

Options & Derivatives

The One Period Model

- time t : $t = 0, 1$
- (deterministic) bond B_t : $B_0 = 1$, $B_1 = 1 + R$
- (stochastic) stock S_t : $S_0 = s > 0$, $S_1 = \begin{cases} s \cdot u & \text{with prob. } p_u \\ s \cdot d & \text{with prob. } p_d \end{cases} \equiv s Z$:
 $u > d$, $p_u + p_d = 1$.
- The value V_t^h of the portfolio $h = (x, y)$, $x, y \in \mathbb{R}$ at time t :
 $V_t^h = x B_t + y S_t$ — $V_0^h = x + y s$, $V_1^h = x(1 + R) + y s Z$
- Arbitrage portfolio h : $V_0^h = 0$, $V_1^h > 0$ with prob. 1.

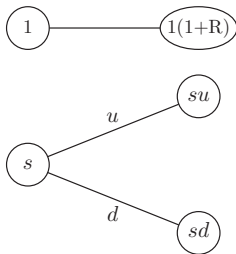


Figure: Asset Dynamics of One Period Model.

Portfolios and Arbitrage I

Theorem

The one period model is arbitrage free $\iff u \geq 1 + R \geq d$.

Proof

(\implies)

- Suppose $u \geq 1 + R \geq d$ does not hold, then $1 + R > u$ or $d > 1 + R$.
- If $1 + R > u$, then $s(1 + R) > su$ and a priori $s(1 + R) > sd$.
- Consider $h = (s, -1)$, then $V_0^h = s \cdot 1 + (-1) \cdot s = 0$,
 $V_1^h = s(1 + R) - s \cdot Z > 0$, an arbitrage.
- If $d > 1 + R$, then $sd > s(1 + R)$ and a priori $su > s(1 + R)$.
- Consider $h = (-s, 1)$, then $V_0^h = (-s) \cdot 1 + 1 \cdot s = 0$,
 $V_1^h = -s(1 + R) + s \cdot Z > 0$, an arbitrage.

Portfolios and Arbitrage II

Theorem

The one period model is arbitrage free $\iff u \geq 1 + R \geq d$.

Proof

(\Leftarrow)

- Arbitrage $h = (x, y)$: $V_0^h = 0$.
- $x + s \cdot y = 0 \implies x = -s \cdot y$.
- $V_1^h = \begin{cases} y s(u - (1 + R)), & Z = u \\ y s(d - (1 + R)), & Z = d \end{cases}$
- If $y > 0$: from $V_1^h > 0 \implies u > 1 + R$ and $d > 1 + R$; a contradiction.
- If $y < 0$: from $V_1^h > 0 \implies u < 1 + R$ and $d < 1 + R$; a contradiction.

Risk-Neutral / Martingale Measure and Probabilities

- Observation: $u \geq 1 + R \geq d \implies 1 + R$ is a convex combination of u and d
- $\exists q_u, q_d \geq 0, q_u + q_d = 1$ s.t. $1 + R = q_u \cdot u + q_d \cdot d$
- Define a new probability measure Q and the associated expectation E^Q s.t.

$$Q(Z = u) = q_u, \quad Q(Z = d) = q_d$$

$$\frac{1}{1 + R} E^Q S_1 = \frac{1}{1 + R} (q_u \cdot s u + q_d \cdot s d) = \frac{1}{1 + R} \cdot s(1 + R) = s$$

Definition

- **Risk-Neutral / Martingale Measure:** A measure Q satisfies

$$S_0 = \frac{1}{1 + R} E^Q S_1.$$

- **Martingale Probabilities:** $q_u = \frac{(1 + R) - d}{u - d}, \quad q_d = \frac{u - (1 + R)}{u - d}$

Contingent Claims I

Definition

- A **contingent claim** X is of the form $X = \Phi(Z)$
- Stochastic Z with **contract function** $\Phi(\cdot)$
- **Price** of X at time t : $\Pi(t; X)$

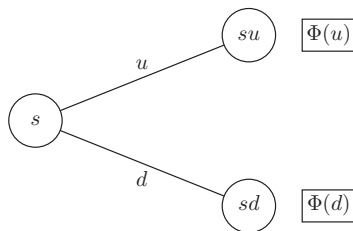


Figure: The Contingent Claim.

Contingent Claims II

Example (European Call Option with Strike K)

Assume $su > K > sd$. At $t = 1$,

- Exercise the option if $S_1 > K$.
 - Pay K to get the stock and sell it at su , thus making net profit $su - K$.
- Do nothing if $S_1 < K$.

$$X = \begin{cases} su - K, & Z = u \\ 0, & Z = d \end{cases}, \quad \begin{cases} \Phi(u) = su - K \\ \Phi(d) = 0 \end{cases}$$

Definition

- A contingent claim X is said to be **reachable** if there exists a portfolio h such that $V_1^h = X$ with probability 1; this portfolio h is called a **hedging** or **replicating** portfolio.
- If all claims can be replicated we say the market is **complete**.

Contingent Claims III

Theorem (Pricing Principle)

If a claim X is reachable with replicating portfolio h , then the “reasonable” price of X is given by $\Pi(t; X) = V_t^h$, $t = 0, 1$.

Theorem

An arbitrage free one period model is complete.

Proof

Fixed any $\Phi(\cdot)$, show that $\exists h = (x, y)$ s.t.

$$V_1^h = \begin{cases} \Phi(u) & Z = u, \\ \Phi(d) & Z = d. \end{cases} \implies x(1+R) + ysu = \Phi(u), \quad x(1+R) + ysd = \Phi(d).$$

$$\text{Solve for } x, y: \quad x = \frac{1}{1+R} \frac{u\Phi(d) - d\Phi(u)}{u-d}, \quad y = \frac{1}{s} \frac{\Phi(u) - \Phi(d)}{u-d}.$$

Risk Neutral Valuation

- From Pricing Principle ($\Pi(t; X) = V_t^h$, $t = 0, 1$)

$$\begin{aligned}\Pi(0; X) &= V_0^h = x + s y \\&= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d} + s \cdot \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d} \\&= \frac{1}{1+R} \left\{ \frac{(1+R) - d}{u-d} \Phi(u) + \frac{u - (1+R)}{u-d} \Phi(d) \right\} \\&= \frac{1}{1+R} \{q_u \Phi(u) + q_d \Phi(d)\} \equiv \frac{1}{1+R} \mathbb{E}^Q X\end{aligned}$$

Theorem (The Risk Neutral Valuation Principle)

If the one period binomial model is arbitrage-free, then the price of X is

$$\Pi(0; X) = \frac{1}{1+R} \mathbb{E}^Q X.$$

The Multiperiod Model

- time t : $t = 0, 1, 2, \dots, T$
- (deterministic) bond B_t with $B_0 = 1$, $B_{n+1} = (1 + R)B_n$
- (stochastic) stock S_t with $S_0 = s > 0$, $S_{n+1} = Z_n S_n$ where $Z_0, Z_1, Z_2, \dots, Z_{T-1}$ are iid with $P(Z_n = u) = p_u$, $P(Z_n = d) = p_d$

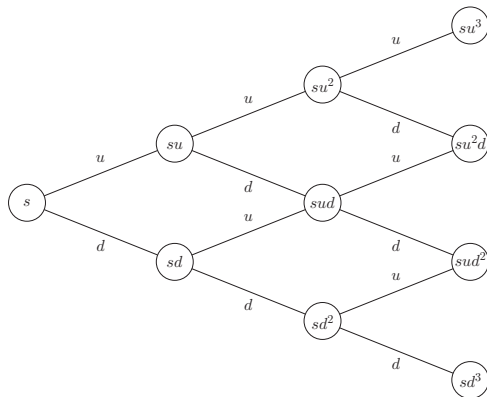


Figure: Asset Dynamics of Multiperiod Model: “Recombining” Tree.