Introduction to Financial Models
Lecture 03: Surprises & Paradoxes III

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- If people maximize expected value, they should be willing to pay any finite amount to play

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- This amounts to $\mathrm{E}\,U(X) \approx \$1.39$, explaining why people would only pay a small amount

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Mathematical Formulation

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- For power utility, $\gamma = 1$ corresponds to logarithmic utility (by L'Hôpital's rule)

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  - The **risk premium** π is the maximum amount they would pay:

$$U(w - \pi) = \mathrm{E}\,U(w + \widetilde{X}) \tag{1}$$

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- Substitute into the risk premium formula (1) $U(w-\pi) = \mathrm{E}\,U(w+\widetilde{X})$,

$$U(w) - \pi U'(w) = U(w) + \frac{1}{2}U''(w)\operatorname{var}\widetilde{X} \implies \pi = \frac{1}{2}\left(-\frac{U''(w)}{U'(w)}\right)\operatorname{var}\widetilde{X}$$

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- The independence axiom is particularly important and controversial
  - It states that preferences between lotteries should not be affected by mixing them with a third lottery
  - This axiom is violated in several famous paradoxes

Allais Paradox

- Game A

$$X = \begin{cases} 101 & \text{prob. } 0.33 \\ 100 & \text{prob. } 0.66 \\ 0 & \text{prob. } 0.01 \end{cases} \qquad Y = 100 \ \text{ with prob. } 1$$

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- There are 10 red balls; total 20 yellow / black balls, but the number of each type unknown
- The agent estimates the probability of drawing yellow as p where $0 < p < \frac{2}{3}$
- A single ball is drawn from the urn

Ellsberg Paradox (Cont'd)

- Game A

$$X = \begin{cases} 100 & \text{if red} \\ 0 & \text{if yellow or black} \end{cases} \qquad Y = \begin{cases} 100 & \text{if yellow} \\ 0 & \text{if red or black} \end{cases}$$

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$$\implies (\frac{1}{3} - p) \cdot U(100) > (\frac{1}{3} - p) \cdot U(0) \quad (4)$$

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Mostly prefer Y to X: from the Expected Utility Hypothesis

$$\frac{2}{3} \cdot U(100) + \frac{1}{3} \cdot U(0) > (1-p) \cdot U(100) + p \cdot U(0)$$

$$\Longrightarrow \left(\frac{1}{3} - p\right) \cdot U(0) > \left(\frac{1}{3} - p\right) \cdot U(100) \quad (5)$$