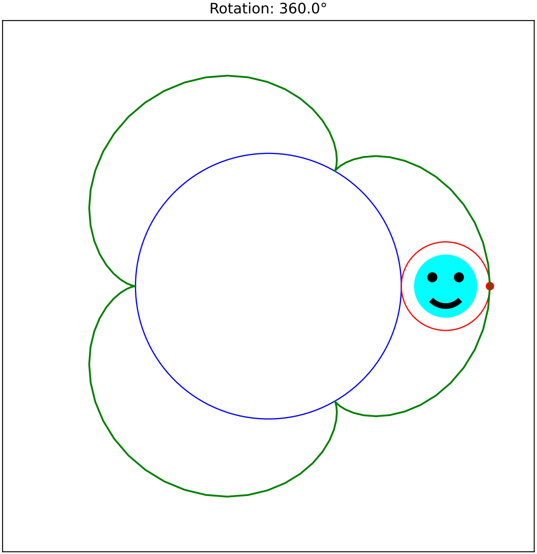
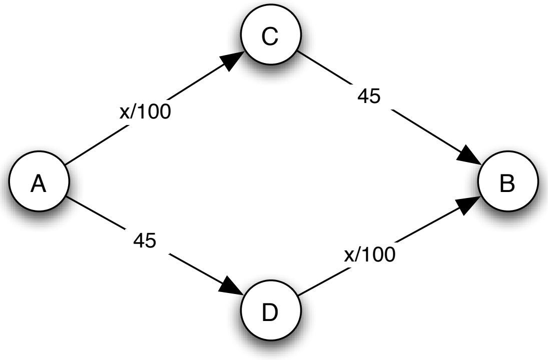


Braess Paradox

A Typical Case



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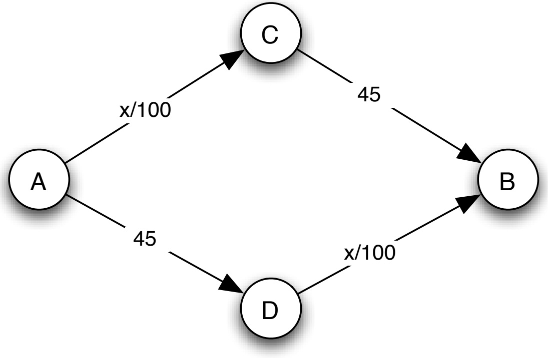

- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it

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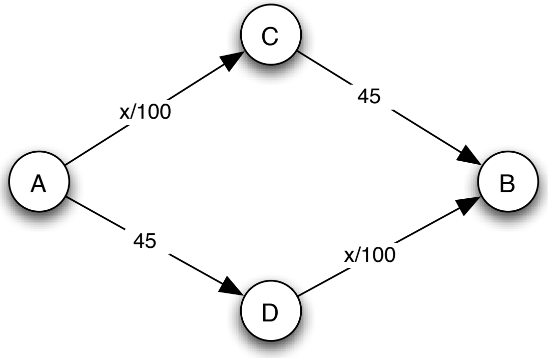

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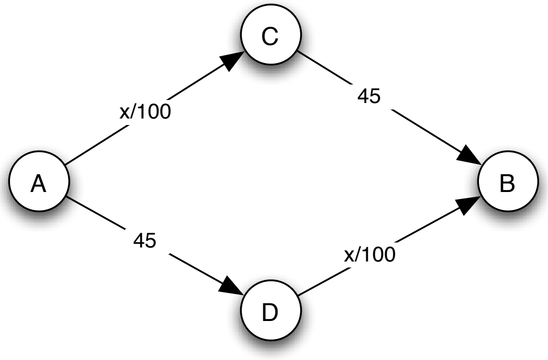
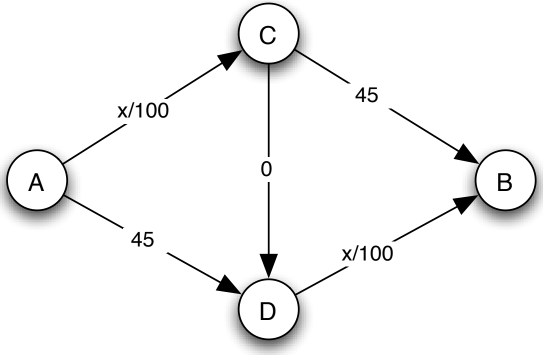


- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it
- Suppose there are 4000 cars need to get from A to B
- They divide evenly over the two routes at equilibrium; the travel time is $45 + 2000/100 = 65$ mins

A Typical Case: Cont'd



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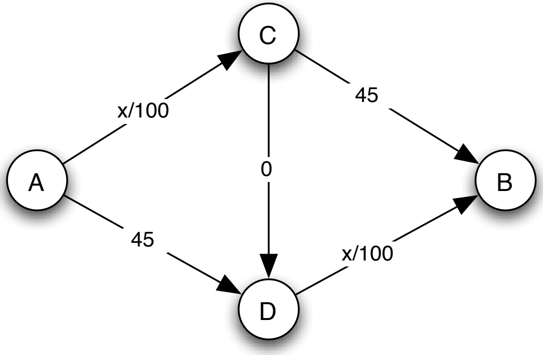

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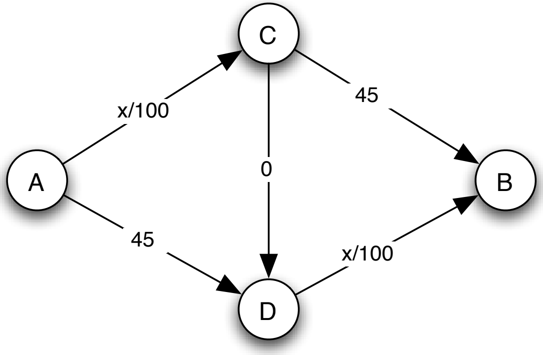

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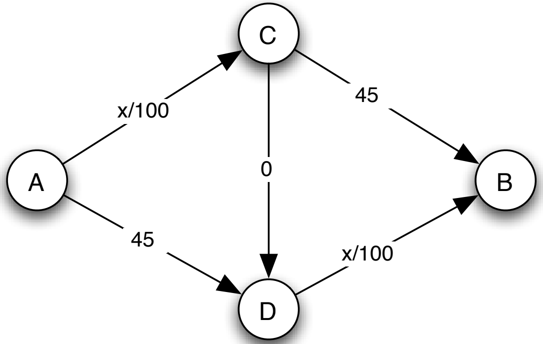
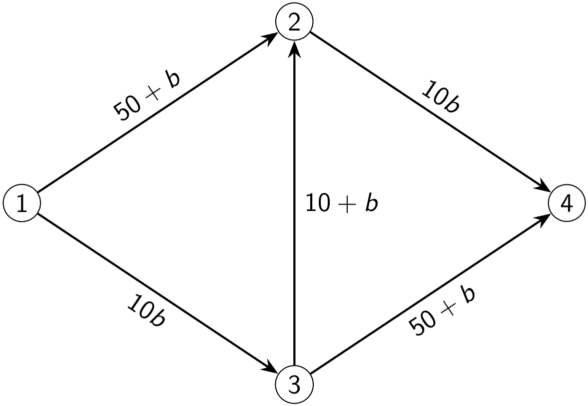


- Now a very fast edge is added from C to D to the previous highway network
- At equilibrium, every user uses the route through C and D
- As a result, the travel time is $4000/10 + 0 + 4000/100 = 80$ mins!

[Braess et al., 2005] Example



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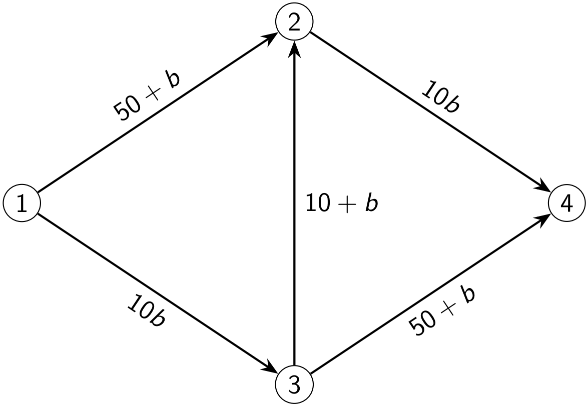

- Assume a flow of 6 units (e.g., 6000 vehicles) must travel from 1 to 4

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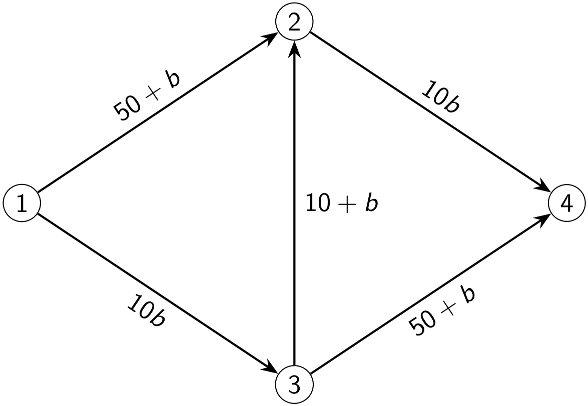

- Assume a flow of 6 units (e.g., 6000 vehicles) must travel from 1 to 4
- Three paths exist: $B_1 = 124$, $B_2 = 1324$, $B_3 = 134$

[Braess et al., 2005] Example: Cont'd

- Case 1: All Paths Open. Split the flow equally (2 units each):

[Braess et al., 2005] Example: Cont'd

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$$b_{12} = 2,\ b_{24} = 4,\ b_{34} = 2,\ b_{13} = 4,\ b_{32} = 2$$
$$d_{12} = 52,\ d_{24} = 40,\ d_{34} = 52,\ d_{13} = 40,\ d_{32} = 12$$
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[Braess et al., 2005] Example: Cont'd

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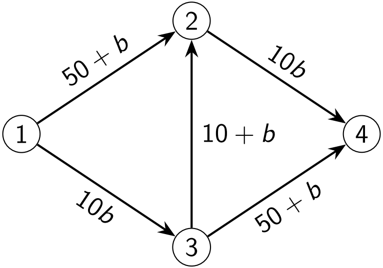
  Despite losing a connection, all paths are shorter!

[Braess et al., 2005] Example: Selfish Deviation



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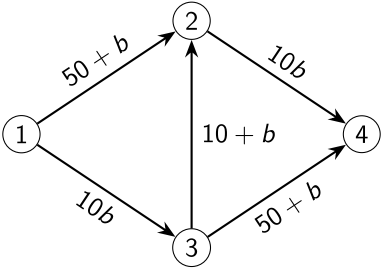

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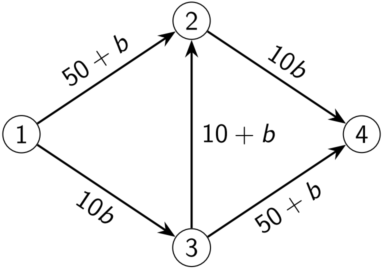

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- A single vehicle ignoring the ban on k_{32} (where $d_{32} = 10$) takes B_2 with $L(B_2) = 70$, gaining an advantage.

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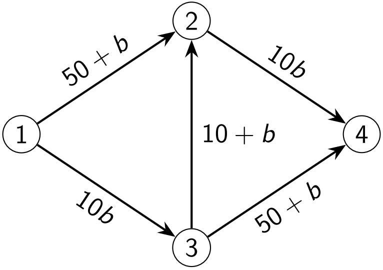

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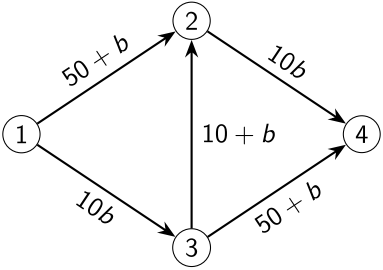

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- Consider an intermediate step: 5 units on B_1 and B_3, 1 on B_2, then $L(B_1) = L(B_3) = 87.5$, $L(B_2) = 82.5$.

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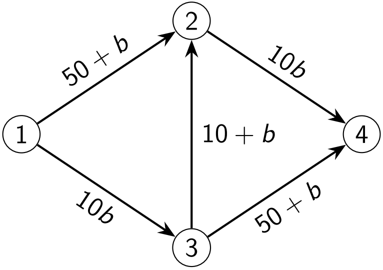

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- A single vehicle ignoring the ban on k_{32} (where $d_{32} = 10$) takes B_2 with $L(B_2) = 70$, gaining an advantage.
- If the ban is lifted, many use B_2, reverting to the original state.
- Consider an intermediate step: 5 units on B_1 and B_3, 1 on B_2, then $L(B_1) = L(B_3) = 87.5$, $L(B_2) = 82.5$.
- B_2 users clog k_{13} and k_{24} (factor 10 in load-time relation), worsening times for B_1 and B_3 beyond 83; Yet B_2 remains shortest, attracting more traffic and degrading the system for all.

Real-Life Examples

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- Road closure in NY: Times and Herald Squares pedestrian plaza (2009 —)

- Cheonggyecheon restoration project (2003 —)
  - Replaced a six lane highway with a five mile long park, traffic flow improved

The Social Cost of Traffic at Equilibrium

Terms

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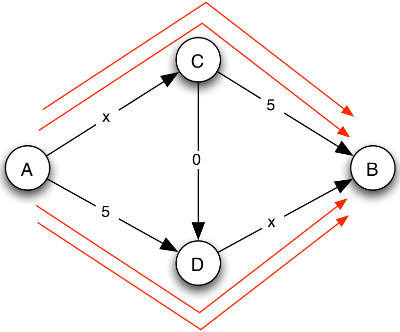
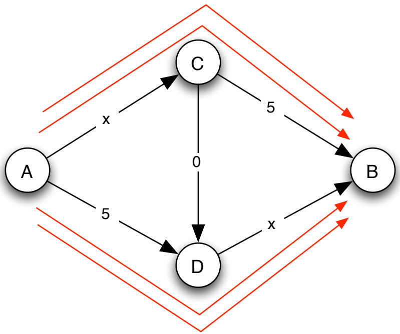
- **Nash Equilibrium**: No driver can reduce their travel time by switching paths, given others' choices.



(a) The social optimum.



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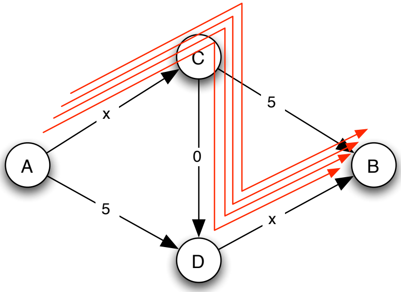

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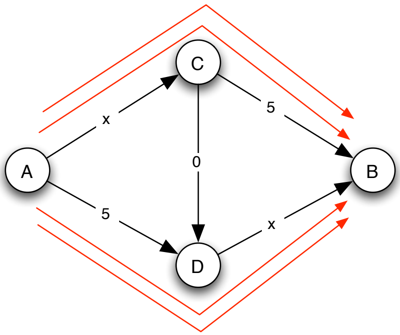

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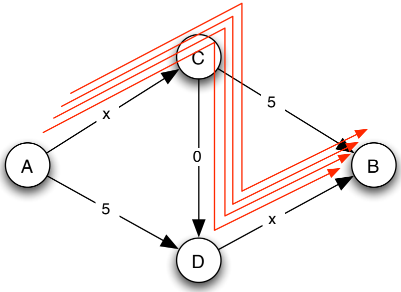

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Figure: A version of Braess's Paradox: In the socially optimal traffic pattern, the social cost is 28, while in the unique Nash equilibrium, the social cost is 32.

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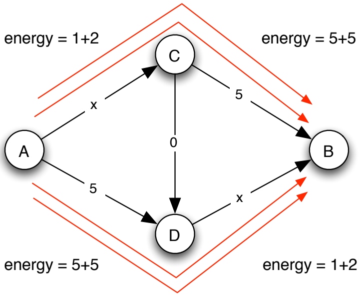
- Social cost can increase or decrease with best-response steps (e.g., from 28 to 32 in the Braess example), but potential energy strictly decreases, serving as a progress measure.

Analyzing Dynamics



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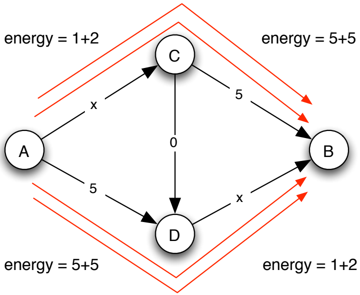

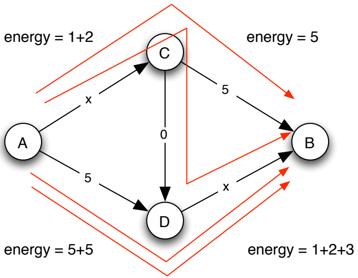

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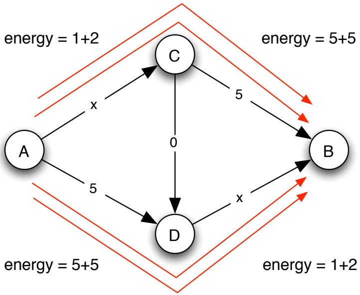

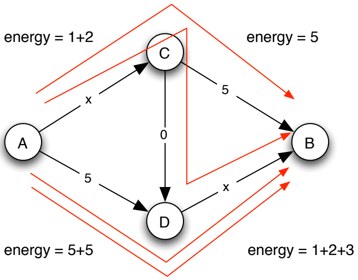

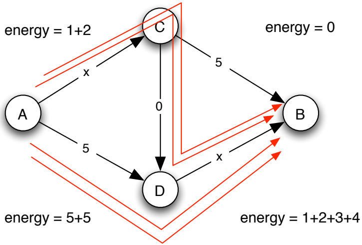

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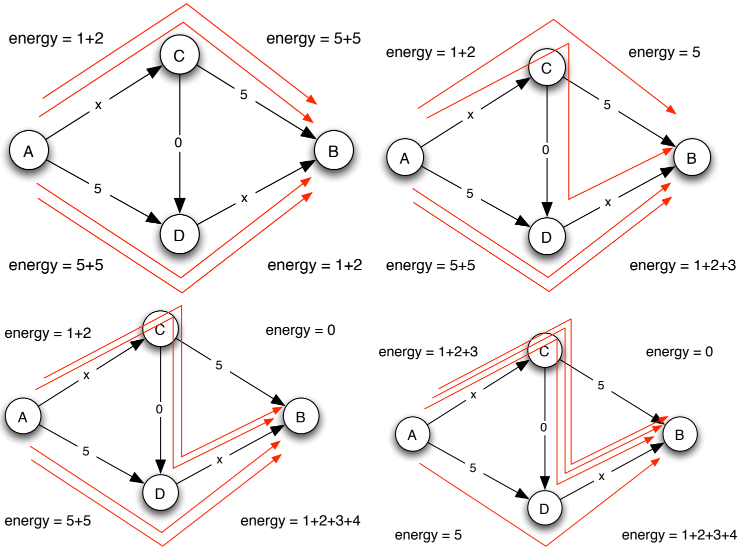

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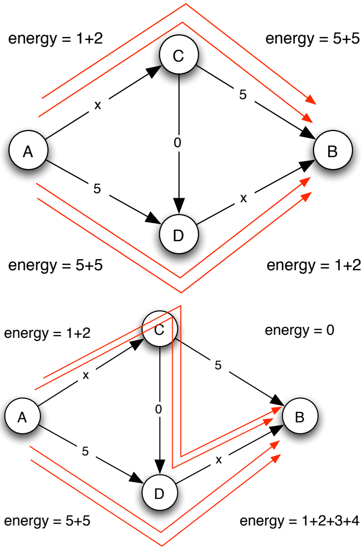

Figure: Steps of best-response dynamics with potential energy changes.

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- With finite patterns, dynamics must stop at an equilibrium.

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- Implications: Network design and tolls can mitigate inefficiencies.

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