Introduction to Voting Theory

Introduction to Voting

Voting for Group Decision-Making

- Voting: A method to aggregate information across a group
- Types of voting applications:
  - AI: Recommendation systems, PageRank algorithms (search engines), algorithmic decision-making, multi-agent systems
  - Economics: Mechanism design, market design, public choice theory
  - Political science: Electoral systems (population choosing candidates), legislative procedures (passing bills)
  - Law: Jury decision rules (determining verdicts), judicial panels
  - Business: Committee decision-making, corporate governance
  - Critics' rankings (best movies, albums, etc.); Prize committees (selecting award recipients)
- Voting can reflect genuine preference differences or different interpretations of information

Individual Preferences

Individual Preferences

- Each voter has a preference relation over alternatives
- Notation: $X \succ_i Y$ means voter i prefers X to Y
- Two key assumptions about rational preferences:
  - **Completeness**: For any two alternatives X and Y, either $X \succ_i Y$ or $Y \succ_i X$ (no abstention or indifference)
  - **Transitivity**: If $X \succ_i Y$ and $Y \succ_i Z$, then $X \succ_i Z$ (consistency across choices)
- Non-transitive preferences can be problematic:
  - Example: If Chocolate $\succ_i$ Vanilla $\succ_i$ Strawberry $\succ_i$ Chocolate
  - No "best" choice exists - each flavor is defeated by another
  - Leads to indecision or exploitation in sequential choices
- If preferences are complete and transitive, they correspond to a ranked list
- Mathematically provable: Any complete and transitive preferences can be represented as a ranked list, and vice versa

Voting Paradox

Voting Paradox (Condorcet Paradox)

- The Simplest Case: Three voters, three alternatives

Voter	First preference	Second preference	Third preference
Voter 1	A	B	C
Voter 2	B	C	A
Voter 3	C	A	B

- Each voter has rational (transitive) preferences
- Majority rule on pairs produces:
  - A beats B (voters 1 and 3)
  - B beats C (voters 1 and 2)
  - C beats A (voters 2 and 3)
- Result: Cyclic group preferences despite transitive individual preferences
- Paradox: Even with rational individuals, the group can be “irrational”
- This creates fundamental problems for democratic decision-making

Voting Paradox in Social Contexts

- A More Complicated Situation: Party preferences over spending priorities

Party	First preference	Second preference	Third preference
Left (3)	education	health	security
Center (4)	health	security	education
Right (5)	security	education	health

- Individual vs. Multi-criteria decision making:
  - The paradox can arise even for a single individual deciding between options with multiple criteria
  - Example: College choice based on ranking, class size, and scholarship money

College	National Ranking	Average Class Size	Scholarship Money
A	4	40	$3000
B	8	18	$1000
C	12	24	$8000

  - When each option wins on different criteria, cycling can occur

Voting Systems: Majority Rule

Voting Systems: Majority Rule

- For two alternatives: the alternative preferred by majority wins
  - Natural, intuitive, and widely accepted
  - Treats all voters equally
  - Produces a complete, transitive ranking
- For three or more alternatives:
  - Create group preferences by majority vote on each pair (pairwise comparisons)
  - Problem: Group preferences may not be transitive (Condorcet Paradox)
  - Cannot simply produce a ranked list when cycles occur
- Elimination tournaments based on majority rule:
  - Structure comparisons as sequential eliminations
  - Arrange alternatives in pairs for voting
  - The winner advances to face the next alternative
  - Continue until overall winner emerges
- Key insight: The ordering of the pairs (the "agenda") affects the outcome
- Raises concerns about fairness and manipulation

Example of Strategic Agenda-Setting

- Individual rankings:
  - Voter 1: $X \succ Y \succ$ Z
  - Voter 2: $Y \succ Z \succ$ X
  - Voter 3: $Z \succ X \succ$ Y
- Pairwise majority votes:
  - X beats Y (voters 1 and 3)
  - Y beats Z (voters 1 and 2)
  - Z beats X (voters 2 and 3)
- Agenda manipulation through different tournament structures:
  - Agenda 1: X vs Y first, then winner vs $Z \implies Z$ wins
  - Agenda 2: Y vs Z first, then winner vs $X \implies X$ wins
  - Agenda 3: X vs Z first, then winner vs $Y \implies Y$ wins
- Power of agenda-setting: The person who controls the order of voting can determine the outcome
- Real-world impact: Parliamentary procedure, committee votes, and meeting agendas all involve this kind of strategic sequencing

Voting Systems: Positional Voting

Positional Voting Systems

- Direct approach: Assign weights based on position in each voter's ranking
- Borda Count: Named after Jean-Charles de Borda (1770)
  - With k alternatives: $k-1$ points for first place, $k-2$ for second, etc.
  - Each alternative receives points based on its positions in all rankings
  - Alternatives ranked by total points received
  - Used in: Heisman Trophy, AP poll rankings, MLB MVP selection
- Plurality voting:
  - 1 point for first place, 0 for all others
  - Special case of positional voting
  - Used in: Most political elections, "first past the post" systems
- Advantages:
  - Always produces a complete, transitive ranking
  - Considers all positions in rankings (Borda)
  - Simple to implement and understand
- Key problems:
  - Results can be manipulated by strategic voting
  - "Irrelevant" alternatives can change the outcome
  - Vulnerable to strategic addition or removal of alternatives

Example of Borda Count

- Example with 4 alternatives (A, B, C, D) and 2 voters:

Voter	**Ranking**
Voter 1	$\mathsf{A} \succ_1 \mathsf{B} \succ_1 \mathsf{C} \succ_1 \mathsf{D}$
Voter 2	$\mathsf{B} \succ_2 \mathsf{C} \succ_2 \mathsf{A} \succ_2 \mathsf{D}$

- Points assigned:
  - A receives: 3 (from voter 1) + 1 (from voter 2) = 4 points
  - B receives: 2 (from voter 1) + 3 (from voter 2) = 5 points
  - C receives: 1 (from voter 1) + 2 (from voter 2) = 3 points
  - D receives: 0 (from voter 1) + 0 (from voter 2) = 0 points
- Group ranking: $\mathsf{B} \succ \mathsf{A} \succ \mathsf{C} \succ \mathsf{D}$
- Note that B wins even though neither voter places the same alternatives in the same positions
- The Borda Count attempts to account for "strength of preference" by including all positions

Example of Strategic Manipulation in Borda Count

- True preferences of five film critics:

Critics	First	Second	Third
1,2,3	Citizen Kane	The Godfather	Pulp Fiction
4,5	The Godfather	Citizen Kane	Pulp Fiction

- Calculating the Borda Count:
  - Citizen Kane receives: $3(2) + 2(1) = 8$ points
  - The Godfather receives: $3(1) + 2(2) = 7$ points
  - Pulp Fiction receives: $3(0) + 2(0) = 0$ points
- Strategic misrepresentation by critics 4,5:

Critics	First	Second	Third
1,2,3	Citizen Kane	The Godfather	Pulp Fiction
4,5	The Godfather	Pulp Fiction	Citizen Kane

  - Citizen Kane receives: $3(2) + 2(0) = 6$ points
  - The Godfather receives: $3(1) + 2(2) = 7$ points
  - The Godfather now wins by strategically “burying” the main competitor
- Strategic voting is rational when voters understand the system
- This undermines the goal of having votes reflect true preferences

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem

- Question: Is there any voting system that avoids all pathologies we've seen?
- Consider voting systems that satisfy three reasonable properties:
  - **Unanimity (Pareto Principle)**: If all voters prefer X to Y, then the group ranking puts X above Y
  - **Independence of Irrelevant Alternatives (IIA)**: The group ranking of X and Y depends only on how each voter ranks X and Y (not on other alternatives)
  - **Non-dictatorship**: No single voter determines the outcome for all profiles
- Arrow's Theorem (Kenneth Arrow, 1950s):
  - If there are at least three alternatives, then no voting system can satisfy all three properties simultaneously
  - Equivalent formulation: Any voting system satisfying Unanimity and IIA must be a dictatorship
- Consequences:
  - All voting systems must violate at least one of these reasonable properties
  - No "perfect" voting system exists
  - Social choice involves fundamental trade-offs
  - Earned Arrow the Nobel Prize in Economics (1972)

Understanding Independence of Irrelevant Alternatives

- Independence of Irrelevant Alternatives (IIA) is subtle but critical:
  - The group ranking of X and Y should depend only on how each individual ranks X and Y
  - Changes in the ranking of other alternatives shouldn't affect X vs Y outcome
- Example: Two profiles with different rankings but same X vs Y preferences

Profile 1		**Profile 2**	
Individual	**Ranking**	**Individual**	**Ranking**
1	$W \succ X \succ Y \succ Z$	1	$X \succ Y \succ W \succ Z$
2	$W \succ Z \succ Y \succ X$	2	$Z \succ Y \succ X \succ W$
3	$X \succ W \succ Z \succ Y$	3	$W \succ X \succ Y \succ Z$

- In both profiles, individual 1 and 3 prefer X to Y, while individual 2 prefers Y to X
- IIA requires the group ranking of X and Y to be the same in both profiles
- Violations of IIA:
  - The Borda Count violates IIA (as we saw in the film critics example)
  - Elimination tournaments violate IIA (through strategic agenda-setting)
- IIA prevents "irrelevant" alternatives from acting as spoilers

Complete Proof: Setting and Terminology

- Let F be a voting system satisfying Unanimity and IIA
- Notation:
  - P: a profile of individual rankings (complete collection of all voters' rankings)
  - $F(P)$: the group ranking produced by applying F to profile P
  - $X \succ_i Y$: voter i prefers X to Y
  - $X \succ Y$: the group ranking places X above Y
- Goal: Show that F must be a dictatorship (i.e., there exists a voter j such that for any profile, the group ranking always matches j's individual ranking)
- Proof approach:
  - Step 1: Show that polarizing alternatives must be ranked first or last
  - Step 2: Identify a voter with decisive power
  - Step 3: Prove this voter is a dictator for all pairs of alternatives

Step 1: Polarizing Alternatives

- **Definition**: An alternative X is polarizing if every voter ranks it either first or last
- **Claim**: In any profile P where X is polarizing, F must place X either first or last in the group ranking $F(P)$
- Proof by contradiction:
    - Suppose X is neither first nor last in $F(P)$
    - Then there exist alternatives Y, Z such that $Y \succ X \succ Z$ in $F(P)$
    - Construct profile P' by moving Z ahead of Y in each ranking where Y was preferred to Z
    - The relative positions of X vs. Y and X vs. Z remain unchanged in each individual ranking
    - By IIA, the group ranking of X vs. Y and X vs. Z must remain the same in $F(P')$
    - So $Y \succ X \succ Z$ still holds in $F(P')$
    - But in P', every voter ranks Z ahead of Y
    - By Unanimity, $Z \succ Y$ must hold in $F(P')$
    - This creates a cycle: $Y \succ X \succ Z \succ$ Y, contradicting transitivity
- Therefore, X must be ranked either first or last in $F(P)$

Step 2: Identifying a Potential Dictator

- Construct a sequence of profiles P_0, P_1, ..., P_k where:
  - P_0: All voters rank alternative X last
  - P_i: The first i voters rank X first, the rest rank X last
  - P_k: All voters rank X first
- By Unanimity:
  - X is ranked last in $F(P_0)$
  - X is ranked first in $F(P_k)$
- Therefore, X must change position from last to first at some point
- Let j be the first index such that X is not last in $F(P_j)$
- Since X is polarizing in P_j, and not last in $F(P_j)$, it must be first in $F(P_j)$
- Voter j has decisive power: changing just j's vote moves X from last to first in the group ranking
- This voter j is our candidate for being the dictator

Step 3a: Proving j is a Dictator

- We must show j is a dictator for all pairs of alternatives
- First, consider any Y, $Z \neq X$ where j ranks Y above Z
- Construct a profile Q' where:
  - X is ranked first by voters $1, \ldots, j$ and last by others
  - In j's ranking, Y is placed just ahead of X
  - All other relative orderings remain the same as in P_j
- Observations:
  - Q' and P_j are identical when restricted to X and Z, so by IIA, $X \succ Z$ in $F(Q')$
  - Q' and P_{j-1} are identical when restricted to X and Y, so by IIA, $Y \succ X$ in $F(Q')$
  - By transitivity of the group ranking, $Y \succ Z$ in $F(Q')$
- For any profile Q where j ranks Y above Z:
  - Q and Q' are identical when restricted to Y and Z
  - By IIA, $Y \succ Z$ in $F(Q)$
- Therefore, j dictates the group ranking for all pairs not involving X

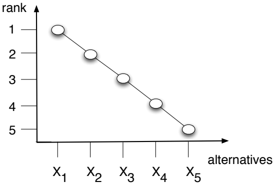
Step 3b: Proving j is a Dictator

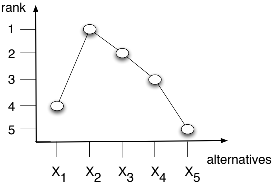
- Now we must show j is also a dictator for pairs involving X
- Proof by contradiction:
  - Suppose there exists another voter $\ell \neq j$ who dictates some pair involving X
  - Apply the same construction using a different alternative W instead of X
  - This would establish ℓ as a dictator for all pairs not involving W
  - Consider X and some third alternative Y different from X and W
  - For profiles P_{j-1} and P_j:
    - These profiles differ only in j's ranking
    - The ordering of X and Y changes in the group ranking
    - But this contradicts ℓ being a dictator for this pair
- Therefore:
  - Voter j must be the dictator for all pairs
  - The only voting systems satisfying Unanimity and IIA are dictatorships
  - This completes the proof of Arrow's Impossibility Theorem

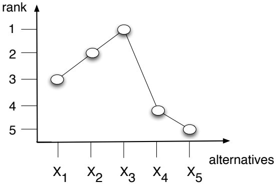
Single-Peaked Preferences

Single-Peaked Preferences I

- Definition: A voter has single-peaked preferences if there is no alternative X_s for which both neighboring alternatives X_{s-1} and X_{s+1} are preferred to X_s
- Intuition: Alternatives are ordered along a spectrum, and each voter has a most preferred point
  - Preferences decrease consistently moving away from that peak in either direction
  - No "valleys" in the preference ranking
- Visual representation examples:







Single-Peaked Preference II

- Natural settings for single-peaked preferences:
  - Political candidates (left to right spectrum)
  - Levels of spending (low to high amounts)
  - Temperature settings (cold to hot)
  - Geographic locations (distance from ideal point)
  - Tax rates (optimal rate somewhere between 0% and 100%)
  - Environmental regulations (balance between economic and ecological concerns)
- Importance: Majority rule works well with single-peaked preferences
- Example of non-single-peaked preferences: Preferring extremes to middle positions
- Real-world implications:
  - Many political preferences naturally follow a single-peaked pattern
  - Economic policy preferences often peak at a voter's ideal point on a spectrum
  - When preferences are single-peaked, voting cycles are less likely to occur
  - This helps explain why many democratic systems work despite Arrow's theorem

The Median Voter Theorem I

- If all preferences are single-peaked, then:
  - Majority rule applied to pairs produces transitive group preferences
  - The "median voter's" favorite alternative defeats all others in pairwise majority votes
- Median voter: The voter whose favorite alternative is the median among all voters' favorites
- Intuition: The median voter's favorite position has majority support against any alternative
- Proof outline:
  - For any alternative to the right of the median, all voters with peaks at or left of the median prefer the median
  - This gives the median position majority support against all alternatives to its right
  - Similarly, the median defeats all alternatives to its left
  - Therefore, the median position wins all pairwise contests

The Median Voter Theorem II

- Consequences:
  - No cycles in majority rule when preferences are single-peaked
  - Political candidates tend to adopt positions near the median voter
  - Explains the tendency toward moderation in two-party systems
  - Economic policy often targets the middle class ("median income voter")
  - Stability of democratic outcomes despite theoretical challenges
- Limitations:
  - Assumes alternatives can be ordered on a single dimension
  - Complex issues often involve multiple dimensions
  - Strategic behavior can still affect outcomes

Voting as Information Aggregation

The Condorcet Jury Theorem

- Setting:
  - Two alternatives, one of which is objectively better
  - Each voter receives an independent signal about which is better
  - Signals favor the correct alternative with probability $q > 1/2$
- Condorcet Jury Theorem:
  - As the number of voters increases, the probability that the majority chooses the correct alternative approaches 1
- Mathematical formulation:
  - Let X_i be 1 if voter i votes correctly, 0 otherwise
  - Each X_i is independent with $\mathbb{P}(X_i = 1) = q > 1/2$
  - By the Law of Large Numbers, as $n \to \infty$:

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{1}{2}\right) \to 1$$

- Implications:
  - "Wisdom of crowds" in situations with objectively correct answers
  - Larger juries may reach more accurate verdicts
  - Statistical foundation for democratic decision-making
  - Provides theoretical justification for polling and aggregating expert opinions
  - Shows how collective intelligence can exceed individual intelligence

Insincere Voting for Information Aggregation

Insincere Voting for Information Aggregation

- Surprising result: Sometimes voters should vote insincerely even when trying to reach the correct group decision
- Example scenario:
  - Urn with either all white marbles or 90% green, 10% white
  - Each person draws one marble, then votes on urn type
  - Group wins only if majority vote is correct
- Strategic insight:
  - Your vote only matters when it breaks a tie
  - In that case, others' signals provide valuable information
  - Optimal strategy may be to vote against your own signal
- Mathematical formulation:
  - Let s_i be voter i's signal
  - A rational voter computes: $\mathbb{P}(\text{correct urn} \mid s_i, \text{my vote matters})$
  - This probability may favor the opposite of what s_i suggests
- Broader implications:
  - Strategic voting can actually improve group accuracy
  - Optimal voting behavior should account for pivotality
  - Simple majority rule may not extract all available information
  - Suggests need for mechanisms that encourage information sharing
  - Relates to "pivotal voter" models in political economy

Jury Decisions and Unanimity Rule

- Criminal trials: Conviction requires unanimous vote
- "Beyond reasonable doubt" standard: High threshold for conviction
- Each juror receives private signals about guilt/innocence
- Paradox with unanimity rule:
  - A juror's vote matters only when everyone else votes to convict
  - This implies strong evidence of guilt, even with an innocence signal
  - Creates incentive to disregard innocence signals
- Game-theoretic analysis (Feddersen & Pesendorfer, 1998):
  - In equilibrium, jurors with "innocent" signals sometimes vote to convict
  - As jury size increases, probabilty of convicting innocent defendants doesn't vanish
  - Supermajority rules (e.g., 10 out of 12) may produce better outcomes than unanimity
- Counterintuitive result: Unanimity rule may lead to more false convictions than majority rule
- Policy implications:
  - Voting rules should account for strategic behavior
  - Deliberation before voting may improve information sharing
  - Legal systems should balance error costs carefully
  - Simple voting rules may have complex strategic consequences

Sequential Voting and Information Cascades

Sequential Voting and Information Cascades

- When voting is sequential rather than simultaneous: Later voters see earlier votes (but not private signals), information cascades can develop
- Process:
  - After two votes for the same alternative, all subsequent voters rationally disregard their own signals; all follow the established pattern
  - Can lead to incorrect group decision even with many voters
- Mathematical formulation:
  - Voter n decides based on prior votes $v_1, \ldots, v_{n-1}$ and private signal s_n
  - Vote according to $\mathbb{P}(\text{correct option} \mid v_1, \ldots, v_{n-1}, s_n)$
  - After certain voting patterns, this probability becomes independent of s_n
- Key differences from Condorcet Jury Theorem:
  - Sequential voting can lead to wrong cascades
  - Adding more voters doesn't guarantee correct outcome
  - Group decision uses only the first few signals
  - Initial voters have disproportionate influence
  - Small changes in initial conditions can lead to different outcomes
- Real-world examples:
  - Primary elections (early states influence later voters)
  - Committee discussions (early speakers shape consensus)
  - Online reviews and ratings (early ratings influence later evaluations)
  - Academic citation patterns (papers with early citations attract more)

Conclusion

Key Takeaways

- Voting systems face fundamental limitations:
  - Condorcet Paradox: Even with rational individuals, group preferences can be cyclical
  - Arrow's Impossibility Theorem: No voting system can satisfy all desired properties simultaneously
  - Sequential voting can lead to information cascades that disregard most available information
- Practical implications:
  - Majority rule works well with single-peaked preferences (Median Voter Theorem)
  - Different voting contexts require different systems with different trade-offs
  - Strategic concerns must be considered in voting system design
  - Institutional design should account for strategic voter behavior
  - Deliberation before voting may improve outcomes
- Information aggregation aspects:
  - Condorcet Jury Theorem shows wisdom of crowds in simple settings
  - But insincere voting and information cascades can limit this wisdom
  - Voting rules like unanimity can have unexpected consequences
  - Optimal aggregation mechanisms depend on information structure
  - Transparency vs. privacy trade-offs in information revelation