Portfolio Optimization

Classical PO: Mean-Variance (MV) Criterion

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- "For some fixed mean rate of return $\mu = \mathsf{E}\{\mathbf{x}^\top \mathbf{R}\}$, try to minimize the variance $\sigma^2 = \operatorname{var}\{\mathbf{x}^\top \mathbf{R}\}$ of the return over portfolios $\mathbf{x}$"

MV: All Risky Assets

$$\min_{\mathbf{x}} \ \frac{1}{2}\,\mathbf{x}^\top \mathbf{V}\mathbf{x} \quad \text{s.t.} \quad \begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \qquad \mathbf{e} \equiv \underbrace{(1, 1, \dots, 1)^\top}_{s \text{ items}}$$

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- $\mathbf{V}$ is symmetric, positive definite, so $\mathbf{V}^{-1}$ also is
- Set $\mathcal{L} \equiv \dfrac{1}{2}\mathbf{x}^\top \mathbf{V}\mathbf{x} + \lambda\,(1 - \mathbf{x}^\top \mathbf{e}) + \nu\,(\mu - \mathbf{x}^\top \mathbf{r})$ with Lagrange multipliers λ, ν

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- Substitute into $\begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \implies \begin{cases} \lambda\,\mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e} + \nu\,\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} = 1 \\ \lambda\,\mathbf{e}^\top \mathbf{V}^{-1}\mathbf{r} + \nu\,\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \mu \end{cases}$

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$$\begin{aligned} &(\mathbf{r} - c\,\mathbf{e})^\top \mathbf{V}^{-1}(\mathbf{r} - c\,\mathbf{e}) > 0 \\ &\implies \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} - c\,\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} - c\,\mathbf{e}\mathbf{V}^{-1}\mathbf{r} + c^2\,\mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e}^\top > 0 \\ &\implies \gamma - 2\,c\,\beta + c^2\,\alpha > 0 \\ &\implies -\delta = \beta^2 - \gamma\alpha < 0 \end{aligned}$$

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- Here we have (σ, μ) with $a = \frac{1}{\sqrt{\alpha}}$, $b = \frac{\sqrt{\delta}}{\alpha}$, $h = 0$, $k = \frac{\beta}{\alpha}$, the asymptotes are $\left(\mu - \frac{\beta}{\alpha}\right) = \pm\frac{\frac{\sqrt{\delta}}{\alpha}}{\frac{1}{\sqrt{\alpha}}}\sigma \implies \mu = \frac{\beta}{\alpha} \pm \sqrt{\frac{\delta}{\alpha}}\sigma$

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  $\mu_d = \mathbf{x}_d^\top \mathbf{r} = \dfrac{1}{\beta}\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \dfrac{\gamma}{\beta}$
- $\mathbf{x} = \lambda\mathbf{V}^{-1}\mathbf{e} + \nu\mathbf{V}^{-1}\mathbf{r} = \lambda\,\alpha\,\mathbf{x}_g + \nu\,\beta\,\mathbf{x}_d$, so **every portfolio is the convex combination of $\mathbf{x}_g$ and $\mathbf{x}_d$**: note that $\lambda\alpha + \nu\beta = 1$ (constraint $\mathbf{x}^\top\mathbf{e} = 1$) !

- Global minimum-variance portfolio $\mathbf{x}_g$
  - First find μ_g that minimizes $\sigma^2 = \dfrac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$:

    By differentiation $2\alpha\mu_g - 2\beta = 0 \implies \mu_g = \dfrac{\beta}{\alpha}$
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Theorem (Mutual Fund Theorem)

Any minimum-variance portfolio is equivalent to investing in the convex combination of $\mathbf{x}_g$ and $\mathbf{x}_d$.

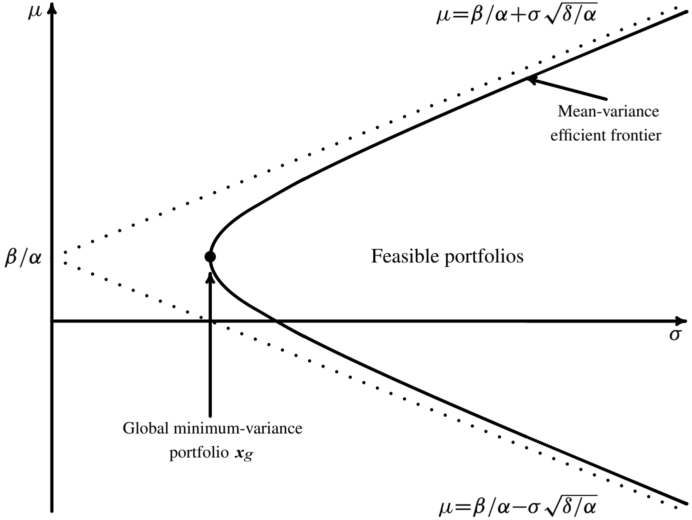

Figure: The Case of All Risky Assets

Theorem

Diversified portfolio $\mathbf{x}_d$ is the portfolio that maximize $s(\mathbf{x}) \equiv \dfrac{\mathbf{x}^\top \mathbf{r}}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$.

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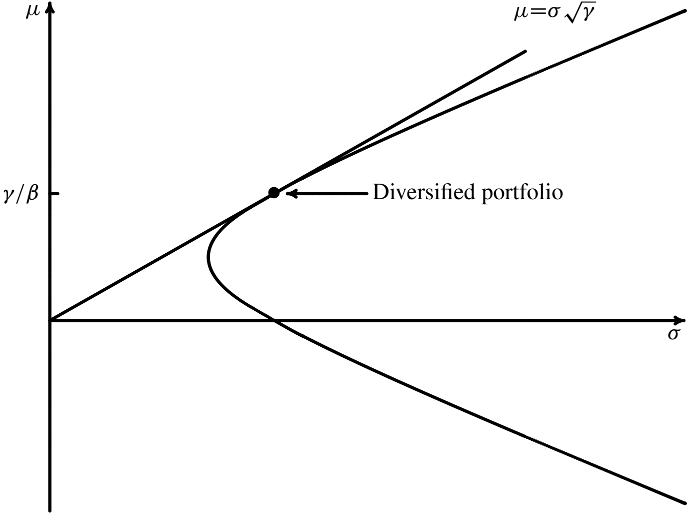

Figure: The Diversified Portfolio

MV: All But One Risky Assets

WLOG add riskless asset 0 with constant return r_0; the portfolio becomes $(x_0, x_1, x_2, \dots, x_s)^\top$

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- By $\dfrac{\partial \overline{\mathcal{L}}}{\partial x_0} = -\overline{\lambda} - \overline{\nu} r_0 = 0 \implies \overline{\nu} = -\dfrac{\overline{\lambda}}{r_0}$

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- Set $\alpha = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e},\ \ \beta = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r},\ \ \gamma = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r},\ \ \delta \equiv \alpha\gamma - \beta^2$, the above becomes

$$\begin{cases} x_0 + \overline{\lambda}\alpha + \overline{\nu}\beta = x_0 + \overline{\lambda}\alpha - \frac{\overline{\lambda}}{r_0}\beta = 1 \\ x_0 r_0 + \overline{\lambda}\beta + \overline{\nu}\gamma = x_0 r_0 + \overline{\lambda}\beta - \frac{\overline{\lambda}}{r_0}\gamma = \mu \end{cases}$$

with solutions $x_0 = \dfrac{\alpha\mu r_0 - \beta r_0 + \gamma - \beta\mu}{\epsilon^2}$, $\overline{\lambda} = \dfrac{(r_0 - \mu) r_0}{\epsilon^2}$,
$\overline{\nu} = -\dfrac{r_0 - \mu}{\epsilon^2}$, where $\epsilon^2 = \alpha r_0^2 - 2\beta r_0 + \gamma = \alpha\Big(r_0 - \dfrac{\beta}{\alpha}\Big)^2 + \dfrac{\delta}{\alpha}$

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- The relation of σ with μ

$$\begin{aligned} \sigma^2 &= \mathbf{x}^\top \mathbf{V}\mathbf{x} = \mathbf{x}^\top \mathbf{V}(\overline{\lambda}\mathbf{V}^{-1}\mathbf{e} + \overline{\nu}\mathbf{V}^{-1}\mathbf{r}) = \overline{\lambda}(\mathbf{x}^\top \mathbf{e}) + \overline{\nu}(\mathbf{x}^\top \mathbf{r}) \\ &= \overline{\lambda}(1 - x_0) + \overline{\nu}(\mu - x_0 r_0) = \overline{\lambda} + \overline{\nu}\mu = \frac{(\mu - r_0)^2}{\epsilon^2} \end{aligned}$$

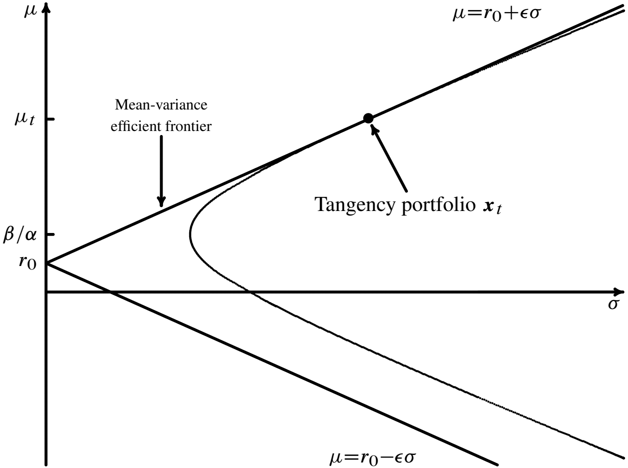

Figure: The Case of All But One Risky Assets

Property

If $r_0 < \dfrac{\beta}{\alpha}$, then $\mu = r_0 + \epsilon\sigma$ touches the hyperbola $\sigma^2 = \dfrac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$ at $\left(\dfrac{\epsilon}{\beta - \alpha r_0}, \dfrac{\gamma - \beta r_0}{\beta - \alpha r_0}\right)$

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Proof

On $\sigma - \mu$ plane the slope of the tangent $\mu'(\sigma)$ is obtained by implicit differentiation of $\sigma^2 = \dfrac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$ w.r.t σ (let $\mu \equiv \mu(\sigma)$):
$2\sigma = \dfrac{2\alpha\mu\mu' - 2\beta\mu'}{\delta} \implies \mu' = \dfrac{\delta\sigma}{\alpha\mu - \beta}$. Solve σ, μ from $\mu = r_0 + \epsilon\sigma$ and $\epsilon = \dfrac{\delta\sigma}{\alpha\mu - \beta}$, we obtain $(\sigma, \mu) = \left(\dfrac{\epsilon}{\beta - \alpha r_0}, \dfrac{\gamma - \beta r_0}{\beta - \alpha r_0}\right)$.

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$$\mathbf{x}_t = \frac{1}{\beta - \alpha r_0}\mathbf{V}^{-1}(\mathbf{r} - r_0\mathbf{e}) = \frac{\beta}{\beta - \alpha r_0}\mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0}\mathbf{x}_g$$

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  $= \dfrac{\beta}{\beta - \alpha r_0}\mu_d - \dfrac{\alpha r_0}{\beta - \alpha r_0}\mu_g = \dfrac{\gamma - \beta r_0}{\beta - \alpha r_0}$ for $\mu_d = \dfrac{\gamma}{\beta}$, $\mu_g = \dfrac{\beta}{\alpha}$

Theorem

Tangency portfolio $\mathbf{x}_t$ is the portfolio that maximize $s(\mathbf{x}) \equiv \dfrac{\mathbf{x}^\top \mathbf{r} - r_0}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$.

Proof

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Set $\boldsymbol{\beta}_t \equiv (\beta_{1,t}, \beta_{2,t}, \dots, \beta_{s,t})^\top$ where $\beta_{i,t} = \dfrac{\mathrm{cov}(R_i, \mathbf{x}^\top \mathbf{R})}{\mathrm{var}(\mathbf{x}^\top \mathbf{R})}$, or

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Mean-Variance Analysis and Expected Utility

$f(\sigma, \mu) = \mathsf{E}\, v(W)$ where $W = (x_0 r_0 + \mathbf{x}^\top \mathbf{R}) w$ with $\mu = x_0 r_0 + \mathbf{x}^\top \mathbf{r}$, $\sigma^2 = \mathbf{x}^\top \mathbf{V} \mathbf{x}$. Assume that $\dfrac{\partial f}{\partial \sigma} < 0$, $\dfrac{\partial f}{\partial \mu} > 0$ with $x_0 + \mathbf{x}^\top \mathbf{e} = 1$,

$\max\limits_{\mathbf{x}} f\big(\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}, r_0 + \mathbf{x}^\top (\mathbf{r} - r_0 \mathbf{e})\big)$.

$$\frac{\partial f}{\partial \mathbf{x}} = \frac{1}{\sigma} \frac{\partial f}{\partial \sigma} \mathbf{V} \mathbf{x} + \frac{\partial f}{\partial \mu} (\mathbf{r} - r_0 \mathbf{e}) = 0 \implies \mathbf{x} = -\frac{\sigma \frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial \sigma}} \mathbf{V}^{-1} (\mathbf{r} - r_0 \mathbf{e}) \propto \mathbf{x}_t$$

quadratic utility: $v(x) = ax + bx^2$ where $a, b \in \mathbb{R}$, $b \leqslant 0$. $\implies$
$\mathsf{E}\, v(W) = \mathsf{E}\, v((x_0 r_0 + \mathbf{x}^\top \mathbf{R}) w) = a w \mu + b w^2 (\mu^2 + \sigma^2) = f(\sigma, \mu)$.
normally distributed returns: $\mathbf{R} \sim N(\mathbf{r}, \mathbf{V}) \implies \mathbf{x}^\top \mathbf{R} \sim N(\mathbf{x}^\top \mathbf{r}, \mathbf{x}^\top V \mathbf{x}) \implies$
$\mathsf{E}\, v(W) = \mathsf{E}\, v((\mu + \sigma Y) w)$, where $Y \sim N(0, 1)$.

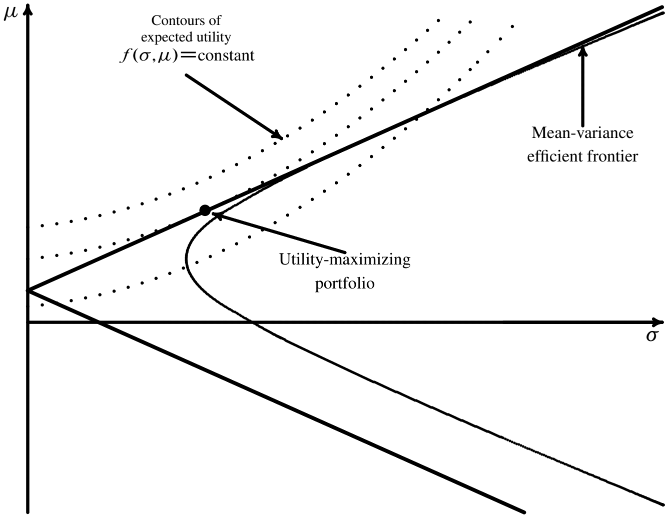

Figure: Determining the Utility-Maximizing Portfolio

Equilibrium: The Capital-Asset Pricing Model

Market with investors indexed by $j \in \mathcal{J}$ with proportions of wealth $x_{0,j}$, $\mathbf{x}_j = (x_{1,j}, x_{2,j}, \dots, x_{s,j})^\top$

When each investor j has the utility function as above, the optimal $\mathbf{x}_j \propto \mathbf{x}_t \implies \mathbf{x}_j = (1 - x_{0,j})\mathbf{x}_t \ \forall j \in \mathcal{J}$.

The total value of the demand for risky asset i:

$$\sum_{j\in\mathcal{J}} w_j x_{i,j} = \Big(\sum_{j\in\mathcal{J}} (1 - x_{0,j}) w_j\Big)(\mathbf{x}_t)_i$$

Market portfolio of risky assets $\mathbf{x}_m$:

$$(\mathbf{x}_m)_i \equiv \frac{\text{The total value of the supply of risky asset } i}{\text{The total value of the supply of all risky assets}}, \ \mathbf{x}_m^\top \mathbf{e} = 1.$$

In equilibrium

$$(\mathbf{x}_m)_i = \frac{\Big(\sum_{j\in\mathcal{J}} (1 - x_{0,j}) w_j\Big)(\mathbf{x}_t)_i}{\sum_{j\in\mathcal{J}} \sum_{k=1}^{s} w_j x_{k,j}} = \frac{\Big(\sum_{j\in\mathcal{J}} (1 - x_{0,j}) w_j\Big)(\mathbf{x}_t)_i}{\Big(\sum_{j\in\mathcal{J}} (1 - x_{0,j}) w_j\Big)\sum_{k=1}^{s} (\mathbf{x}_t)_k} = (\mathbf{x}_t)_i$$

since $\mathbf{x}^\top \mathbf{e} = 1$. $\implies$ $\mathbf{r} = r_0 \mathbf{e} + (\mu_m - r_0)\boldsymbol{\beta}_m$, $\boldsymbol{\beta}_m \equiv (\beta_{1,m}, \beta_{2,m}, \dots, \beta_{s,m})^\top$,

$$\beta_{i,m} = \frac{\operatorname{cov}(R_i, \mathbf{x}_m^\top \mathbf{R})}{\operatorname{var}(\mathbf{x}_m^\top \mathbf{R})}$$ — capital-asset-pricing equation

Problem

Suppose that an investment X has either (i) the uniform distribution $U[0, 2\mu]$ or (ii) the exponential distribution with $\mathsf{E}\, X = \mu$, and the investor has a utility function which is either (a) logarithmic, $v(x) = \log x$ (b) power form, $v(x) = x^\theta$. Show that both the compensatory risk premium and the investment risk premium are proportional to μ in all 4 possible cases.

Solution I

For distributions (i)(ii) of X, the r.v. $Y \equiv \dfrac{X}{\mu}$ does not depend on μ, so the equation $\mathsf{E}\, v(X+\alpha) = v(\mu)$ for the compensatory risk premium α reduces to $\mathsf{E}\, v(Y+c) = v(1)$ in cases (a)(b) when $\alpha = c\mu$. For the insurance risk premium when $\beta = d\mu$, d is the solution of $\mathsf{E}\, v(Y) = v(1-d)$.
For case (i)(a),
$\mathsf{E}\, v(Y+c) = \displaystyle\int_0^2 \frac{\log(y+c)}{2}\,\mathsf{d}y = \frac{1}{2}((2+c)\log(2+c) - c\log c - 2)$, and $v(1) = \log 1 = 0$, so $\alpha = c\mu$ where c is the unique positive root of $(2+c)\log(2+c) - c\log c - 2 = 0$. Using `rmaxima` we have $c = 0.176965531$.
For the insurance premium $\beta = d\mu$, $\mathsf{E}\log Y = \log 2 - 1 = \log(1-d)$, so $d = 1 - \dfrac{2}{e} = 0.264$.

Problem

An investor has a utility function $v(x) = \sqrt{x}$ and is considering three investments with random outcomes X, Y, Z. Here X has the uniform distribution $U[0, a]$, Y has the gamma distribution $\Gamma(\gamma, \lambda)$ with probability density function $\dfrac{e^{-\lambda y}\lambda^{\gamma}y^{\gamma-1}}{\Gamma(\gamma)}$ for $y > 0$, where $\gamma > 0$, $\lambda > 0$, and Z is log-normal, i.e $Z \sim N(\nu, \sigma^2)$. The parameter of the distributions are such that $\mathsf{E}\,X = \mathsf{E}\,Y = \mathsf{E}\,Z = \mu$ and $\operatorname{var} X = \operatorname{var} Y = \operatorname{var} Z$. Recall that the gamma function $\Gamma(\gamma) = \displaystyle\int_0^\infty u^{\gamma-1}e^{-u}\,\mathrm{d}u$ that satisfies $\Gamma(\gamma+1) = \gamma\,\Gamma(\gamma)$ and $\Gamma(1/2) = \sqrt{\pi}$. Determine the investor's preference ordering of X, Y, Z for all values of μ.

Solution I

- $X \sim U[0,a] \implies \mathsf{E}\, X = \frac{a}{2}$, $\operatorname{var} X = \frac{a^2}{12}$
- $Y \sim \Gamma(\gamma,\lambda) \implies \mathsf{E}\, Y = \frac{\gamma}{\lambda}$, $\operatorname{var} Y = \frac{\gamma}{\lambda^2}$
- $Z \sim$ lognormal$(\nu,\sigma^2) \implies \mathsf{E}\, Z = e^{\nu+\frac{\sigma^2}{2}}$, $\operatorname{var} Z = e^{2\nu+\sigma^2}(e^{\sigma^2}-1)$ by the formula $\mathsf{E}\, e^{\theta W} = e^{\mu\theta+\frac{\theta^2\sigma^2}{2}}$ for $W \sim N(\mu,\sigma^2)$

$\mathsf{E}\, e^{\theta W} = e^{\mu\theta+\frac{\theta^2\sigma^2}{2}}$ as $W \sim N(\mu,\sigma^2)$: $\sqrt{2\pi}\sigma\ \mathsf{E}\, e^{\theta W} = \int_{-\infty}^{\infty} e^{\theta x}\cdot e^{-\frac12\frac{x^2-2\mu+\mu^2}{\sigma^2}}\,\mathsf{d}x$

$$= \int_{-\infty}^{\infty} e^{-\frac12\frac{x^2-2(\mu+\theta\sigma^2)+\mu^2}{\sigma^2}}\,\mathsf{d}x = \int_{-\infty}^{\infty} e^{-\frac12\frac{(x-(\mu+\theta\sigma^2))^2+\mu^2-(\mu+\theta\sigma^2)^2}{\sigma^2}}\,\mathsf{d}x$$

$$= \int_{-\infty}^{\infty} e^{-\frac12\frac{(x-(\mu+\theta\sigma^2))^2-2\mu\theta\sigma^2-(\theta\sigma^2)^2}{\sigma^2}}\,\mathsf{d}x = e^{\mu\theta+\frac{\theta^2\sigma^2}{2}}\cdot\int_{-\infty}^{\infty} e^{-\frac12\frac{(x-(\mu+\theta\sigma^2))^2}{\sigma^2}}\,\mathsf{d}x$$

$= e^{\mu\theta+\frac{\theta^2\sigma^2}{2}}\cdot\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,\mathsf{d}x = \sqrt{2\pi}\sigma\cdot e^{\mu\theta+\frac{\theta^2\sigma^2}{2}}$ by $\int_{-\infty}^{\infty} e^{-x^2}\,\mathsf{d}x = \sqrt{\pi}$.

The conditions $\mathsf{E}\, X = \mathsf{E}\, Y = \mathsf{E}\, Z = \mu$ and $\operatorname{var} X = \operatorname{var} Y = \operatorname{var} Z$ imply

- $a = 2\mu$, so that $\operatorname{var} X = \frac{\mu^2}{3}$.

Solution II

- $\mathsf{E}\,Y = \dfrac{\gamma}{\lambda} = \mu$, so that $\operatorname{var} Y = \dfrac{\gamma}{\lambda^2} = \dfrac{\mu^2}{\gamma} = \operatorname{var} X = \dfrac{\mu^2}{3} \implies \gamma = 3$
- $\mathsf{E}\,Z = e^{\nu + \frac{\sigma^2}{2}} = \mu$, $\operatorname{var} Z = e^{2\nu+\sigma^2}(e^{\sigma^2} - 1) = \mu^2(e^{\sigma^2} - 1) = \operatorname{var} X = \dfrac{\mu^2}{3}$
  $\implies \sigma^2 = \log \dfrac{4}{3}$.
- $\mathsf{E}\,\sqrt{X} = \displaystyle\int_0^{2\mu} \frac{\sqrt{x}}{2\mu}\,\mathsf{d}x = \frac{2^{\frac{3}{2}}}{3}\sqrt{\mu} \approx 0.943\sqrt{\mu}$
- $\mathsf{E}\,\sqrt{Y} = \displaystyle\int_0^{\infty} \sqrt{y}\,\frac{1}{2}e^{-\lambda y}\lambda^3 y^2\,\mathsf{d}y = \frac{\Gamma(\frac{7}{2})}{2\sqrt{\lambda}} = \frac{\Gamma(\frac{7}{2})}{2\sqrt{3}}\sqrt{\mu} = \frac{5\sqrt{3\pi}}{16}\sqrt{\mu} \approx 0.959\sqrt{\mu}$
- $\mathsf{E}\,\sqrt{Z} = e^{\frac{\nu}{2} + \frac{\sigma^2}{8}} = e^{-\frac{\sigma^2}{8}}\sqrt{\mu} = \left(\dfrac{3}{4}\right)^{\frac{1}{8}}\sqrt{\mu} \approx 0.965\sqrt{\mu}$

So $Z \succ Y \succ X$.

Problem

Suppose that an investor has the utility function $v(x) = 1 - e^{-ax}$ with $a > 0$, and the outcome of an investment is a r.v. X with mean μ, finite variance and finite moment-generating function $\psi(a) = \mathsf{E}\{e^{-ax}\}$ for $a > 0$.

1. Show that the compensatory risk premium and the insurance risk premium have the same value α, and express α in terms of μ and the moment generating function ψ.
2. Both the Arrow-Pratt and global risk aversions are a. Confirm directly that as $a \downarrow 0$, $\alpha = a\dfrac{\operatorname{var} X}{2} + o(a)$. Under what circumstances is it true that $\alpha = a\dfrac{\operatorname{var} X}{2}$ for all $a > 0$?
3. Prove that $\psi''\psi - (\psi')^2 \geqslant 0$ and hence α is an increasing function of a. This shows that the more risk-averse the investor is, the higher the value of the premium that is required.

Solution I

The compensatory risk premium α solves $\mathsf{E}\, v(\alpha + X) = v(\mu)$ while the insurance risk premium β solves $\mathsf{E}\, v(X) = v(\mu - \beta)$ giving the common value:

$$\alpha = \beta = \mu + \frac{1}{a}\ln(\psi(a)).$$

The expansion for small a is straightforward; when $\alpha = \frac{a \operatorname{var} X}{2}$ for all $a > 0$ we have:

$$\psi(a) = \mathsf{E}\, e^{-aX} = e^{-a\mu + \frac{a^2 \operatorname{var} X}{2}}$$

which is true only when X has a normal distribution, using (A.16). For the final part:

$$\psi''\psi - (\psi')^2 = \mathsf{E}\, X^2 e^{-aX}\, \mathsf{E}\, e^{-aX} - (\mathsf{E}\, X e^{-aX})^2 \geqslant 0$$

by the Cauchy-Schwarz inequality applied to the random variables $A = Xe^{-aX/2}$ and $B = e^{-aX/2}$. To see that α is increasing:

$$\frac{d\alpha}{da} = \frac{1}{a^2}\left[\frac{a\psi'}{\psi} - \ln(\psi)\right] = \frac{1}{a^2} f(a), \text{ say.}$$

Solution II

But $f(0) = 0$ and $f' = a[\psi''\psi - (\psi')^2]/\psi^2 \geqslant 0$ and the conclusion follows.

Problem

Consider a one-period investment model in which there are only two risky assets. The returns on these assets have means 3, 4 respectively and variances 2, 3 respectively with the covariance between the returns being 2.

1. Calculate the mean-variance efficient frontier and the minimum-variance portfolio in terms of the mean return.
2. Calculate the mean return of the global minimum-variance portfolio and of diversified portfolio.
3. Suppose that in addition to the two risky assets there is a riskless asset with return $3/2$. Find the minimum-variance portfolio in terms of the mean return and hence calculate the mean return of the tangency portfolio.

Solution I

The inverse matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so if $V = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$, $V^{-1} = \dfrac{1}{2}\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$. $\alpha = \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e} = \dfrac{1}{2}$, $\beta = \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{r} = \dfrac{3}{2}$, $\gamma = \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \dfrac{11}{2}$, $\delta = \alpha\gamma - \beta^2 = \dfrac{1}{2}$.

1. $\min\limits_{x_1, x_2} \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \min\limits_{x_1, x_2} 2x_1^2 + 4x_1x_2 + 3x_2^2$ s.t. $\begin{cases} x_1 + x_2 = 1 \\ 3x_1 + 4x_2 = \mu \end{cases}$. From constraints $x_1 = 4 - \mu$, $x_2 = \mu - 3$, so the mean-variance efficient frontier is $\sigma^2 = \mu^2 - 6\mu + 11$.
2. μ_g is the root of $\dfrac{\mathrm{d}\sigma^2}{\mathrm{d}\mu} = 0$, so $2\mu_g - 6 = 0 \implies \mu_g = 3$. $\mu_d = \dfrac{\gamma}{\beta} = \dfrac{11}{3}$.

Solution II

3. Now the problem is $\min\limits_{x_0,x_1,x_2} 2x_1^2 + 4x_1x_2 + 3x_2^2$ s.t.
$\begin{cases} x_0 + x_1 + x_2 = 1 \\ \frac{3}{2}x_0 + 3x_1 + 4x_2 = \mu \end{cases}$. Form the Lagrangian
$\mathcal{L} = 2x_1^2 + 4x_1x_2 + 3x_2^2 + \lambda(1 - x_0 - x_1 - x_2) + \nu(\mu - \dfrac{3}{2}x_0 - 3x_1 - 4x_2)$.
By solving $\dfrac{\partial \mathcal{L}}{\partial x_0} = 0$, $\nu = -\dfrac{2\lambda}{3}$. From $\dfrac{\partial \mathcal{L}}{\partial x_1} = 0$ and $\dfrac{\partial \mathcal{L}}{\partial x_2} = 0$ we have
$4x_1 + 4x_2 - \lambda - 3\nu = 0$ and $4x_1 + 6x_2 - \lambda - 4\nu = 0$; so $x_1 = \dfrac{\lambda}{12}$, $x_2 = -\dfrac{\lambda}{3}$.
Substitute into the constraints yields $\lambda = \dfrac{12(3-2\mu)}{17}$, and so $x_0 = \dfrac{26-6\mu}{17}$,
$x_1 = \dfrac{3-2\mu}{17}$, $x_2 = -\dfrac{4(3-2\mu)}{17}$. The tangency portfolio corresponds to
$x_0 = 0$ or $\mu_t = \dfrac{13}{3}$.

Problem

Suppose that v is concave, $X \sim N(\mu, \sigma^2)$ and $f(\sigma, \mu) = \mathsf{E}\, v(X)$.

1. Show that $\dfrac{\partial f}{\partial \mu} > 0$ when v is strictly increasing, and $\dfrac{\partial f}{\partial \sigma} \leqslant 0$. Hence show in the context of mean-variance analysis that, when all returns are jointly normally distributed, an investor maximizing the expected utility of his final wealth will choose a mean-variance-efficient optimal portfolio.
2. Show that f is concave in μ and σ. Deduce that this optimal portfolio corresponds to a point in the (σ, μ) plane where an indifference contour is tangent to the efficient frontier.

Solution I

Write $X = \mu + \sigma Y$ where $Y \sim N(0,1)$. Then it follows that:

$$\frac{\partial f}{\partial \mu} = \mathsf{E}\{v'(\mu + \sigma Y)\} > 0 \text{ when } v' > 0,$$

and using the relation (A.14):

$$\frac{\partial f}{\partial \sigma} = \mathsf{E}\{Y v'(\mu + \sigma Y)\} = \sigma\,\mathsf{E}\{v''(\mu + \sigma Y)\} \leqslant 0,$$

by the concavity of v. Now when returns are normally distributed then the wealth created by each portfolio has a normal distribution; this argument shows that maximizing in σ for fixed μ gives a value of (σ, μ) on the efficient frontier. To see the concavity of f, note that:

$$\frac{\partial^2 f}{\partial \mu^2} = \mathsf{E}\{v''(\mu + \sigma Y)\} \leqslant 0 \text{ and } \frac{\partial^2 f}{\partial \sigma^2} = \mathsf{E}\left\{Y^2 v''(\mu + \sigma Y)\right\} \leqslant 0,$$

$$\frac{\partial^2 f}{\partial \mu \partial \sigma} = \mathsf{E}\{Y v''(\mu + \sigma Y)\},$$

Solution II

and then:

$$\frac{\partial^2 f}{\partial \mu^2}\frac{\partial^2 f}{\partial \sigma^2} \geqslant \left(\frac{\partial^2 f}{\partial \mu \partial \sigma}\right)^2$$

follows by applying the Cauchy-Schwarz inequality to the random variables $A = Y\sqrt{-v''(\mu + \sigma Y)}$ and $B = \sqrt{-v''(\mu + \sigma Y)}$; this shows that the 2×2 matrix of second derivatives has non-positive diagonal entries and a non-negative determinant which is sufficient for the matrix to be negative semi-definite. The fact that f is concave means that sets of the form $\{(\sigma, \mu) : f(\sigma, \mu) > c\}$ are convex which gives the last statement.

Problem

Suppose that an investor has a concave utility function v. The investor seeks to maximize $\mathsf{E}\, v(W)$ where $W = (x_0 r_0 + \mathbf{x}^\top \mathbf{R})w$ is his final wealth.

1. Show that, when $\overline{W}$ is his optimal final wealth, then $\mathsf{E}\{v'(\overline{W})(R_j - r_0)\} = 0$, $\forall\, j = 1, 2, \dots, s$.
2. Show that, when $\mathbf{R}$ has a multivariate normal distribution, then $r_j - r_0 = \alpha \operatorname{cov}(\overline{W}, R_j)$, $\forall\, j = 1, 2, \dots, s$, where $\alpha = -\dfrac{\mathsf{E}\{v''(\overline{W})\}}{\mathsf{E}\{v'(\overline{W})\}}$ is his global risk aversion.
3. Now suppose that the market is determined by investors $i = 1, 2, \dots, n$, where investor i has concave utility v_i, initial wealth w_i, optimal final wealth $\overline{W}_i$ and global risk aversion α_i. With the normality assumption, show that $\mathsf{E}\, M - r_0 = \overline{w}\,\overline{\alpha} \operatorname{var} M$, where $M = \dfrac{\sum_{i=1}^n \overline{W}_i}{\sum_{i=1}^n w_i}$ is the market rate of return, $\overline{w} = \dfrac{\sum_{i=1}^n w_i}{n}$ is the average initial wealth of investors, and $\overline{\alpha}$ is the harmonic mean of α_i.

Solution I

The objective function to maximize is:

$$f(\mathbf{x}) = \mathsf{E}\, v\left(w\left(r_0 + \sum_{j=1}^{s} x_j(R_j - r_0)\right)\right)$$

where $\mathbf{x} = (x_1, \ldots, x_s)^\top$ and we have used the condition that $x_0 + \sum_{j=1}^{s} x_j = 1$. The first-order conditions give:

$$\frac{\partial f}{\partial x_j} = w\,\mathsf{E}\, v'(\overline{W})(R_j - r_0) = 0, \text{ for } 1 \leqslant j \leqslant s.$$

Since $r_j = \mathsf{E}\, R_j$ and the fact that $\overline{W}$ and R_j have a joint normal distribution we have that:

$$\begin{aligned} 0 &= \mathsf{E}\, v'(\overline{W})(R_j - r_0) = \mathsf{E}\{v'(\overline{W})(R_j - r_j)\} + \mathsf{E}\{v'(\overline{W})\}(r_j - r_0) \\ &= \operatorname{cov}(v'(\overline{W}), R_j) + \mathsf{E}\{v'(\overline{W})\}(r_j - r_0) \\ &= \mathsf{E}\{v''(\overline{W})\}\operatorname{cov}(\overline{W}, R_j) + \mathsf{E}\{v'(\overline{W})\}(r_j - r_0), \end{aligned}$$

Solution II

where the last equality uses (A.21), and this now gives the relation:

$$r_j - r_0 = \alpha \operatorname{cov}(\overline{W}, R_j),$$

as required.
For the final part, recall that for random variables X and Y and a a constant $\operatorname{cov}(X, Y + a) = \operatorname{cov}(X, Y)$ and $\operatorname{cov}(aX, Y) = a \operatorname{cov}(X, Y)$. Now for each i:

$$\alpha_i^{-1}(r_j - r_0) = \operatorname{cov}(\overline{W}_i, R_j)$$

and summing this on i yields:

$$\left(\sum_{i=1}^{n} \alpha_i^{-1} \right) (r_j - r_0) = \left(\sum_{i=1}^{n} w_i \right) \operatorname{cov}(M, R_j).$$

Divide through this relation by n and multiply by $\overline{\alpha}$, where $(\overline{\alpha})^{-1} = \sum_{i=1}^{n} \alpha_i^{-1}/n$, to obtain:

$$\mathsf{E}\, R_j - r_0 = w\overline{\alpha} \operatorname{cov}(M, R_j).$$

Solution III

When $\overline{x}_{ij}$ is the optimal proportion invested by investor i in asset j then:

$$\overline{W}_i = w_i \left[r_0 + \sum_{j=1}^{s} \overline{x}_{ij}(R_j - r_0) \right]$$

which when summed on i gives:

$$(M - r_0) \left(\sum_{i=1}^{n} w_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{s} w_i \overline{x}_{ij}(R_j - r_0).$$

Take the expectation in (B.3), multiply (B.2) by $w_i\overline{x}_{ij}$, sum on i and j, rearrange the expression using the two properties of covariance mentioned above and the result (1.21) follows. This shows that the risk premium for the market is proportional to $\overline{\alpha}$ which is a measure of the risk aversion in the economy.

Problem

Consider an investor with the utility function $v(x) = 1 - e^{-ax}$, $a > 0$, who is faced with a riskless asset with return r_0 and s risky assets with returns $\mathbf{R} \sim N(\mathbf{r}, \mathbf{V})$.

1. Show that when he seeks to maximize the expected utility of his final wealth, he will hold the risky assets in the same proportion as the tangency portfolio.
2. Show that when $\beta > \alpha\, r_0$, the more risk averse that he is, the smaller amount of his wealth that he invests in the risky assets.

Solution I

Suppose that the investor's initial wealth is $w > 0$ and that he wishes to minimize $\mathsf{E}\, e^{-aW}$ where:

$$W = w\left(r_0 + \sum_{j=1}^{s} x_j(R_j - r_0)\right) = w[r_0(1 - \mathbf{x}^\top\mathbf{e}) + \mathbf{x}^\top\mathbf{R}],$$

where $\mathbf{x} = (x_1, \dots, x_s)^\top$ and $\mathbf{e} = (1, \dots, 1)^\top$ as usual; the proportion of his wealth in the riskless asset is $x_0 = 1 - \mathbf{x}^\top\mathbf{e}$. Note that the linear combination $\mathbf{x}^\top\mathbf{R}$ has the $N(\mathbf{r}^\top\mathbf{x}, \mathbf{x}^\top V\mathbf{x})$-distribution, then use the expression (A.16) for the moment-generating function of a normally distributed random variable to see that:

$$\mathsf{E}\, e^{-aW} = \exp\left(-awr_0(1 - \mathbf{x}^\top\mathbf{e}) - aw\mathbf{r}^\top\mathbf{x} + \frac{1}{2}a^2w^2\mathbf{x}^\top V\mathbf{x}\right).$$

It is necessary to minimize the expression:

$$\frac{1}{2}aw\mathbf{x}^\top V\mathbf{x} - \mathbf{x}^\top(\mathbf{r} - r_0\mathbf{e}),$$

Solution II

for which the minimum occurs when $\mathbf{x} = (1/aw)V^{-1}(\mathbf{r} - r_0\mathbf{e})$, and the conclusion follows from (1.17). The amount of his wealth invested in the risky assets is $(\mathbf{x}^\top \mathbf{e})w = (\beta - \alpha r_0)/a$, which decreases in $a > 0$ when $\beta > \alpha r_0$.

Problem

Consider an investor with $\mathbf{R} = (R_1, R_2, \ldots, R_s)^\top$ where R_is are independent r.v. with R_i having gamma distribution, $\mathsf{E}\, R_i = r_i$ and $\operatorname{var} R_i = \sigma_i^2$. Suppose that he has the utility function $v(x) = 1 - e^{-ax}$, $a > 0$, and he seeks to maximize the expected utility of his final wealth.

1. Show that he divides his wealth between the diversified portfolio and a second portfolio which should be identified; determine the amounts that he invests in each.
2. If he may invest in a risky asset with return r_0, show that he will again divide his wealth between these two portfolios and the riskless asset; determine the amounts that he invests in each.
3. Give a necessary and sufficient condition, expressed in terms of the parameters r_i, $i = 0, 1, 2, \ldots, s$ and σ_i^2, $i = 1, 2, \ldots, s$, that he is long in the risky assets.

Solution I

When R_i has the gamma distribution $\Gamma(\gamma_i, \lambda_i)$ we have that $\mathsf{E}\, R_i = r_i = \gamma_i/\lambda_i$ and $\operatorname{var} R_i = \gamma_i/\lambda_i^2$, from which it follows that $\gamma_i = r_i^2/\sigma_i^2$ and $\lambda_i = r_i/\sigma_i^2$. For $\phi + \lambda_i > 0$, note that:

$$\mathsf{E}\left(e^{-\phi R_i}\right) = \int_0^\infty \frac{e^{-\phi x} e^{-\lambda_i x} \lambda_i^{\gamma_i} x^{\gamma_i - 1}}{\Gamma(\gamma_i)} dx = \left(\frac{\lambda_i}{\phi + \lambda_i}\right)^{\gamma_i} \int_0^\infty \frac{e^{-\phi x} e^{-\lambda_i x} (\phi + \lambda_i)^{\gamma_i} x^{\gamma_i}}{\Gamma(\gamma_i)}$$

because the integrand in the latter integral is a probability density function, and so the value of the integral is 1. The investor wishes to solve the constrained optimization problem:

$$\begin{aligned} \text{maximize} \quad & \mathsf{E}\left(1 - e^{-aw(\mathbf{x}^\top \mathbf{R})}\right) \\ \text{subject to} \quad & \mathbf{x}^\top \mathbf{e} = 1, \end{aligned}$$

but this is equivalent to minimizing:

$$\mathsf{E}\left(e^{-aw(\mathbf{x}^\top \mathbf{R})}\right) = \prod_{i=1}^{s} \mathsf{E}\left(e^{-awx_i R_i}\right) = \prod_{i=1}^{s} \left(\frac{\lambda_i}{awx_i + \lambda_i}\right)^{\gamma_i},$$

Solution II

subject to the constraint. Taking logarithms, we need to:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{s} \gamma_i \ln(awx_i + \lambda_i) \\ \text{subject to} \quad & \sum_{i=1}^{s} x_i = 1. \end{aligned}$$

Maximizing the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{s} \gamma_i \ln(awx_i + \lambda_i) + \theta \left(1 - \sum_{i=1}^{s} x_i \right)$$

in x_i gives $x_i = (\gamma_i/\theta) - \lambda_i/(aw)$. Substituting back into the constraint shows that the Lagrange multiplier is given as:

$$\theta = \frac{\sum_{j=1}^{s} \gamma_j}{1 + (aw)^{-1} \sum_{j=1}^{s} \lambda_j},$$

from which it follows that the optimal portfolio may be expressed as:

Solution III

$$\mathbf{x} = \left(1 + (aw)^{-1} \sum_{j=1}^{s} \lambda_j\right) \overline{\mathbf{x}} - \left((aw)^{-1} \sum_{j=1}^{s} \lambda_j\right) \mathbf{x}_d,$$

where the two portfolios $\overline{\mathbf{x}}$ and $\mathbf{x}_d$ are:

$$(\overline{\mathbf{x}})_i = \frac{\gamma_i}{\sum_j \gamma_j} = \frac{r_i^2/\sigma_i^2}{\sum_j r_j^2/\sigma_j^2} \text{ and } (\mathbf{x}_d)_i = \frac{\lambda_i}{\sum_j \lambda_j} = \frac{r_i/\sigma_i^2}{\sum_j r_j/\sigma_j^2},$$

with the latter portfolio being the diversified portfolio (see Example 1.1). As his initial wealth is w, the investor invests the amount $w + \sum_j \lambda_j/a$ in $\overline{\mathbf{x}}$ and the amount $-\sum_j \lambda_j/a$ in the diversified portfolio.

Note that in the case when the random variables R_i have exponential distributions, then $\gamma_i = 1$, or $r_i^2 = \sigma_i^2$, for each $1 \leqslant i \leqslant s$, so that the portfolio $\overline{\mathbf{x}}$ is just the uniform portfolio $\overline{\mathbf{x}} = (1/s, \dots, 1/s)^\top$ which apportions wealth equally between the s risky assets.

For the final part, when there is a riskless asset and we set $x_0 = 1 - \mathbf{x}^\top \mathbf{e}$, we see that we wish to minimize the expression:

Solution IV

$$\mathsf{E}\left(e^{-aw(r_0(1-\mathbf{x}^\top\mathbf{e})+\mathbf{x}^\top\mathbf{R})}\right) = e^{awr_0(\sum_j x_j-1)}\prod_{i=1}^{s}\mathsf{E}\left(e^{-awx_iR_i}\right) = e^{awr_0(\sum_j x_j-1)}\prod_{i=1}^{s}\left(\frac{\quad}{awx\ldots}\right.$$

in $\mathbf{x} = (x_1, \dots, x_s)^\top$, which is equivalent to maximizing:

$$\sum_{i=1}^{s}\gamma_i\ln(awx_i+\lambda_i) - awr_0\sum_{i=1}^{s}x_i.$$

Deduce that for $1 \leqslant i \leqslant s$, the optimal $x_i = (aw)^{-1}((\gamma_i/r_0)-\lambda_i)$, and the optimal investment in the risky assets is determined by:

$$\mathbf{x} = \left((awr_0)^{-1}\sum_{j=1}^{s}\gamma_j\right)\overline{\mathbf{x}} - \left((aw)^{-1}\sum_{j=1}^{s}\lambda_j\right)\mathbf{x}_d.$$

The investor is long in the particular risky asset i when $x_i > 0$, which is true if and only if $r_i > r_0$; he is long overall in risky assets if and only if $\sum_{j=1}^{s} x_j > 0$ which is equivalent to the condition that $r_0^{-1} > \sum_{j=1}^{s}(r_j/\sigma_j^2)/\sum_{j=1}^{s}(r_j^2/\sigma_j^2)$.