The Social Cost of Traffic at Equilibrium: A Game-Theoretic Approach

Extracted from Chapter 8: Modeling Network Traffic Using Game Theory

1 Introduction

This section explores how game theory models network traffic, focusing on the social cost at equilibrium versus the social optimum. A key phenomenon, the *Braess Paradox*, shows that adding roads can worsen traffic, challenging intuition that network upgrades always improve outcomes.

1.1 The Braess Paradox

Consider a network where drivers choose paths selfishly, leading to a Nash equilibrium. Adding an edge might increase travel times for all, as seen in a simple example:

- **Before**: Equilibrium travel time is 60 minutes.
- **After Adding Edge**: Equilibrium jumps to 80 minutes, a 4/3 increase.

Roughgarden and Tardos [18, 353] prove this is the worst-case increase with linear travel-time functions $T_e(x) = a_e x + b_e$, where $a_e, b_e \geq 0$.

Figure 1: Network before and after adding an edge, illustrating Braess's Paradox (to be inserted).

1.2 Broader Context

Traffic at equilibrium may not optimize social welfare (total travel time). We aim to:

1. Prove equilibria exist in any network with linear travel times.
2. Quantify how far equilibrium social cost deviates from the optimum.

2 Network Model

A traffic network is a directed graph with:

- **Nodes**: Start and destination points for drivers.
- **Edges**: Roads with travel-time functions $T_e(x) = a_e x + b_e$, where x is the number of drivers.

- **Traffic Pattern**: Path choices for all drivers.
- **Social Cost**: Social-Cost(Z) $= \sum_{\text{drivers}}$ travel time, summed over all drivers in pattern Z.
- **Social Optimum**: Pattern minimizing social cost.
- **Nash Equilibrium**: No driver can reduce their travel time by switching paths, given others' choices.

Figure 2: A network with travel-time functions $T_e(x)$ on edges (to be inserted).

2.1 Example Network

In Figure 3, with 4 drivers from A to B:

- **Social Optimum**: Social cost = 28 (each driver takes 7 units).
- **Nash Equilibrium**: Social cost = 32 (each takes 8 units).

Figure 3: Social optimum (left) vs. Nash equilibrium (right) (to be inserted).

3 Existence of Equilibrium

3.1 Best-Response Dynamics

To find an equilibrium:

1. Start with any traffic pattern.
2. If not an equilibrium, some driver can switch to a path with less travel time.
3. Update the pattern and repeat until no driver wants to switch.

This process, *best-response dynamics*, raises the question: does it always converge?

3.2 Potential Energy Concept

Define *potential energy* for an edge e with x drivers:

$$\text{Energy}(e) = T_e(1) + T_e(2) + \cdots + T_e(x)$$

Total potential energy of a pattern Z is:

$$\text{Energy}(Z) = \sum_e \text{Energy}(e)$$

If no drivers use e, Energy(e) = 0.

3.2.1 Why Potential Energy?

Social cost can increase or decrease with best-response steps (e.g., from 28 to 32 in the Braess example), but potential energy strictly decreases, serving as a progress measure.

3.3 Analyzing Dynamics

Consider a driver switching paths:

- **Old Path**: Travel time = 7.
- **New Path**: Travel time = 5.

Potential energy change:

- **Released**: $T_e(x)$ on each edge of the old path (total = 7).
- **Added**: $T_e(x+1)$ on each edge of the new path (total = 5).
- **Net Change**: $5 - 7 = -2$ (decreases).

Figure 4: Steps of best-response dynamics with potential energy changes (to be inserted).

3.3.1 General Proof

For any edge e:

- Driver leaves: Energy(e) drops by $T_e(x)$, their old travel time.
- Driver joins: Energy(e) rises by $T_e(x+1)$, their new travel time.

Net change in Energy(Z) = new time - old time. Since drivers switch only to improve (new < old), Energy(Z) decreases. With finite patterns, dynamics must stop at an equilibrium.

4 Comparing Equilibrium to Optimum

4.1 Potential Energy vs. Travel Time

For edge e with x drivers:

- **Total Travel Time**: $xT_e(x)$.
- **Potential Energy**: $T_e(1) + \cdots + T_e(x)$.

Since $T_e(x) = a_e x + b_e$:

$$\text{Energy}(e) = a_e(1 + 2 + \cdots + x) + b_e x = \frac{a_e x(x+1)}{2} + b_e x$$

$$xT_e(x) = x(a_e x + b_e) = a_e x^2 + b_e x$$

Compare:

$$\frac{1}{2}xT_e(x) \leq \text{Energy}(e) \leq xT_e(x)$$

Proof:

$$\frac{a_e x(x+1)}{2} + b_e x \geq \frac{1}{2}(a_e x^2 + b_e x) \quad \text{and} \quad \leq a_e x^2 + b_e x$$

Figure 5: Potential energy (shaded) vs. total travel time (rectangle) (to be inserted).

4.2 Bounding Social Cost

For a pattern Z:

$$\frac{1}{2} \cdot \text{Social-Cost}(Z) \leq \text{Energy}(Z) \leq \text{Social-Cost}(Z)$$

From social optimum Z to equilibrium Z':

- $\text{Energy}(Z') \leq \text{Energy}(Z)$ (decreases in dynamics).
- $\text{Social-Cost}(Z') \leq 2 \cdot \text{Energy}(Z') \leq 2 \cdot \text{Energy}(Z) \leq 2 \cdot \text{Social-Cost}(Z)$.

Thus, some equilibrium has social cost at most twice the optimum. Stronger results show 4/3 is the tight bound [18, 353].

5 Conclusion

- Equilibria exist due to decreasing potential energy in best-response dynamics.
- Social cost at equilibrium is bounded (up to 2x, or 4/3x with refinement) relative to the optimum.
- Practical implications: Network design and tolls can mitigate inefficiencies.