Options & Derivatives

The One Period Model

- time t: $t = 0, 1$
- (deterministic) bond B_t: $B_0 = 1,\ B_1 = 1 + R$
- (stochastic) stock S_t: $S_0 = s > 0,\ S_1 = \begin{cases} s \cdot u & \text{with prob. } p_u \\ s \cdot d & \text{with prob. } p_d \end{cases} \equiv s\,Z$:
  $u > d$, $p_u + p_d = 1$.
- The value V_t^h of the portfolio $h = (x, y)$, $x, y \in \mathbb{R}$ at time t:
  $V_t^h = x\,B_t + y\,S_t$ — $V_0^h = x + y\,s,\ V_1^h = x(1+R) + y\,s\,Z$
- Arbitrage portfolio h: $V_0^h = 0$, $V_1^h > 0$ with prob. 1.

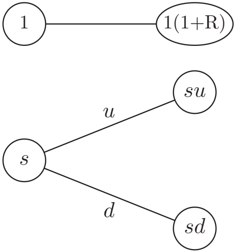


Figure: Asset Dynamics of One Period Model.

Portfolios and Arbitrage I

Theorem

The one period model is arbitrage free $\iff$ $u \geqslant 1 + R \geqslant d$.

Proof

($\Longrightarrow$)

- Suppose $u \geqslant 1 + R \geqslant d$ does not hold, then $1 + R > u$ or $d > 1 + R$.
- If $1 + R > u$, then $s(1 + R) > s\,u$ and a priori $s(1 + R) > s\,d$.
- Consider $h = (s, -1)$, then $V_0^h = s \cdot 1 + (-1) \cdot s = 0$,
  $V_1^h = s(1 + R) - s \cdot Z > 0$, an arbitrage.
- If $d > 1 + R$, then $s\,d > s(1 + R)$ and a priori $s\,u > s(1 + R)$.
- Consider $h = (-s, 1)$, then $V_0^h = (-s) \cdot 1 + 1 \cdot s = 0$,
  $V_1^h = -s(1 + R) + s \cdot Z > 0$, an arbitrage.

Portfolios and Arbitrage II

Theorem

The one period model is arbitrage free $\iff u \geqslant 1 + R \geqslant d$.

Proof

$(\Longleftarrow)$

- Arbitrage $h = (x, y)$: $V_0^h = 0$.
- $x + s \cdot y = 0 \implies x = -s \cdot y$.
- $V_1^h = \begin{cases} y\, s(u - (1+R)), & Z = u \\ y\, s(d - (1+R)), & Z = d \end{cases}$
- If $y > 0$: from $V_1^h > 0 \implies u > 1 + R$ and $d > 1 + R$; a contradiction.
- If $y < 0$: from $V_1^h > 0 \implies u < 1 + R$ and $d < 1 + R$; a contradiction.

Risk-Neutral / Martingale Measure and Probabilities

- Observation: $u \geqslant 1+R \geqslant d \implies 1+R$ is a convex combination of u and d
- $\exists\, q_u, q_d \geqslant 0,\ q_u + q_d = 1$ s.t. $1+R = q_u \cdot u + q_d \cdot d$
- Define a new probability measure Q and the associated expectation E^Q s.t.

$$Q(Z=u) = q_u, \quad Q(Z=d) = q_d$$
$$\frac{1}{1+R}\,\mathsf{E}^Q\, S_1 = \frac{1}{1+R}(q_u \cdot s\,u + q_d \cdot s\,d) = \frac{1}{1+R} \cdot s(1+R) = s$$

Definition

- **Risk-Neutral / Martingale Measure**: A measure Q satisfies $S_0 = \dfrac{1}{1+R}\,\mathsf{E}^Q\, S_1.$
- **Martingale Probabilities**: $q_u = \dfrac{(1+R)-d}{u-d},\ q_d = \dfrac{u-(1+R)}{u-d}$

Contingent Claims I

Definition

- A **contingent claim** X is of the form $X = \Phi(Z)$
- Stochastic Z with **contract function** $\Phi(\cdot)$
- **Price** of X at time t: $\Pi(t; X)$

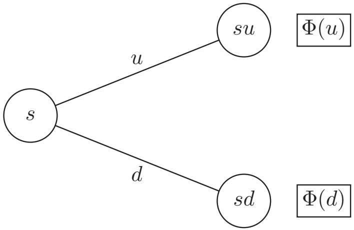


Figure: The Contingent Claim.

Contingent Claims II

Example (European Call Option with Strike K)

Assume $s\,u > K > s\,d$. At $t = 1$,

- Exercise the option if $S_1 > K$.
  - Pay K to get the stock and sell it at $s\,u$, thus making net profit $s\,u - K$.
- Do nothing if $S_1 < K$.

$$X = \begin{cases} s\,u - K, & Z = u \\ 0, & Z = d \end{cases}, \qquad \begin{cases} \Phi(u) = s\,u - K \\ \Phi(d) = 0 \end{cases}$$

Definition

- A contingent claim X is said to be **reachable** if there exists a portfolio h such that $V_1^h = X$ with probability 1; this portfolio h is called a **hedging** or **replicating** portfolio.
- If all claims can be replicated we say the market is **complete**.

Contingent Claims III

Theorem (Pricing Principle)

If a claim X is reachable with replicating portfolio h, then the "reasonable" price of X is given by $\Pi(t; X) = V_t^h,\ t = 0, 1.$

Theorem

An arbitrage free one period model is complete.

Proof

Fixed any $\Phi(\cdot)$, show that $\exists\, h = (x, y)$ s.t.

$$V_1^h = \begin{cases} \Phi(u) & Z = u, \\ \Phi(d) & Z = d. \end{cases} \implies x(1+R) + y\, s\, u = \Phi(u),\ x(1+R) + y\, s\, d = \Phi(d).$$

Solve for x, y: $x = \dfrac{1}{1+R}\, \dfrac{u\Phi(d) - d\,\Phi(u)}{u-d}, \quad y = \dfrac{1}{s}\, \dfrac{\Phi(u) - \Phi(d)}{u-d}.$

Risk Neutral Valuation

- From Pricing Principle ($\Pi(t; X) = V_t^h$, $t = 0, 1$)

$$\begin{aligned}
\Pi(0; X) = V_0^h &= x + s\,y \\
&= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\,\Phi(u)}{u-d} + s \cdot \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d} \\
&= \frac{1}{1+R} \left\{ \frac{(1+R) - d}{u-d}\,\Phi(u) + \frac{u - (1+R)}{u-d}\,\Phi(d) \right\} \\
&= \frac{1}{1+R} \{ q_u\,\Phi(u) + q_d\,\Phi(d) \} \equiv \frac{1}{1+R}\,\mathsf{E}^Q\,X
\end{aligned}$$

Theorem (The Risk Neutral Valuation Principle)

If the one period binomial model is arbitrage-free, then the price of X is $\Pi(0; X) = \dfrac{1}{1+R}\,\mathsf{E}^Q\,X$.

The Multiperiod Model

- time t: $t = 0, 1, 2, \dots, T$
- (deterministic) bond B_t with $B_0 = 1,\ B_{n+1} = (1+R)B_n$
- (stochastic) stock S_t with $S_0 = s > 0,\ S_{n+1} = Z_n\, S_n$ where $Z_0, Z_1, Z_2, \dots, Z_{T-1}$ are iid with $\mathsf{P}(Z_n = u) = p_u,\ \mathsf{P}(Z_n = d) = p_d$

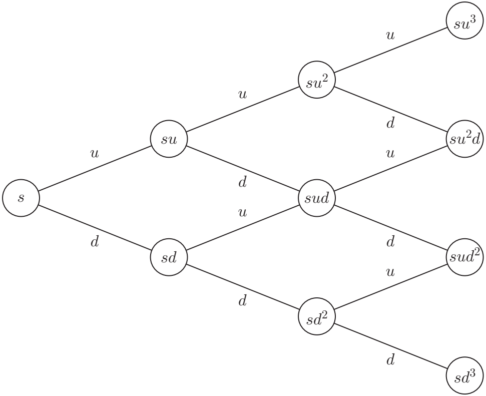


Figure: Asset Dynamics of Multiperiod Model: “Recombining” Tree.

Portfolios and Arbitrage

Definition

The portfolio $h_t \equiv (x_t, y_t)$; The value $V_t^{h_t}$ of portfolio h_t at time t is $V_t^{h_t} = x_t\, B_t + y_t\, S_t$.

- Hereafter we write V_t^h instead of the cumbersome $V_t^{h_t}$.
- x_t is the amount which we invest in the bank at time $t-1$ and keep until t.

Definition

Self-financing portfolio $h_t = (x_t, y_t)$:
$x_t\,(1+R) + y_t\,S_t = x_{t+1} + y_{t+1}\,S_t, \quad \forall t = 0, 1, \ldots, T-1.$

Contingent Claims

Definition

- Arbitrage: there exists a self-financing portfolio h_t with $V_0^h = 0$, $\mathsf{P}(V_T^h \geqslant 0) = 1$, $\mathsf{P}(V_T^h > 0) > 0$.
- A contingent claim X is said to be **reachable** if there exists a self-financing portfolio h such that $V_T^h = X$ with probability 1; this portfolio h is called a **hedging** or **replicating** portfolio.
- If all claims can be replicated we say the market is **complete**.

Theorem (Pricing Principle)

If a claim X is reachable with replicating (and self-financing) portfolio h, then the "reasonable" price process of X is given by $\Pi(t; X) = V_t^h,\ t = 0, 1, 2, \ldots T$.

Theorem

An arbitrage-free multiperiod model is complete.

Theorem (Binomial Algorithms)

- Given a contingent claim $X = \Phi(S_T)$; let $V_t(k)$ denotes the value of the replicating portfolio at node (t,k), then $V_t(k)$ is computed recursively by

$$V_T(k) = \Phi(s\, u^k\, d^{T-k})$$
$$V_t(k) = \frac{1}{1+R}\left\{q_u\, V_{t+1}(k+1) + q_d\, V_{t+1}(k)\right\}$$

- The martingale probabilities q_u, q_d are $q_u = \dfrac{(1+R)-d}{u-d}$, $q_d = \dfrac{u-(1+R)}{u-d}$
- The replicating portfolio $h_t = (x_t, y_t)$ is

$$x_t(k) = \frac{1}{1+R}\,\frac{u\,V_t(k) - d\,V_t(k+1)}{u-d}, \quad y_t(k) = \frac{1}{S_{t-1}}\,\frac{V_t(k+1) - V_t(k)}{u-d}$$

- The arbitrage-free price of a contingent claim X at $t=0$ is

$$\Pi(0;X) = \frac{1}{(1+R)^T}\,\mathsf{E}^Q\, X = \frac{1}{(1+R)^T}\cdot\sum_{k=0}^{T}\binom{T}{k} q_u^k\, q_d^{T-k}\,\Phi(s\,u^k\,d^{T-k})$$

Example

Given $T = 3, S_0 = 80, K = 80, u = 1.5, d = 0.5, p_u = 0.6, p_d = 0.4, R = 0$, compute the European call option price and the replicating portfolio of each node.

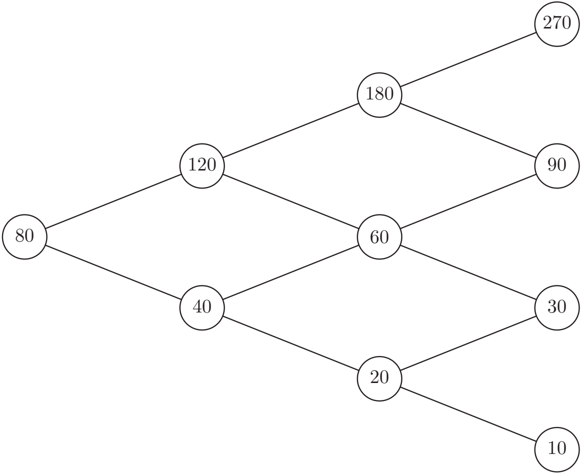


Figure: Asset Dynamics of the Example.

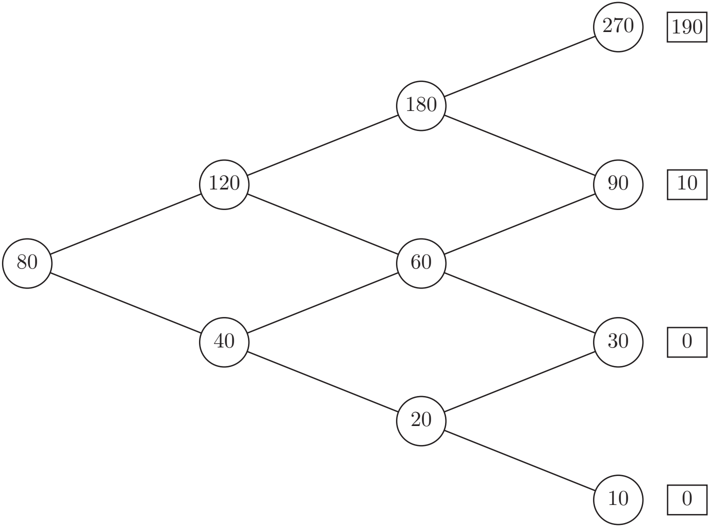


Figure: Payoff at the End of Terms.

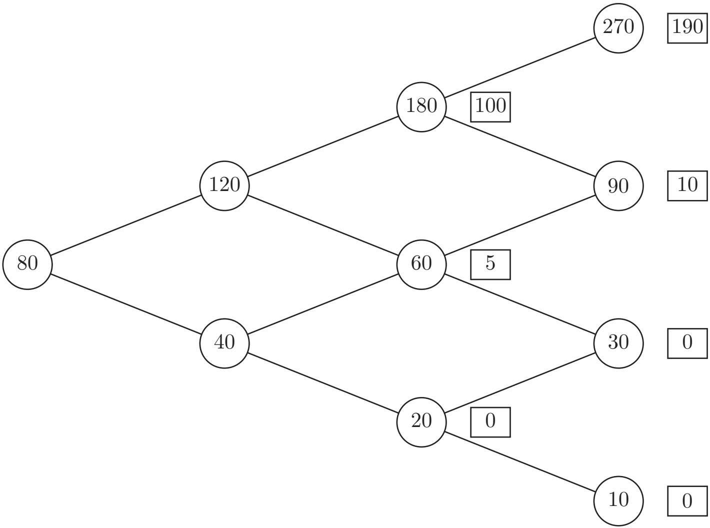


Figure: Iterated Computation of $\Pi(t;X):\ \Pi(t-1;X) \equiv \dfrac{1}{1+R}\,\mathsf{E}^{Q}\{\Pi(t;X)\}$.

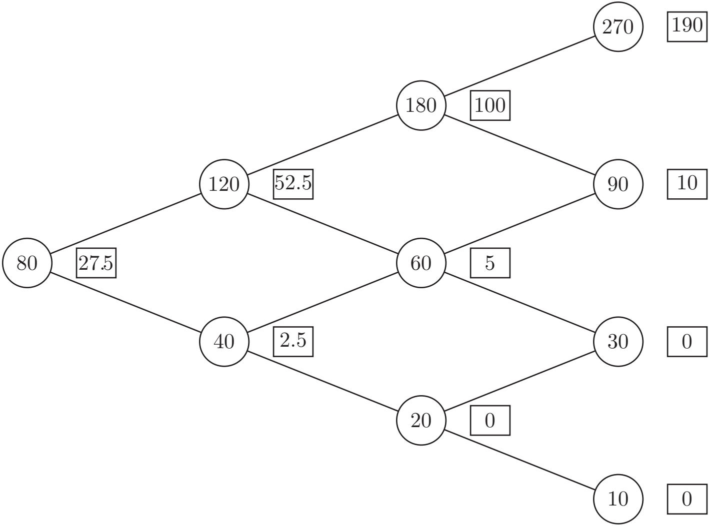


Figure: The Completed $\Pi(t; X)$.

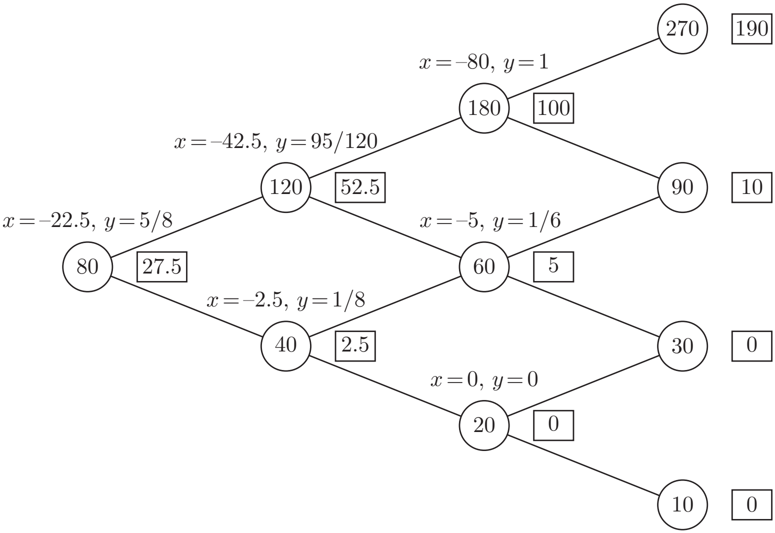


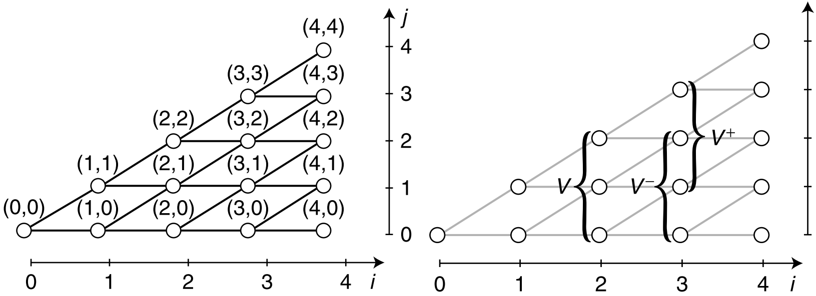
Figure: Replicating $h_t = (x_t, y_t)$: $x_t(k) = \frac{1}{1+R} \frac{u\,V_t(k) - d\,V_t(k+1)}{u-d}$, $y_t(k) = \frac{1}{S_{t-1}} \frac{V_t(k+1) - V_t(k)}{u-d}$

Algorithmic Considerations

$$\Pi(0;X) = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^{T} \binom{T}{k} q_u^k\, q_d^{T-k} \Phi(s\, u^k\, d^{T-k})$$

For big T the formula can't be directly used because of the binomial coefficient

$$V_T(k) = \Phi(s\, u^k\, d^{T-k}), \quad V_t(k) = \frac{1}{1+R}\left\{q_u\, V_{t+1}(k+1) + q_d\, V_{t+1}(k)\right\}$$



Python Code Illustration: Common Parts

```python
import numpy as np

S0 = 80; r = 0; K = 80; u = 1.5; d = 0.5;
q = (1 - d) / (u - d); M = 3;
df = 1     # discount factor per time interval
# exhibit stock paths
S = np.zeros((M + 1, M + 1), dtype=np.float)
S[0, 0] = S0
for j in range(1, M + 1, 1):
    for i in range(j + 1):
        S[i, j] = S[0, 0] * (u ** (j - i)) * (d ** i)
```

Python Codes: Traditional Loops

```
iv = np.zeros((M + 1, M + 1), dtype=np.float); z = 0  # inner values
for j in range(0, M + 1, 1):
    for i in range(z + 1):
        iv[i, j] = round(max(S[i, j] - K, 0), 8)
    z += 1

pv = np.zeros((M + 1, M + 1), dtype=np.float)         # present values
pv[:, M] = iv[:, M]
z = M + 1
for j in range(M - 1, -1, -1):
    z -= 1
    for i in range(z):
        pv[i, j] = (q * pv[i, j + 1] + (1 - q) * pv[i + 1, j + 1]) * df
```

Python Codes: Vectorized Loops

```
import numpy as np
from params import *
import time

mu = np.arange(M + 1)
mu = np.resize(mu, (M + 1, M + 1))
md = np.transpose(mu)
mu = u ** (mu - md)
md = d ** md
S = S0 * mu * md

start_time = time.time()

# present value array initialized with inner values
pv = np.maximum(S - K, 0)
z = 0
for i in range(M - 1, -1, -1): # backwards induction
    pv[0:M-z, i] = (q * pv[0:M-z, i+1] + (1 - q) * pv[1:M-z+1, i+1]) * df
    z += 1

print(pv)
print('Value of European call option is %8.3f' % pv[0, 0])
print('vector elapsed: %f seconds.' % (time.time() - start_time,))
```

Example

Stock A has the following characteristics:

- The current price is 40.
- The price of a 35-strike 1-year European call option is 9.12.
- The price of a 40-strike 1-year European call option is 6.22.
- The price of a 45-strike 1-year European call option is 4.08.

The annual risk-free interest rate is 8%. Let S be the price of the stock one year from now. All call positions being compared are long. Determine the range of S such that the 45-strike call produces a higher profit than the 40-strike call but a lower profit than the 35-strike call.

Solution

Denote by p_K the profit of a K-strike 1-year European call. We first express each p_K in terms of S:

$$
\begin{aligned}
p_{35} &= (S-35)_+ - 9.12 \cdot 1.08 = (S-35)_+ - 9.8496 \\
p_{40} &= (S-40)_+ - 6.22 \cdot 1.08 = (S-40)_+ - 6.7176 \\
p_{45} &= (S-45)_+ - 4.08 \cdot 1.08 = (S-45)_+ - 4.4064
\end{aligned}
$$

To find the range for S such that $p_{40} < p_{45} < p_{35}$, consider the followings:

- If $S < 35$ then all the $(\cdot)_+$ vanish, we have $p_{35} < p_{40} < p_{45}$, a contradiction.
- If $35 \leqslant S < 40$, then $p_{35} = S - 44.8496$, $p_{40} = -6.7176$, $p_{45} = -4.4064$. To make $p_{35} > p_{45}$, $S - 44.8496 > -4.4064 \implies S > 40.4432$, a contradiction.
- If $40 \leqslant S < 45$, then $p_{35} = S - 44.8496$, $p_{40} = S - 46.7176$, $p_{45} = -4.4064$. To make $p_{35} > p_{45}$, $S - 44.8496 > -4.4064 \implies S > 40.4432$. To make $p_{45} > p_{40}$, $-4.4064 > S - 46.7176 \implies 42.3112 > S$.
- If $S \geqslant 45$, $p_{40} < p_{45} \implies S - 46.71716 < S - 49.4064$ cannot hold.

So $40.4432 < S < 42.3112$.

Example

Investor A wrote a 104-strike 1-year call option whose price is 2. Investor B entered into a 1-year forward with a forward price of 105. The anually compounded risk-free interest rate is 5%, and it turns out that investors A and B earn the same profit. What is the year 1 stock price?

Solution

Equating the profits of the short call and long forward,

$$2 \cdot 1.05 - (S_1 - 104)_+ = S_1 - 105$$

Consider the followings:

- If $S_1 < 104$ then $2 \cdot 1.05 = S_1 - 105 \implies S_1 = 107.1$, a contradiction to $S_1 < 104$.
- If $S_1 \geqslant 104$ then $2 \cdot 1.05 - (S_1 - 104) = S_1 - 105 \implies S_1 = 105.55$.

So $S_1 = 105.55$.

Example

For a certain stock, Investor A purchases a 45-strike call option while Investor B purchases a 135-strike put option. Both options are European with the same expiration date. Assume that there are no transaction costs. If the final stock price at expiration is S, Investor A's payoff will be 12. Calculate Investor B's payoff at expiration, if the final stock price is S.

Solution

$S - 45 = 12 \implies S = 57$, so investor B's payoff at expiration is $(135 - S)_+ = 135 - 57 = 78$.

Example

John bought three separate 6-month options on the same stock.

- Option I was an American-style put with strike price 20.
- Option II was a Bermudan-style call with strike price 25, where exercise was allowed at any time following an initial 3-month period of call protection.
- Option III was a European-style put with strike price 30.

When the options were bought, the stock price was 20. When the options expired, the stock price was 26. The table below gives the maximum and minimum stock prices during the 6-month period:

Time Period	1st 3 months of Term	2nd 3 months of Term
Maximum Stock Price	24	28
Minimum Stock Price	18	22

John exercised each option at the optimal time. Rank the three options, from highest to lowest payoff.

Solution

III (30 - 26) $>$ II (28 - 25) $>$ I (20 - 18)

Example

A customer buys a 50-strike put on an index when the market price of the index is also 50. The premium for the put is 5. Assume that the option contract is for an underlying 100 units of the index. Calculate the customer's profit if the index declines to 45 at expiration.

Solution

$100\,(50 - 45 - 5) = 0$

Example

Consider a European put option on a stock index without dividends, with 6 months to expiration and a strike price of 1,000. Suppose that the annual nominal risk-free rate is 4% convertible semiannually, and that the put costs 74.20 today. Calculate the price that the index must be in 6 months so that being long in the put would produce the same profit as being short in the put.

Solution

$(1000 - S_{0.5})_+ - 74.2 \cdot 1.02 = 74.2 \cdot 1.02 - (1000 - S_{0.5})_+ \implies S_{0.5} = 924.32$

Example

Determine which, if any, of the following positions has or have an unlimited loss potential from adverse price movements in the underlying asset, regardless of the initial premium received.

1. Short 1 forward contract
2. Short 1 call option
3. Short 1 put option

Solution

1 and 2: Plot the diagram.

Example

The price of an asset will either rise by 25% or fall by 40% in 1 year, with equal probability. A European put option on this asset matures after 1 year. Assume the following:

- Price of the asset today: 100
- Strike price of the put option: 130
- Put option premium: 7
- Annual effective risk free rate: 3%

Calculate the expected profit of the put option.

Solution

$(130 - 125) \cdot 0.5 + (130 - 60) \cdot 0.5 - 7 \cdot 1.03 = 30.29$

Example

The market price of Stock A is 50. A customer buys a 50-strike put contract on Stock A for 500. The put contract is for 100 shares of A. Calculate the customer's maximum possible loss.

Solution

Because the customer is long a put option, his maximum loss is attained when the price of stock A at maturity is above 50. In that case, the payoff is 0 and the customer will have lost the initial investment of 500.

Elementary Option Strategies

- Insuring a long position: floors
  - At time 0 you own a share of a stock and want to sell in time T
  - Additionally buy a put option with maturity T and strike K
  - Payoff at T: $S_T + (K - S_T)_+ = \max\{S_T, K\}$, with floor K
- Insuring a short position: caps
  - At time 0 you short sell a share of a stock and want to buy back in time T
  - Additionally buy a call option with maturity T and strike K
  - Payoff at T: $-S_T + (S_T - K)_+ = -\min\{S_T, K\}$, with cap $-K$
- Short covered calls
  - At time 0 you own a share of a stock
  - Additionally sell a call option with maturity T and strike K
  - Payoff at T: $S_T - (S_T - K)_+ = \min\{S_T, K\}$
  - Earn call premium at 0 and interests
- Synthetic forwards
  - Turn "options" into "obligations"
  - Buy a European call and sell a European put; both options have the same underlying, strike K and maturity T
  - Payoff at T: $(S_T - K)_+ - (K - S_T)_+ = S_T - K$
  - At T: if $S_T \geqslant K$, exercise the call option (to buy K); else buy K as forced by the put option holder — All have to buy K

Bull Spreads

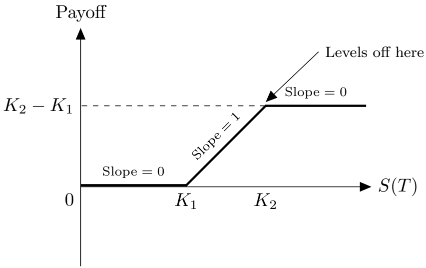
Let $K_2 > K_1$.

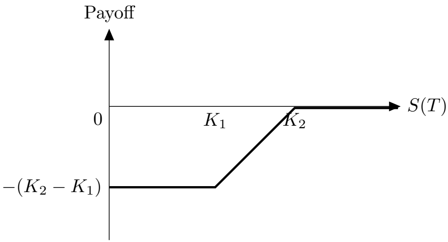
- Call bull spreads: The payoff at T is

$$\underbrace{(S_T - K_1)_+}_{\text{Long } K_1 \text{ strike call}} - \underbrace{(S_T - K_2)_+}_{\text{Short } K_2 \text{ strike call}} = \begin{cases} 0 & \text{if } S_T < K_1 \\ S_T - K_1 & \text{if } K_1 \leqslant S_T < K_2 \\ K_2 - K_1 & \text{if } K_2 \leqslant S_T \end{cases}$$

- Put Bull spreads: The payoff at T is

$$\underbrace{(K_1 - S_T)_+}_{\text{Long } K_1 \text{ strike put}} - \underbrace{(K_2 - S_T)_+}_{\text{Short } K_2 \text{ strike put}} = \begin{cases} -(K_2 - K_1) & \text{if } S_T < K_1 \\ S_T - K_2 & \text{if } K_1 \leqslant S_T < K_2 \\ 0 & \text{if } K_2 \leqslant S_T \end{cases}$$





Bear Spreads

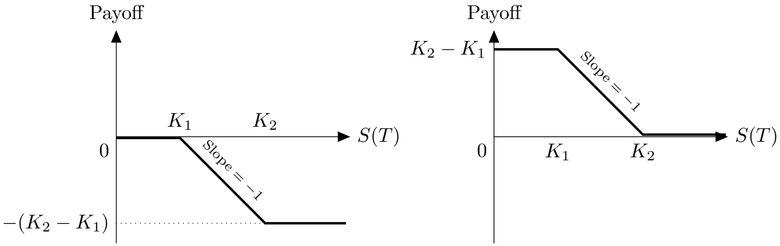
Let $K_2 > K_1$.

- Call bear spreads: The payoff at T is

$$\underbrace{-(S_T - K_1)_+}_{\text{Short } K_1 \text{ strike call}} + \underbrace{(S_T - K_2)_+}_{\text{Long } K_2 \text{ strike call}} = \begin{cases} 0 & \text{if } S_T < K_1 \\ K_1 - S_T & \text{if } K_1 \leqslant S_T < K_2 \\ -(K_2 - K_1) & \text{if } K_2 \leqslant S_T \end{cases}$$

- Put bear spreads: The payoff at T is

$$\underbrace{-(K_1 - S_T)_+}_{\text{Short } K_1 \text{ strike put}} + \underbrace{(K_2 - S_T)_+}_{\text{Long } K_2 \text{ strike put}} = \begin{cases} K_2 - K_1 & \text{if } S_T < K_1 \\ K_2 - S_T & \text{if } K_1 \leqslant S_T < K_2 \\ 0 & \text{if } K_2 \leqslant S_T \end{cases}$$



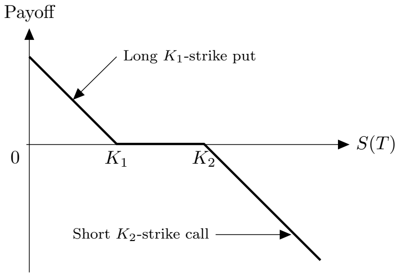
Collars

- Collars: a long put with strike K_1 and a short call with strike K_2, $K_2 > K_1$; same underlying and maturity. The payoff at T is

$$\underbrace{(K_1 - S_T)_+}_{\text{Long } K_1 \text{ strike put}} - \underbrace{(S_T - K_2)_+}_{\text{short } K_2 \text{ strike call}} = \begin{cases} K_1 - S_T & \text{if } S_T < K_1 \\ 0 & \text{if } K_1 \leqslant S_T < K_2 \\ K_2 - S_T & \text{if } K_2 \leqslant S_T \end{cases}$$

- Collared stock: a long stock with a long collar, with payoff at T

$$S_T + \underbrace{(K_1 - S_T)_+ - (S_T - K_2)_+}_{\text{Long collar}} = \begin{cases} K_1 & \text{if } S_T < K_1 \\ S_T & \text{if } K_1 \leqslant S_T < K_2 \\ K_2 & \text{if } K_2 \leqslant S_T \end{cases}$$

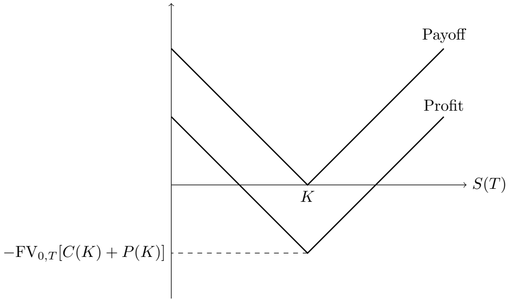


Straddles

- Straddles: a long put and a long call with the same underlying, strike, and maturity. The payoff at T is

$$\underbrace{(K-S_T)_+}_{\text{Long put}} + \underbrace{(S_T-K)_+}_{\text{Long call}} = \begin{cases} K-S_T & \text{if } S_T < K \\ S_T-K & \text{if } S_T \geqslant K \end{cases} = |S_T-K|$$

- Holding a straddle: a bet on the volatility of the underlying being higher than that perceived by the market
- Let $\Delta = \mathrm{FV}_{0,T}(C(0,K)+P(0,K))$, then the profit of a long K strike straddle is zero when $S_T = K \pm \Delta$

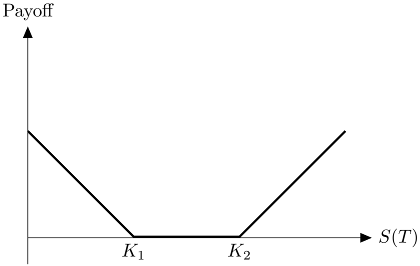


Strangles

- Strangles: a long put with strike K_1 and a long call with strike K_2, all of the same underlying S_t, maturity, and $K_1 < S_0 < K_2$. The payoff at T is

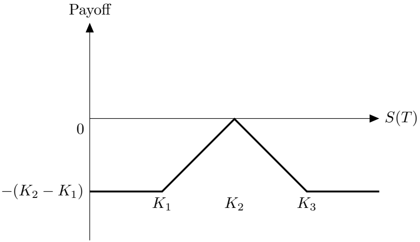
$$\underbrace{(K_1 - S_T)_+}_{\text{Long strike } K_1 \text{ put}} + \underbrace{(S_T - K_2)_+}_{\text{Long strike } K_2 \text{ call}} = \begin{cases} K_1 - S_T & \text{if } S_T < K_1 \\ 0 & \text{if } K_1 \leqslant S_T < K_2 \\ S_T - K_2 & \text{if } K_2 \leqslant S_T \end{cases}$$

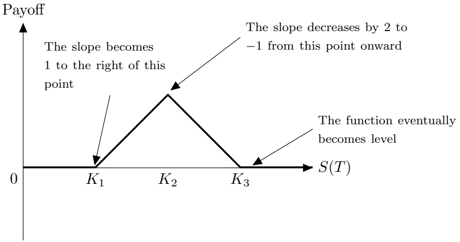
- The long strangle premium is minimized with smaller K_1 and greater K_2



Butterfly Spreads

- Motivation: The loss of a short straddle with strike K_2 is limited if add
  - a long out-of-the-money put with strike K_1 — protection on the downside
  - a long out-of-the-money call with strike K_3 — protection on the upside
- Equivalent combinations:
  - 1 long K_1 call + 2 short K_2 calls + 1 long K_3 call
  - 1 long K_1 put + 2 short K_2 puts + 1 long K_3 put
  - 1 long K_1-K_2 (call / put) bull spread + 1 long K_2-K_3 (call / put) bear spread
- Asymmetric butterfly spreads can be constructed by varying the number of involved options





Option Pricing in Continuous Time

- Option pricing in discrete time: for contract X

$$\Pi(0;\, X) = \frac{1}{(1+R)^T}\, \mathsf{E}^Q\, X_T$$

- Discretize each interval further into m sections, then the compounding factor $(1+R)^T$ becomes $(1+\frac{R}{m})^{mT}$
- Let $m \to \infty$ (continuous time), $(1+\frac{R}{m})^{mT} \to e^{RT}$
- So option pricing in continuous time: for contract X

$$\Pi(0;\, X) = e^{-RT}\, \mathsf{E}^Q\, X_T$$

- Hereafter r, instead of R, is the underlying interest rate

Option Pricing: The Black-Scholes Formula I

- Under the risk-neutral probability measure Q, the stock S evolves as $S(t) = S(0)\, \exp\left\{(r - \delta - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z\right\}$, where $Z \sim N(0,1)$.
- For the European call option with strike K, the contract is $X(t) = \max\{S(t) - K,\, 0\} \equiv (S(t) - K)_+$.
- So the price of the call option at $t = 0$ is

$$
\begin{aligned}
\Pi_c(0;\, X) &= e^{-rT}\, \mathsf{E}^Q\{X(T)\} = e^{-rT}\, \mathsf{E}^Q\{(S(T) - K)_+\} \\
&= e^{-rT}\, \mathsf{E}^Q\{(S(T) - K)_+ \,|\, S(T) > K\}\, \mathsf{P}^Q\{S(T) > K\} \\
&\qquad + e^{-rT}\, \underbrace{\mathsf{E}^Q\{(S(T) - K)_+ \,|\, S(T) < K\}}_{=0}\, \mathsf{P}^Q\{S(T) < K\} \\
&= e^{-rT}\, \mathsf{E}^Q\{(S(T) - K)_+ \,|\, S(T) > K\}\, \mathsf{P}^Q\{S(T) > K\} \\
&= e^{-rT}\, \mathsf{E}^Q\{S(T) - K \,|\, S(T) > K\}\, \mathsf{P}^Q\{S(T) > K\} \\
&= e^{-rT}\big(\, \mathsf{E}^Q\{S(T) \,|\, S(T) > K\} - K\big)\, \mathsf{P}^Q\{S(T) > K\}
\end{aligned}
$$

Option Pricing: The Black-Scholes Formula II

- As $S(T) = S(0)\,\exp\{(r-\delta-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\}$, evaluate $\mathsf{P}^Q\{S(T) > K\}$ and $\mathsf{E}^Q\{S(T)\,|\,S(T) > K\}$
- Let $\Phi(\cdot)$ be the CDF of $N(0,1)$, then

$$
\begin{aligned}
\mathsf{P}^Q\{S(T) > K\} &= \mathsf{P}^Q\Big\{S(0)\,\exp\Big\{\Big(r-\delta-\frac{\sigma^2}{2}\Big)T + \sigma\sqrt{T}Z\Big\} > K\Big\} \\
&= \mathsf{P}^Q\Big\{\exp\Big\{\Big(r-\delta-\frac{\sigma^2}{2}\Big)T + \sigma\sqrt{T}Z\Big\} > \frac{K}{S(0)}\Big\} \\
&= \mathsf{P}^Q\Big\{\Big(r-\delta-\frac{\sigma^2}{2}\Big)T + \sigma\sqrt{T}Z > \ln\frac{K}{S(0)}\Big\} \\
&= \mathsf{P}^Q\Big\{Z > \frac{\ln\frac{K}{S(0)} - (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Big\} \\
&= 1 - \Phi\Big(\frac{\ln\frac{K}{S(0)} - (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Big) \\
&= \Phi\Big(\frac{\ln\frac{S(0)}{K} + (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Big) \equiv \Phi(d_2)
\end{aligned}
$$

Option Pricing: The Black-Scholes Formula III

- Define $d_2 = \dfrac{\ln \frac{S(0)}{K} + (r - \delta - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_1 = \dfrac{\ln \frac{S(0)}{K} + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_2 + \sigma\sqrt{T}$; $\mathsf{E}^Q\{S(T) \mid S(T) > K\} = \dfrac{\mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{S(T)>K\}}\right\}}{\mathsf{P}^Q\{S(T) > K\}}$ and

$$
\begin{aligned}
\mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{S(T)>K\}}\right\} &= \mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{Z>-d_2\}}\right\} \\
&= \int_{-d_2}^{\infty} S(0)\, e^{\left(r-\delta-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2+\sigma\sqrt{T}z-\frac{1}{2}\sigma^2 T}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_2-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,\mathrm{d}z \\
&= S(0)\, e^{(r-\delta)T} \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,\mathrm{d}z = S(0)\, e^{(r-\delta)T}\Phi(d_1)
\end{aligned}
$$

Option Pricing: The Black-Scholes Formula IV

- The price of the call option with strike K at $t = 0$ is

$$\begin{aligned}
\Pi_c(0;\, X) &= e^{-rT}\big(\,\mathsf{E}^Q\{S(T)\,|\,S(T) > K\} - K\big)\,\mathsf{P}^Q\{S(T) > K\} \\
&= e^{-rT}\,\mathsf{E}^Q\left\{S(T)\,\mathbb{1}_{\{S(T)>K\}}\right\} - Ke^{-rT}\,\mathsf{P}^Q\{S(T) > K\} \\
&= e^{-rT}S(0)\,e^{(r-\delta)T}\Phi(d_1) - K e^{-rT}\Phi(d_2) \\
&= S(0)\,e^{-\delta T}\Phi(d_1) - K e^{-rT}\Phi(d_2)
\end{aligned}$$

- Note that

$$\begin{aligned}
(S(T) - K)_+ - (K - S(T))_+ &= \max\{S(T) - K, 0\} - \max\{K - S(T), 0\} \\
&= \max\{S(T) - K, 0\} + \min\{S(T) - K, 0\} \\
&= S(T) - K
\end{aligned}$$

- Let the price of the put option with strike K at $t = 0$ be $\Pi_p(0;\, X)$, then

$$\Pi_c(0;\, X) - \Pi_p(0;\, X) = e^{-rT}\,\mathsf{E}^Q\{S(T) - K\}$$

Option Pricing: The Black-Scholes Formula V

- Note that $\mathsf{E}^Q\{e^{kz}\}$ for $z \sim N(0,1)$ is $e^{\frac{1}{2}k^2}$, then

$$\begin{aligned} e^{-rT}\,\mathsf{E}^Q\{S(T)-K\} &= e^{-rT}S(0)\,e^{(r-\delta-\frac{1}{2}\sigma^2)T}\,\mathsf{E}^Q\left\{e^{\sigma\sqrt{T}Z}\right\} - Ke^{-rT} \\ &= S(0)\,e^{(-\delta-\frac{1}{2}\sigma^2)T}\underbrace{\mathsf{E}^Q\left\{e^{\sigma\sqrt{T}Z}\right\}}_{=e^{\frac{1}{2}\sigma^2 T}} - Ke^{-rT} \\ &= S(0)\,e^{-\delta T} - Ke^{-rT} \end{aligned}$$

- By $\Phi(x) + \Phi(-x) = 1$,

$$\begin{aligned} \Pi_p(0;\,X) &= \Pi_c(0;\,X) - S(0)\,e^{-\delta T} + Ke^{-rT} \\ &= S(0)\,e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2) - S(0)\,e^{-\delta T} + Ke^{-rT} \\ &= -S(0)\,e^{-\delta T}(1-\Phi(d_1)) + Ke^{-rT}(1-\Phi(d_2)) \\ &= -S(0)\,e^{-\delta T}\Phi(-d_1) + Ke^{-rT}\Phi(-d_2) \end{aligned}$$

Example

You are asked to determine the price of a European put option on a stock. Assuming the Black-Scholes model, you are given

- The stock price now is 100.
- The option expires in 6 months.
- The strike price is 98.
- The interest rate $r = 0.055$.
- $\delta = 0.01$.
- $\sigma = 0.5$.

What is the price?

Solution

Note that $S(0) = 100$, $T = 0.5$, $K = 98$, $d_1 = \dfrac{\ln \frac{100}{98} + (0.055 - 0.01 + \frac{0.5^2}{2})\, 0.5}{0.5\sqrt{0.5}}$ $= 0.29756$, $d_2 = d_1 - 0.5\sqrt{0.5} = -0.056$, $\Phi(-d_1) = 0.38302$, $\Phi(-d_2) = 0.52233$.
The price of the put is

$$K e^{-rT} \Phi(-d_2) - S(0)\, e^{-\delta T} \Phi(-d_1)$$
$$= 98\, e^{-0.055 \cdot 0.5} \cdot 0.52233 - 100\, e^{-0.01 \cdot 0.5} \cdot 0.38302 = 11.6889.$$