Operations Research
08. Portfolio Optimization

Classical PO: Mean-Variance (MV) Criterion

- Assets evolve from time 0 to time 1 for one period
- s: # of risky assets
- $\mathbf{S}_0 \equiv (S_{1,0}, S_{2,0}, \dots, S_{s,0})^\top \neq \mathbf{0}$: the constant price vector at time 0
- $\mathbf{S}_1 \equiv (S_{1,1}, S_{2,1}, \dots, S_{s,1})^\top$: the random price vector at time 1
- $\mathbf{x} \equiv (x_1, x_2, \dots, x_s)^\top$: the proportion vector of the time-0 wealth invested in each asset; $\displaystyle\sum_{i=1}^{s} x_i = 1$.
- $\mathbf{R} \equiv (R_1, R_2, \dots, R_s)^\top$: the random vector representing the rate of return on the assets; $R_i = \dfrac{S_{i,1}}{S_{i,0}}$
- w: the (constant) wealth at time 0
- W: the (random) wealth at time 1; $W = \left(\displaystyle\sum_{i=1}^{s} x_i R_i\right) w =$

$\mathbf{x}^\top \mathbf{R}\, w$ (For asset S_i, $\dfrac{x_i w}{S_{i,0}}$ denotes the "quantity" allocated at time 0; so at time 1 this part of wealth becomes $\dfrac{x_i w}{S_{i,0}} S_{i,1} = x_i R_i w$)

- $\mathbf{r} \equiv \mathsf{E}\,\mathbf{R} = (r_1, r_2, \dots, r_s)^\top$: the (constant) mean vector of $\mathbf{R}$; $r_i = \mathsf{E}\,R_i$
- $\mathbf{V} \equiv \operatorname{cov} \mathbf{R} \equiv \mathsf{E}\{(\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top\}$: the (constant) covariance matrix of $\mathbf{R}$; $\mathbf{V}$ is symmetric positive semidefinite $s \times s$ matrix
- $\mathsf{E}\,W = \mathsf{E}\{\mathbf{x}^\top \mathbf{R}\} = \mathbf{x}^\top \mathbf{r} = \mu$
- $\sigma^2 = \operatorname{var} W = \operatorname{var}\{\mathbf{x}^\top \mathbf{R}\} = \mathsf{E}\{\mathbf{x}^\top (\mathbf{R} - \mathbf{r})(\mathbf{R} - \mathbf{r})^\top \mathbf{x}\} = \mathbf{x}^\top \mathbf{V} \mathbf{x}$

(Classical) Mean-Variance Portfolio Optimization

"For some fixed mean rate of return $\mu = \mathsf{E}\{\mathbf{x}^\top \mathbf{R}\}$, try to minimize the variance $\sigma^2 = \operatorname{var}\{\mathbf{x}^\top \mathbf{R}\}$ of the return over portfolios $\mathbf{x}$"

MV: All Risky Assets

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^\top \mathbf{V}\mathbf{x} \quad \text{s.t.} \quad \begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \qquad \mathbf{e} \equiv \underbrace{(1, 1, \ldots, 1)^\top}_{s \text{ items}}$$

- $\mathbf{V}$ is symmetric, positive definite, so $\mathbf{V}^{-1}$ also is
- Set $\mathcal{L} \equiv \frac{1}{2}\mathbf{x}^\top \mathbf{V}\mathbf{x} + \lambda\,(1 - \mathbf{x}^\top \mathbf{e}) + \nu\,(\mu - \mathbf{x}^\top \mathbf{r})$ with Lagrange multipliers λ, ν
- By $\dfrac{\partial \mathcal{L}}{\partial \mathbf{x}} = \mathbf{V}\mathbf{x} - \lambda\,\mathbf{e} - \nu\,\mathbf{r} = 0 \implies \mathbf{x} = \lambda\,\mathbf{V}^{-1}\mathbf{e} + \nu\,\mathbf{V}^{-1}\mathbf{r}$
$\implies \mathbf{x}^\top = \lambda\,\mathbf{e}^\top \left(V^{-1}\right)^\top + \nu\,\mathbf{r}^\top \left(V^{-1}\right)^\top = \lambda\,\mathbf{e}^\top \mathbf{V}^{-1} + \nu\,\mathbf{r}^\top \mathbf{V}^{-1}$
- Substitute into $\begin{cases} \mathbf{x}^\top \mathbf{e} = 1 \\ \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \implies \begin{cases} \lambda\,\mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e} + \nu\,\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} = 1 \\ \lambda\,\mathbf{e}^\top \mathbf{V}^{-1}\mathbf{r} + \nu\,\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \mu \end{cases}$

- Set $\alpha = \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e}$, $\beta = \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} = \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{r}$, $\gamma = \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r}$, $\delta \equiv \alpha\gamma - \beta^2$, then

$$\begin{cases} \lambda\, \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e} + \nu\, \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} = 1 \\ \lambda\, \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{r} + \nu\, \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \mu \end{cases}$$

  becomes

$$\begin{cases} \lambda\alpha + \nu\beta = 1 \\ \lambda\beta + \nu\gamma = \mu \end{cases}$$

  Solutions: $\lambda = \dfrac{\gamma - \beta\mu}{\delta}$, $\gamma = \dfrac{\alpha\mu - \beta}{\delta}$

- If $\mathbf{r} \neq c\,\mathbf{e}$, $c \in \mathbb{R}$, then from the positive-definiteness of $\mathbf{V}^{-1}$

$$\begin{aligned} &(\mathbf{r} - c\,\mathbf{e})^\top \mathbf{V}^{-1}(\mathbf{r} - c\,\mathbf{e}) > 0 \\ &\implies \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} - c\,\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} - c\,\mathbf{e}\mathbf{V}^{-1}\mathbf{r} + c^2\,\mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e}^\top > 0 \\ &\implies \gamma - 2\,c\,\beta + c^2\,\alpha > 0 \\ &\implies -\delta = \beta^2 - \gamma\alpha < 0 \end{aligned}$$

- The relation of σ with μ:

$$\sigma^2 = \mathbf{x}^\top \mathbf{V}\mathbf{x} = \mathbf{x}^\top \mathbf{V}(\lambda \mathbf{V}^{-1}\mathbf{e} + \nu \mathbf{V}^{-1}\mathbf{r}) = \lambda(\mathbf{x}^\top \mathbf{e}) + \nu(\mathbf{x}^\top \mathbf{r})$$

$$= \lambda + \nu\mu = \frac{\gamma - \beta\mu}{\delta} + \nu\frac{\alpha\mu - \beta}{\delta} = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$$

$$\implies \frac{\sigma^2}{\left(\frac{1}{\sqrt{\alpha}}\right)^2} - \frac{\left(\mu - \frac{\beta}{\alpha}\right)^2}{\left(\frac{\sqrt{\delta}}{\alpha}\right)^2} = 1$$

- Hyperbola (x, y)

$$\text{equation:} \quad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\text{asymptotes:} \quad (y-k) = \pm\frac{b}{a}(x-h)$$

- Here we have (σ, μ) with $a = \dfrac{1}{\sqrt{\alpha}}$, $b = \dfrac{\sqrt{\delta}}{\alpha}$, $h = 0$, $k = \dfrac{\beta}{\alpha}$, the asymptotes are $\left(\mu - \dfrac{\beta}{\alpha}\right) = \pm\dfrac{\frac{\sqrt{\delta}}{\alpha}}{\frac{1}{\sqrt{\alpha}}}\sigma \implies \mu = \dfrac{\beta}{\alpha} \pm \sqrt{\dfrac{\delta}{\alpha}}\sigma$

- Global minimum-variance portfolio $\mathbf{x}_g$
  - First find μ_g that minimizes $\sigma^2 = \dfrac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$:

    By differentiation $2\alpha\mu_g - 2\beta = 0 \implies \mu_g = \dfrac{\beta}{\alpha}$
  - $\lambda_g = \dfrac{\gamma - \beta\mu_g}{\delta} = \dfrac{\gamma - \beta\frac{\beta}{\alpha}}{\delta} = \dfrac{\gamma\alpha - \beta^2}{\alpha\delta} = \dfrac{1}{\alpha}$

    $\nu_g = \dfrac{\alpha\mu_g - \beta}{\delta} = \dfrac{\beta - \beta}{\delta} = 0$

    so $\mathbf{x}_g = \lambda_g \mathbf{V}^{-1}\mathbf{e} + \nu_g \mathbf{r}^\top\mathbf{V}^{-1} = \dfrac{1}{\alpha}\mathbf{V}^{-1}\mathbf{e}$
- Diversified portfolio: $\mathbf{x}_d \equiv \dfrac{1}{\beta}\mathbf{V}^{-1}\mathbf{r}$, then the expected return $\mu_d = \mathbf{x}_d^\top\mathbf{r} = \dfrac{1}{\beta}\mathbf{r}^\top\mathbf{V}^{-1}\mathbf{r} = \dfrac{\gamma}{\beta}$

$\mathbf{x} = \lambda\mathbf{V}^{-1}\mathbf{e} + \nu\mathbf{V}^{-1}\mathbf{r} = \lambda\,\alpha\,\mathbf{x}_g + \nu\,\beta\,\mathbf{x}_d$, so **every portfolio is the convex combination of $\mathbf{x}_g$ and $\mathbf{x}_d$**: note that $\lambda\alpha + \nu\beta = 1$ (constraint $\mathbf{x}^\top\mathbf{e} = 1$) !

Theorem (Mutual Fund Theorem)**.** Any minimum-variance portfolio is equivalent to investing in the convex combination of $\mathbf{x}_g$ and $\mathbf{x}_d$.

Theorem. Diversified portfolio $\mathbf{x}_d$ is the portfolio that maximize $s(\mathbf{x}) \equiv \dfrac{\mathbf{x}^\top \mathbf{r}}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$.

Proof.

- $\max\ s(\mathbf{x}) \equiv \max\ \log(s(\mathbf{x}))$ s.t. $\mathbf{x}^\top \mathbf{e} = 1$
- Change of variable: $\mathbf{x}^\top \mathbf{r} = \mu \implies$ $\log(s(\mathbf{x})) = \log \dfrac{\mu}{\sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}} \equiv f(\mu)$ with $\mu > 0$
- $f'(\mu) = \dfrac{\gamma - \beta\mu}{\mu\left(\alpha\left(\mu - \frac{\beta}{\alpha}\right)^2 + \frac{\delta}{\alpha}\right)} = 0$ at $\mu = \dfrac{\gamma}{\beta} = \mu_d$

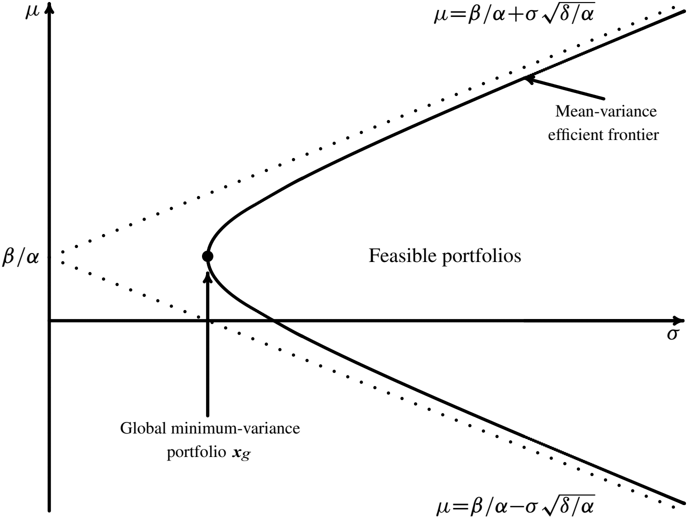

Figure 1: The Case of All Risky Assets

All But One Risky Assets

WLOG add riskless asset 0 with constant return r_0; the portfolio becomes $(x_0, x_1, x_2, \dots, x_s)^\top$

$$\min_{x_0,\mathbf{x}} \frac{1}{2}\mathbf{x}^\top \mathbf{V}\mathbf{x} \quad \text{s.t.} \quad \begin{cases} x_0 + \mathbf{x}^\top \mathbf{e} = 1 \\ x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \qquad \mathbf{e} \equiv \underbrace{(1, 1, \dots, 1)^\top}_{s \text{ items}}$$

- Set $\overline{\mathcal{L}} \equiv \dfrac{1}{2}\mathbf{x}^\top \mathbf{V}\mathbf{x} + \overline{\lambda}\,(1 - x_0 - \mathbf{x}^\top \mathbf{e}) + \overline{\nu}\,(\mu - x_0 r_0 \mathbf{x}^\top \mathbf{r})$ with Lagrange multipliers $\overline{\lambda}$, $\overline{\nu}$
- By $\dfrac{\partial \overline{\mathcal{L}}}{\partial \mathbf{x}} = \mathbf{V}\mathbf{x} - \overline{\lambda}\,\mathbf{e} - \overline{\nu}\,\mathbf{r} = 0 \implies \mathbf{x} = \overline{\lambda}\,\mathbf{V}^{-1}\mathbf{e} + \overline{\nu}\,\mathbf{V}^{-1}\mathbf{r}$,
  so $\mathbf{x}^\top = \overline{\lambda}\,\mathbf{e}^\top \left(V^{-1}\right)^\top + \overline{\nu}\,\mathbf{r}^\top \left(V^{-1}\right)^\top = \overline{\lambda}\,\mathbf{e}^\top \mathbf{V}^{-1} + \overline{\nu}\,\mathbf{r}^\top \mathbf{V}^{-1}$
- By $\dfrac{\partial \overline{\mathcal{L}}}{\partial x_0} = -\overline{\lambda} - \overline{\nu} r_0 = 0 \implies \overline{\nu} = -\dfrac{\overline{\lambda}}{r_0}$

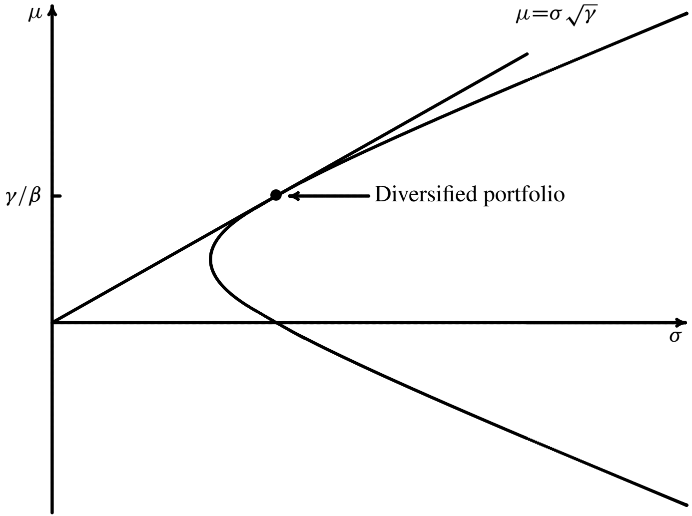

Figure 2: The Diversified Portfolio

- $\begin{cases} x_0 + \mathbf{x}^\top \mathbf{e} = 1 \\ x_0 r_0 + \mathbf{x}^\top \mathbf{r} = \mu \end{cases} \implies \begin{cases} x_0 + \overline{\lambda}\, \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e} + \overline{\nu}\, \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} = 1 \\ x_0 r_0 + \overline{\lambda}\, \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{r} + \overline{\nu}\, \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r} = \mu \end{cases}$
- Set $\alpha = \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{e}$, $\beta = \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{e} = \mathbf{e}^\top \mathbf{V}^{-1}\mathbf{r}$, $\gamma = \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r}$, $\delta \equiv \alpha\gamma - \beta^2$, the above becomes

$$\begin{cases} x_0 + \overline{\lambda}\alpha + \overline{\nu}\beta = x_0 + \overline{\lambda}\alpha - \frac{\overline{\lambda}}{r_0}\beta = 1 \\ x_0 r_0 + \overline{\lambda}\beta + \overline{\nu}\gamma = x_0 r_0 + \overline{\lambda}\beta - \frac{\overline{\lambda}}{r_0}\gamma = \mu \end{cases}$$

  with solutions $x_0 = \dfrac{\alpha\mu r_0 - \beta r_0 + \gamma - \beta\mu}{\epsilon^2}$, $\overline{\lambda} = \dfrac{(r_0 - \mu) r_0}{\epsilon^2}$, $\overline{\nu} = -\dfrac{r_0 - \mu}{\epsilon^2}$, where $\epsilon^2 = \alpha r_0^2 - 2\beta r_0 + \gamma = \alpha\Big(r_0 - \dfrac{\beta}{\alpha}\Big)^2 + \dfrac{\delta}{\alpha}$
- The relation of σ with μ

$$\sigma^2 = \mathbf{x}^\top \mathbf{V}\mathbf{x} = \mathbf{x}^\top \mathbf{V}(\overline{\lambda}\mathbf{V}^{-1}\mathbf{e} + \overline{\nu}\mathbf{V}^{-1}\mathbf{r}) = \overline{\lambda}(\mathbf{x}^\top \mathbf{e}) + \overline{\nu}(\mathbf{x}^\top \mathbf{r})$$

$$= \overline{\lambda}(1 - x_0) + \overline{\nu}(\mu - x_0 r_0) = \overline{\lambda} + \overline{\nu}\mu = \frac{(\mu - r_0)^2}{\epsilon^2}$$

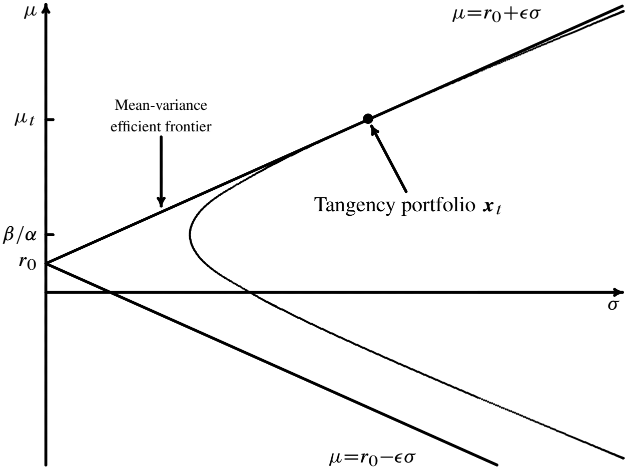

Figure 3: The Case of All But One Risky Assets

Property. If $r_0 < \dfrac{\beta}{\alpha}$, then $\mu = r_0 + \epsilon\sigma$ touches the hyperbola $\sigma^2 = \dfrac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$ at $\left(\dfrac{\epsilon}{\beta - \alpha r_0}, \dfrac{\gamma - \beta r_0}{\beta - \alpha r_0}\right)$

Proof. On $\sigma - \mu$ plane the slope of the tangent $\mu'(\sigma)$ is obtained by implicit differentiation of $\sigma^2 = \dfrac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$ w.r.t σ (let $\mu \equiv \mu(\sigma)$): $2\sigma = \dfrac{2\alpha\mu\mu' - 2\beta\mu'}{\delta} \implies \mu' = \dfrac{\delta\sigma}{\alpha\mu - \beta}$. Solve σ, μ from $\mu = r_0 + \epsilon\sigma$ and $\epsilon = \dfrac{\delta\sigma}{\alpha\mu - \beta}$, we obtain $(\sigma, \mu) = \left(\dfrac{\epsilon}{\beta - \alpha r_0}, \dfrac{\gamma - \beta r_0}{\beta - \alpha r_0}\right)$.

- Define the tangency portfolio

$$\mathbf{x}_t = \frac{1}{\beta - \alpha r_0}\mathbf{V}^{-1}(\mathbf{r} - r_0\mathbf{e}) = \frac{\beta}{\beta - \alpha r_0}\mathbf{x}_d - \frac{\alpha r_0}{\beta - \alpha r_0}\mathbf{x}_g$$

- $\mathbf{x} = \overline{\lambda}\mathbf{V}^{-1}\mathbf{e} + \overline{\nu}\mathbf{V}^{-1}\mathbf{r} = \overline{\nu}\mathbf{V}^{-1}(\mathbf{r} - r_0\mathbf{e}) \equiv (1 - x_0)\mathbf{x}_t$

- $\mathbf{e}^\top \mathbf{x}_t = \dfrac{\beta}{\beta - \alpha r_0}\mathbf{e}^\top \mathbf{x}_d - \dfrac{\alpha r_0}{\beta - \alpha r_0}\mathbf{e}^\top \mathbf{x}_g = \dfrac{\beta}{\beta - \alpha r_0} - \dfrac{\alpha r_0}{\beta - \alpha r_0} = 1$
- $\mu_t = \mathbf{x}_t^\top \mathbf{r} = \mathbf{r}^\top \mathbf{x}_t = \dfrac{\beta}{\beta - \alpha r_0}\mathbf{r}^\top \mathbf{x}_d - \dfrac{\alpha r_0}{\beta - \alpha r_0}\mathbf{r}^\top \mathbf{x}_g$
  $= \dfrac{\beta}{\beta - \alpha r_0}\mu_d - \dfrac{\alpha r_0}{\beta - \alpha r_0}\mu_g = \dfrac{\gamma - \beta r_0}{\beta - \alpha r_0}$ for $\mu_d = \dfrac{\gamma}{\beta}$, $\mu_g = \dfrac{\beta}{\alpha}$

Theorem. Tangency portfolio $\mathbf{x}_t$ is the portfolio that maximize $s(\mathbf{x}) \equiv \dfrac{\mathbf{x}^\top \mathbf{r} - r_0}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$.

Proof.

- $\max\ s(\mathbf{x}) \equiv \max\ \log(s(\mathbf{x}))$ s.t. $\mathbf{x}^\top \mathbf{e} = 1$
- Change of variable $\mathbf{x}^\top \mathbf{r} = \mu \implies$
  $\log(s(\mathbf{x})) = \log \frac{\mu - r_0}{\sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}} \equiv f(\mu)$ with $\mu > r_0$
- $f'(\mu) = \dfrac{(\gamma - \beta r_0) - (\beta - \alpha r_0)\mu}{(\mu - r_0)\,(\alpha\mu^2 - 2\beta\mu + \gamma)} = 0$ at $\mu = \dfrac{\gamma - \beta r_0}{\beta - \alpha r_0} = \mu_t$.