

Portfolio Choice

Utility

Preferences and Utility

- Let Γ be a sample space of possible outcomes for gambles with random payoffs
- Let \mathbb{P} be a set of probabilities on Γ , where $\mathcal{A} \in \mathbb{P}$ satisfies:
 - (i) $0 \leq \mathcal{A}(G) \leq 1$, for all $G \subseteq \Gamma$
 - (ii) $\mathcal{A}(\Gamma) = 1$
 - (iii) For disjoint events $\{G_i\}_i$: $\mathcal{A}(\bigcup_i G_i) = \sum_i \mathcal{A}(G_i)$
- $\mathcal{A} \in \mathbb{P}$ is a *gamble* - probability distribution of the outcome
- \mathbb{P} is closed under convex combinations: $p\mathcal{A} + (1-p)\mathcal{B} \in \mathbb{P} \forall 0 \leq p \leq 1$
- The gamble $p\mathcal{A} + (1-p)\mathcal{B}$ corresponds to tossing a coin with probability p of “heads”, choosing \mathcal{A} for heads and \mathcal{B} for tails
- By induction, for $p_i \geq 0$ with $\sum_{i=1}^k p_i = 1$: $p_1\mathcal{A}_1 + \dots + p_k\mathcal{A}_k \in \mathbb{P}$
- An investor has a preference relation \succ on \mathbb{P}
- $\mathcal{A} \succ \mathcal{B}$ means “ \mathcal{A} is preferred to \mathcal{B} ”
- Define the indifference relation \sim on \mathbb{P} by setting $\mathcal{A} \sim \mathcal{B}$ when $\mathcal{A} \not\succ \mathcal{B}$ and $\mathcal{B} \not\succ \mathcal{A}$; $\mathcal{A} \sim \mathcal{B}$ means “investor is indifferent between \mathcal{A} and \mathcal{B} ”
- The relations \succ and \sim satisfy rational axioms as follows.

Rational Axioms

- 1 (Completeness) For any $\mathcal{A}, \mathcal{B} \in \mathbb{P}$ exactly one of the following holds:
 - (i) $\mathcal{A} \succ \mathcal{B}$
 - (ii) $\mathcal{B} \succ \mathcal{A}$
 - (iii) $\mathcal{A} \sim \mathcal{B}$
- 2 (Equivalence Relation) The relation \sim is an equivalence relation on \mathbb{P} :
 - (i) $\mathcal{A} \sim \mathcal{A}$ for all $\mathcal{A} \in \mathbb{P}$
 - (ii) If $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ then $\mathcal{A} \sim \mathcal{C}$
 - (iii) If $\mathcal{A} \sim \mathcal{B}$ then $\mathcal{B} \sim \mathcal{A}$
- 3 (Transitivity of Preference) For any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$, if $\mathcal{A} \succ \mathcal{B}$ and $\mathcal{B} \succ \mathcal{C}$ then $\mathcal{A} \succ \mathcal{C}$.
- 4 (Mixed Transitivity) For any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$,
 - (i) If $\mathcal{A} \succ \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ then $\mathcal{A} \succ \mathcal{C}$
 - (ii) If $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \succ \mathcal{C}$ then $\mathcal{A} \succ \mathcal{C}$
- 5 (Independence – Indifference) For any $\mathcal{A}, \mathcal{C} \in \mathbb{P}$ and $p \in [0, 1]$, if $\mathcal{A} \sim \mathcal{C}$ and $\mathcal{B} \in \mathbb{P}$ then $p\mathcal{A} + (1-p)\mathcal{B} \sim p\mathcal{C} + (1-p)\mathcal{B}$.
- 6 (Independence – Preference) For any $\mathcal{A}, \mathcal{C} \in \mathbb{P}$ and $p \in (0, 1]$, if $\mathcal{A} \succ \mathcal{C}$ and $\mathcal{B} \in \mathbb{P}$ then $p\mathcal{A} + (1-p)\mathcal{B} \succ p\mathcal{C} + (1-p)\mathcal{B}$.
- 7 (Continuity) For any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$, if $\mathcal{A} \succ \mathcal{C} \succ \mathcal{B}$ then there exists $p \in [0, 1]$ with $p\mathcal{A} + (1-p)\mathcal{B} \sim \mathcal{C}$.

Uniqueness of Probability Values

Lemma

Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{P}$ with $\mathcal{A} \succ \mathcal{C} \succ \mathcal{B}$ and $p\mathcal{A} + (1-p)\mathcal{B} \sim \mathcal{C}$, then $0 < p < 1$ and p is unique.

Proof

- Trivially $p \neq 0, 1$; Suppose p is not unique: $\exists q$ with $q\mathcal{A} + (1-q)\mathcal{B} \sim \mathcal{C}$
- WLOG assume $q < p$, so $0 < p - q < 1 - q$
- Note that $\mathcal{B} = \left(\frac{p-q}{1-q}\right)\mathcal{B} + \left(\frac{1-p}{1-q}\right)\mathcal{B}$ and $\mathcal{A} \succ \mathcal{B}$
- By Independence – Preference Axiom $\left(\frac{p-q}{1-q}\right)\mathcal{A} + \left(\frac{1-p}{1-q}\right)\mathcal{B} \succ \mathcal{B}$
- However $p\mathcal{A} + (1-p)\mathcal{B} = q\mathcal{A} + (1-q)\left(\left(\frac{p-q}{1-q}\right)\mathcal{A} + \left(\frac{1-p}{1-q}\right)\mathcal{B}\right)$
- By Independence – Preference Axiom again $p\mathcal{A} + (1-p)\mathcal{B} \succ q\mathcal{A} + (1-q)\mathcal{B}$
- But this contradicts that both expressions are indifferent to \mathcal{C}

Existence of Utility Function

Theorem

There exists a real-valued function $f : \mathbb{P} \rightarrow \mathbb{R}$ with

$$f(\mathcal{A}) > f(\mathcal{B}) \quad \text{if and only if} \quad \mathcal{A} \succ \mathcal{B},$$

and

$$f(p\mathcal{A} + (1-p)\mathcal{B}) = pf(\mathcal{A}) + (1-p)f(\mathcal{B})$$

for any $\mathcal{A}, \mathcal{B} \in \mathbb{P}$ and $0 \leq p \leq 1$. Furthermore, f is unique up to affine transformations.

Proof of Utility Function Existence (Part 1)

Proof

- If $\mathcal{A} \sim \mathcal{B}$ for all $\mathcal{A}, \mathcal{B} \in \mathbb{P}$, take $f(\mathcal{A}) \equiv 0$
- Otherwise, \exists a pair $\mathcal{C}, \mathcal{D} \in \mathbb{P}$ with $\mathcal{C} \succ \mathcal{D}$
- For any $\mathcal{A} \in \mathbb{P}$, five possibilities:

$$\textcircled{a} \quad \mathcal{A} \succ \mathcal{C}$$

$$\textcircled{c} \quad \mathcal{C} \succ \mathcal{A} \succ \mathcal{D}$$

$$\textcircled{e} \quad \mathcal{D} \succ \mathcal{A}$$

$$\textcircled{b} \quad \mathcal{A} \sim \mathcal{C}$$

$$\textcircled{d} \quad \mathcal{A} \sim \mathcal{D}$$

- Define $f(\mathcal{C}) = 1$ and $f(\mathcal{D}) = 0$
- For case (a): \exists unique $p \in (0, 1)$ with $p\mathcal{A} + (1-p)\mathcal{D} \sim \mathcal{C}$; define $f(\mathcal{A}) = \frac{1}{p}$
- For case (b): set $f(\mathcal{A}) = 1$
- For case (c): \exists unique $q \in (0, 1)$ with $q\mathcal{C} + (1-q)\mathcal{D} \sim \mathcal{A}$; define $f(\mathcal{A}) = q$
- For case (d): set $f(\mathcal{A}) = 0$
- For case (e): \exists unique $r \in (0, 1)$ with $r\mathcal{C} + (1-r)\mathcal{A} \sim \mathcal{D}$; define
$$f(\mathcal{A}) = \frac{-r}{1-r}$$

Proof of Utility Function Existence (Part 2)

Proof (continued)

- To verify f satisfies the conditions requires checking 15 cases: 5 where both \mathcal{A} and \mathcal{B} are in the same case, and 10 where they're in different cases
- Consider one example: both \mathcal{A} and \mathcal{B} satisfy case (c): $\mathcal{C} \succ \mathcal{A} \succ \mathcal{D}$ and $\mathcal{C} \succ \mathcal{B} \succ \mathcal{D}$
- We have $f(\mathcal{A}) = q_1$ and $f(\mathcal{B}) = q_2$ where $\mathcal{A} \sim q_1 \mathcal{C} + (1 - q_1) \mathcal{D}$ and $\mathcal{B} \sim q_2 \mathcal{C} + (1 - q_2) \mathcal{D}$
- When $q_1 = q_2$: $\mathcal{A} \sim \mathcal{B}$ and condition is satisfied
- When $q_1 > q_2$: $q_1 \mathcal{C} + (1 - q_1) \mathcal{D} \succ q_2 \mathcal{C} + (1 - q_2) \mathcal{D}$, so $\mathcal{A} \succ \mathcal{B}$ and $f(\mathcal{A}) > f(\mathcal{B})$ as required
- Similarly when $q_1 < q_2$: $\mathcal{B} \succ \mathcal{A}$
- For linearity, let $p \in (0, 1)$ and apply Independence – Indifference Axiom:

$$\begin{aligned} p \mathcal{A} + (1 - p) \mathcal{B} &\sim p(q_1 \mathcal{C} + (1 - q_1) \mathcal{D}) + (1 - p)(q_2 \mathcal{C} + (1 - q_2) \mathcal{D}) \\ &\sim (pq_1 + (1 - p)q_2)\mathcal{C} + (p(1 - q_1) + (1 - p)(1 - q_2))\mathcal{D} \end{aligned}$$

Proof of Utility Function Existence (Part 3)

Proof (continued)

- From definition of f :
$$f(p\mathcal{A} + (1-p)\mathcal{B}) = pq_1 + (1-p)q_2 = pf(\mathcal{A}) + (1-p)f(\mathcal{B})$$
 - To verify f is unique up to affine transformations, suppose g also satisfies the conditions
 - Since $\mathcal{C} \succ \mathcal{D}$, we have $g(\mathcal{C}) > g(\mathcal{D})$
 - Define $\beta = g(\mathcal{D})$ and $\alpha = g(\mathcal{C}) - g(\mathcal{D}) > 0$
 - For an \mathcal{A} in case (c) with $f(\mathcal{A}) = q$, we have $\mathcal{A} \sim q\mathcal{C} + (1-q)\mathcal{D}$
 - Therefore $g(\mathcal{A}) = g(q\mathcal{C} + (1-q)\mathcal{D}) = qg(\mathcal{C}) + (1-q)g(\mathcal{D}) = q(\alpha + \beta) + (1-q)\beta = q\alpha + \beta = \alpha f(\mathcal{A}) + \beta$
 - The other cases follow similarly, proving $g(\mathcal{A}) = \alpha f(\mathcal{A}) + \beta$ for all $\mathcal{A} \in \mathbb{P}$
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- For an investor with consistent preferences, there exists a function f , unique up to affine transformations, which quantifies preference ordering
 - For $p_i \geq 0$ with $\sum_{i=1}^k p_i = 1$: $f\left(\sum_{i=1}^k p_i \mathcal{A}_i\right) = \sum_{i=1}^k p_i f(\mathcal{A}_i)$

Expected Utility

- In finance, an investor faces investments yielding random payoffs
- Let Ω be a probability space with measure P
- Let \mathcal{X} be the set of real-valued random variables on Ω
- For $X \in \mathcal{X}$, let P^X be the probability distribution on \mathbb{R} induced by X
- Take $\Gamma = \mathbb{R}$ and $\mathbb{P} = \{P^X : X \in \mathcal{X}\}$
- If X takes values $\{x_1, \dots, x_m\}$, then:

$$P^X(\{x\}) = \begin{cases} P(X = x) & \text{for } x \in \{x_1, \dots, x_m\}, \\ 0 & \text{otherwise.} \end{cases}$$

- Define a utility function $v : \mathbb{R} \rightarrow \mathbb{R}$ by $v(x) = f(P^x)$, where P^x assigns probability 1 to value x
- Then $f(P^X) = \sum_{i=1}^m f(P^{x_i}) P(X = x_i) = \sum_{i=1}^m v(x_i) P(X = x_i) = E\{v(X)\}$
- This gives us $E\{v(X)\} > E\{v(Y)\} \iff X \succ Y$

Risk Attitudes

- Consider an investment with outcome described by random variable X on (Ω, P)
- Investor has utility function $v : \mathbb{R} \rightarrow \mathbb{R}$ and prefers higher expected utility
- Let E_P denote expectation with respect to P
- An investor is *risk averse* when $E_P v(X) \leq v(E_P X)$ for all random variable X and all probabilities P
- This is equivalent to v being concave
- Given $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$ with $P(X = x) = \lambda$, $P(X = y) = 1 - \lambda$:
 $\lambda v(x) + (1 - \lambda)v(y) \leq v(\lambda x + (1 - \lambda)y)$
- Risk aversion implies preferring certain outcome μ to random investment with mean μ
- An investor is *risk neutral* when $E_P v(X) = v(E_P X)$ for all P and X
- Risk neutrality corresponds to v being affine (linear)
- An investor is *risk preferring* when $E_P v(X) > v(E_P X)$, corresponding to v being convex

Risk Premiums

- *Compensatory risk premium* α is the amount that must be added to make an investor indifferent between a risky investment and a certain outcome:

$$E\{v(\alpha + X)\} = v(\mu) \quad \text{with} \quad \mu = E X$$

- *Insurance risk premium* β is the amount an investor would pay to avoid risk:

$$E\{v(X)\} = v(\mu - \beta) \quad \text{with} \quad \mu = E X$$

- The insurance premium β satisfies: if X and Y have same mean μ and v is strictly increasing, then $X \succ Y \iff \beta_X < \beta_Y$
- Using the Taylor expansion of $E\{v(X)\}$ about $\mu = E X$, we have $E\{v(X)\} = E \left\{ v(\mu) + (X - \mu)v'(\mu) + \frac{(X - \mu)^2}{2}v''(\mu) + \dots \right\} = v(\mu) + \frac{\text{var } X}{2}v''(\mu) + \dots$
- Expanding $v(\mu - \beta)$ and equating yields $\beta \approx \frac{1}{2} \left(\frac{-v''(\mu)}{v'(\mu)} \right) \text{var } X$
- $-\frac{v''(\mu)}{v'(\mu)}$ is the *Arrow-Pratt absolute risk aversion* at μ
- $-\frac{E\{v''(X)\}}{E\{v'(X)\}}$ is the *global absolute risk aversion* for investment X

HARA Utility Functions

- *Hyperbolic absolute risk aversion* (HARA) functions have form:

$$v(x) = \frac{1-\gamma}{\gamma} \left(\frac{ax}{1-\gamma} + b \right)^\gamma$$

for constants a, b, γ with $\frac{ax}{1-\gamma} + b \geq 0$ (usually $b \geq 0$)

- Arrow-Pratt risk aversion for HARA functions:

$$-\frac{v''(x)}{v'(x)} = \left(\frac{x}{1-\gamma} + \frac{b}{a} \right)^{-1}$$

- Special cases (possibly with affine transformations):

- Ⓐ *Quadratic*: $v(x) = x - \frac{1}{2}\theta x^2$; take $\gamma = 2$, $a = \sqrt{\theta}$, $ab = 1$
- Ⓑ *Exponential*: $v(x) = -e^{-ax}$; let $\gamma \rightarrow -\infty$. Has absolute risk aversion a
- Ⓒ *Power*: $v(x) = x^\gamma$ with $\gamma > 0$. Strictly concave only when $\gamma < 1$. Case $\gamma = 1$ gives risk-neutral utility
- Ⓓ *Logarithmic*: $v(x) = \ln x$. Follows from HARA as $\gamma \rightarrow 0$, using l'Hôpital's rule:
$$\frac{x^\gamma - 1}{\gamma} \rightarrow \ln x$$

Mean-Variance Analysis

Classical PO: Mean-Variance (MV) Criterion

- Assets evolve from time 0 to time 1 for one period
- s : # of risky assets
- $\mathbf{S}_0 \equiv (S_{1,0}, S_{2,0}, \dots, S_{s,0})^\top \neq \mathbf{0}$: the constant price vector at time 0
- $\mathbf{S}_1 \equiv (S_{1,1}, S_{2,1}, \dots, S_{s,1})^\top$: the random price vector at time 1
- $\mathbf{x} \equiv (x_1, x_2, \dots, x_s)^\top$: the proportion vector of the time-0 wealth invested in each asset; $\sum_{i=1}^s x_i = 1$.
- $\mathbf{R} \equiv (R_1, R_2, \dots, R_s)^\top$: the random vector representing the rate of return on the assets; $R_i = \frac{S_{i,1}}{S_{i,0}}$
- w : the (constant) wealth at time 0

Classical PO: Mean-Variance (MV) Criterion

- W : the (random) wealth at time 1; $W = \left(\sum_{i=1}^s x_i R_i \right) w = \mathbf{x}^\top \mathbf{R} w$
(For asset S_i , $\frac{x_i w}{S_{i,0}}$ denotes the “quantity” allocated at time 0; so at time 1 this part of wealth becomes $\frac{x_i w}{S_{i,0}} S_{i,1} = x_i R_i w$)
- $\mathbf{r} \equiv \mathbb{E} \mathbf{R} = (r_1, r_2, \dots, r_s)^\top$: the (constant) mean vector of \mathbf{R} ; $r_i = \mathbb{E} R_i$
- $\mathbf{V} \equiv \text{cov} \mathbf{R} \equiv \mathbb{E}\{(\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top\}$: the (constant) covariance matrix of \mathbf{R} ; \mathbf{V} is symmetric positive definite $s \times s$ matrix
- $\mathbb{E} W = \mathbb{E}\{\mathbf{x}^\top \mathbf{R}\} = \mathbf{x}^\top \mathbf{r} = \mu$
- $\sigma^2 = \text{var} W = \text{var}\{\mathbf{x}^\top \mathbf{R}\} = \mathbb{E}\{\mathbf{x}^\top (\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top \mathbf{x}\} = \mathbf{x}^\top \mathbf{V} \mathbf{x}$
- “For some fixed mean rate of return $\mu = \mathbb{E}\{\mathbf{x}^\top \mathbf{R}\}$, try to minimize the variance $\sigma^2 = \text{var}\{\mathbf{x}^\top \mathbf{R}\}$ of the return over portfolios \mathbf{x} ”

MV: All Risky Assets

$$\min_{\mathbf{x}} \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} \quad \text{s.t.} \quad \begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \quad \mathbf{e} \equiv \underbrace{(1, 1, \dots, 1)^\top}_{s \text{ items}}$$

- \mathbf{V} is symmetric, positive definite, so \mathbf{V}^{-1} also is
- Set $\mathcal{L} \equiv \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} + \lambda (1 - \mathbf{x}^\top \mathbf{e}) + \nu (\mu - \mathbf{x}^\top \mathbf{r})$ with Lagrange multipliers λ, ν
- By $\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \mathbf{V} \mathbf{x} - \lambda \mathbf{e} - \nu \mathbf{r} = 0 \implies \mathbf{x} = \lambda \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{V}^{-1} \mathbf{r}$
 $\implies \mathbf{x}^\top = \lambda \mathbf{e}^\top (\mathbf{V}^{-1})^\top + \nu \mathbf{r}^\top (\mathbf{V}^{-1})^\top = \lambda \mathbf{e}^\top \mathbf{V}^{-1} + \nu \mathbf{r}^\top \mathbf{V}^{-1}$
- Substitute into $\begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \implies \begin{cases} \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = 1 \\ \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \mu \end{cases}$

- Set $\alpha = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e}$, $\beta = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r}$, $\gamma = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r}$, $\delta \equiv \alpha\gamma - \beta^2$, then

$$\begin{cases} \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = 1 \\ \lambda \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + \nu \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \mu \end{cases}$$

becomes

$$\begin{cases} \lambda\alpha + \nu\beta = 1 \\ \lambda\beta + \nu\gamma = \mu \end{cases}$$

Solutions: $\lambda = \frac{\gamma - \beta\mu}{\delta}$, $\nu = \frac{\alpha\mu - \beta}{\delta}$

- If $\mathbf{r} \neq c\mathbf{e}$, $c \in \mathbb{R}$, then from the positive-definiteness of \mathbf{V}^{-1}

$$\begin{aligned} &(\mathbf{r} - c\mathbf{e})^\top \mathbf{V}^{-1} (\mathbf{r} - c\mathbf{e}) > 0 \\ \implies &\mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} - c \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} - c \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + c^2 \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} > 0 \\ \implies &\gamma - 2c\beta + c^2\alpha > 0 \\ \implies &-\delta = \beta^2 - \gamma\alpha < 0 \end{aligned}$$

- The relation of σ with μ :

$$\begin{aligned}\sigma^2 &= \mathbf{x}^\top \mathbf{V} \mathbf{x} = \mathbf{x}^\top \mathbf{V} (\lambda \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{V}^{-1} \mathbf{r}) = \lambda (\mathbf{x}^\top \mathbf{e}) + \nu (\mathbf{x}^\top \mathbf{r}) \\ &= \lambda + \nu \mu = \frac{\gamma - \beta \mu}{\delta} + \frac{\alpha \mu - \beta}{\delta} \mu = \frac{\alpha \mu^2 - 2\beta \mu + \gamma}{\delta} \\ \Rightarrow \frac{\sigma^2}{\left(\frac{1}{\sqrt{\alpha}}\right)^2} - \frac{\left(\mu - \frac{\beta}{\alpha}\right)^2}{\left(\frac{\sqrt{\delta}}{\alpha}\right)^2} &= 1\end{aligned}$$

- Recall the standard form of hyperbola (x, y)

$$\text{equation: } \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\text{asymptotes: } (y-k) = \pm \frac{b}{a}(x-h)$$

- Here we have (σ, μ) with $a = \frac{1}{\sqrt{\alpha}}$, $b = \frac{\sqrt{\delta}}{\alpha}$, $h = 0$, $k = \frac{\beta}{\alpha}$, the asymptotes are $\left(\mu - \frac{\beta}{\alpha}\right) = \pm \frac{\frac{\sqrt{\delta}}{\alpha}}{\frac{1}{\sqrt{\alpha}}} \sigma \Rightarrow \mu = \frac{\beta}{\alpha} \pm \sqrt{\frac{\delta}{\alpha}} \sigma$

- Global minimum-variance portfolio \mathbf{x}_g

- First find μ_g that minimizes $\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$: By differentiation

$$2\alpha\mu_g - 2\beta = 0 \implies \mu_g = \frac{\beta}{\alpha}$$

- $\lambda_g = \frac{\gamma - \beta\mu_g}{\delta} = \frac{\gamma - \beta\frac{\beta}{\alpha}}{\delta} = \frac{\gamma\alpha - \beta^2}{\alpha\delta} = \frac{1}{\alpha}$

$$\nu_g = \frac{\alpha\mu_g - \beta}{\delta} = \frac{\beta - \beta}{\delta} = 0$$

$$\text{so } \mathbf{x}_g = \lambda_g \mathbf{V}^{-1}\mathbf{e} + \nu_g \mathbf{r}^\top \mathbf{V}^{-1} = \frac{1}{\alpha} \mathbf{V}^{-1}\mathbf{e}$$

- Diversified portfolio: define $\mathbf{x}_d \equiv \frac{1}{\beta} \mathbf{V}^{-1}\mathbf{r}$, then the expected return

$$\mu_d = \mathbf{x}_d^\top \mathbf{r} = \frac{1}{\beta} \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \frac{\gamma}{\beta}$$

- $\mathbf{x} = \lambda \mathbf{V}^{-1}\mathbf{e} + \nu \mathbf{V}^{-1}\mathbf{r} = \lambda \alpha \mathbf{x}_g + \nu \beta \mathbf{x}_d$, so **every portfolio is the convex combination of \mathbf{x}_g and \mathbf{x}_d** : note that $\lambda\alpha + \nu\beta = 1$ (constraint $\mathbf{x}^\top \mathbf{e} = 1$) !

Theorem (Mutual Fund Theorem)

Any minimum-variance portfolio is equivalent to investing in the convex combination of \mathbf{x}_g and \mathbf{x}_d .

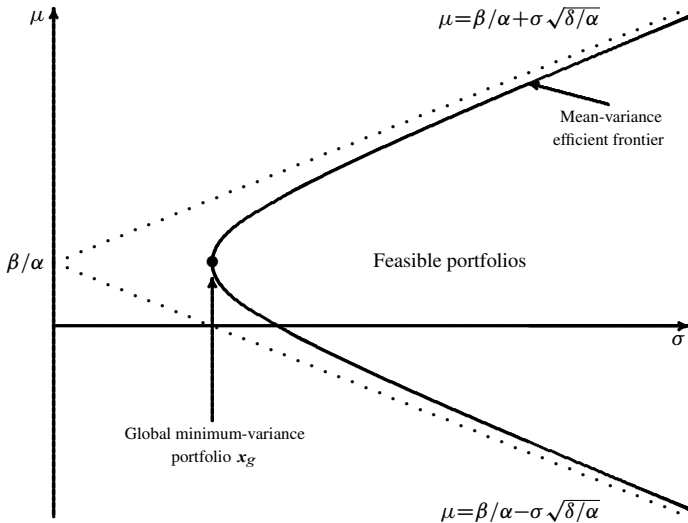


Figure: The Case of All Risky Assets

Theorem

Diversified portfolio \mathbf{x}_d is the portfolio that maximize $s(\mathbf{x}) \equiv \frac{\mathbf{x}^\top \mathbf{r}}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$.

Proof

- Maximize $s(\mathbf{x}) \equiv \text{maximize } \log(s(\mathbf{x}))$ s.t. $\mathbf{x}^\top \mathbf{e} = 1$
- Change of variable: $\mathbf{x}^\top \mathbf{r} = \mu \implies \log(s(\mathbf{x})) = \log \frac{\mu}{\sqrt{\frac{\alpha \mu^2 - 2\beta \mu + \gamma}{\delta}}} \equiv f(\mu)$

with $\mu > 0$

- $f'(\mu) = \frac{\gamma - \beta \mu}{\mu \left(\alpha \left(\mu - \frac{\beta}{\alpha} \right)^2 + \frac{\delta}{\alpha} \right)} = 0$ at $\mu = \frac{\gamma}{\beta} = \mu_d$

- The covariance between the return of the global minimum-variance portfolio and other minimum-variance portfolio is constant:

$$\begin{aligned} \text{cov}(\mathbf{x}_g^\top \mathbf{R}, \mathbf{x}^\top \mathbf{R}) &= \mathbf{x}_g^\top \mathbf{V} \mathbf{x} = \mathbf{x}_g^\top \mathbf{V} (\lambda \mathbf{V}^{-1} \mathbf{e} + \nu \mathbf{V}^{-1} \mathbf{r}) = \lambda \mathbf{x}_g^\top \mathbf{e} + \nu \mathbf{x}_g^\top \mathbf{r} \\ &= \frac{\lambda}{\alpha} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \frac{\nu}{\alpha} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} = \frac{\lambda \alpha + \nu \beta}{\alpha} = \frac{1}{\alpha} \end{aligned}$$

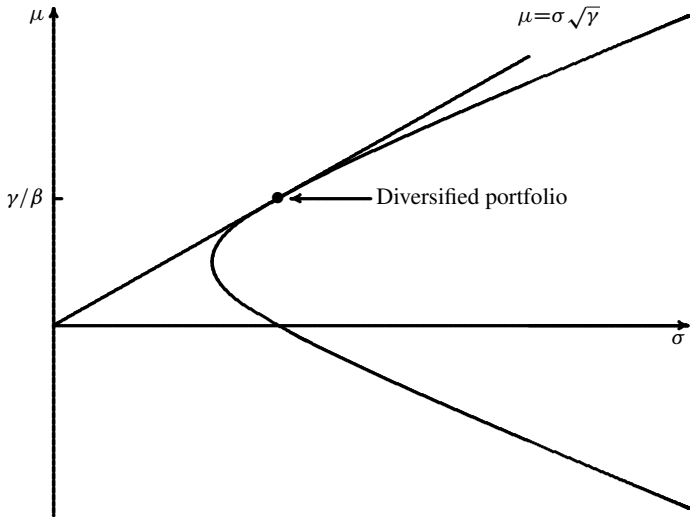


Figure: The Diversified Portfolio

MV: All But One Risky Assets

WLOG add riskless asset 0 with constant return r_0 ; the portfolio becomes $(x_0, x_1, x_2, \dots, x_s)^\top$

$$\min_{x_0, \mathbf{x}} \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} \quad \text{s.t.} \quad \begin{cases} x_0 + \mathbf{x}^\top \mathbf{e} = 1 \\ x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \quad \mathbf{e} \equiv \underbrace{(1, 1, \dots, 1)^\top}_{s \text{ items}}$$

- Set $\bar{\mathcal{L}} \equiv \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} + \bar{\lambda} (1 - x_0 - \mathbf{x}^\top \mathbf{e}) + \bar{\nu} (\mu - x_0 r_0 - \mathbf{x}^\top \mathbf{r})$ with Lagrange multipliers $\bar{\lambda}, \bar{\nu}$
- By $\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{x}} = \mathbf{V} \mathbf{x} - \bar{\lambda} \mathbf{e} - \bar{\nu} \mathbf{r} = 0 \implies \mathbf{x} = \bar{\lambda} \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{V}^{-1} \mathbf{r}$,
so $\mathbf{x}^\top = \bar{\lambda} \mathbf{e}^\top (\mathbf{V}^{-1})^\top + \bar{\nu} \mathbf{r}^\top (\mathbf{V}^{-1})^\top = \bar{\lambda} \mathbf{e}^\top \mathbf{V}^{-1} + \bar{\nu} \mathbf{r}^\top \mathbf{V}^{-1}$
- By $\frac{\partial \bar{\mathcal{L}}}{\partial x_0} = -\bar{\lambda} - \bar{\nu} r_0 = 0 \implies \bar{\nu} = -\frac{\bar{\lambda}}{r_0}$

- $\begin{cases} x_0 + \mathbf{x}^\top \mathbf{e} = 1 \\ x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \implies \begin{cases} x_0 + \bar{\lambda} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = 1 \\ x_0 r_0 + \bar{\lambda} \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} + \bar{\nu} \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \mu \end{cases}$
- Set $\alpha = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e}$, $\beta = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{e} = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r}$, $\gamma = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r}$, $\delta \equiv \alpha\gamma - \beta^2$, the above becomes

$$\begin{cases} x_0 + \bar{\lambda}\alpha + \bar{\nu}\beta = x_0 + \bar{\lambda}\alpha - \frac{\bar{\lambda}}{r_0}\beta = 1 \\ x_0 r_0 + \bar{\lambda}\beta + \bar{\nu}\gamma = x_0 r_0 + \bar{\lambda}\beta - \frac{\bar{\lambda}}{r_0}\gamma = \mu \end{cases}$$

with solutions $x_0 = \frac{\alpha\mu r_0 - \beta r_0 + \gamma - \beta\mu}{\epsilon^2}$, $\bar{\lambda} = \frac{(r_0 - \mu)r_0}{\epsilon^2}$,

$\bar{\nu} = -\frac{r_0 - \mu}{\epsilon^2}$, where $\epsilon^2 = \alpha r_0^2 - 2\beta r_0 + \gamma = \alpha \left(r_0 - \frac{\beta}{\alpha}\right)^2 + \frac{\delta}{\alpha}$

- The relation of σ with μ

$$\begin{aligned} \sigma^2 &= \mathbf{x}^\top \mathbf{V} \mathbf{x} = \mathbf{x}^\top \mathbf{V} (\bar{\lambda} \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{V}^{-1} \mathbf{r}) = \bar{\lambda} (\mathbf{x}^\top \mathbf{e}) + \bar{\nu} (\mathbf{x}^\top \mathbf{r}) \\ &= \bar{\lambda} (1 - x_0) + \bar{\nu} (\mu - x_0 r_0) = \bar{\lambda} + \bar{\nu} \mu = \frac{(\mu - r_0)^2}{\epsilon^2} \end{aligned}$$

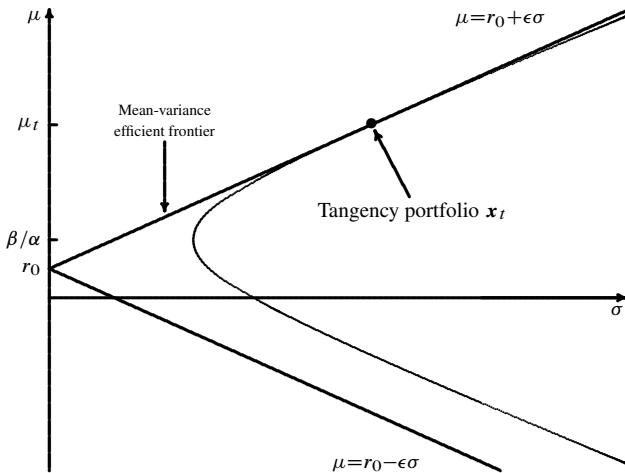


Figure: The Case of All But One Risky Assets

Property

If $r_0 < \frac{\beta}{\alpha}$, then $\mu = r_0 + \epsilon\sigma$ touches the hyperbola $\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$ at $\left(\frac{\epsilon}{\beta - \alpha r_0}, \frac{\gamma - \beta r_0}{\beta - \alpha r_0}\right)$

Proof

On $\sigma - \mu$ plane the slope of the tangent is obtained by implicit differentiation of $\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$ w.r.t σ (let $\mu \equiv \mu(\sigma)$): $2\sigma = \frac{2\alpha\mu\mu' - 2\beta\mu'}{\delta} \Rightarrow \mu' = \frac{\delta\sigma}{\alpha\mu - \beta}$. The tangent line is $\mu = r_0 + \epsilon\sigma$ with slope ϵ , so $\epsilon = \frac{\delta\sigma}{\alpha\mu - \beta} \Rightarrow \delta\sigma = \alpha\mu\epsilon - \beta\epsilon \Rightarrow \delta\sigma = \alpha\epsilon(r_0 + \epsilon\sigma) - \beta\epsilon \Rightarrow (\delta - \alpha\epsilon^2)\sigma = \epsilon(\alpha r_0 - \beta)$. Note that $\epsilon^2 = \alpha r_0^2 - 2\beta r_0 + \gamma = \alpha\left(r_0 - \frac{\beta}{\alpha}\right)^2 + \frac{\delta}{\alpha}$, so $\sigma = \frac{\epsilon(\alpha r_0 - \beta)}{\delta - \alpha\epsilon^2} = \frac{\epsilon(\alpha r_0 - \beta)}{-\alpha^2\left(r_0 - \frac{\beta}{\alpha}\right)^2} = \frac{\epsilon}{\beta - \alpha r_0}$, $\mu = r_0 + \epsilon\sigma = \frac{\beta r_0 - \alpha r_0^2 + \epsilon^2}{\beta - \alpha r_0} = \frac{\gamma - \beta r_0}{\beta - \alpha r_0}$.

- Define the tangency portfolio

$$\mathbf{x}_t = \frac{1}{\beta - \alpha r_0} \mathbf{V}^{-1}(\mathbf{r} - r_0 \mathbf{e}) = \frac{\beta}{\beta - \alpha r_0} \mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mathbf{x}_g$$

- $\mathbf{x} = \bar{\lambda} \mathbf{V}^{-1} \mathbf{e} + \bar{\nu} \mathbf{V}^{-1} \mathbf{r} = \bar{\nu} \mathbf{V}^{-1}(\mathbf{r} - r_0 \mathbf{e}) \equiv (1 - x_0) \mathbf{x}_t$

- $\mathbf{e}^\top \mathbf{x}_t = \frac{\beta}{\beta - \alpha r_0} \mathbf{e}^\top \mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mathbf{e}^\top \mathbf{x}_g = \frac{\beta}{\beta - \alpha r_0} - \frac{\alpha r_0}{\beta - \alpha r_0} = 1$

- $\mu_t = \mathbf{x}_t^\top \mathbf{r} = \mathbf{r}^\top \mathbf{x}_t = \frac{\beta}{\beta - \alpha r_0} \mathbf{r}^\top \mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mathbf{r}^\top \mathbf{x}_g$
 $= \frac{\beta}{\beta - \alpha r_0} \mu_d - \frac{\alpha r_0}{\beta - \alpha r_0} \mu_g = \frac{\gamma - \beta r_0}{\beta - \alpha r_0}$ for $\mu_d = \frac{\gamma}{\beta}$, $\mu_g = \frac{\beta}{\alpha}$

Theorem

Tangency portfolio \mathbf{x}_t is the portfolio that maximize $s(\mathbf{x}) \equiv \frac{\mathbf{x}^\top \mathbf{r} - r_0}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$.

Proof

- Maximize $s(\mathbf{x}) \equiv \text{maximize } \log(s(\mathbf{x}))$ s.t. $\mathbf{x}^\top \mathbf{e} = 1$
- Change of variable $\mathbf{x}^\top \mathbf{r} = \mu \implies \log(s(\mathbf{x})) = \log \frac{\mu - r_0}{\sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}} \equiv f(\mu)$ with $\mu > r_0$
- $f'(\mu) = \frac{(\gamma - \beta r_0) - (\beta - \alpha r_0)\mu}{(\mu - r_0)(\alpha\mu^2 - 2\beta\mu + \gamma)} = 0$ at $\mu = \frac{\gamma - \beta r_0}{\beta - \alpha r_0} = \mu_t$.

Mean-Variance Pricing Equation

- $\mathbf{V} = \mathbb{E} \{ (\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top \} = \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top - \mathbf{R} \mathbf{r}^\top - \mathbf{r} \mathbf{R}^\top + \mathbf{r} \mathbf{r}^\top \} = \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top - \mathbf{R} \mathbf{r}^\top \}$
- $\text{cov}(R_i, \mathbf{x}_t^\top \mathbf{R}) = \mathbb{E} \{ (R_i - r_i)(\mathbf{x}_t^\top \mathbf{R} - \mathbf{x}_t^\top \mathbf{r}) \} = \mathbb{E} \{ R_i \mathbf{x}_t^\top \mathbf{R} - R_i \mathbf{x}_t^\top \mathbf{r} - r_i \mathbf{x}_t^\top \mathbf{R} + r_i \mathbf{x}_t^\top \mathbf{r} \} = \mathbb{E} \{ R_i \mathbf{x}_t^\top \mathbf{R} - R_i \mathbf{x}_t^\top \mathbf{r} \} = \mathbb{E} \{ R_i \mathbf{R}^\top \mathbf{x}_t - R_i \mathbf{r}^\top \mathbf{x}_t \}$
- $\mathbf{V} \mathbf{x}_t = \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top \mathbf{x}_t - \mathbf{R} \mathbf{r}^\top \mathbf{x}_t \}$
- $(\mathbf{V} \mathbf{x}_t)_i = \frac{1}{\beta - \alpha r_0} (r_i - r_0);$
- $\text{var}(\mathbf{x}_t^\top \mathbf{R}) = \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \cdot (\mathbf{x}_t^\top \mathbf{R})^\top \} - (\mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \})^2 = \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \mathbf{R}^\top \mathbf{x}_t \} - \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \} \mathbb{E} \{ \mathbf{R}^\top \mathbf{x}_t \} = \mathbb{E} \{ \mathbf{x}_t^\top \mathbf{R} \mathbf{R}^\top \mathbf{x}_t \} - \mathbf{x}_t^\top \mathbf{r} \mathbf{r}^\top \mathbf{x}_t = \mathbf{x}_t^\top \mathbb{E} \{ \mathbf{R} \mathbf{R}^\top - \mathbf{r} \mathbf{r}^\top \} \mathbf{x}_t = \mathbf{x}_t^\top \mathbf{V} \mathbf{x}_t = \frac{\mu_t - r_0}{\beta - \alpha r_0}.$
- $\beta_{i,t} = \frac{\text{cov}(R_i, \mathbf{x}_t^\top \mathbf{R})}{\text{var}(\mathbf{x}_t^\top \mathbf{R})} = \text{cor}(R_i, \mathbf{x}_t^\top \mathbf{R}) \sqrt{\frac{\text{var } R_i}{\text{var}(\mathbf{x}_t^\top \mathbf{R})}}; \text{ define}$
 $\boldsymbol{\beta}_t \equiv (\beta_{1,t}, \beta_{2,t}, \dots, \beta_{s,t})^\top$
- $\boldsymbol{\beta}_t = \frac{1}{\mu_t - r_0} (\mathbf{r} - r_0 \mathbf{e}) \implies \mathbf{r} = r_0 \mathbf{e} + (\mu_t - r_0) \boldsymbol{\beta}_t$

Mean-Variance Analysis and Expected Utility

- Define $f(\sigma, \mu) = \mathbb{E} v(W)$ where $W = (x_0 r_0 + \mathbf{x}^\top \mathbf{R})w$, $\sigma^2 = \mathbf{x}^\top \mathbf{V} \mathbf{x}$, $\mu = x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mathbf{x}^\top (\mathbf{r} - r_0 \mathbf{e})$
- Assume that $\frac{\partial f}{\partial \sigma} < 0$, $\frac{\partial f}{\partial \mu} > 0$ with $x_0 + \mathbf{x}^\top \mathbf{e} = 1$, perform
$$\max_{\mathbf{x}} f\left(\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}, r_0 + \mathbf{x}^\top (\mathbf{r} - r_0 \mathbf{e})\right)$$
- $$\frac{\partial f}{\partial \mathbf{x}} = \frac{1}{\sigma} \frac{\partial f}{\partial \sigma} \mathbf{V} \mathbf{x} + \frac{\partial f}{\partial \mu} (\mathbf{r} - r_0 \mathbf{e}) = 0 \implies \mathbf{x} = -\frac{\sigma \frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial \sigma}} \mathbf{V}^{-1} (\mathbf{r} - r_0 \mathbf{e}) \propto \mathbf{x}_t$$
- Example:
 - For quadratic utility $v(x) = ax + bx^2$ where $a, b \in \mathbb{R}$, $b \leq 0$:
 $\mathbb{E} v(W) = \mathbb{E} v((x_0 r_0 + \mathbf{x}^\top \mathbf{R})w) = aw\mu + bw^2(\mu^2 + \sigma^2) = f(\sigma, \mu)$
 - For normally distributed returns $\mathbf{R} \sim N(\mathbf{r}, \mathbf{V})$, $\mathbf{x}^\top \mathbf{R} \sim N(\mathbf{x}^\top \mathbf{r}, \mathbf{x}^\top \mathbf{V} \mathbf{x})$:
 $\mathbb{E} v(W) = \mathbb{E} v((\mu + \sigma Y)w)$, where $Y \sim N(0, 1)$

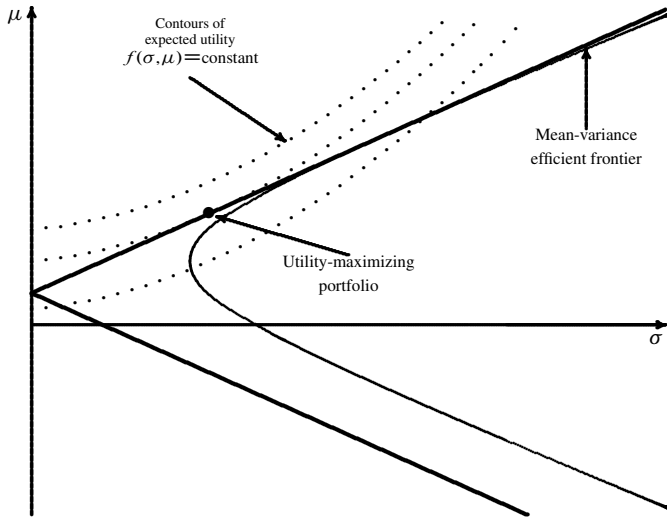


Figure: Determining the Utility-Maximizing Portfolio

Equilibrium: The Capital-Asset Pricing Model

- Investors indexed by $j \in \mathcal{J}$, each with proportions of wealth $x_{0,j}$ and $\mathbf{x}_j = (x_{1,j}, x_{2,j}, \dots, x_{s,j})^\top$
- When each investor j has the utility function as above, the optimal $\mathbf{x}_j \propto \mathbf{x}_t$
 $\implies \mathbf{x}_j = (1 - x_{0,j}) \mathbf{x}_t \quad \forall j \in \mathcal{J}$
- The total value of the demand for risky asset i :

$$\sum_{j \in \mathcal{J}} w_j x_{i,j} = \left(\sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) (\mathbf{x}_t)_i$$
- Market portfolio* of risky assets \mathbf{x}_m :

$$(\mathbf{x}_m)_i \equiv \frac{\text{The total value of the supply of risky asset } i}{\text{The total value of the supply of all risky assets}}; \quad \mathbf{x}_m^\top \mathbf{e} = 1$$
- In equilibrium $(\mathbf{x}_m)_i = \frac{\left(\sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) (\mathbf{x}_t)_i}{\sum_{j \in \mathcal{J}} \sum_{k=1}^s w_j x_{k,j}} =$

$$\frac{\left(\sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) (\mathbf{x}_t)_i}{\left(\sum_{j \in \mathcal{J}} (1 - x_{0,j}) w_j \right) \sum_{k=1}^s (\mathbf{x}_t)_k} = (\mathbf{x}_t)_i, \text{ since } \mathbf{x}_t^\top \mathbf{e} = 1$$
- $\mathbf{r} = r_0 \mathbf{e} + (\mu_m - r_0) \boldsymbol{\beta}_m$, $\boldsymbol{\beta}_m \equiv (\beta_{1,m}, \beta_{2,m}, \dots, \beta_{s,m})^\top$,

$$\beta_{i,m} = \frac{\text{cov}(R_i, \mathbf{x}_m^\top \mathbf{R})}{\text{var}(\mathbf{x}_m^\top \mathbf{R})} \text{ — capital-asset-pricing equation}$$

Problems and Solutions

Problem 1

Suppose that an investment X has either (i) the uniform distribution $U[0, 2\mu]$ or (ii) the exponential distribution with $E X = \mu$, and the investor has a utility function which is either (a) logarithmic, $v(x) = \log x$ (b) power form, $v(x) = x^\theta$. Show that both the compensatory risk premium and the investment risk premium are proportional to μ in all 4 possible cases.

Problem 1 Solution

- For distributions (i)(ii) of X , the random variable $Y \equiv \frac{X}{\mu}$ does not depend on μ , so $E v(X + \alpha) = v(\mu)$ for the compensatory risk premium α reduces to $E v(Y + c) = v(1)$ in cases (a)(b) when $\alpha = c\mu$. For the insurance risk premium when $\beta = d\mu$, d is the solution of $E v(Y) = v(1 - d)$.
- For case (i)(a),

$$E v(Y + c) = \int_0^2 \frac{\log(y + c)}{2} dy = \frac{1}{2}((2 + c) \log(2 + c) - c \log c - 2), \text{ and}$$

$v(1) = \log 1 = 0$, so $\alpha = c\mu$ where c is the unique positive root of $(2 + c) \log(2 + c) - c \log c - 2 = 0$. Using `rmaxima`

```
find_root((2 + x) * log(2 + x) - x * log(x) - 2, x, 0.01, 20);
```

we have $c = 0.176965531$. For the insurance premium $\beta = d\mu$,

$$E \log Y = \log 2 - 1 = \log(1 - d), \text{ so } d = 1 - \frac{2}{e} = 0.264.$$

Problem 2

An investor has a utility function $v(x) = \sqrt{x}$ and is considering three investments with random outcomes X, Y, Z . Here X has the uniform distribution $U[0, a]$, Y has the gamma distribution $\Gamma(\gamma, \lambda)$ with probability density function $\frac{e^{-\lambda y} \lambda^\gamma y^{\gamma-1}}{\Gamma(\gamma)}$ for $y > 0$, where $\gamma > 0$, $\lambda > 0$, and Z is log-normal, i.e. $Z \sim N(\nu, \sigma^2)$. The parameter of the distributions are such that $E X = E Y = E Z = \mu$ and $\text{var } X = \text{var } Y = \text{var } Z$. Recall that the gamma function $\Gamma(\gamma) = \int_0^\infty u^{\gamma-1} e^{-u} du$ that satisfies $\Gamma(\gamma + 1) = \gamma \Gamma(\gamma)$ and $\Gamma(1/2) = \sqrt{\pi}$. Determine the investor's preference ordering of X, Y, Z for all values of μ .

Problem 2 Solution I

- $X \sim U[0, a] \implies \mathbb{E} X = \frac{a}{2}, \text{ var } X = \frac{a^2}{12}$
- $Y \sim \Gamma(\gamma, \lambda) \implies \mathbb{E} Y = \frac{\gamma}{\lambda}, \text{ var } Y = \frac{\gamma}{\lambda^2}$
- $Z \sim \text{lognormal}(\nu, \sigma^2) \implies \mathbb{E} Z = e^{\nu + \frac{\sigma^2}{2}}, \text{ var } Z = e^{2\nu + \sigma^2}(e^{\sigma^2} - 1)$ by the formula $\mathbb{E} e^{\theta W} = e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}}$ for $W \sim N(\mu, \sigma^2)$

$$\begin{aligned}\mathbb{E} e^{\theta W} &= e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \text{ as } W \sim N(\mu, \sigma^2): \sqrt{2\pi}\sigma \mathbb{E} e^{\theta W} = \int_{-\infty}^{\infty} e^{\theta x} \cdot e^{-\frac{1}{2} \frac{x^2 - 2\mu x + \mu^2}{\sigma^2}} dx \\&= \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{x^2 - 2(\mu + \theta\sigma^2)x + \mu^2}{\sigma^2}} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x - (\mu + \theta\sigma^2))^2 + \mu^2 - (\mu + \theta\sigma^2)^2}{\sigma^2}} dx \\&= \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x - (\mu + \theta\sigma^2))^2 - 2\mu\theta\sigma^2 - (\theta\sigma^2)^2}{\sigma^2}} dx = e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2} \frac{(x - (\mu + \theta\sigma^2))^2}{\sigma^2}} dx \\&= e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi}\sigma \cdot e^{\mu\theta + \frac{\theta^2 \sigma^2}{2}} \text{ by } \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.\end{aligned}$$

The conditions $\mathbb{E} X = \mathbb{E} Y = \mathbb{E} Z = \mu$ and $\text{var } X = \text{var } Y = \text{var } Z$ imply

- $a = 2\mu$, so that $\text{var } X = \frac{\mu^2}{3}$.

Problem 2 Solution II

- $EY = \frac{\gamma}{\lambda} = \mu$, so that $\text{var } Y = \frac{\gamma}{\lambda^2} = \frac{\mu^2}{\gamma} = \text{var } X = \frac{\mu^2}{3} \implies \gamma = 3$
- $EZ = e^{\nu + \frac{\sigma^2}{2}} = \mu$, $\text{var } Z = e^{2\nu + \sigma^2}(e^{\sigma^2} - 1) = \mu^2(e^{\sigma^2} - 1) = \text{var } X = \frac{\mu^2}{3}$
 $\implies \sigma^2 = \log \frac{4}{3}$.
- $E\sqrt{X} = \int_0^{2\mu} \frac{\sqrt{x}}{2\mu} dx = \frac{2^{\frac{3}{2}}}{3} \sqrt{\mu} \approx 0.943\sqrt{\mu}$
- $E\sqrt{Y} = \int_0^\infty \sqrt{y} \frac{1}{2} e^{-\lambda y} \lambda^3 y^2 dy = \frac{\Gamma(\frac{7}{2})}{2\sqrt{\lambda}} = \frac{\Gamma(\frac{7}{2})}{2\sqrt{3}} \sqrt{\mu} = \frac{5\sqrt{3}\pi}{16} \sqrt{\mu} \approx 0.959\sqrt{\mu}$
- $E\sqrt{Z} = e^{\frac{\nu}{2} + \frac{\sigma^2}{8}} = e^{-\frac{\sigma^2}{8}} \sqrt{\mu} = \left(\frac{3}{4}\right)^{\frac{1}{8}} \sqrt{\mu} \approx 0.965\sqrt{\mu}$

So $Z \succ Y \succ X$.

Problem 3

Suppose that an investor has the utility function $v(x) = 1 - e^{-ax}$ with $a > 0$, and the outcome of an investment is a random variable X with mean μ , finite variance and finite moment-generating function $\psi(a) = E\{e^{-aX}\}$ for $a > 0$. Show that

- ① The compensatory risk premium and the insurance risk premium have the same value α , and express α in terms of μ and the moment generating function ψ .
- ② Both the Arrow-Pratt and global risk aversions are a .
- ③ As $a \downarrow 0$, $\alpha = \frac{a}{2} \text{var } X + \mathcal{O}(a)$. Under what circumstances is it true that $\alpha = \frac{a}{2} \text{var } X$ for $a > 0$?
- ④ $\psi''\psi - (\psi')^2 \geq 0$ and hence α is an increasing function of a . This shows that the more risk-averse the investor is, the higher the value of the premium that is required.

Problem 3 Solution I

- ①
- The compensatory risk premium α solves $E\{v(\alpha + X)\} = v(\mu) \implies E\{1 - e^{-a(\alpha + X)}\} = 1 - e^{-a\mu} \implies 1 - e^{-a\alpha} E\{e^{-aX}\} = 1 - e^{-a\mu} \implies -a\alpha + \ln \psi(a) = -a\mu \implies \alpha = \mu + \frac{1}{a} \ln(\psi(a))$
 - The insurance risk premium β solves $E v(X) = v(\mu - \beta) \implies E\{1 - e^{-aX}\} = 1 - e^{-a(\mu - \beta)} \implies 1 - \psi(a) = 1 - e^{-a(\mu - \beta)} \implies \ln \psi(a) = -a(\mu - \beta) \implies \beta = \mu + \frac{1}{a} \ln(\psi(a))$

So $\alpha = \beta = \mu + \frac{1}{a} \ln(\psi(a))$.

- ② Note that $v'(x) = ae^{-ax}$, $v''(x) = -a^2e^{-ax}$, the Arrow-Pratt absolute risk aversion is $-\frac{v''(\mu)}{v'(\mu)} = \frac{a^2e^{-a\mu}}{ae^{-a\mu}} = a$, the global absolute risk aversion is $-\frac{E\{v''(X)\}}{E\{v'(X)\}} = \frac{a^2 E\{e^{-aX}\}}{a E\{e^{-aX}\}} = a$.

Problem 3 Solution II

- ③ From $\alpha(a) = \mu + \frac{1}{a} \ln(\psi(a)) \implies \psi(a) = e^{a(\alpha(a)-\mu)}$; Differentiation yields $\psi'(a) = e^{a(\alpha(a)-\mu)}(\alpha(a) - \mu + a\alpha'(a))$, $\psi''(a) = e^{a(\alpha(a)-\mu)}(2\alpha'(a) + a\alpha''(a)) + e^{a(\alpha(a)-\mu)}(\alpha(a) - \mu + a\alpha'(a))^2$. Note that $\psi(0) = 1$, $\psi'(0) = -\mu$, $\psi''(0) = \mathbb{E} X^2$; setting $a = 0$ yields $\psi'(0) = e^{0(\alpha(0)-\mu)}(\alpha(0) - \mu + 0\alpha'(0)) \implies -\mu = \alpha(0) - \mu \implies \alpha(0) = 0$, $\psi''(0) = e^{0(\alpha(0)-\mu)}(2\alpha'(0) + 0\alpha''(0)) + e^{0(\alpha(0)-\mu)}(\alpha(0) - \mu + 0\alpha'(0))^2 \implies \mathbb{E} X^2 = 2\alpha'(0) + \mu^2 \implies \alpha'(0) = \frac{1}{2}(\mathbb{E} X^2 - \mu^2) = \frac{1}{2} \text{var } X$. For small $a > 0$, the Taylor expansion of $\alpha(a) = \alpha(0) + a\alpha'(0) + \mathcal{O}(a^2) = \frac{a}{2} \text{var } X + \mathcal{O}(a^2)$. When $\alpha(a) = \frac{a}{2} \text{var } X$ exactly for $a > 0$, then $\psi(a) = \mathbb{E} \{e^{-aX}\} = e^{-a\mu + \frac{a^2}{2} \text{var } X}$, which is true only when X is normally distributed.
- ④ $\psi''\psi - (\psi')^2 = \mathbb{E} \{X^2 e^{-aX}\} \mathbb{E} \{e^{-aX}\} - (\mathbb{E} \{X e^{-aX}\})^2 \geq 0$ by the Cauchy-Schwarz inequality applied to random variables $A = X e^{-\frac{a}{2}X}$ and $B = e^{-\frac{a}{2}X}$. To see that α is increasing $\frac{d\alpha}{da} = \frac{1}{a^2} \left(\frac{a\psi'}{\psi} - \ln(\psi) \right) \equiv \frac{1}{a^2} f(a)$, but $f(0) = 0$ and $f' = \frac{a(\psi''\psi - (\psi')^2)}{\psi^2} \geq 0$ and the conclusion follows.

Problem 4

Consider a one-period investment model in which there are only two risky assets. The returns on these assets have means 3, 4 respectively and variances 2, 3 respectively with the covariance between the returns being 2.

- 1 Calculate the mean-variance efficient frontier and the minimum-variance portfolio in terms of the mean return.
- 2 Calculate the mean return of the global minimum-variance portfolio and of diversified portfolio.
- 3 Suppose that in addition to the two risky assets there is a riskless asset with return $3/2$. Find the minimum-variance portfolio in terms of the mean return and hence calculate the mean return of the tangency portfolio.

Problem 4 Solution I

The inverse matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so if $V = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$,
 $V^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$. $\alpha = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e} = \frac{1}{2}$, $\beta = \mathbf{e}^\top \mathbf{V}^{-1} \mathbf{r} = \frac{3}{2}$, $\gamma = \mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r} = \frac{11}{2}$,
 $\delta = \alpha\gamma - \beta^2 = \frac{1}{2}$.

$$\textcircled{1} \min_{x_1, x_2} (x_1 \quad x_2) \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \min_{x_1, x_2} 2x_1^2 + 4x_1x_2 + 3x_2^2 \quad \text{s.t.} \\ \begin{cases} x_1 + x_2 = 1 \\ 3x_1 + 4x_2 = \mu \end{cases} \quad \text{From constraints } x_1 = 4 - \mu, x_2 = \mu - 3, \text{ so the} \\ \text{mean-variance efficient frontier is } \sigma^2 = \mu^2 - 6\mu + 11.$$

$$\textcircled{2} \mu_g \text{ is the root of } \frac{d\sigma^2}{d\mu} = 0, \text{ so } 2\mu_g - 6 = 0 \implies \mu_g = 3. \quad \mu_d = \frac{\gamma}{\beta} = \frac{11}{3}.$$

Problem 4 Solution II

③ Now the problem is $\min_{x_0, x_1, x_2} 2x_1^2 + 4x_1x_2 + 3x_2^2$ s.t.

$$\begin{cases} x_0 + x_1 + x_2 = 1 \\ \frac{3}{2}x_0 + 3x_1 + 4x_2 = \mu \end{cases}. \text{ Form the Lagrangian}$$

$$\mathcal{L} = 2x_1^2 + 4x_1x_2 + 3x_2^2 + \lambda(1 - x_0 - x_1 - x_2) + \nu(\mu - \frac{3}{2}x_0 - 3x_1 - 4x_2).$$

By solving $\frac{\partial \mathcal{L}}{\partial x_0} = 0$, $\nu = -\frac{2\lambda}{3}$. From $\frac{\partial \mathcal{L}}{\partial x_1} = 0$ and $\frac{\partial \mathcal{L}}{\partial x_2} = 0$ we have

$$4x_1 + 4x_2 - \lambda - 3\nu = 0 \text{ and } 4x_1 + 6x_2 - \lambda - 4\nu = 0; \text{ so } x_1 = \frac{\lambda}{12}, x_2 = -\frac{\lambda}{3}.$$

Substitute into the constraints yields $\lambda = \frac{12(3 - 2\mu)}{17}$, and so $x_0 = \frac{26 - 6\mu}{17}$,

$$x_1 = \frac{3 - 2\mu}{17}, x_2 = -\frac{4(3 - 2\mu)}{17}. \text{ The tangency portfolio corresponds to}$$

$$x_0 = 0 \text{ or } \mu_t = \frac{13}{3}.$$

Problem 5

Suppose that v is concave, $X \sim N(\mu, \sigma^2)$ and $f(\sigma, \mu) = \mathbb{E} v(X)$.

- 1 Show that $\frac{\partial f}{\partial \mu} > 0$ when v is strictly increasing, and $\frac{\partial f}{\partial \sigma} \leq 0$. Hence show in the context of mean-variance analysis that, when all returns are jointly normally distributed, an investor maximizing the expected utility of his final wealth will choose a mean-variance-efficient optimal portfolio.
- 2 Show that f is concave in μ and σ . Deduce that this optimal portfolio corresponds to a point in the (σ, μ) plane where an indifference contour is tangent to the efficient frontier.

Problem 5 Solution I

Note that, when $W \sim N(\mu, \sigma^2)$, then for $a, b \in \mathbb{R}$, $a \neq 0$, the random variable $Z = aW + b \sim N(a\mu + b, a^2\sigma^2)$. So if $X \sim N(\mu, \sigma^2)$, X can be written as $X = \mu + \sigma Y$ where $Y \sim N(0, 1)$. Moreover $E\{f(X)(X - \mu)\} = \sigma^2 E\{f'(X)\}$ for any differentiable f and if both sides of the equation are finite: first note that for the standard normal density function ϕ , $\phi'(y) = -y\phi(y)$, so

$$\begin{aligned} E\{f(X)(X - \mu)\} &= \sigma E\{f(\mu + \sigma Y)Y\} = \int_{-\infty}^{\infty} yf(\mu + \sigma y)\phi(y) dy = \\ &= - \int_{-\infty}^{\infty} f(\mu + \sigma y) d\phi(y) = -\sigma f(\mu + \sigma y)\phi(y) \Big|_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} f'(\mu + \sigma y)\phi(y) dy = \\ &= \sigma^2 \int_{-\infty}^{\infty} f'(\mu + \sigma y)\phi(y) dy = \sigma^2 E\{f'(X)\}. \end{aligned}$$

- ④ Note that $\frac{\partial f}{\partial \mu} = E\{v'(\mu + \sigma Y)\} > 0$ when $v' > 0$, $\frac{\partial f}{\partial \sigma} = E\{Yv'(\mu + \sigma Y)\} = \sigma E\{v''(\mu + \sigma Y)\} \leq 0$ by the concavity of v ($v'' < 0$). When returns are normally distributed, the wealth created by each portfolio is normally distributed; this shows that maximizing in σ for fixed μ gives a value of (σ, μ) on the efficient frontier.

Problem 5 Solution II

② To see the concavity of f , note that $\frac{\partial^2 f}{\partial \mu^2} = E\{v''(\mu + \sigma Y)\} \leq 0$,

$$\frac{\partial^2 f}{\partial \sigma^2} = E\{Y^2 v''(\mu + \sigma Y)\} \leq 0, \quad \frac{\partial^2 f}{\partial \mu \partial \sigma} = E\{Y v''(\mu + \sigma Y)\}, \text{ and then}$$

$$\frac{\partial^2 f}{\partial \mu^2} \frac{\partial^2 f}{\partial \sigma^2} \geq \left(\frac{\partial^2 f}{\partial \mu \partial \sigma} \right)^2 \text{ follows by applying the Cauchy-Schwarz inequality to}$$

the random variables $A = Y \sqrt{-v''(\mu + \sigma Y)}$ and $B = \sqrt{-v''(\mu + \sigma Y)}$. This shows that the 2×2 matrix of second derivatives has non-positive diagonal entries and a non-negative determinant which is sufficient for the matrix to be negative semi-definite: it is clear that the quadratic form

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2cxy + by^2 = a \left(x + \frac{c}{a} y \right)^2 + \left(\frac{ab - c^2}{a} \right) y^2$$

≤ 0 if $a, b \leq 0$ and $c^2 - ab \geq 0$. The fact that f is concave means that sets of the form $\{(\sigma, \mu) : f(\sigma, \mu) \geq c\}$ are convex which gives the last statement.

Problem 6

Suppose that an investor has a concave utility function v . The investor seeks to maximize $E v(W)$ where $W = (x_0 r_0 + \mathbf{x}^\top \mathbf{R}) w$ is his final wealth.

- ① Show that, when \bar{W} is his optimal final wealth, then

$$E \{v'(\bar{W})(R_j - r_0)\} = 0, \quad \forall j = 1, 2, \dots, s.$$

- ② Show that, when \mathbf{R} has a multivariate normal distribution, then

$$r_j - r_0 = \alpha \operatorname{cov}(\bar{W}, R_j), \quad \forall j = 1, 2, \dots, s, \text{ where } \alpha = -\frac{E \{v''(\bar{W})\}}{E \{v'(\bar{W})\}}.$$

- ③ Now suppose that the market is determined by investors $i = 1, 2, \dots, n$, where investor i has concave utility v_i , initial wealth w_i , optimal final wealth \bar{W}_i and global risk aversion α_i . With the normality assumption, show that

$$E M - r_0 = \bar{w} \bar{\alpha} \operatorname{var} M, \text{ where } M = \frac{\sum_{i=1}^n \bar{W}_i}{\sum_{i=1}^n w_i} \text{ is the market rate of return,}$$

$$\bar{w} = \frac{\sum_{i=1}^n w_i}{n} \text{ is the average initial wealth of investors, and } \bar{\alpha} \text{ is the harmonic mean of } \alpha_i.$$

Problem 6 Solution I

- 1 The objective function to maximize is

$$f(\mathbf{x}) = \mathbb{E} \left\{ v \left(w \left(r_0 + \sum_{j=1}^s x_j (R_j - r_0) \right) \right) \right\}$$

where $\mathbf{x} = (x_1, \dots, x_s)^\top$ and we have used $x_0 + \sum_{j=1}^s x_j = 1$. The first-order conditions give

$$\frac{\partial f}{\partial x_j} = w \mathbb{E} \{ v'(\bar{W})(R_j - r_0) \} = 0, \text{ for } 1 \leq j \leq s$$

Since $r_j = \mathbb{E} R_j$ and the fact that \bar{W} and R_j have a joint normal distribution we have that

$$\begin{aligned} 0 &= \mathbb{E} \{ v'(\bar{W})(R_j - r_0) \} = \mathbb{E} \{ v'(\bar{W})(R_j - r_j) \} + \mathbb{E} \{ v'(\bar{W}) \} (r_j - r_0) \\ &= \text{cov}(v'(\bar{W}), R_j) + \mathbb{E} \{ v'(\bar{W}) \} (r_j - r_0) \\ &= \mathbb{E} \{ v''(\bar{W}) \} \text{cov}(\bar{W}, R_j) + \mathbb{E} \{ v'(\bar{W}) \} (r_j - r_0) \end{aligned}$$

Problem 6 Solution II

where the last equality uses $\text{cov}(f(X), Y) = E \{f'(X)\} \text{cov}(X, Y)$, and

$$r_j - r_0 = \alpha \text{cov}(\overline{W}, R_j)$$

Note that for random variables X and Y and constant a , $\text{cov}(X, Y + a) = \text{cov}(X, Y)$ and $\text{cov}(aX, Y) = a \text{cov}(X, Y)$. Now for each i

$$\frac{1}{\alpha_i}(r_j - r_0) = \text{cov}(\overline{W}_i, R_j)$$

and summing this on i yields

$$\left(\sum_{i=1}^n \frac{1}{\alpha_i} \right) (r_j - r_0) = \left(\sum_{i=1}^n w_i \right) \text{cov}(M, R_j)$$

Divide through by n and multiply by $\bar{\alpha}$, where $\frac{1}{\bar{\alpha}} = \frac{\sum_{i=1}^n \frac{1}{\alpha_i}}{n}$, to obtain

$$E R_j - r_0 = w \bar{\alpha} \text{cov}(M, R_j) \quad (1)$$

Problem 6 Solution III

When \bar{x}_{ij} is the optimal proportion invested by investor i in asset j then

$$\bar{W}_i = w_i \left(r_0 + \sum_{j=1}^s \bar{x}_{ij} (R_j - r_0) \right)$$

which when summed on i gives

$$(M - r_0) \left(\sum_{i=1}^n w_i \right) = \sum_{i=1}^n \sum_{j=1}^s w_i \bar{x}_{ij} (R_j - r_0) \quad (2)$$

- ② Take the expectation in (2), multiply (1) by $w_i \bar{x}_{ij}$, sum on i and j , rearrange the expression using the two properties of covariance mentioned above and $E M - r_0 = \bar{w} \bar{\alpha} \text{var } M$ follows. This shows that the risk premium for the market is proportional to $\bar{\alpha}$ which is a measure of the risk aversion in the economy.

Problem 7

Consider an investor with the utility function $v(x) = 1 - e^{-ax}$, $a > 0$, who is faced with a riskless asset with return r_0 and s risky assets with returns $\mathbf{R} \sim N(\mathbf{r}, \mathbf{V})$.

- 1 Show that when he seeks to maximize the expected utility of his final wealth, he will hold the risky assets in the same proportion as the tangency portfolio.
- 2 Show that when $\beta > \alpha r_0$, the more risk averse that he is, the smaller amount of his wealth that he invests in the risky assets.

Problem 7 Solution

- ① Suppose that the investor's initial wealth is $w > 0$ and that he wishes to minimize $E e^{-aW}$ where

$$W = w \left(r_0 + \sum_{j=1}^s x_j (R_j - r_0) \right) = w (r_0(1 - \mathbf{x}^\top \mathbf{e}) + \mathbf{x}^\top \mathbf{R})$$

and $\mathbf{x} = (x_1, \dots, x_s)^\top$, $\mathbf{e} = (1, \dots, 1)^\top$, $x_0 = 1 - \mathbf{x}^\top \mathbf{e}$. Note that $\mathbf{x}^\top \mathbf{R} \sim N(\mathbf{r}^\top \mathbf{x}, \mathbf{x}^\top \mathbf{V} \mathbf{x})$, so

$$E e^{-aW} = \exp \left\{ -aw r_0(1 - \mathbf{x}^\top \mathbf{e}) - aw \mathbf{r}^\top \mathbf{x} + \frac{1}{2} a^2 w^2 \mathbf{x}^\top \mathbf{V} \mathbf{x} \right\}$$

It amounts to minimize $\frac{1}{2} aw \mathbf{x}^\top \mathbf{V} \mathbf{x} - \mathbf{x}^\top (\mathbf{r} - r_0 \mathbf{e})$ for which the minimum occurs when $\mathbf{x} = \frac{1}{aw} \mathbf{V}^{-1} (\mathbf{r} - r_0 \mathbf{e})$, and the conclusion follows.

- ② The amount of his wealth invested in the risky assets is $(\mathbf{x}^\top \mathbf{e})w = \frac{\beta - \alpha r_0}{a}$, which decreases in $a > 0$ when $\beta > \alpha r_0$.

Problem 8

Consider an investor with $\mathbf{R} = (R_1, R_2, \dots, R_s)^\top$ where R_i s are independent random variables with R_i having gamma distribution, $E R_i = r_i$ and $\text{var } R_i = \sigma_i^2$. Suppose that he has the utility function $v(x) = 1 - e^{-ax}$, $a > 0$, and he seeks to maximize the expected utility of his final wealth.

- 1 Show that he divides his wealth between the diversified portfolio and a second portfolio which should be identified; determine the amounts that he invests in each.
- 2 If he may invest in a risky asset with return r_0 , show that he will again divide his wealth between these two portfolios and the riskless asset; determine the amounts that he invests in each.
- 3 Give a necessary and sufficient condition, expressed in terms of the parameters r_i , $i = 0, 1, 2, \dots, s$ and σ_i^2 , $i = 1, 2, \dots, s$, that he is long in the risky assets.

Problem 8 Solution I

- ① When $R_i \sim \Gamma(\gamma_i, \lambda_i)$, $E R_i = r_i = \frac{\gamma_i}{\lambda_i}$ and $\text{var } R_i = \sigma_i^2 = \frac{\gamma_i}{\lambda_i^2} \implies \gamma_i = \frac{r_i^2}{\sigma_i^2}$, $\lambda_i = \frac{r_i}{\sigma_i^2}$. For $\phi + \lambda_i > 0$, note that

$$\begin{aligned} E e^{-\phi R_i} &= \int_0^\infty \frac{e^{-\phi x} e^{-\lambda_i x} \lambda_i^{\gamma_i} x^{\gamma_i-1}}{\Gamma(\gamma_i)} dx \\ &= \left(\frac{\lambda_i}{\phi + \lambda_i} \right)^{\gamma_i} \int_0^\infty \frac{e^{-\phi x} e^{-\lambda_i x} (\phi + \lambda_i)^{\gamma_i} x^{\gamma_i-1}}{\Gamma(\gamma_i)} dx = \left(\frac{\lambda_i}{\phi + \lambda_i} \right)^{\gamma_i} \end{aligned}$$

because the integrand in the latter integral is a probability density function, and so the value of the integral is 1. The investor wishes to solve the constrained optimization problem

$$\text{maximize} \quad E \left\{ 1 - e^{-aw(\mathbf{x}^\top \mathbf{R})} \right\} \quad \text{subject to} \quad \mathbf{x}^\top \mathbf{e} = 1$$

Problem 8 Solution II

which is equivalent to minimizing

$$\mathbb{E} \left\{ e^{-aw(\mathbf{x}^\top \mathbf{R})} \right\} = \prod_{i=1}^s \mathbb{E} \left\{ e^{-awx_i R_i} \right\} = \prod_{i=1}^s \left(\frac{\lambda_i}{awx_i + \lambda_i} \right)^{\gamma_i}$$

subject to the constraint. Taking logarithms, we need to

$$\text{maximize} \quad \sum_{i=1}^s \gamma_i \ln(awx_i + \lambda_i) \quad \text{subject to} \quad \sum_{i=1}^s x_i = 1$$

Maximizing the Lagrangian

$$\mathcal{L} = \sum_{i=1}^s \gamma_i \ln(awx_i + \lambda_i) + \theta \left(1 - \sum_{i=1}^s x_i \right)$$

Problem 8 Solution III

in x_i gives $x_i = \frac{\gamma_i}{\theta} - \frac{\lambda_i}{aw}$. Substituting back into the constraint shows that the Lagrange multiplier is given as

$$\theta = \frac{\sum_{j=1}^s \gamma_j}{1 + \frac{1}{aw} \sum_{j=1}^s \lambda_j}$$

from which it follows that the optimal portfolio may be expressed as

$$\mathbf{x} = \left(1 + \frac{1}{aw} \sum_{j=1}^s \lambda_j\right) \bar{\mathbf{x}} - \left(\frac{1}{aw} \sum_{j=1}^s \lambda_j\right) \mathbf{x}_d$$

where the two portfolios $\bar{\mathbf{x}}$ and \mathbf{x}_d are

$$(\bar{\mathbf{x}})_i = \frac{\gamma_i}{\sum_{j=1}^s \gamma_j} = \frac{\frac{r_i^2}{\sigma_i^2}}{\sum_{j=1}^s \frac{r_j^2}{\sigma_j^2}} \text{ and } (\mathbf{x}_d)_i = \frac{\lambda_i}{\sum_{j=1}^s \lambda_j} = \frac{\frac{r_i}{\sigma_i^2}}{\sum_{j=1}^s \frac{r_j}{\sigma_j^2}}$$

Problem 8 Solution IV

with the latter portfolio being the diversified portfolio. As his initial wealth is w , the investor invests the amount $w + \frac{1}{a} \sum_{j=1}^s \lambda_j$ in $\bar{\mathbf{x}}$ and the amount

$-\frac{1}{a} \sum_{j=1}^s \lambda_j$ in the diversified portfolio. Note that in the case when the random variables R_i have exponential distributions, then $\gamma_i = 1$, or $r_i^2 = \sigma_i^2$, for each $1 \leq i \leq s$, so that the portfolio $\bar{\mathbf{x}}$ is just the uniform portfolio

$\bar{\mathbf{x}} = \left(\frac{1}{s}, \dots, \frac{1}{s} \right)^\top$ which apportions wealth equally between the s risky assets.

- ② When there is a riskless asset, set $x_0 = 1 - \mathbf{x}^\top \mathbf{e}$ and we wish to minimize

$$\begin{aligned} \mathbb{E} \left\{ e^{-aw(r_0(1-\mathbf{x}^\top \mathbf{e}) + \mathbf{x}^\top \mathbf{R})} \right\} &= e^{awr_0(\sum_{j=1}^s x_j - 1)} \prod_{i=1}^s \mathbb{E} \left\{ e^{-awx_i R_i} \right\} \\ &= e^{awr_0(\sum_{j=1}^s x_j - 1)} \prod_{i=1}^s \left(\frac{\lambda_i}{awx_i + \lambda_i} \right)^{\gamma_i} \end{aligned}$$

Problem 8 Solution V

in $\mathbf{x} = (x_1, \dots, x_s)^\top$, which is equivalent to maximizing

$$\sum_{i=1}^s \gamma_i \ln(awx_i + \lambda_i) - awr_0 \sum_{i=1}^s x_i$$

Deduce that for $1 \leq i \leq s$, the optimal $x_i = \frac{1}{aw} \left(\frac{\gamma_i}{r_0} - \lambda_i \right)$, and the optimal investment in the risky assets is determined by

$$\mathbf{x} = \left(\frac{1}{awr_0} \sum_{j=1}^s \gamma_j \right) \bar{\mathbf{x}} - \left(\frac{1}{aw} \sum_{j=1}^s \lambda_j \right) \mathbf{x}_d$$

- ④ The investor is long in the particular risky asset i when $x_i > 0$, which is true if and only if $r_i > r_0$; he is long overall in risky assets if and only if

$$\sum_{j=1}^s x_j > 0 \text{ which is equivalent to the condition that } \frac{1}{r_0} > \frac{\sum_{j=1}^s \frac{r_j}{\sigma_j^2}}{\sum_{j=1}^s \frac{r_j^2}{\sigma_j^2}}.$$