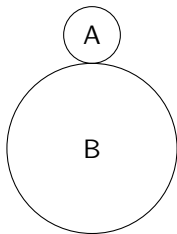


Surprises & Paradoxes II

- 1 Coin Rotation Paradox
- 2 Braess Paradox
- 3 The Social Cost of Traffic at Equilibrium

Coin Rotation Paradox

The 1982 SAT Question Everyone Got Wrong



The radius of circle A is $\frac{1}{3}$ of the radius of circle B. Circle A rolls around circle B one trip back to its starting point. How many times will circle A revolve in total?

- (a) $\frac{3}{2}$ (b) 3 (c) 6 (d) $\frac{9}{2}$ (e) 9

Animation of Coin Rotation Paradox: Python Code I

```
import math
import matplotlib.pyplot as plt
from matplotlib.patches import Circle, Wedge
from matplotlib.transforms import Affine2D
from matplotlib.backends.backend_pdf import PdfPages
import os; os.chdir(os.path.dirname(__file__))

# Set the radii
R = 1.5 # Radius of the outer (stationary) coin
r = 0.5 # Radius of the inner (rolling) coin

def create_logo(ax, x, y, radius, angle):
    # Create a simple smiley face logo
    face = Circle((0, 0), radius * 0.7, fill=True, color="cyan")
    left_eye = Circle((-radius * 0.3, radius * 0.2), radius *
        ↪ 0.1, fill=True, color="black")
    right_eye = Circle((radius * 0.3, radius * 0.2), radius *
        ↪ 0.1, fill=True, color="black")
```

Animation of Coin Rotation Paradox: Python Code II

```
mouth = Wedge((0, 0), radius * 0.5, 225, 315, width=radius *  
    ↪ 0.1, color="black")
```

```
# Create a transformation
```

```
t = Affine2D().rotate(-angle).translate(x, y) + ax.transData
```

```
# Apply the transformation to all elements
```

```
face.set_transform(t)
```

```
left_eye.set_transform(t)
```

```
right_eye.set_transform(t)
```

```
mouth.set_transform(t)
```

```
ax.add_artist(face)
```

```
ax.add_artist(left_eye)
```

```
ax.add_artist(right_eye)
```

```
ax.add_artist(mouth)
```

```
def create_frame(angle):
```

Animation of Coin Rotation Paradox: Python Code III

```
# Create figure and axis
```

```
fig, ax = plt.subplots(figsize=(8, 8))
```

```
# Set limits and aspect ratio
```

```
ax.set_xlim(-(R + r + 1), R + r + 1)
```

```
ax.set_ylim(-(R + r + 1), R + r + 1)
```

```
ax.set_aspect("equal")
```

```
# Remove axis ticks
```

```
ax.set_xticks([])
```

```
ax.set_yticks([])
```

```
# Calculate position of the rolling coin's center
```

```
x = (R + r) * math.cos(angle)
```

```
y = (R + r) * math.sin(angle)
```

```
# Calculate position of the point on the edge of the moving  
↪ coin
```

Animation of Coin Rotation Paradox: Python Code IV

```
point_x = x + r * math.cos((R + r) / r * angle)
point_y = y + r * math.sin((R + r) / r * angle)
```

```
# Draw stationary coin
```

```
stationary_coin = Circle((0, 0), R, fill=False, color="blue")
ax.add_artist(stationary_coin)
```

```
# Draw rotating coin
```

```
rotating_coin = Circle((x, y), r, fill=False, color="red")
ax.add_artist(rotating_coin)
```

```
# Add logo to the rotating coin
```

```
create_logo(ax, x, y, r, (R + r) / r * angle)
```

```
# Draw point on the edge of the moving coin
```

```
ax.plot(point_x, point_y, "ro")
```

```
# Draw cardioid path
```


Animation of Coin Rotation Paradox: Python Code V

```
theta = [i * 2 * math.pi / 100 for i in range(int(100 * angle
↪ / (2 * math.pi)) + 1)]
cardioid_x = [(R + r) * math.cos(t) + r * math.cos((R + r) /
↪ r * t) for t in theta]
cardioid_y = [(R + r) * math.sin(t) + r * math.sin((R + r) /
↪ r * t) for t in theta]
ax.plot(cardioid_x, cardioid_y, "g-")
```

```
# Add title
```

```
ax.set_title(f"Rotation: {angle/(2*math.pi)*360:.1f}°")
```

```
return fig
```

```
# Generate 512 frames and save them into a single PDF
```

```
num_frames = 512
```

```
pdf_filename = 'coin_rotation_paradox.pdf'
```

```
tex = []
```

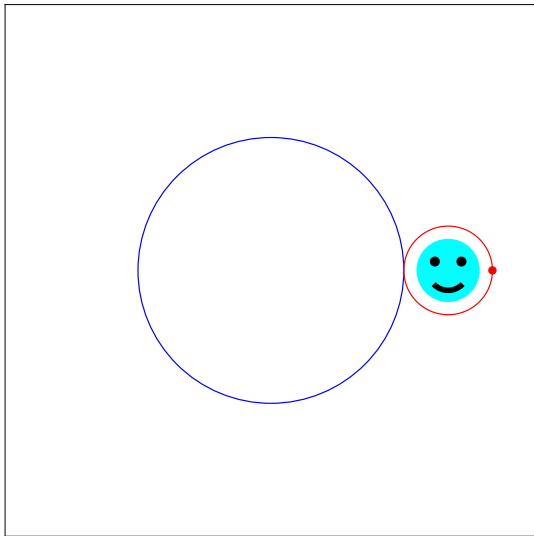
Animation of Coin Rotation Paradox: Python Code VI

```
with PdfPages(pdf_filename) as pdf:
    for i in range(num_frames + 1):
        angle = i * 2 * math.pi / num_frames
        fig = create_frame(angle)
        pdf.savefig(fig, bbox_inches='tight', pad_inches=0.1,
            ↪ dpi=300)
        plt.close(fig)
        # Generate LaTeX commands referencing pages (1-based
        ↪ indexing)
        tex.append(f'\\only<{i +
            ↪ 1}>{{\\includegraphics[width=.6\\textwidth, page={i
            ↪ + 1]}}{{fig/note02/coin_rotation_paradox.pdf}}\\noind
            ↪ ent}}')

# Write the LaTeX overlayarea content to a .tex file
open('../..//coin.tex',
    ↪ 'w').write(f"\\hspace*{{\\dimexpr(\\textwidth - .6\\textwidt
    ↪ h)/2\\relax}}\\n\\begin{{overlayarea}}{{\\textwidth}}{{0.7\\t
    ↪ extheight}}\\n{'\\n'.join(tex)}\\n\\end{{overlayarea}}")
```

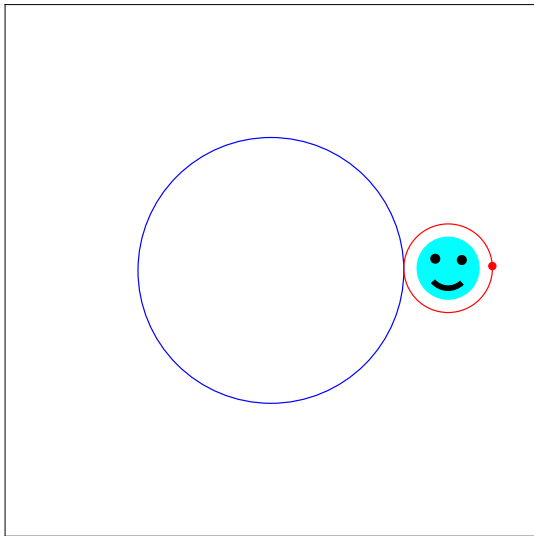
Animation of Coin Rotation Paradox

Rotation: 0.0°



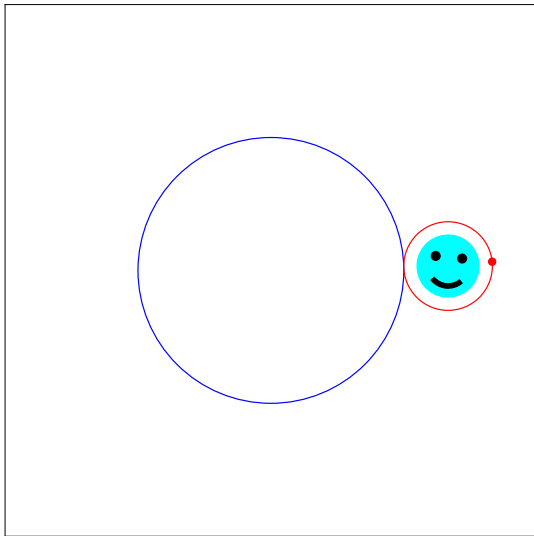
Animation of Coin Rotation Paradox

Rotation: 0.7°



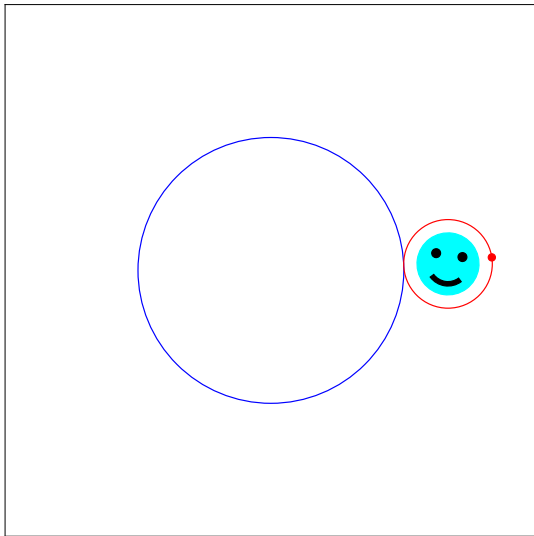
Animation of Coin Rotation Paradox

Rotation: 1.4°



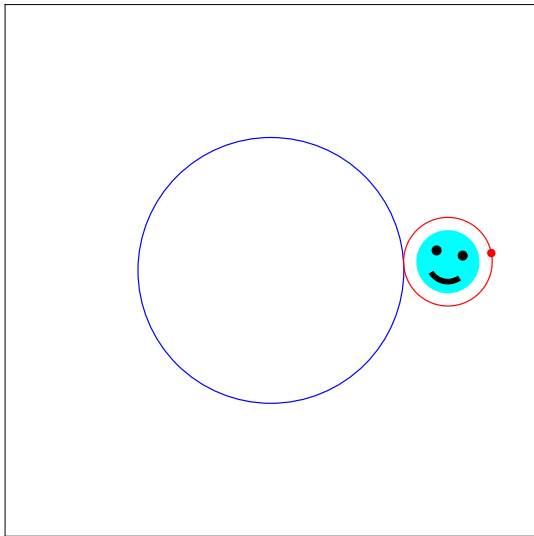
Animation of Coin Rotation Paradox

Rotation: 2.1°



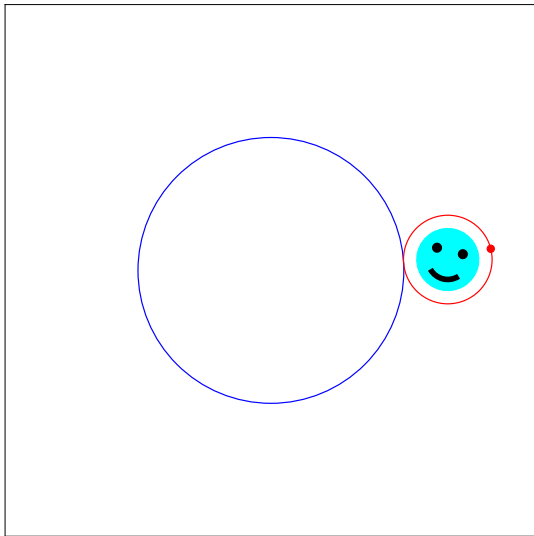
Animation of Coin Rotation Paradox

Rotation: 2.8°



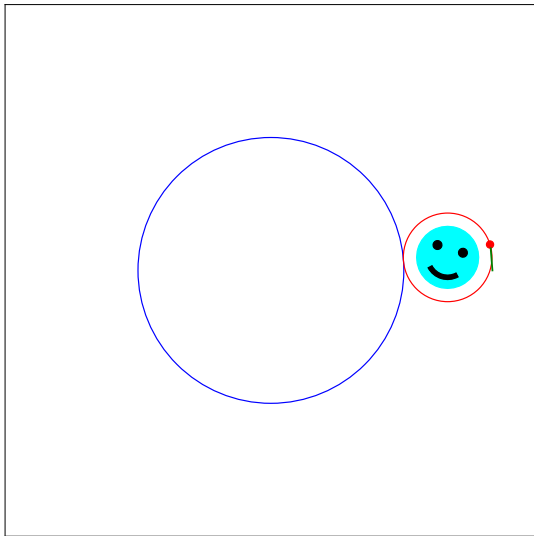
Animation of Coin Rotation Paradox

Rotation: 3.5°



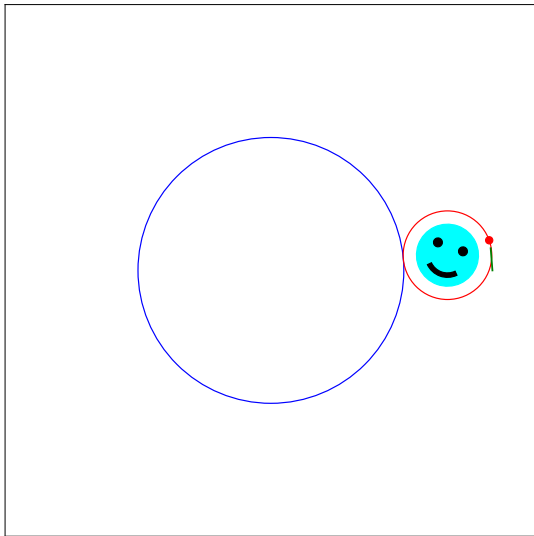
Animation of Coin Rotation Paradox

Rotation: 4.2°



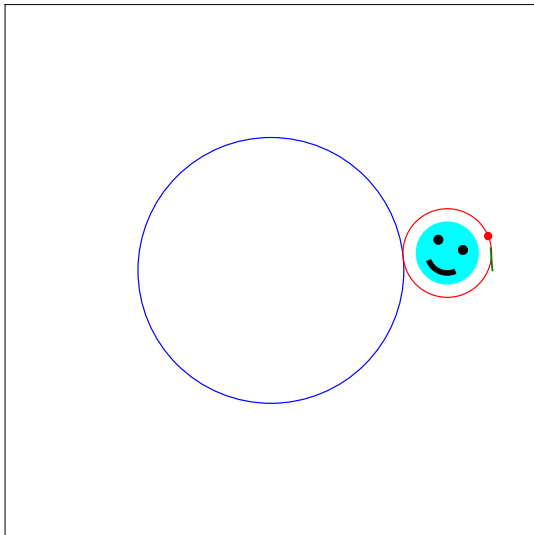
Animation of Coin Rotation Paradox

Rotation: 4.9°



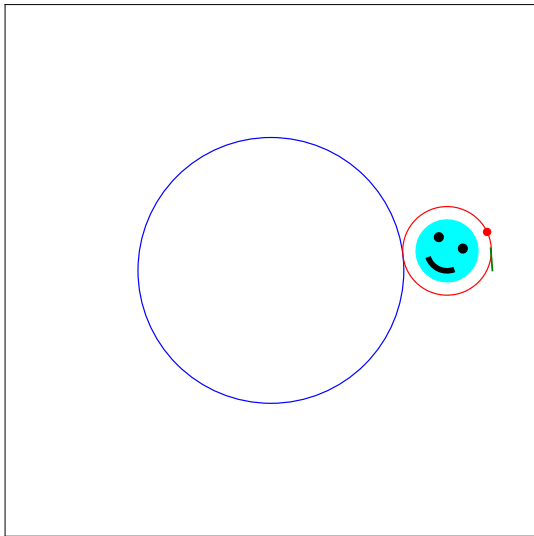
Animation of Coin Rotation Paradox

Rotation: 5.6°



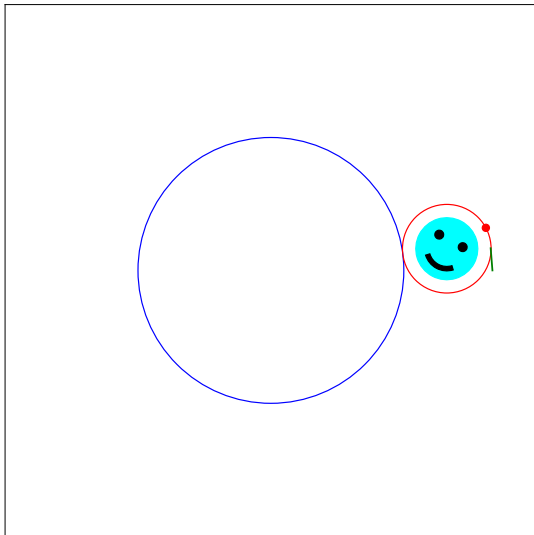
Animation of Coin Rotation Paradox

Rotation: 6.3°



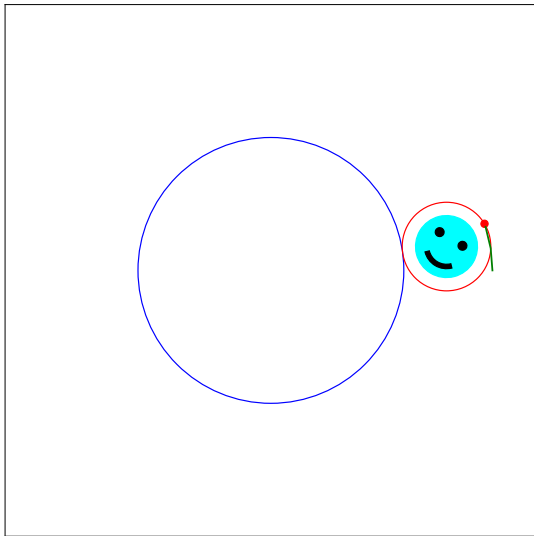
Animation of Coin Rotation Paradox

Rotation: 7.0°



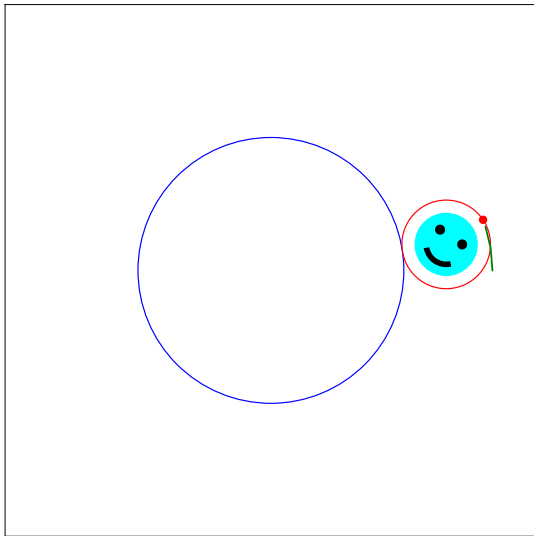
Animation of Coin Rotation Paradox

Rotation: 7.7°



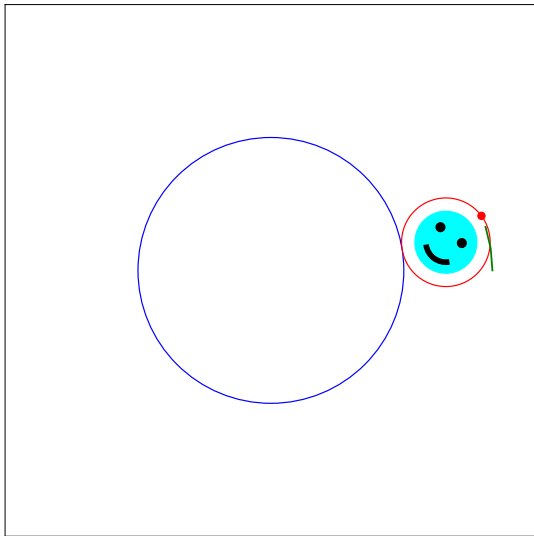
Animation of Coin Rotation Paradox

Rotation: 8.4°



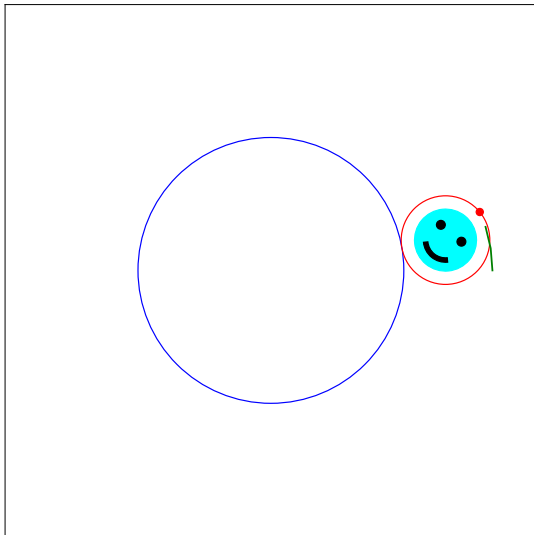
Animation of Coin Rotation Paradox

Rotation: 9.1°



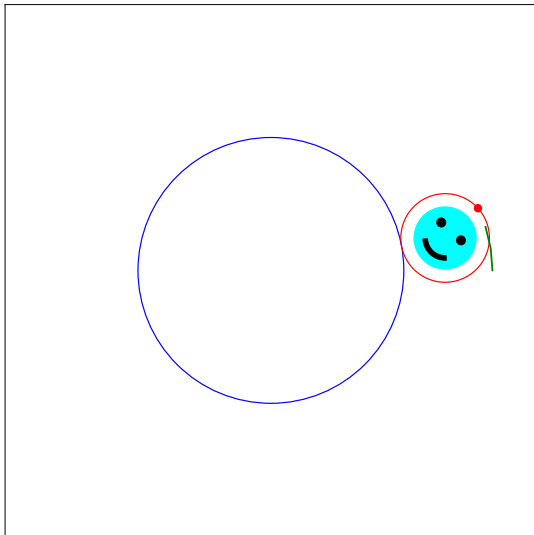
Animation of Coin Rotation Paradox

Rotation: 9.8°



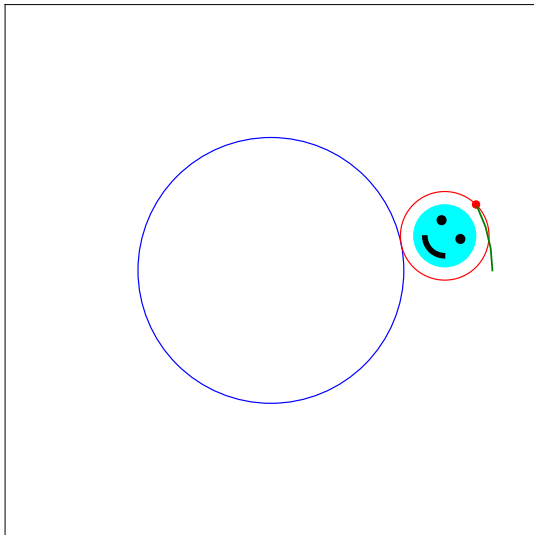
Animation of Coin Rotation Paradox

Rotation: 10.5°



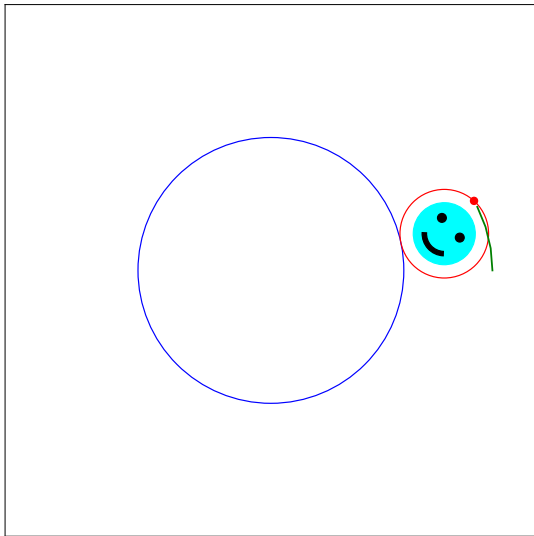
Animation of Coin Rotation Paradox

Rotation: 11.2°



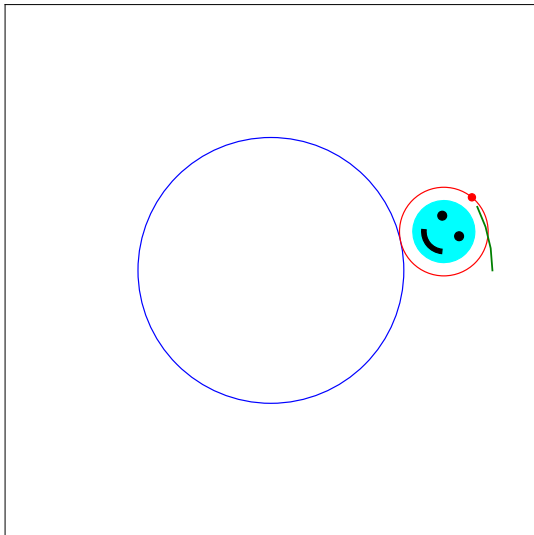
Animation of Coin Rotation Paradox

Rotation: 12.0°



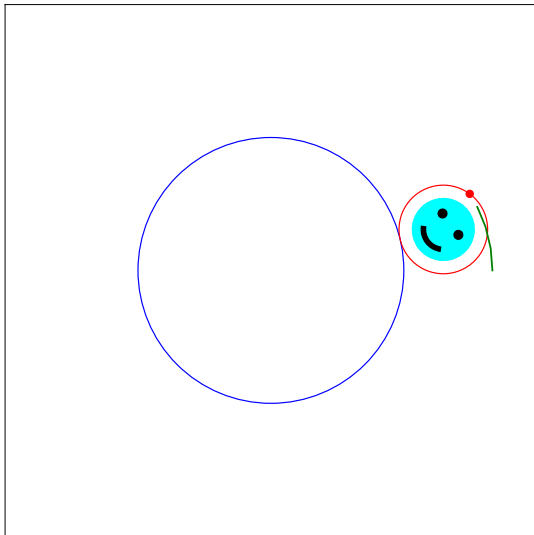
Animation of Coin Rotation Paradox

Rotation: 12.7°



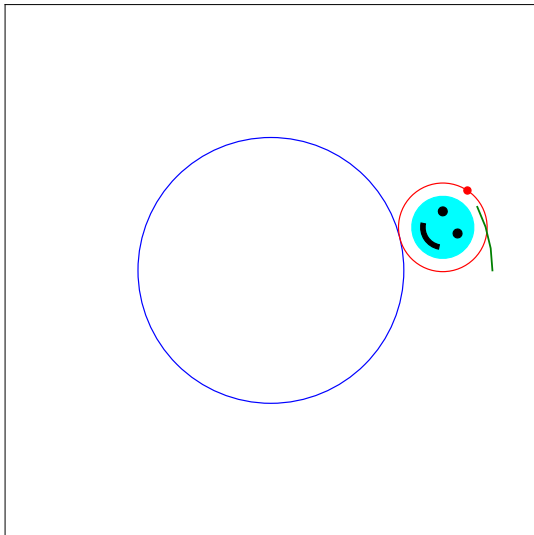
Animation of Coin Rotation Paradox

Rotation: 13.4°



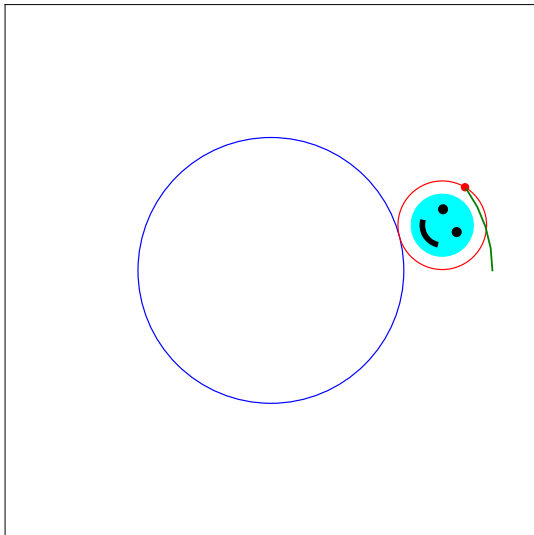
Animation of Coin Rotation Paradox

Rotation: 14.1°



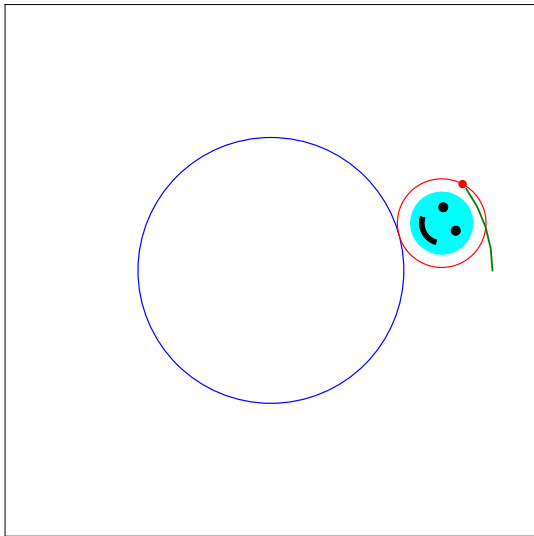
Animation of Coin Rotation Paradox

Rotation: 14.8°



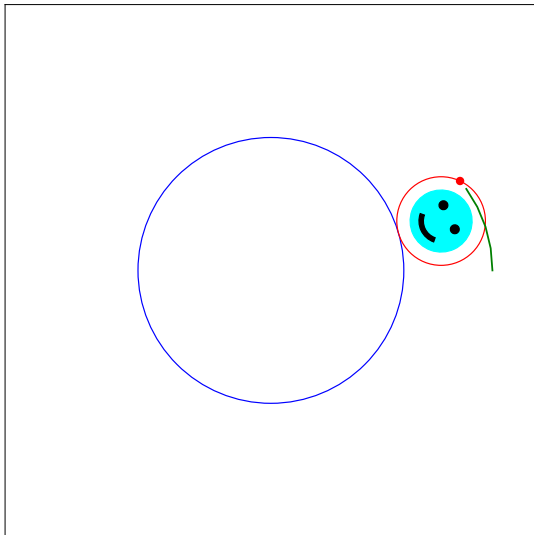
Animation of Coin Rotation Paradox

Rotation: 15.5°



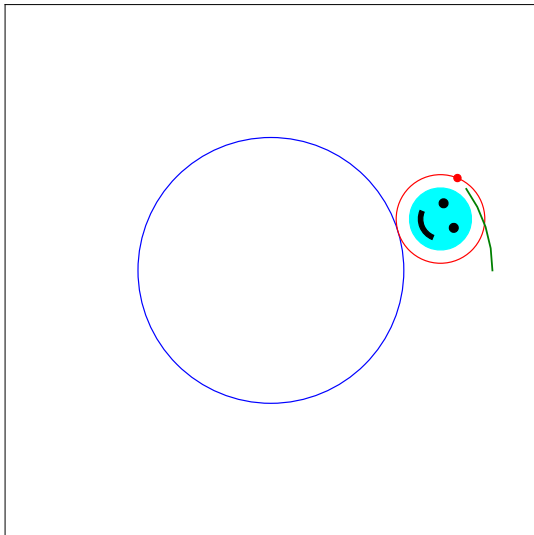
Animation of Coin Rotation Paradox

Rotation: 16.2°



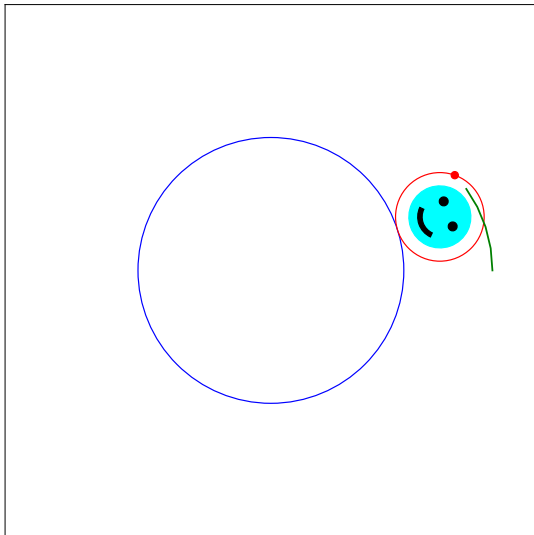
Animation of Coin Rotation Paradox

Rotation: 16.9°



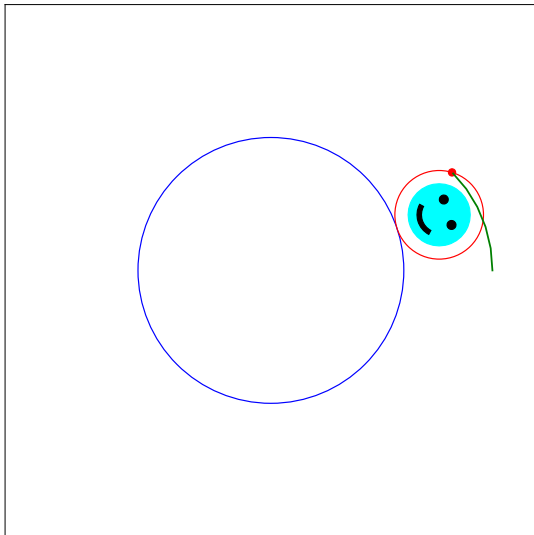
Animation of Coin Rotation Paradox

Rotation: 17.6°



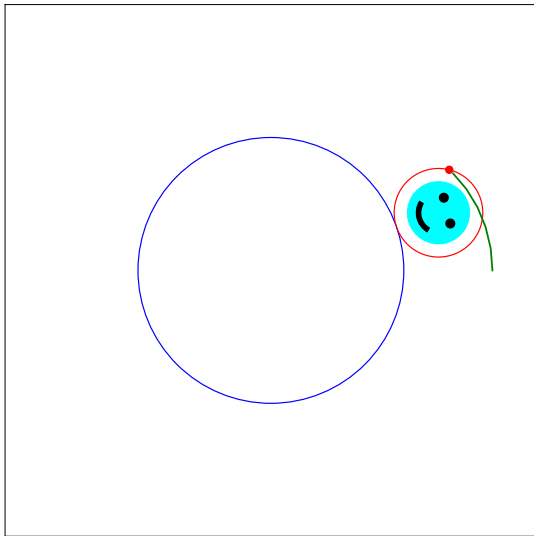
Animation of Coin Rotation Paradox

Rotation: 18.3°



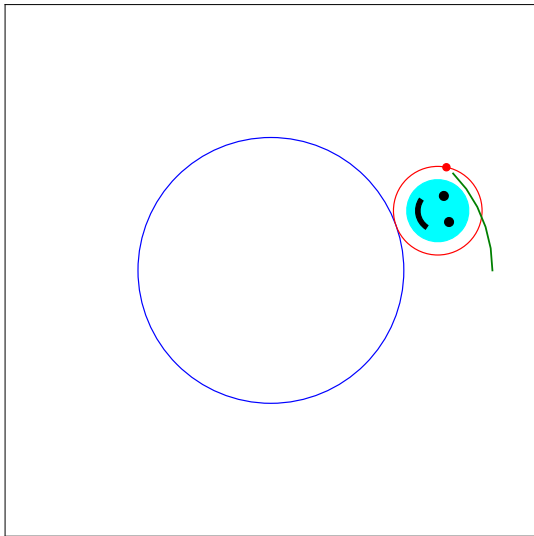
Animation of Coin Rotation Paradox

Rotation: 19.0°



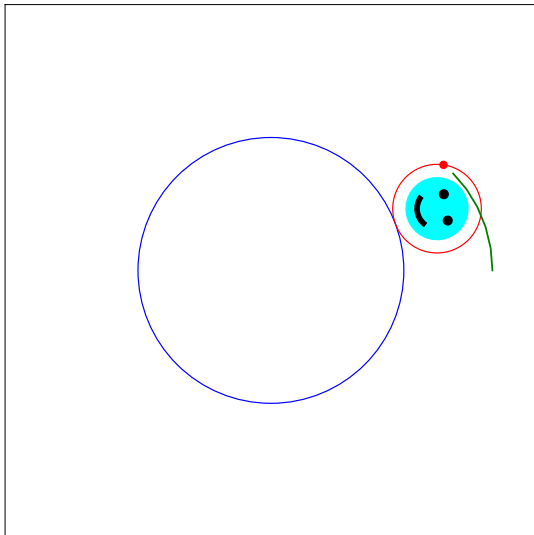
Animation of Coin Rotation Paradox

Rotation: 19.7°



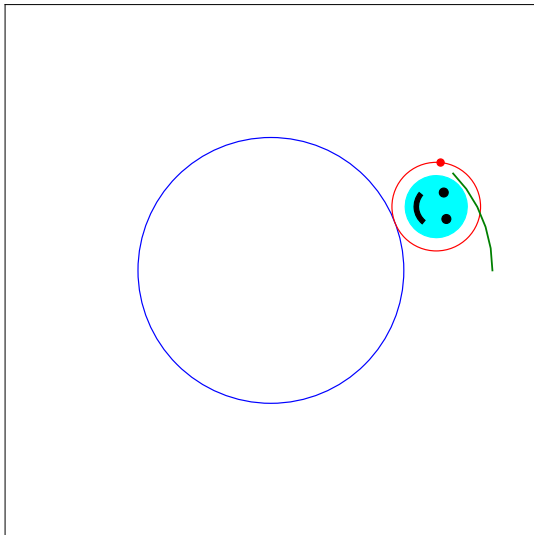
Animation of Coin Rotation Paradox

Rotation: 20.4°



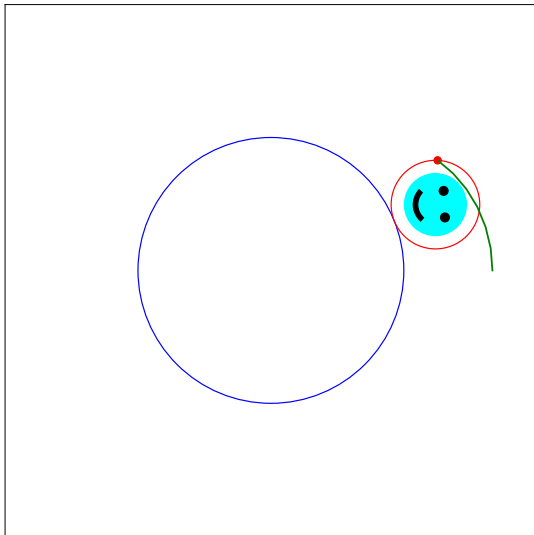
Animation of Coin Rotation Paradox

Rotation: 21.1°



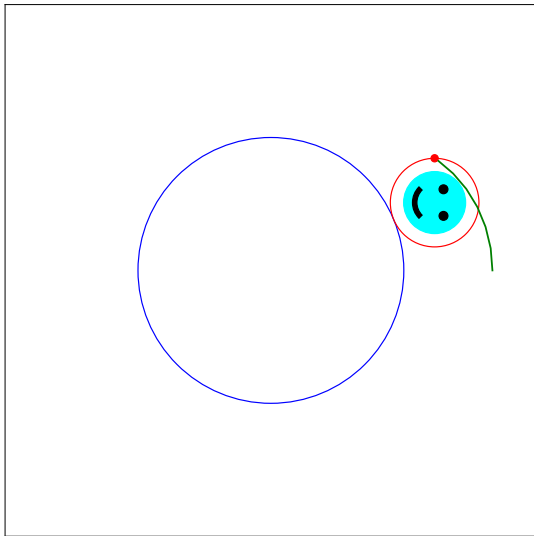
Animation of Coin Rotation Paradox

Rotation: 21.8°



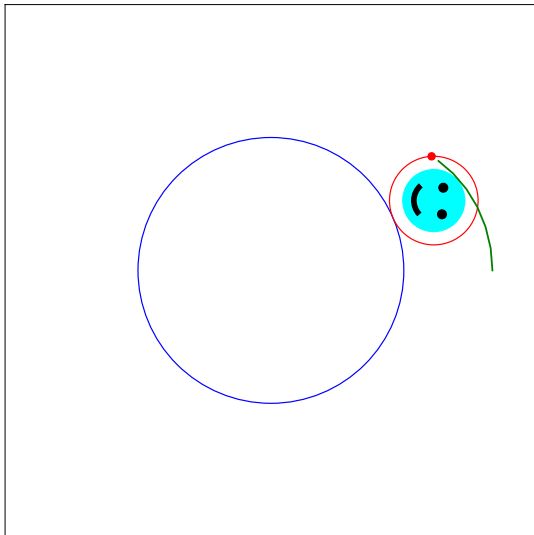
Animation of Coin Rotation Paradox

Rotation: 22.5°



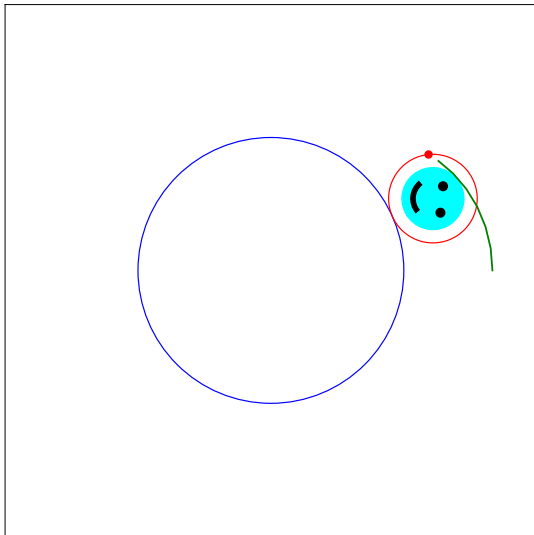
Animation of Coin Rotation Paradox

Rotation: 23.2°



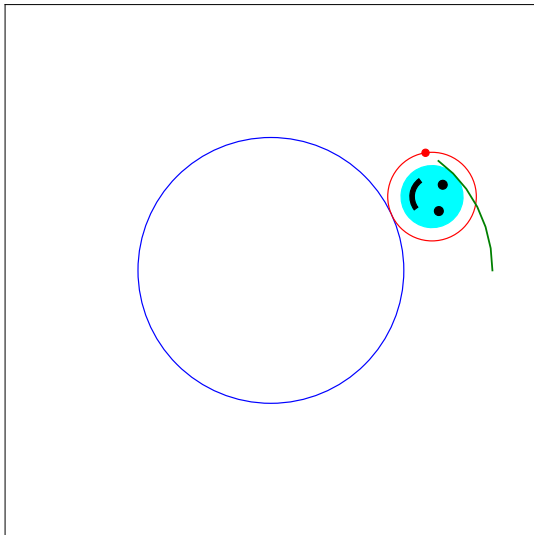
Animation of Coin Rotation Paradox

Rotation: 23.9°



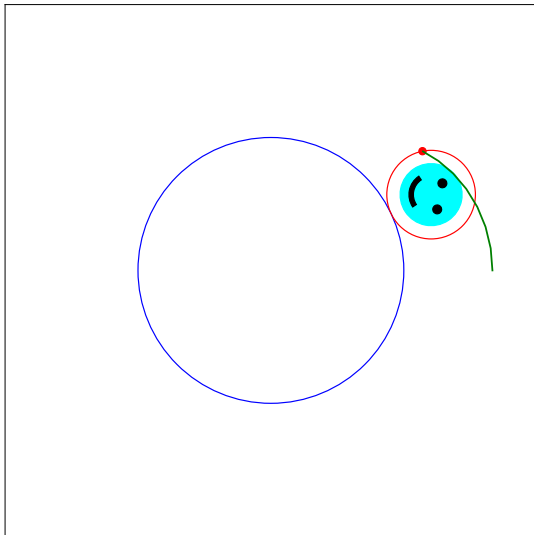
Animation of Coin Rotation Paradox

Rotation: 24.6°



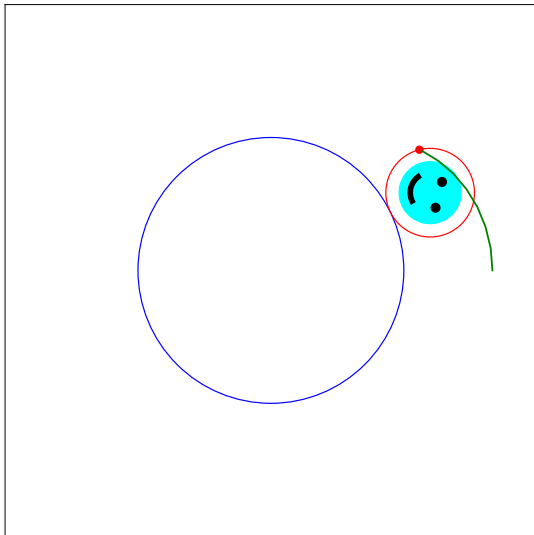
Animation of Coin Rotation Paradox

Rotation: 25.3°



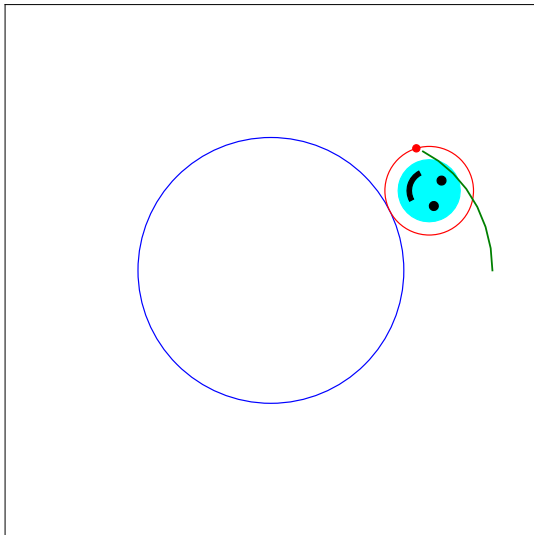
Animation of Coin Rotation Paradox

Rotation: 26.0°



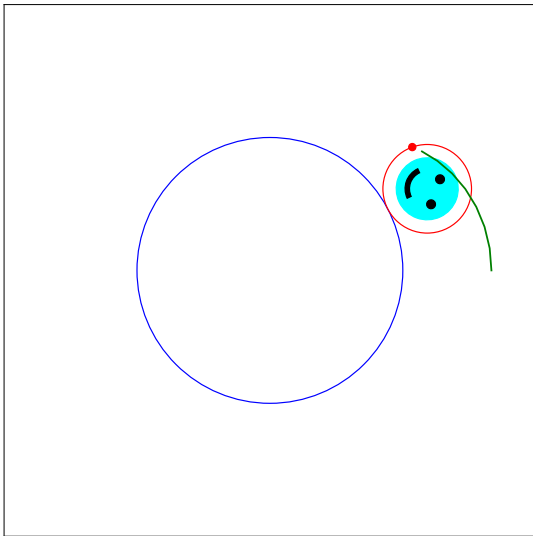
Animation of Coin Rotation Paradox

Rotation: 26.7°



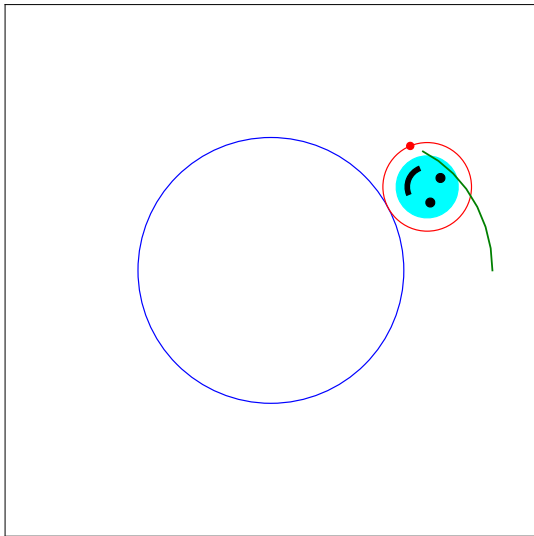
Animation of Coin Rotation Paradox

Rotation: 27.4°



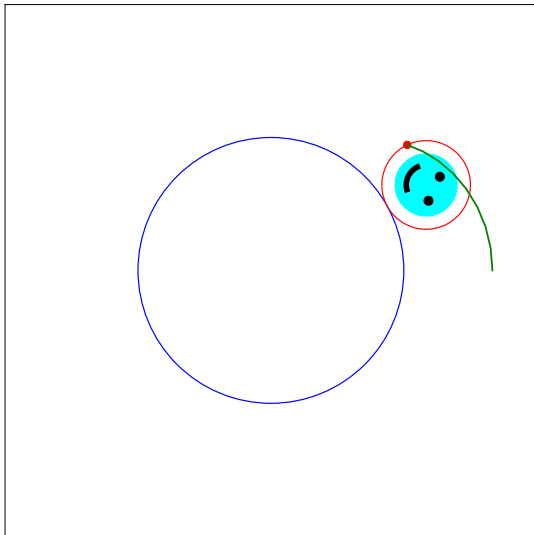
Animation of Coin Rotation Paradox

Rotation: 28.1°



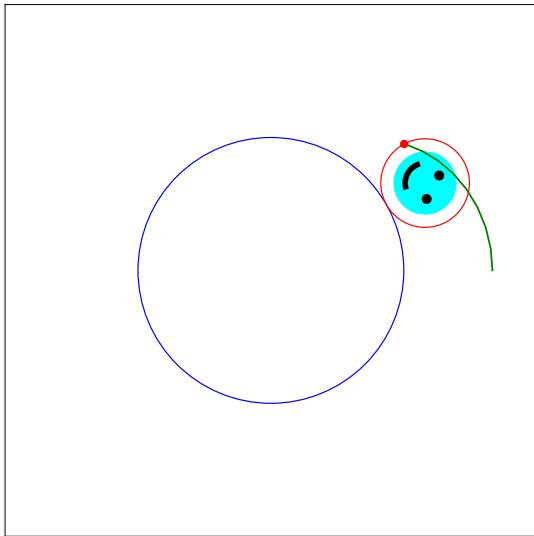
Animation of Coin Rotation Paradox

Rotation: 28.8°



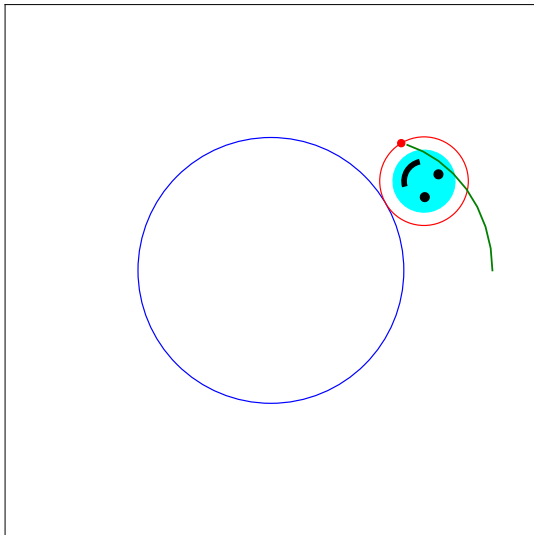
Animation of Coin Rotation Paradox

Rotation: 29.5°



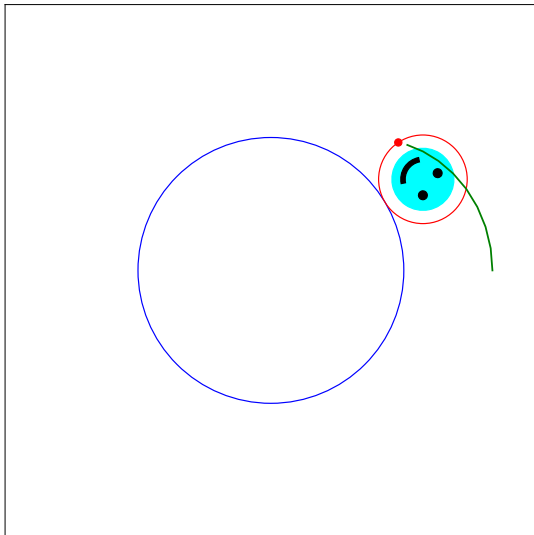
Animation of Coin Rotation Paradox

Rotation: 30.2°



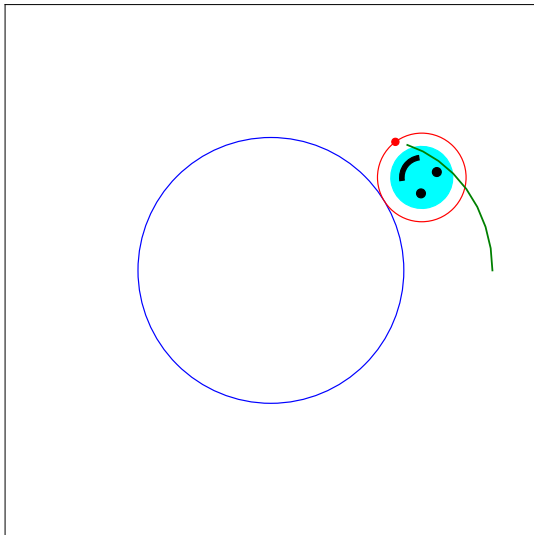
Animation of Coin Rotation Paradox

Rotation: 30.9°



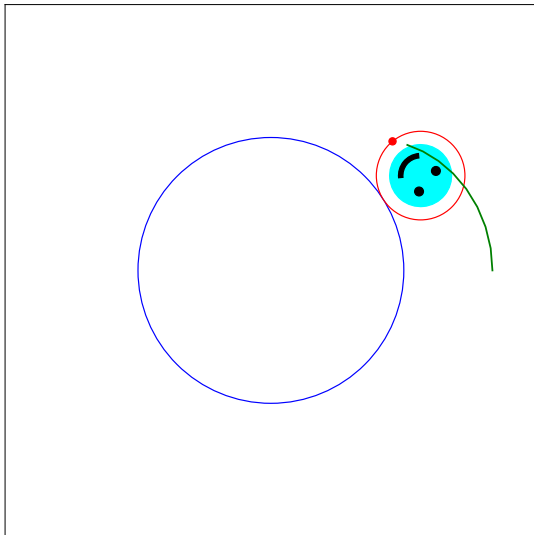
Animation of Coin Rotation Paradox

Rotation: 31.6°



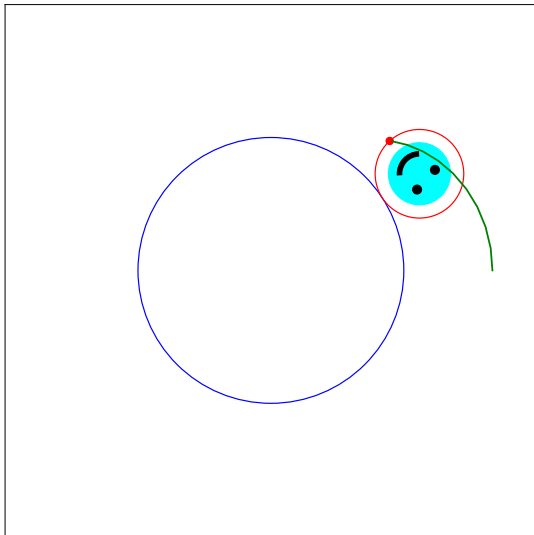
Animation of Coin Rotation Paradox

Rotation: 32.3°



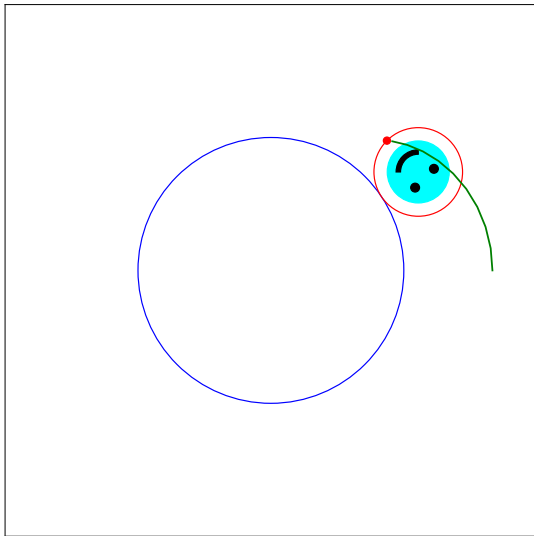
Animation of Coin Rotation Paradox

Rotation: 33.0°



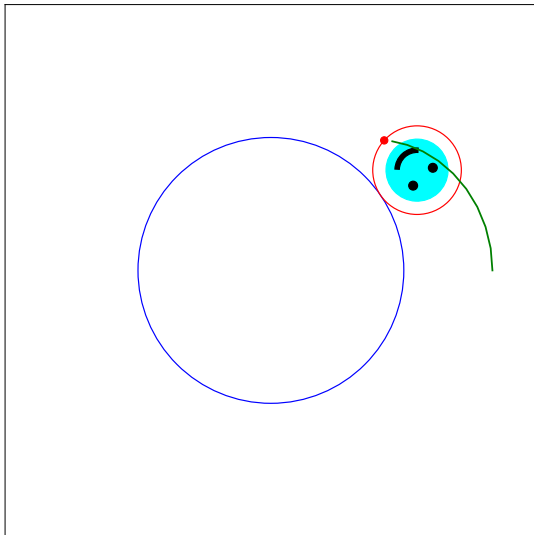
Animation of Coin Rotation Paradox

Rotation: 33.8°



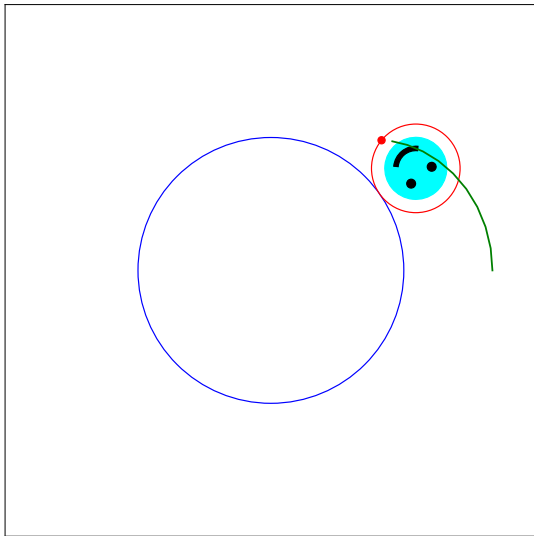
Animation of Coin Rotation Paradox

Rotation: 34.5°



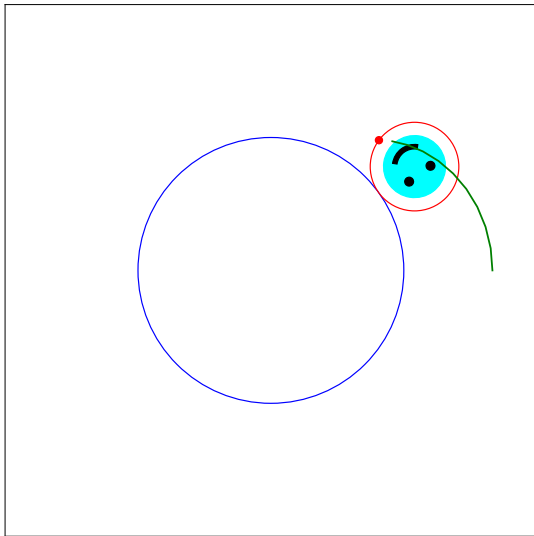
Animation of Coin Rotation Paradox

Rotation: 35.2°



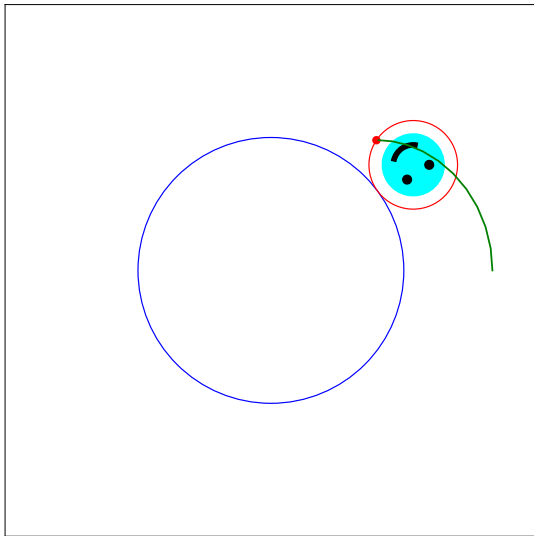
Animation of Coin Rotation Paradox

Rotation: 35.9°



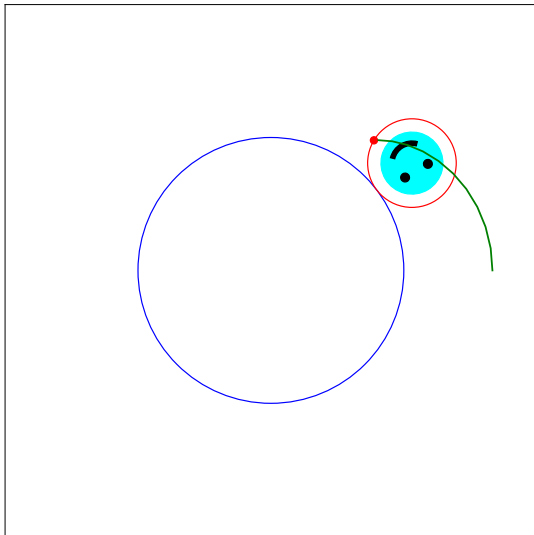
Animation of Coin Rotation Paradox

Rotation: 36.6°



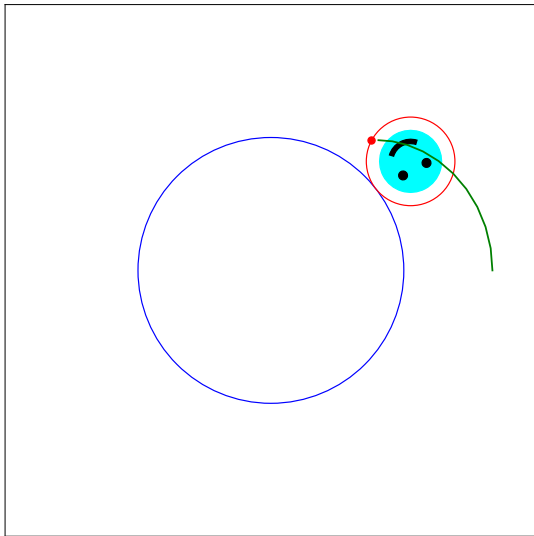
Animation of Coin Rotation Paradox

Rotation: 37.3°



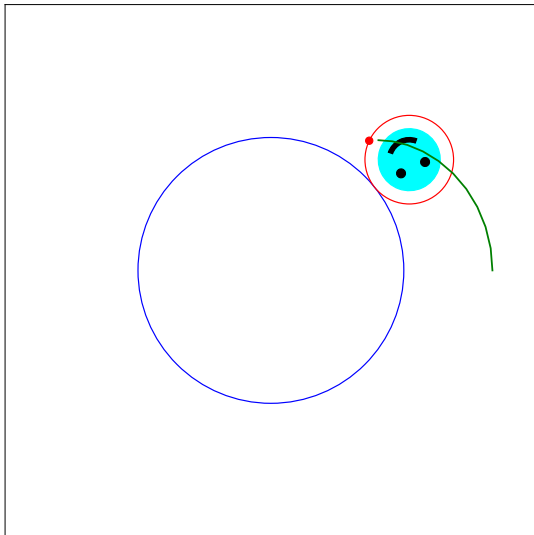
Animation of Coin Rotation Paradox

Rotation: 38.0°



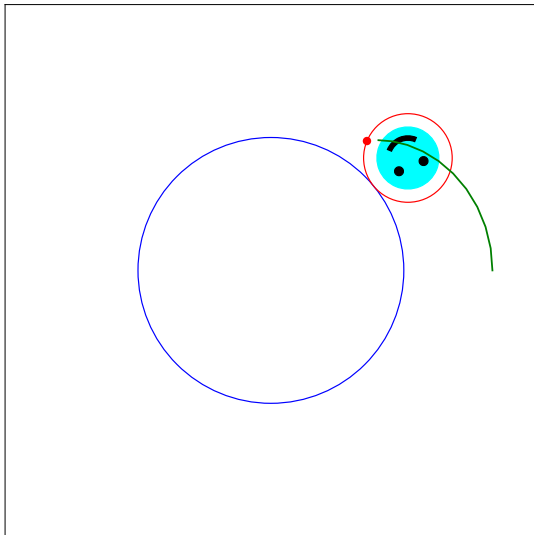
Animation of Coin Rotation Paradox

Rotation: 38.7°



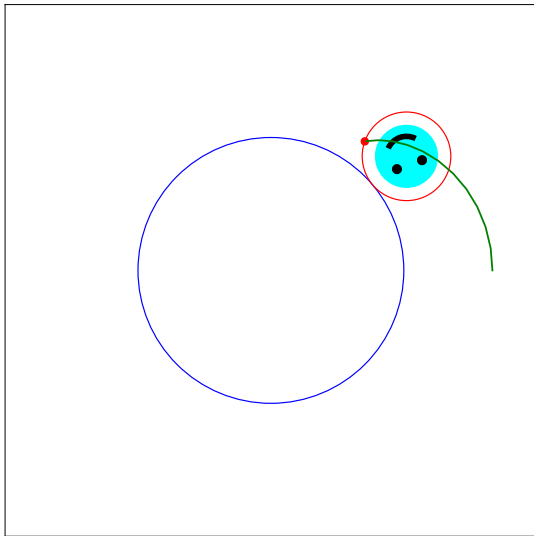
Animation of Coin Rotation Paradox

Rotation: 39.4°



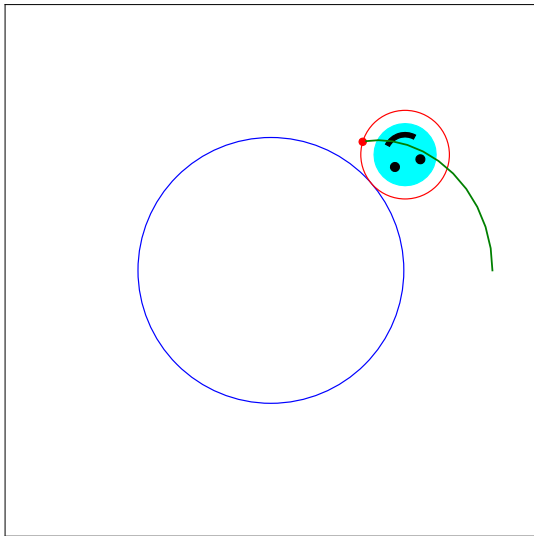
Animation of Coin Rotation Paradox

Rotation: 40.1°



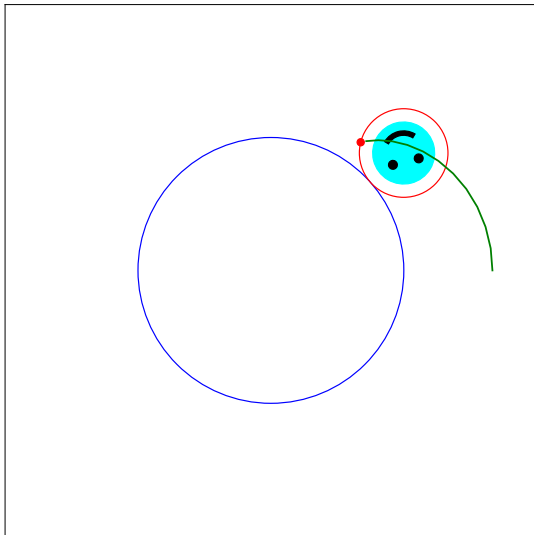
Animation of Coin Rotation Paradox

Rotation: 40.8°



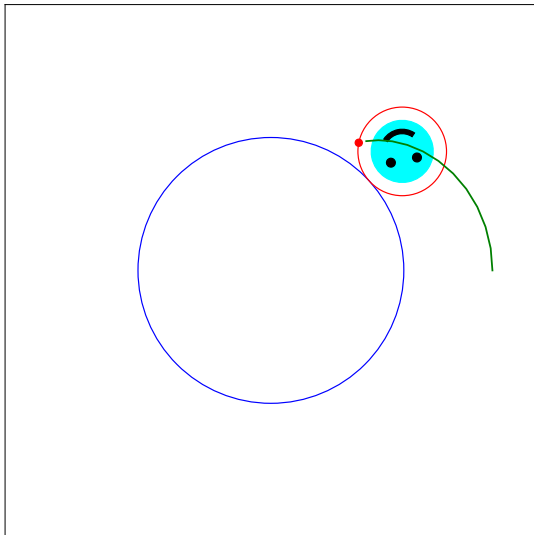
Animation of Coin Rotation Paradox

Rotation: 41.5°



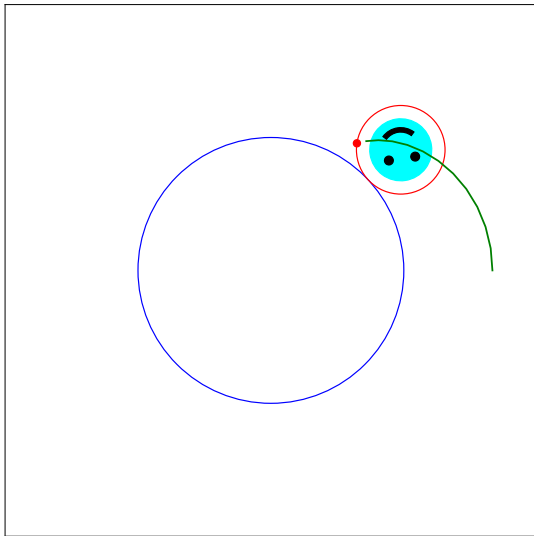
Animation of Coin Rotation Paradox

Rotation: 42.2°



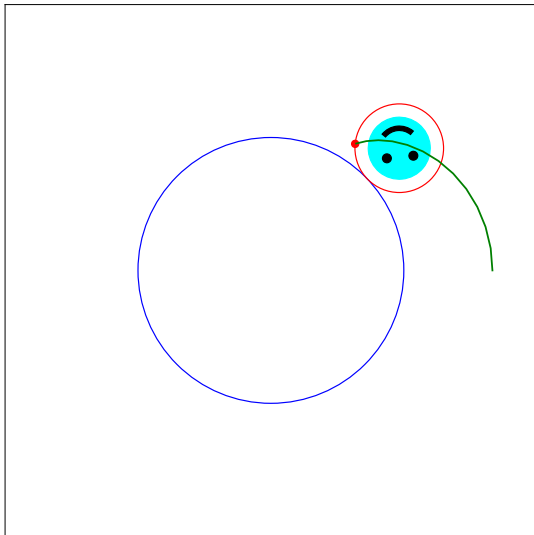
Animation of Coin Rotation Paradox

Rotation: 42.9°



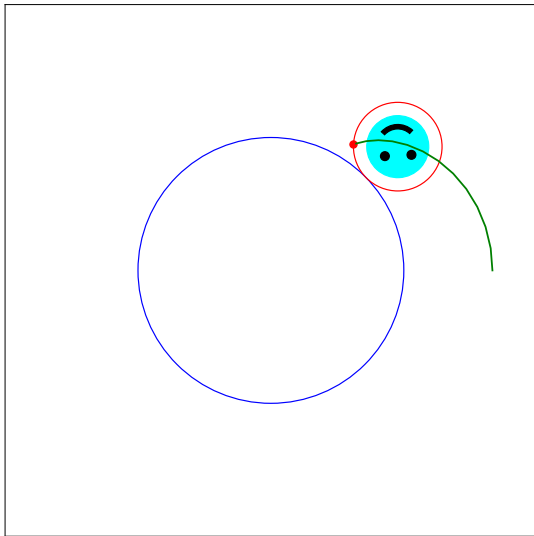
Animation of Coin Rotation Paradox

Rotation: 43.6°



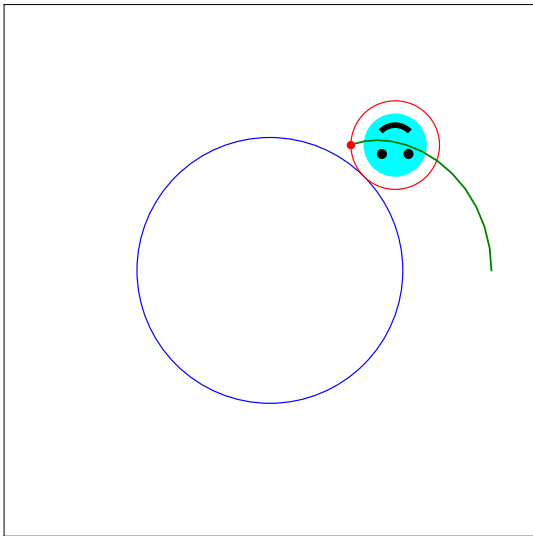
Animation of Coin Rotation Paradox

Rotation: 44.3°



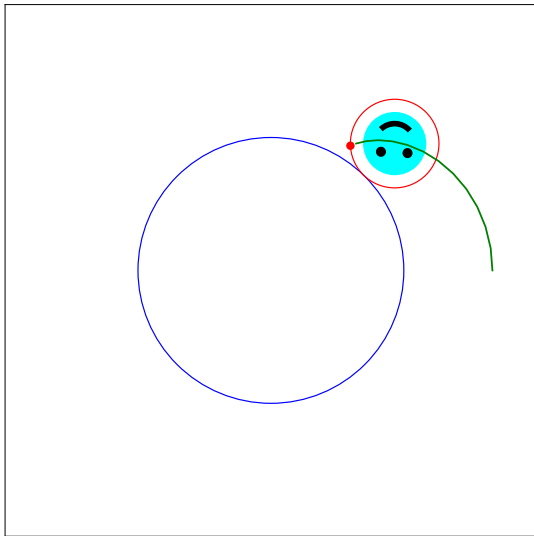
Animation of Coin Rotation Paradox

Rotation: 45.0°



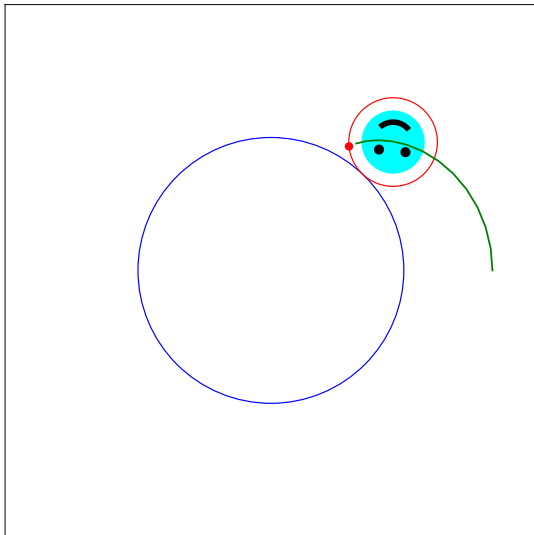
Animation of Coin Rotation Paradox

Rotation: 45.7°



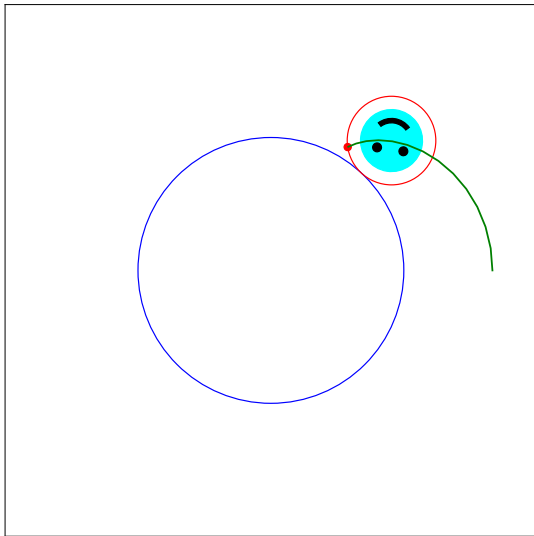
Animation of Coin Rotation Paradox

Rotation: 46.4°



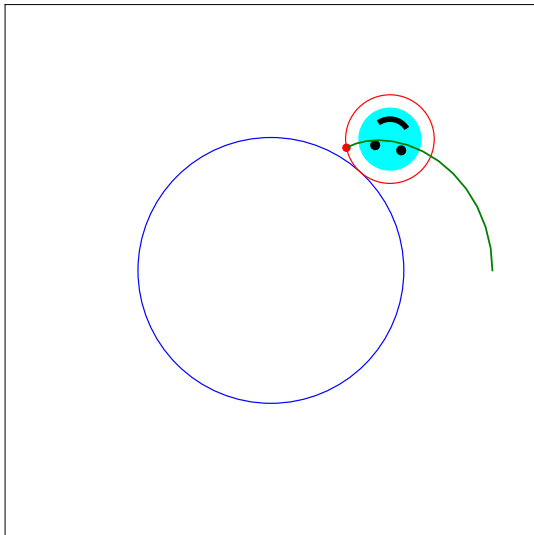
Animation of Coin Rotation Paradox

Rotation: 47.1°



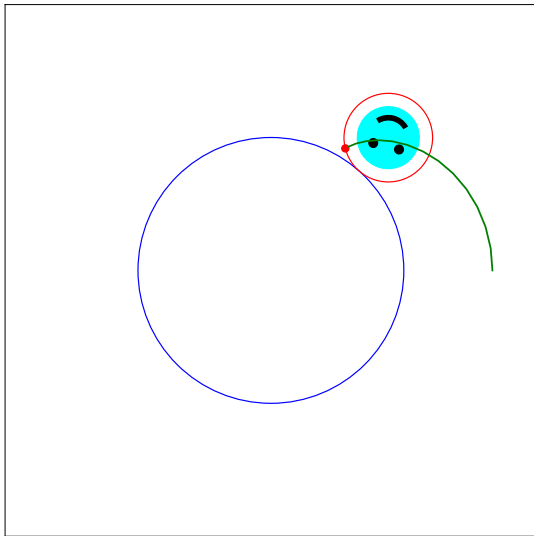
Animation of Coin Rotation Paradox

Rotation: 47.8°



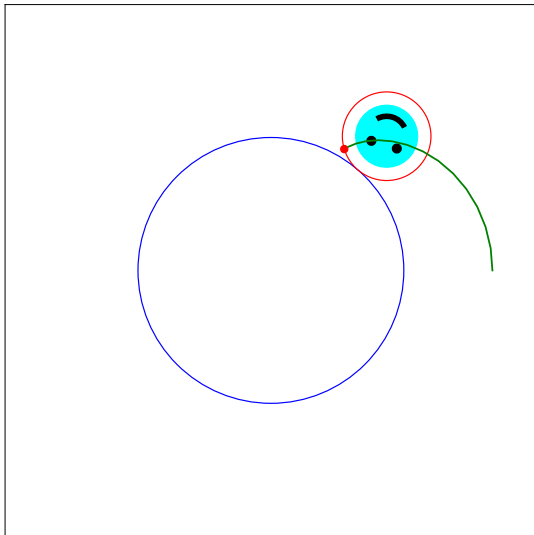
Animation of Coin Rotation Paradox

Rotation: 48.5°



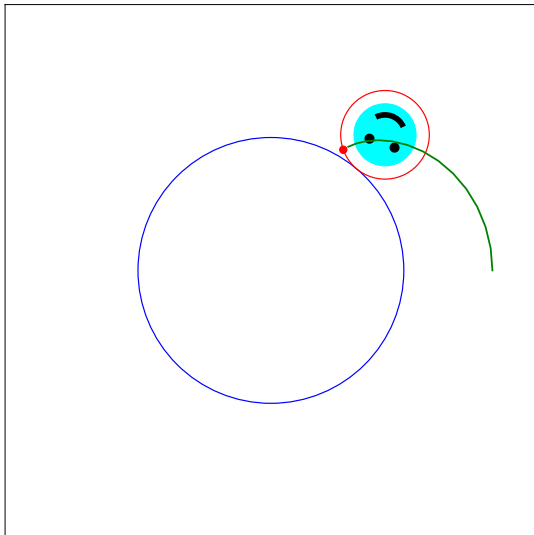
Animation of Coin Rotation Paradox

Rotation: 49.2°



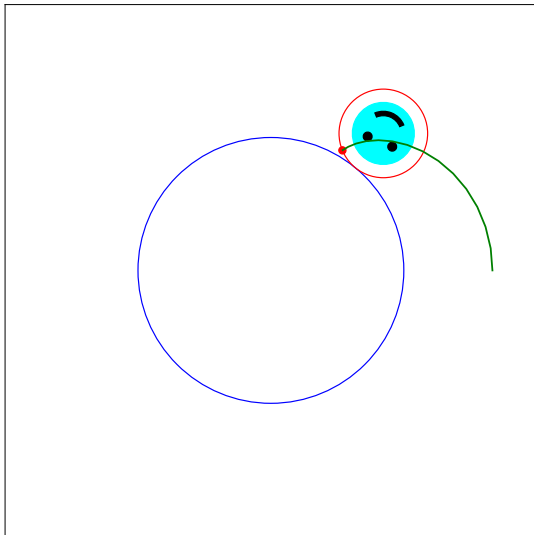
Animation of Coin Rotation Paradox

Rotation: 49.9°



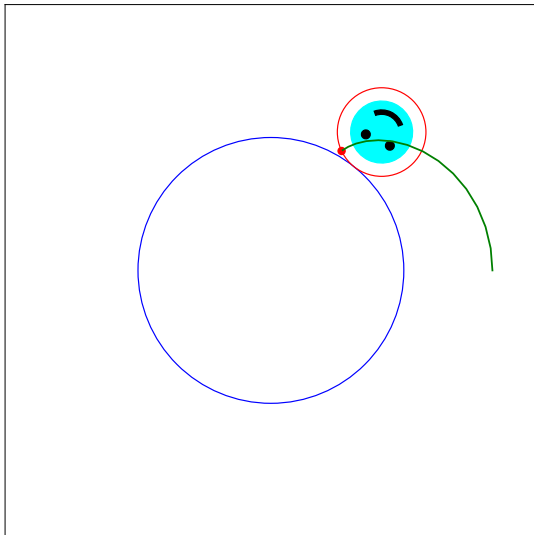
Animation of Coin Rotation Paradox

Rotation: 50.6°

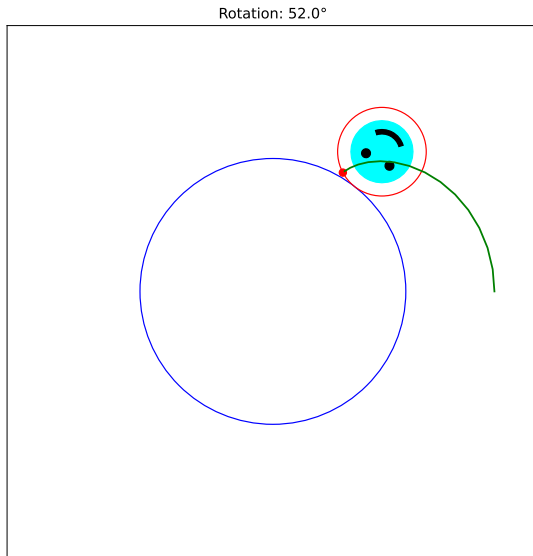


Animation of Coin Rotation Paradox

Rotation: 51.3°

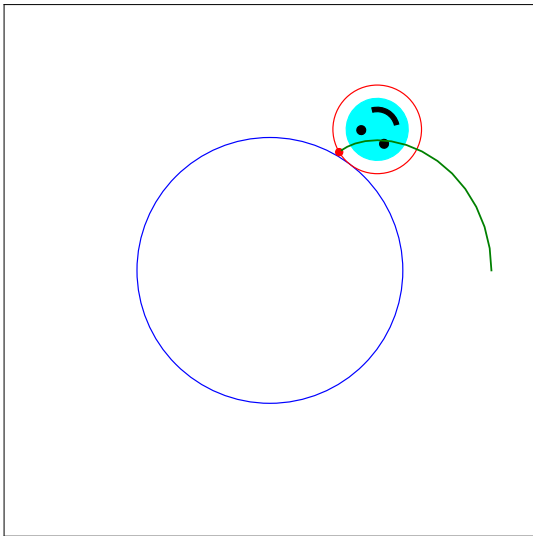


Animation of Coin Rotation Paradox



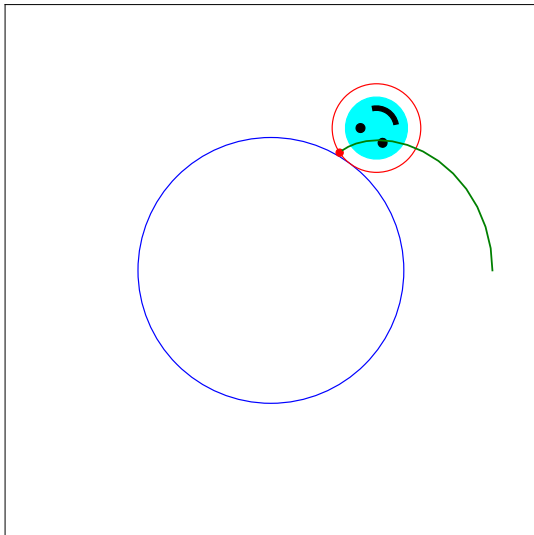
Animation of Coin Rotation Paradox

Rotation: 52.7°



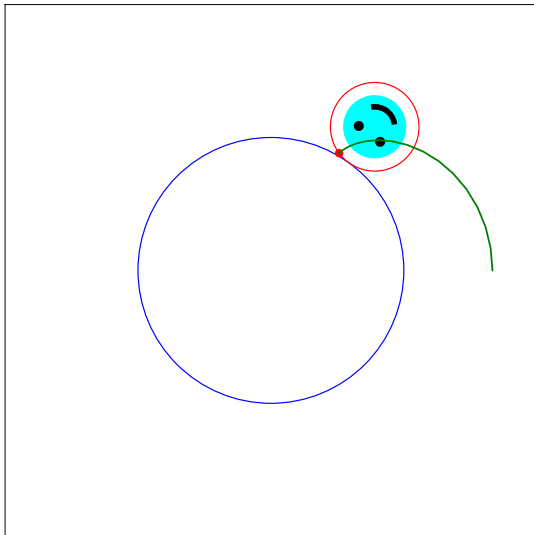
Animation of Coin Rotation Paradox

Rotation: 53.4°



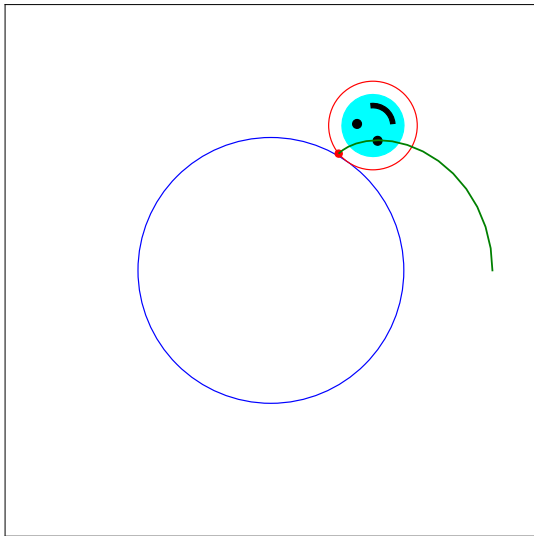
Animation of Coin Rotation Paradox

Rotation: 54.1°



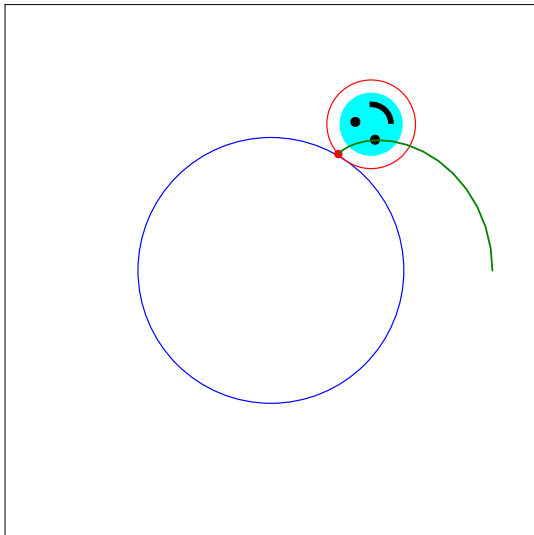
Animation of Coin Rotation Paradox

Rotation: 54.8°



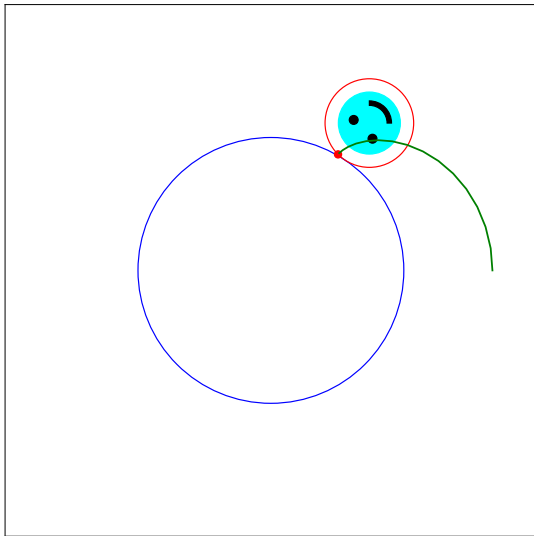
Animation of Coin Rotation Paradox

Rotation: 55.5°



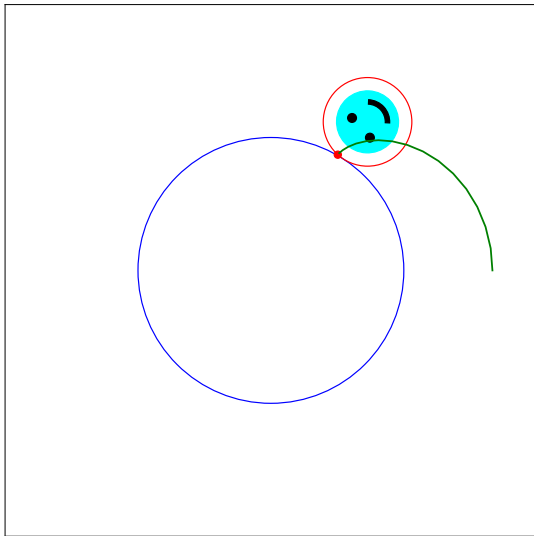
Animation of Coin Rotation Paradox

Rotation: 56.2°



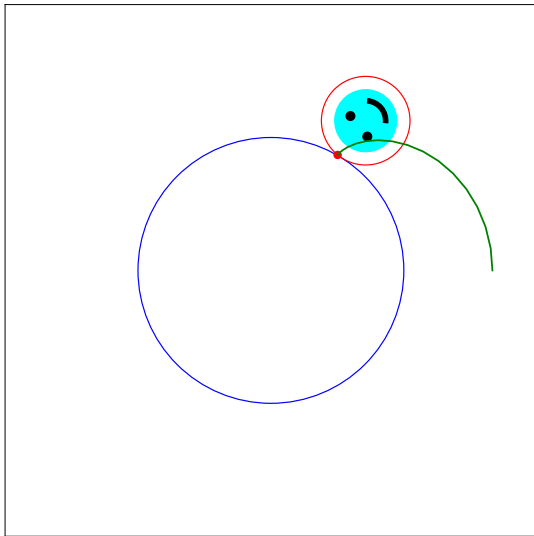
Animation of Coin Rotation Paradox

Rotation: 57.0°



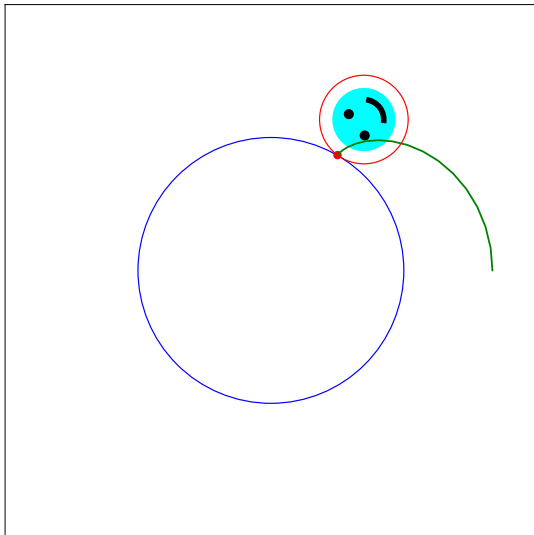
Animation of Coin Rotation Paradox

Rotation: 57.7°



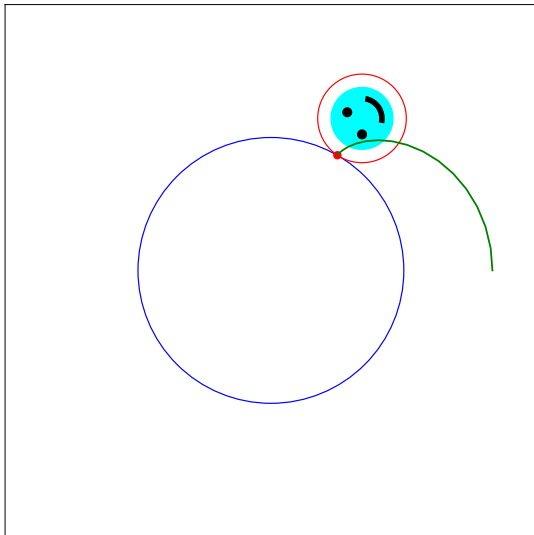
Animation of Coin Rotation Paradox

Rotation: 58.4°



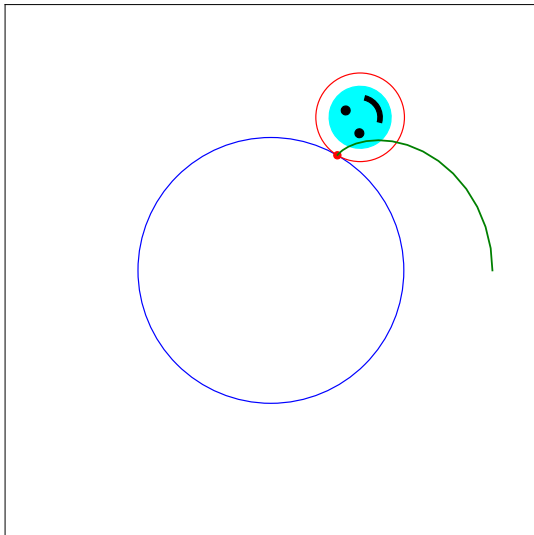
Animation of Coin Rotation Paradox

Rotation: 59.1°



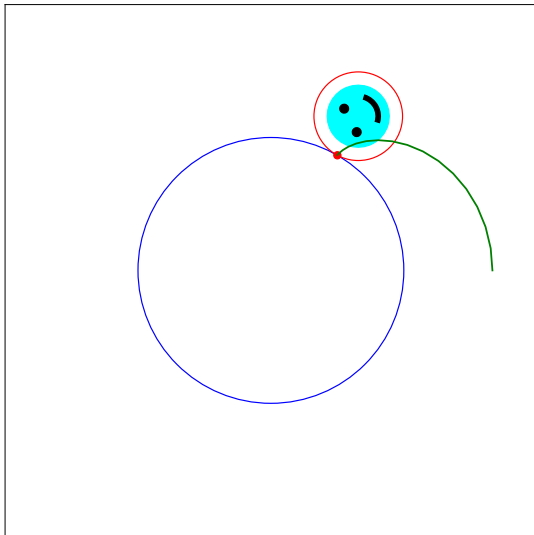
Animation of Coin Rotation Paradox

Rotation: 59.8°



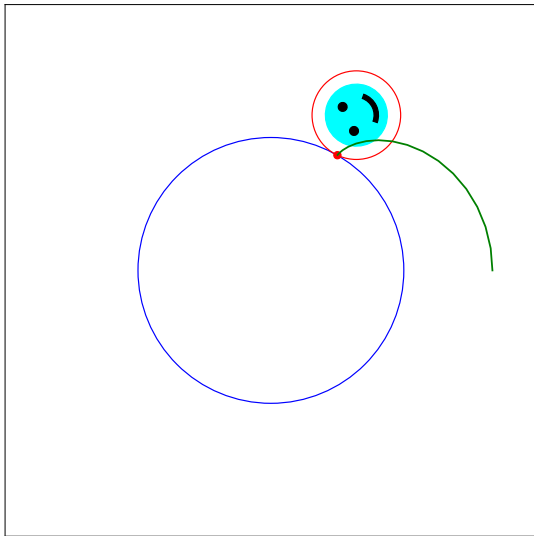
Animation of Coin Rotation Paradox

Rotation: 60.5°



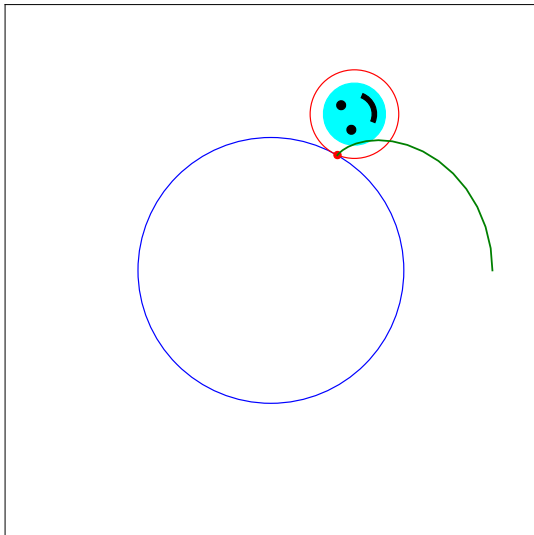
Animation of Coin Rotation Paradox

Rotation: 61.2°



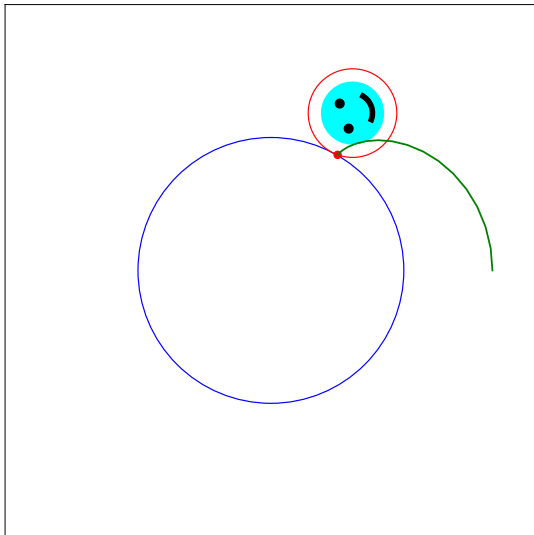
Animation of Coin Rotation Paradox

Rotation: 61.9°



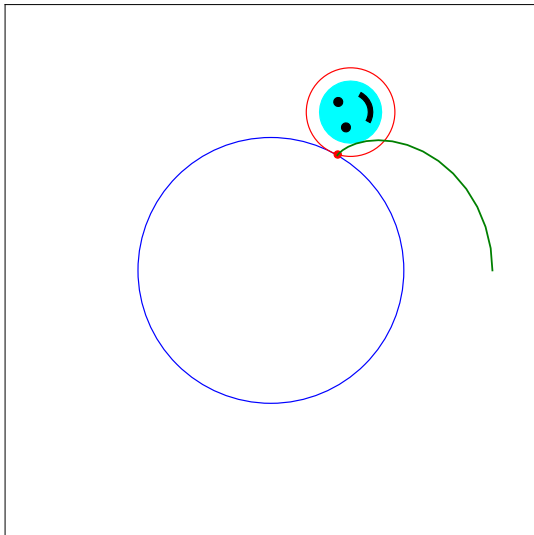
Animation of Coin Rotation Paradox

Rotation: 62.6°



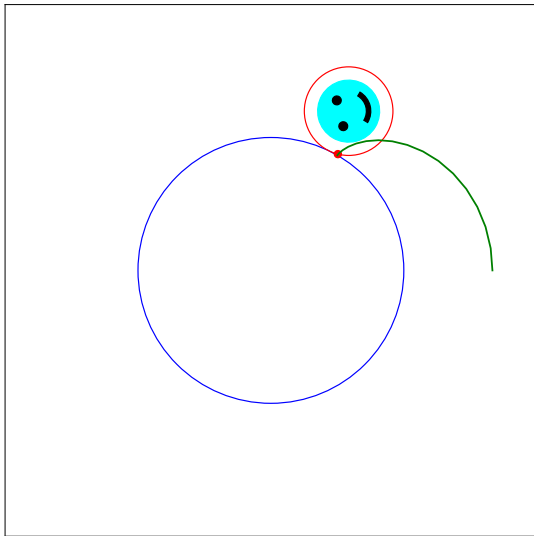
Animation of Coin Rotation Paradox

Rotation: 63.3°



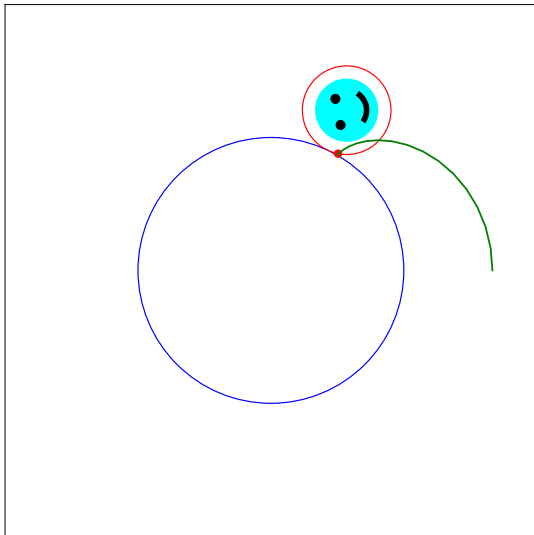
Animation of Coin Rotation Paradox

Rotation: 64.0°



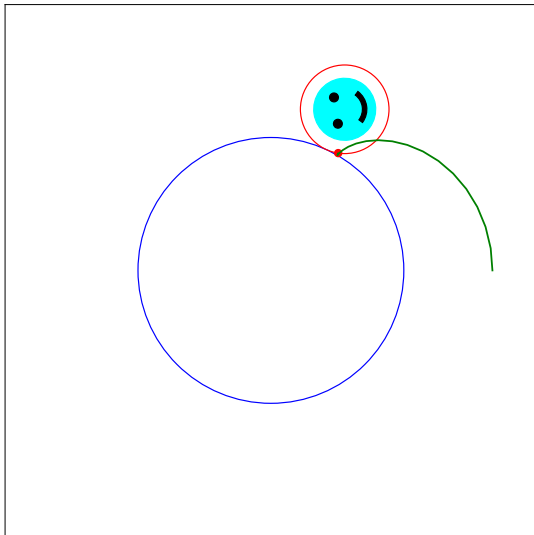
Animation of Coin Rotation Paradox

Rotation: 64.7°



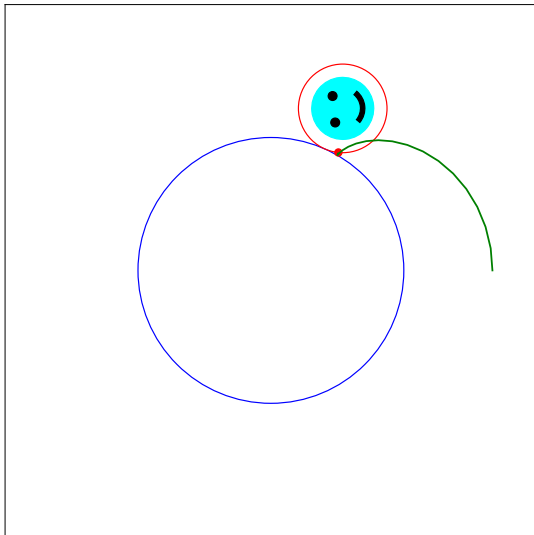
Animation of Coin Rotation Paradox

Rotation: 65.4°



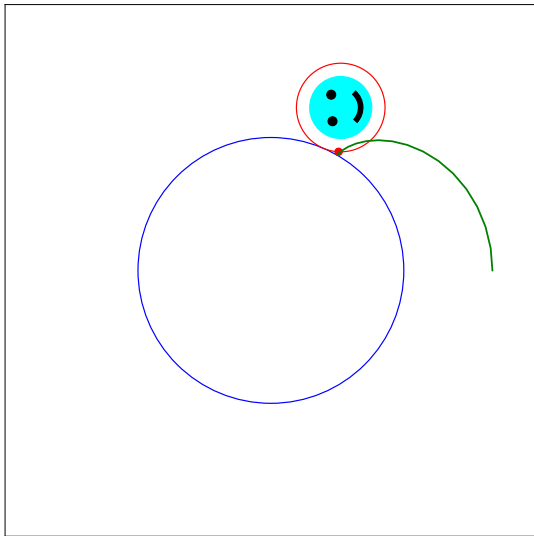
Animation of Coin Rotation Paradox

Rotation: 66.1°



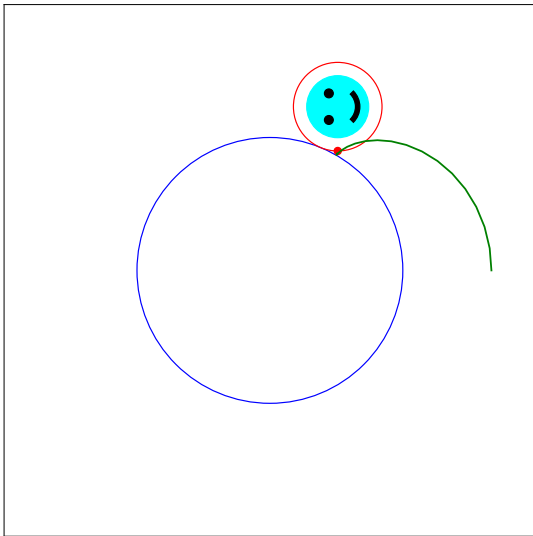
Animation of Coin Rotation Paradox

Rotation: 66.8°



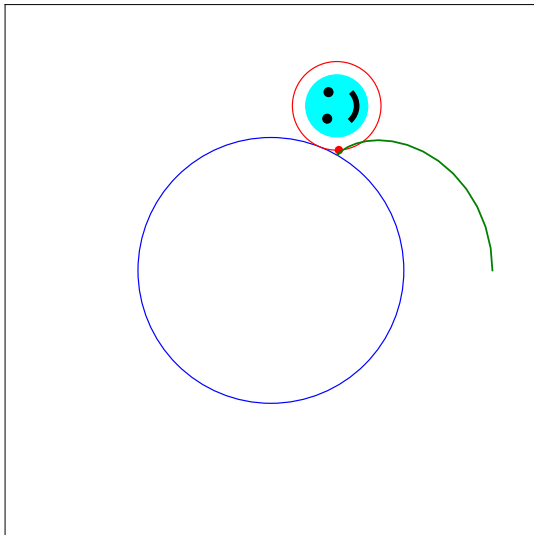
Animation of Coin Rotation Paradox

Rotation: 67.5°



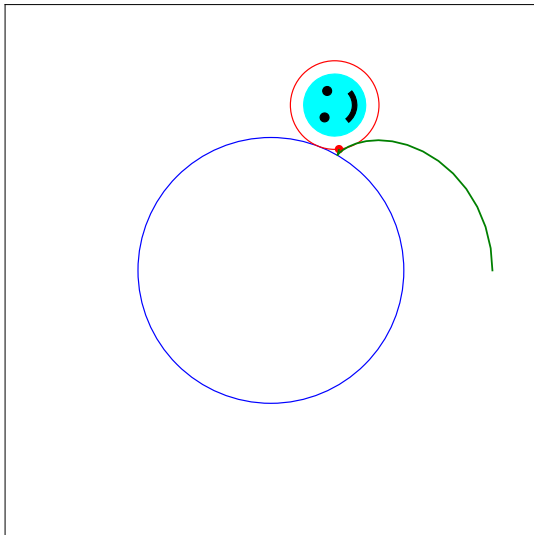
Animation of Coin Rotation Paradox

Rotation: 68.2°



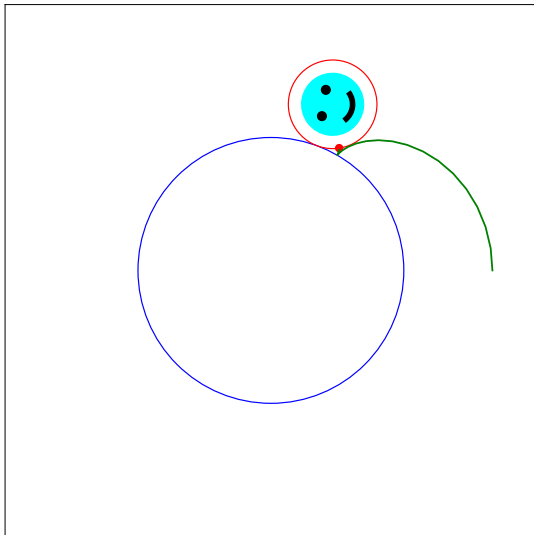
Animation of Coin Rotation Paradox

Rotation: 68.9°



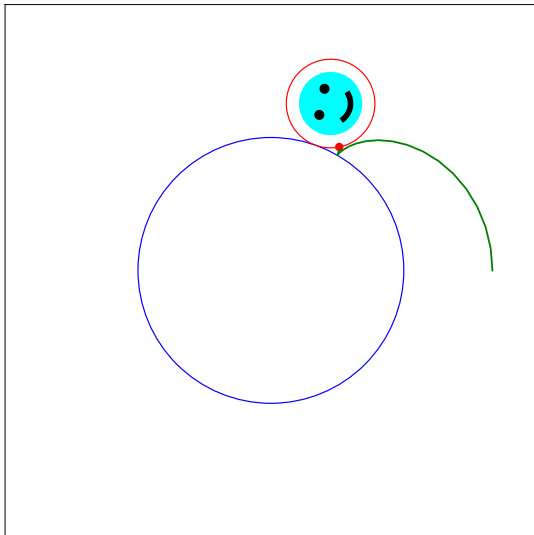
Animation of Coin Rotation Paradox

Rotation: 69.6°



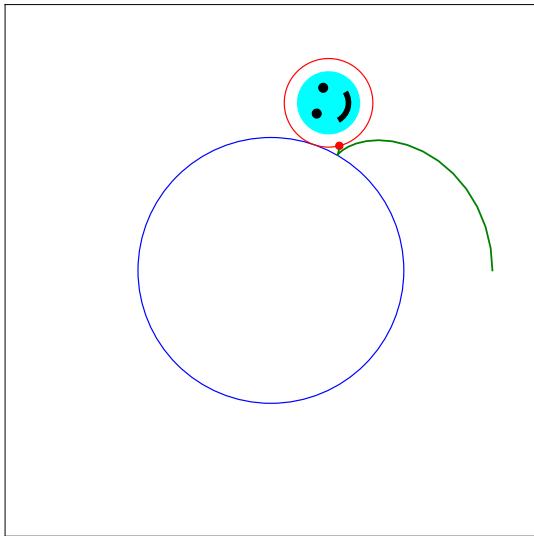
Animation of Coin Rotation Paradox

Rotation: 70.3°



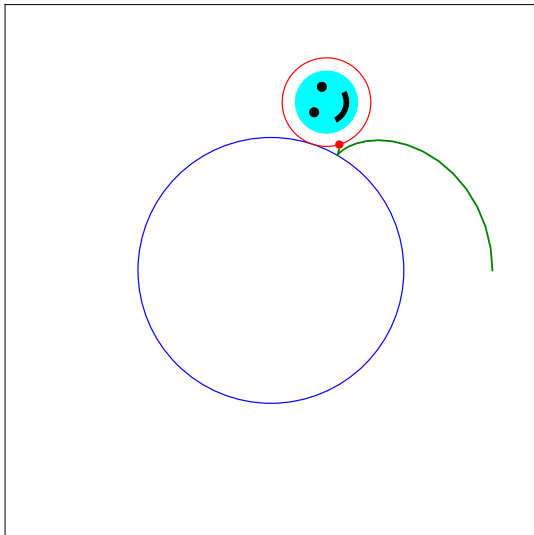
Animation of Coin Rotation Paradox

Rotation: 71.0°



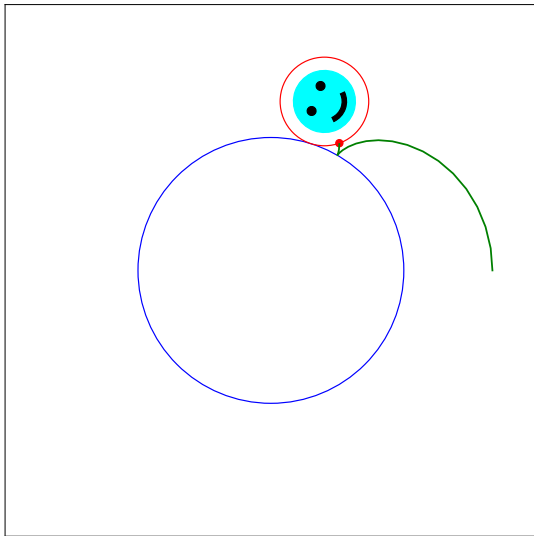
Animation of Coin Rotation Paradox

Rotation: 71.7°



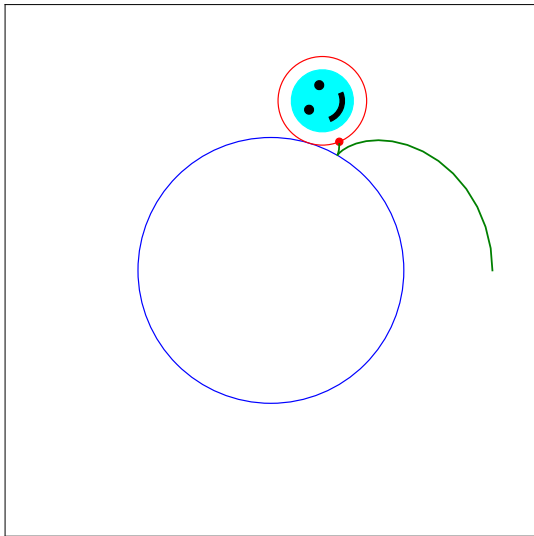
Animation of Coin Rotation Paradox

Rotation: 72.4°



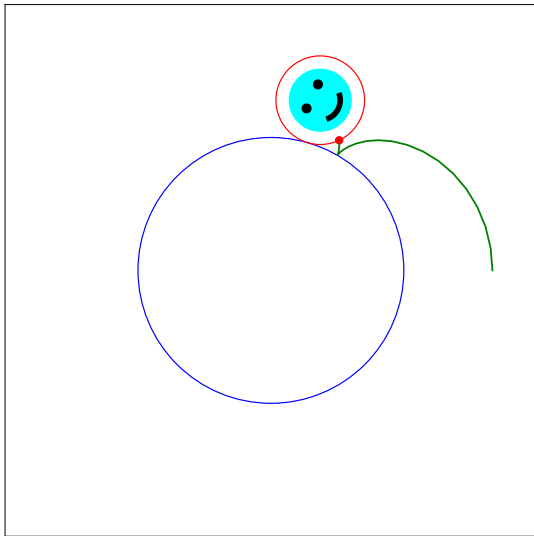
Animation of Coin Rotation Paradox

Rotation: 73.1°



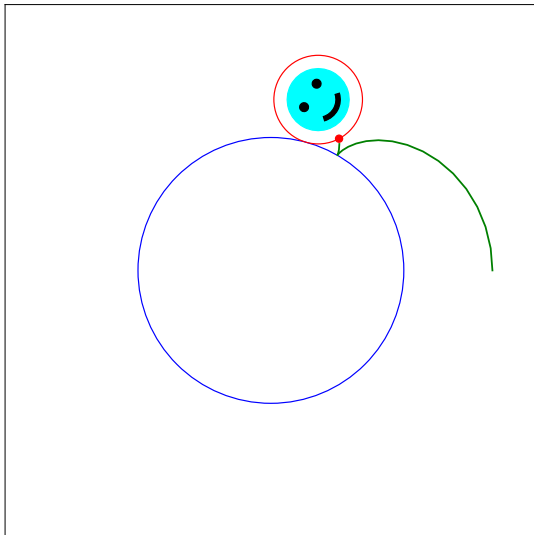
Animation of Coin Rotation Paradox

Rotation: 73.8°



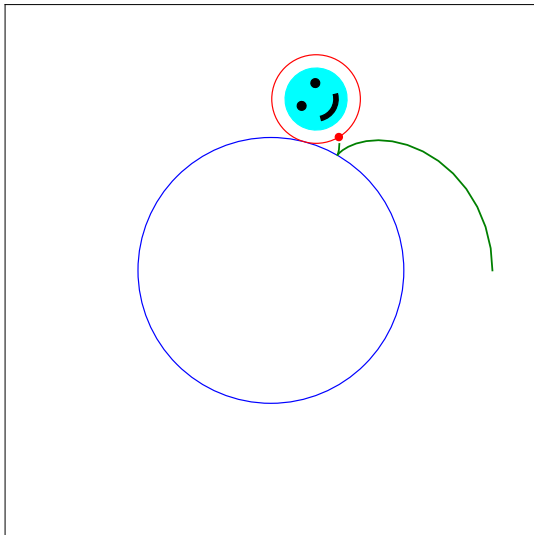
Animation of Coin Rotation Paradox

Rotation: 74.5°



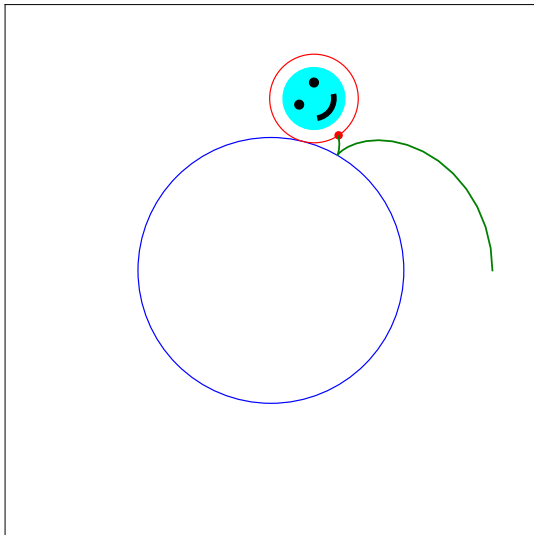
Animation of Coin Rotation Paradox

Rotation: 75.2°



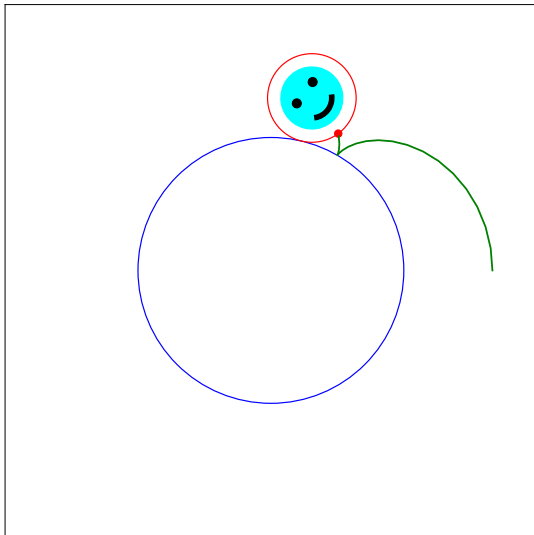
Animation of Coin Rotation Paradox

Rotation: 75.9°



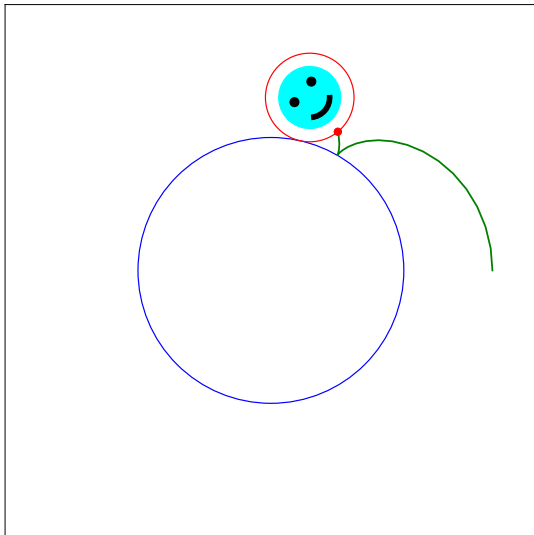
Animation of Coin Rotation Paradox

Rotation: 76.6°



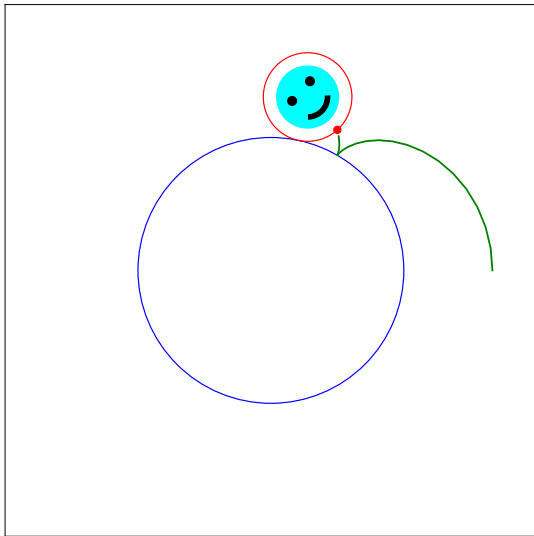
Animation of Coin Rotation Paradox

Rotation: 77.3°



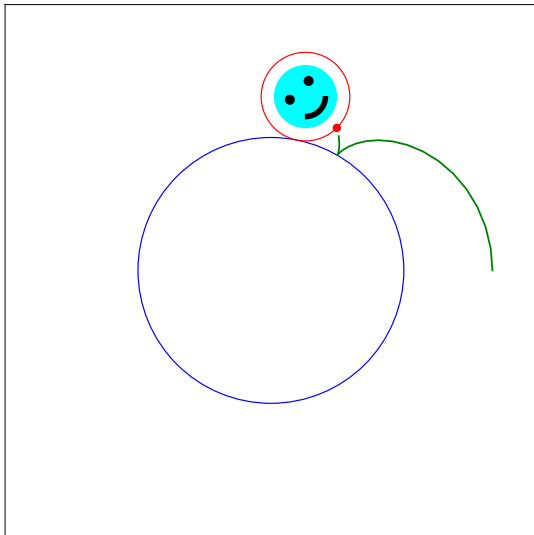
Animation of Coin Rotation Paradox

Rotation: 78.0°



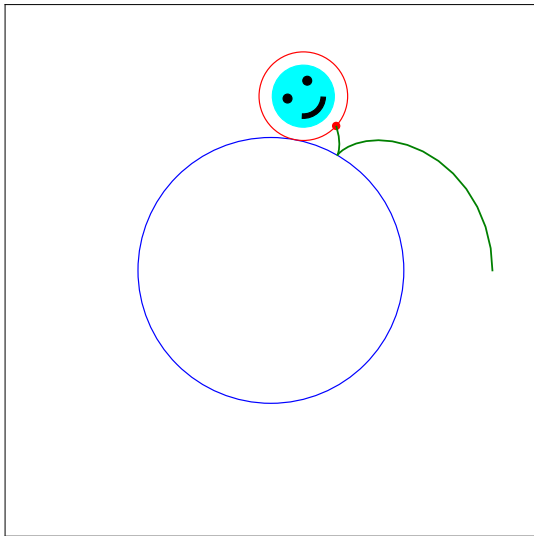
Animation of Coin Rotation Paradox

Rotation: 78.8°



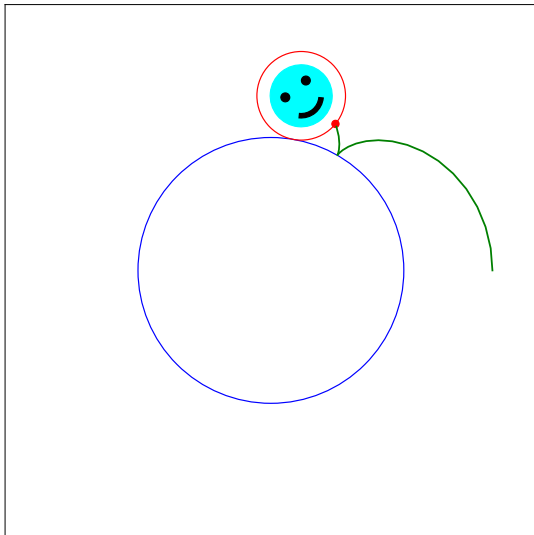
Animation of Coin Rotation Paradox

Rotation: 79.5°



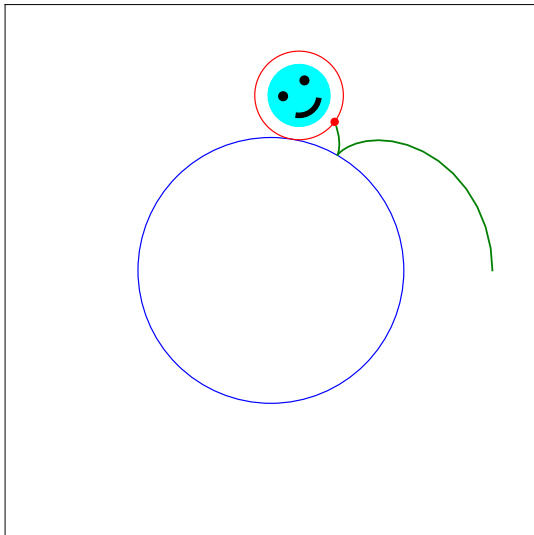
Animation of Coin Rotation Paradox

Rotation: 80.2°



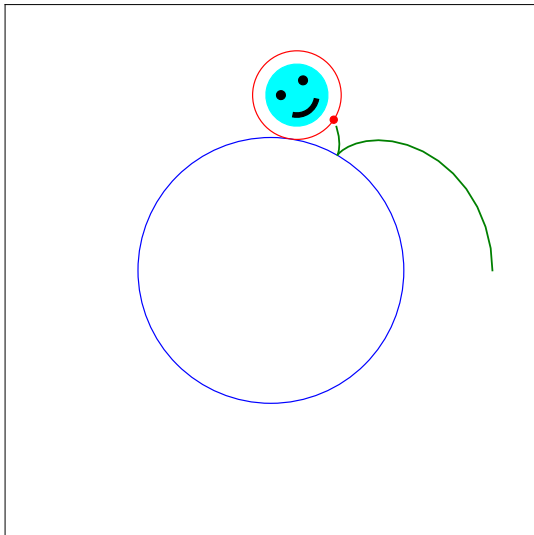
Animation of Coin Rotation Paradox

Rotation: 80.9°



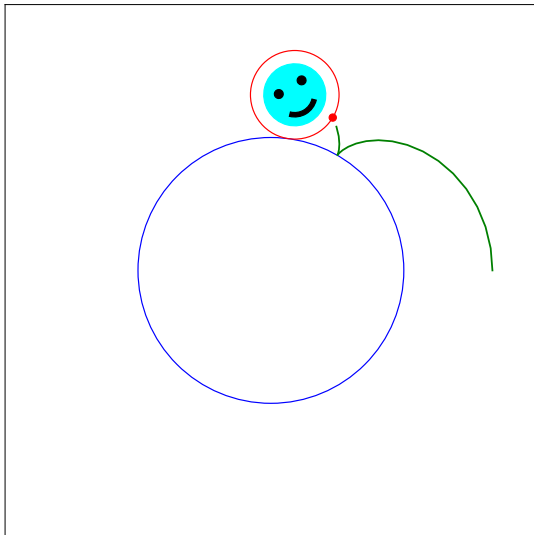
Animation of Coin Rotation Paradox

Rotation: 81.6°



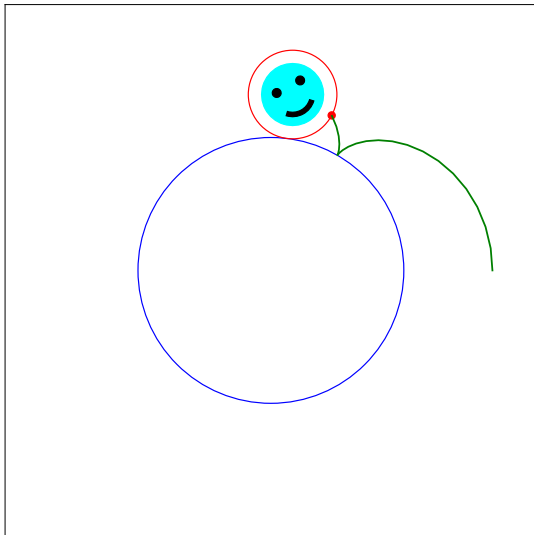
Animation of Coin Rotation Paradox

Rotation: 82.3°



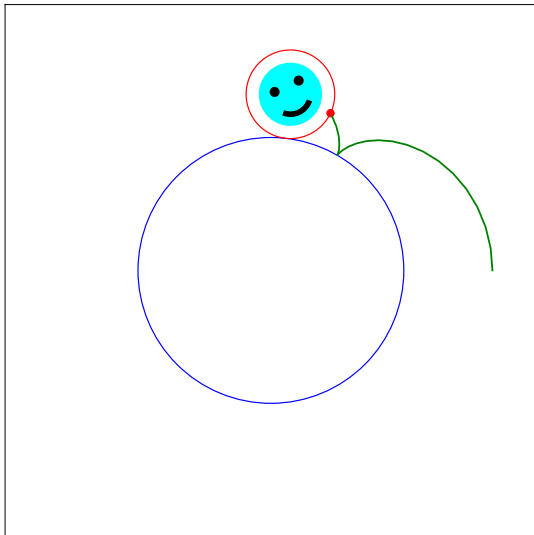
Animation of Coin Rotation Paradox

Rotation: 83.0°



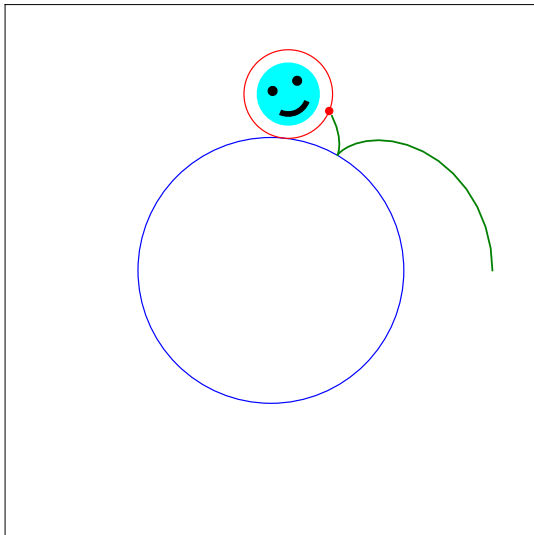
Animation of Coin Rotation Paradox

Rotation: 83.7°



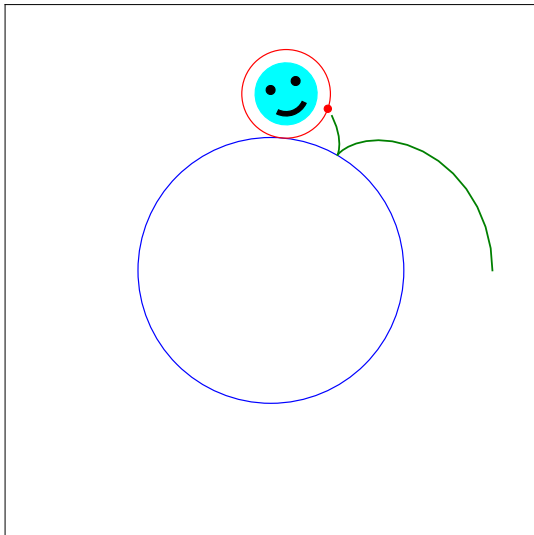
Animation of Coin Rotation Paradox

Rotation: 84.4°



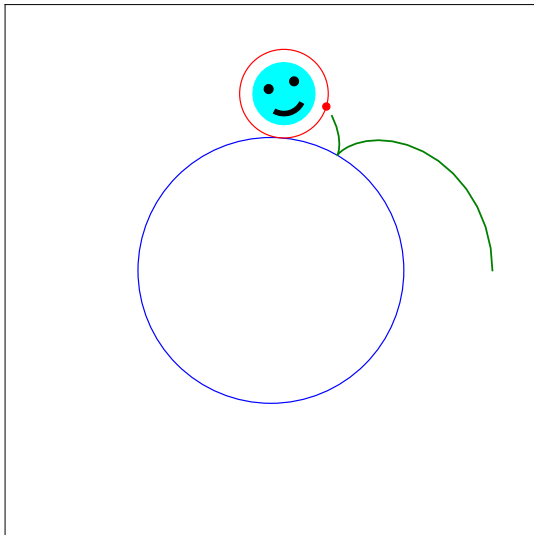
Animation of Coin Rotation Paradox

Rotation: 85.1°



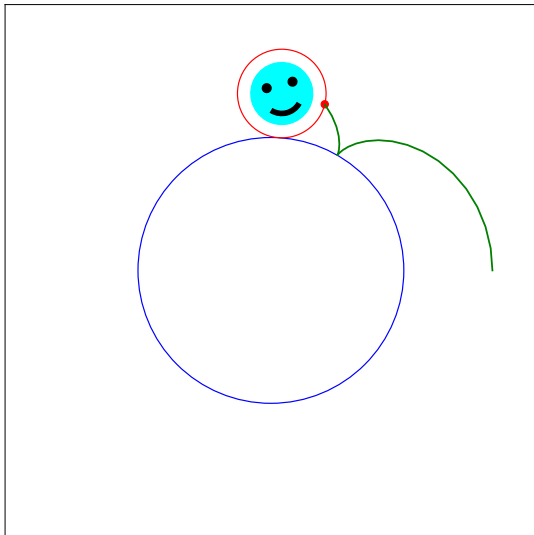
Animation of Coin Rotation Paradox

Rotation: 85.8°



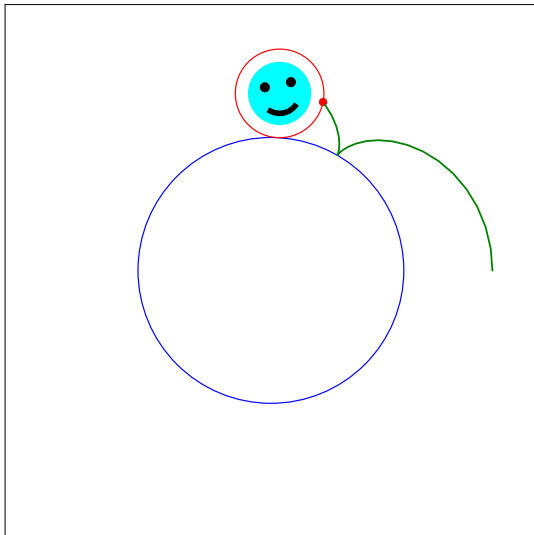
Animation of Coin Rotation Paradox

Rotation: 86.5°



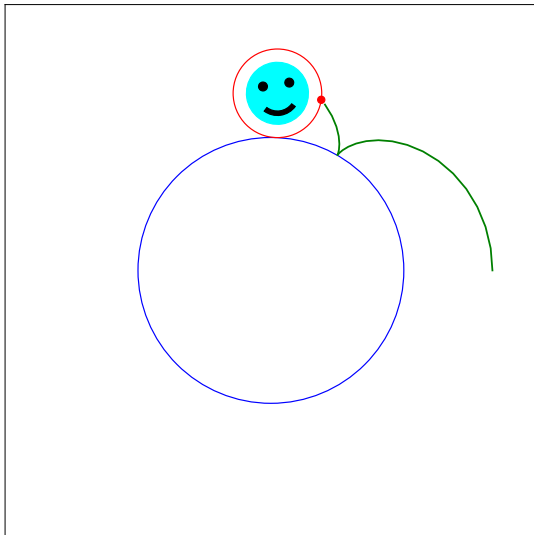
Animation of Coin Rotation Paradox

Rotation: 87.2°



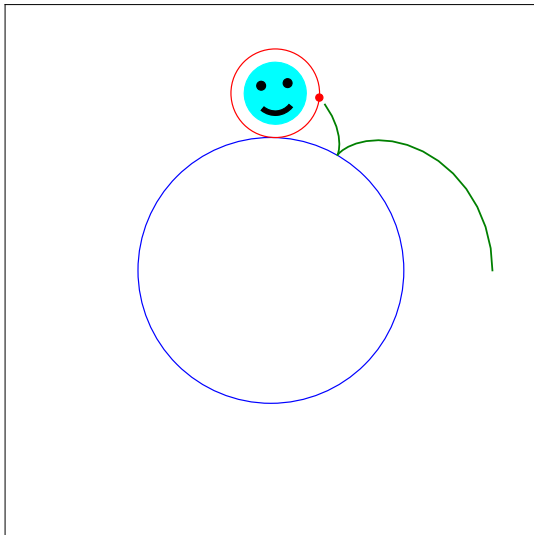
Animation of Coin Rotation Paradox

Rotation: 87.9°



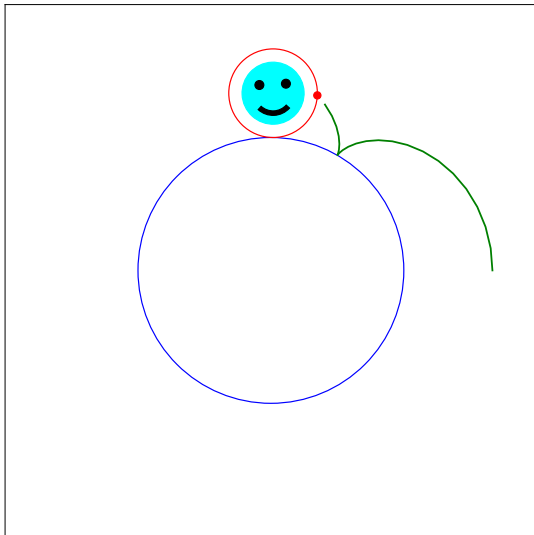
Animation of Coin Rotation Paradox

Rotation: 88.6°



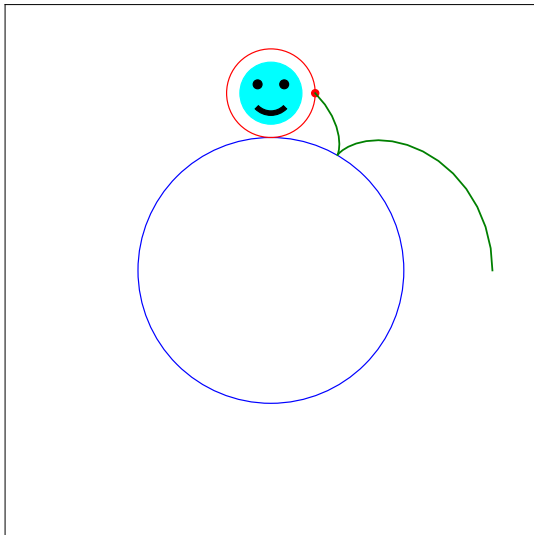
Animation of Coin Rotation Paradox

Rotation: 89.3°



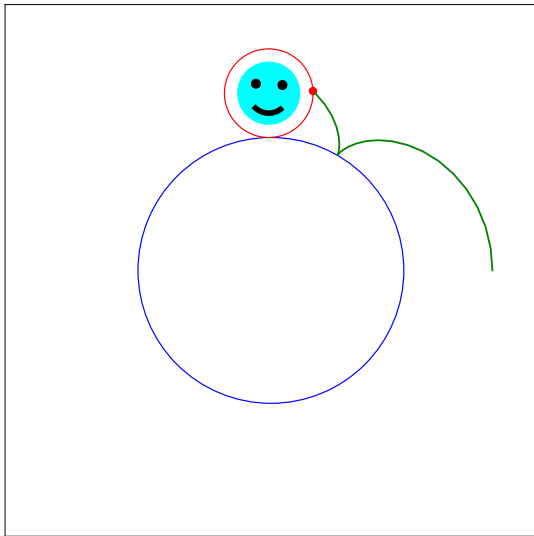
Animation of Coin Rotation Paradox

Rotation: 90.0°



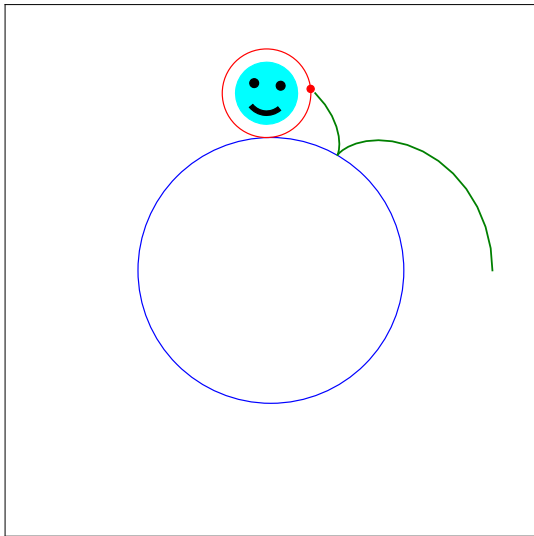
Animation of Coin Rotation Paradox

Rotation: 90.7°



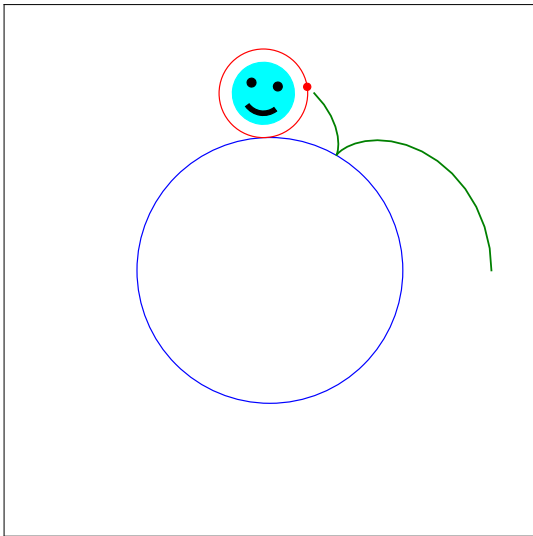
Animation of Coin Rotation Paradox

Rotation: 91.4°



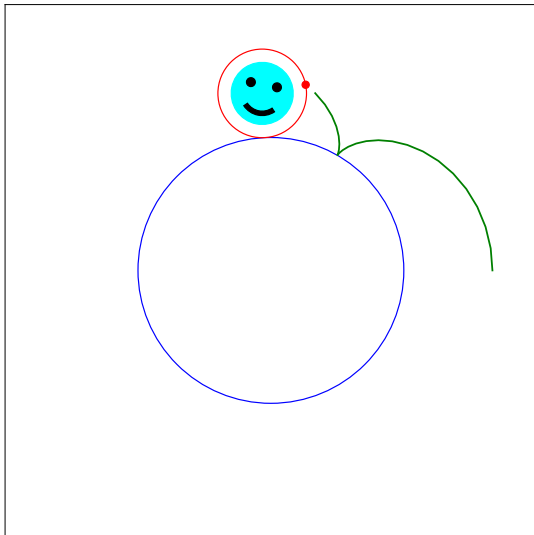
Animation of Coin Rotation Paradox

Rotation: 92.1°



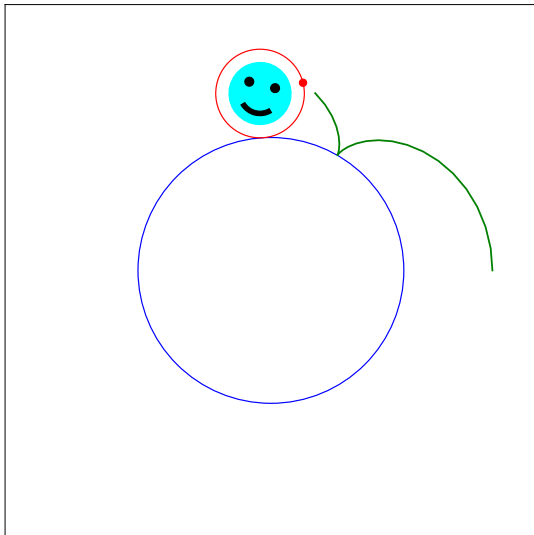
Animation of Coin Rotation Paradox

Rotation: 92.8°



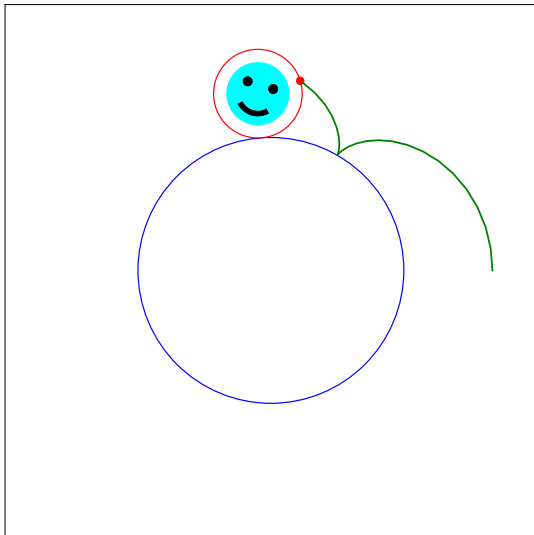
Animation of Coin Rotation Paradox

Rotation: 93.5°



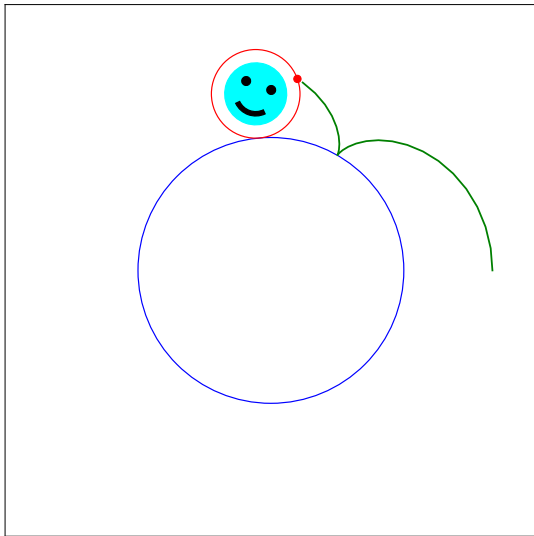
Animation of Coin Rotation Paradox

Rotation: 94.2°



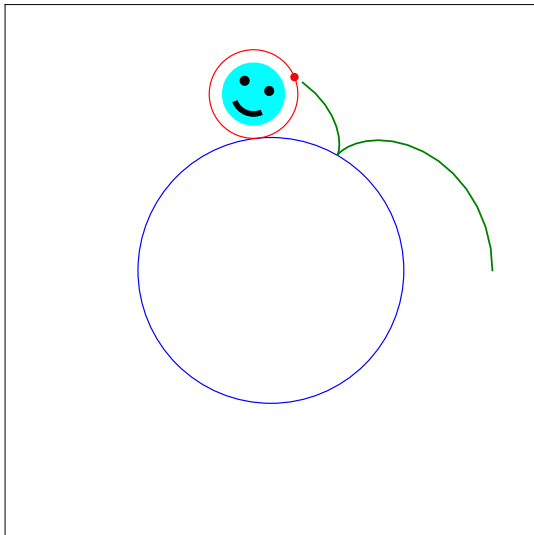
Animation of Coin Rotation Paradox

Rotation: 94.9°



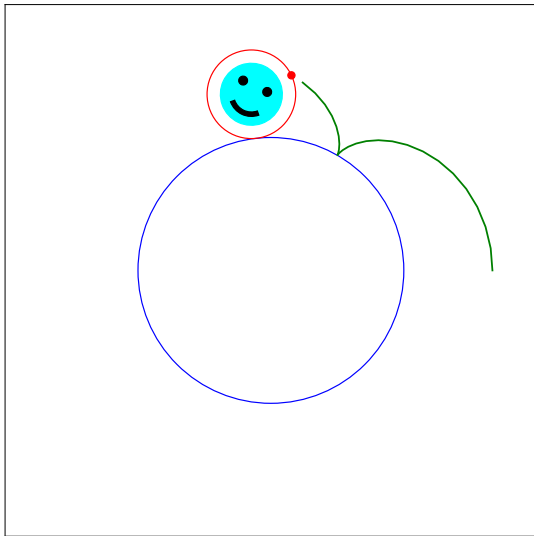
Animation of Coin Rotation Paradox

Rotation: 95.6°



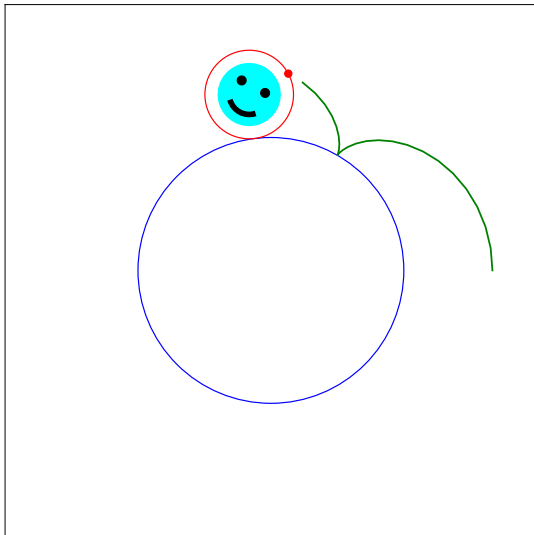
Animation of Coin Rotation Paradox

Rotation: 96.3°



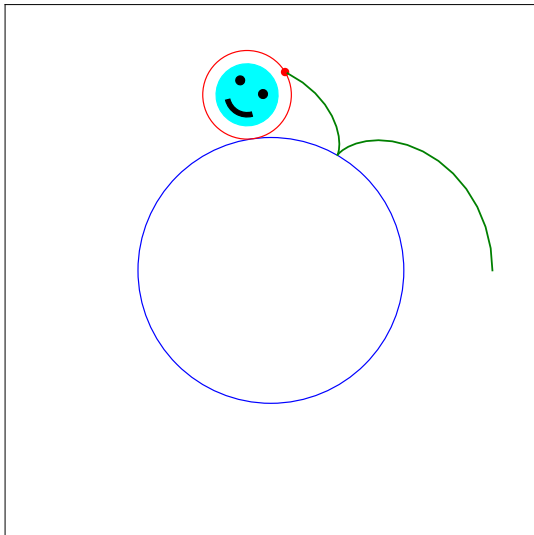
Animation of Coin Rotation Paradox

Rotation: 97.0°



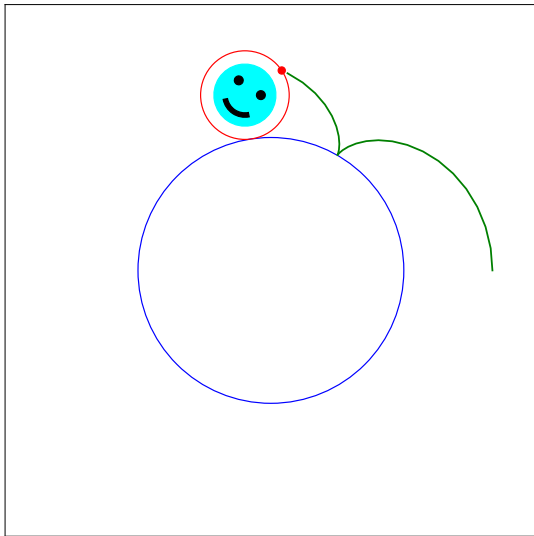
Animation of Coin Rotation Paradox

Rotation: 97.7°



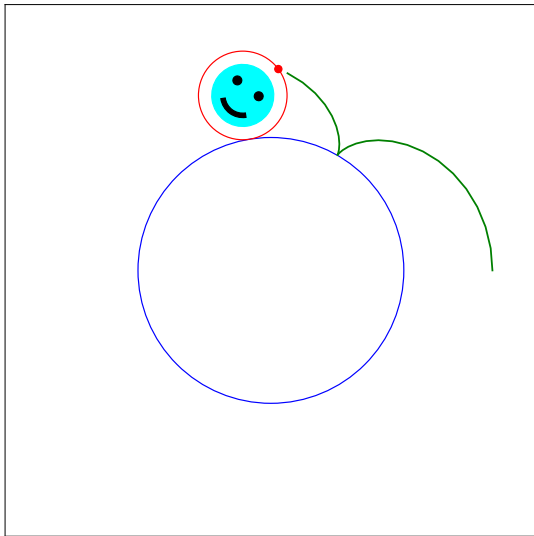
Animation of Coin Rotation Paradox

Rotation: 98.4°



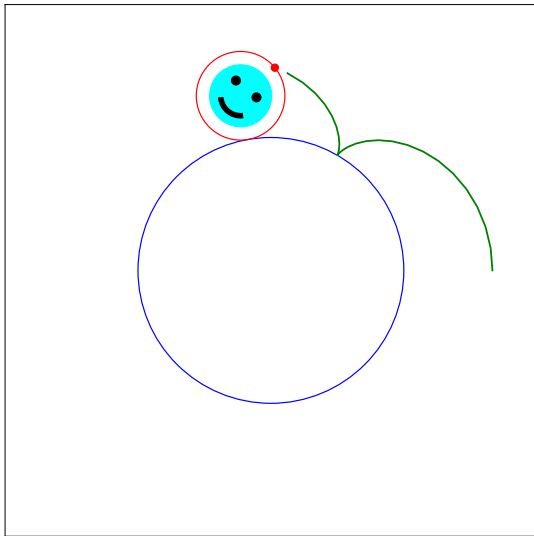
Animation of Coin Rotation Paradox

Rotation: 99.1°

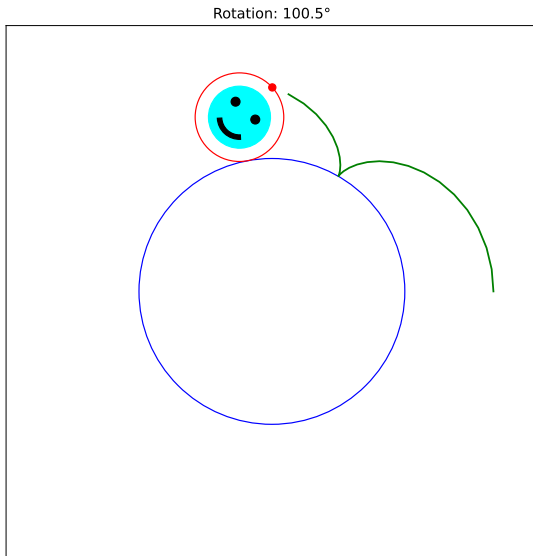


Animation of Coin Rotation Paradox

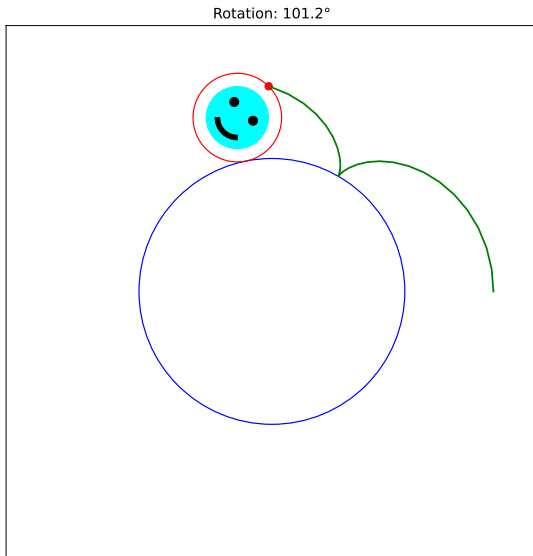
Rotation: 99.8°



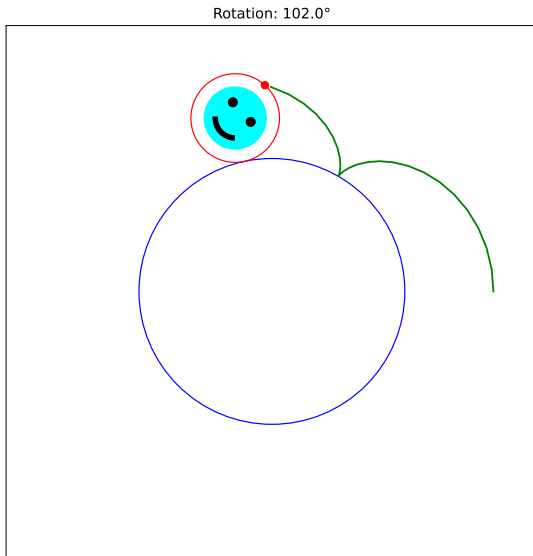
Animation of Coin Rotation Paradox



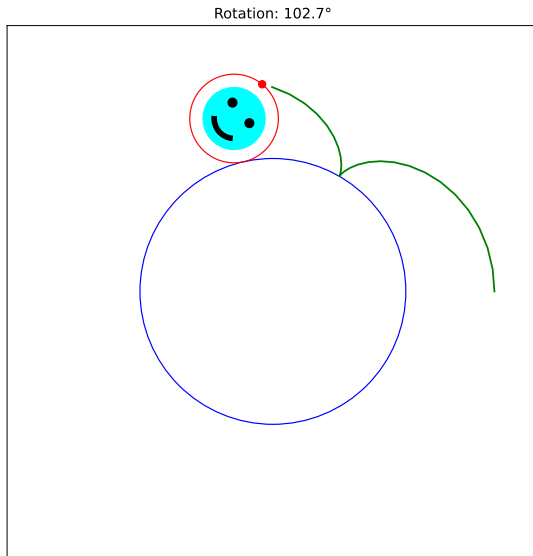
Animation of Coin Rotation Paradox



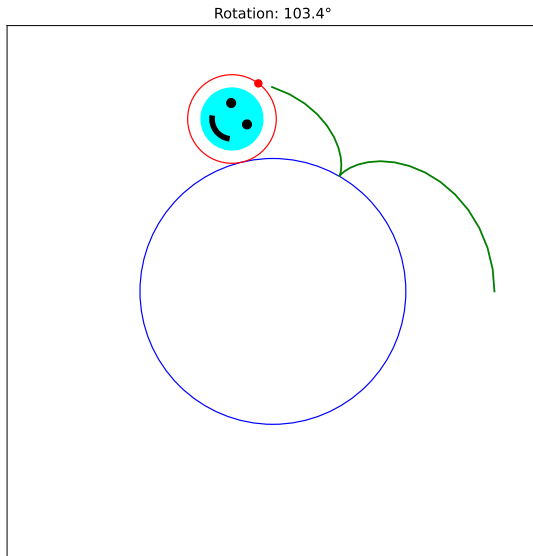
Animation of Coin Rotation Paradox



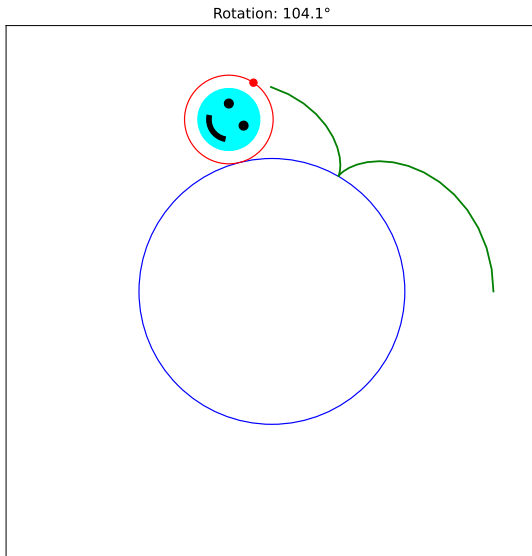
Animation of Coin Rotation Paradox



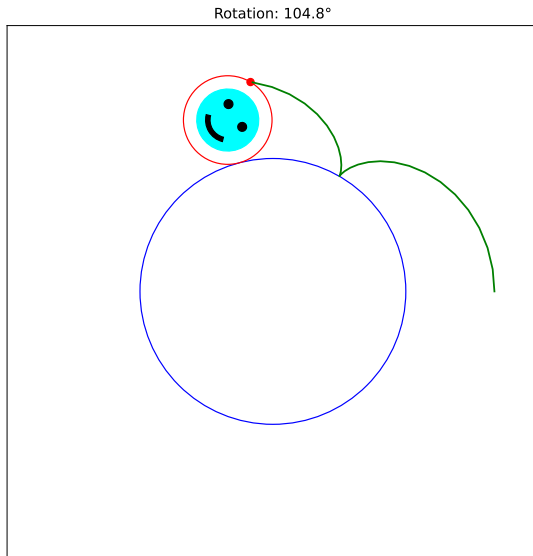
Animation of Coin Rotation Paradox



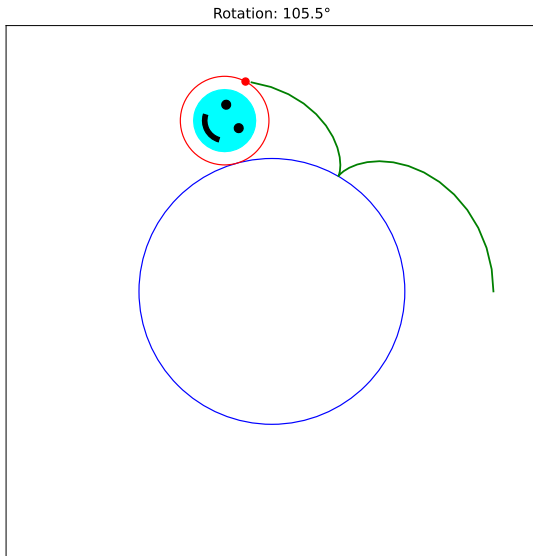
Animation of Coin Rotation Paradox



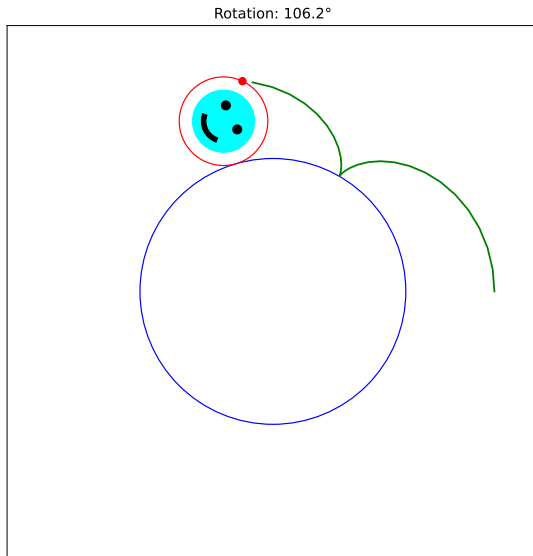
Animation of Coin Rotation Paradox



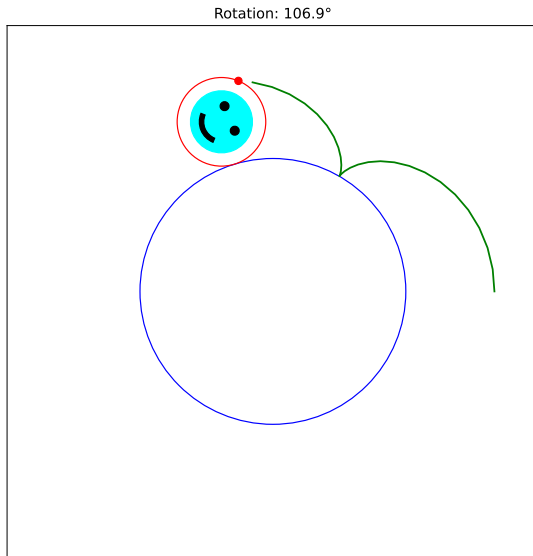
Animation of Coin Rotation Paradox



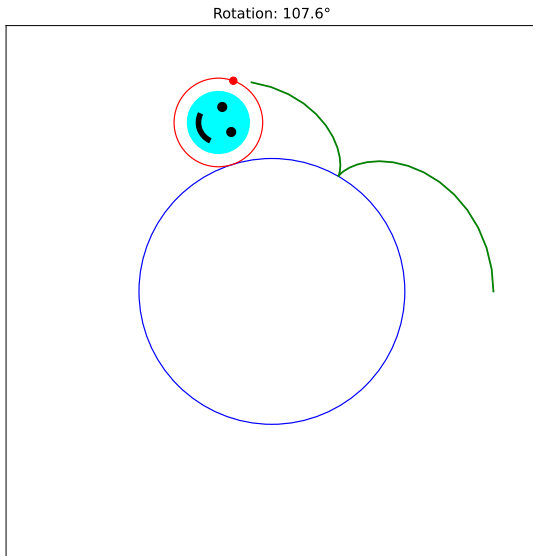
Animation of Coin Rotation Paradox



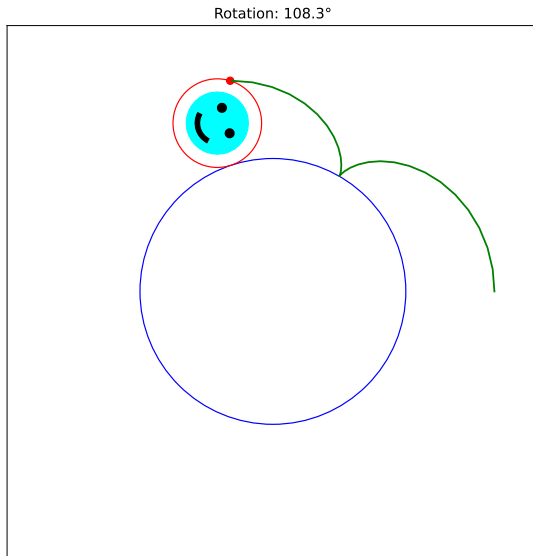
Animation of Coin Rotation Paradox



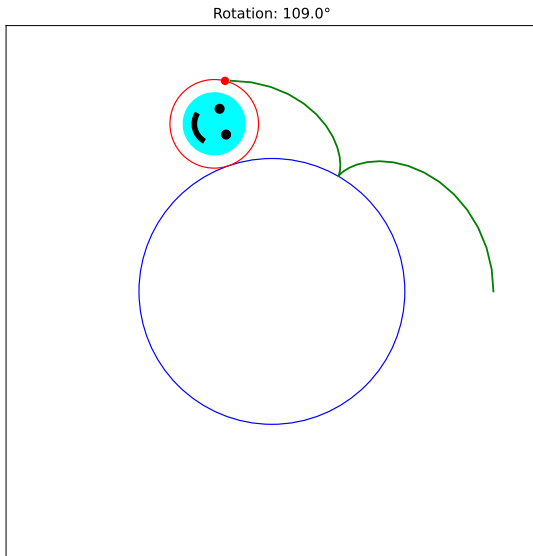
Animation of Coin Rotation Paradox



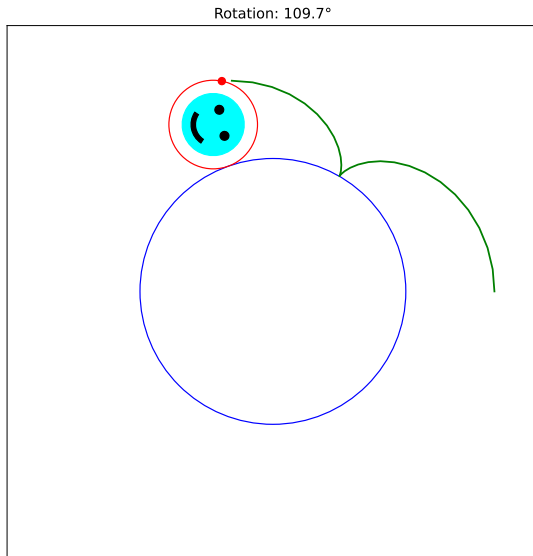
Animation of Coin Rotation Paradox



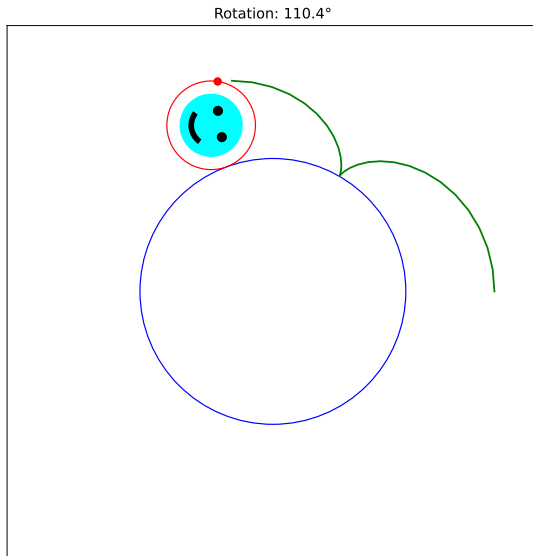
Animation of Coin Rotation Paradox



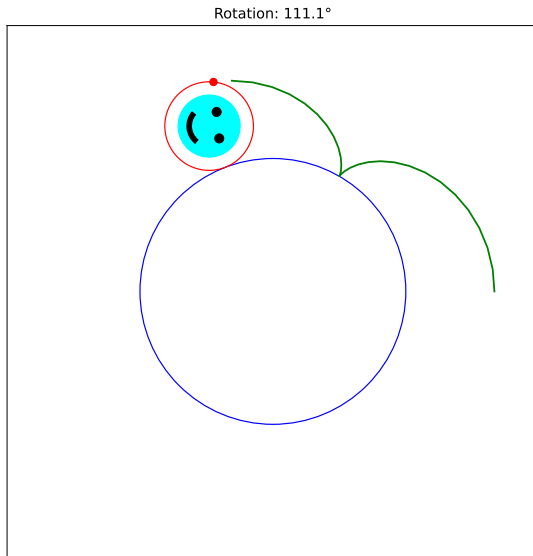
Animation of Coin Rotation Paradox



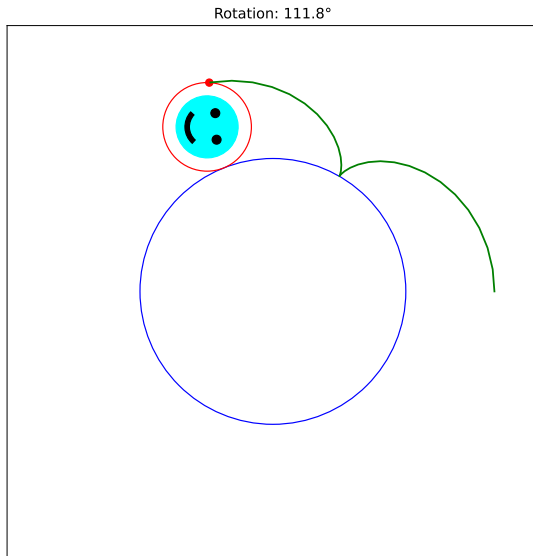
Animation of Coin Rotation Paradox



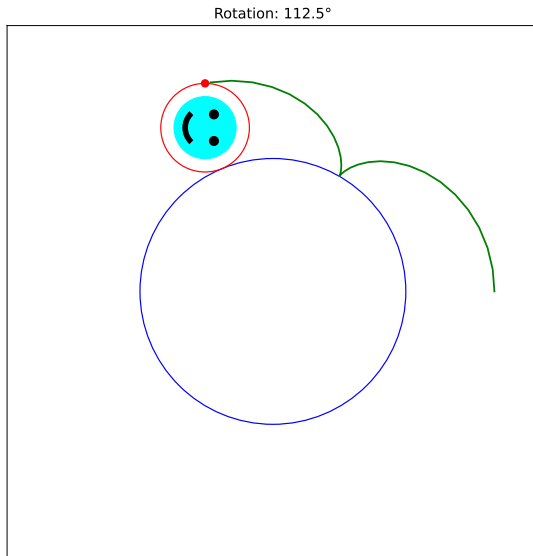
Animation of Coin Rotation Paradox



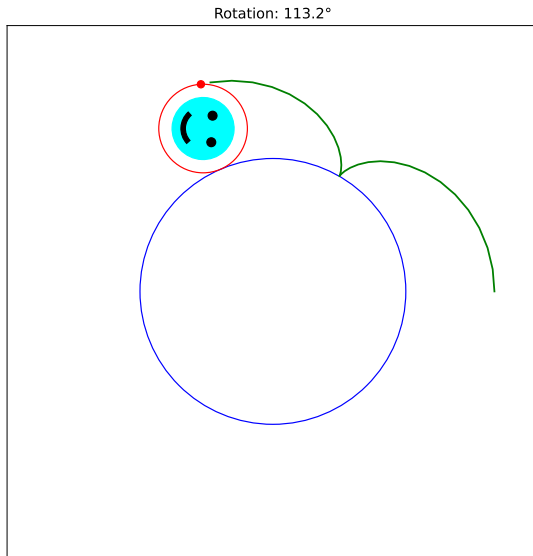
Animation of Coin Rotation Paradox



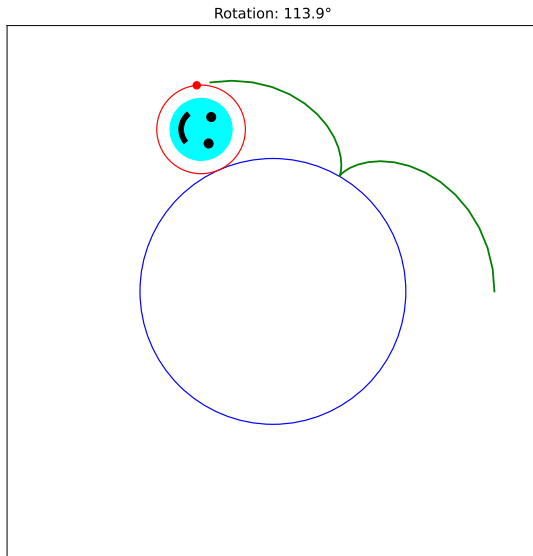
Animation of Coin Rotation Paradox



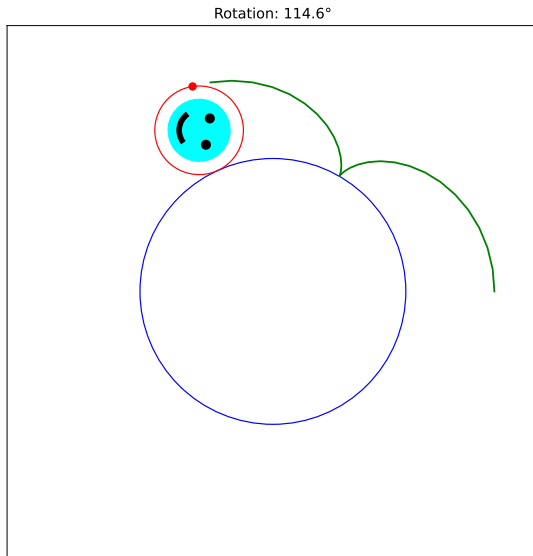
Animation of Coin Rotation Paradox



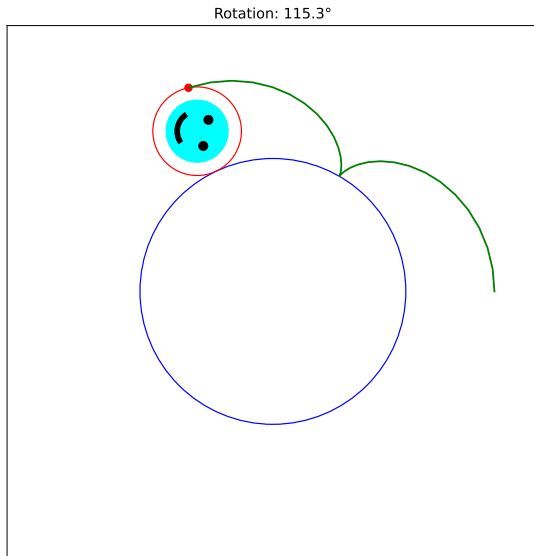
Animation of Coin Rotation Paradox



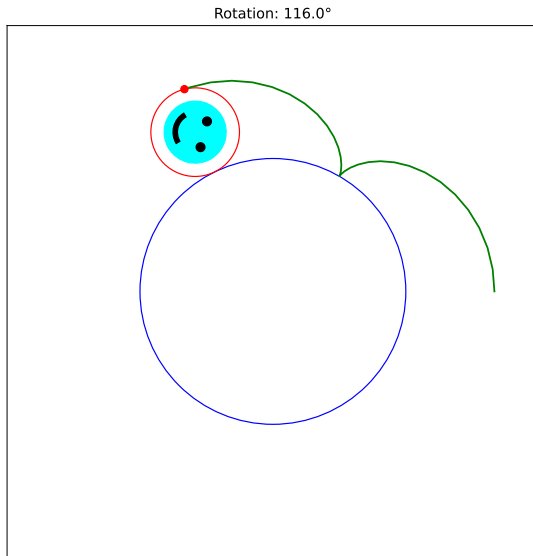
Animation of Coin Rotation Paradox



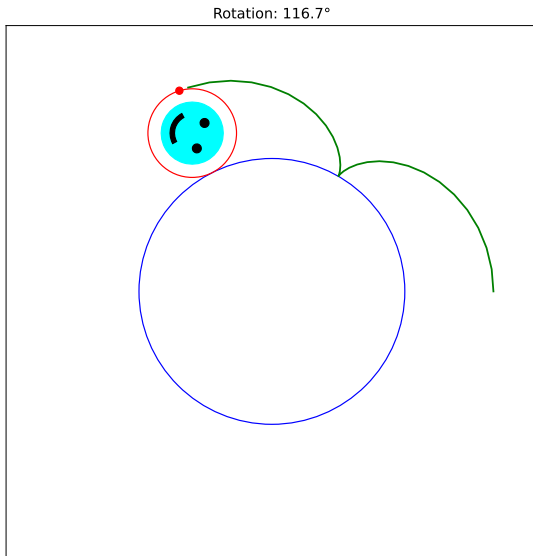
Animation of Coin Rotation Paradox



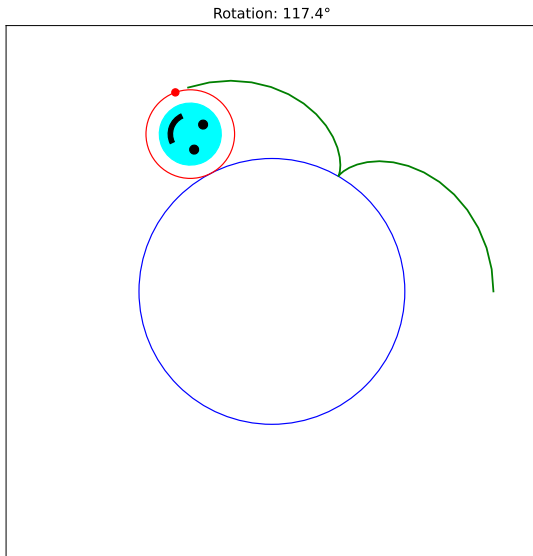
Animation of Coin Rotation Paradox



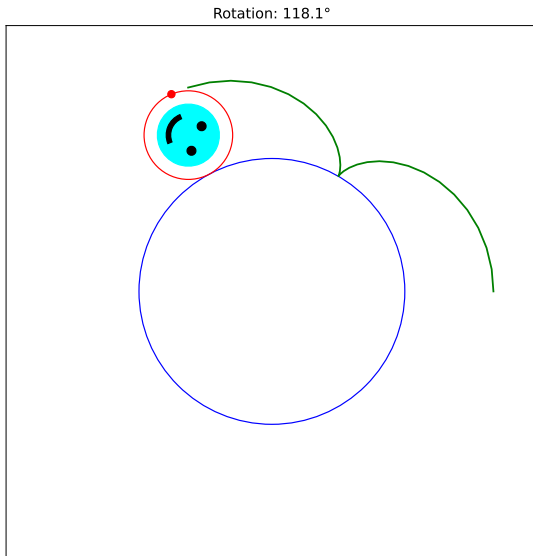
Animation of Coin Rotation Paradox



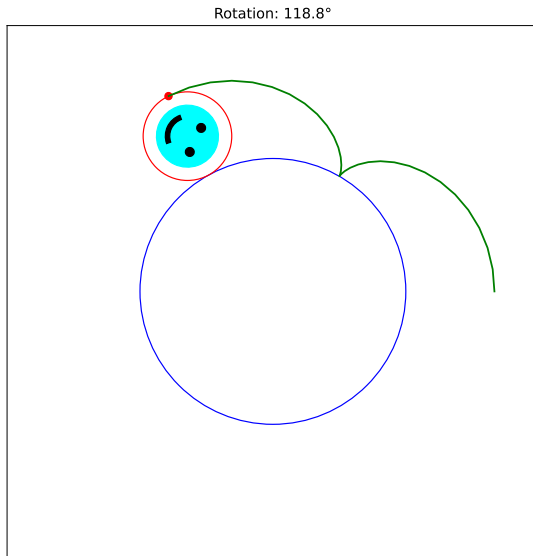
Animation of Coin Rotation Paradox



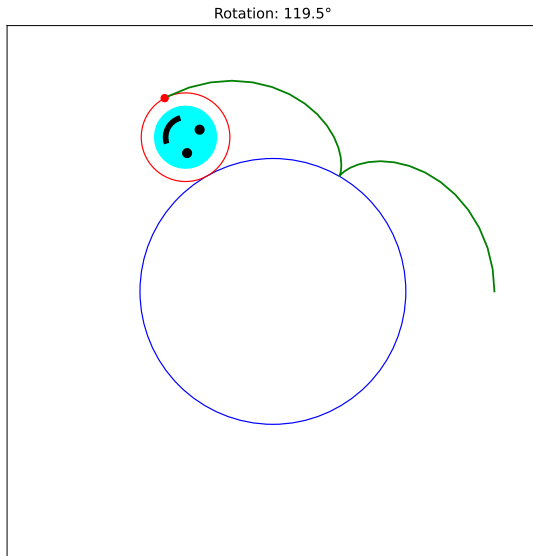
Animation of Coin Rotation Paradox



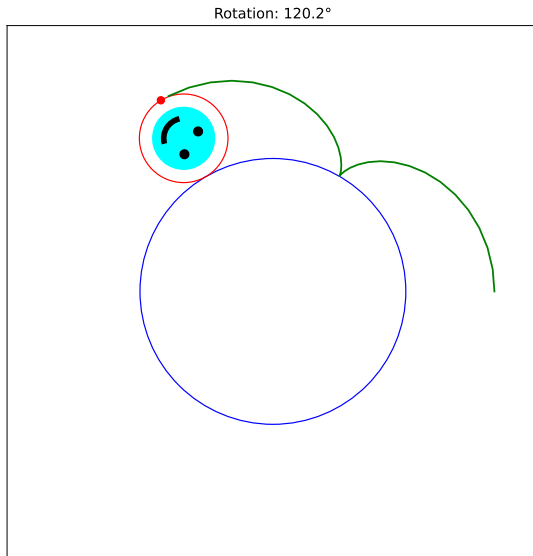
Animation of Coin Rotation Paradox



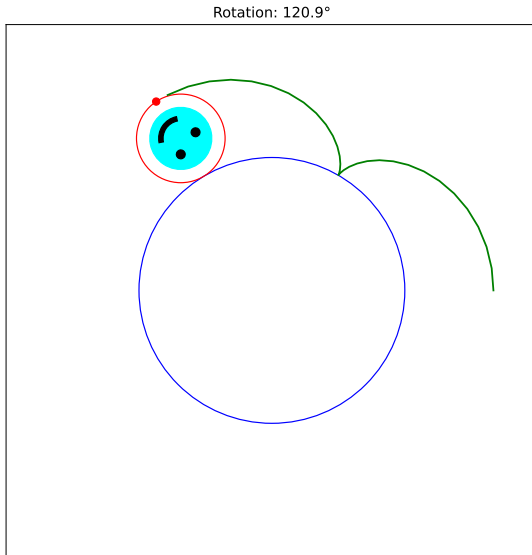
Animation of Coin Rotation Paradox



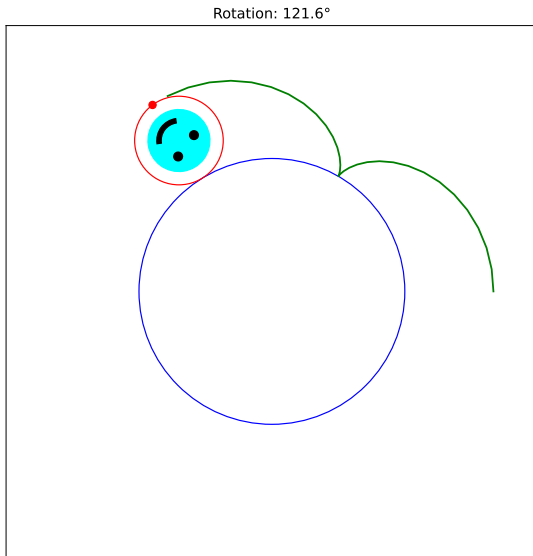
Animation of Coin Rotation Paradox



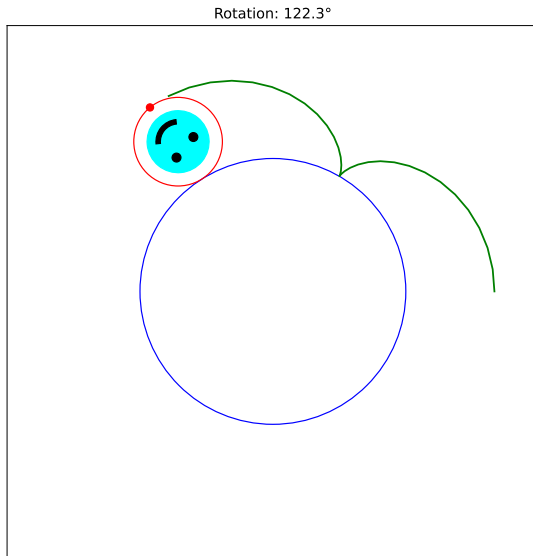
Animation of Coin Rotation Paradox



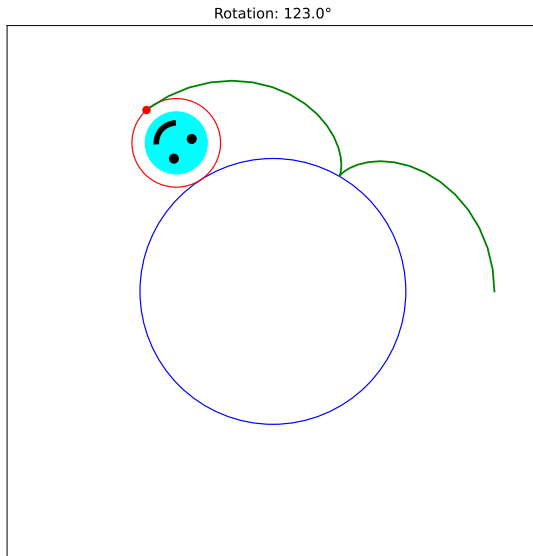
Animation of Coin Rotation Paradox



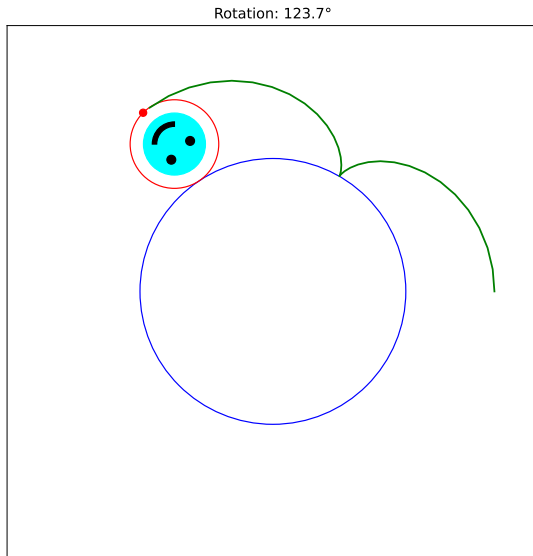
Animation of Coin Rotation Paradox



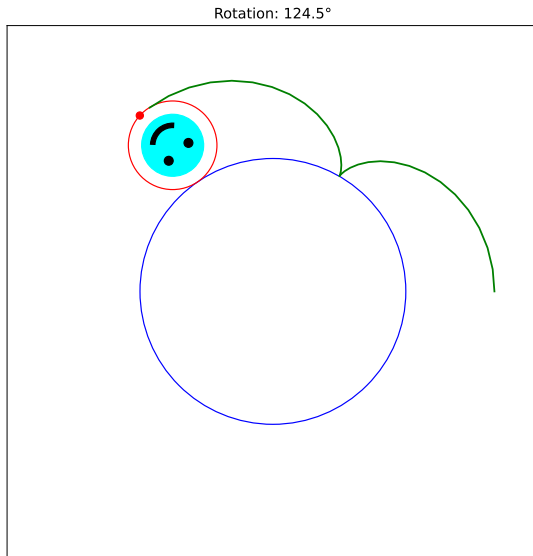
Animation of Coin Rotation Paradox



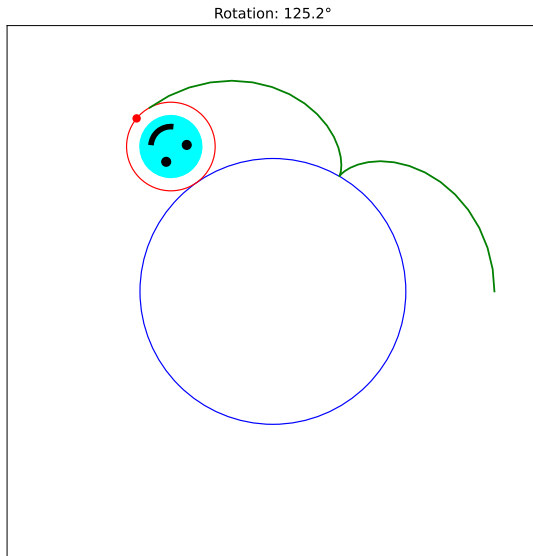
Animation of Coin Rotation Paradox



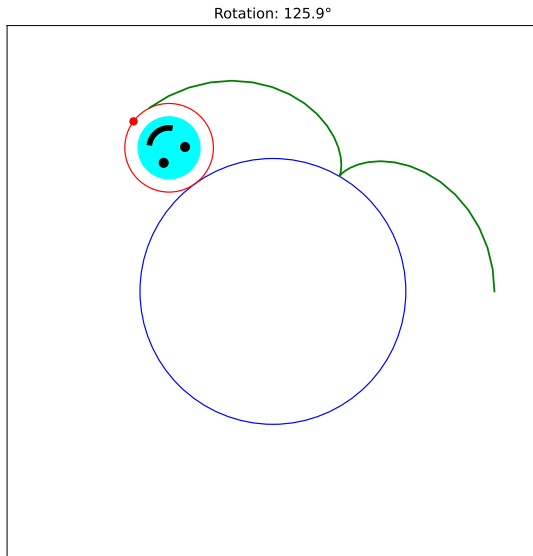
Animation of Coin Rotation Paradox



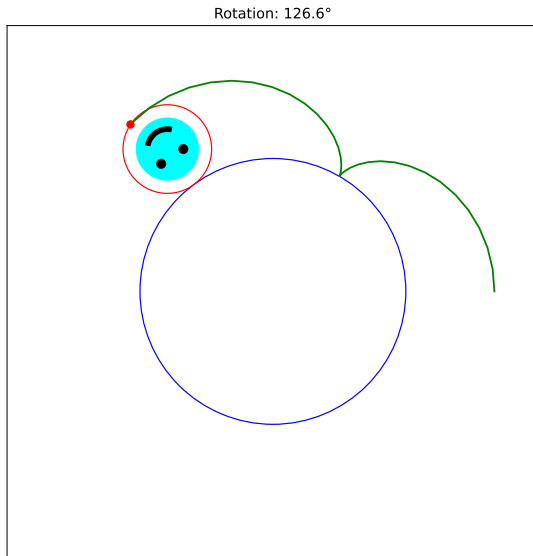
Animation of Coin Rotation Paradox



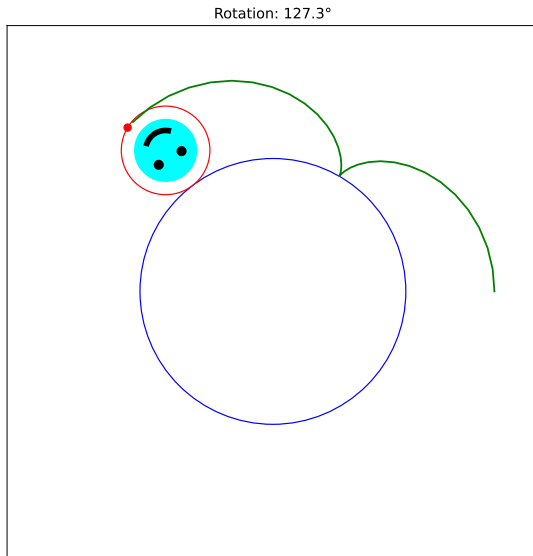
Animation of Coin Rotation Paradox



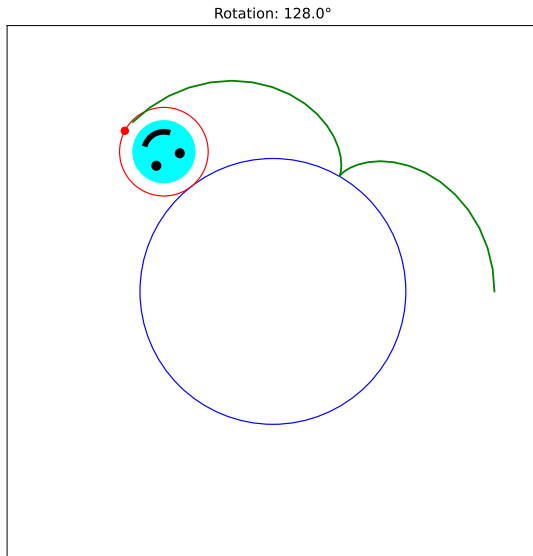
Animation of Coin Rotation Paradox



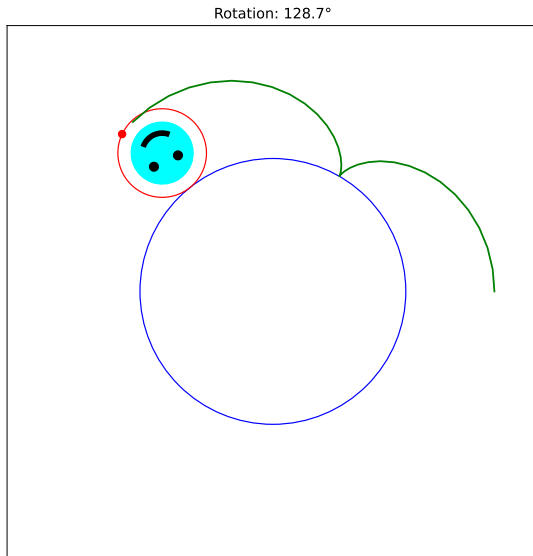
Animation of Coin Rotation Paradox



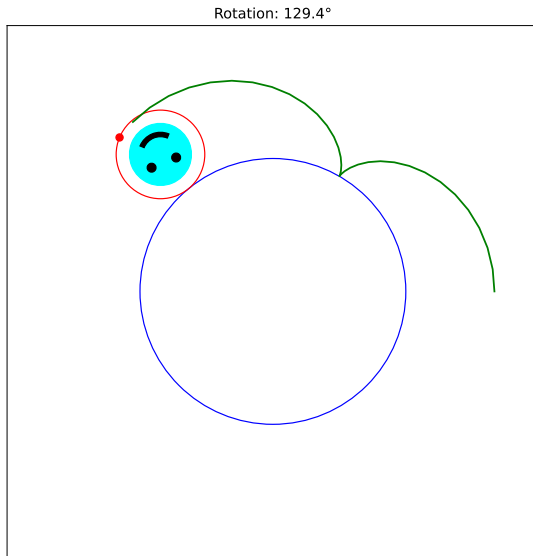
Animation of Coin Rotation Paradox



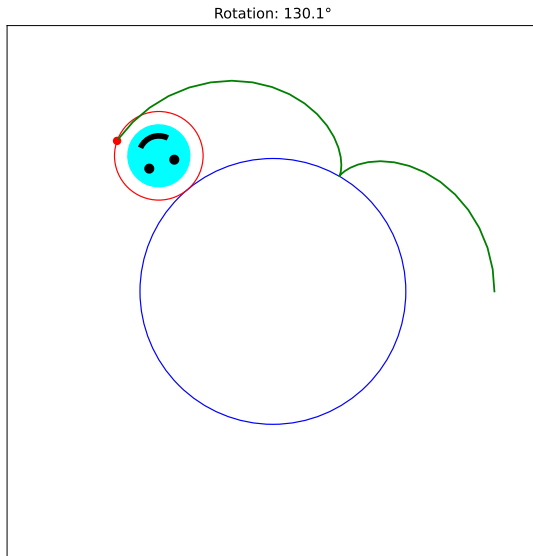
Animation of Coin Rotation Paradox



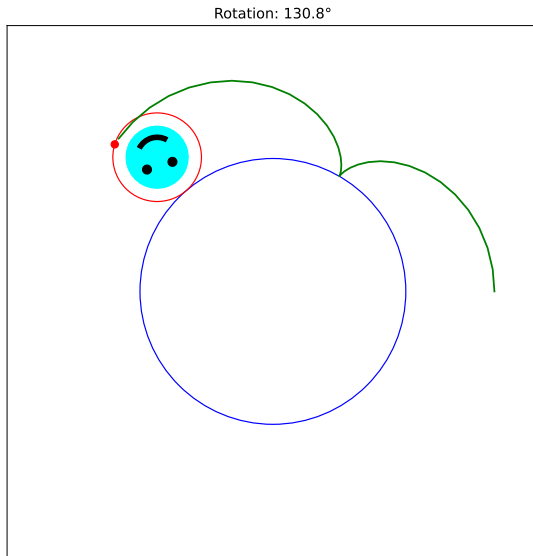
Animation of Coin Rotation Paradox



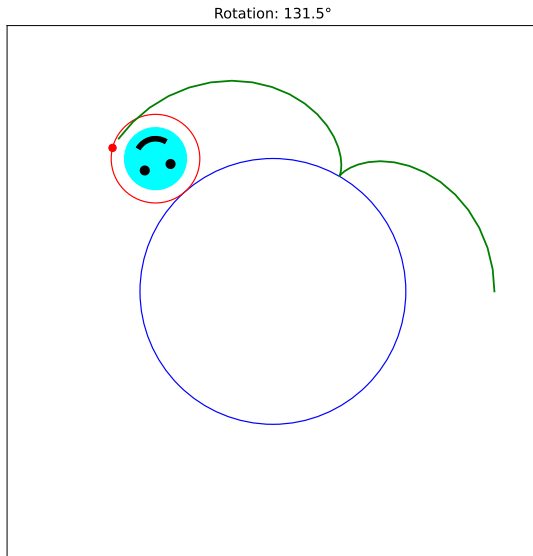
Animation of Coin Rotation Paradox



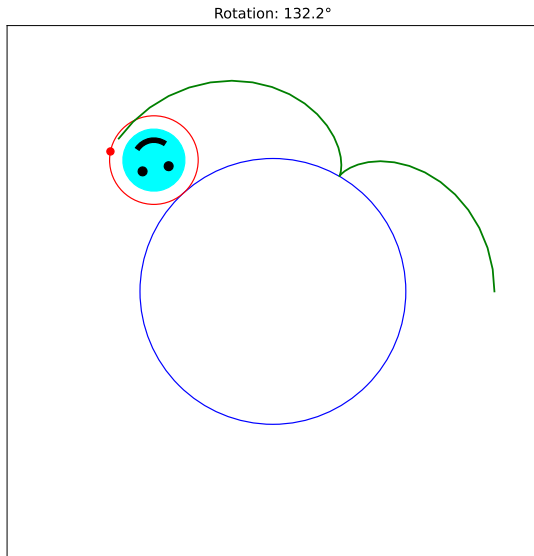
Animation of Coin Rotation Paradox



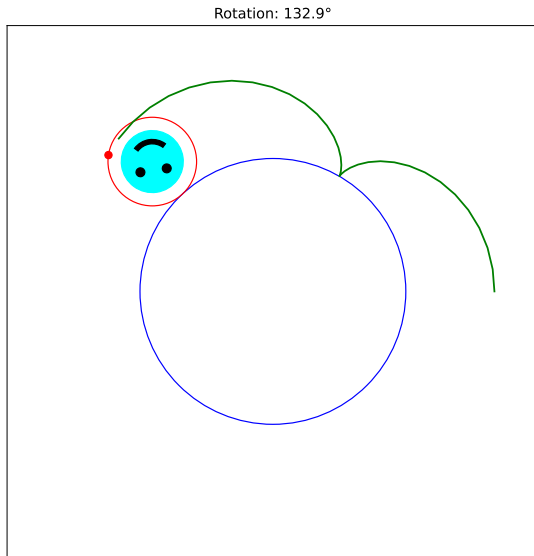
Animation of Coin Rotation Paradox



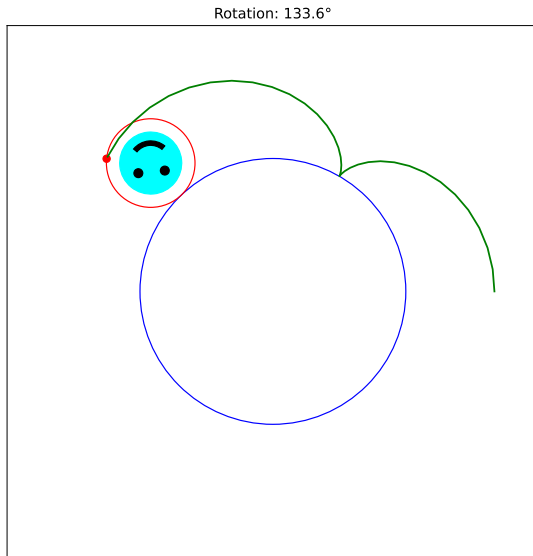
Animation of Coin Rotation Paradox



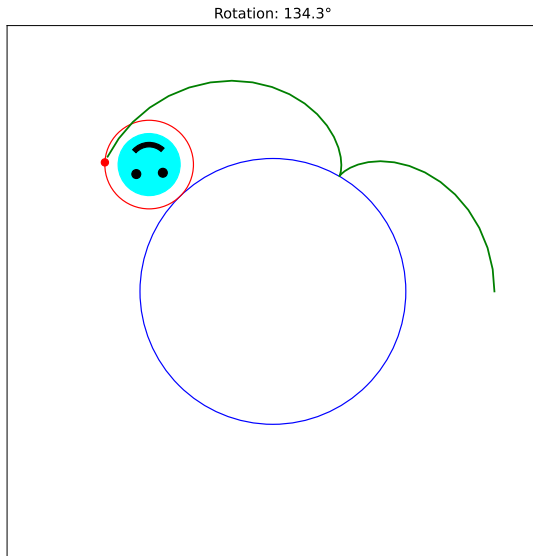
Animation of Coin Rotation Paradox



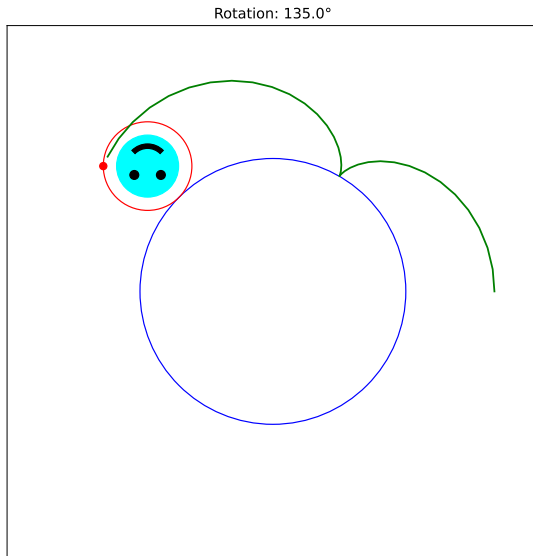
Animation of Coin Rotation Paradox



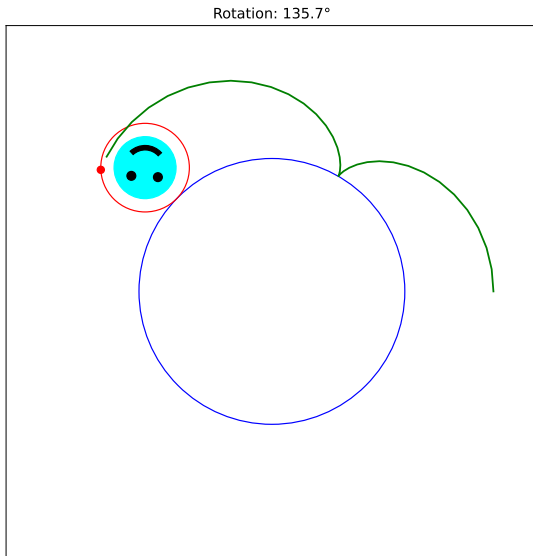
Animation of Coin Rotation Paradox



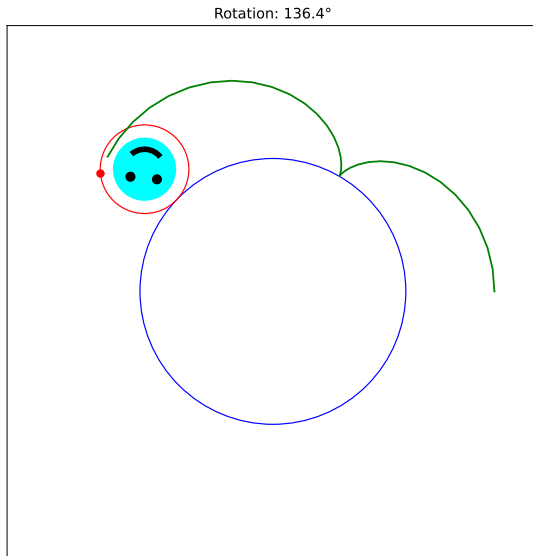
Animation of Coin Rotation Paradox



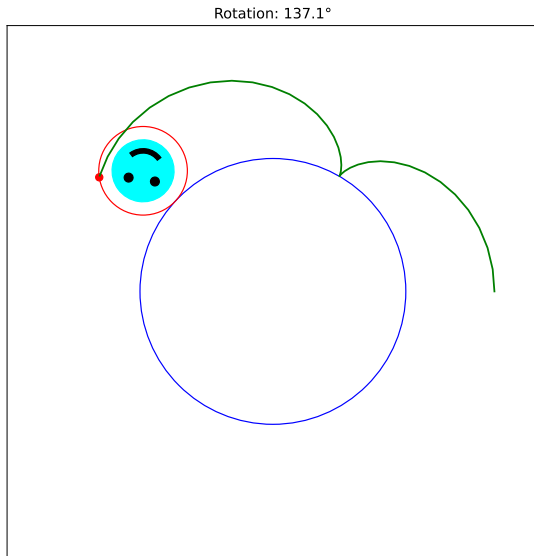
Animation of Coin Rotation Paradox



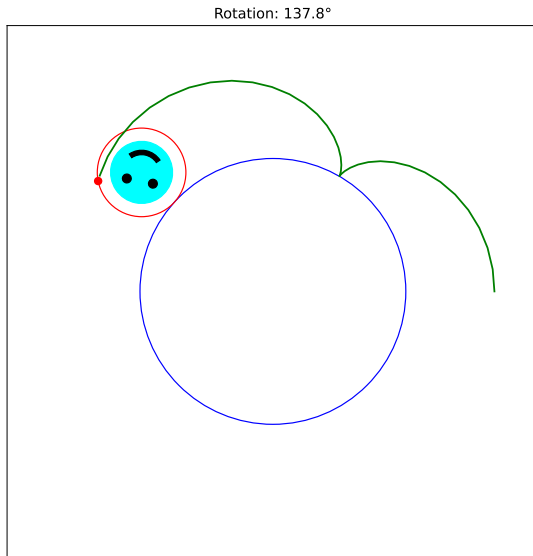
Animation of Coin Rotation Paradox



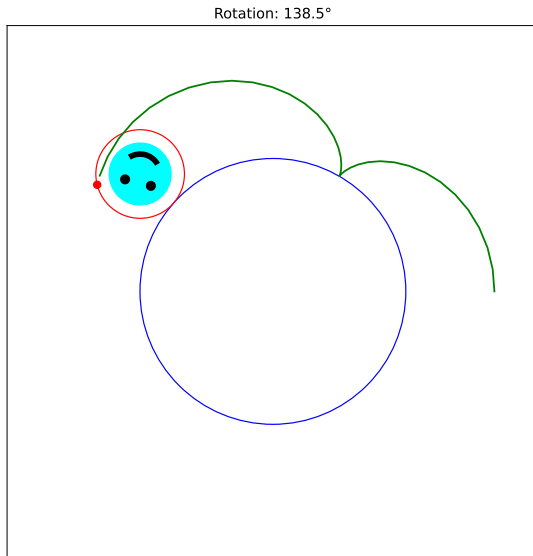
Animation of Coin Rotation Paradox



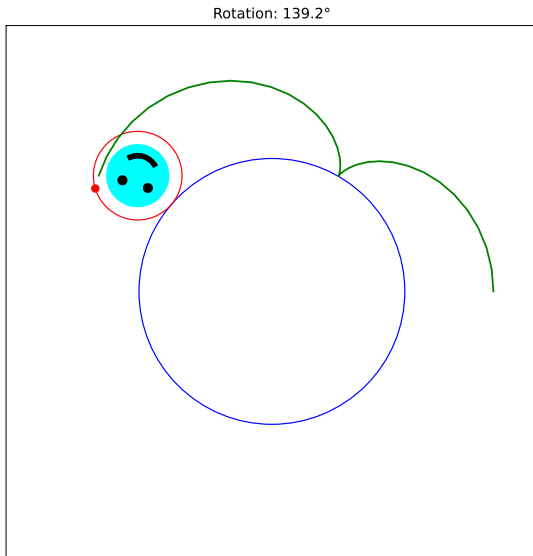
Animation of Coin Rotation Paradox



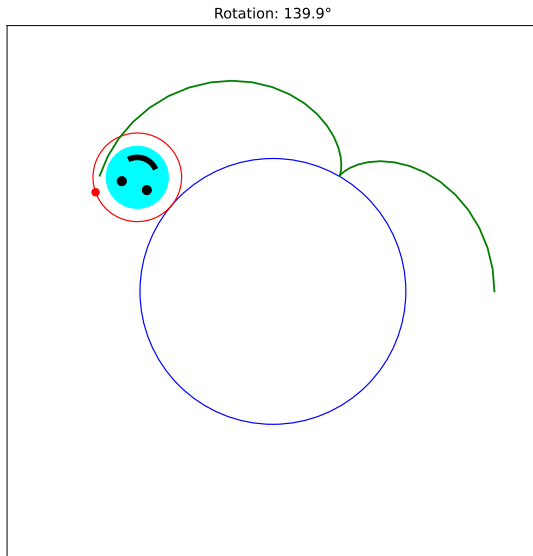
Animation of Coin Rotation Paradox



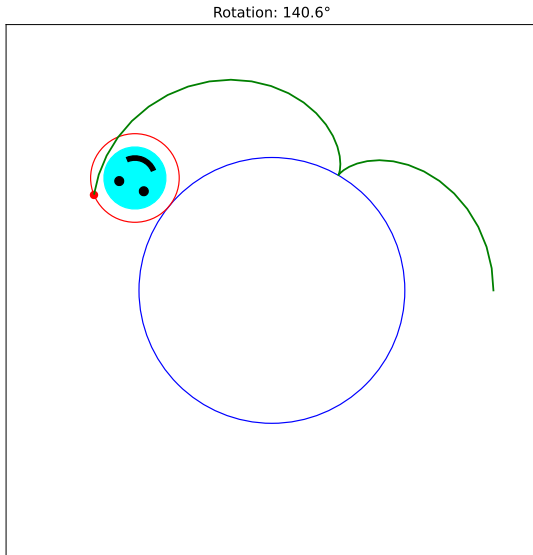
Animation of Coin Rotation Paradox



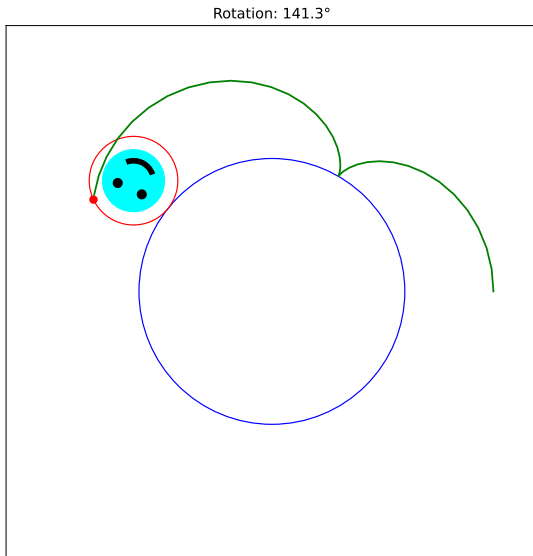
Animation of Coin Rotation Paradox



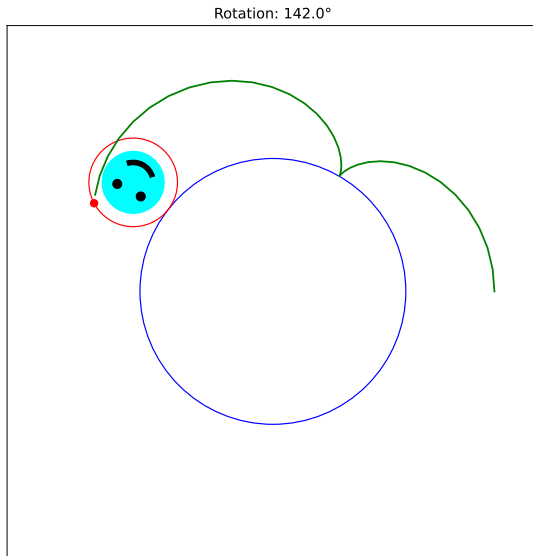
Animation of Coin Rotation Paradox



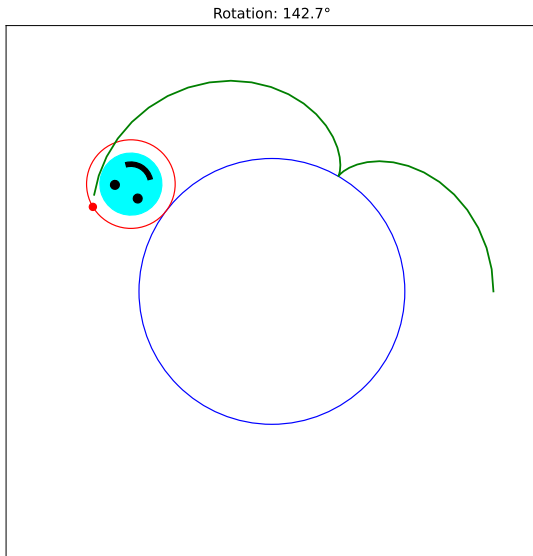
Animation of Coin Rotation Paradox



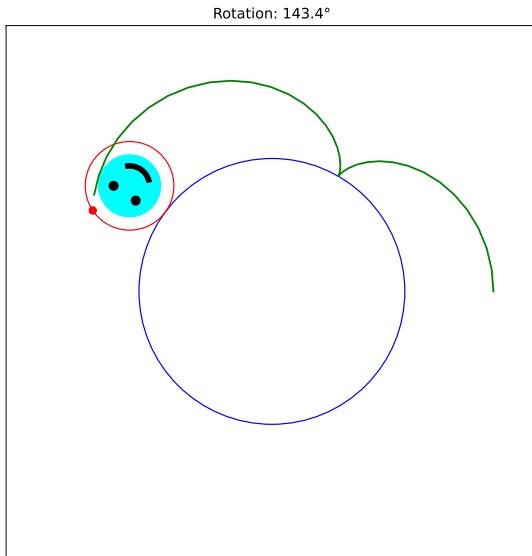
Animation of Coin Rotation Paradox



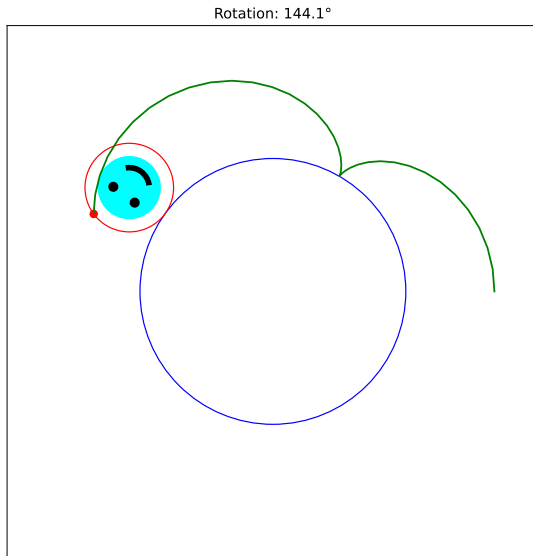
Animation of Coin Rotation Paradox



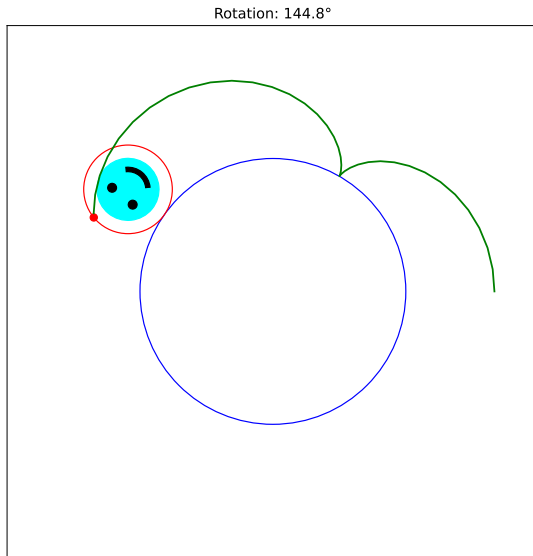
Animation of Coin Rotation Paradox



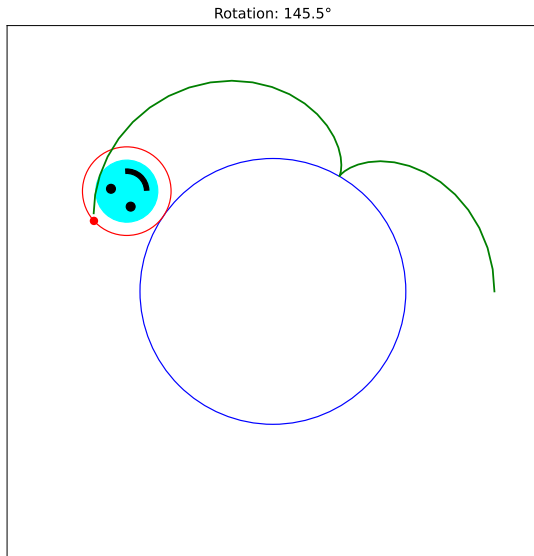
Animation of Coin Rotation Paradox



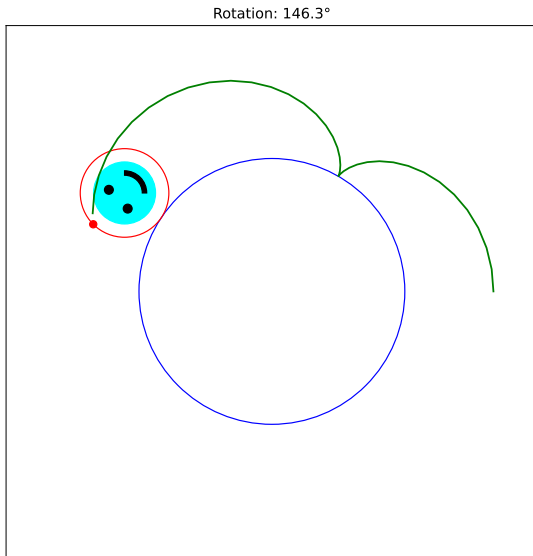
Animation of Coin Rotation Paradox



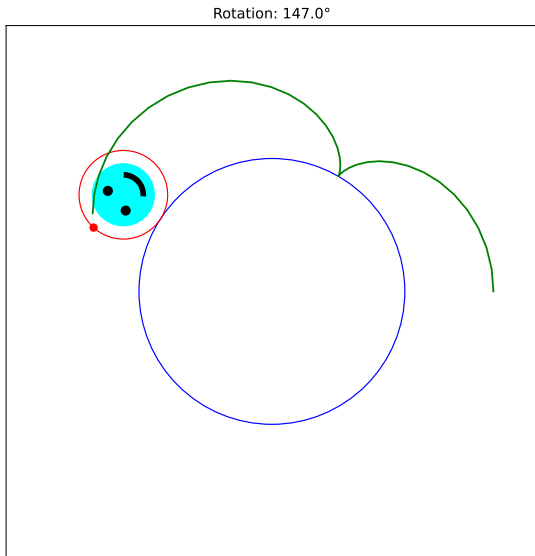
Animation of Coin Rotation Paradox



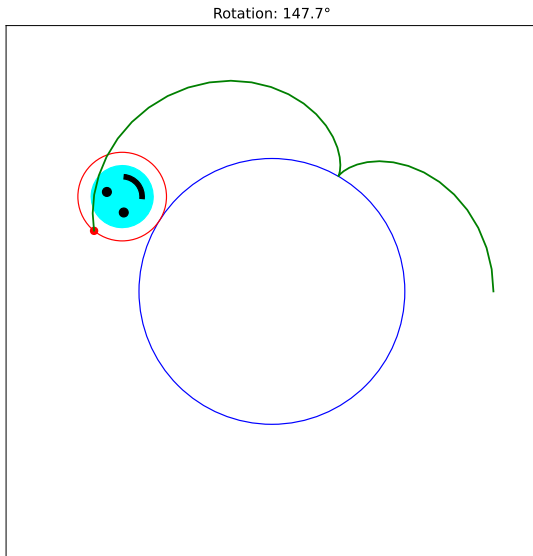
Animation of Coin Rotation Paradox



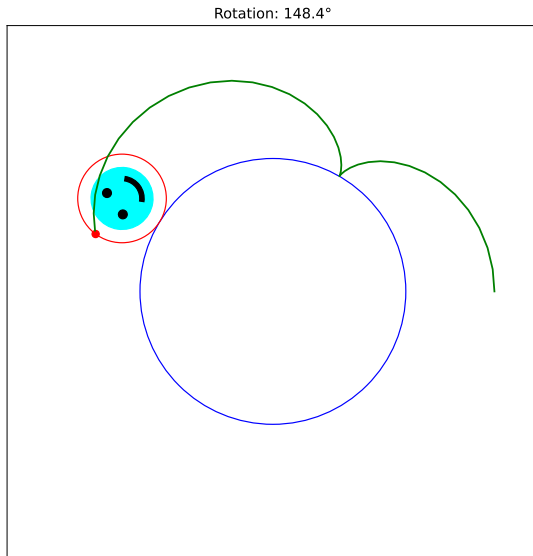
Animation of Coin Rotation Paradox



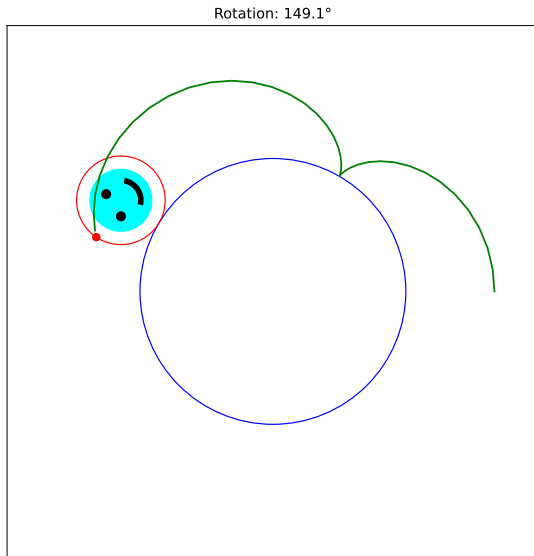
Animation of Coin Rotation Paradox



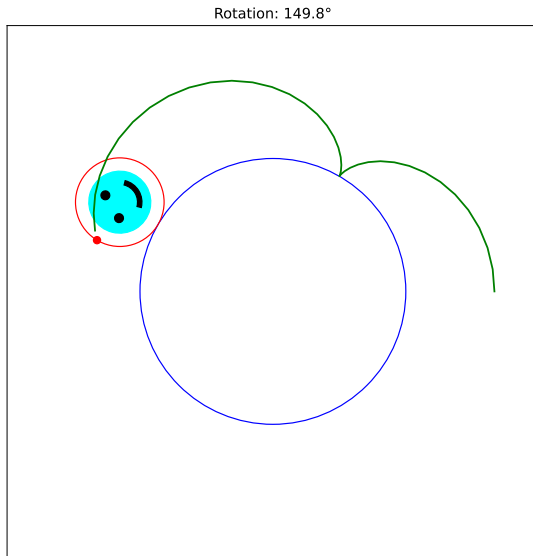
Animation of Coin Rotation Paradox



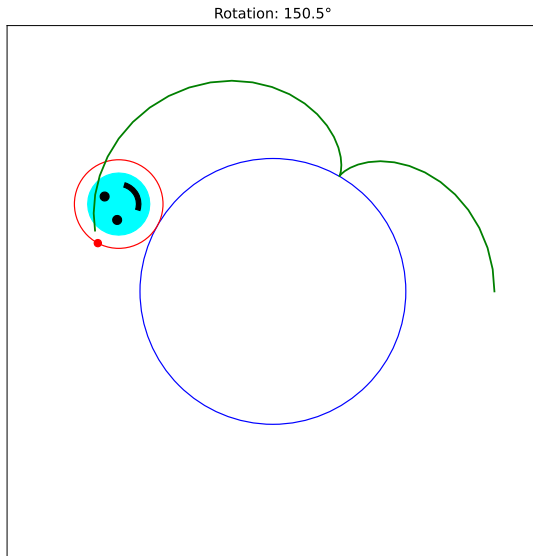
Animation of Coin Rotation Paradox



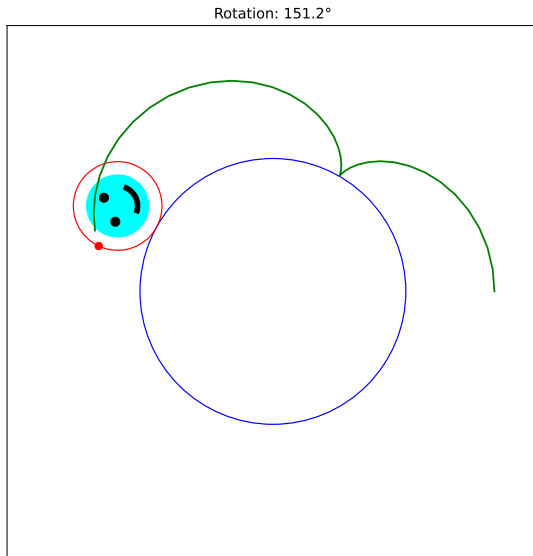
Animation of Coin Rotation Paradox



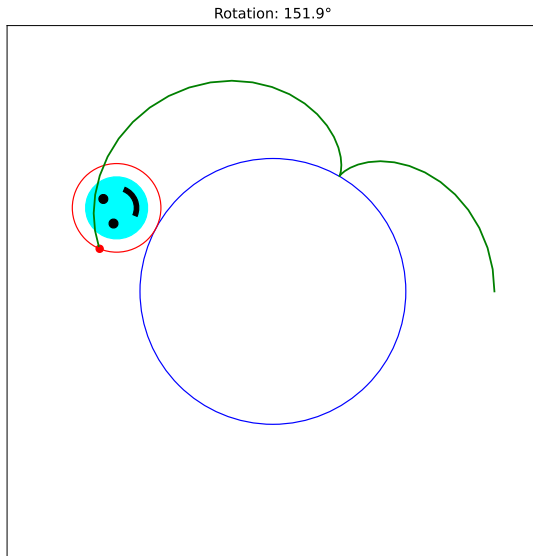
Animation of Coin Rotation Paradox



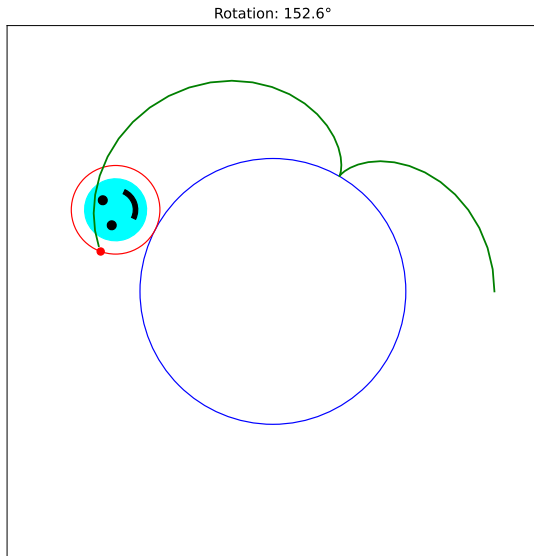
Animation of Coin Rotation Paradox



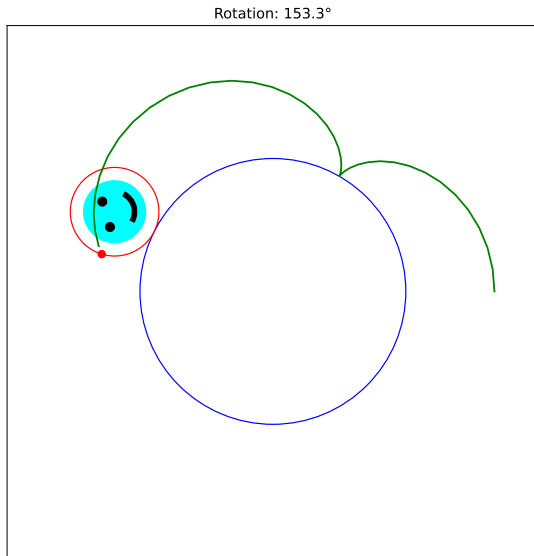
Animation of Coin Rotation Paradox



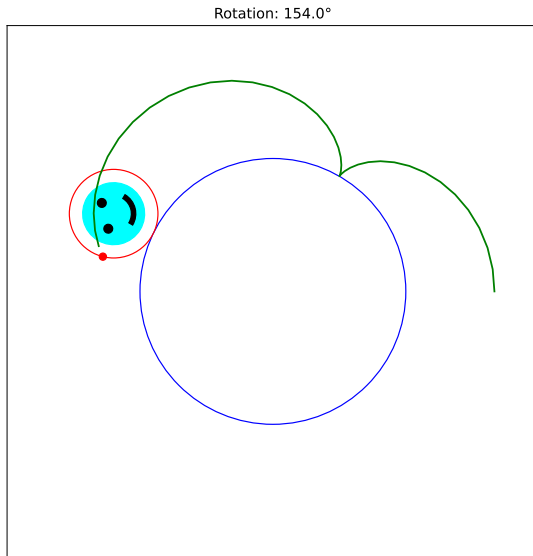
Animation of Coin Rotation Paradox



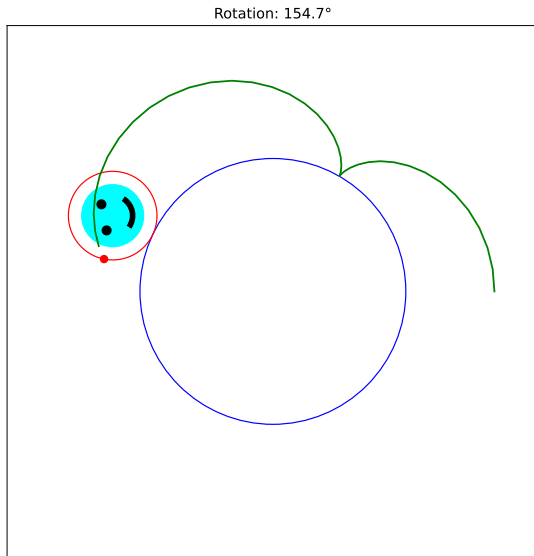
Animation of Coin Rotation Paradox



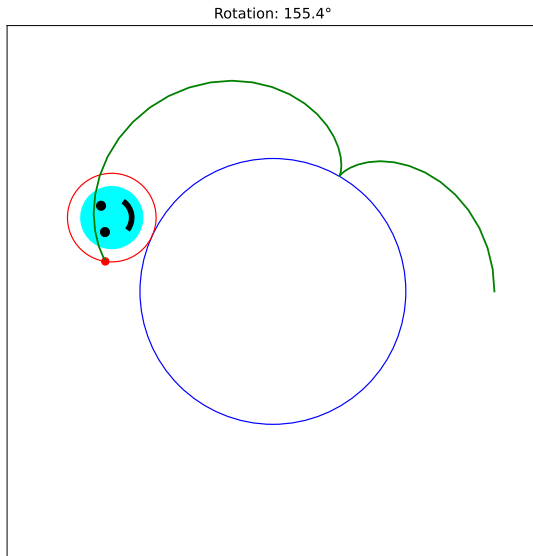
Animation of Coin Rotation Paradox



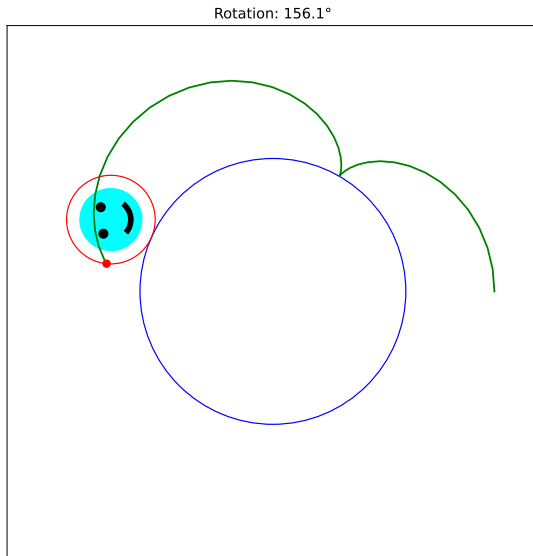
Animation of Coin Rotation Paradox



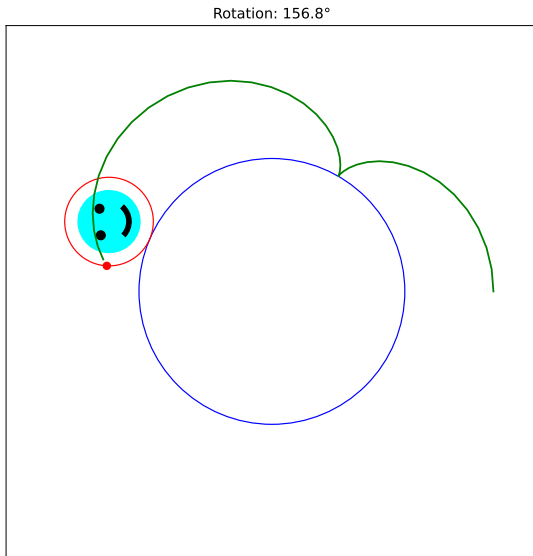
Animation of Coin Rotation Paradox



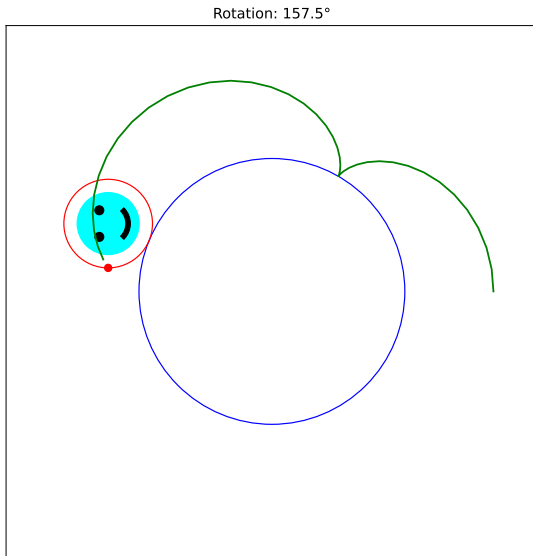
Animation of Coin Rotation Paradox



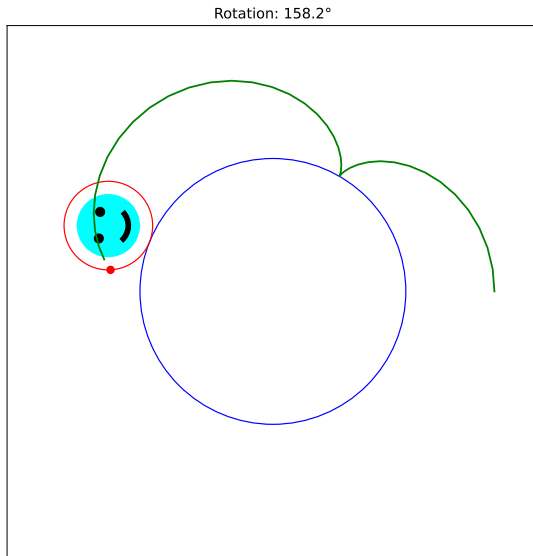
Animation of Coin Rotation Paradox



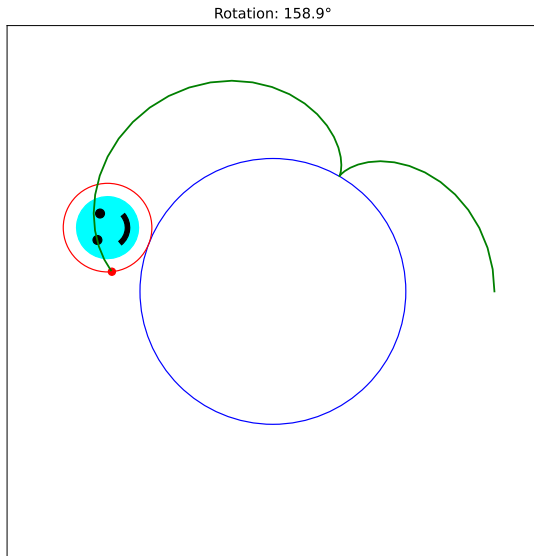
Animation of Coin Rotation Paradox



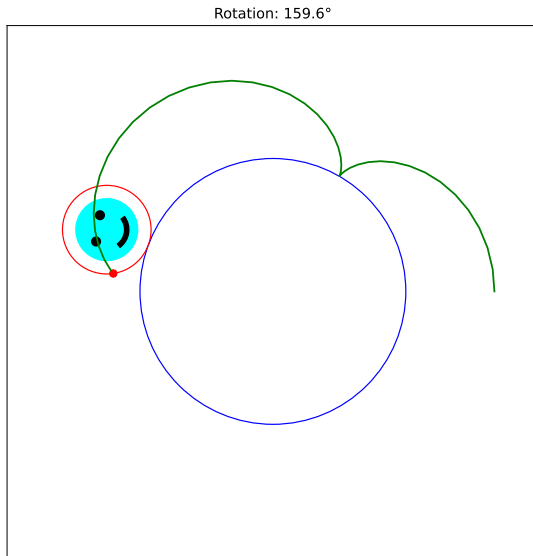
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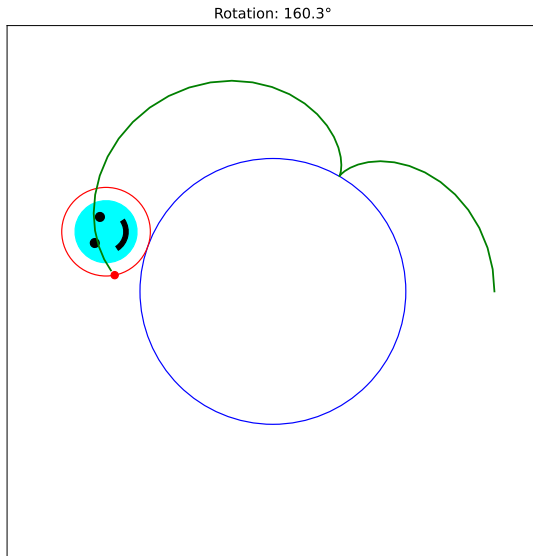
Animation of Coin Rotation Paradox



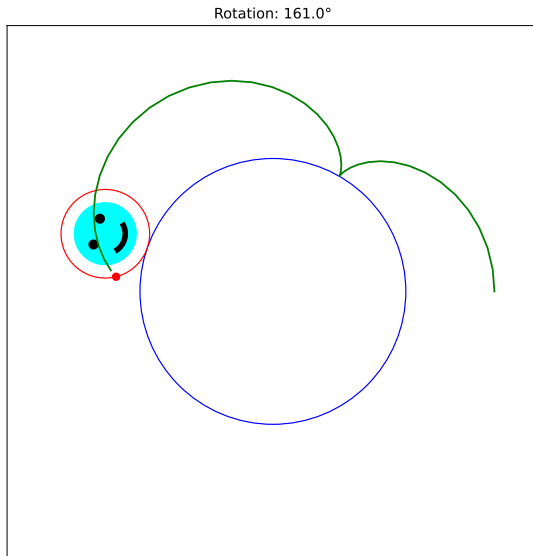
Animation of Coin Rotation Paradox



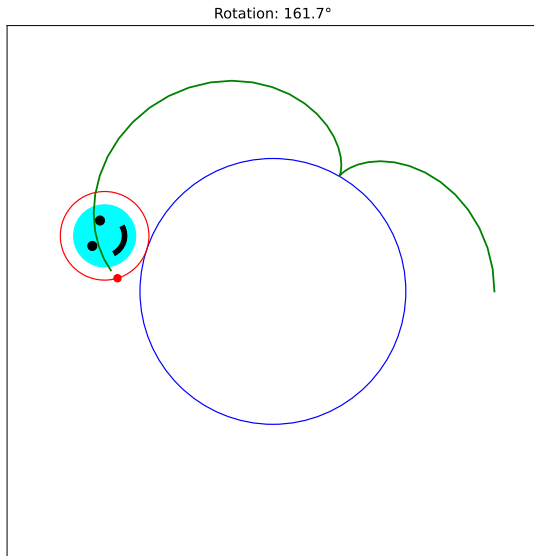
Animation of Coin Rotation Paradox



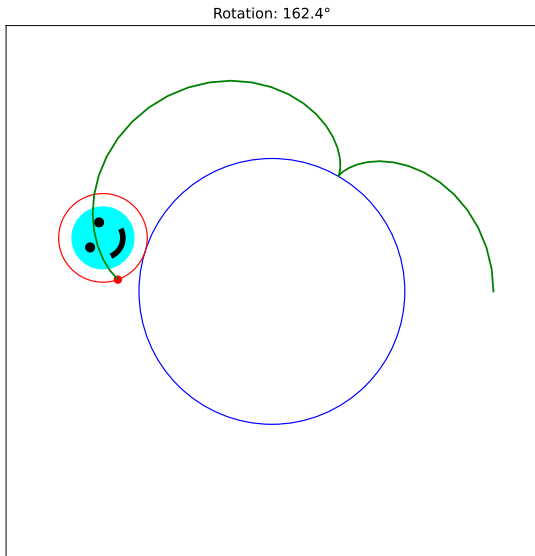
Animation of Coin Rotation Paradox



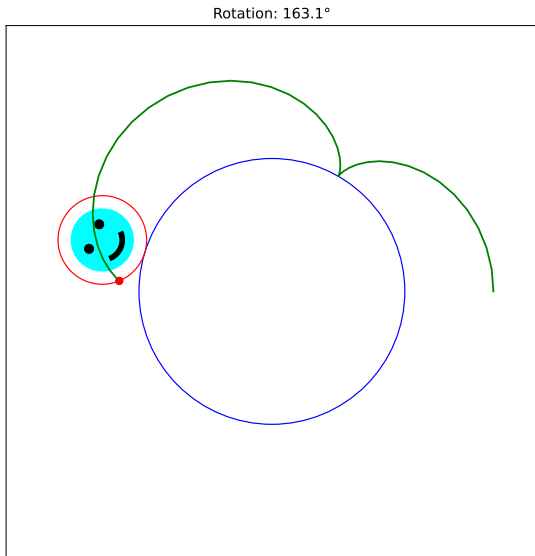
Animation of Coin Rotation Paradox



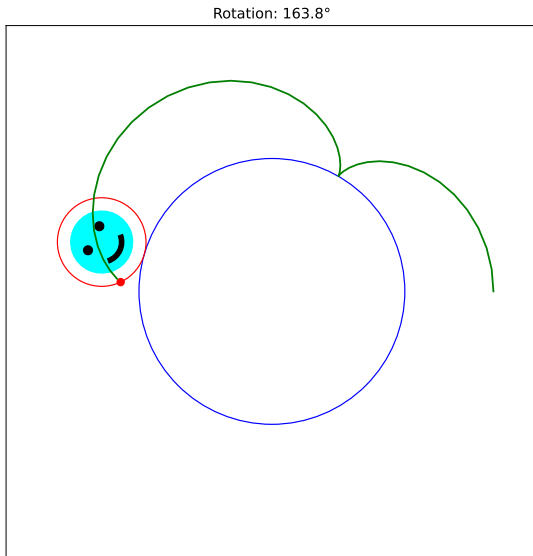
Animation of Coin Rotation Paradox



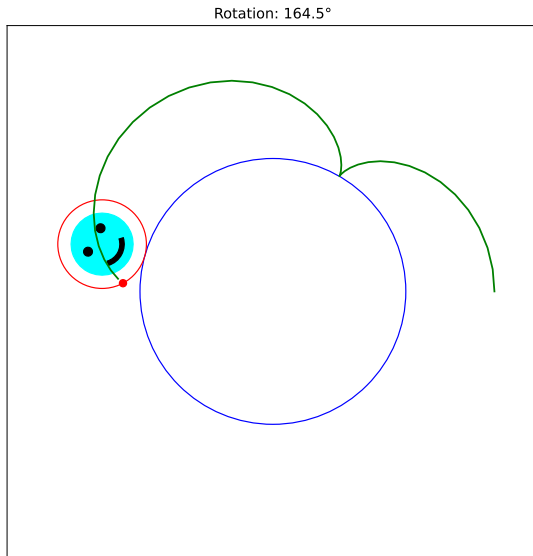
Animation of Coin Rotation Paradox



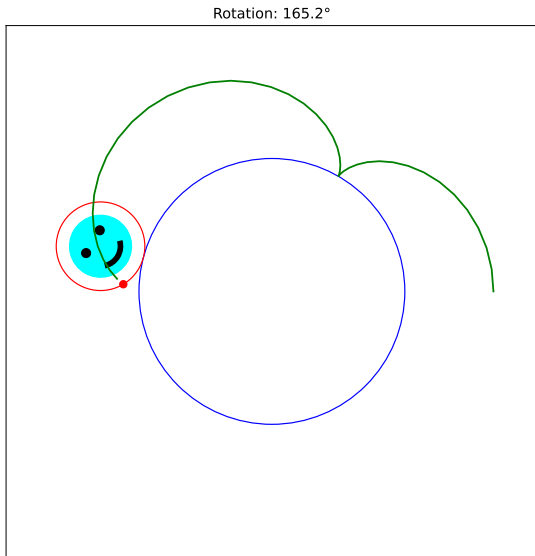
Animation of Coin Rotation Paradox



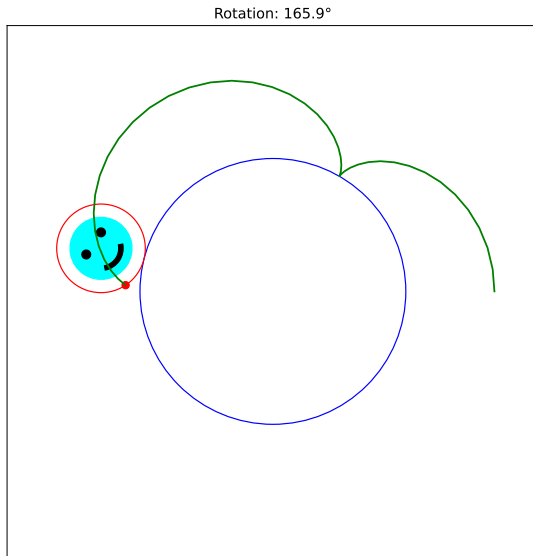
Animation of Coin Rotation Paradox



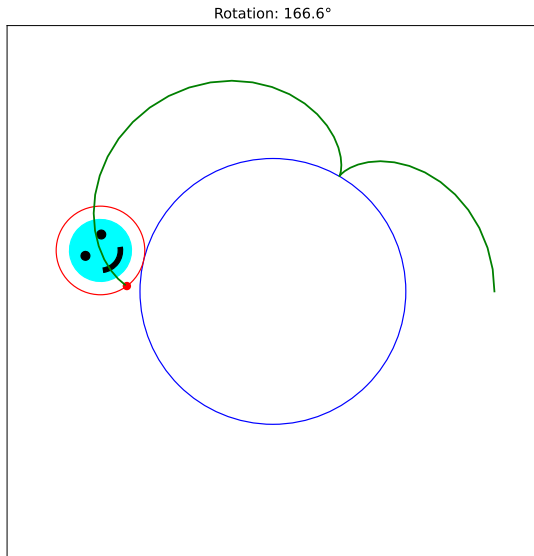
Animation of Coin Rotation Paradox



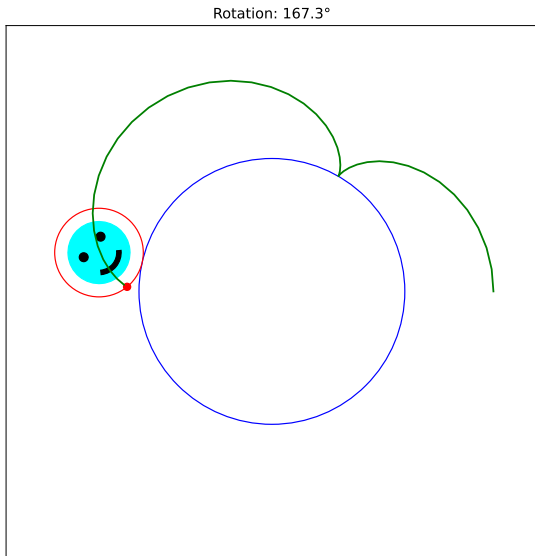
Animation of Coin Rotation Paradox



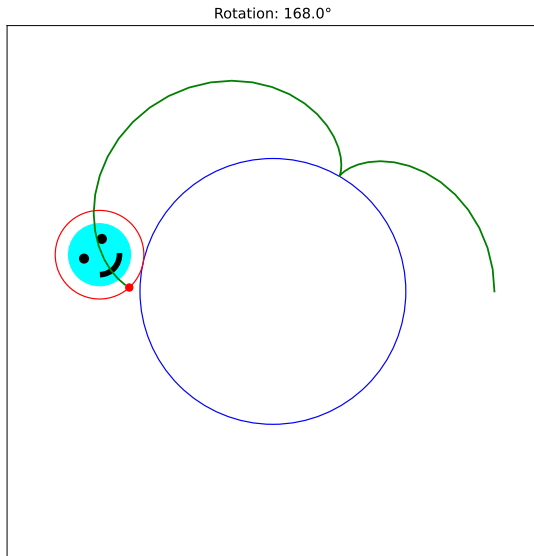
Animation of Coin Rotation Paradox



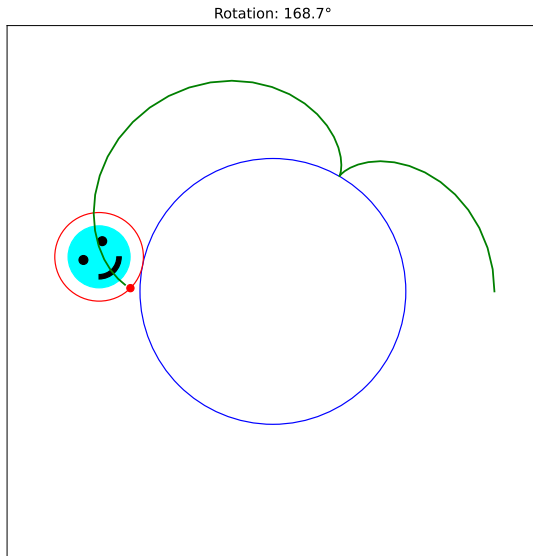
Animation of Coin Rotation Paradox



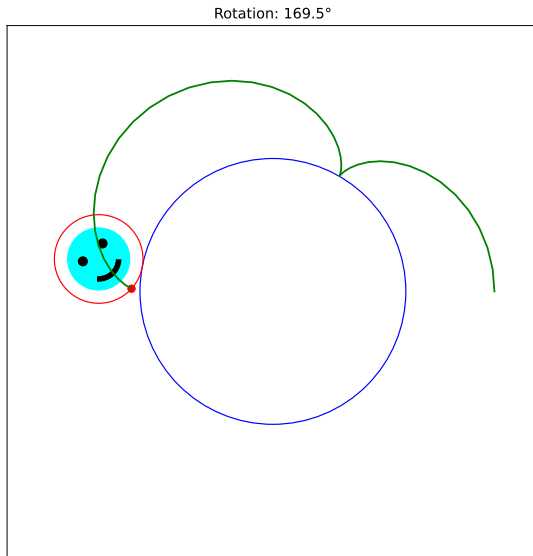
Animation of Coin Rotation Paradox



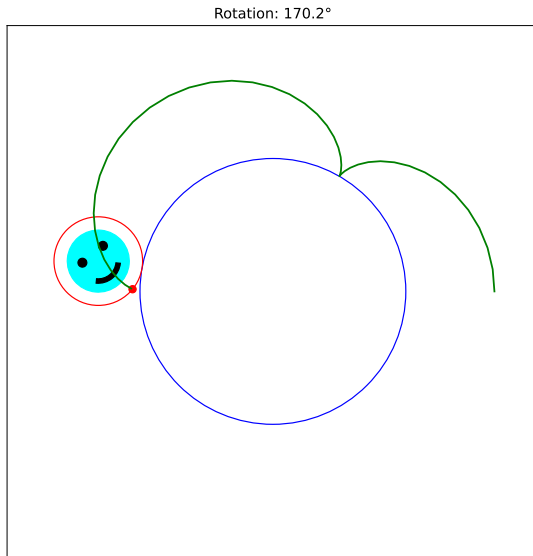
Animation of Coin Rotation Paradox



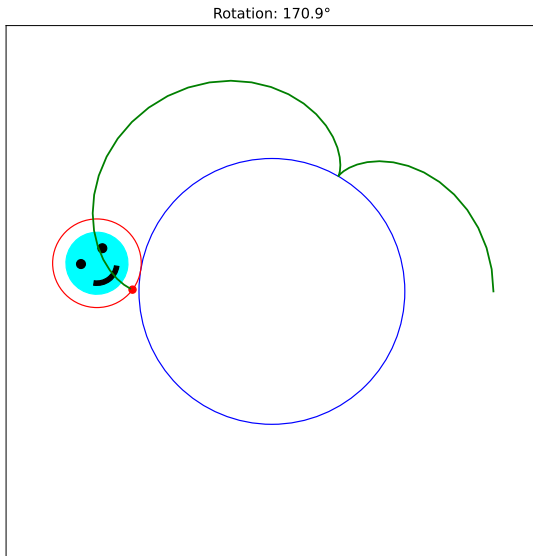
Animation of Coin Rotation Paradox



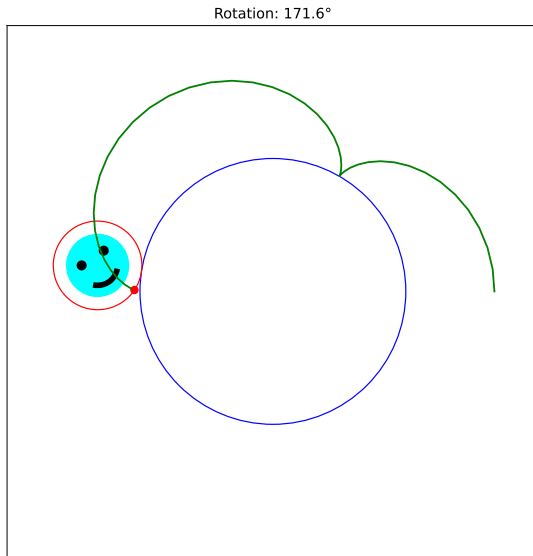
Animation of Coin Rotation Paradox



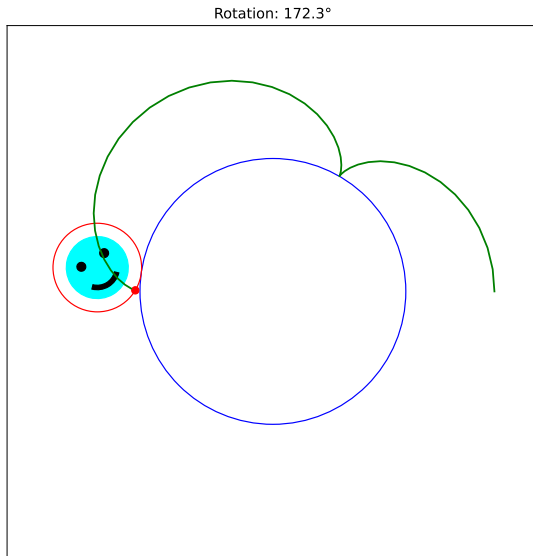
Animation of Coin Rotation Paradox



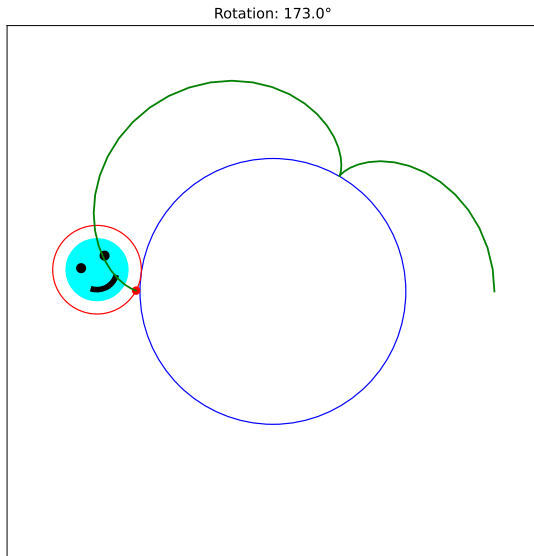
Animation of Coin Rotation Paradox



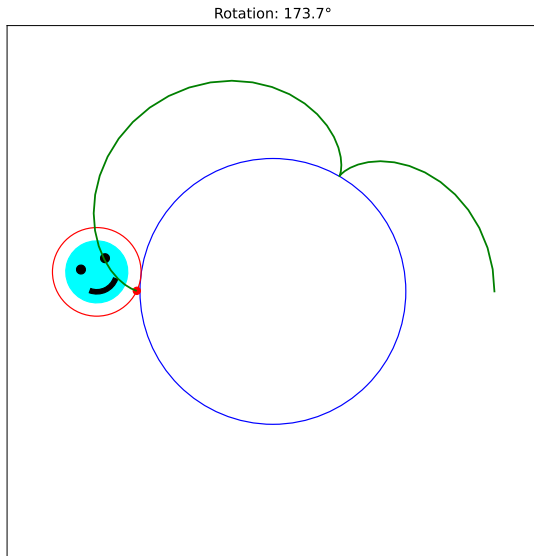
Animation of Coin Rotation Paradox



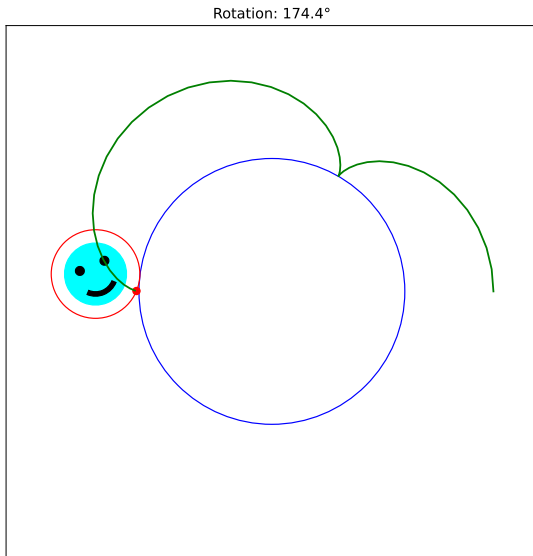
Animation of Coin Rotation Paradox



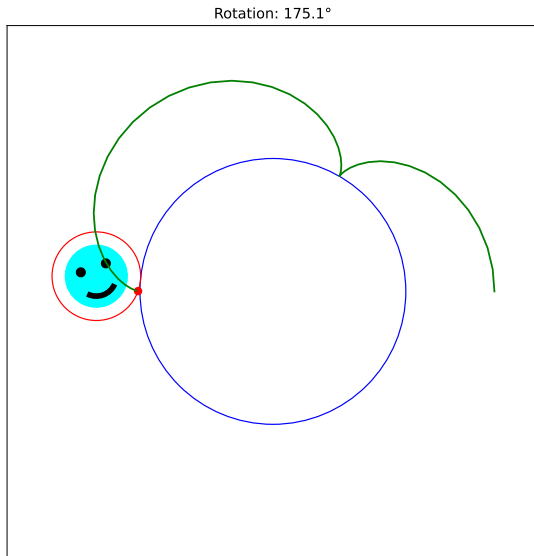
Animation of Coin Rotation Paradox



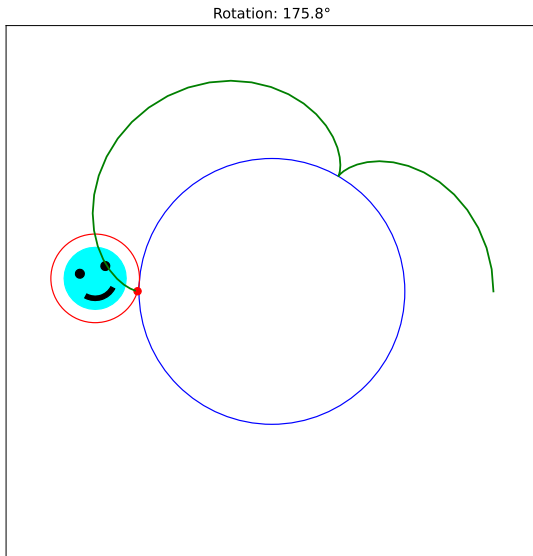
Animation of Coin Rotation Paradox



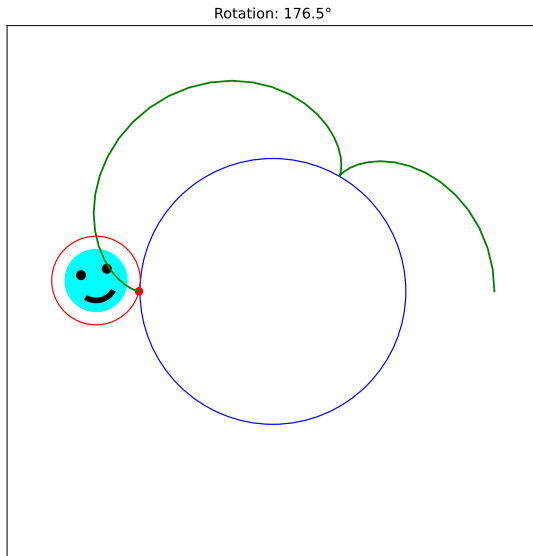
Animation of Coin Rotation Paradox



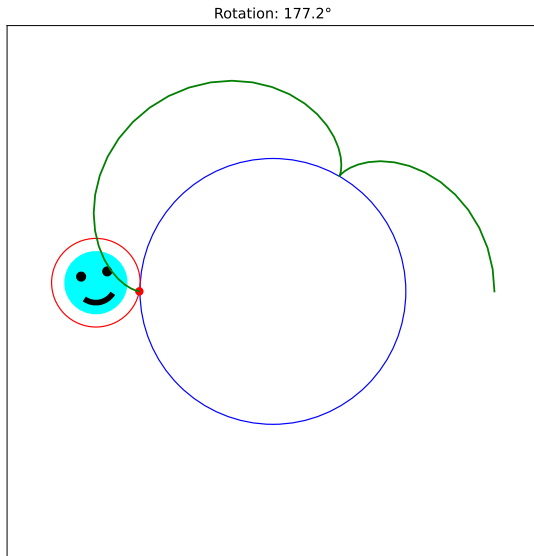
Animation of Coin Rotation Paradox



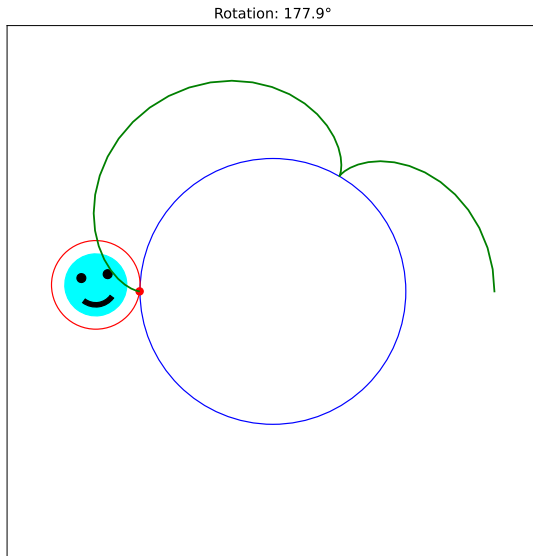
Animation of Coin Rotation Paradox



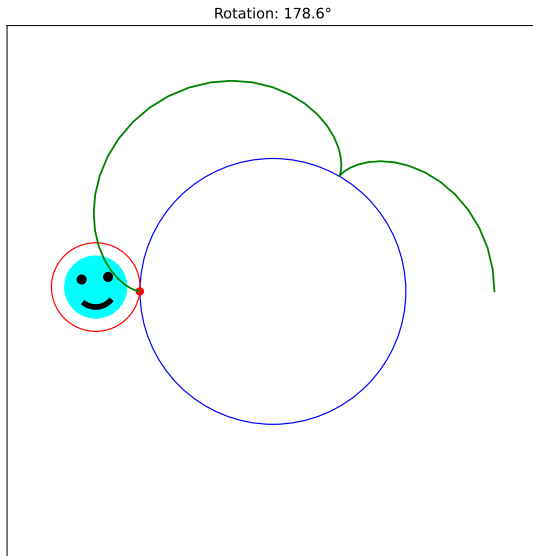
Animation of Coin Rotation Paradox



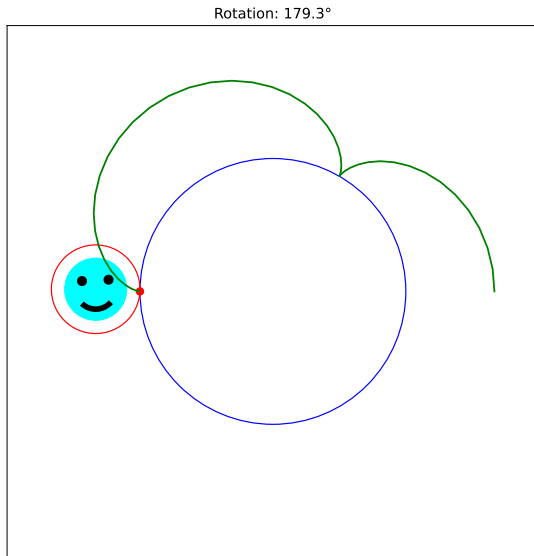
Animation of Coin Rotation Paradox



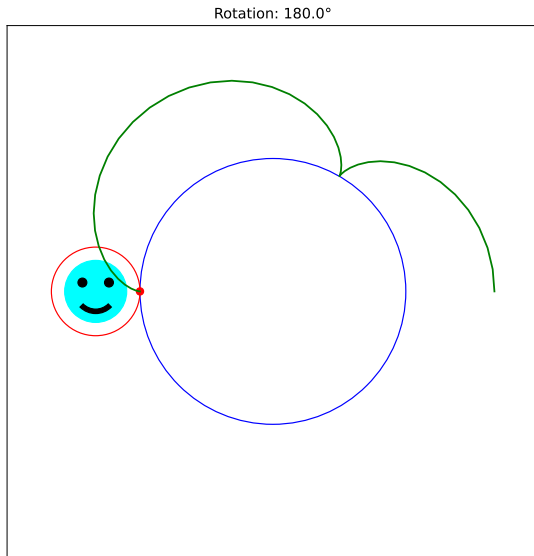
Animation of Coin Rotation Paradox



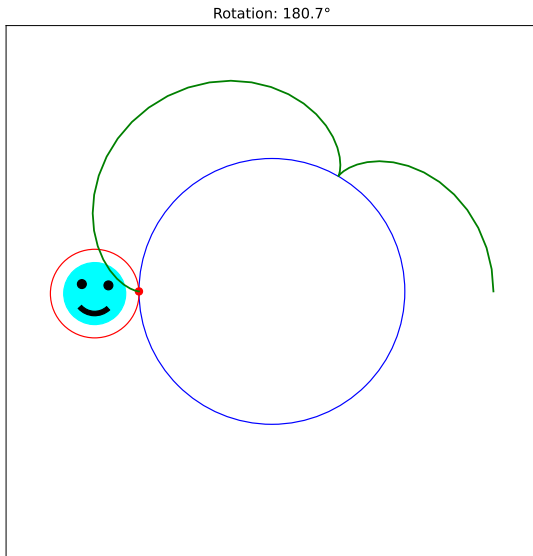
Animation of Coin Rotation Paradox



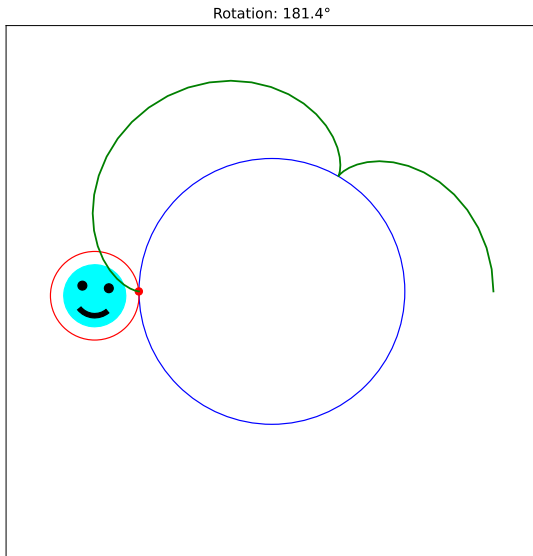
Animation of Coin Rotation Paradox



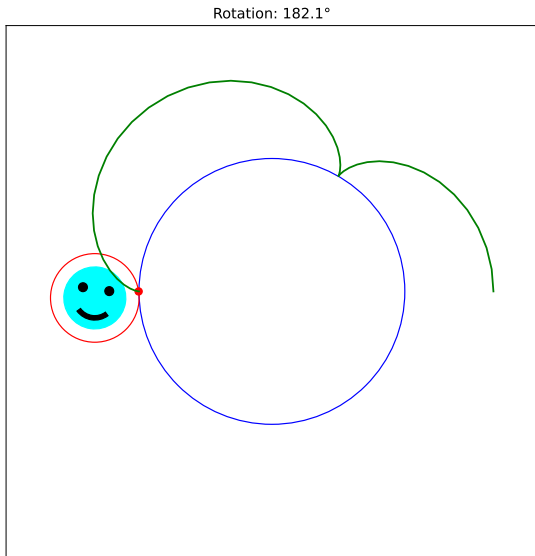
Animation of Coin Rotation Paradox



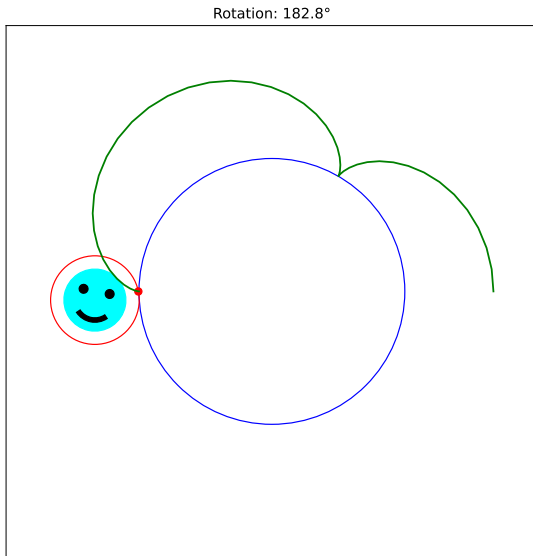
Animation of Coin Rotation Paradox



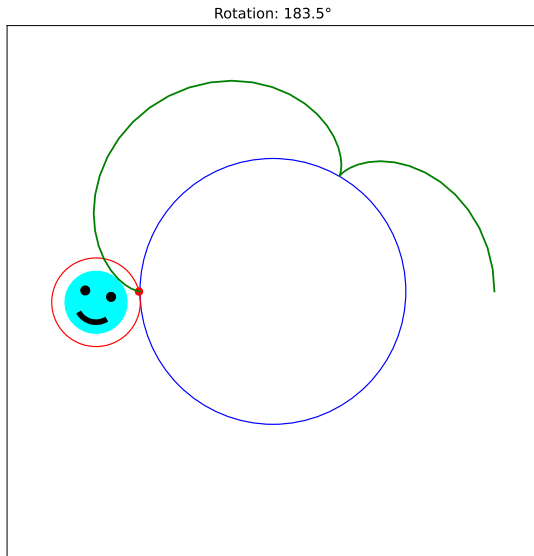
Animation of Coin Rotation Paradox



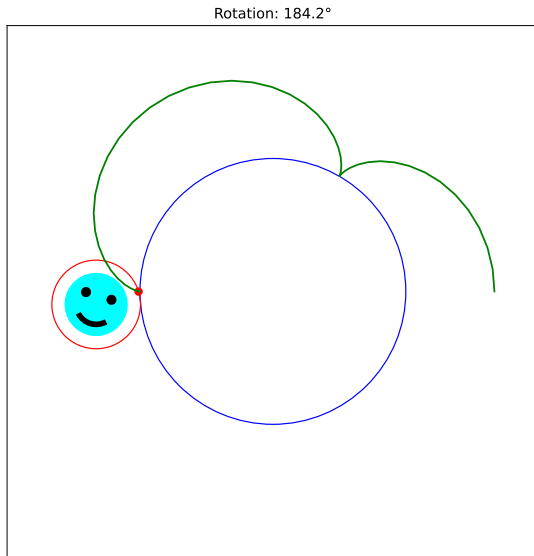
Animation of Coin Rotation Paradox



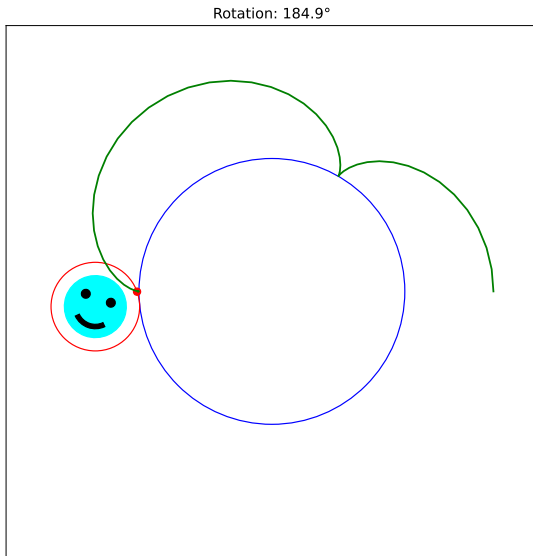
Animation of Coin Rotation Paradox



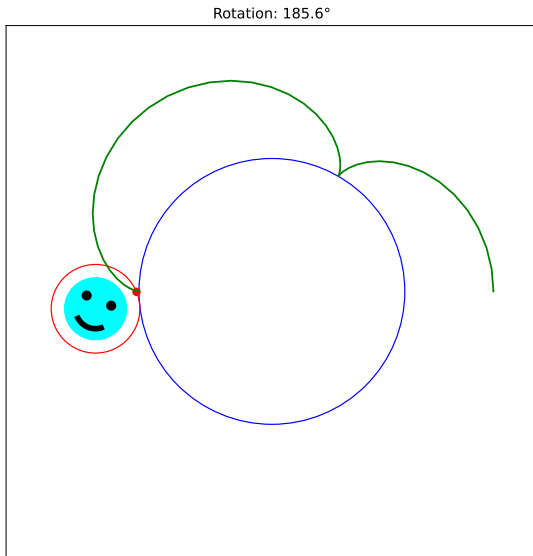
Animation of Coin Rotation Paradox



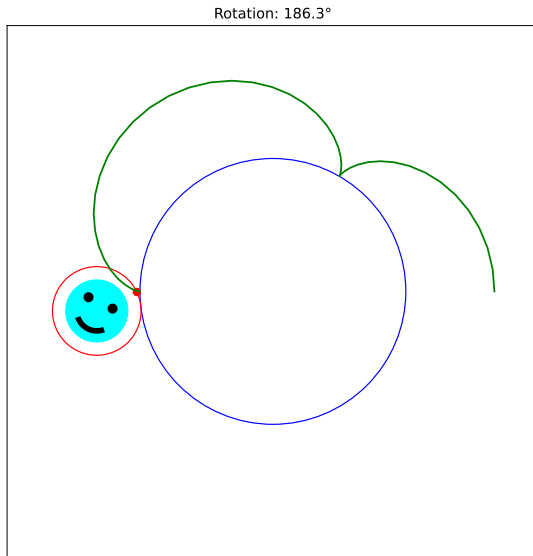
Animation of Coin Rotation Paradox



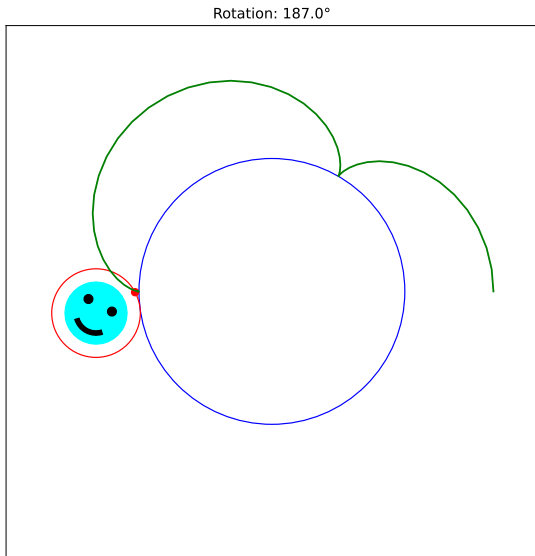
Animation of Coin Rotation Paradox



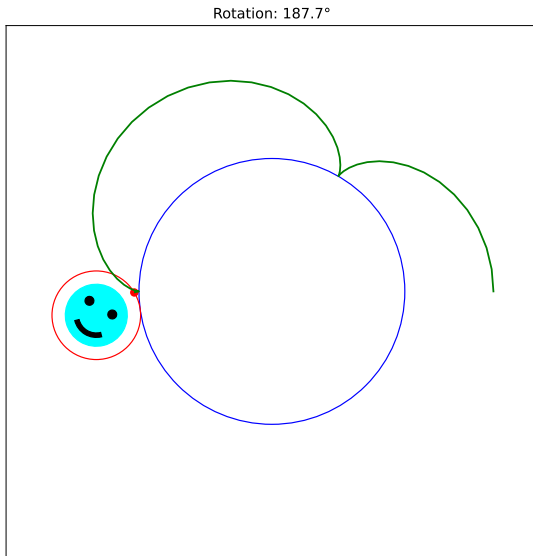
Animation of Coin Rotation Paradox



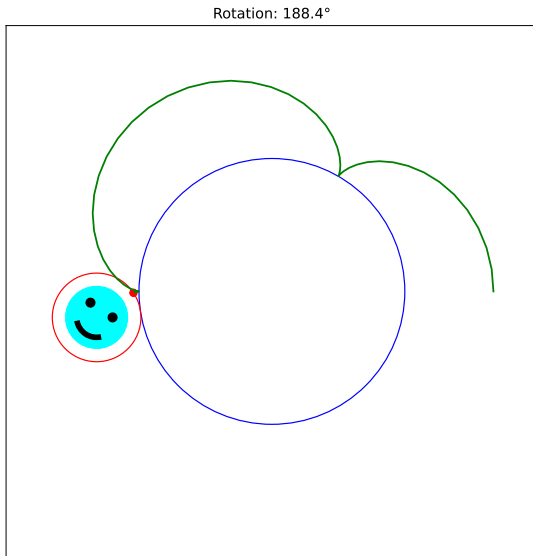
Animation of Coin Rotation Paradox



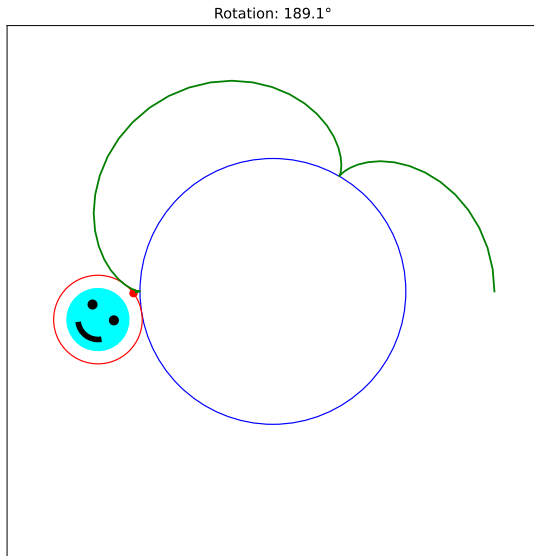
Animation of Coin Rotation Paradox



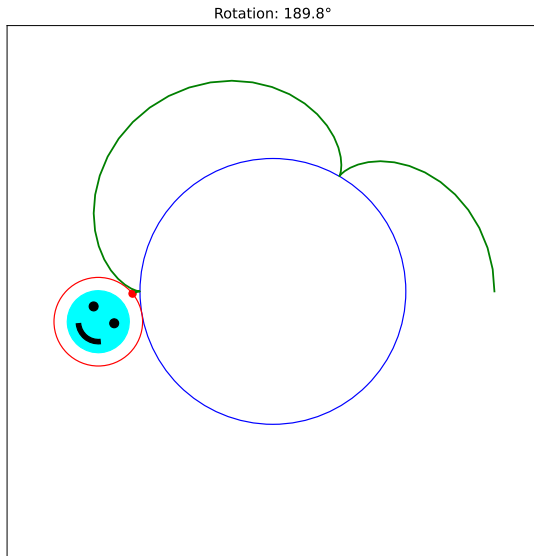
Animation of Coin Rotation Paradox



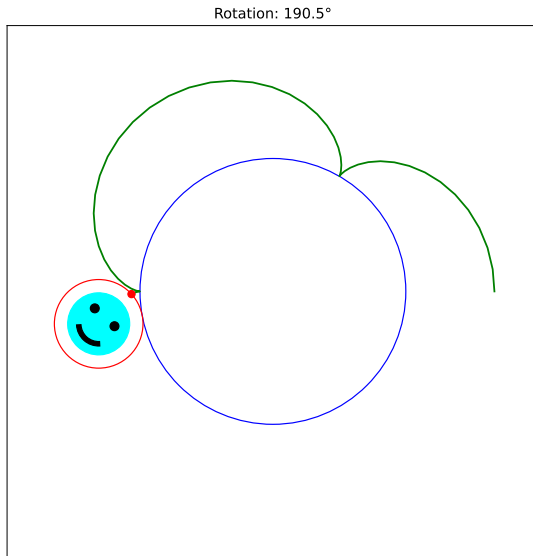
Animation of Coin Rotation Paradox



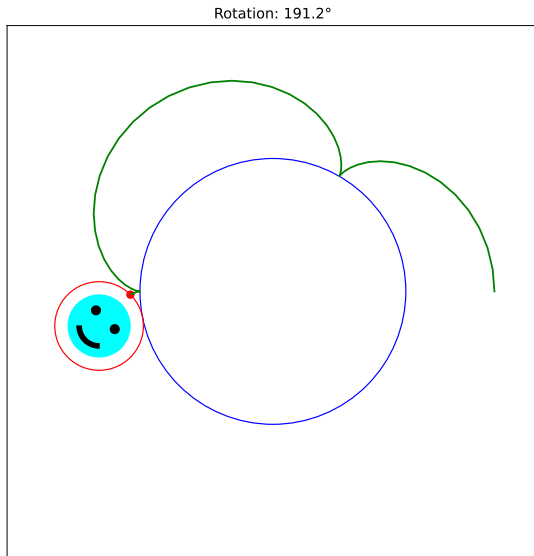
Animation of Coin Rotation Paradox



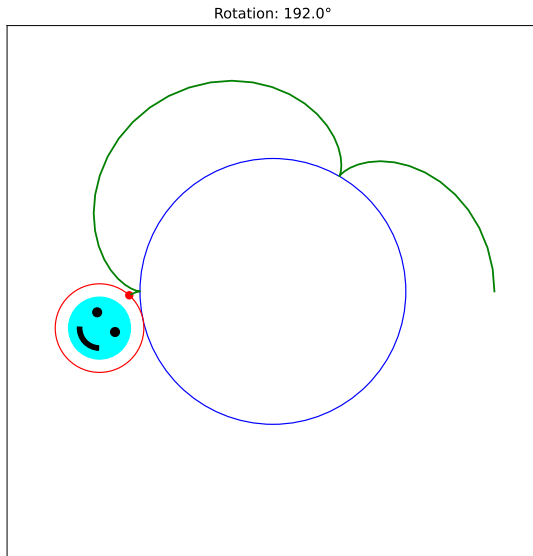
Animation of Coin Rotation Paradox



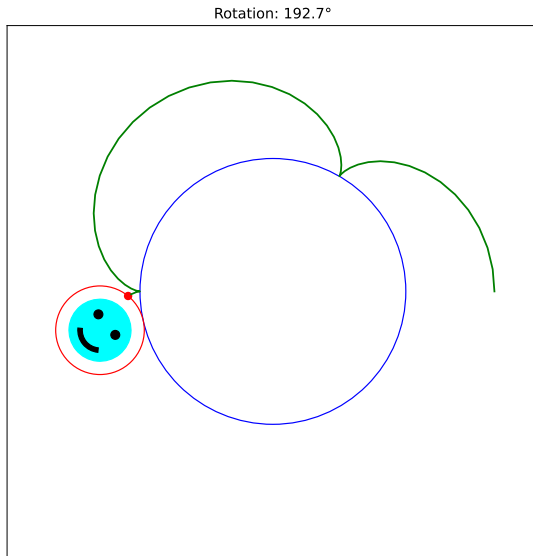
Animation of Coin Rotation Paradox



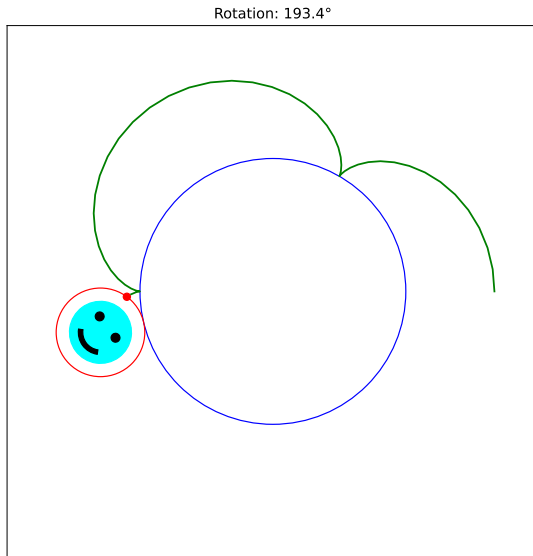
Animation of Coin Rotation Paradox



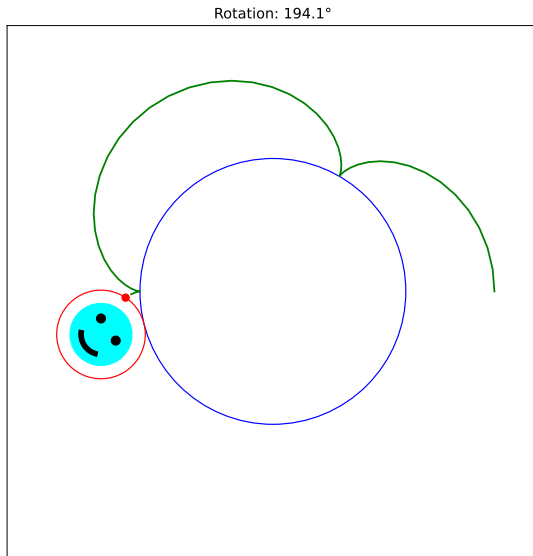
Animation of Coin Rotation Paradox



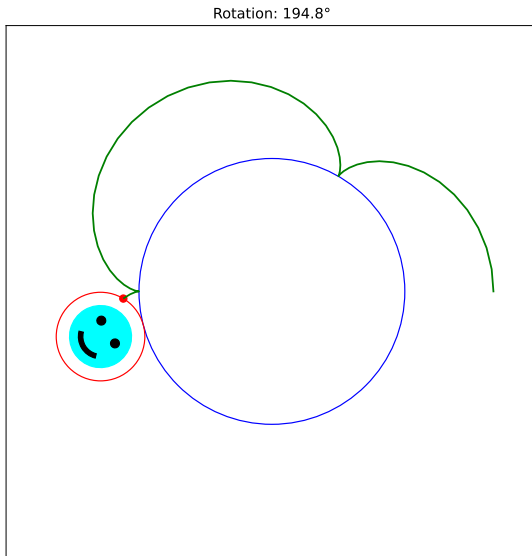
Animation of Coin Rotation Paradox



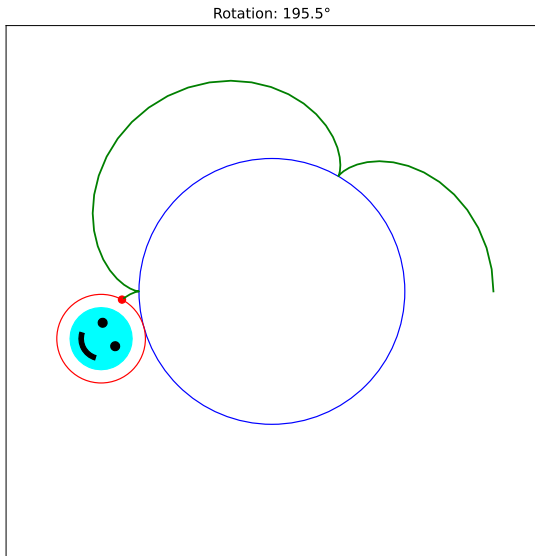
Animation of Coin Rotation Paradox



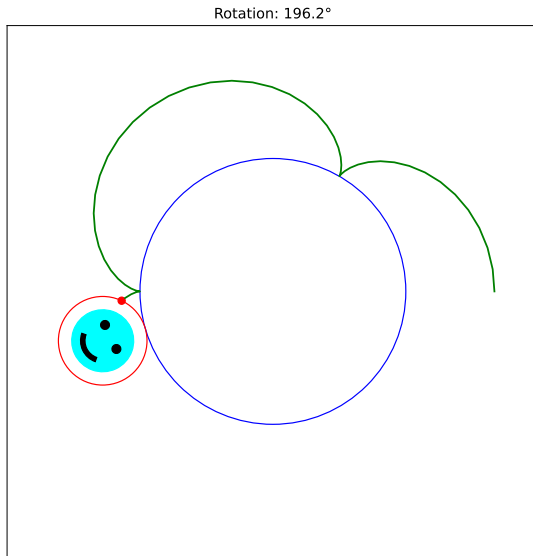
Animation of Coin Rotation Paradox



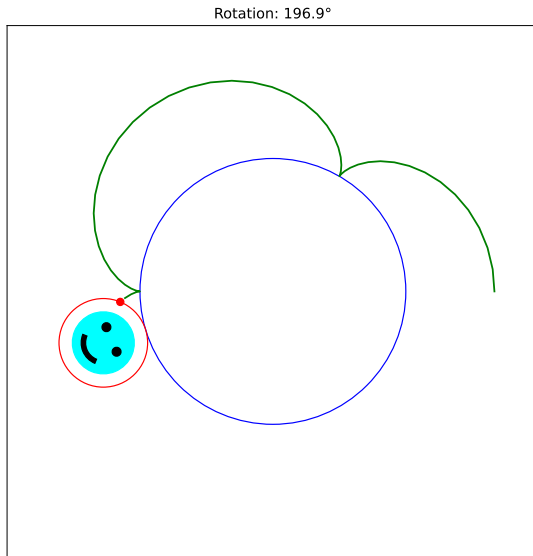
Animation of Coin Rotation Paradox



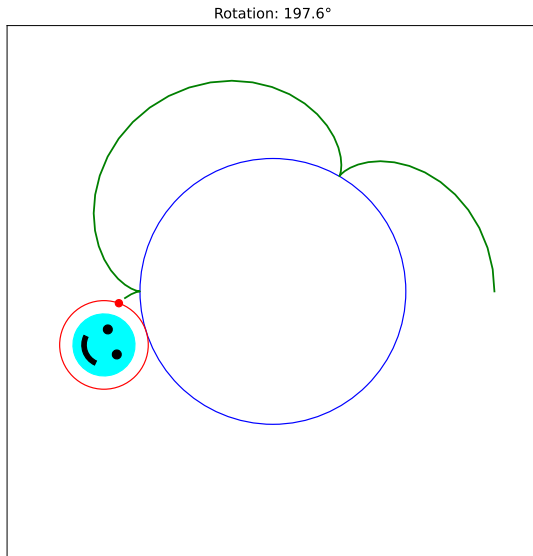
Animation of Coin Rotation Paradox



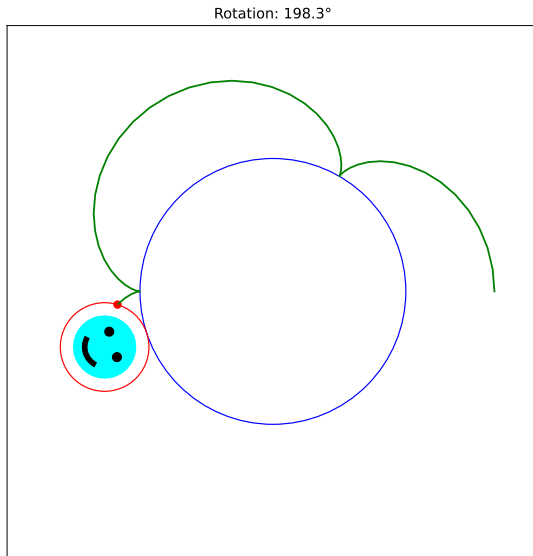
Animation of Coin Rotation Paradox



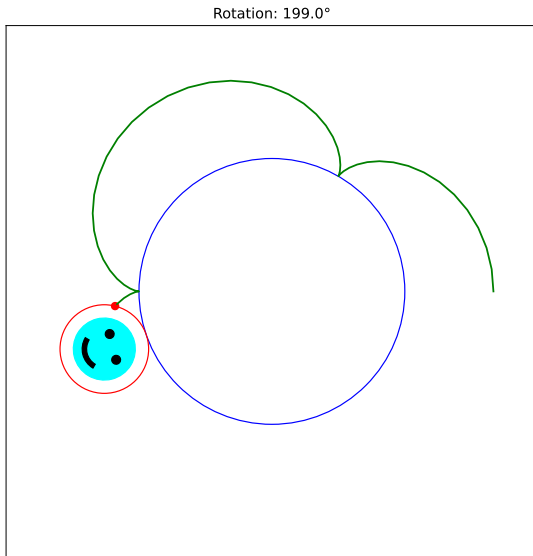
Animation of Coin Rotation Paradox



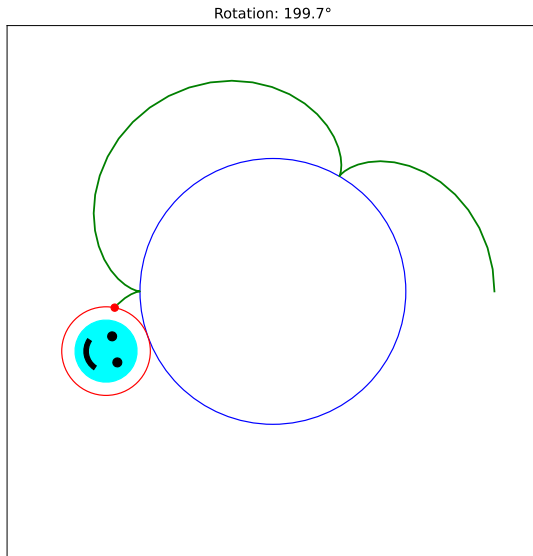
Animation of Coin Rotation Paradox



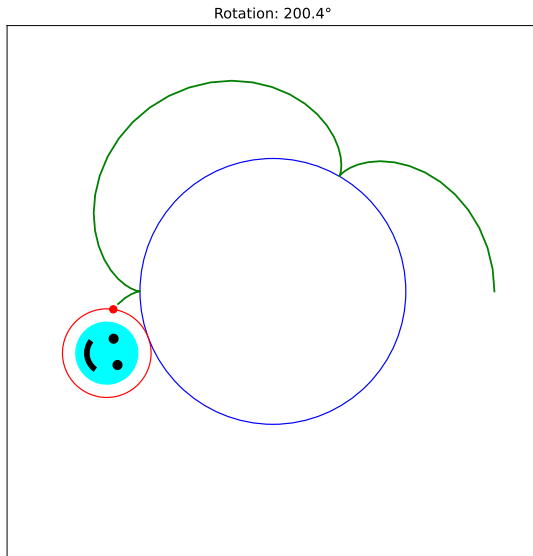
Animation of Coin Rotation Paradox



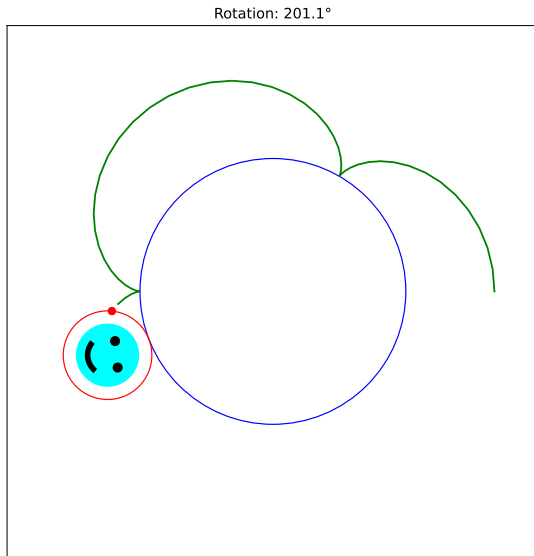
Animation of Coin Rotation Paradox



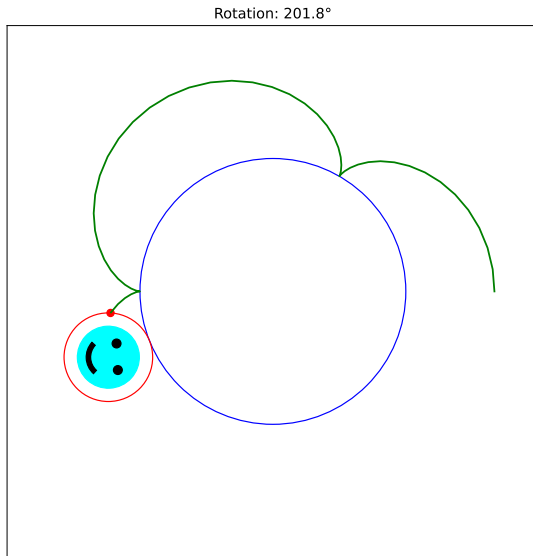
Animation of Coin Rotation Paradox



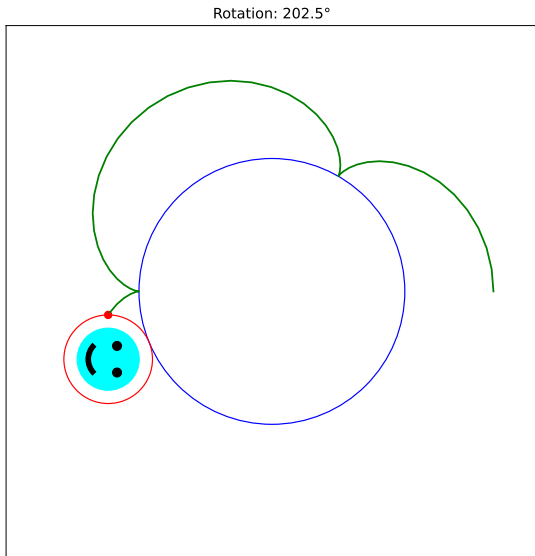
Animation of Coin Rotation Paradox



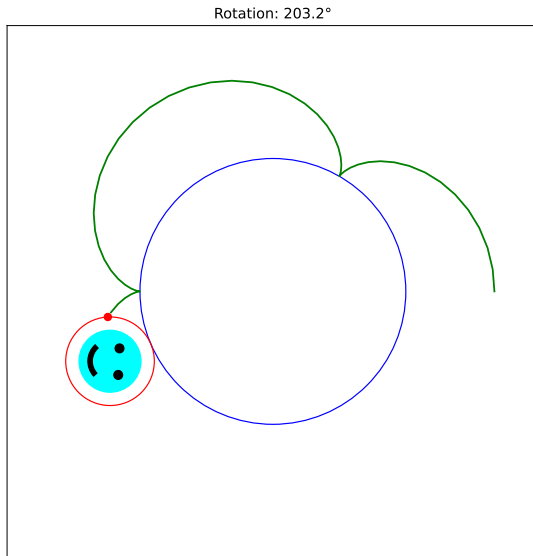
Animation of Coin Rotation Paradox



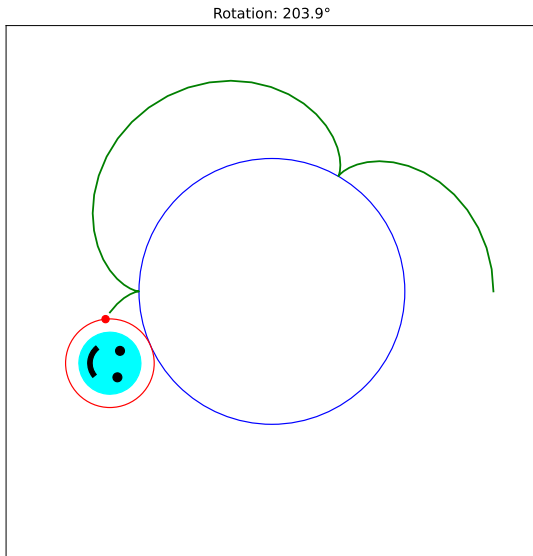
Animation of Coin Rotation Paradox



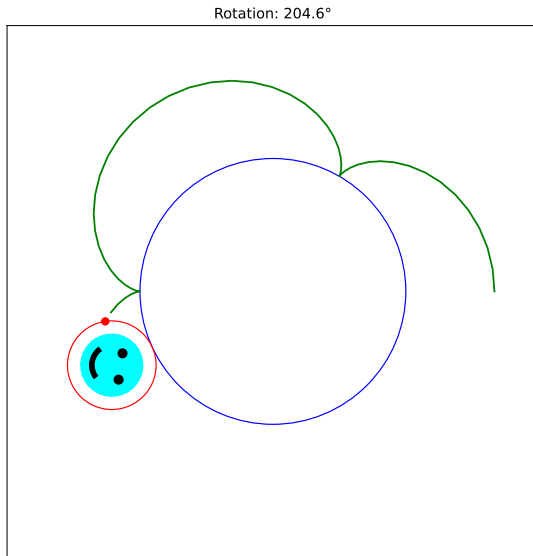
Animation of Coin Rotation Paradox



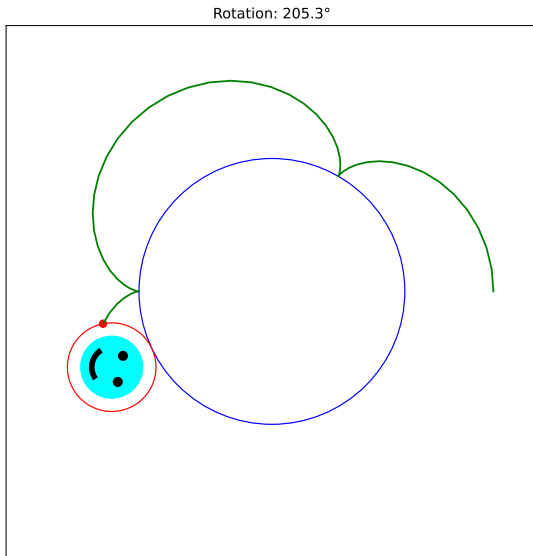
Animation of Coin Rotation Paradox



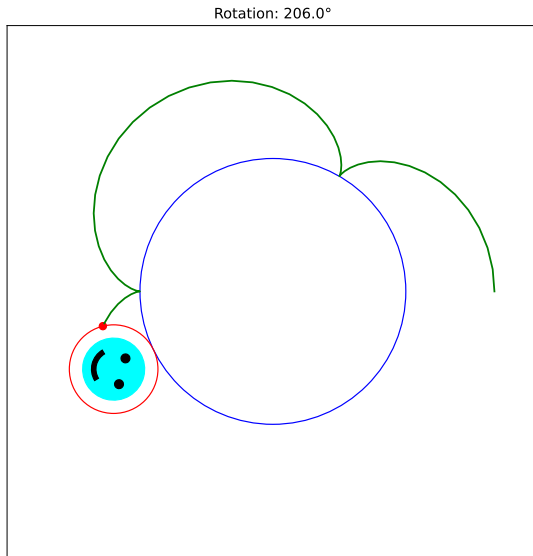
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

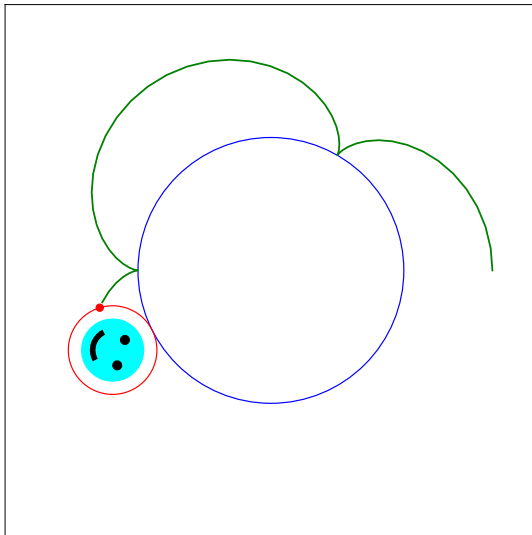


Animation of Coin Rotation Paradox



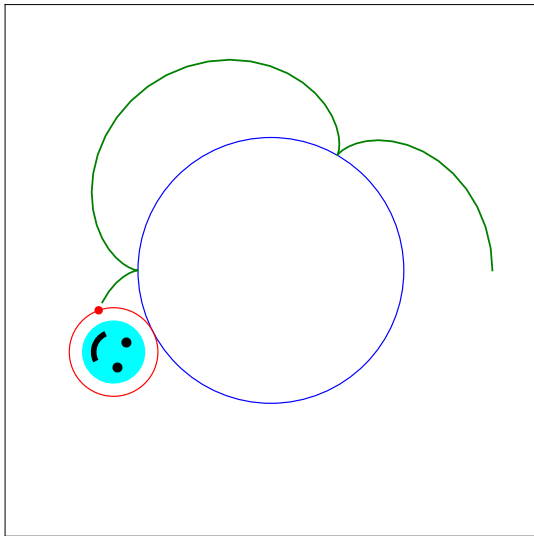
Animation of Coin Rotation Paradox

Rotation: 206.7°

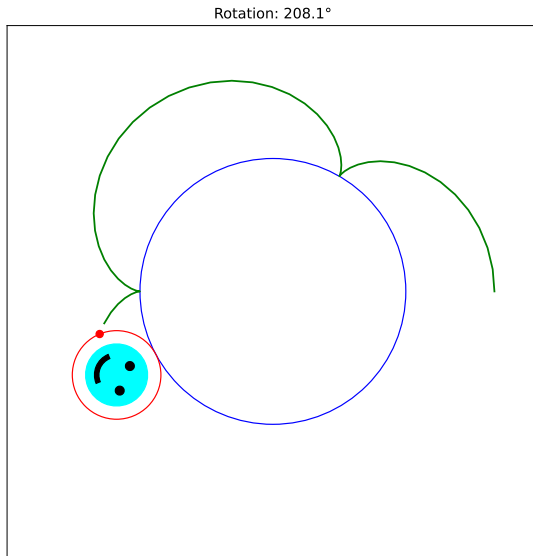


Animation of Coin Rotation Paradox

Rotation: 207.4°

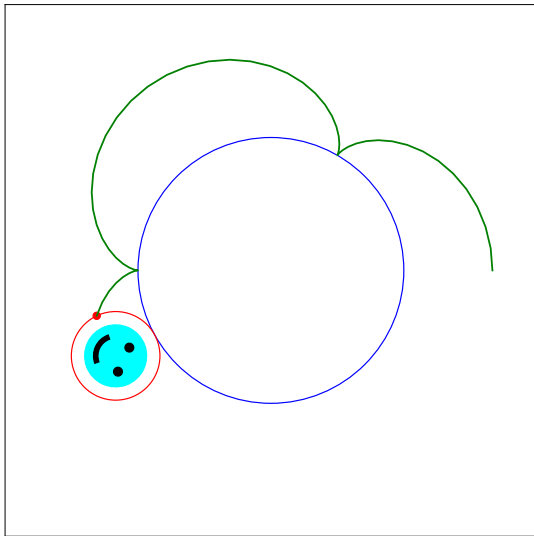


Animation of Coin Rotation Paradox



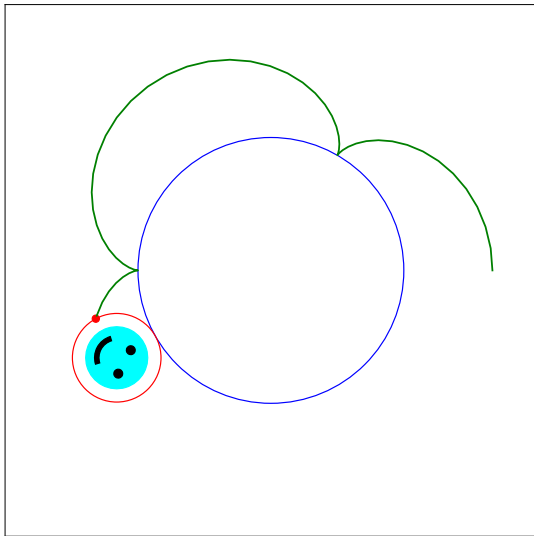
Animation of Coin Rotation Paradox

Rotation: 208.8°

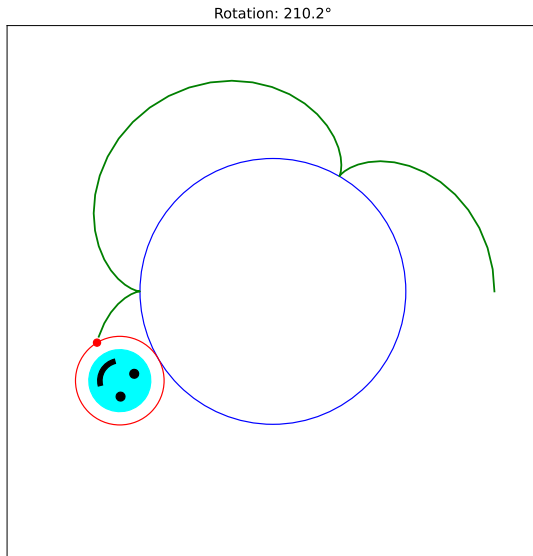


Animation of Coin Rotation Paradox

Rotation: 209.5°

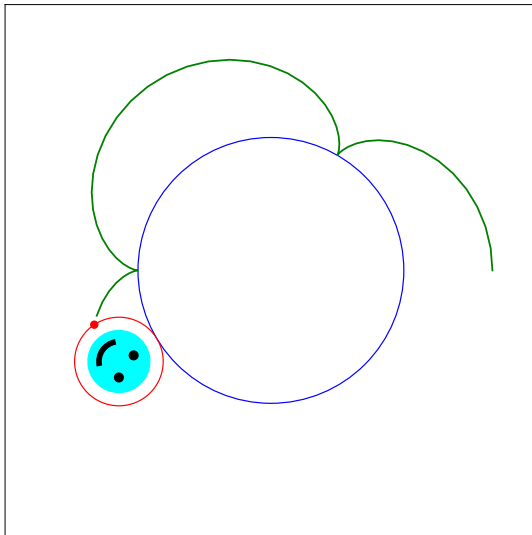


Animation of Coin Rotation Paradox



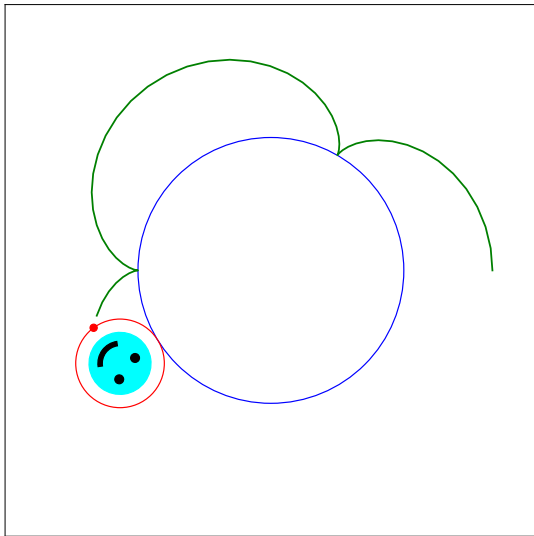
Animation of Coin Rotation Paradox

Rotation: 210.9°



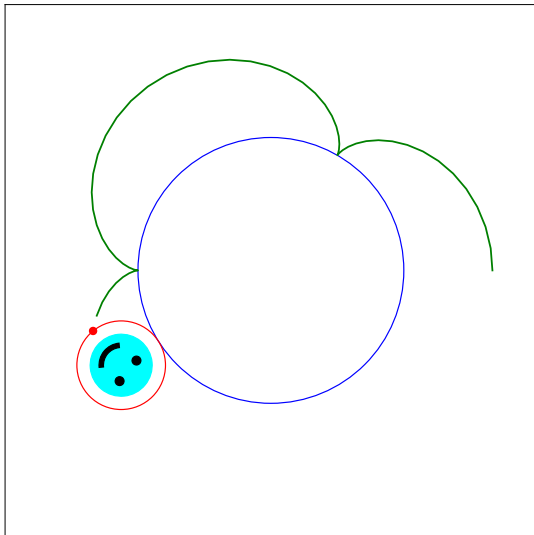
Animation of Coin Rotation Paradox

Rotation: 211.6°

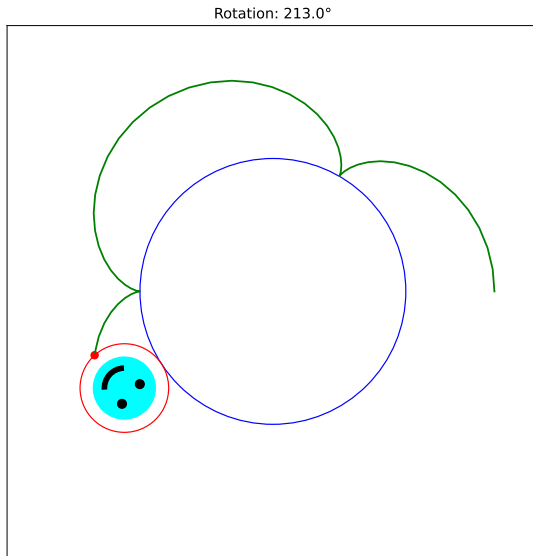


Animation of Coin Rotation Paradox

Rotation: 212.3°

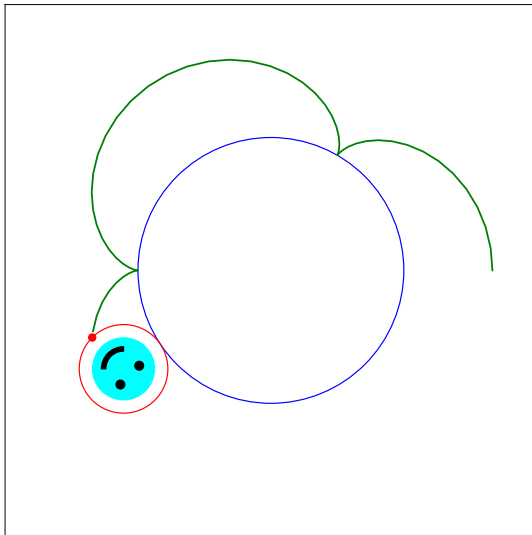


Animation of Coin Rotation Paradox



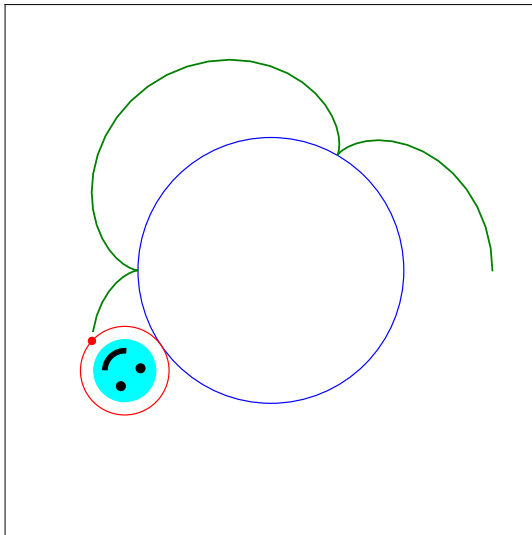
Animation of Coin Rotation Paradox

Rotation: 213.8°

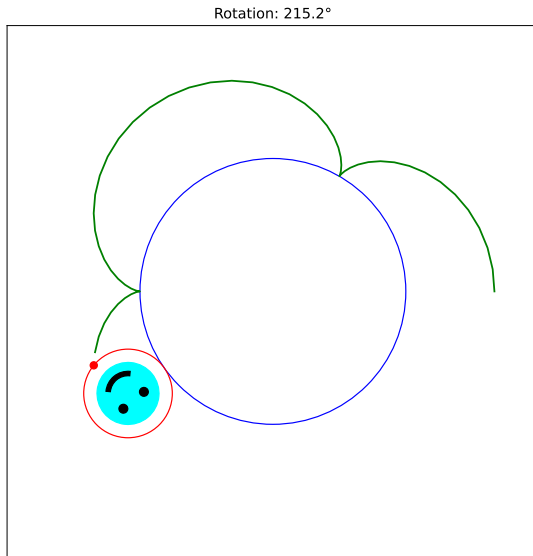


Animation of Coin Rotation Paradox

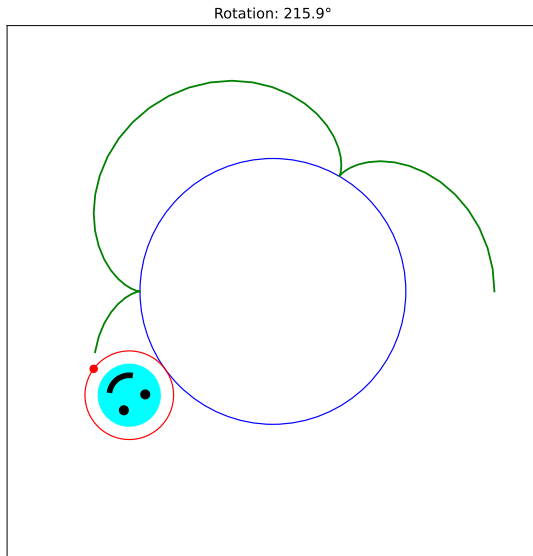
Rotation: 214.5°



Animation of Coin Rotation Paradox

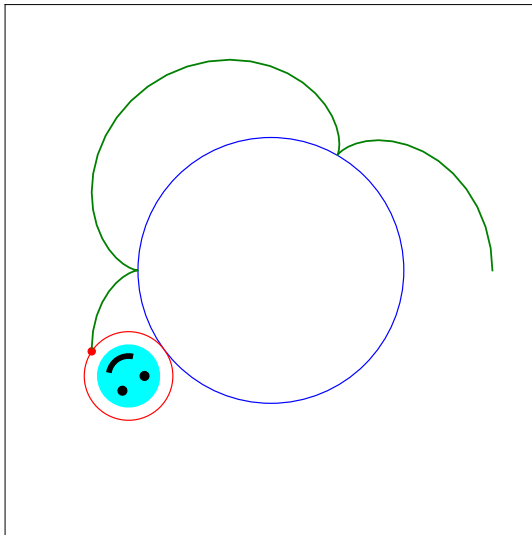


Animation of Coin Rotation Paradox



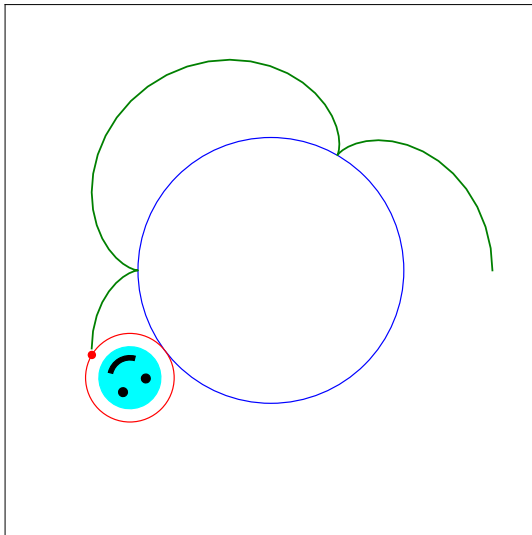
Animation of Coin Rotation Paradox

Rotation: 216.6°

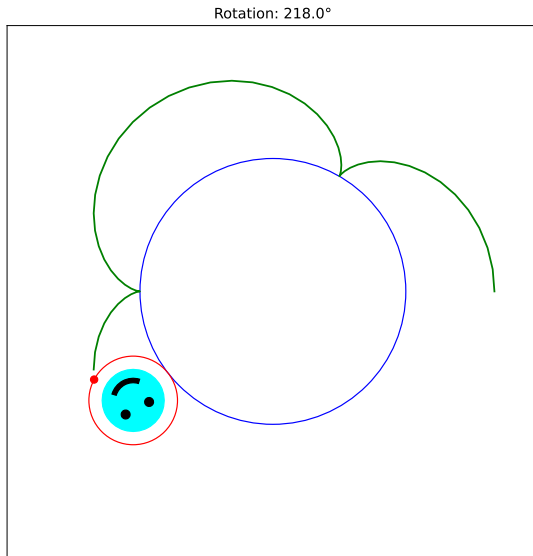


Animation of Coin Rotation Paradox

Rotation: 217.3°

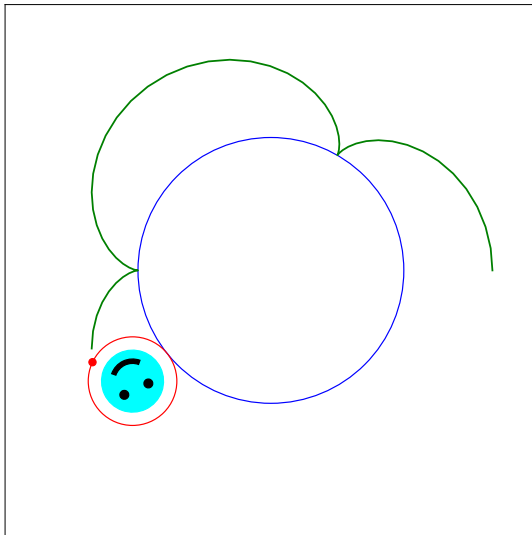


Animation of Coin Rotation Paradox



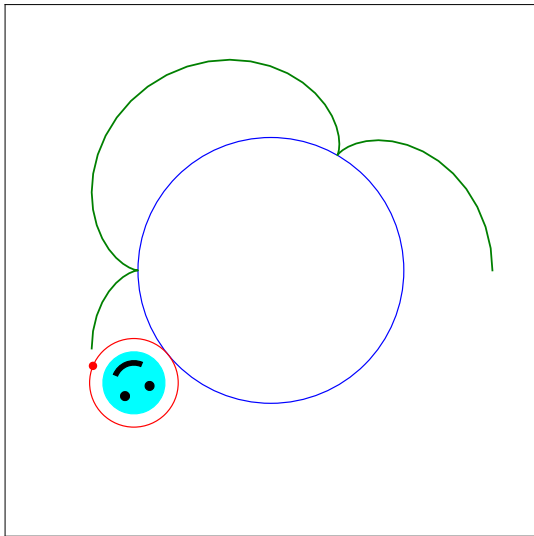
Animation of Coin Rotation Paradox

Rotation: 218.7°

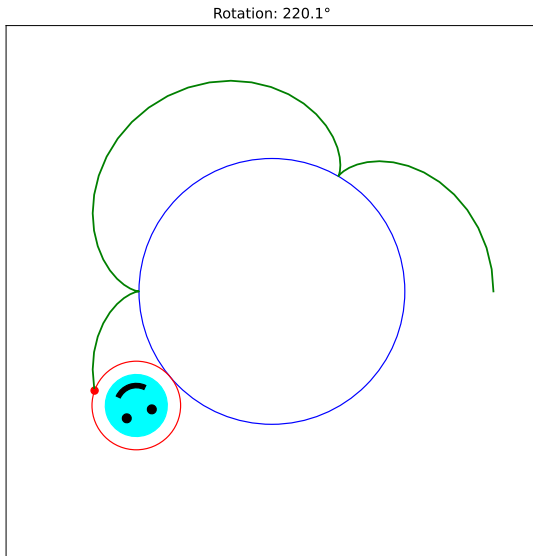


Animation of Coin Rotation Paradox

Rotation: 219.4°

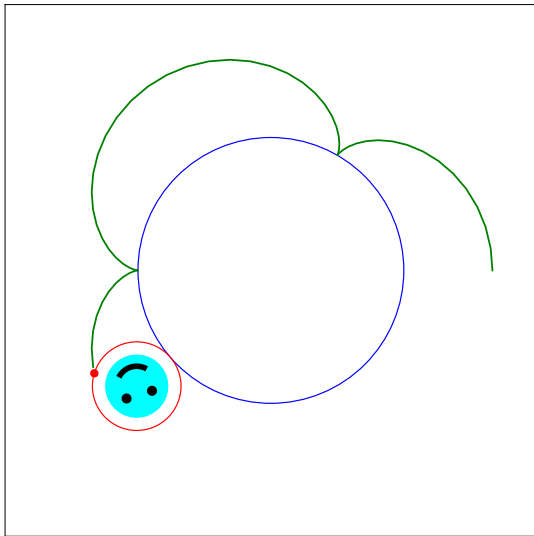


Animation of Coin Rotation Paradox



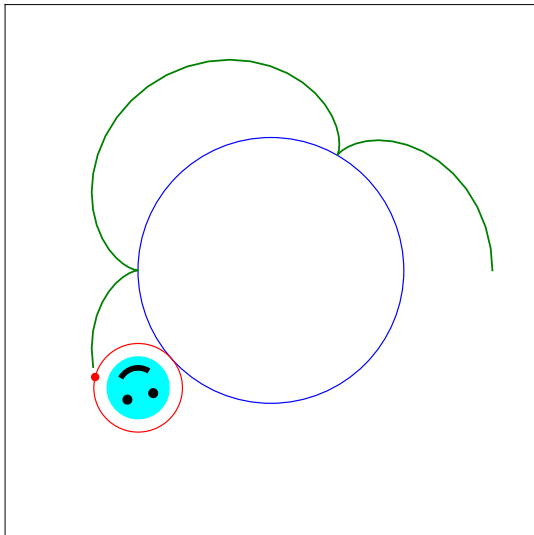
Animation of Coin Rotation Paradox

Rotation: 220.8°



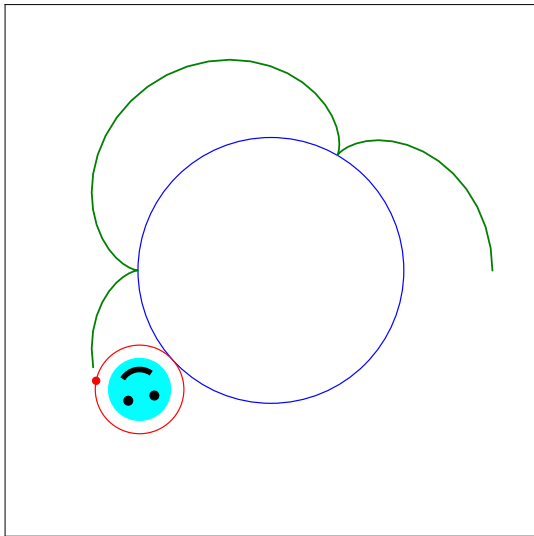
Animation of Coin Rotation Paradox

Rotation: 221.5°



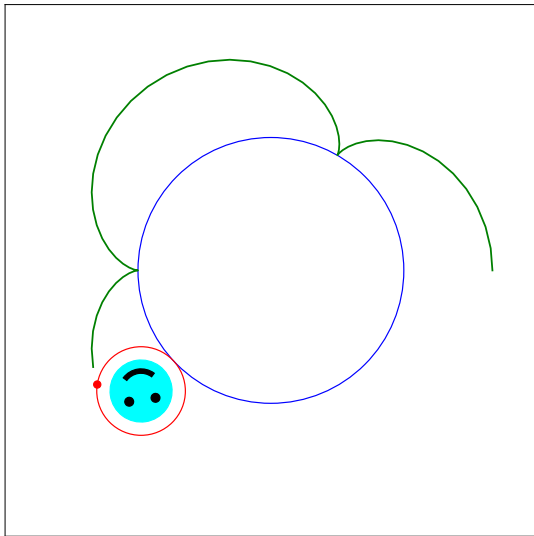
Animation of Coin Rotation Paradox

Rotation: 222.2°



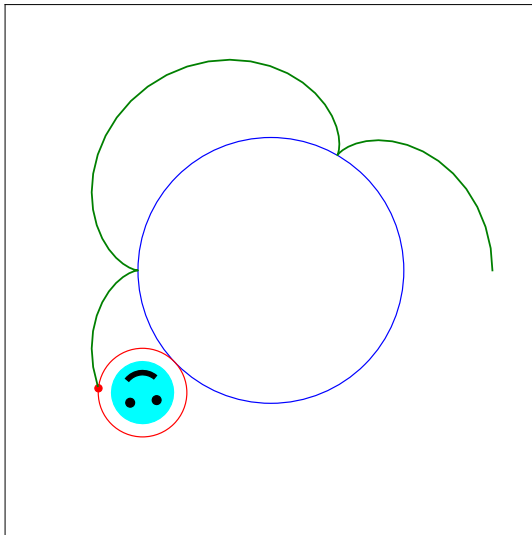
Animation of Coin Rotation Paradox

Rotation: 222.9°



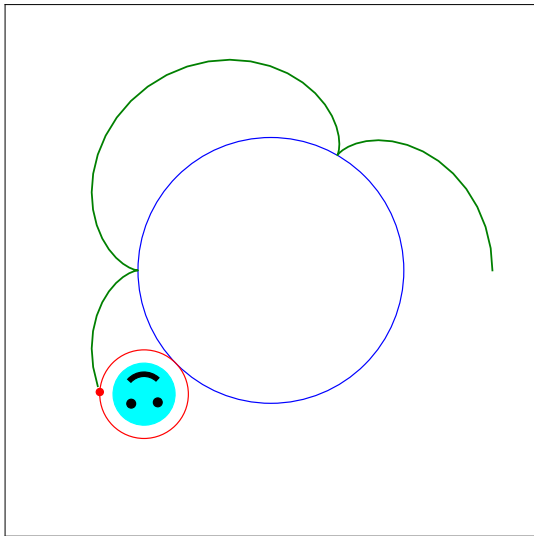
Animation of Coin Rotation Paradox

Rotation: 223.6°



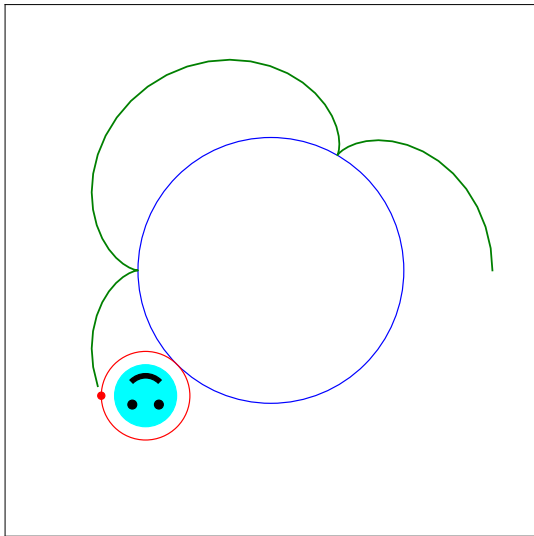
Animation of Coin Rotation Paradox

Rotation: 224.3°



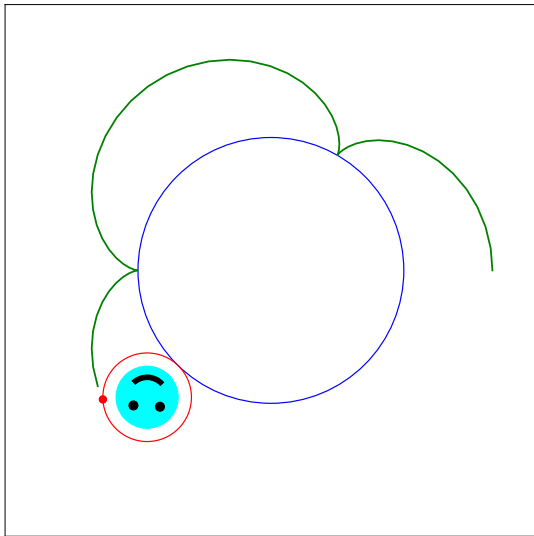
Animation of Coin Rotation Paradox

Rotation: 225.0°



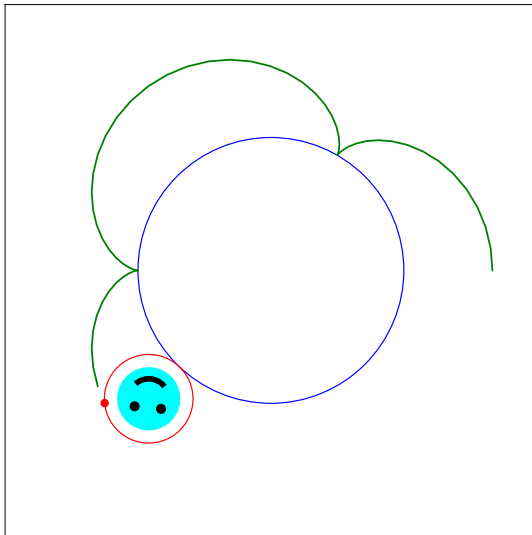
Animation of Coin Rotation Paradox

Rotation: 225.7°

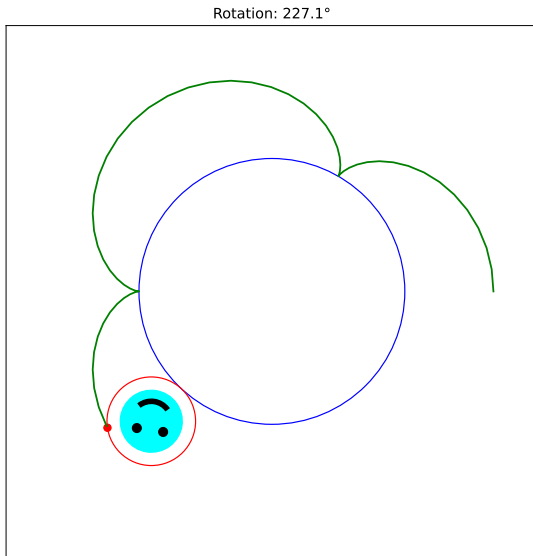


Animation of Coin Rotation Paradox

Rotation: 226.4°

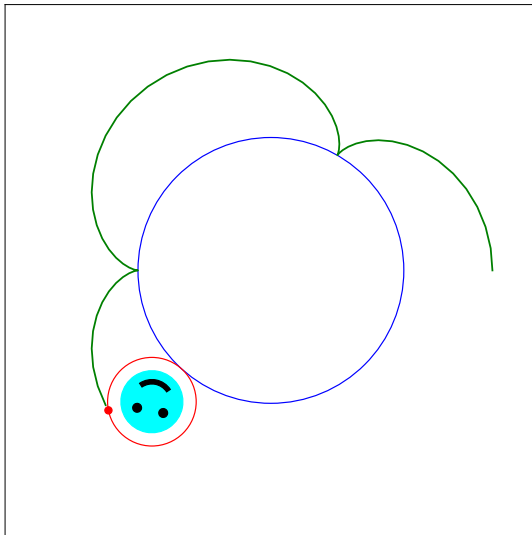


Animation of Coin Rotation Paradox



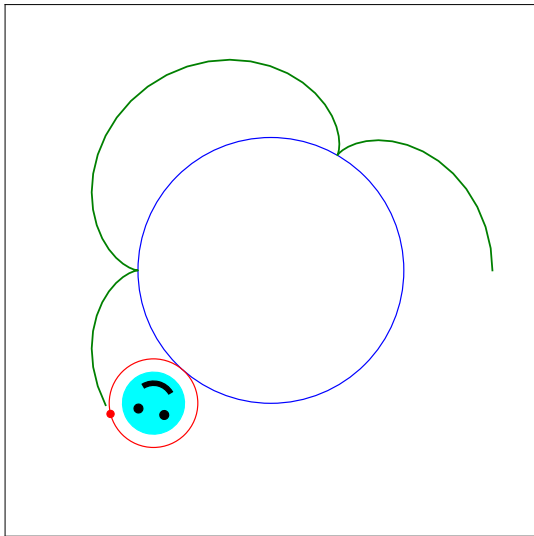
Animation of Coin Rotation Paradox

Rotation: 227.8°



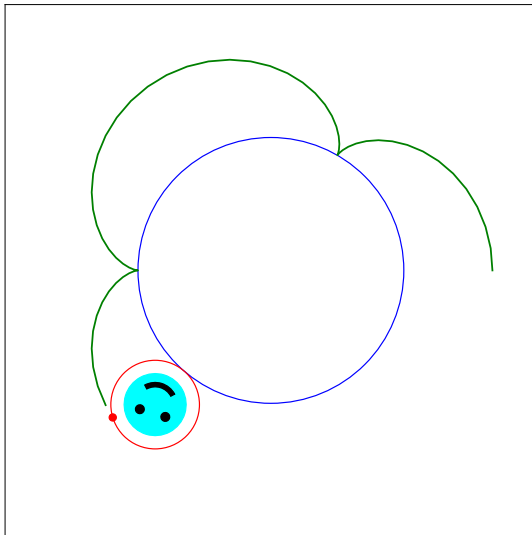
Animation of Coin Rotation Paradox

Rotation: 228.5°



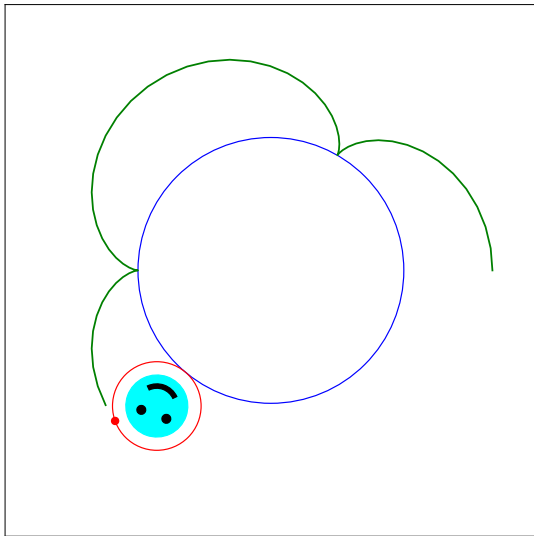
Animation of Coin Rotation Paradox

Rotation: 229.2°



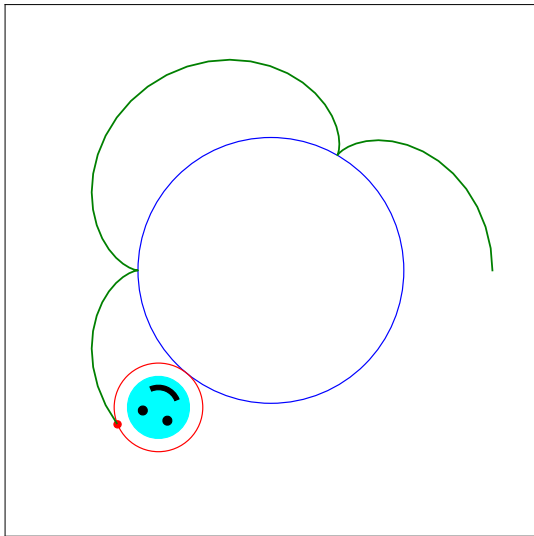
Animation of Coin Rotation Paradox

Rotation: 229.9°

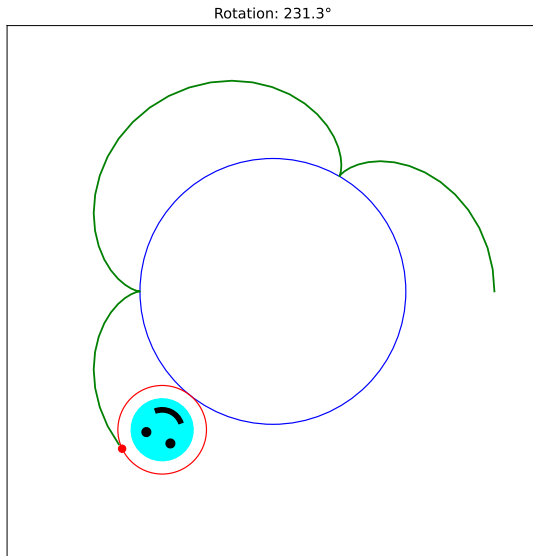


Animation of Coin Rotation Paradox

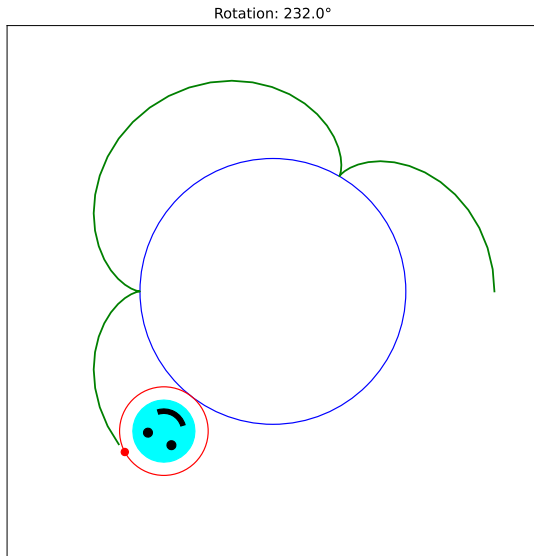
Rotation: 230.6°



Animation of Coin Rotation Paradox

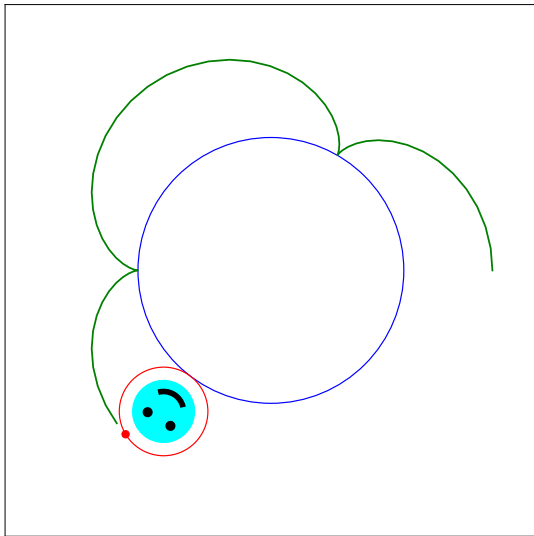


Animation of Coin Rotation Paradox



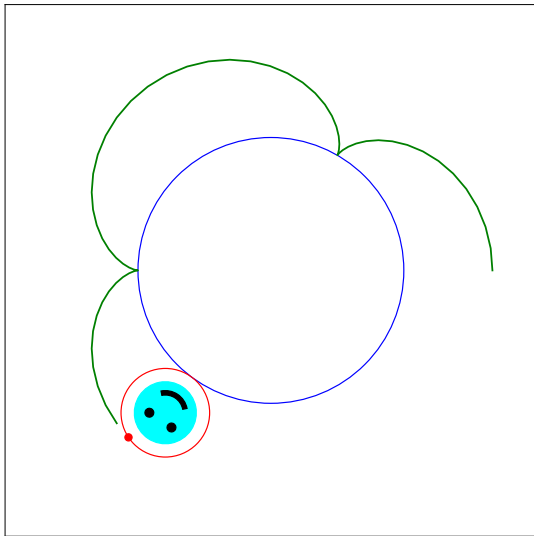
Animation of Coin Rotation Paradox

Rotation: 232.7°

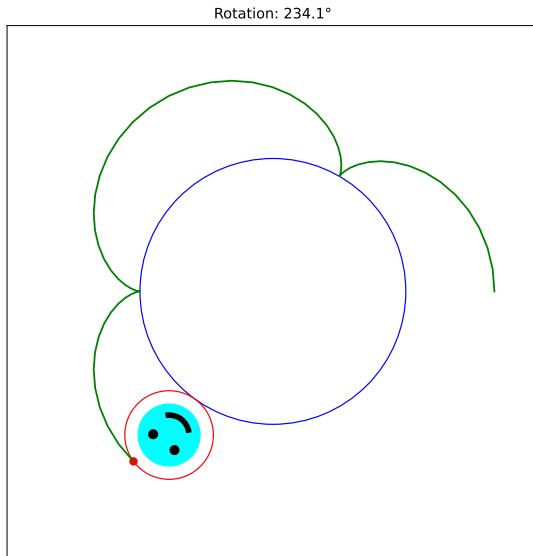


Animation of Coin Rotation Paradox

Rotation: 233.4°

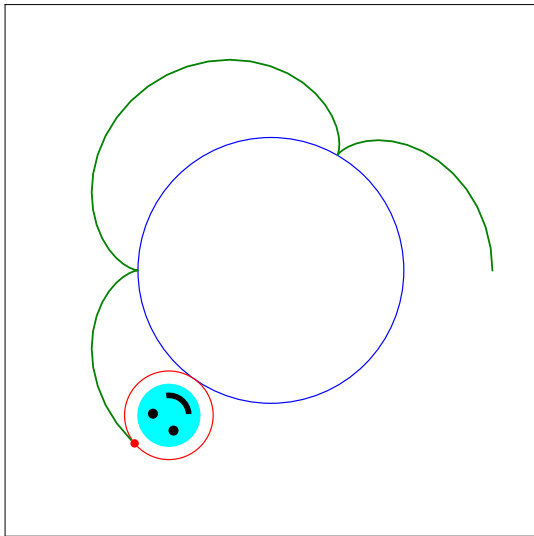


Animation of Coin Rotation Paradox



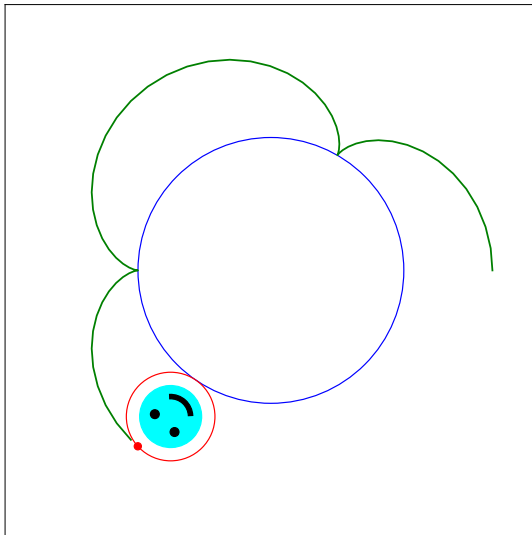
Animation of Coin Rotation Paradox

Rotation: 234.8°



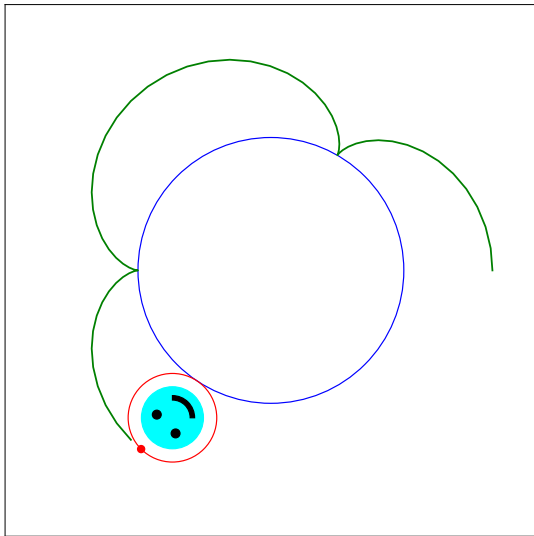
Animation of Coin Rotation Paradox

Rotation: 235.5°



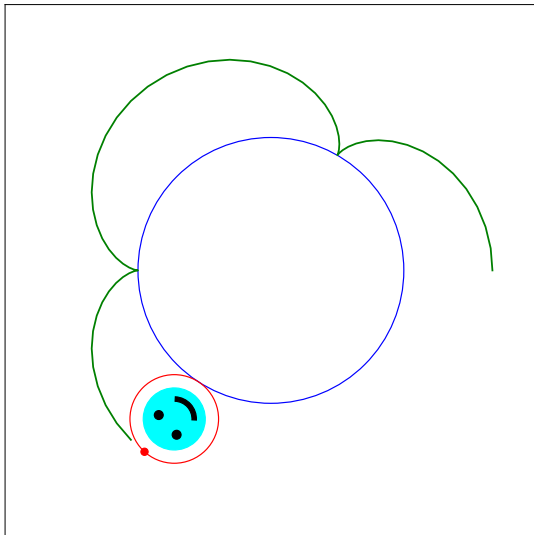
Animation of Coin Rotation Paradox

Rotation: 236.2°



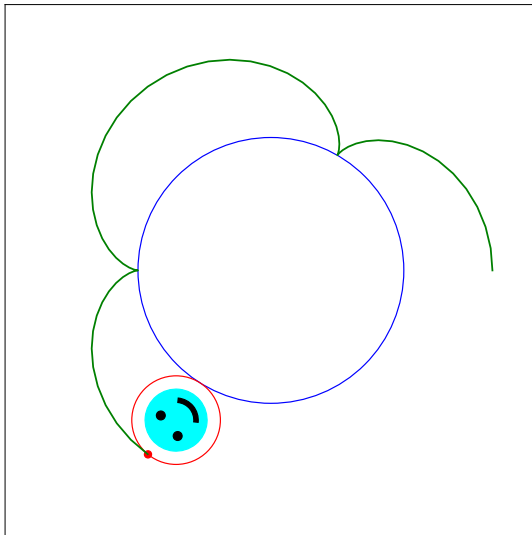
Animation of Coin Rotation Paradox

Rotation: 237.0°



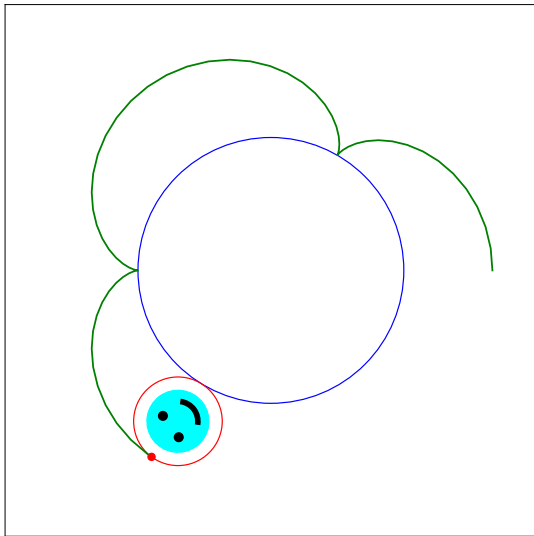
Animation of Coin Rotation Paradox

Rotation: 237.7°

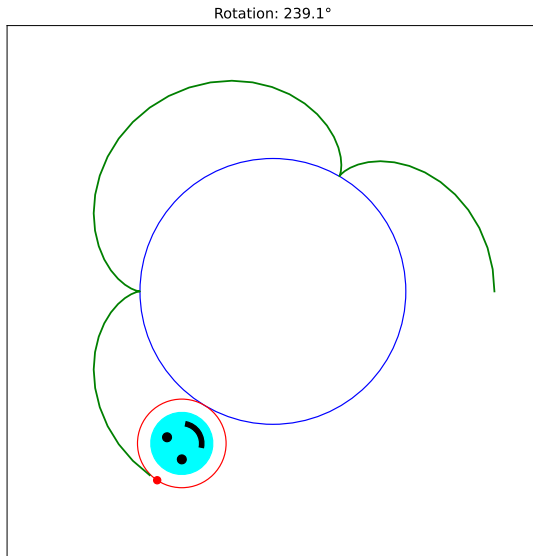


Animation of Coin Rotation Paradox

Rotation: 238.4°

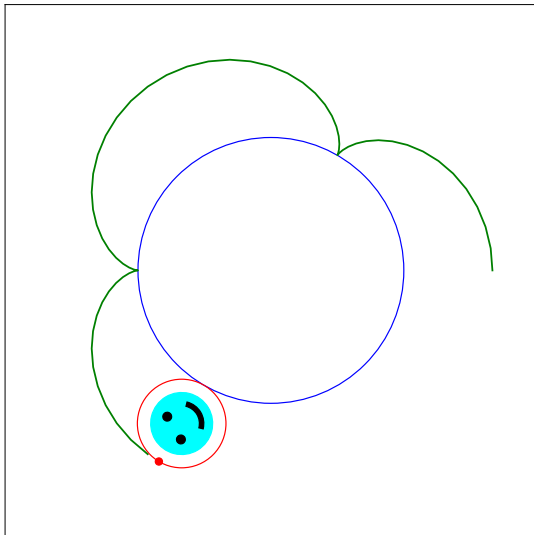


Animation of Coin Rotation Paradox



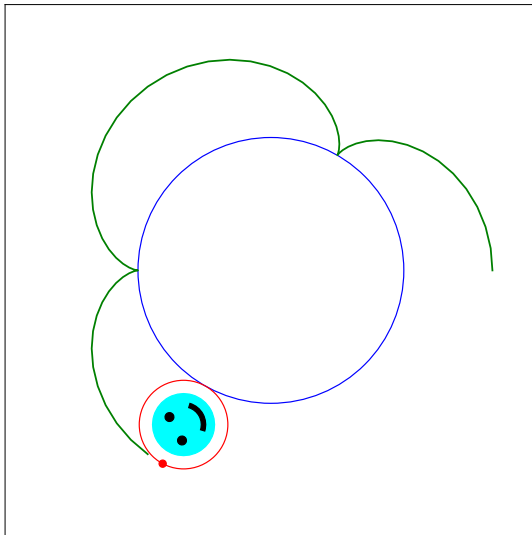
Animation of Coin Rotation Paradox

Rotation: 239.8°

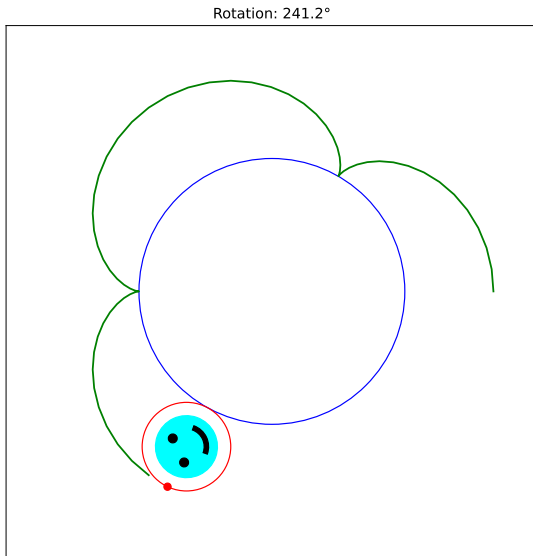


Animation of Coin Rotation Paradox

Rotation: 240.5°

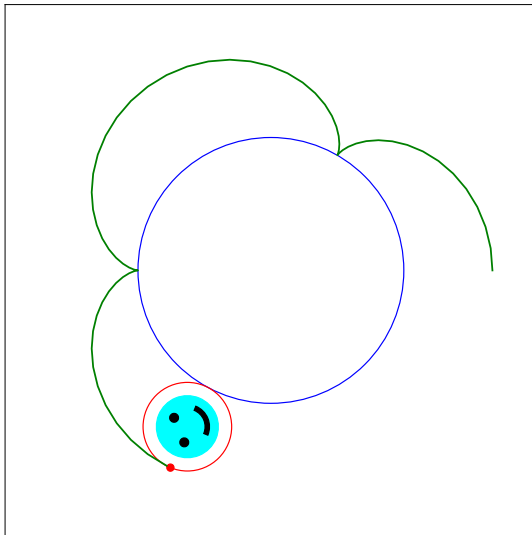


Animation of Coin Rotation Paradox



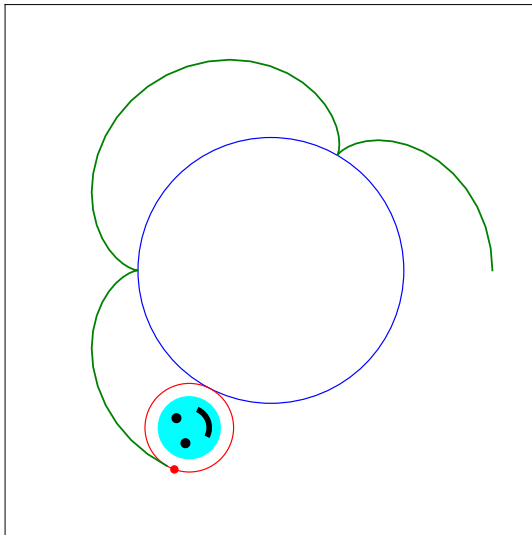
Animation of Coin Rotation Paradox

Rotation: 241.9°

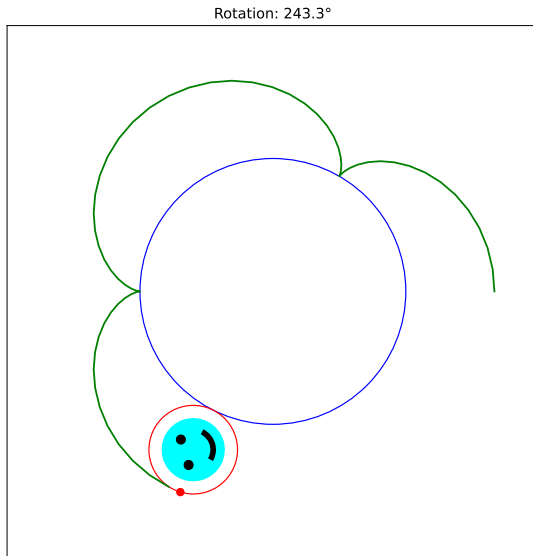


Animation of Coin Rotation Paradox

Rotation: 242.6°

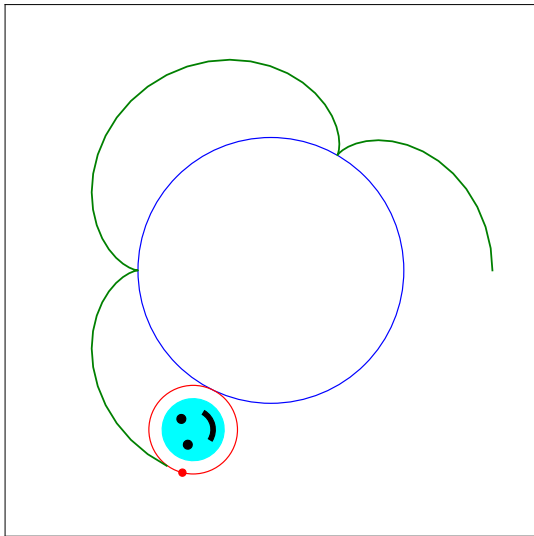


Animation of Coin Rotation Paradox



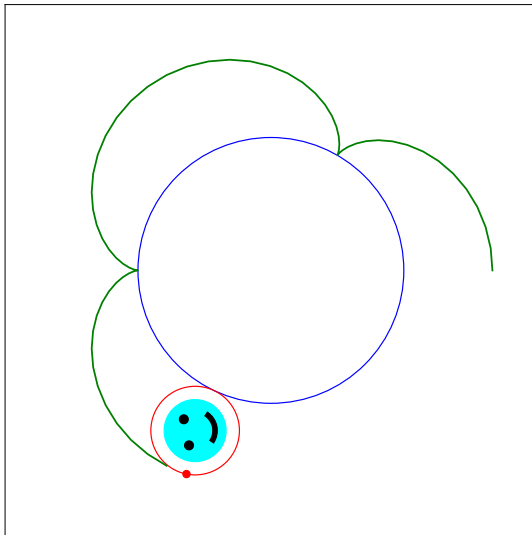
Animation of Coin Rotation Paradox

Rotation: 244.0°



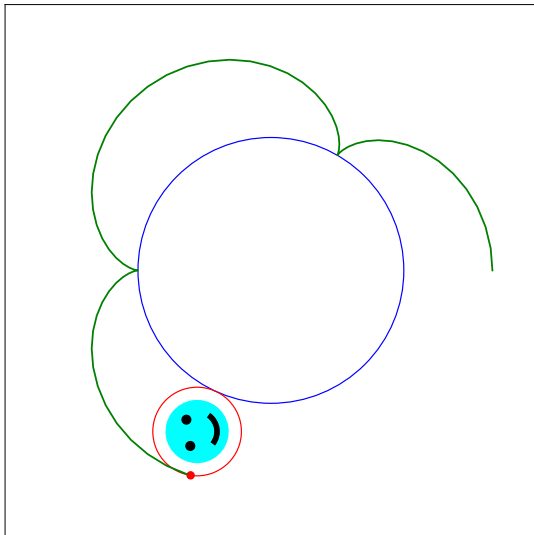
Animation of Coin Rotation Paradox

Rotation: 244.7°



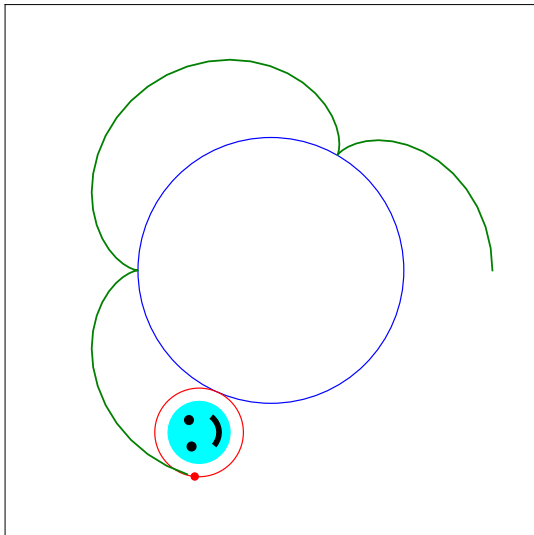
Animation of Coin Rotation Paradox

Rotation: 245.4°



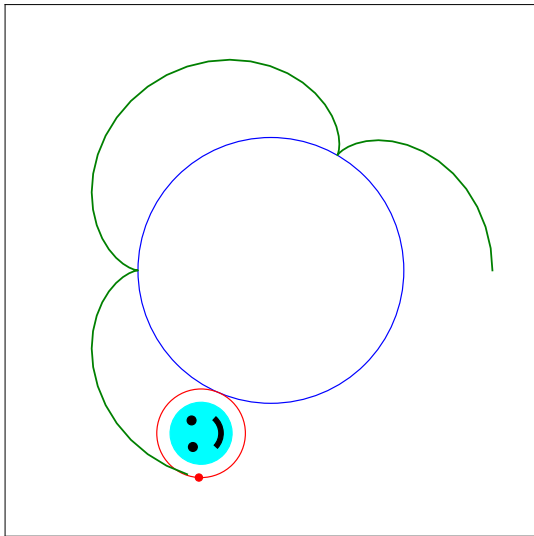
Animation of Coin Rotation Paradox

Rotation: 246.1°

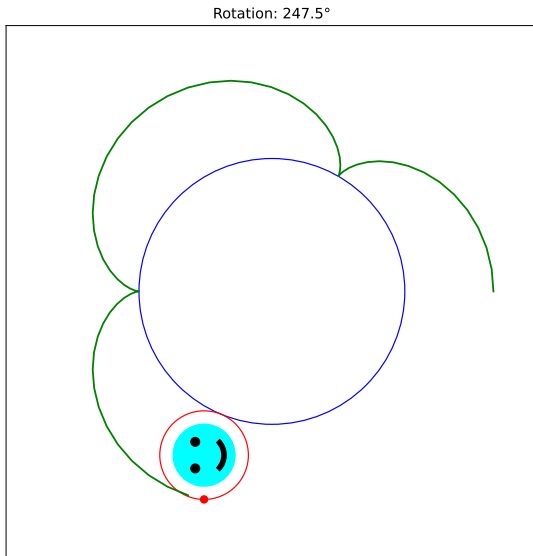


Animation of Coin Rotation Paradox

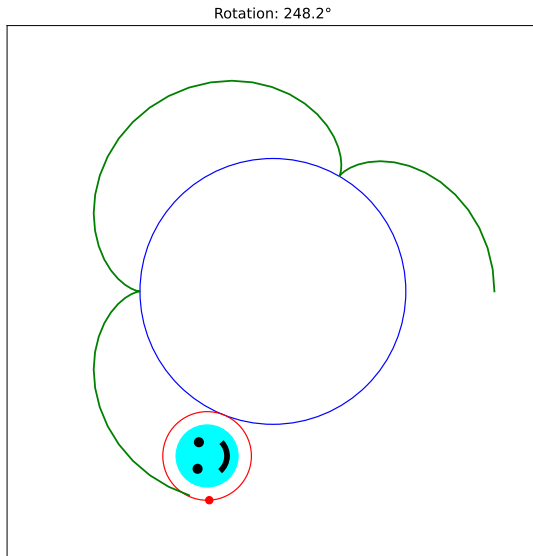
Rotation: 246.8°



Animation of Coin Rotation Paradox

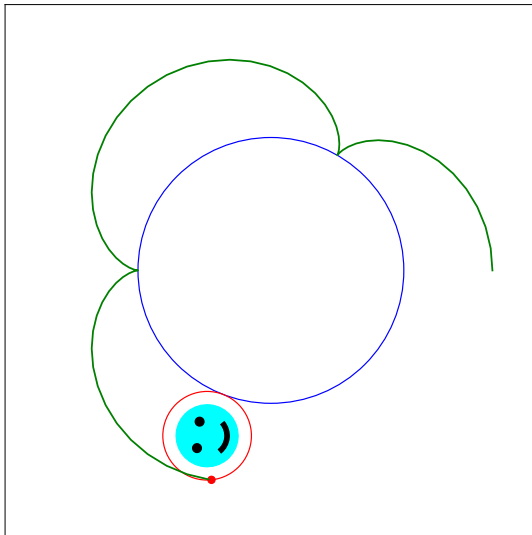


Animation of Coin Rotation Paradox



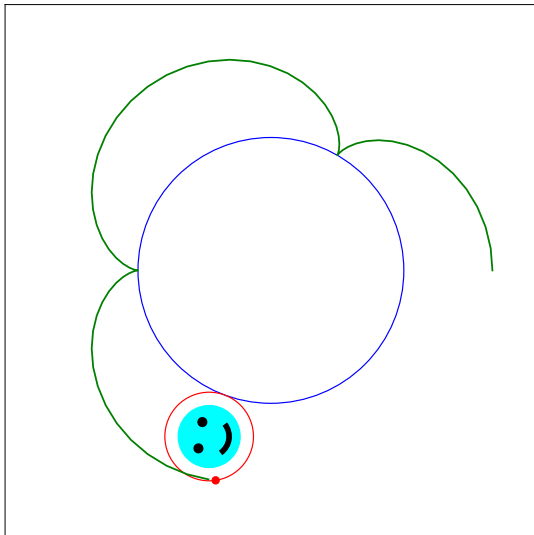
Animation of Coin Rotation Paradox

Rotation: 248.9°

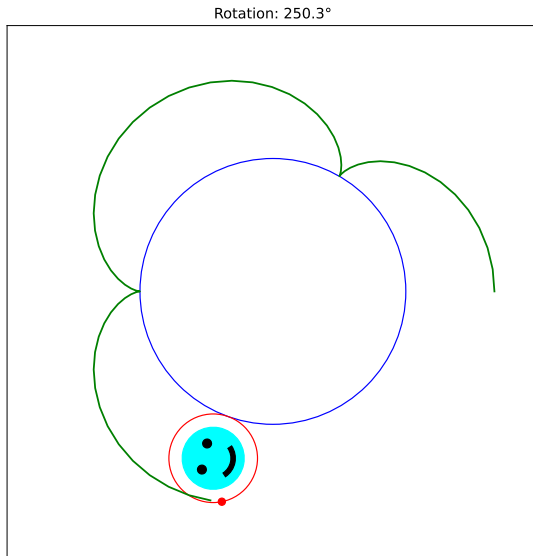


Animation of Coin Rotation Paradox

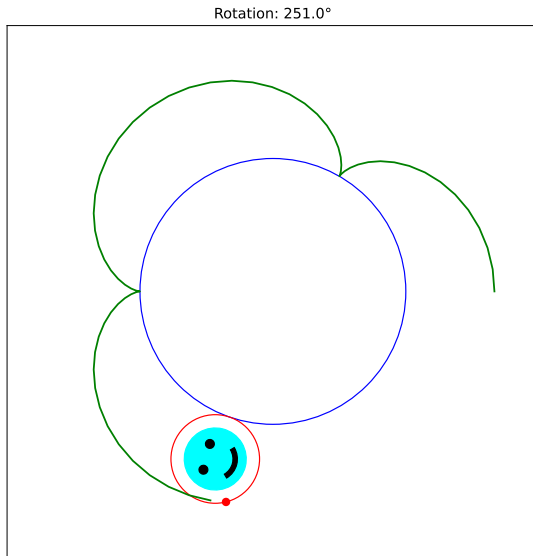
Rotation: 249.6°



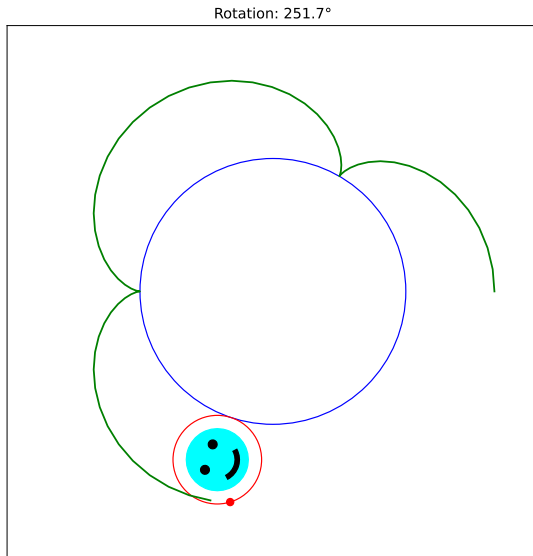
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

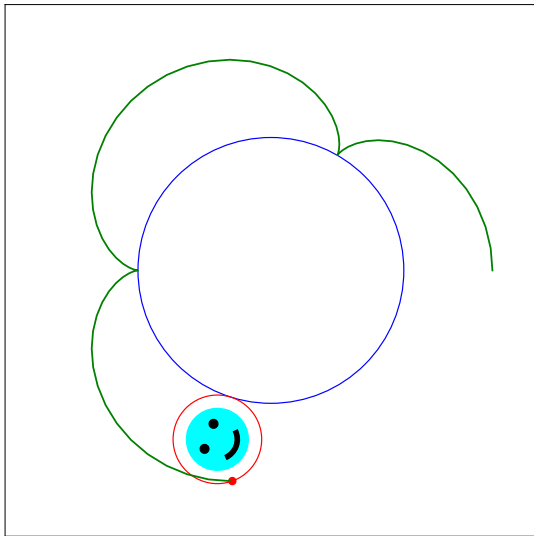


Animation of Coin Rotation Paradox



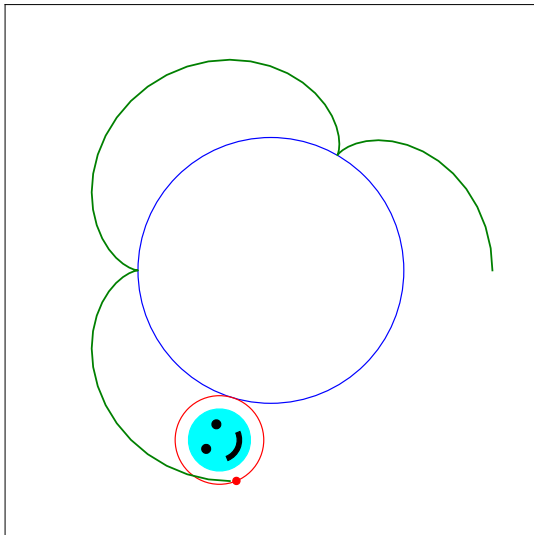
Animation of Coin Rotation Paradox

Rotation: 252.4°



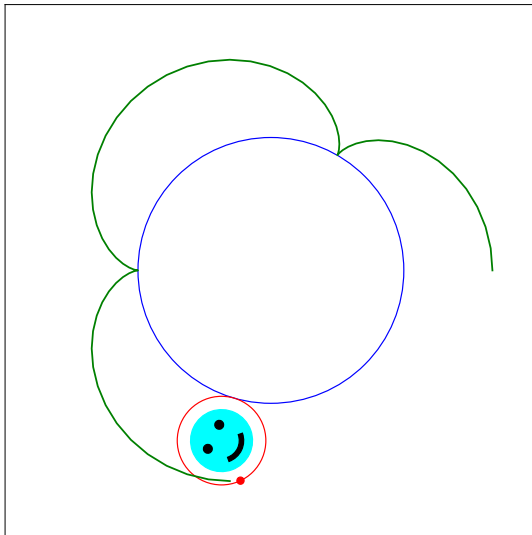
Animation of Coin Rotation Paradox

Rotation: 253.1°



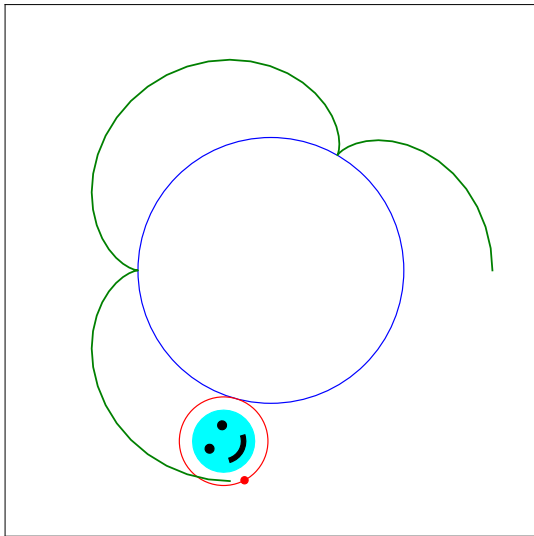
Animation of Coin Rotation Paradox

Rotation: 253.8°



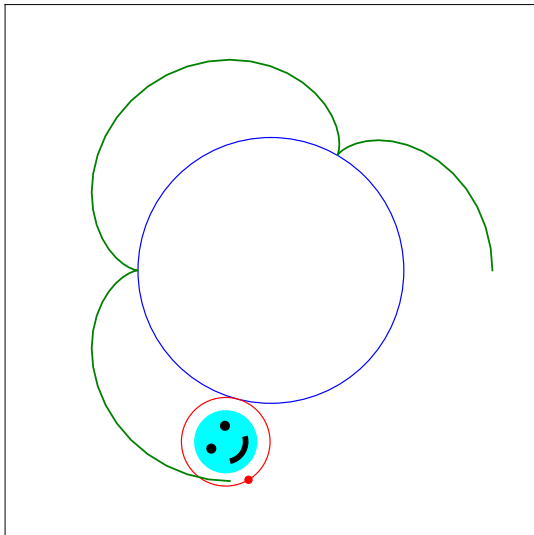
Animation of Coin Rotation Paradox

Rotation: 254.5°



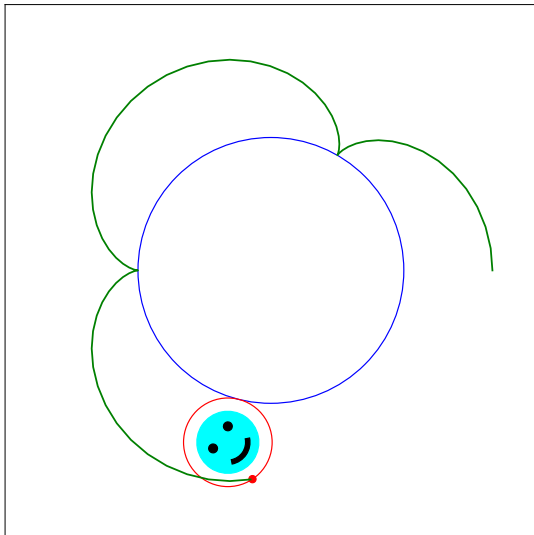
Animation of Coin Rotation Paradox

Rotation: 255.2°



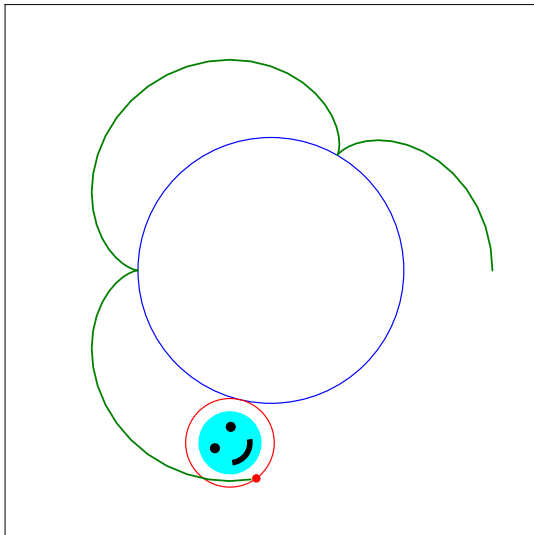
Animation of Coin Rotation Paradox

Rotation: 255.9°



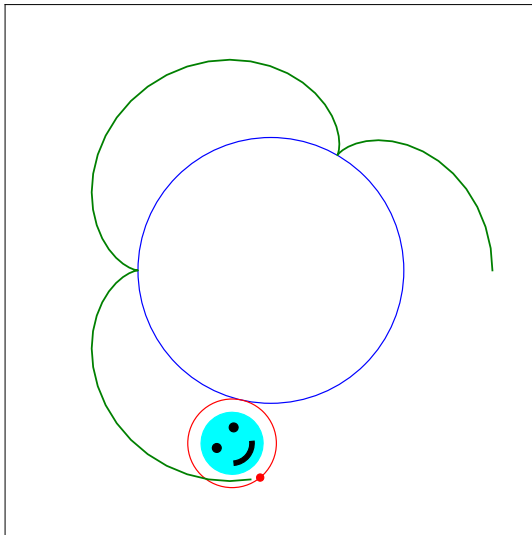
Animation of Coin Rotation Paradox

Rotation: 256.6°



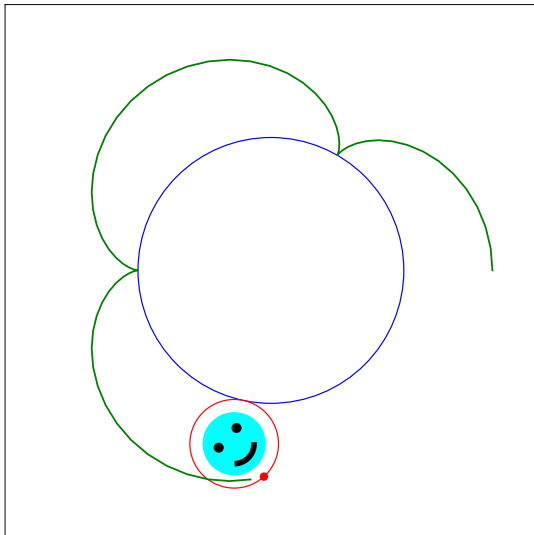
Animation of Coin Rotation Paradox

Rotation: 257.3°



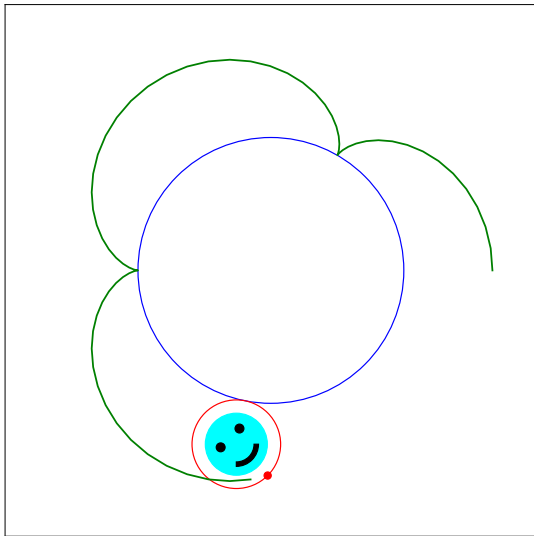
Animation of Coin Rotation Paradox

Rotation: 258.0°



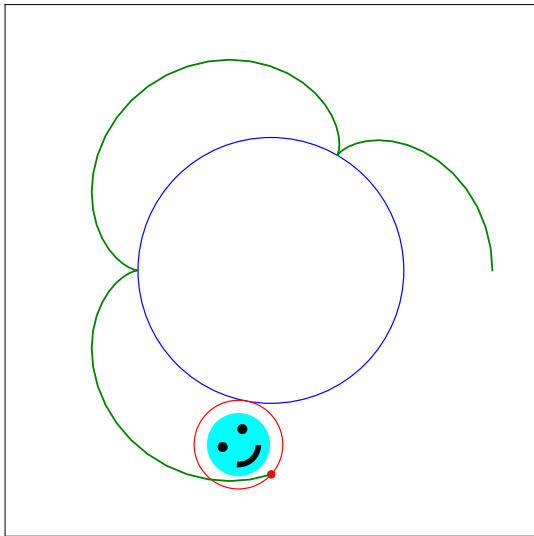
Animation of Coin Rotation Paradox

Rotation: 258.8°

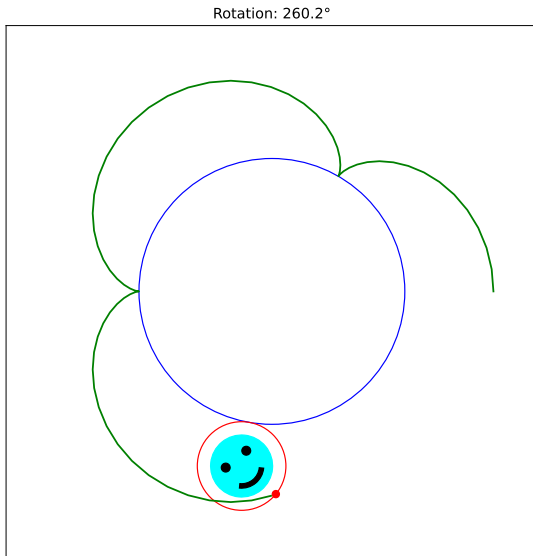


Animation of Coin Rotation Paradox

Rotation: 259.5°

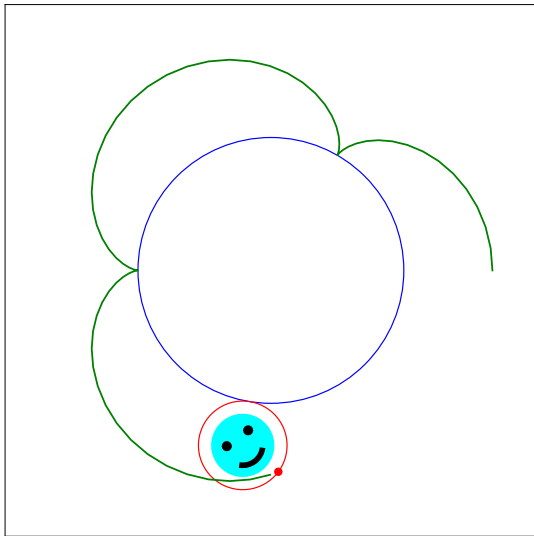


Animation of Coin Rotation Paradox



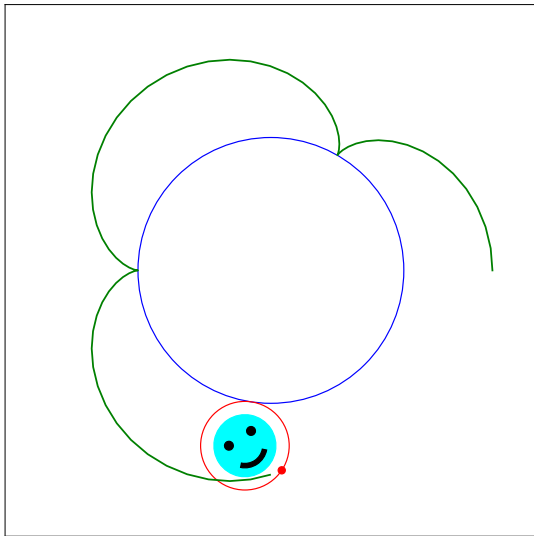
Animation of Coin Rotation Paradox

Rotation: 260.9°



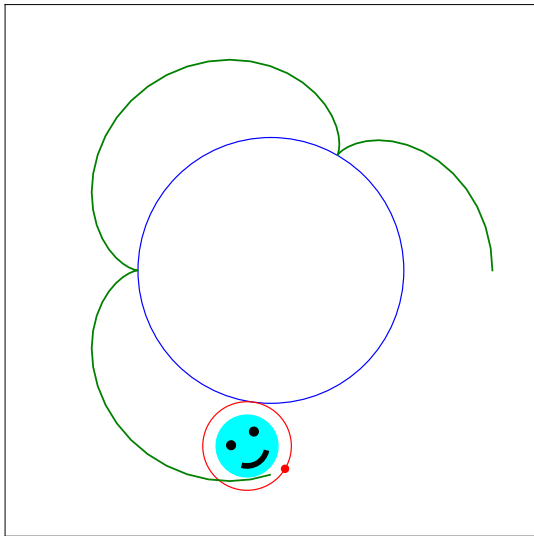
Animation of Coin Rotation Paradox

Rotation: 261.6°



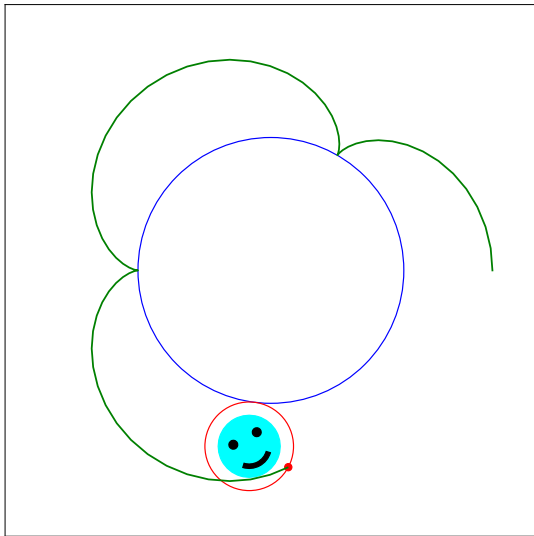
Animation of Coin Rotation Paradox

Rotation: 262.3°



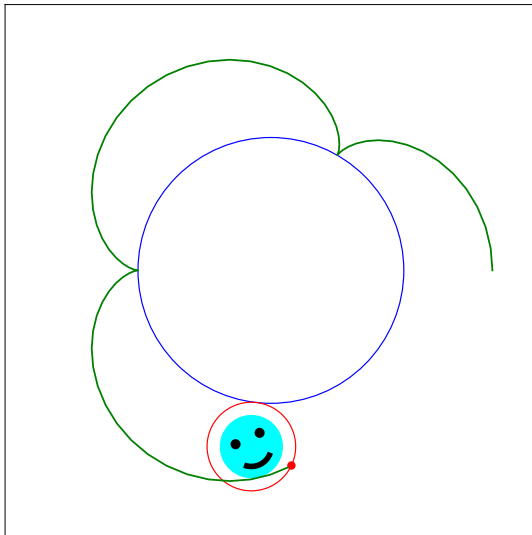
Animation of Coin Rotation Paradox

Rotation: 263.0°



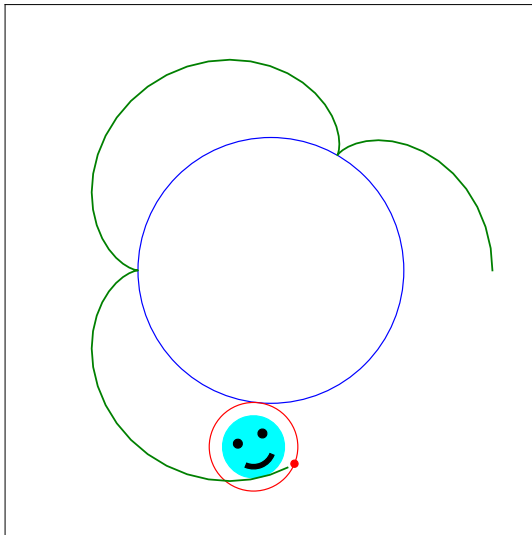
Animation of Coin Rotation Paradox

Rotation: 263.7°



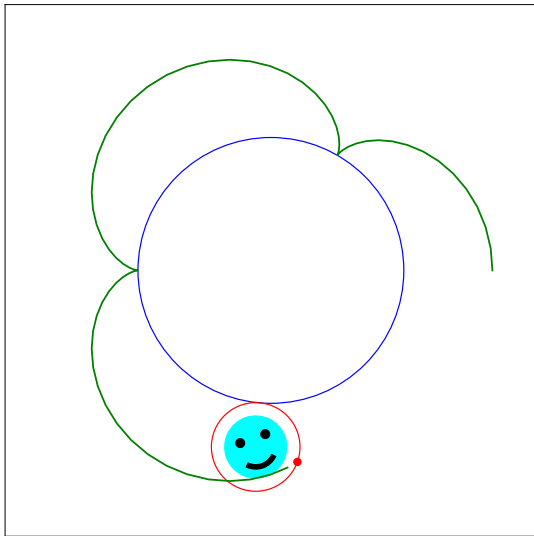
Animation of Coin Rotation Paradox

Rotation: 264.4°



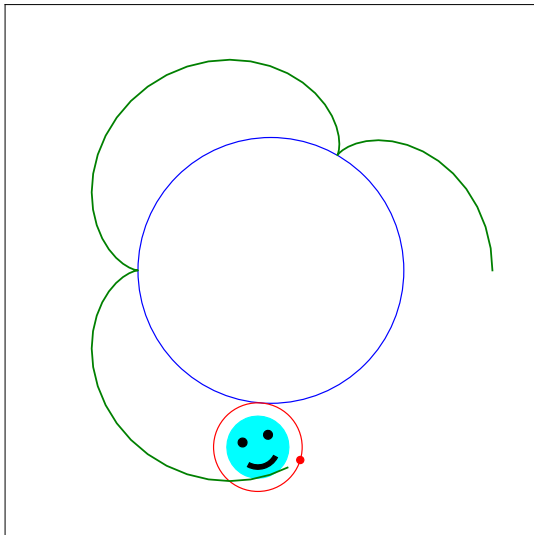
Animation of Coin Rotation Paradox

Rotation: 265.1°



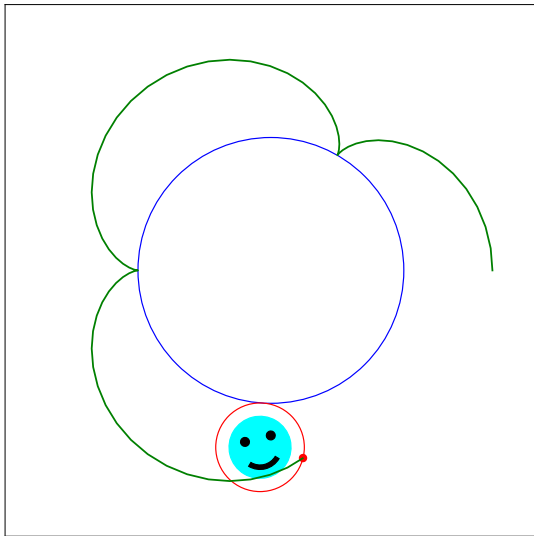
Animation of Coin Rotation Paradox

Rotation: 265.8°



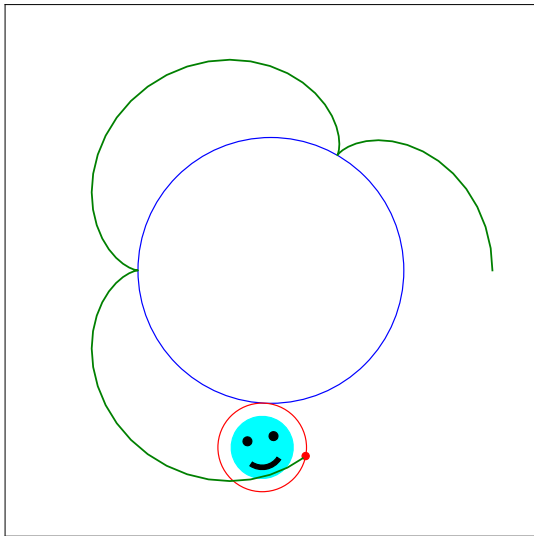
Animation of Coin Rotation Paradox

Rotation: 266.5°



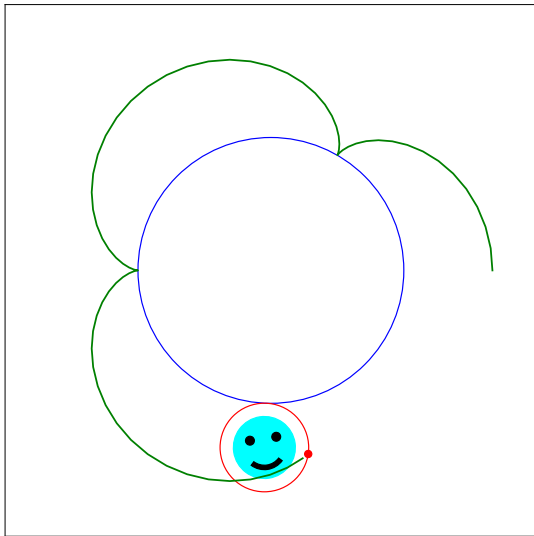
Animation of Coin Rotation Paradox

Rotation: 267.2°



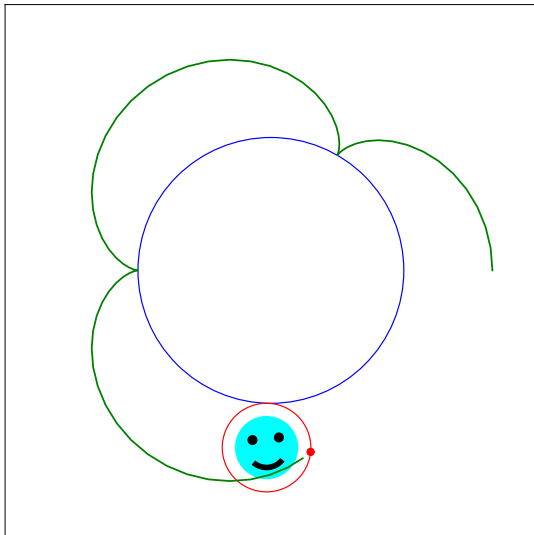
Animation of Coin Rotation Paradox

Rotation: 267.9°



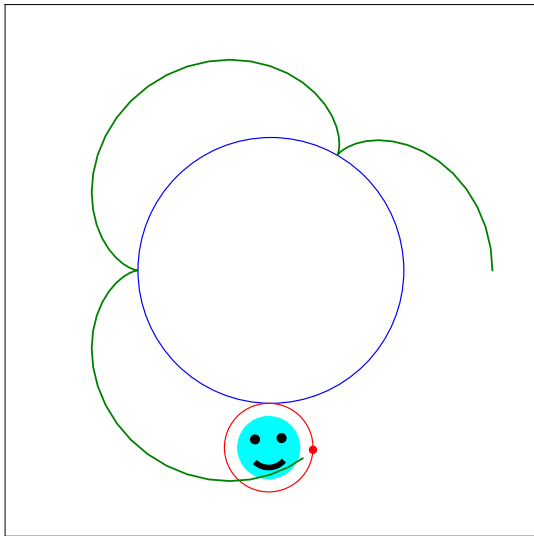
Animation of Coin Rotation Paradox

Rotation: 268.6°



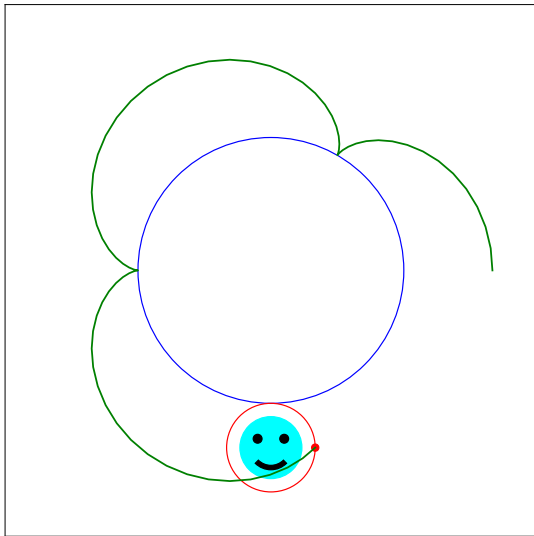
Animation of Coin Rotation Paradox

Rotation: 269.3°



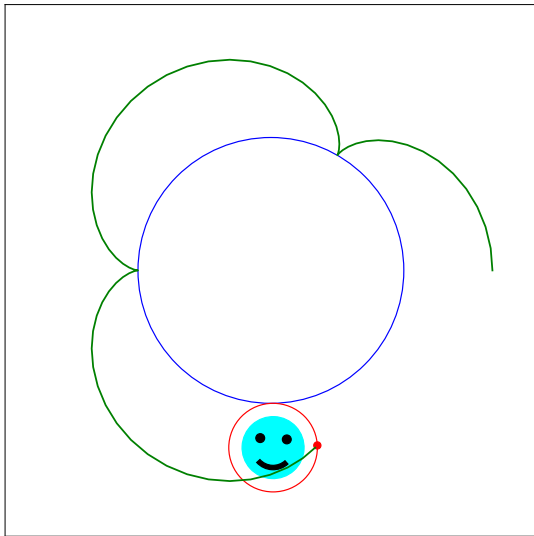
Animation of Coin Rotation Paradox

Rotation: 270.0°



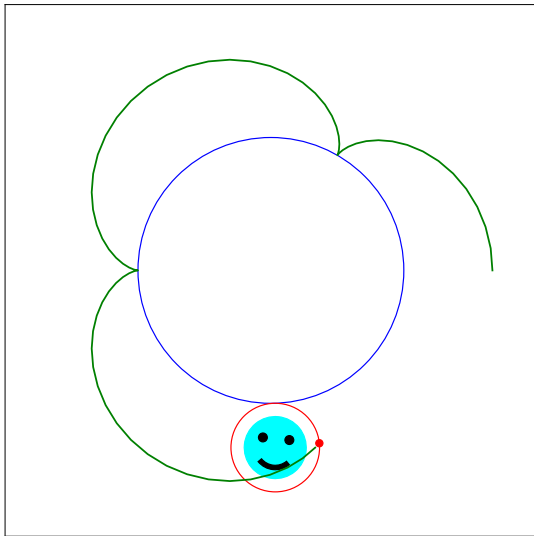
Animation of Coin Rotation Paradox

Rotation: 270.7°



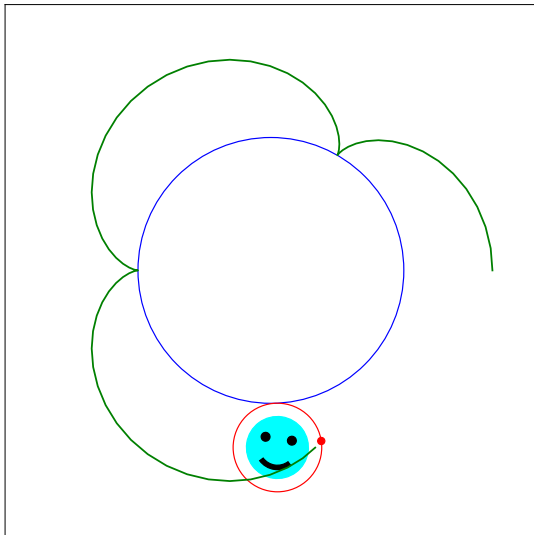
Animation of Coin Rotation Paradox

Rotation: 271.4°



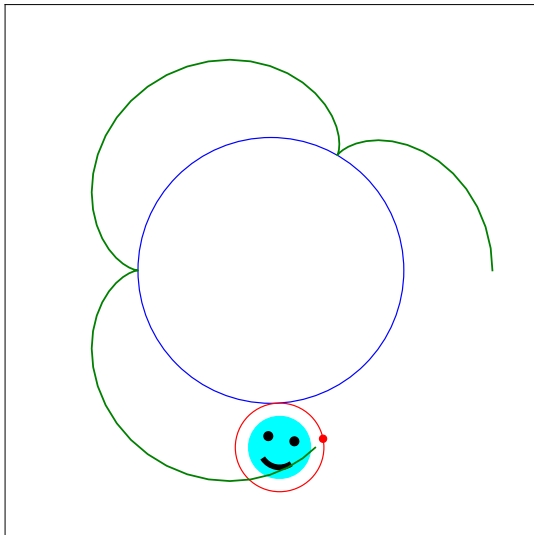
Animation of Coin Rotation Paradox

Rotation: 272.1°



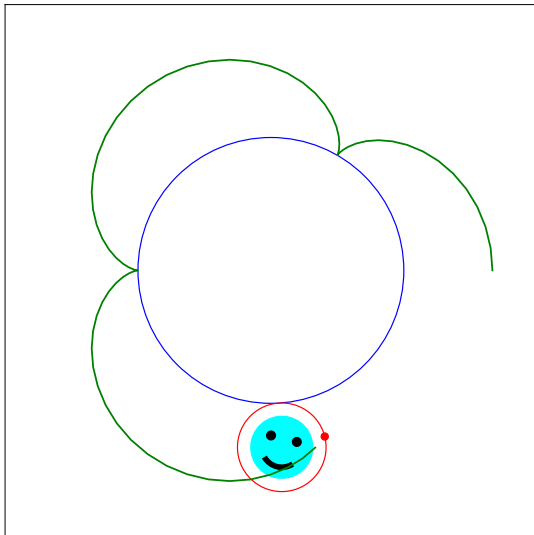
Animation of Coin Rotation Paradox

Rotation: 272.8°



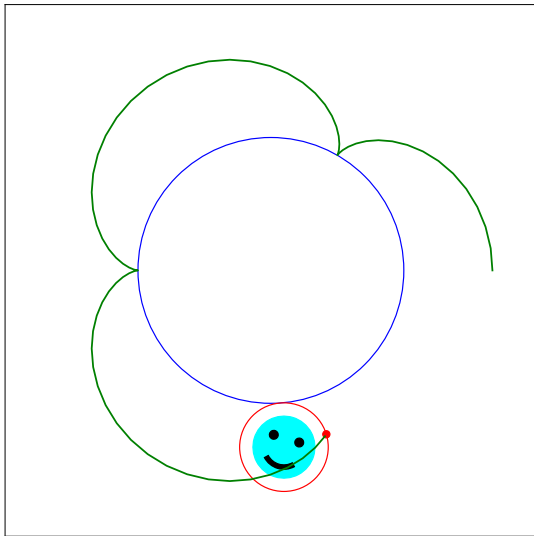
Animation of Coin Rotation Paradox

Rotation: 273.5°



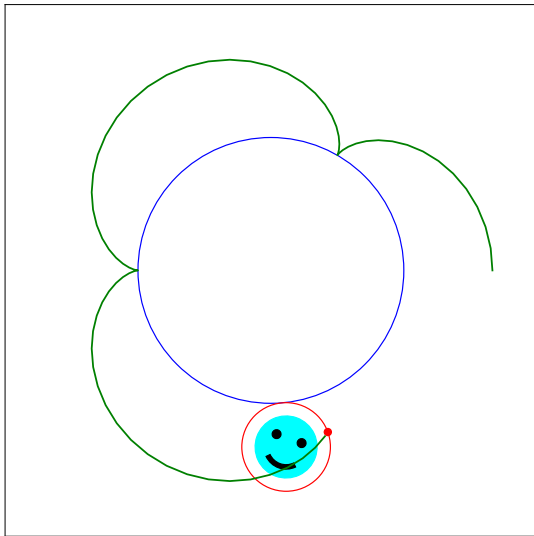
Animation of Coin Rotation Paradox

Rotation: 274.2°



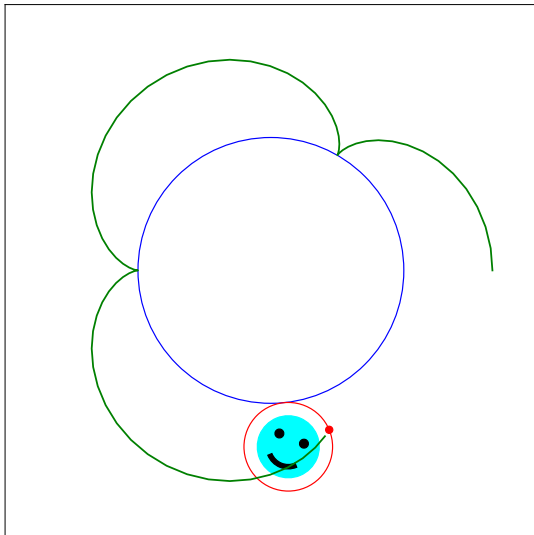
Animation of Coin Rotation Paradox

Rotation: 274.9°



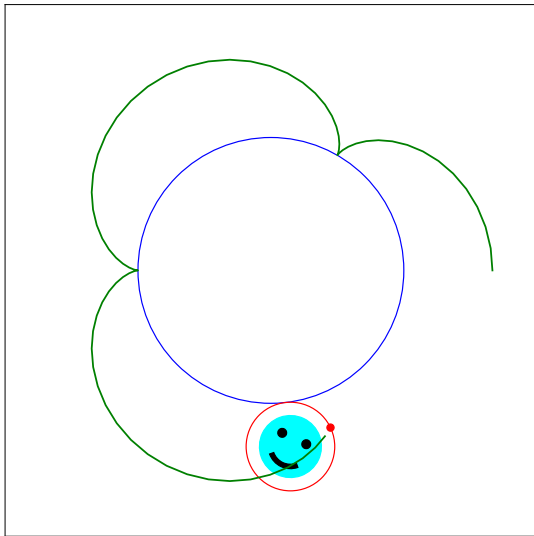
Animation of Coin Rotation Paradox

Rotation: 275.6°



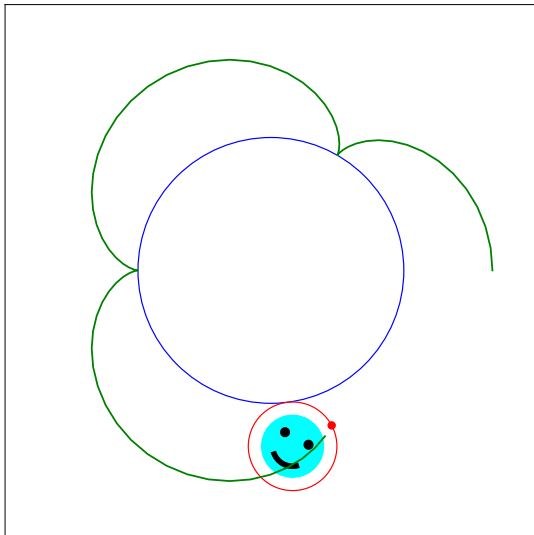
Animation of Coin Rotation Paradox

Rotation: 276.3°



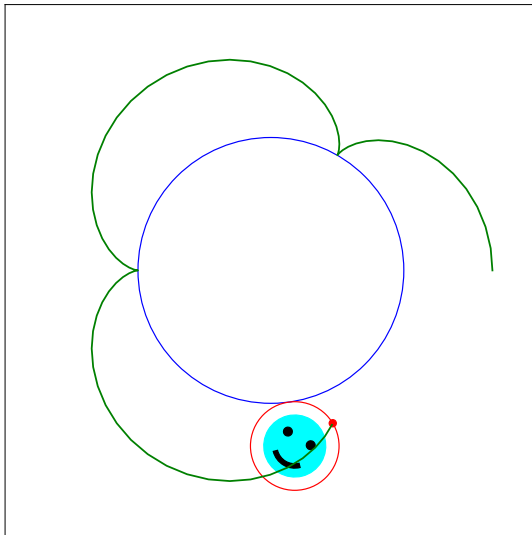
Animation of Coin Rotation Paradox

Rotation: 277.0°



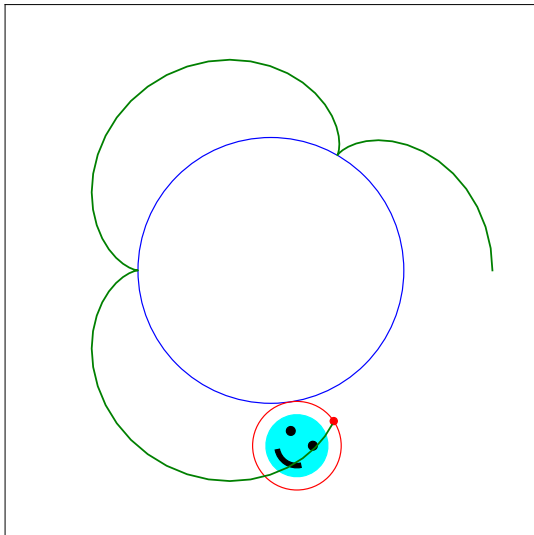
Animation of Coin Rotation Paradox

Rotation: 277.7°



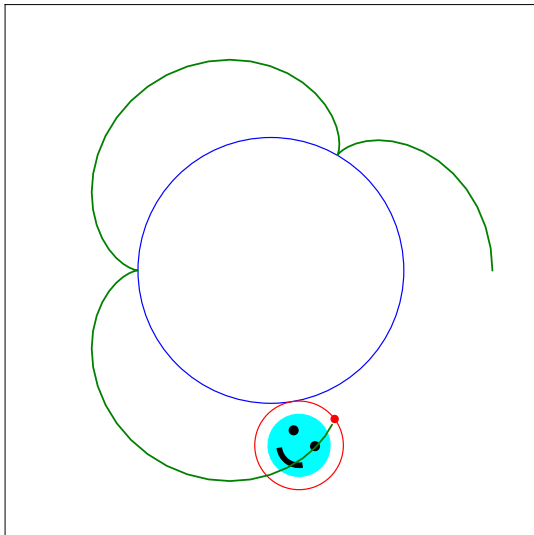
Animation of Coin Rotation Paradox

Rotation: 278.4°



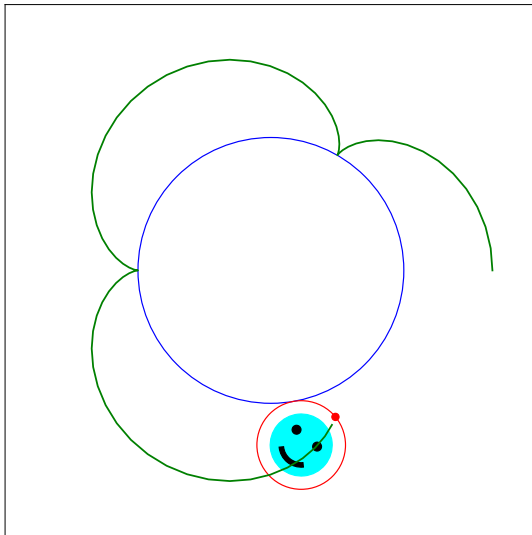
Animation of Coin Rotation Paradox

Rotation: 279.1°



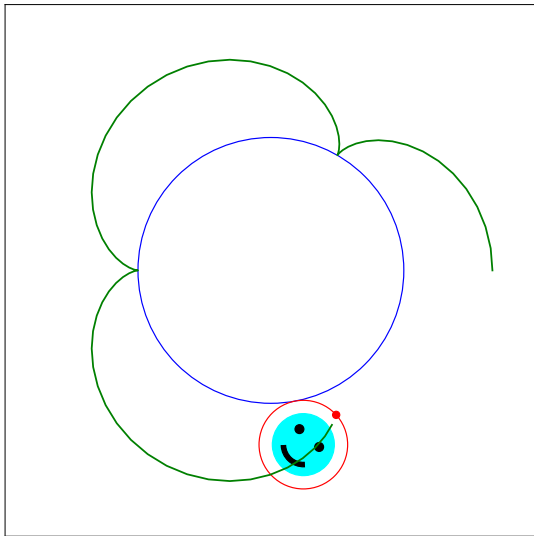
Animation of Coin Rotation Paradox

Rotation: 279.8°



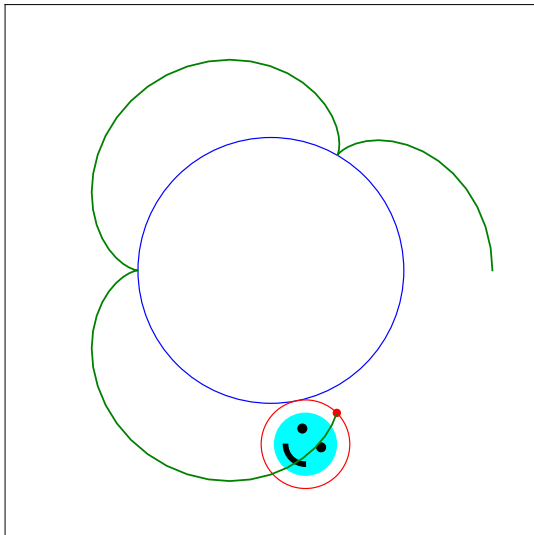
Animation of Coin Rotation Paradox

Rotation: 280.5°

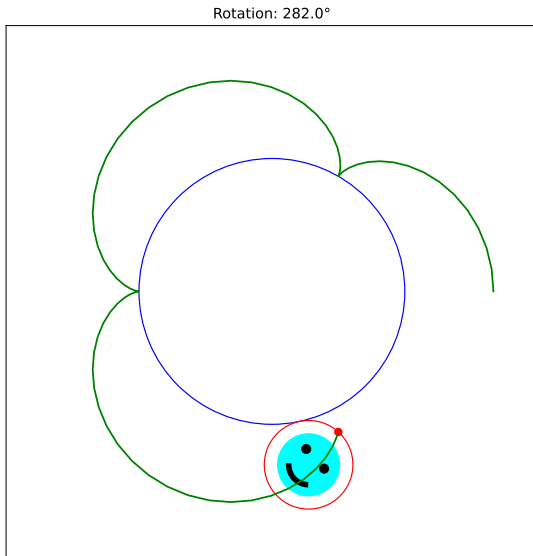


Animation of Coin Rotation Paradox

Rotation: 281.2°

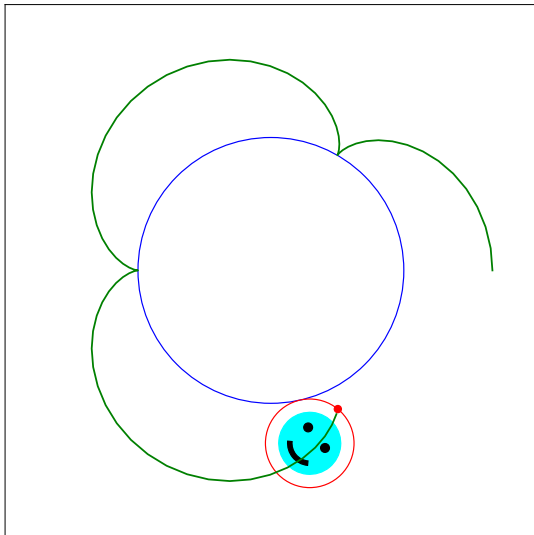


Animation of Coin Rotation Paradox



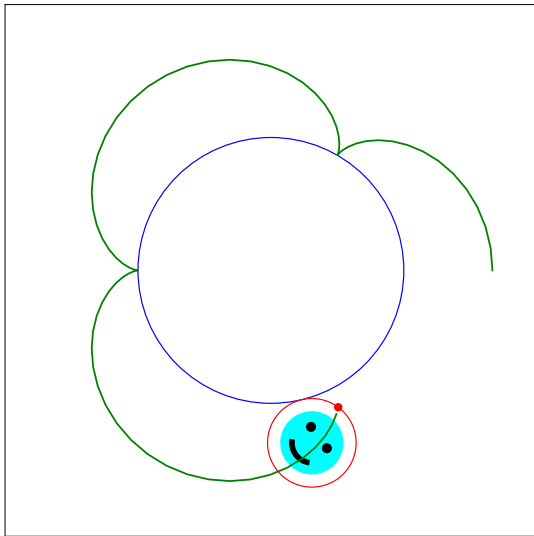
Animation of Coin Rotation Paradox

Rotation: 282.7°



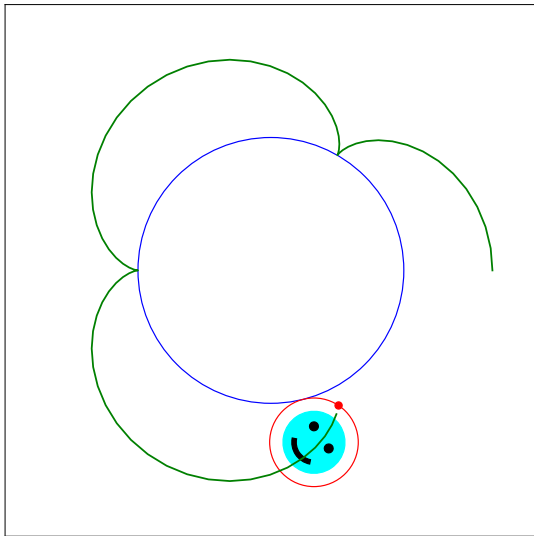
Animation of Coin Rotation Paradox

Rotation: 283.4°



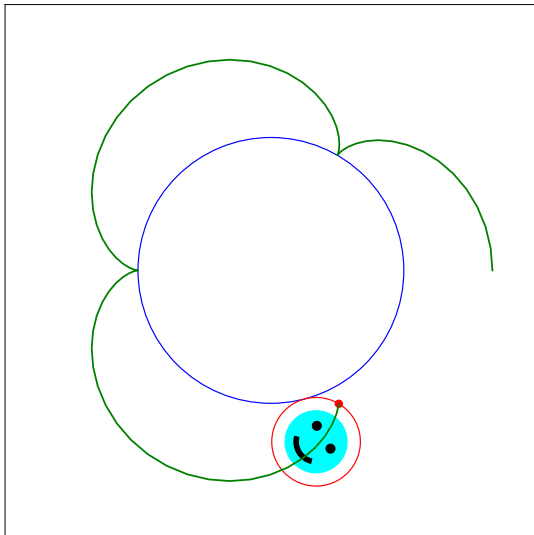
Animation of Coin Rotation Paradox

Rotation: 284.1°



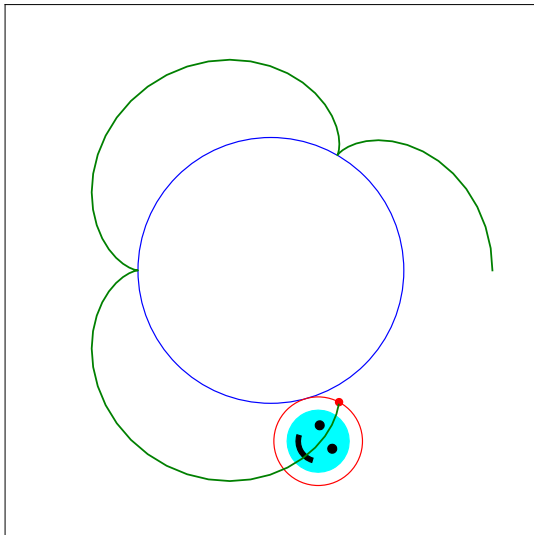
Animation of Coin Rotation Paradox

Rotation: 284.8°



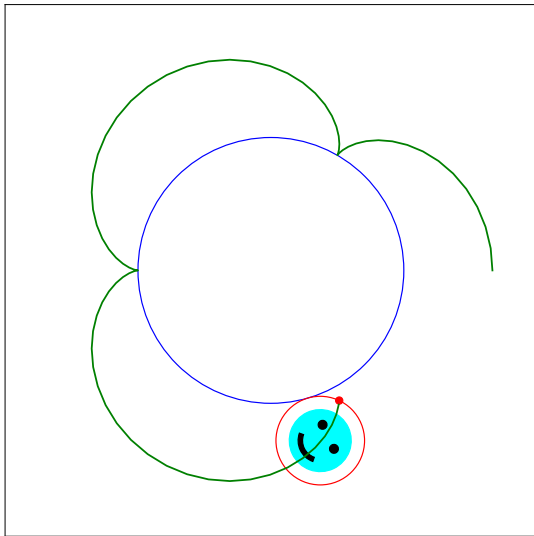
Animation of Coin Rotation Paradox

Rotation: 285.5°



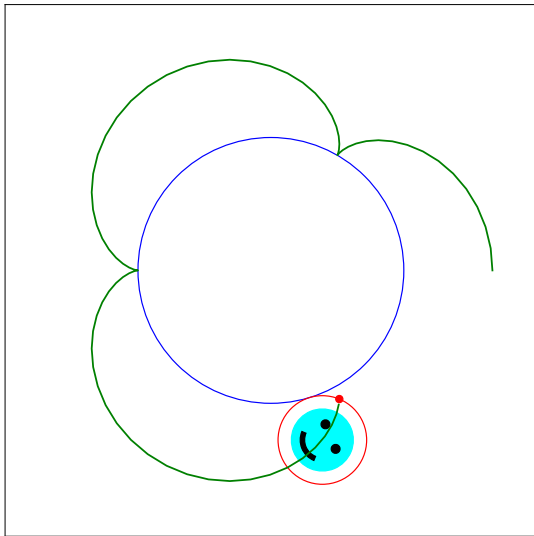
Animation of Coin Rotation Paradox

Rotation: 286.2°



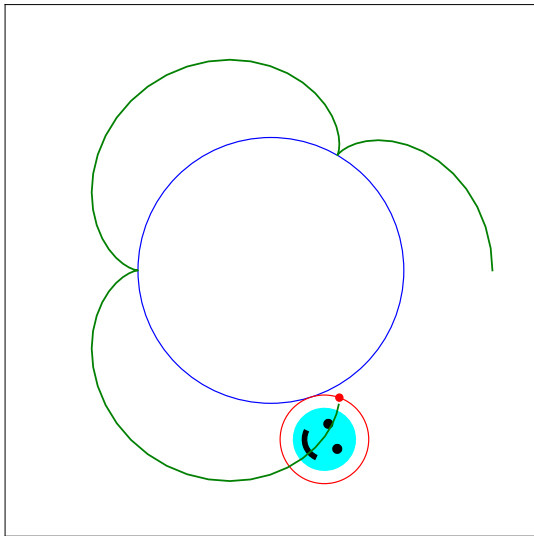
Animation of Coin Rotation Paradox

Rotation: 286.9°



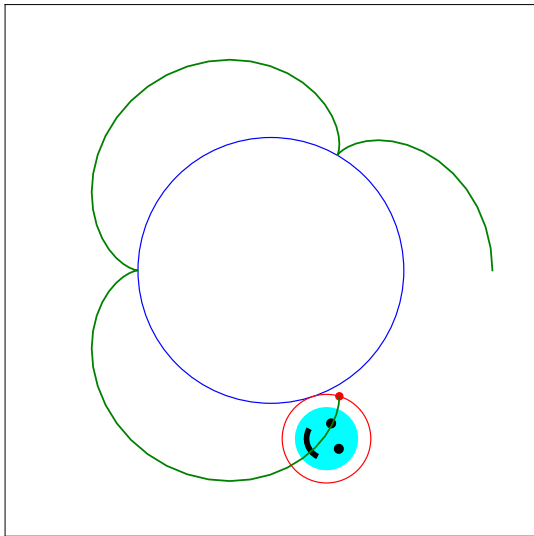
Animation of Coin Rotation Paradox

Rotation: 287.6°

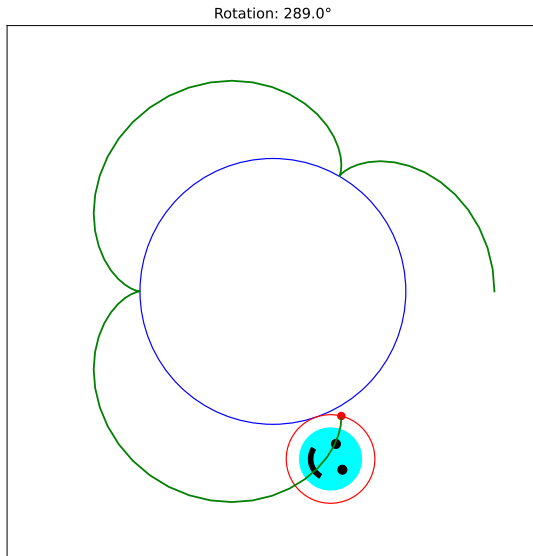


Animation of Coin Rotation Paradox

Rotation: 288.3°

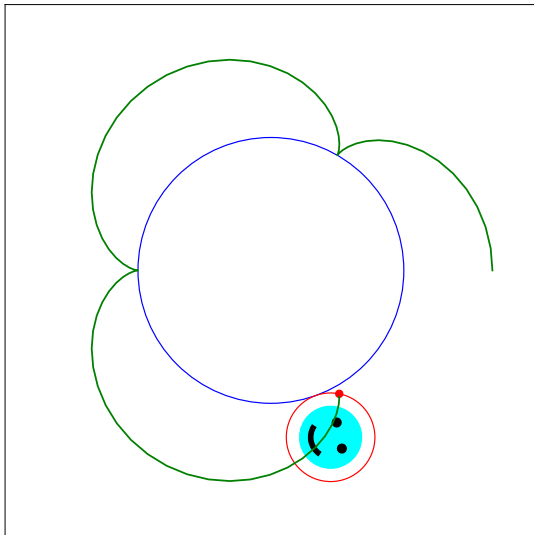


Animation of Coin Rotation Paradox

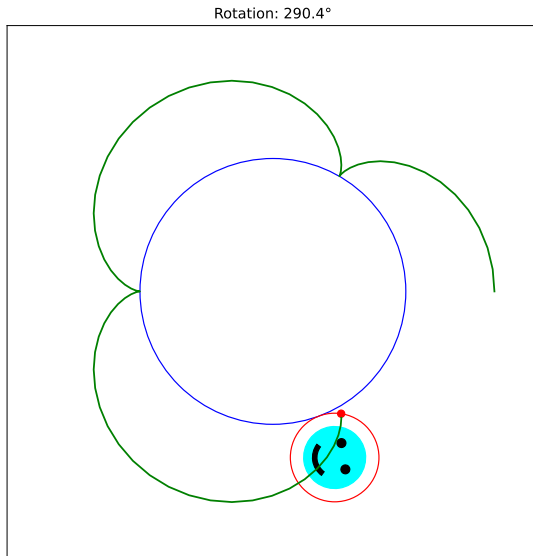


Animation of Coin Rotation Paradox

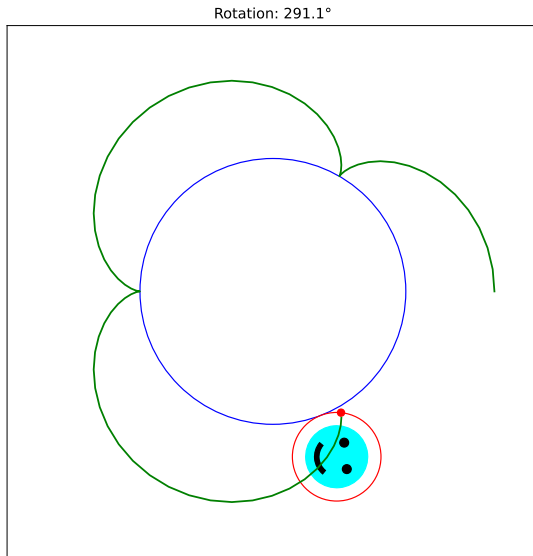
Rotation: 289.7°



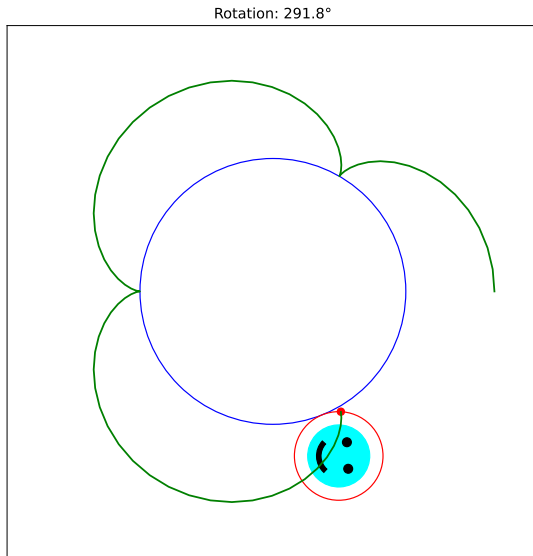
Animation of Coin Rotation Paradox



Animation of Coin Rotation Paradox

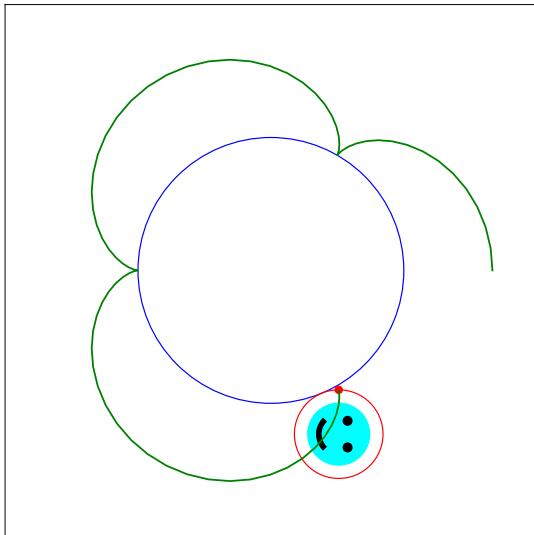


Animation of Coin Rotation Paradox



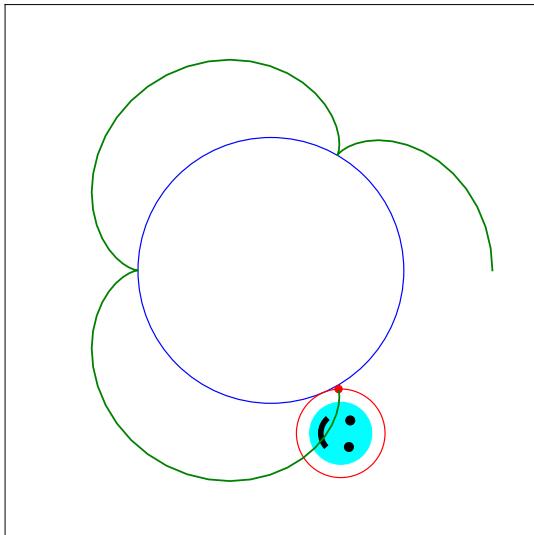
Animation of Coin Rotation Paradox

Rotation: 292.5°



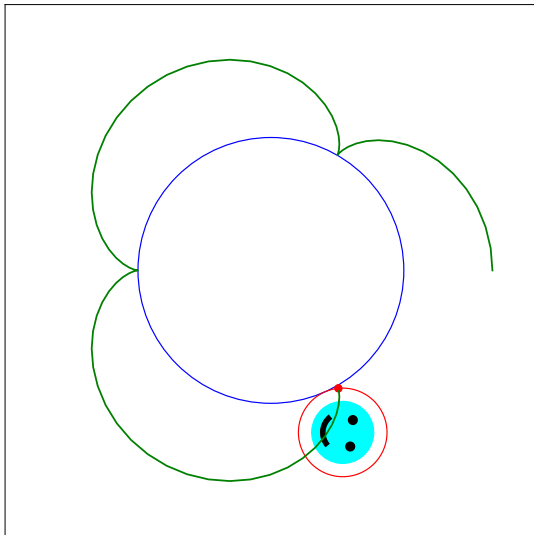
Animation of Coin Rotation Paradox

Rotation: 293.2°



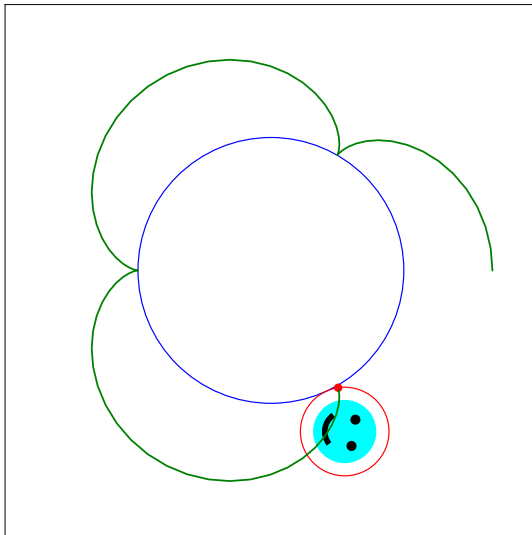
Animation of Coin Rotation Paradox

Rotation: 293.9°



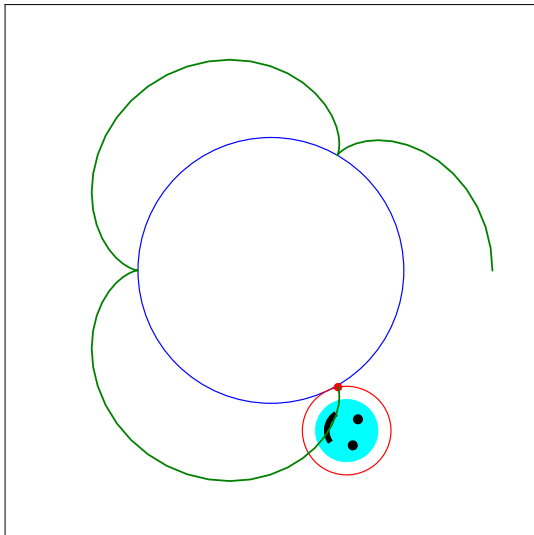
Animation of Coin Rotation Paradox

Rotation: 294.6°

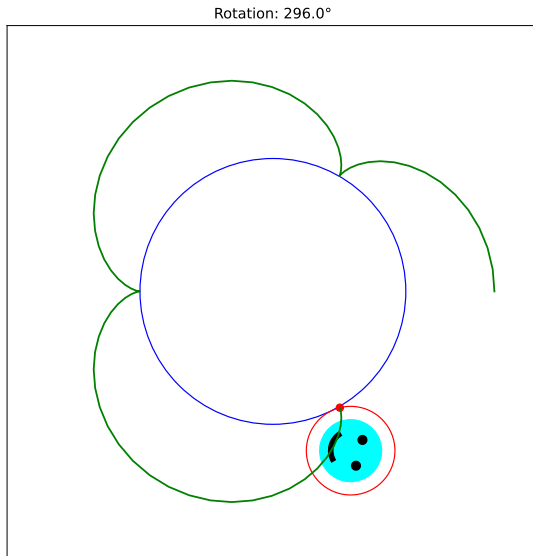


Animation of Coin Rotation Paradox

Rotation: 295.3°

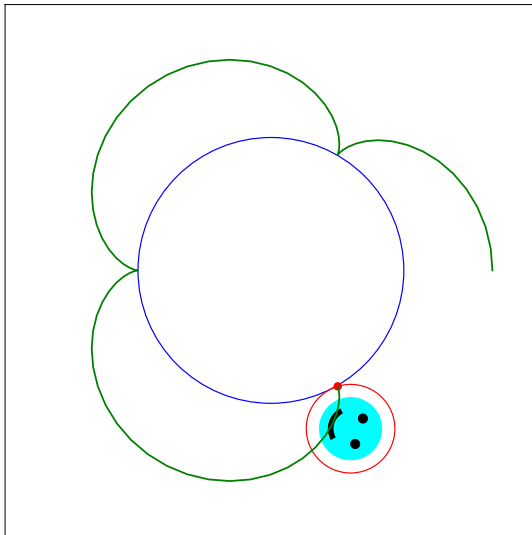


Animation of Coin Rotation Paradox



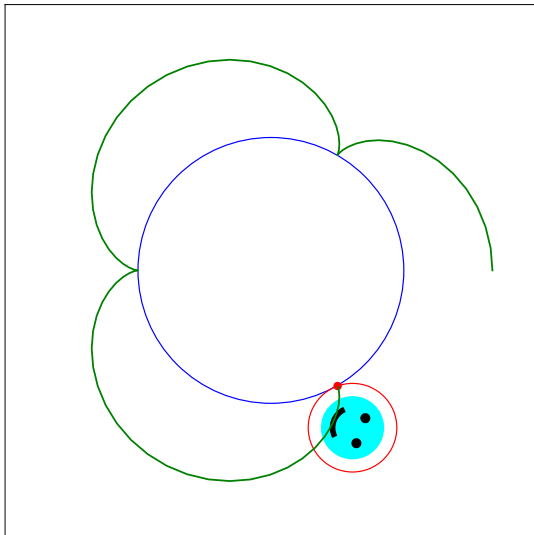
Animation of Coin Rotation Paradox

Rotation: 296.7°



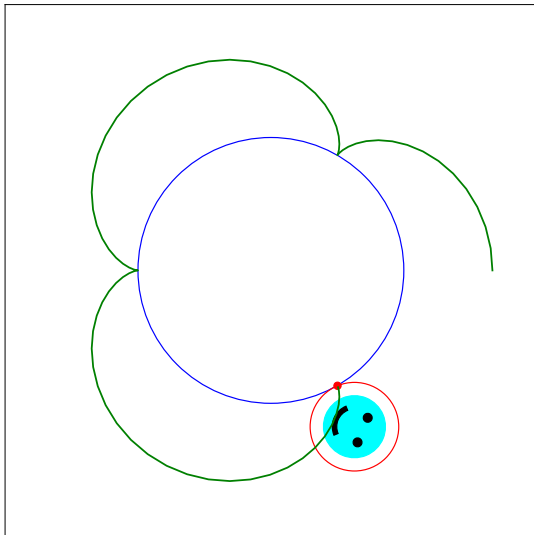
Animation of Coin Rotation Paradox

Rotation: 297.4°



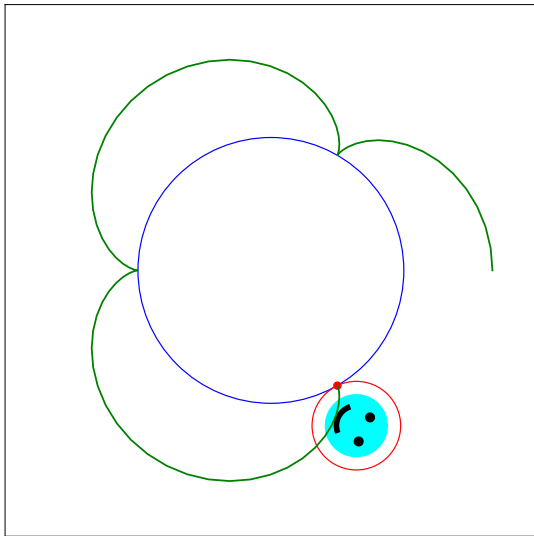
Animation of Coin Rotation Paradox

Rotation: 298.1°



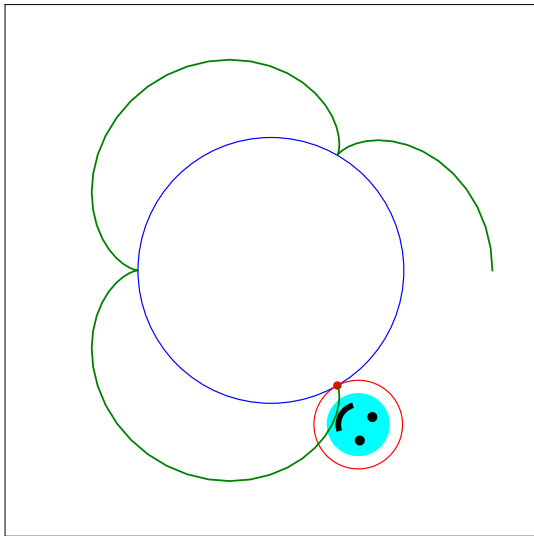
Animation of Coin Rotation Paradox

Rotation: 298.8°

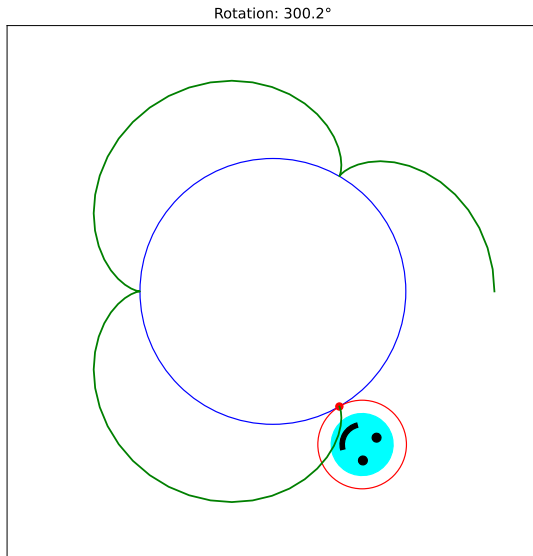


Animation of Coin Rotation Paradox

Rotation: 299.5°

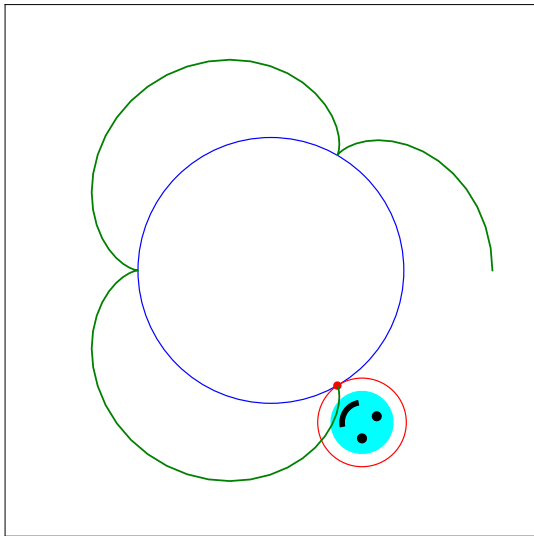


Animation of Coin Rotation Paradox

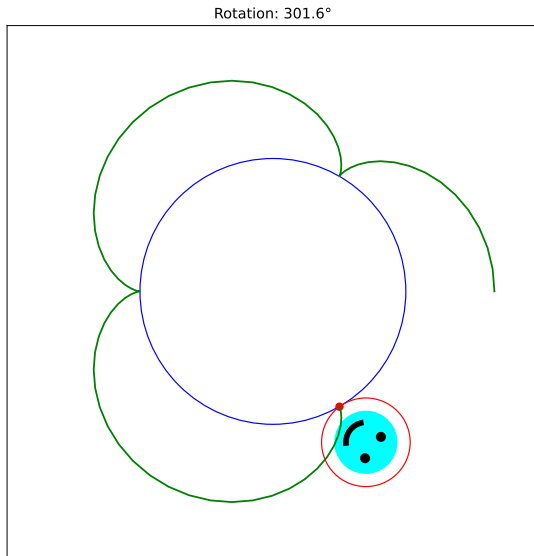


Animation of Coin Rotation Paradox

Rotation: 300.9°

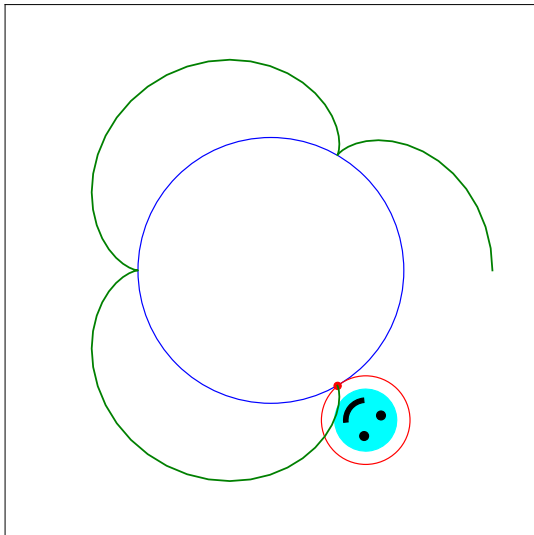


Animation of Coin Rotation Paradox

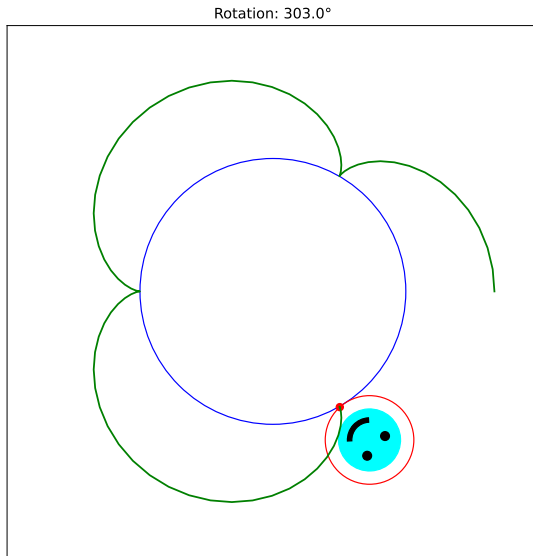


Animation of Coin Rotation Paradox

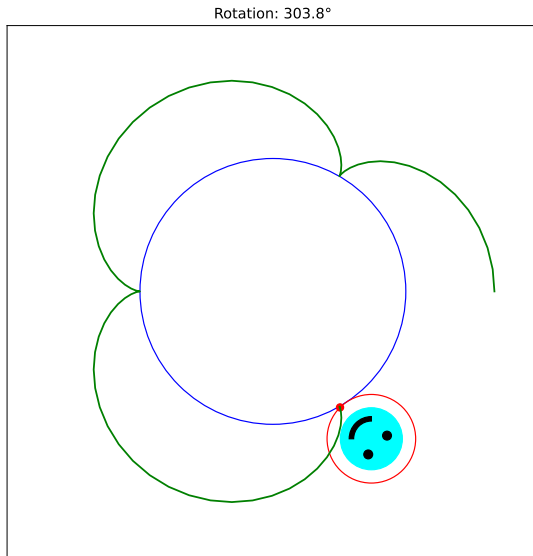
Rotation: 302.3°



Animation of Coin Rotation Paradox

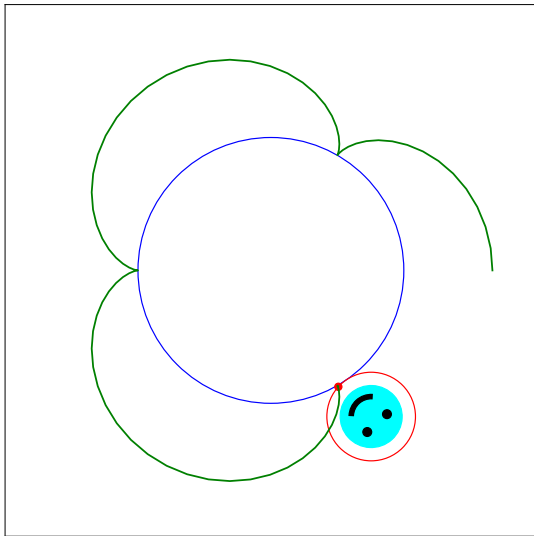


Animation of Coin Rotation Paradox



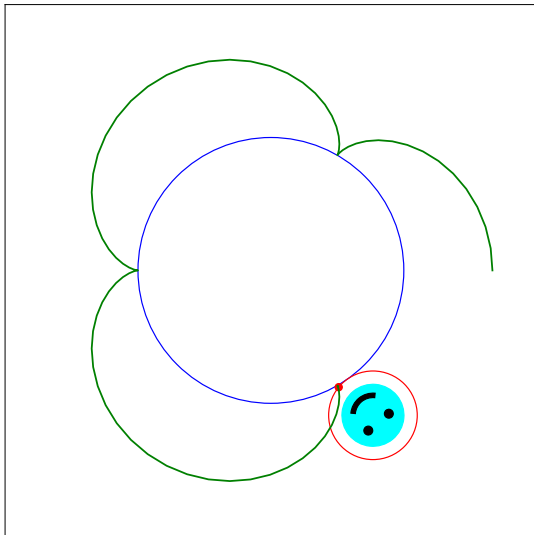
Animation of Coin Rotation Paradox

Rotation: 304.5°



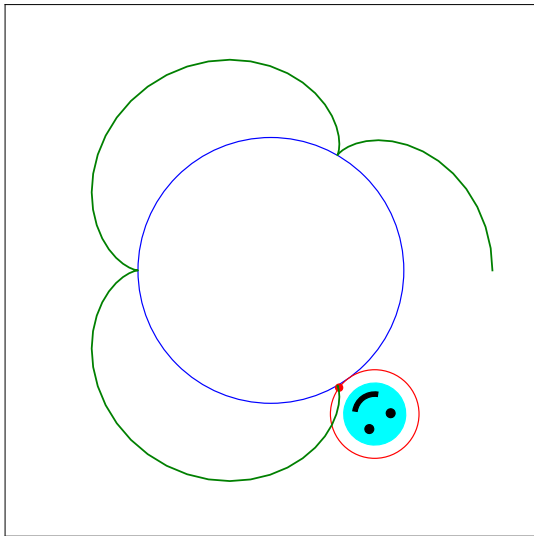
Animation of Coin Rotation Paradox

Rotation: 305.2°



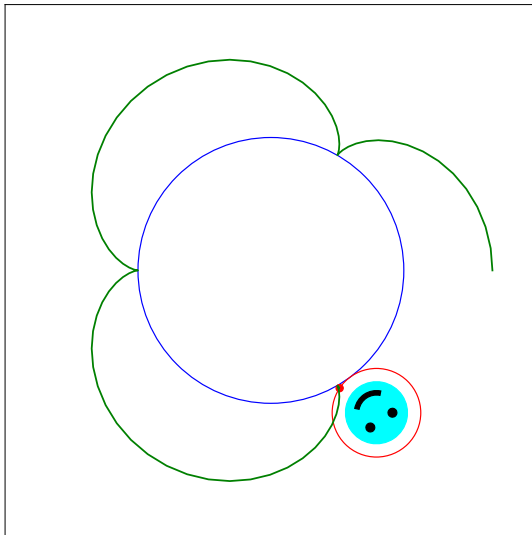
Animation of Coin Rotation Paradox

Rotation: 305.9°



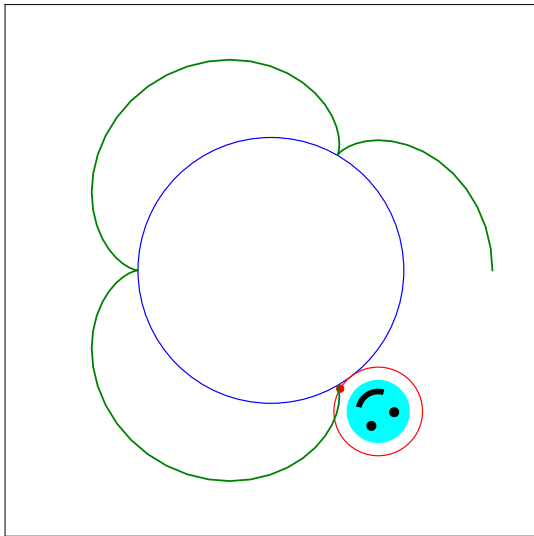
Animation of Coin Rotation Paradox

Rotation: 306.6°



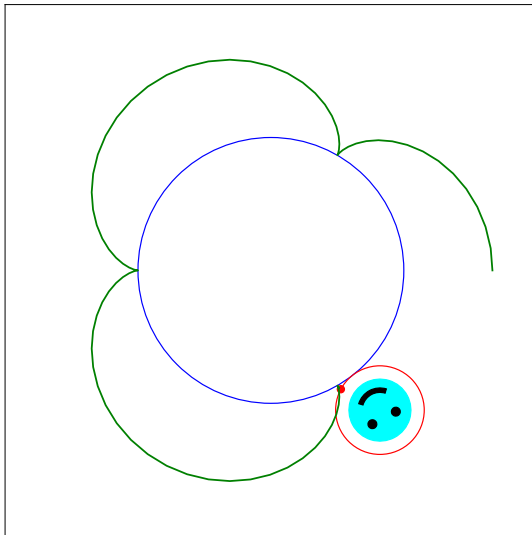
Animation of Coin Rotation Paradox

Rotation: 307.3°



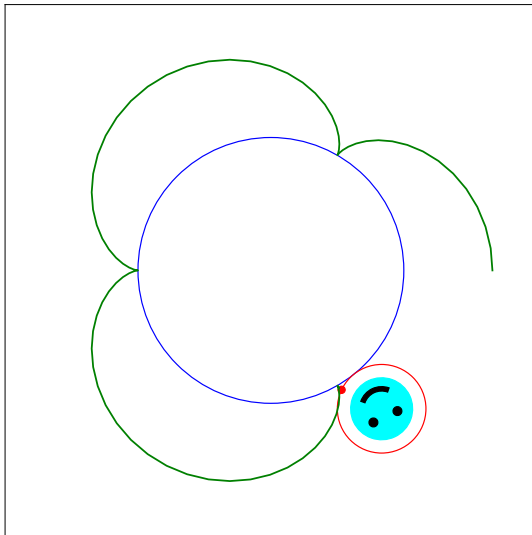
Animation of Coin Rotation Paradox

Rotation: 308.0°



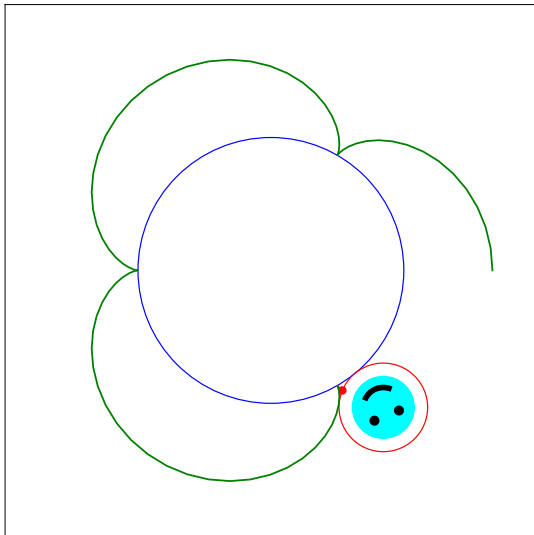
Animation of Coin Rotation Paradox

Rotation: 308.7°

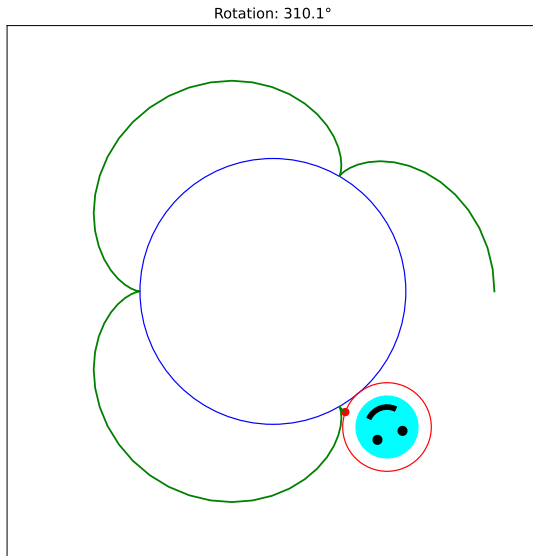


Animation of Coin Rotation Paradox

Rotation: 309.4°

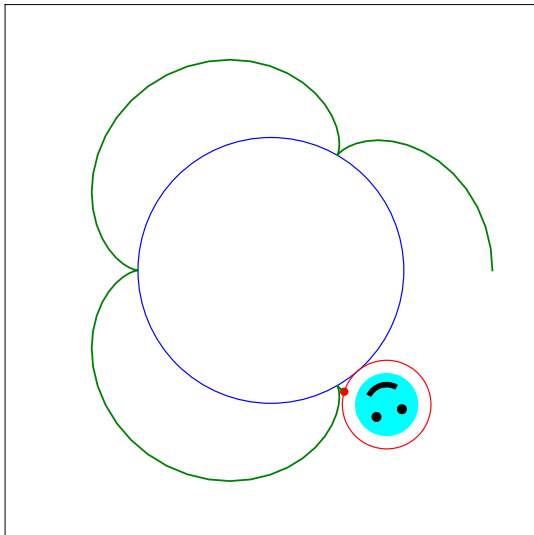


Animation of Coin Rotation Paradox



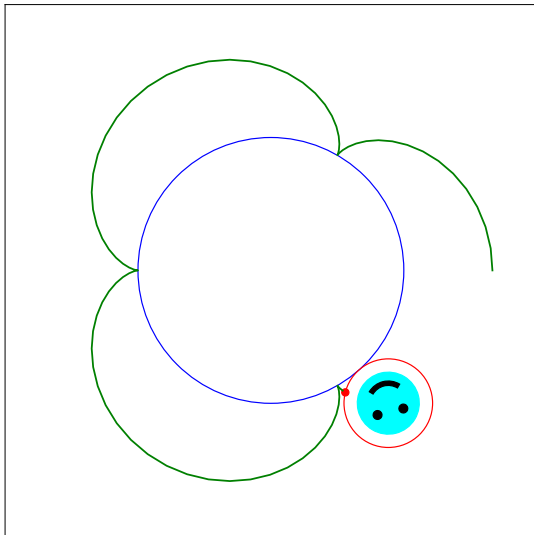
Animation of Coin Rotation Paradox

Rotation: 310.8°



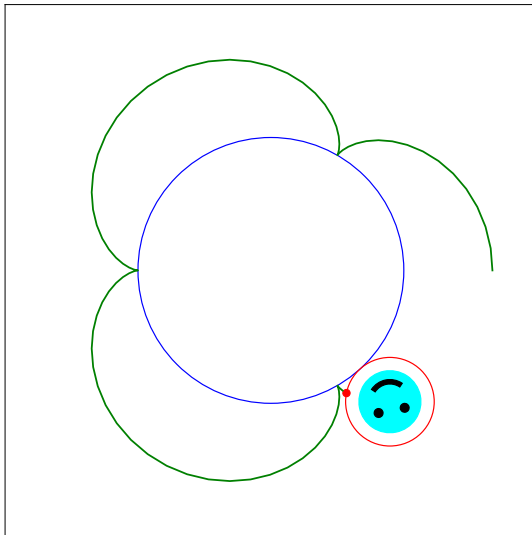
Animation of Coin Rotation Paradox

Rotation: 311.5°



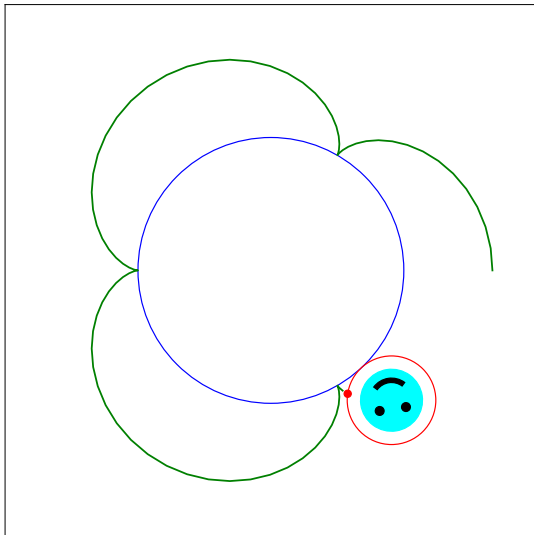
Animation of Coin Rotation Paradox

Rotation: 312.2°



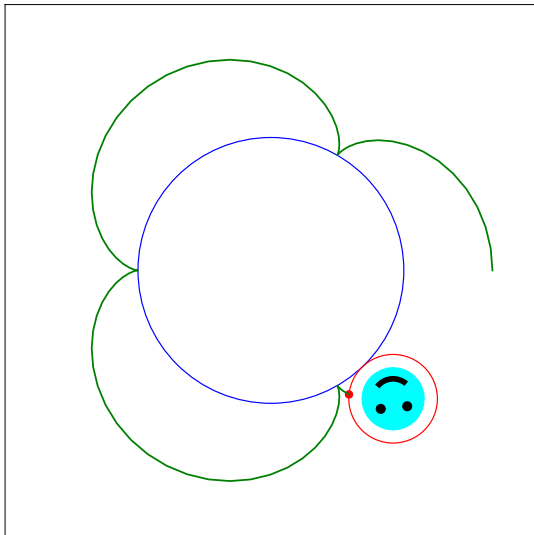
Animation of Coin Rotation Paradox

Rotation: 312.9°



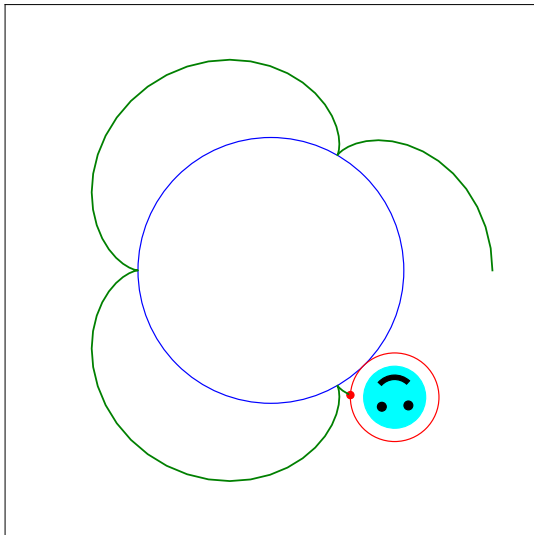
Animation of Coin Rotation Paradox

Rotation: 313.6°



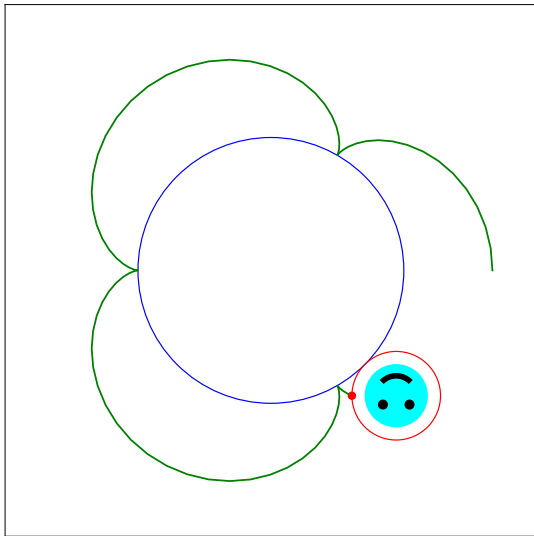
Animation of Coin Rotation Paradox

Rotation: 314.3°



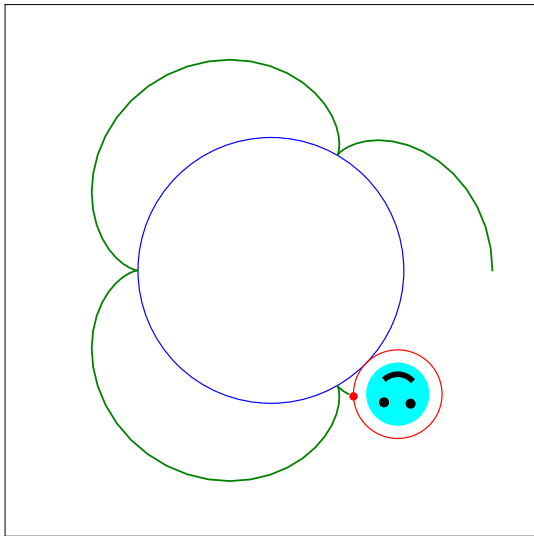
Animation of Coin Rotation Paradox

Rotation: 315.0°



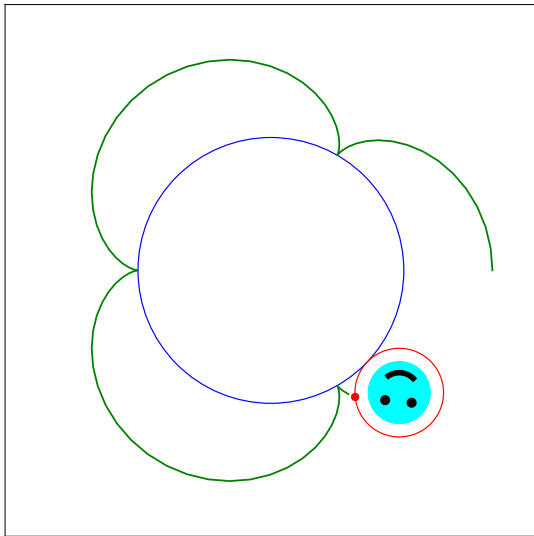
Animation of Coin Rotation Paradox

Rotation: 315.7°



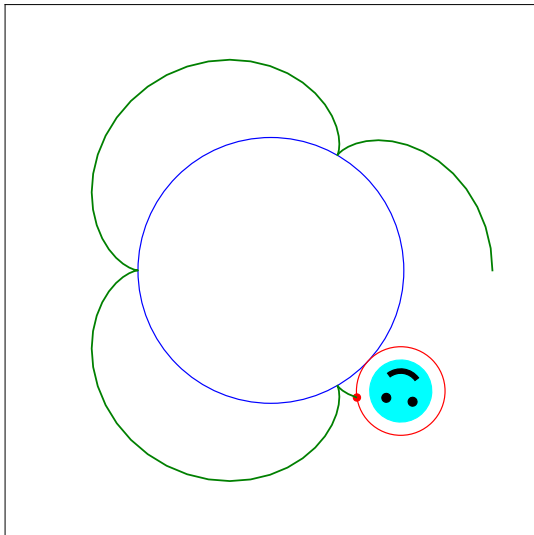
Animation of Coin Rotation Paradox

Rotation: 316.4°



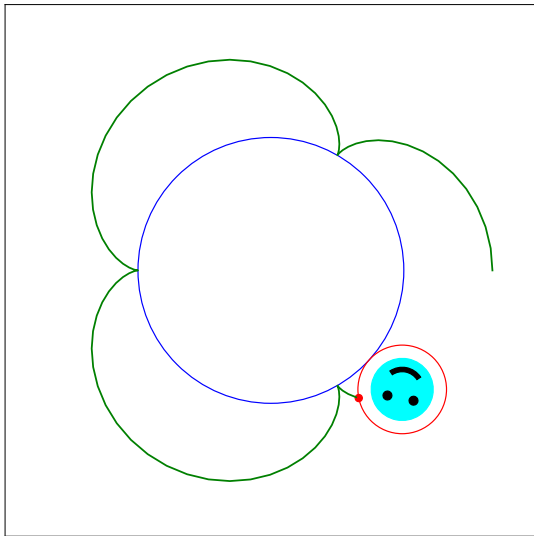
Animation of Coin Rotation Paradox

Rotation: 317.1°



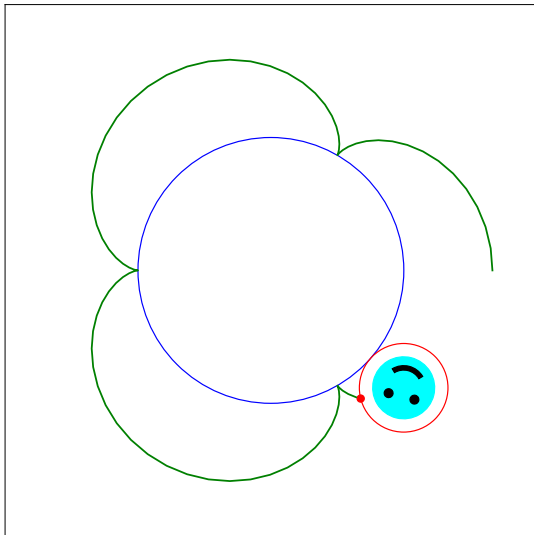
Animation of Coin Rotation Paradox

Rotation: 317.8°



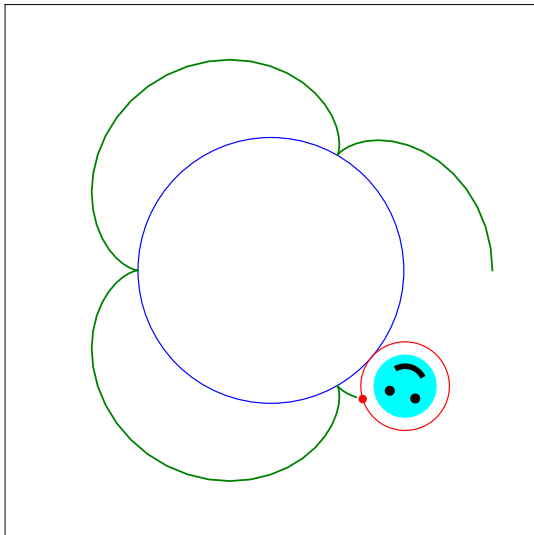
Animation of Coin Rotation Paradox

Rotation: 318.5°



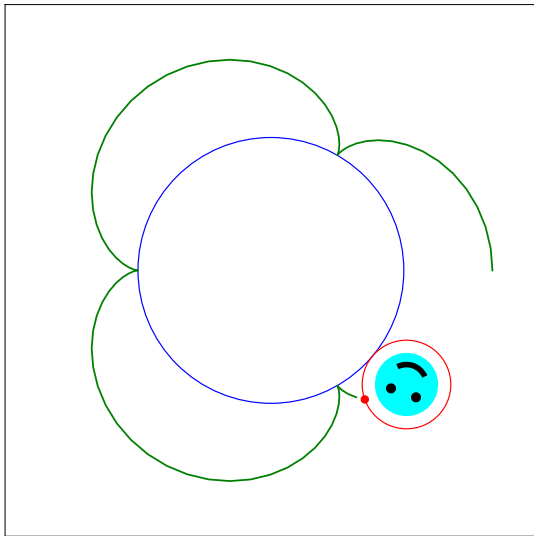
Animation of Coin Rotation Paradox

Rotation: 319.2°



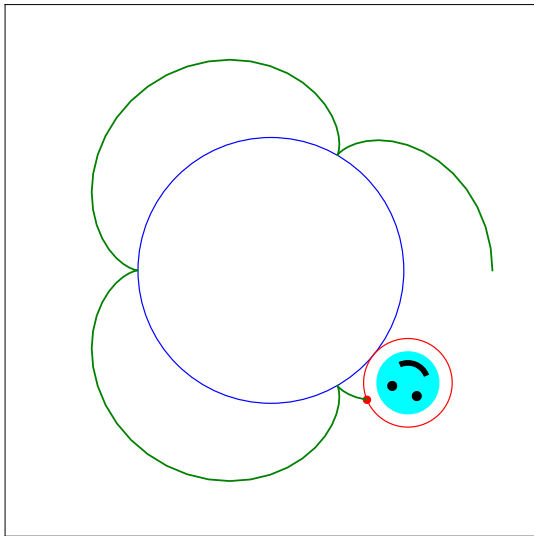
Animation of Coin Rotation Paradox

Rotation: 319.9°



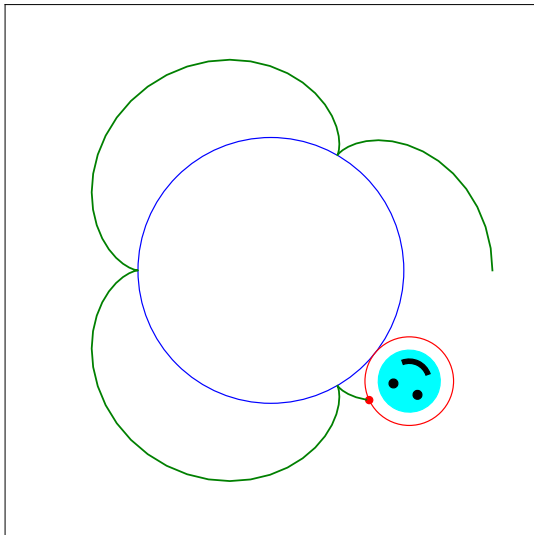
Animation of Coin Rotation Paradox

Rotation: 320.6°



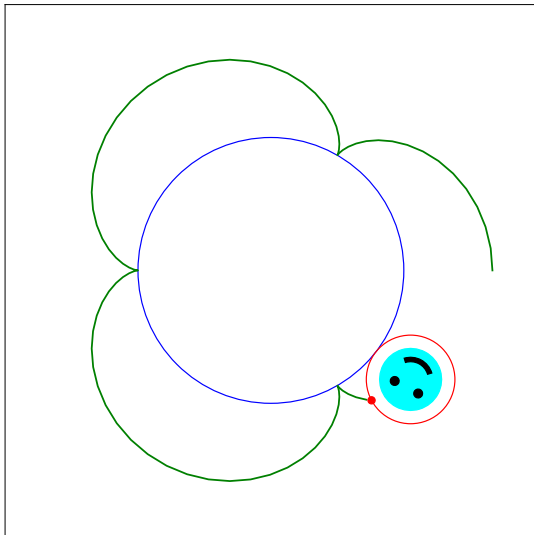
Animation of Coin Rotation Paradox

Rotation: 321.3°



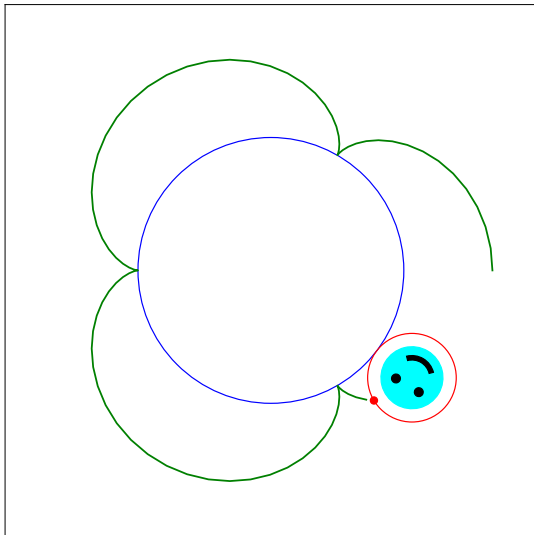
Animation of Coin Rotation Paradox

Rotation: 322.0°



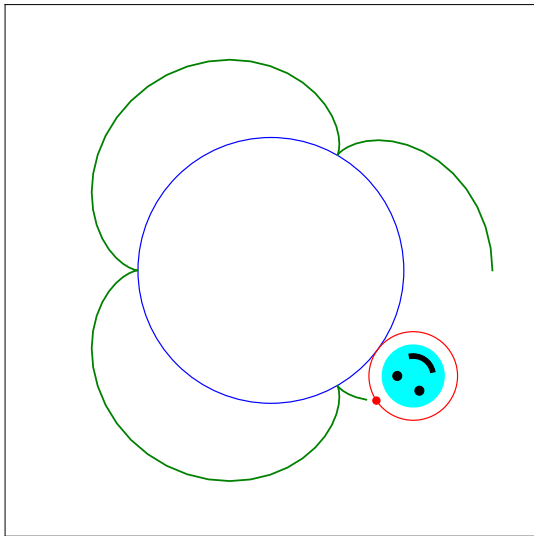
Animation of Coin Rotation Paradox

Rotation: 322.7°



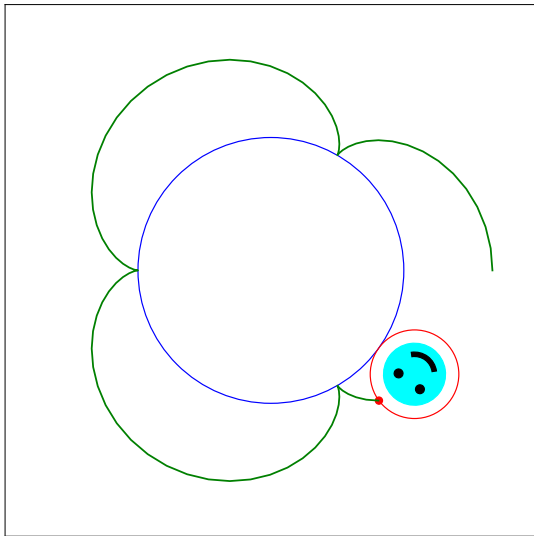
Animation of Coin Rotation Paradox

Rotation: 323.4°



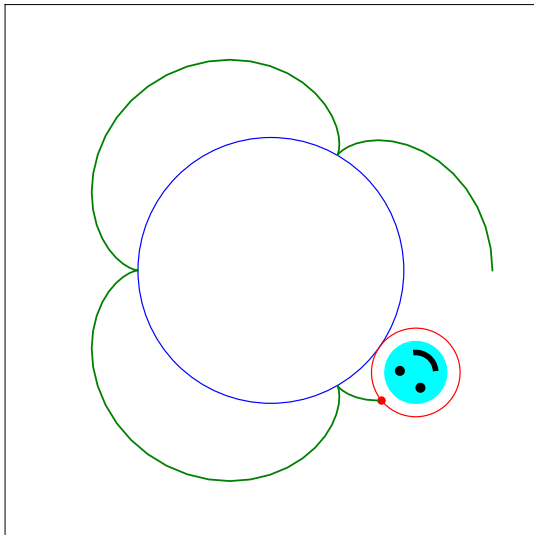
Animation of Coin Rotation Paradox

Rotation: 324.1°



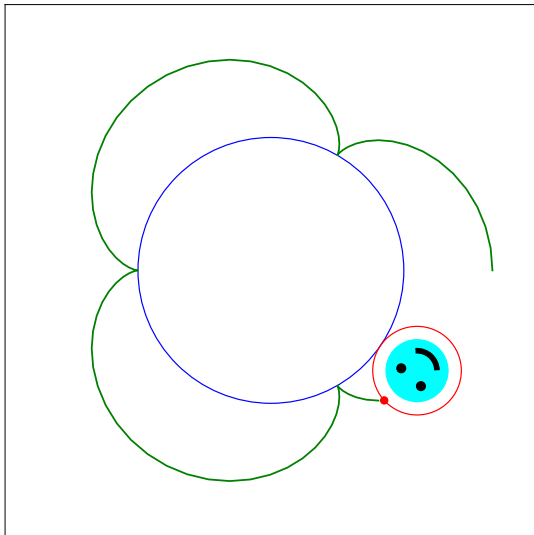
Animation of Coin Rotation Paradox

Rotation: 324.8°



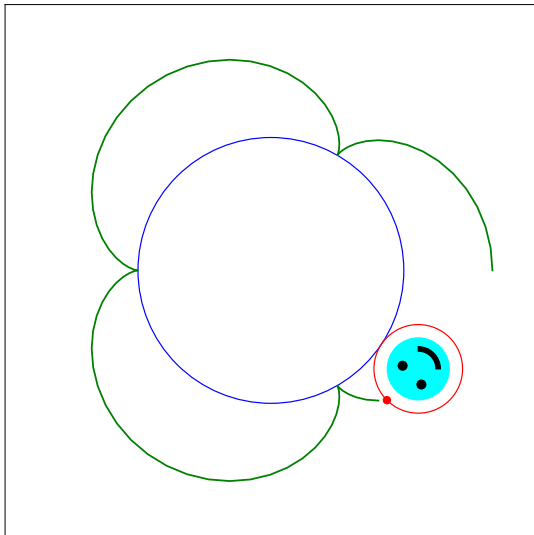
Animation of Coin Rotation Paradox

Rotation: 325.5°



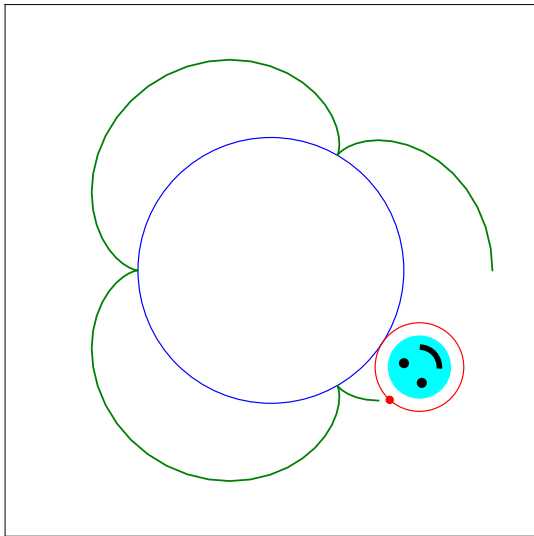
Animation of Coin Rotation Paradox

Rotation: 326.2°



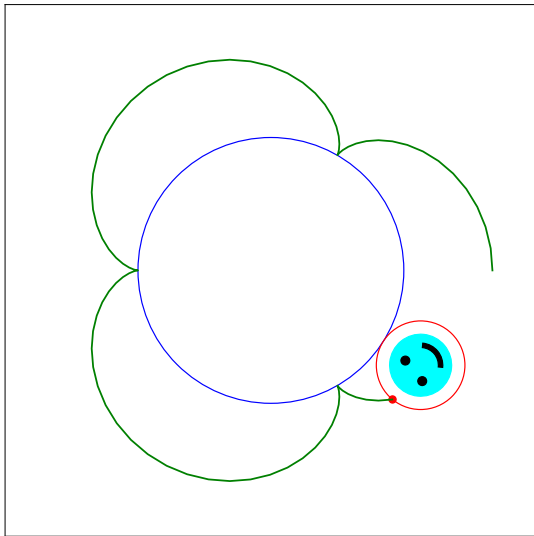
Animation of Coin Rotation Paradox

Rotation: 327.0°



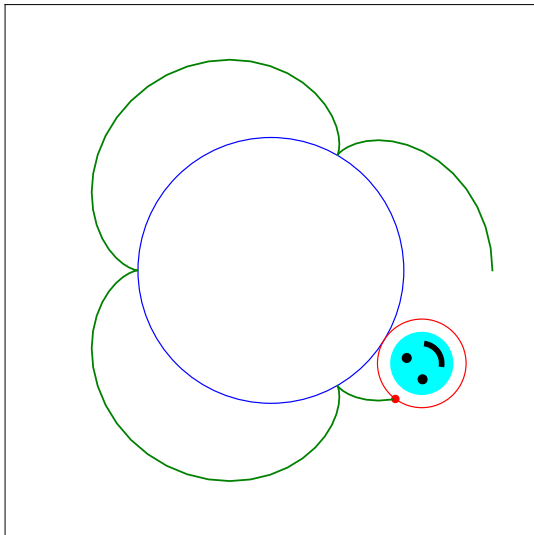
Animation of Coin Rotation Paradox

Rotation: 327.7°



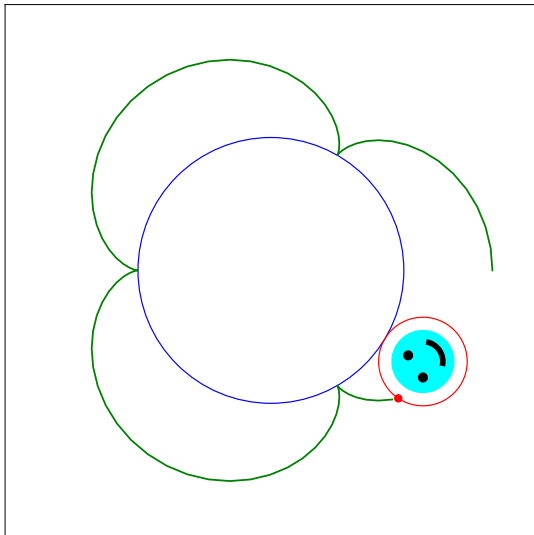
Animation of Coin Rotation Paradox

Rotation: 328.4°



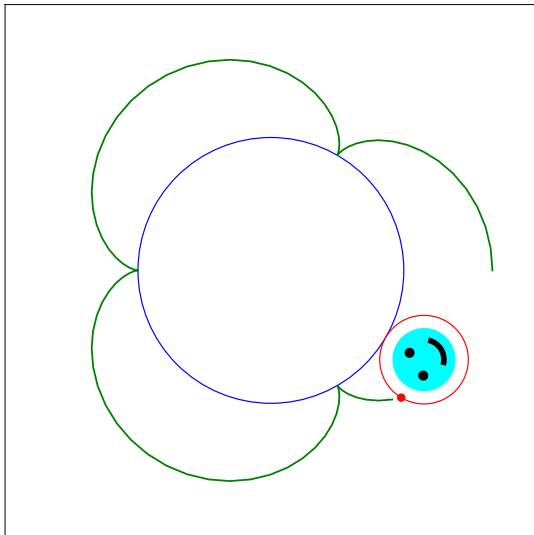
Animation of Coin Rotation Paradox

Rotation: 329.1°



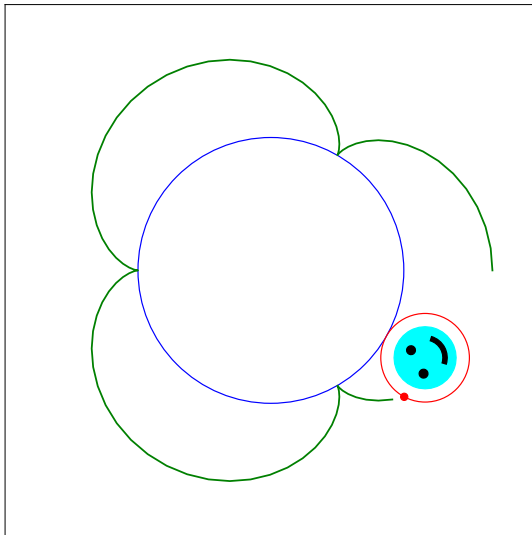
Animation of Coin Rotation Paradox

Rotation: 329.8°



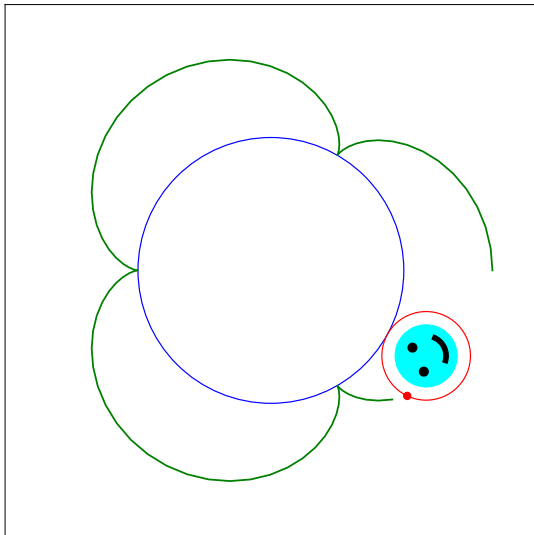
Animation of Coin Rotation Paradox

Rotation: 330.5°



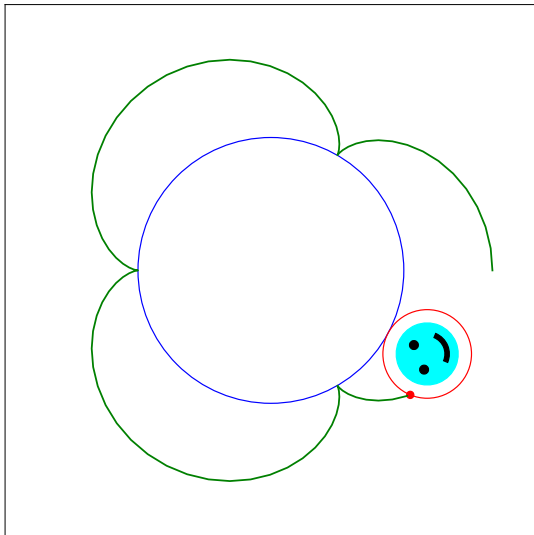
Animation of Coin Rotation Paradox

Rotation: 331.2°



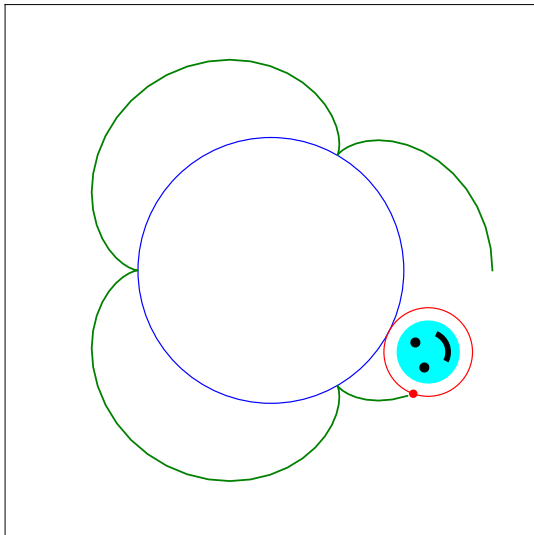
Animation of Coin Rotation Paradox

Rotation: 331.9°



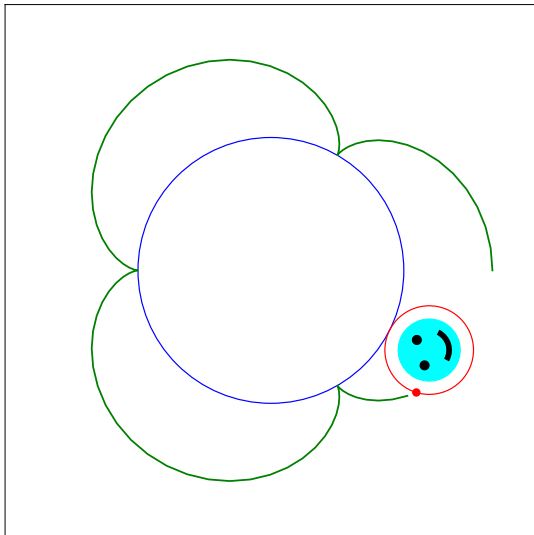
Animation of Coin Rotation Paradox

Rotation: 332.6°



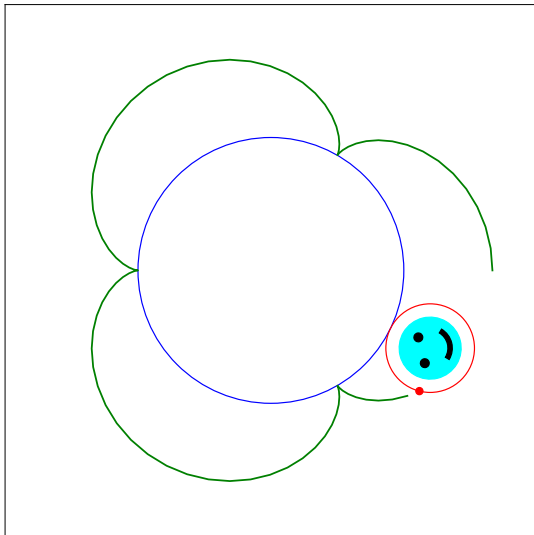
Animation of Coin Rotation Paradox

Rotation: 333.3°



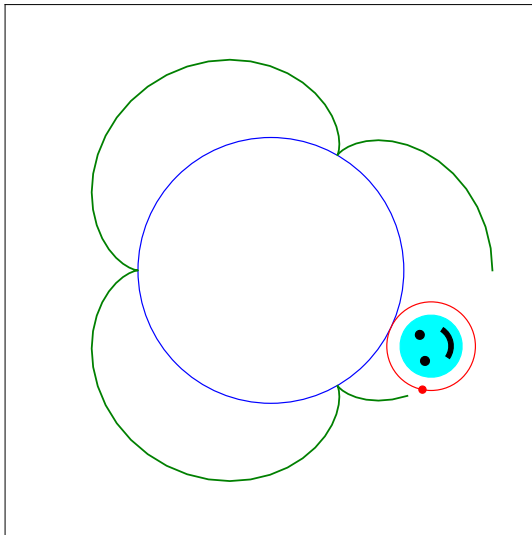
Animation of Coin Rotation Paradox

Rotation: 334.0°



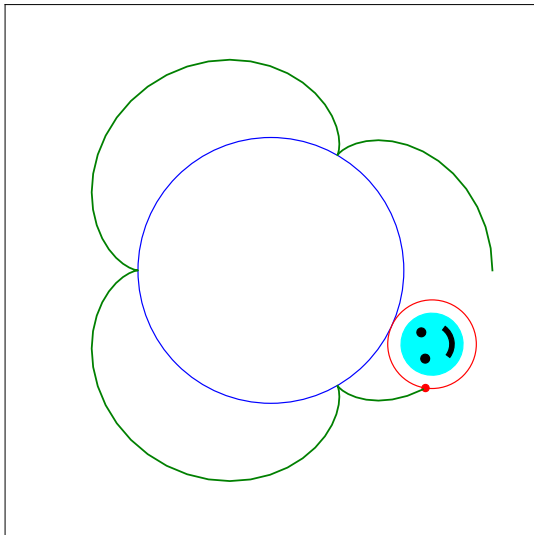
Animation of Coin Rotation Paradox

Rotation: 334.7°



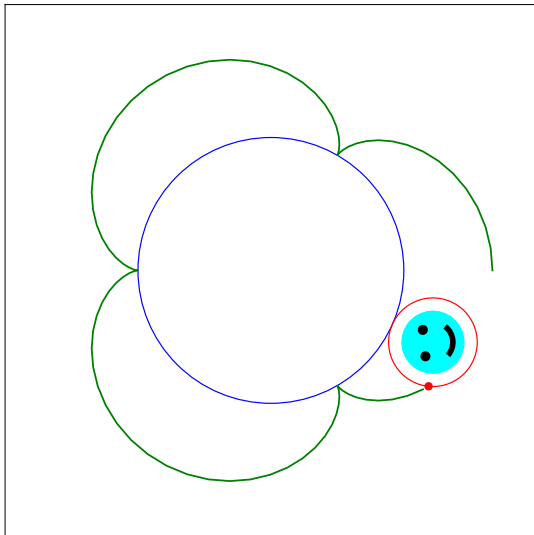
Animation of Coin Rotation Paradox

Rotation: 335.4°



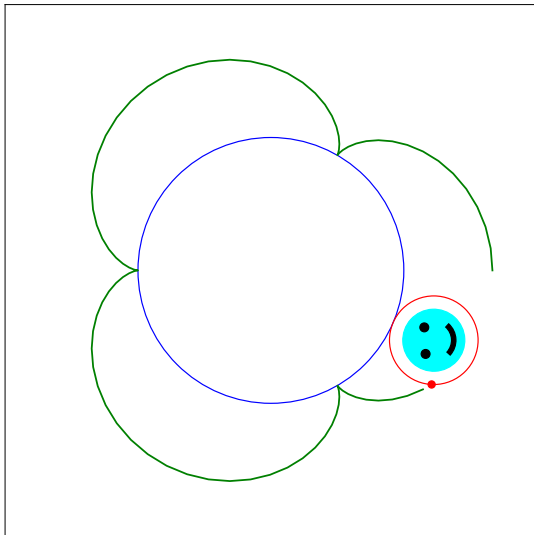
Animation of Coin Rotation Paradox

Rotation: 336.1°



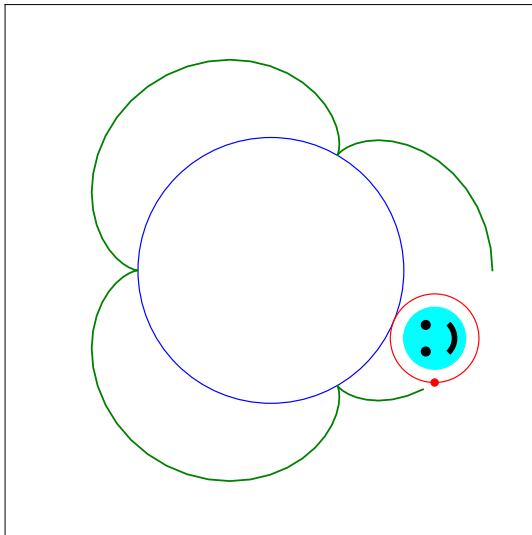
Animation of Coin Rotation Paradox

Rotation: 336.8°



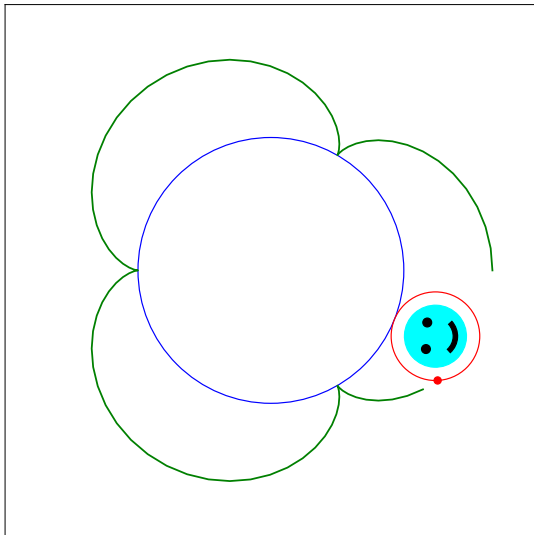
Animation of Coin Rotation Paradox

Rotation: 337.5°



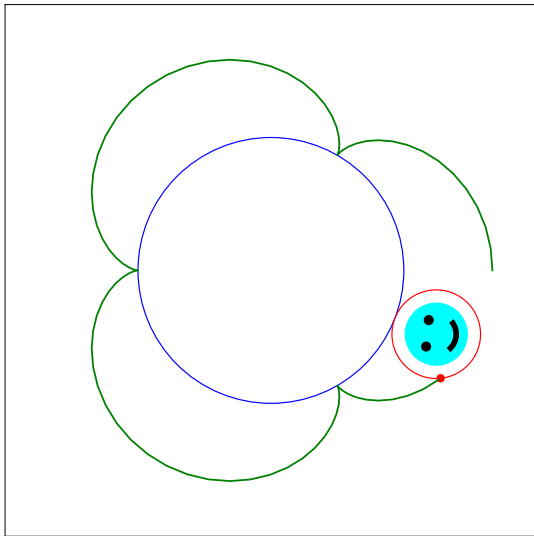
Animation of Coin Rotation Paradox

Rotation: 338.2°



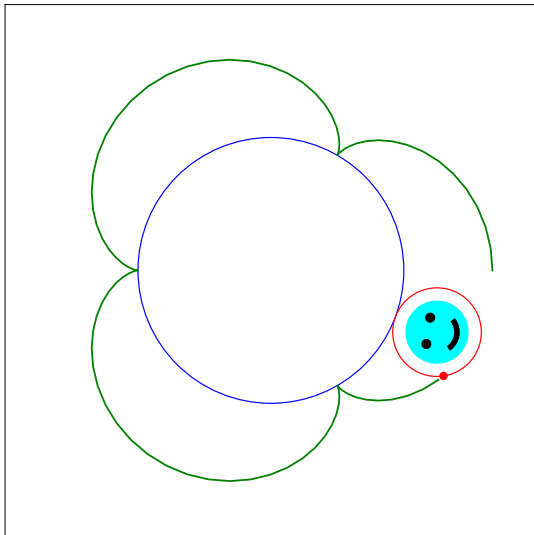
Animation of Coin Rotation Paradox

Rotation: 338.9°



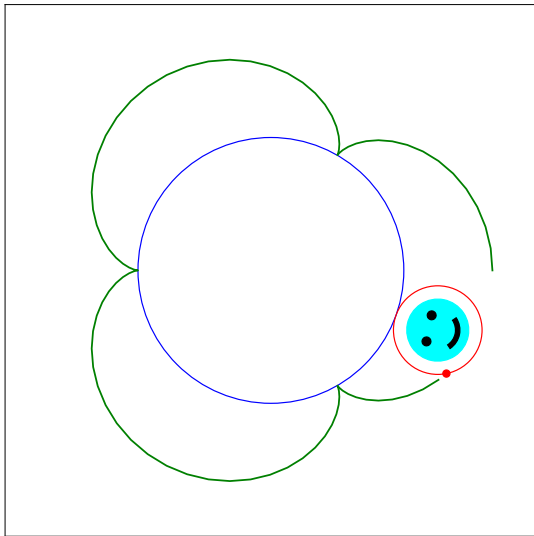
Animation of Coin Rotation Paradox

Rotation: 339.6°



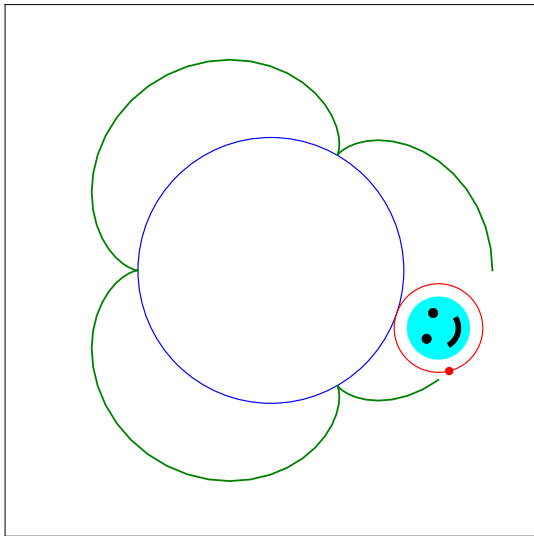
Animation of Coin Rotation Paradox

Rotation: 340.3°



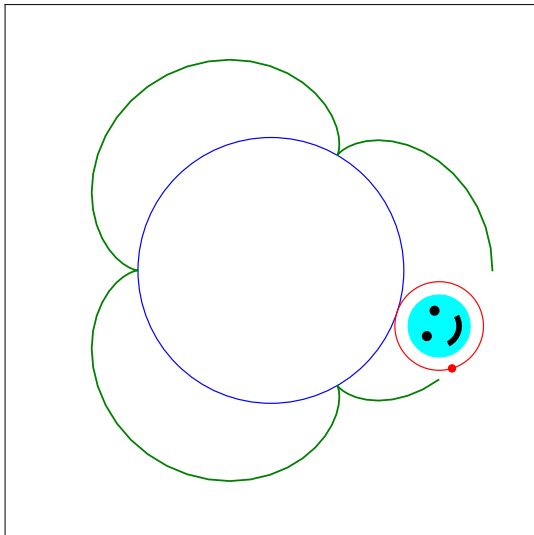
Animation of Coin Rotation Paradox

Rotation: 341.0°



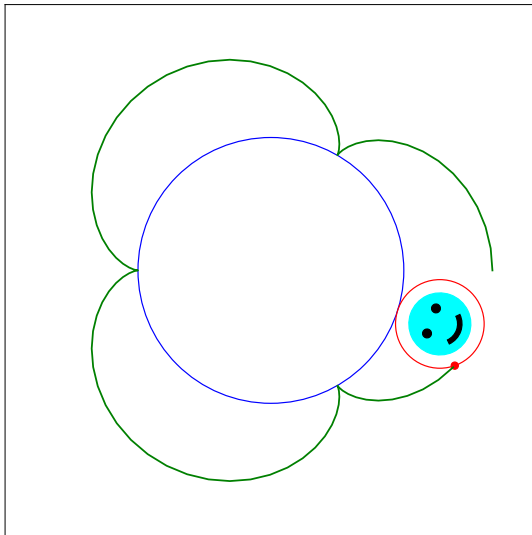
Animation of Coin Rotation Paradox

Rotation: 341.7°



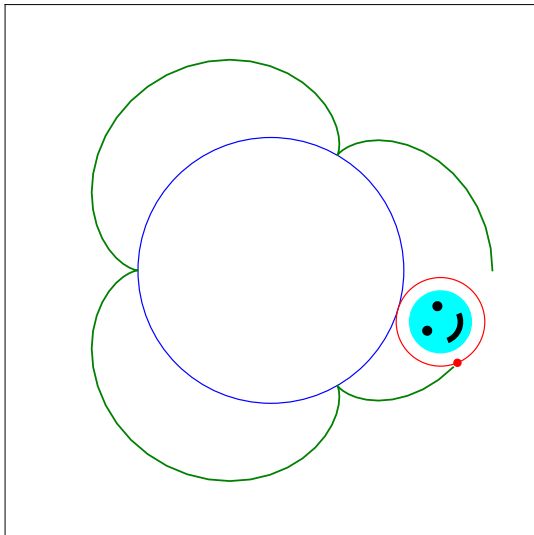
Animation of Coin Rotation Paradox

Rotation: 342.4°



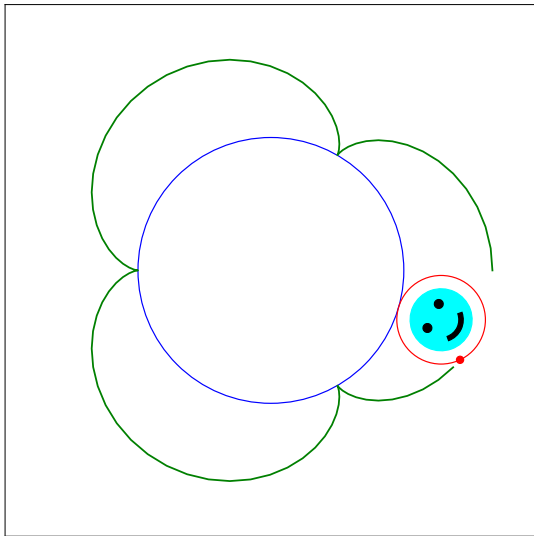
Animation of Coin Rotation Paradox

Rotation: 343.1°



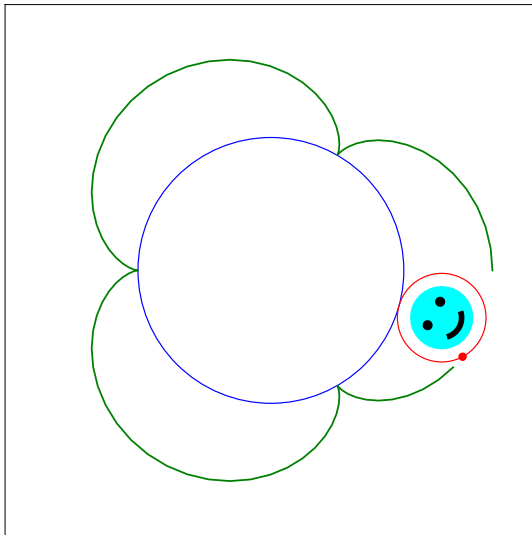
Animation of Coin Rotation Paradox

Rotation: 343.8°



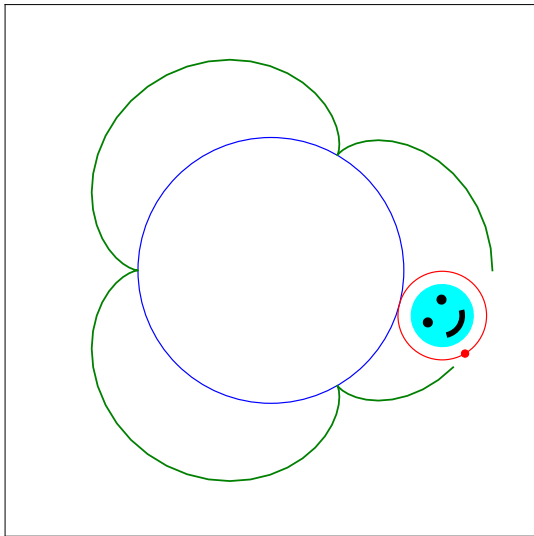
Animation of Coin Rotation Paradox

Rotation: 344.5°



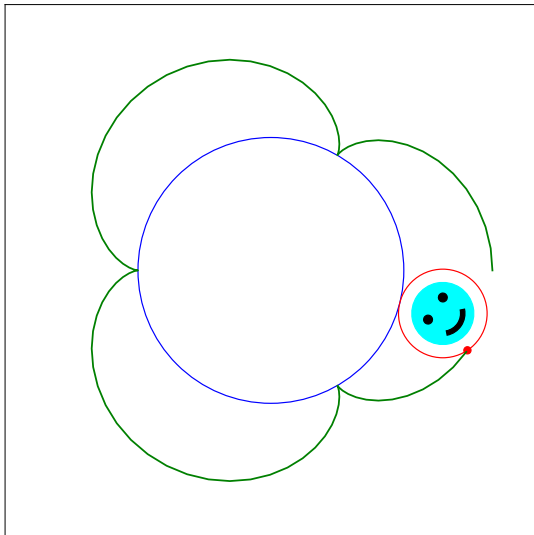
Animation of Coin Rotation Paradox

Rotation: 345.2°



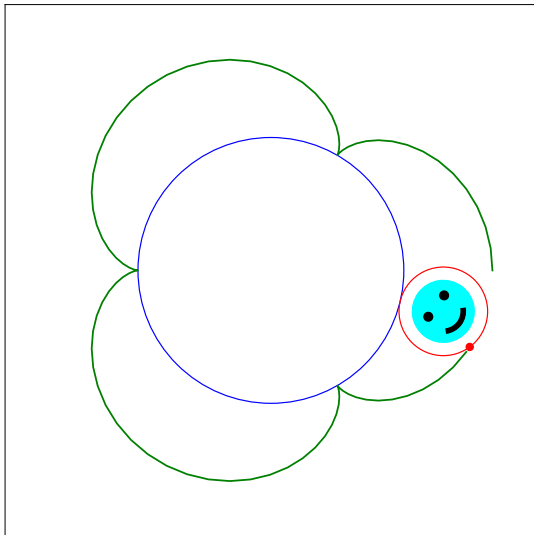
Animation of Coin Rotation Paradox

Rotation: 345.9°



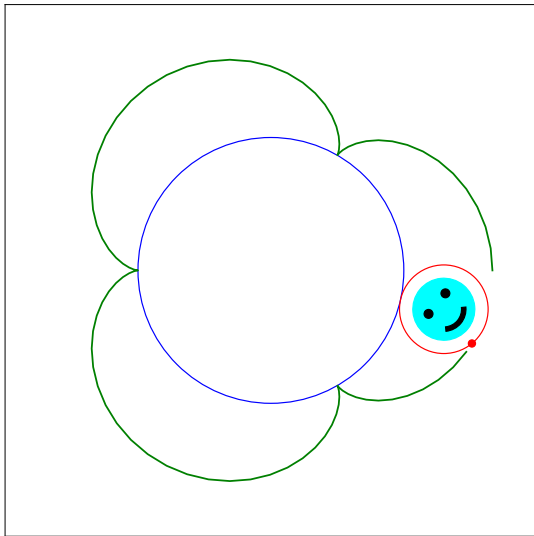
Animation of Coin Rotation Paradox

Rotation: 346.6°



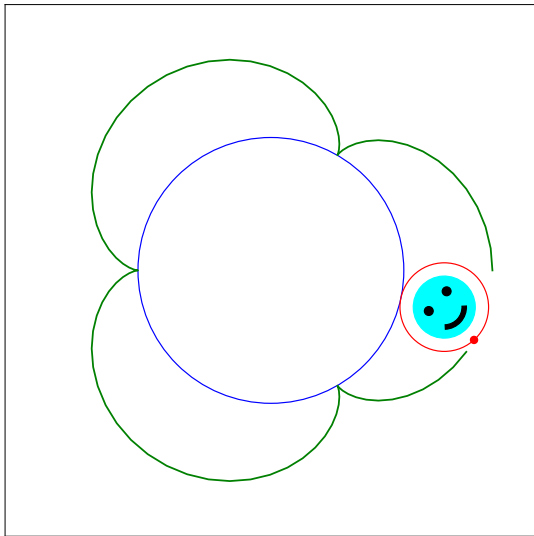
Animation of Coin Rotation Paradox

Rotation: 347.3°



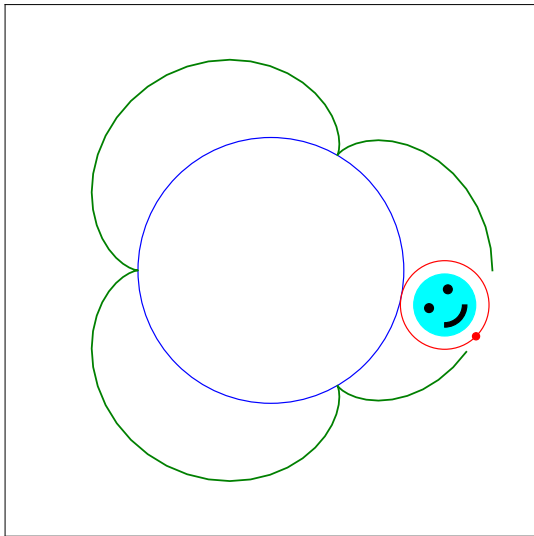
Animation of Coin Rotation Paradox

Rotation: 348.0°



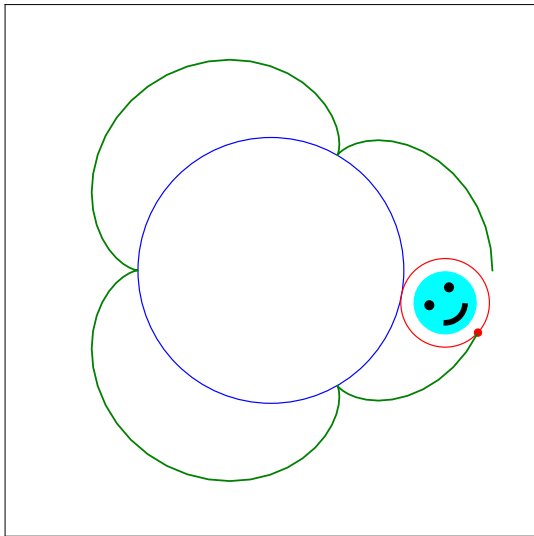
Animation of Coin Rotation Paradox

Rotation: 348.8°



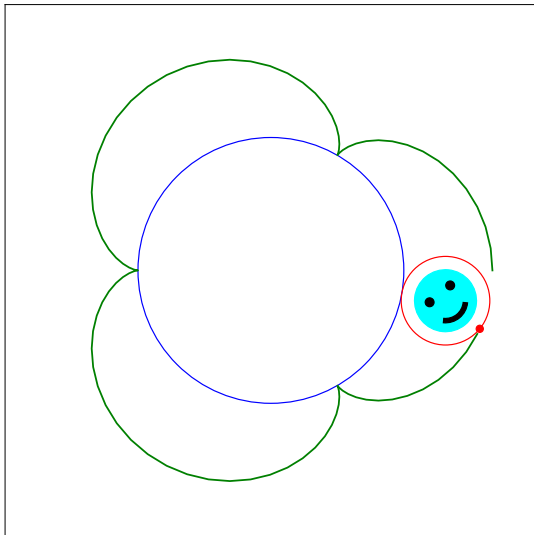
Animation of Coin Rotation Paradox

Rotation: 349.5°



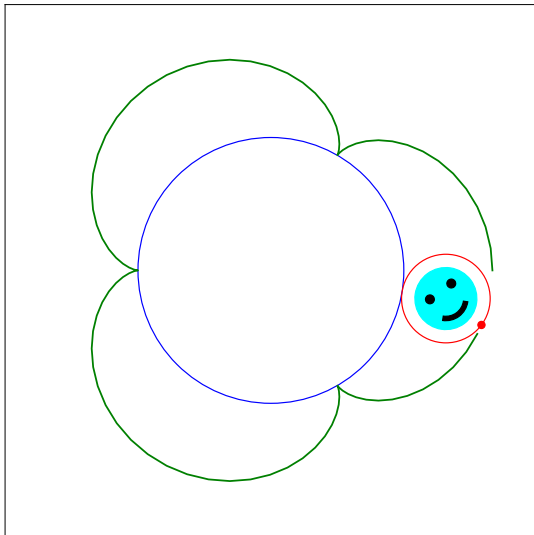
Animation of Coin Rotation Paradox

Rotation: 350.2°



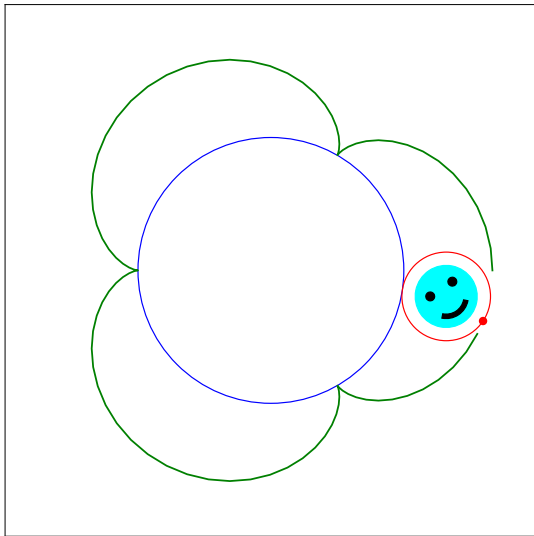
Animation of Coin Rotation Paradox

Rotation: 350.9°



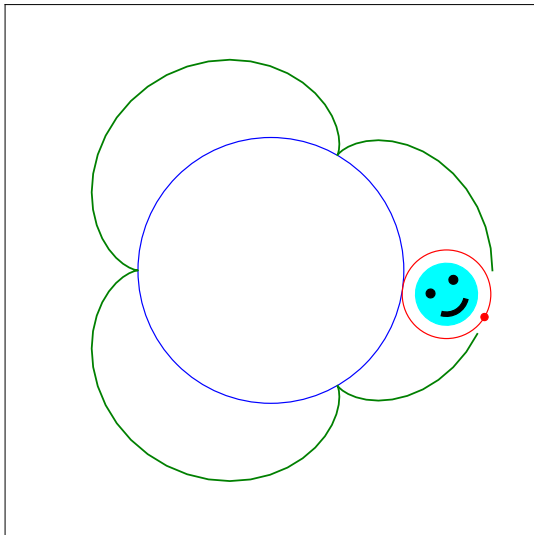
Animation of Coin Rotation Paradox

Rotation: 351.6°



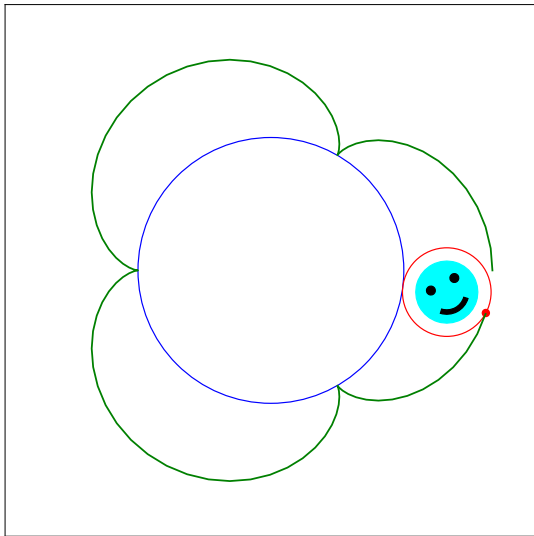
Animation of Coin Rotation Paradox

Rotation: 352.3°



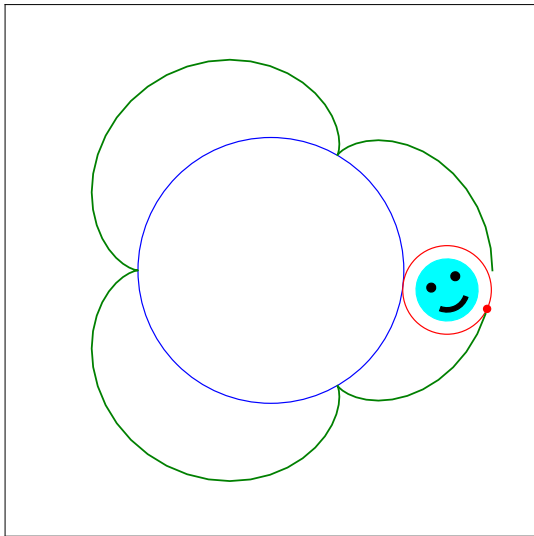
Animation of Coin Rotation Paradox

Rotation: 353.0°



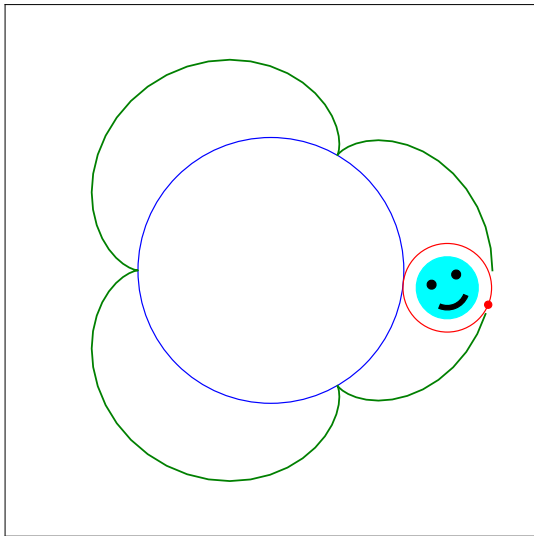
Animation of Coin Rotation Paradox

Rotation: 353.7°



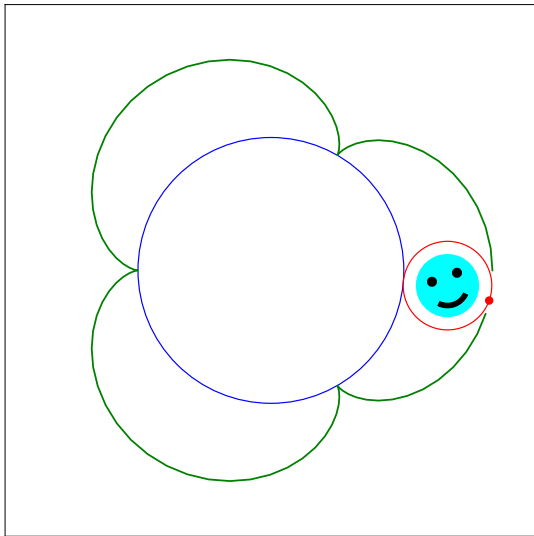
Animation of Coin Rotation Paradox

Rotation: 354.4°



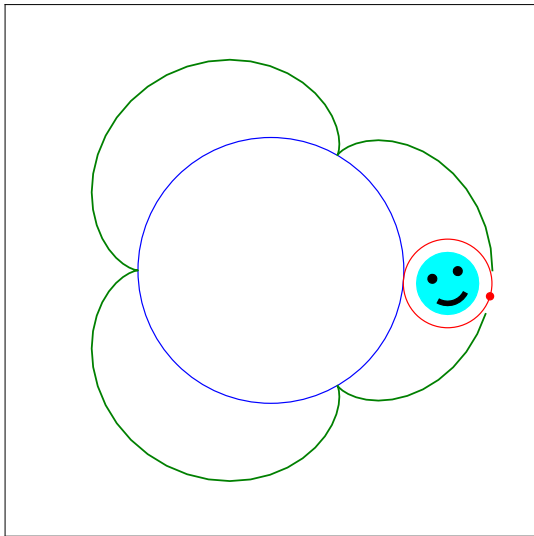
Animation of Coin Rotation Paradox

Rotation: 355.1°



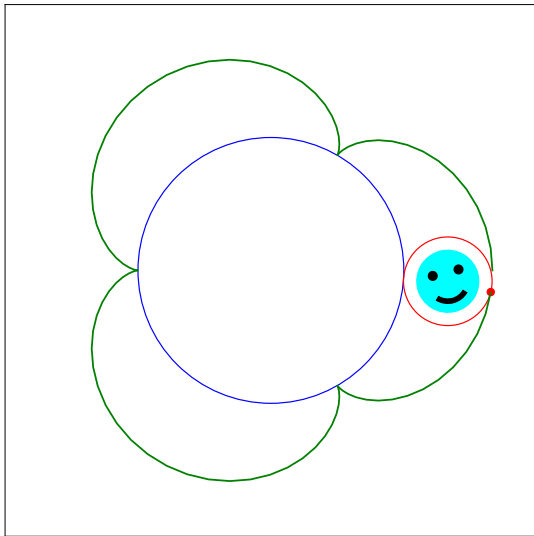
Animation of Coin Rotation Paradox

Rotation: 355.8°



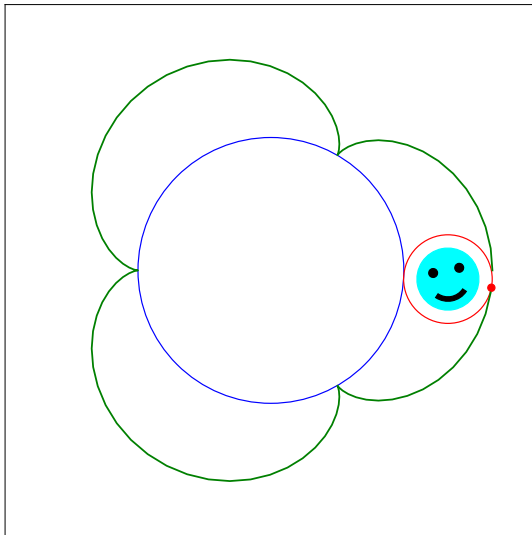
Animation of Coin Rotation Paradox

Rotation: 356.5°



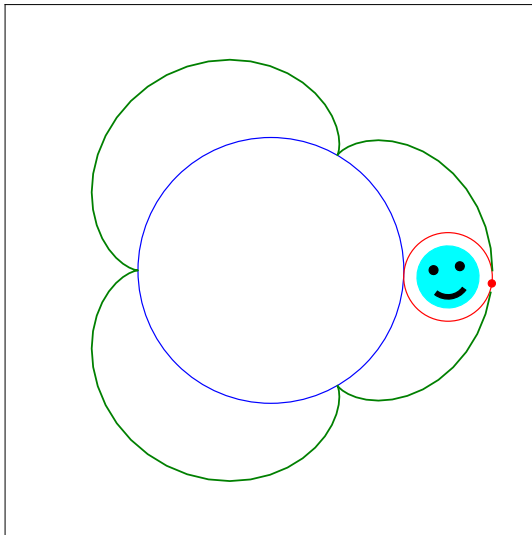
Animation of Coin Rotation Paradox

Rotation: 357.2°



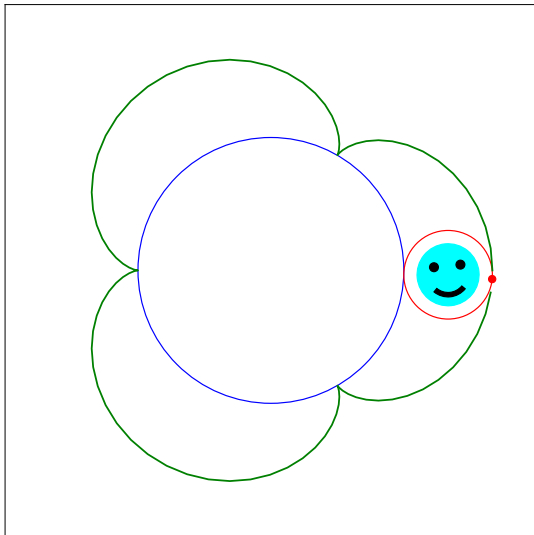
Animation of Coin Rotation Paradox

Rotation: 357.9°



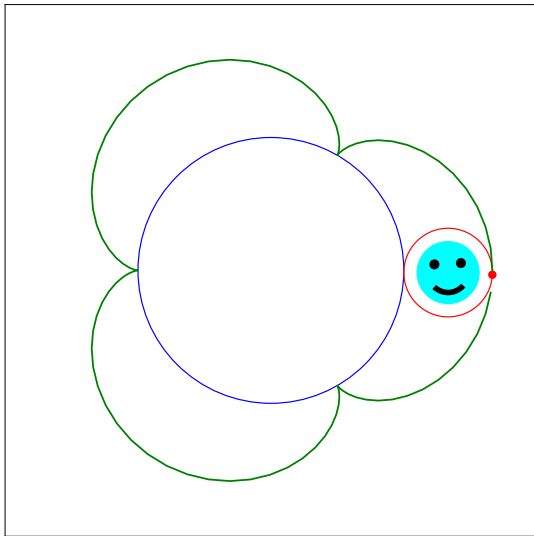
Animation of Coin Rotation Paradox

Rotation: 358.6°



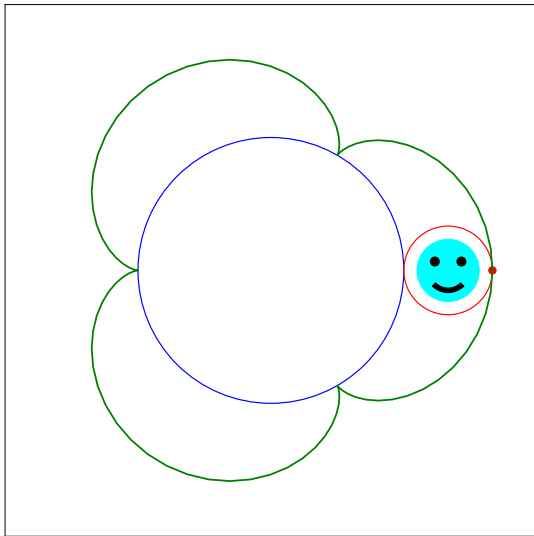
Animation of Coin Rotation Paradox

Rotation: 359.3°



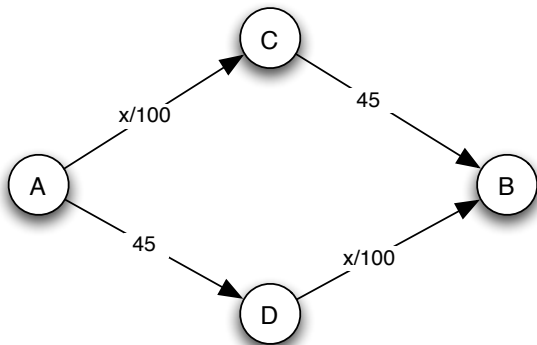
Animation of Coin Rotation Paradox

Rotation: 360.0°

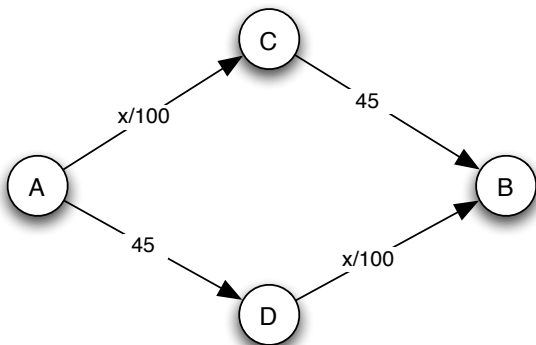


Braess Paradox

A Typical Case

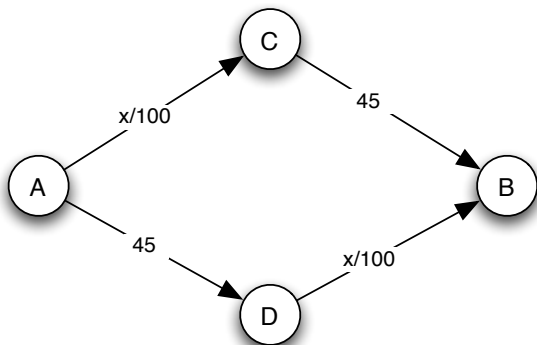


A Typical Case



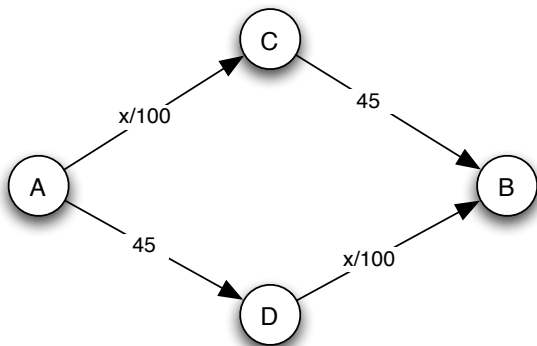
- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it

A Typical Case



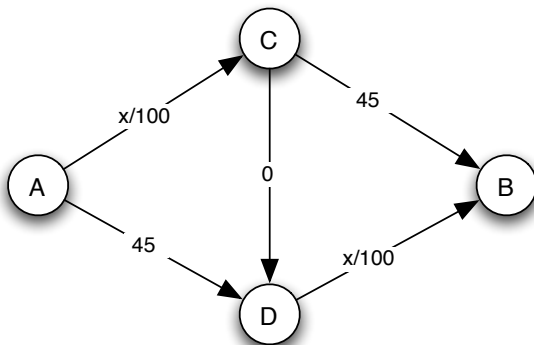
- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it
- Suppose there are 4000 cars need to get from A to B

A Typical Case

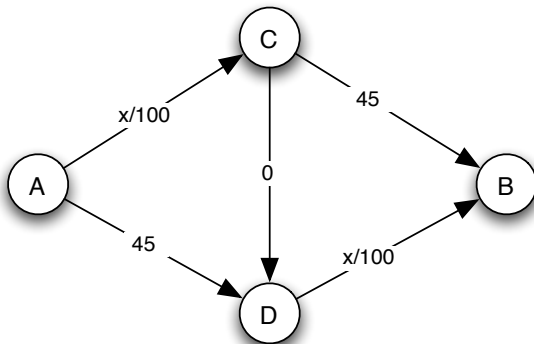


- A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it
- Suppose there are 4000 cars need to get from A to B
- They divide evenly over the two routes at equilibrium; the travel time is $45 + 2000/100 = 65$ mins

A Typical Case: Cont'd

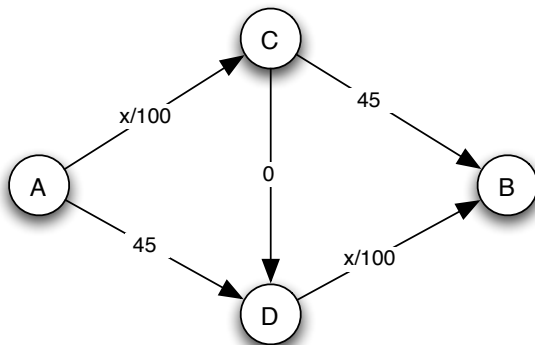


A Typical Case: Cont'd



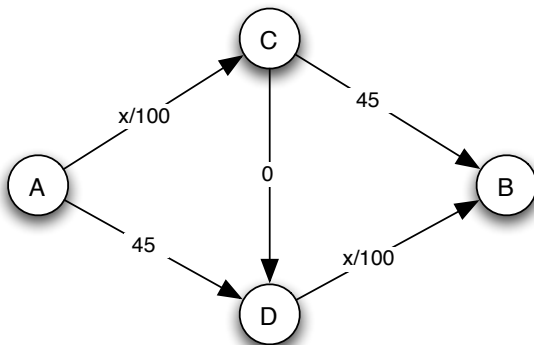
- Now a very fast edge is added from C to D to the previous highway network

A Typical Case: Cont'd



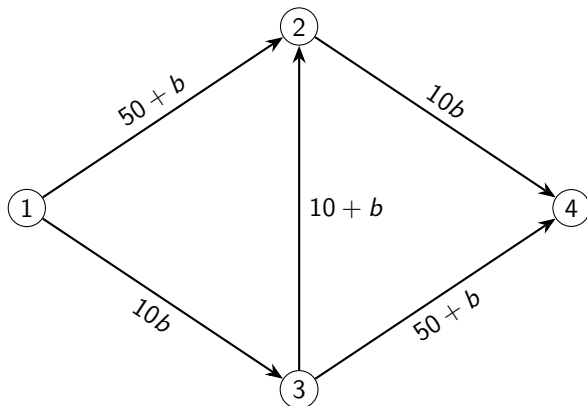
- Now a very fast edge is added from C to D to the previous highway network
- At equilibrium, every user uses the route through C and D

A Typical Case: Cont'd

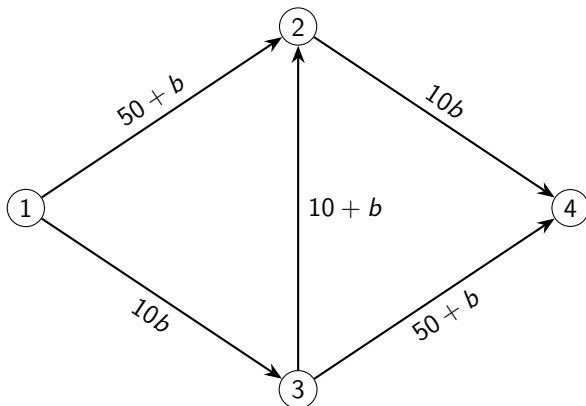


- Now a very fast edge is added from C to D to the previous highway network
- At equilibrium, every user uses the route through C and D
- As a result, the travel time is $4000/10 + 0 + 4000/100 = 80$ mins!

[Braess et al., 2005] Example

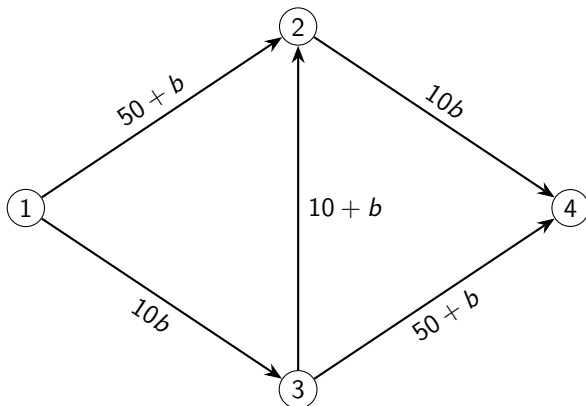


[Braess et al., 2005] Example



- Assume a flow of 6 units (e.g., 6000 vehicles) must travel from 1 to 4

[Braess et al., 2005] Example



- Assume a flow of 6 units (e.g., 6000 vehicles) must travel from 1 to 4
- Three paths exist: $B_1 = 124$, $B_2 = 1324$, $B_3 = 134$

[Braess et al., 2005] Example: Cont'd

- Case 1: All Paths Open. Split the flow equally (2 units each):

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$$b_{12} = 2, b_{24} = 4, b_{34} = 2, b_{13} = 4, b_{32} = 2$$

$$d_{12} = 52, d_{24} = 40, d_{34} = 52, d_{13} = 40, d_{32} = 12$$

$$L(B_1) = L(B_2) = L(B_3) = 92$$

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$$\begin{aligned}b_{12} &= 2, \quad b_{24} = 4, \quad b_{34} = 2, \quad b_{13} = 4, \quad b_{32} = 2 \\d_{12} &= 52, \quad d_{24} = 40, \quad d_{34} = 52, \quad d_{13} = 40, \quad d_{32} = 12 \\L(B_1) &= L(B_2) = L(B_3) = 92\end{aligned}$$

All paths are critical (maximal length). This distribution is stable: Switching paths increases load and travel time beyond 92.

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All paths are critical (maximal length). This distribution is stable: Switching paths increases load and travel time beyond 92.

- Case 2: Ban Edge k_{32} . Split flow equally between B_1 and B_3 (3 units each):

$$\begin{aligned}b_{12} &= b_{24} = b_{13} = b_{34} = 3 \\d_{12} &= 53, \quad d_{24} = 30, \quad d_{13} = 30, \quad d_{34} = 53 \\L(B_1) &= L(B_3) = 83\end{aligned}$$

[Braess et al., 2005] Example: Cont'd

- Case 1: All Paths Open. Split the flow equally (2 units each):

$$\begin{aligned}b_{12} &= 2, \quad b_{24} = 4, \quad b_{34} = 2, \quad b_{13} = 4, \quad b_{32} = 2 \\d_{12} &= 52, \quad d_{24} = 40, \quad d_{34} = 52, \quad d_{13} = 40, \quad d_{32} = 12 \\L(B_1) &= L(B_2) = L(B_3) = 92\end{aligned}$$

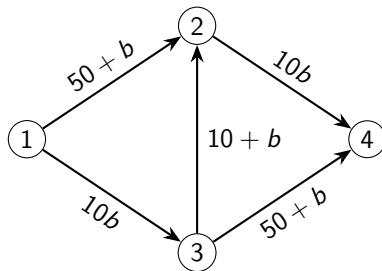
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- Case 2: Ban Edge k_{32} . Split flow equally between B_1 and B_3 (3 units each):

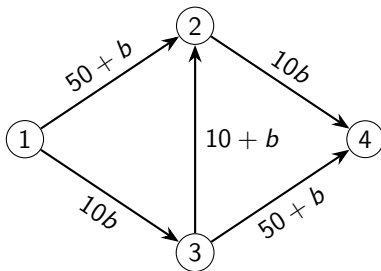
$$\begin{aligned}b_{12} &= b_{24} = b_{13} = b_{34} = 3 \\d_{12} &= 53, \quad d_{24} = 30, \quad d_{13} = 30, \quad d_{34} = 53 \\L(B_1) &= L(B_3) = 83\end{aligned}$$

Despite losing a connection, all paths are shorter!

[Braess et al., 2005] Example: Selfish Deviation

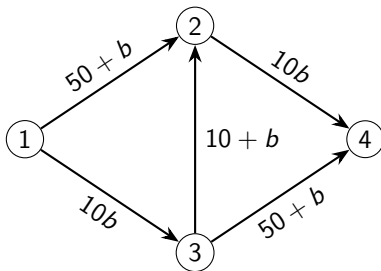


[Braess et al., 2005] Example: Selfish Deviation



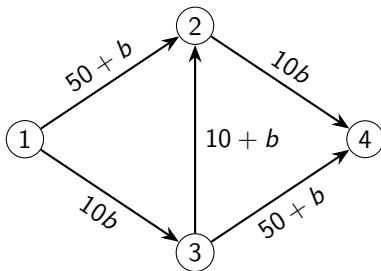
- Recall that $B_1 = 124$, $B_2 = 1324$, $B_3 = 134$

[Braess et al., 2005] Example: Selfish Deviation



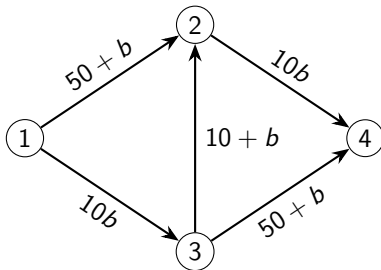
- Recall that $B_1 = 124$, $B_2 = 1324$, $B_3 = 134$
- A single vehicle ignoring the ban on k_{32} (where $d_{32} = 10$) takes B_2 with $L(B_2) = 70$, gaining an advantage.

[Braess et al., 2005] Example: Selfish Deviation



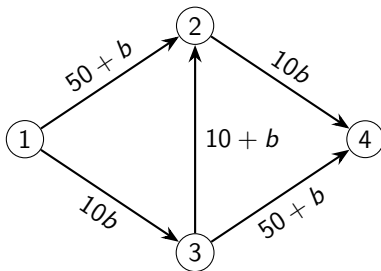
- Recall that $B_1 = 124$, $B_2 = 1324$, $B_3 = 134$
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[Braess et al., 2005] Example: Selfish Deviation



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- A single vehicle ignoring the ban on k_{32} (where $d_{32} = 10$) takes B_2 with $L(B_2) = 70$, gaining an advantage.
- If the ban is lifted, many use B_2 , reverting to the original state.
- Consider an intermediate step: 5 units on B_1 and B_3 , 1 on B_2 , then $L(B_1) = L(B_3) = 87.5$, $L(B_2) = 82.5$.

[Braess et al., 2005] Example: Selfish Deviation



- Recall that $B_1 = 124$, $B_2 = 1324$, $B_3 = 134$
- A single vehicle ignoring the ban on k_{32} (where $d_{32} = 10$) takes B_2 with $L(B_2) = 70$, gaining an advantage.
- If the ban is lifted, many use B_2 , reverting to the original state.
- Consider an intermediate step: 5 units on B_1 and B_3 , 1 on B_2 , then $L(B_1) = L(B_3) = 87.5$, $L(B_2) = 82.5$.
- B_2 users clog k_{13} and k_{24} (factor 10 in load-time relation), worsening times for B_1 and B_3 beyond 83; Yet B_2 remains shortest, attracting more traffic and degrading the system for all.

Real-Life Examples

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- The mechanical analogy: the spring paradox

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- Road closure in NY: Times and Herald Squares pedestrian plaza (2009 —)



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- Cheonggyecheon restoration project (2003 —)

Real-Life Examples

- The mechanical analogy: the spring paradox
- Road closure in NY: Times and Herald Squares pedestrian plaza (2009 —)



- Cheonggyecheon restoration project (2003 —)
 - Replaced a six lane highway with a five mile long park, traffic flow improved

The Social Cost of Traffic at Equilibrium

A traffic network is a directed graph with

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- **Nodes:** Start and destination points for drivers.

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- **Edges:** Roads with travel-time functions $T_e(x) = a_e x + b_e$, where x is the number of drivers.

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- **(Traffic) Pattern:** Path choices for all drivers.

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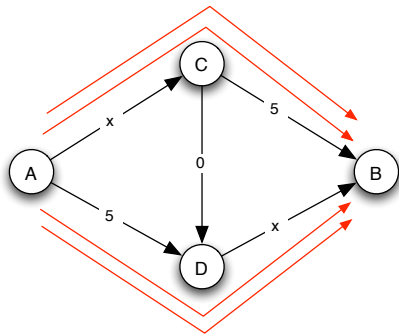
- **Nodes:** Start and destination points for drivers.
- **Edges:** Roads with travel-time functions $T_e(x) = a_e x + b_e$, where x is the number of drivers.
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- **Social Cost:** $\text{Social-Cost}(Z) = \sum_{\text{drivers}} (\text{travel time})$, summed over all drivers in pattern Z .

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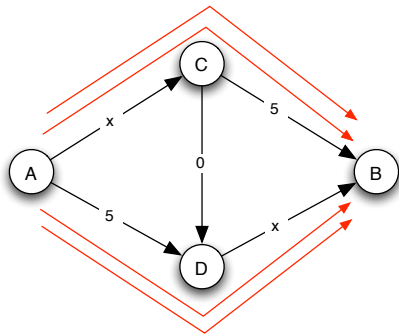
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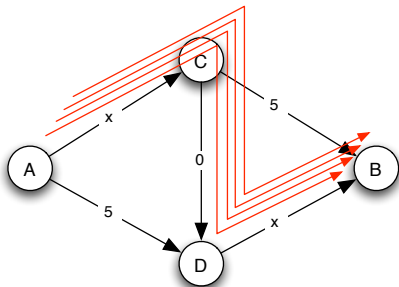
- **Nodes:** Start and destination points for drivers.
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- **(Traffic) Pattern:** Path choices for all drivers.
- **Social Cost:** $\text{Social-Cost}(Z) = \sum_{\text{drivers}} (\text{travel time})$, summed over all drivers in pattern Z .
- **Social Optimum:** Pattern minimizing social cost.
- **Nash Equilibrium:** No driver can reduce their travel time by switching paths, given others' choices.



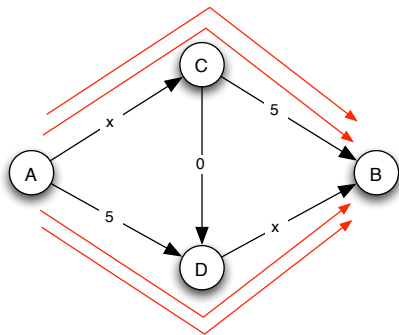
(a) The social optimum.



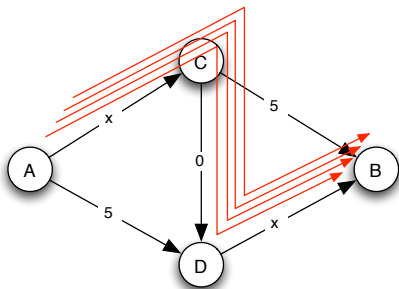
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(b) The Nash equilibrium.



(a) The social optimum.



(b) The Nash equilibrium.

Figure: A version of Braess's Paradox: In the socially optimal traffic pattern, the social cost is 28, while in the unique Nash equilibrium, the social cost is 32.

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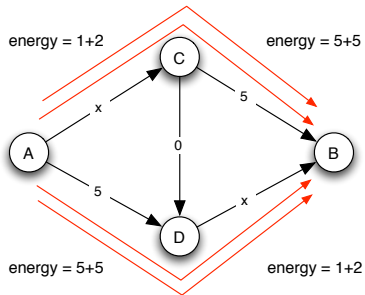
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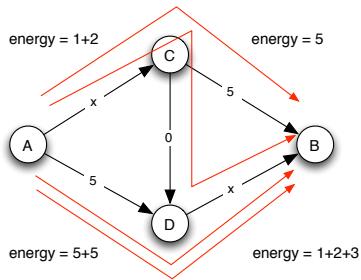
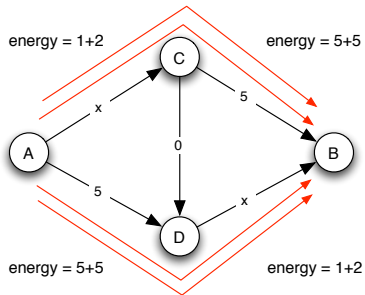
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- Social cost can increase or decrease with best-response steps (e.g., from 28 to 32 in the Braess example), but potential energy strictly decreases, serving as a progress measure.

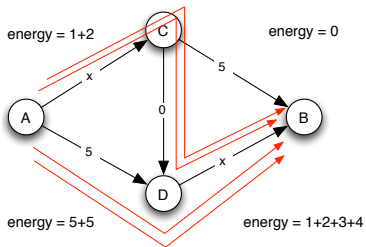
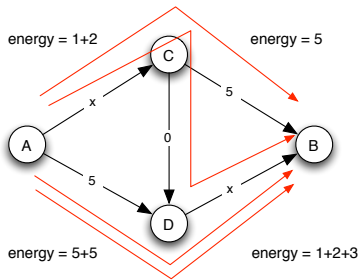
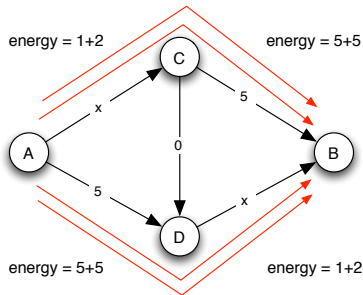
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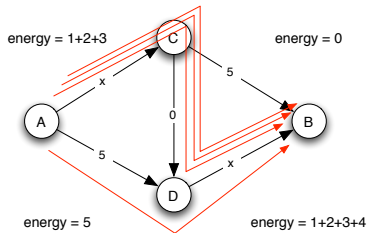
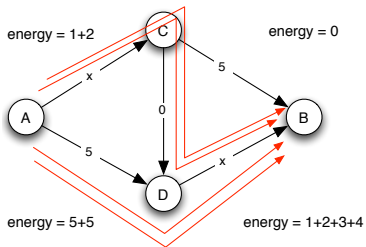
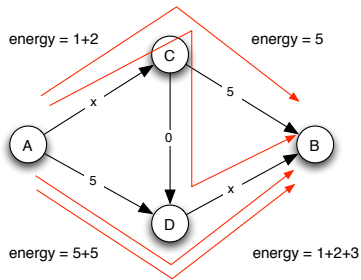
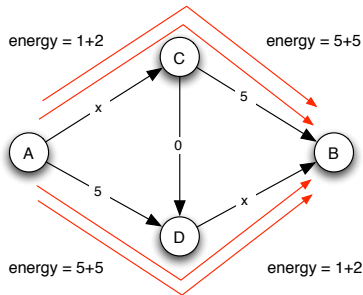
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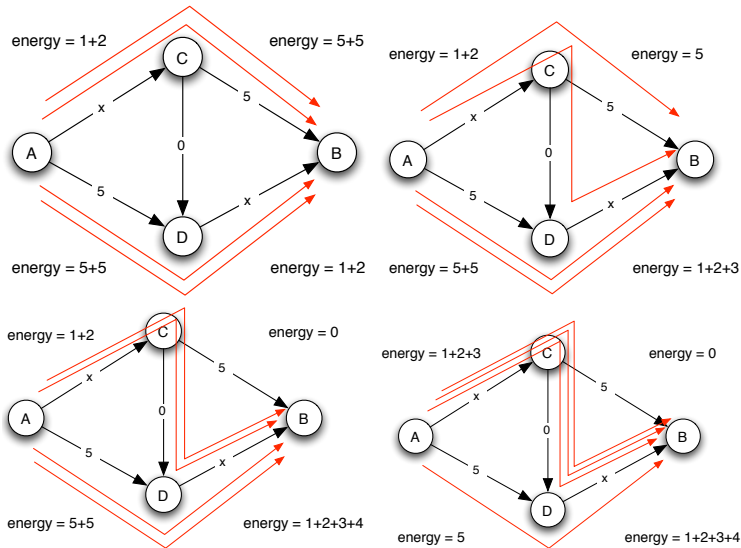


Figure: Steps of best-response dynamics with potential energy changes.

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- Implications: Network design and tolls can mitigate inefficiencies.

References

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