Operations Research
11. The Simplex Method. Dual of Linear Program

An iterative process starting with suboptimal solution and stopping at a feasible solution which cannot be improved

$$
\begin{aligned}
\text{maximize} \quad & 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leqslant 5 \\
& 4x_1 + x_2 + 2x_3 \leqslant 11 \\
& 3x_1 + 4x_2 + 2x_3 \leqslant 8 \\
& x_1 \geqslant 0, \ x_2 \geqslant 0, \ x_3 \geqslant 0
\end{aligned}
$$

$$
\begin{array}{lrcrcrcrcr}
\hline
\text{maximize} & \zeta & = & & + & 5\,x_1 & + & 4\,x_2 & + & 3\,x_3 \\
\hline
\text{subject to} & w_1 & = & 5 & - & 2\,x_1 & - & 3\,x_2 & - & 1\,x_3 \\
& w_2 & = & 11 & - & 4\,x_1 & - & 1\,x_2 & - & 2\,x_3 \\
& w_3 & = & 8 & - & 3\,x_1 & - & 4\,x_2 & - & 2\,x_3 \\
\hline
\end{array}
$$

$$x_1, \ x_2, \ x_3, \ w_1, \ w_2, \ w_3 \geqslant 0$$

Turn inequalities to equalities: add nonnegative slack variables w_1, w_2, w_3

$$\begin{array}{llrrrrrrr}
\text{maximize} & \zeta = & & \boxed{+\ 5\ x_1} & + 4\ x_2 & + 3\ x_3 \\
\hline
\text{subject to} & w_1 = & 5 & - 2\ x_1 & - 3\ x_2 & - 1\ x_3 \\
& w_2 = & 11 & - 4\ x_1 & - 1\ x_2 & - 2\ x_3 \\
& w_3 = & 8 & - 3\ x_1 & - 4\ x_2 & - 2\ x_3
\end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

$$\begin{array}{llrrrr}
\text{maximize} & \zeta = & \frac{25}{2} & - \frac{5}{2}\ w_1 & - \frac{7}{2}\ x_2 & + \frac{1}{2}\ x_3 \\
\hline
\text{subject to} & x_1 = & \frac{5}{2} & - \frac{1}{2}\ w_1 & - \frac{3}{2}\ x_2 & - \frac{1}{2}\ x_3 \\
& w_2 = & 1 & + 2\ w_1 & + 5\ x_2 & \\
& w_3 = & \frac{1}{2} & + \frac{3}{2}\ w_1 & + \frac{1}{2}\ x_2 & - \frac{1}{2}\ x_3
\end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

1st step: take $x_1 = 0,\ x_2 = 0,\ x_3 = 0$, then $w_1 = 5,\ w_2 = 11,\ w_3 = 8$

Find in ζ the variable with the greatest positive coefficient: here $+5x_1$

Increase x_1 and keep $x_2 = x_3 = 0$; Note the constraints $w_1,\ w_2,\ w_3 \geqslant 0$.

$$w_1 = 5 - 2x_1 \geqslant 0 \implies x_1 \leqslant \frac{5}{2}; \quad w_2 = 11 - 4x_1 \geqslant 0 \implies x_1 \leqslant \frac{11}{4}; \quad w_3 = 8 - 3x_1 \geqslant 0 \implies x_1 \leqslant \frac{8}{3}$$

2nd step: take $x_1 = \frac{5}{2},\ x_2 = 0,\ x_3 = 0$, then $w_1 = 0,\ w_2 = 1,\ w_3 = \frac{1}{2}$

Swap the roles of x_1 and w_1, then the first constraint becomes $x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3$

Rewrite all other constraints and ζ with $x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3$

$$
\begin{array}{lrcl}
\text{maximize} & \zeta & = & \frac{25}{2} - \frac{5}{2} w_1 - \frac{7}{2} x_2 \boxed{+ \frac{1}{2} x_3} \\
\hline
\text{subject to} & x_1 & = & \frac{5}{2} - \frac{1}{2} w_1 - \frac{3}{2} x_2 - \frac{1}{2} x_3 \\
& w_2 & = & 1 + 2 w_1 + 5 x_2 \\
& w_3 & = & \frac{1}{2} + \frac{3}{2} w_1 + \frac{1}{2} x_2 - \frac{1}{2} x_3
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

$$
\begin{array}{lrcl}
\text{maximize} & \zeta & = & 13 - 1\, w_1 - 3\, x_2 - 1\, w_3 \\
\hline
\text{subject to} & x_1 & = & 2 - 2\, w_1 - 2\, x_2 + 1\, w_3 \\
& w_2 & = & 1 + 2\, w_1 + 5\, x_2 \\
& x_3 & = & 1 + 3\, w_1 + 1\, x_2 - 2\, w_3
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

Find in ζ the variable with the greatest positive coefficient: here $+\frac{1}{2}x_3$

Increase x_3 and keep $w_1 = x_2 = 0$; Note the constraints $x_1,\ w_3 \geqslant 0$.

$$x_1 = \frac{5}{2} - \frac{1}{2}x_3 \geqslant 0 \implies x_3 \leqslant 5; \quad w_3 = \frac{1}{2} - \frac{1}{2}x_3 \geqslant 0 \implies x_3 \leqslant 1$$

3rd step: take $x_1 = 2,\ x_2 = 0,\ x_3 = 1$, then $w_1 = 0,\ w_2 = 1,\ w_3 = 0$

Swap the roles of x_3 and w_3, then the last constraint becomes $x_3 = 1 + 3w_1 + x_2 - 2w_3$

Rewrite all other constraints and ζ with $x_3 = 1 + 3w_1 + x_2 - 2w_3$

Now ζ has no variable with positive coefficient; stop. $\max \zeta = 13$ with $x_1 = 2,\ x_2 = 0,\ x_3 = 1$.

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + 3x_2 + 4x_3 \\
\text{subject to} & 2x_2 + 3x_3 \leqslant 5 \\
& x_1 + x_2 + 2x_3 \leqslant 4 \\
& x_1 + 2x_2 + 3x_3 \leqslant 7 \\
& x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0
\end{array}
$$

$$
\begin{array}{lrcrrrrrrr}
\hline
\text{maximize} & \zeta & = & & + & 2\,x_1 & + & 3\,x_2 & + & 4\,x_3 \\
\hline
\text{subject to} & w_1 & = & 5 & & & - & 2\,x_2 & - & 3\,x_3 \\
& w_2 & = & 4 & - & 1\,x_1 & - & 1\,x_2 & - & 2\,x_3 \\
& w_3 & = & 7 & - & 1\,x_1 & - & 2\,x_2 & - & 3\,x_3 \\
\hline
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

Turn inequalities to equalities: add nonnegative slack variables w_1, w_2, w_3

$$\begin{array}{llrrrr} \text{maximize} & \zeta = & & +\,2\,x_1 & +\,3\,x_2 & \boxed{+\,4\,x_3} \\ \hline \text{subject to} & w_1 = & 5 & & -\,2\,x_2 & -\,3\,x_3 \\ & w_2 = & 4 & -\,1\,x_1 & -\,1\,x_2 & -\,2\,x_3 \\ & w_3 = & 7 & -\,1\,x_1 & -\,2\,x_2 & -\,3\,x_3 \end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

$$\begin{array}{llrrrr} \text{maximize} & \zeta = & \frac{20}{3} & +\,2\,x_1 & +\,\frac{1}{3}\,x_2 & -\,\frac{4}{3}\,w_1 \\ \hline \text{subject to} & x_3 = & \frac{5}{3} & & -\,\frac{2}{3}\,x_2 & -\,\frac{1}{3}\,w_1 \\ & w_2 = & \frac{2}{3} & -\,1\,x_1 & +\,\frac{1}{3}\,x_2 & +\,\frac{2}{3}\,w_1 \\ & w_3 = & 2 & -\,1\,x_1 & & +\,1\,w_1 \end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

1st step: take $x_1 = 0,\ x_2 = 0,\ x_3 = 0,$ then $w_1 = 5,\ w_2 = 4,\ w_3 = 7$

Find in ζ the variable with the greatest positive coefficient: here $+4\,x_3$

Increase x_3 and keep $x_1 = x_2 = 0$; Note the constraints $w_1, w_2, w_3 \geqslant 0$.

$$w_1 = 5 - 3x_3 \geqslant 0 \implies x_3 \leqslant \frac{5}{3}; \quad w_2 = 4 - 2x_3 \geqslant 0 \implies x_3 \leqslant 2; \quad w_3 = 7 - 3x_3 \geqslant 0 \implies x_3 \leqslant \frac{7}{3}$$

2nd step: take $x_1 = 0,\ x_2 = 0,\ x_3 = \frac{5}{3},$ then $w_1 = 0,\ w_2 = \frac{2}{3},\ w_3 = 2$

Swap the roles of x_3 and w_1, then the first constraint becomes $x_3 = \frac{5}{3} - \frac{2}{3}x_2 - \frac{1}{3}w_1$

Rewrite all other constraints and ζ with $x_3 = \frac{5}{3} - \frac{2}{3}x_2 - \frac{1}{3}w_1$

$$\begin{array}{llrrrrrrrr}
\text{maximize} & \zeta & = & \frac{20}{3} & \boxed{+\ 2\ x_1} & + & \frac{1}{3}\ x_2 & - & \frac{4}{3}\ w_1 \\
\hline
\text{subject to} & x_3 & = & \frac{5}{3} & & - & \frac{2}{3}\ x_2 & - & \frac{1}{3}\ w_1 \\
& w_2 & = & \frac{2}{3} & -\ 1\ x_1 & + & \frac{1}{3}\ x_2 & + & \frac{2}{3}\ w_1 \\
& w_3 & = & 2 & -\ 1\ x_1 & & & + & 1\ w_1
\end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

$$\begin{array}{llrrrrrrrr}
\text{maximize} & \zeta & = & 8 & -\ 2\ w_2 & + & 1\ x_2 & & \\
\hline
\text{subject to} & x_3 & = & \frac{5}{3} & & - & \frac{2}{3}\ x_2 & - & \frac{1}{3}\ w_1 \\
& x_1 & = & \frac{2}{3} & -\ 1\ w_2 & + & \frac{1}{3}\ x_2 & + & \frac{2}{3}\ w_1 \\
& w_3 & = & \frac{4}{3} & +\ 1\ w_2 & - & \frac{1}{3}\ x_2 & + & \frac{1}{3}\ w_1
\end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

Find in ζ the variable with the greatest positive coefficient: here $+2x_1$

Increase x_1 and keep $x_2 = w_1 = 0$; Note the constraints $w_2, w_3 \geqslant 0$.

$$w_2 = \frac{2}{3} - 1x_1 \geqslant 0 \implies x_1 \leqslant \frac{2}{3}; \quad w_3 = 2 - 1x_1 \geqslant 0 \implies x_1 \leqslant 2$$

3rd step: take $x_1 = \frac{2}{3},\ x_2 = 0,\ x_3 = \frac{5}{3}$, then $w_1 = 0,\ w_2 = 0,\ w_3 = \frac{4}{3}$

Swap the roles of x_1 and w_2, then the second constraint becomes $x_1 = \frac{2}{3} - w_2 + \frac{1}{3}x_2 + \frac{2}{3}w_1$

Rewrite all other constraints and ζ with $x_1 = \frac{2}{3} - w_2 + \frac{1}{3}x_2 + \frac{2}{3}w_1$

$$\begin{array}{llcrlrlrl} \text{maximize} & \zeta & = & 8 & - & 2\,w_2 & \boxed{+\ 1\,x_2} & & \\ \hline \text{subject to} & x_3 & = & \frac{5}{3} & & & -\ \frac{2}{3}\,x_2 & - & \frac{1}{3}\,w_1 \\ & x_1 & = & \frac{2}{3} & - & 1\,w_2 & +\ \frac{1}{3}\,x_2 & + & \frac{2}{3}\,w_1 \\ & w_3 & = & \frac{4}{3} & + & 1\,w_2 & -\ \frac{1}{3}\,x_2 & + & \frac{1}{3}\,w_1 \end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

$$\begin{array}{llcrlrlrl} \text{maximize} & \zeta & = & \frac{21}{2} & - & 2\,w_2 & -\ \frac{3}{2}\,x_3 & - & \frac{1}{2}\,w_1 \\ \hline \text{subject to} & x_2 & = & \frac{5}{2} & & & -\ \frac{3}{2}\,x_3 & - & \frac{1}{2}\,w_1 \\ & x_1 & = & \frac{3}{2} & - & 1\,w_2 & -\ \frac{1}{2}\,x_3 & + & \frac{1}{2}\,w_1 \\ & w_3 & = & \frac{1}{2} & + & 1\,w_2 & +\ \frac{1}{2}\,x_3 & + & \frac{1}{2}\,w_1 \end{array}$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3 \geqslant 0$$

Find in ζ the variable with the greatest positive coefficient: here $+1x_2$

Increase x_2 and keep $w_1, w_2 = 0$; Note the constraints $x_3,\ w_3 \geqslant 0$.

$$x_3 = \frac{5}{3} - \frac{2}{3}x_2 \geqslant 0 \implies x_2 \leqslant \frac{5}{2}; \quad w_3 = \frac{4}{3} - \frac{1}{3}x_2 \geqslant 0 \implies x_2 \leqslant 4$$

3rd step: take $x_1 = \frac{3}{2},\ x_2 = \frac{5}{2},\ x_3 = 0$, and $w_1 = 0,\ w_2 = 0,\ w_3 = \frac{1}{2}$

Swap the roles of x_2 and x_3, then the first constraint becomes $x_2 = \frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}w_1$

Rewrite all other constraints and ζ with $x_2 = \frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}w_1$

Now ζ has no variable with positive coefficient; stop. $\max \zeta = \frac{21}{2}$ with $x_1 = \frac{3}{2},\ x_2 = \frac{5}{2},\ x_3 = 0$.

LP in standard form:

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \quad i = 1, 2, \ldots, m \\ & x_j \geqslant 0, \quad j = 1, 2, \ldots, n \end{array}$$

Introduce slack variables w_i, $i = 1, 2, \ldots, m$ and ζ for the objective function value:

$$\zeta = \sum_{j=1}^{n} c_j x_j$$

$$w_i = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m$$

$$\begin{array}{lrcrcrcrcr} \text{maximize} & \zeta & = & \frac{25}{2} & - & \frac{5}{2}\, w_1 & - & \frac{7}{2}\, x_2 & + & \frac{1}{2}\, x_3 \\ \hline \text{subject to} & x_1 & = & \frac{5}{2} & - & \frac{1}{2}\, w_1 & - & \frac{3}{2}\, x_2 & - & \frac{1}{2}\, x_3 \\ & w_2 & = & 1 & + & 2\, w_1 & + & 5\, x_2 & & \\ & w_3 & = & \frac{1}{2} & + & \frac{3}{2}\, w_1 & + & \frac{1}{2}\, x_2 & - & \frac{1}{2}\, x_3 \end{array}$$

dictionary

basic variables (lhs): x_1, w_2, w_3

nonbasic variables (rhs): w_1, x_2, x_3

entering variable: the variable nonbasic $\Longrightarrow$ basic

leaving variable: the variable basic $\Longrightarrow$ nonbasic

rename $(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_m) \Longrightarrow (x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m})$, so

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m$$

The starting dictionary

$$\zeta = \sum_{j=1}^{n} c_j x_j$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m$$

Each dictionary has m basic variables and n nonbasic variables

$\mathcal{B}$: subset from $\{1, 2, \ldots, n+m\}$ that of the basic variables

$\mathcal{N}$: subset from $\{1, 2, \ldots, n+m\}$ that of the nonbasic variables

Initially $\mathcal{N} = \{1, 2, \ldots, n\}$ and $\mathcal{B} = \{n+1, n+2, \ldots, n+m\}$

The current dictionary is of the form

$$\zeta = \overline{\zeta} + \sum_{j \in \mathcal{N}} \overline{c_j} x_j$$

$$x_i = \overline{b_i} - \sum_{j \in \mathcal{N}} \overline{a_{ij}} x_j, \quad i \in \mathcal{B}$$

The entering variable is chosen to increase ζ: pick $k \in \{j \in \mathcal{N} : \overline{c_j} > 0\}$. If no such k, then the current solution is optimal.

The leaving variable is chosen to preserve nonnegativity of the current basic variables, so

$$x_i = \overline{b_i} - \overline{a_{ik}} x_k \implies \overline{b_i} - \overline{a_{ik}} x_k \geqslant 0, \quad i \in \mathcal{B}$$

The rule for selecting leaving variable: pick $l \in \{i \in \mathcal{B} : \overline{a_{ik}} > 0 \ \wedge \ \frac{\overline{b_i}}{\overline{a_{ik}}} \text{ is minimal}\}$

Previously in standard form LP

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \quad i = 1, 2, \ldots, m \\ & x_j \geqslant 0, \quad j = 1, 2, \ldots, n \end{aligned}$$

We assume $b_i \geqslant 0, \forall\, i = 1,\ 2,\ \ldots,\ m$.

If not the case, then consider the auxiliary problem

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leqslant b_i, \quad i = 1, 2, \ldots, m \\ & x_j \geqslant 0, \quad j = 0, 1, 2, \ldots, n \end{aligned}$$

The original problem has a feasible solution iff the optimal solution of the auxiliary problem is zero.

First try to convert the initial infeasible dictionary with into a feasible one by one pivot of the "most infeasible variable".

Proceed with the usual simplex steps.

Discard all x_0 terms; reintroduce the original objective with the substituted nonbasic variables.

An example:

$$\begin{aligned} \text{maximize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leqslant -1 \\ & -x_1 - 2x_2 \leqslant -2 \\ & x_2 \leqslant 1 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0 \end{aligned}$$

The original problem:

$$\begin{aligned} \text{maximize} \quad & -2x_1 - x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leqslant -1 \\ & -x_1 - 2x_2 \leqslant -2 \\ & x_2 \leqslant 1 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0 \end{aligned}$$

The auxiliary problem:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{subject to} \quad & -x_1 + x_2 - x_0 \leqslant -1 \\ & -x_1 - 2x_2 - x_0 \leqslant -2 \\ & x_2 - x_0 \leqslant 1 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_0 \geqslant 0 \end{aligned}$$

Setup the initial infeasible dictionary:

$$\begin{array}{llrl} \text{maximize} & \zeta = & & -\ 1\ x_0 \\ \hline \text{subject to} & w_1 = & -1 + 1\ x_1 - 1\ x_2 & +\ 1\ x_0 \\ & \boxed{w_2} = & -2 + 1\ x_1 + 2\ x_2 & +\ 1\ x_0 \\ & w_3 = & 1 \qquad\quad - 1\ x_2 & +\ 1\ x_0 \\ \hline \end{array}$$

$$x_1,\ x_2,\ x_0,\ w_1,\ w_2,\ w_3 \geqslant 0$$

The "most infeasible variable" is w_2;
Substitute x_0 with $w_2 + 2 - x_1 - x_2$:

$$\begin{array}{llrl} \text{maximize} & \zeta = & -2 + 1\ x_1 + 2\ x_2 & -\ 1\ w_2 \\ \hline \text{subject to} & w_1 = & 1 \qquad\quad - 3\ x_2 & +\ 1\ w_2 \\ & x_0 = & 2 - 1\ x_1 - 2\ x_2 & +\ 1\ w_2 \\ & w_3 = & 3 - 1\ x_1 - 3\ x_2 & +\ 1\ w_2 \\ \hline \end{array}$$

$$x_1,\ x_2,\ x_0,\ w_1,\ w_2,\ w_3 \geqslant 0$$

Proceed with ordinary simplex method:

$$
\begin{array}{llcl}
\text{maximize} & \zeta & = & 0 - 1\,x_0 \\
\hline
\text{subject to} & x_2 & = & \frac{1}{3} \qquad\quad - \frac{1}{3}\,w_1 + \frac{1}{3}\,w_2 \\
& x_1 & = & \frac{4}{3} - 1\,x_0 + \frac{2}{3}\,w_1 + \frac{1}{3}\,w_2 \\
& w_3 & = & \frac{2}{3} + 1\,x_0 + \frac{1}{3}\,w_1 - \frac{1}{3}\,w_2 \\
\hline
& \multicolumn{3}{c}{x_1,\ x_2,\ x_0,\ w_1,\ w_2,\ w_3 \geqslant 0}
\end{array}
$$

Discard all x_0 terms; Reintroduce the original objective

$$\text{maximize} \quad -2x_1 - x_2 \equiv -3 - w_1 - w_2$$

with the substituted nonbasic variables

$$x_1 = \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2$$
$$x_2 = \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

Combine the auxiliary dictionary with the substituted original object, we have the dictionary

$$
\begin{array}{llcl}
\text{maximize} & \zeta & = & -3 - 1\,w_1 - 1\,w_2 \\
\hline
\text{subject to} & x_2 & = & \frac{1}{3} - \frac{1}{3}\,w_1 + \frac{1}{3}\,w_2 \\
& x_1 & = & \frac{4}{3} + \frac{2}{3}\,w_1 + \frac{1}{3}\,w_2 \\
& w_3 & = & \frac{2}{3} + \frac{1}{3}\,w_1 - \frac{1}{3}\,w_2 \\
\hline
& \multicolumn{3}{c}{x_1,\ x_2,\ w_1,\ w_2,\ w_3 \geqslant 0}
\end{array}
$$

This is the final form, for the coefficients of all nonbasic variables are negative.

The final maximum of the original problem is -3.

Using Simple Pivot Tool to find the extrema of the following LP problems:

$$\begin{aligned} \text{maximize} \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 + 3x_4 \leqslant 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leqslant 3 \\ & x_1,\ x_2,\ x_3,\ x_4 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & -x_1 - 3x_2 - x_3 \\ \text{subject to} \quad & 2x_1 - 5x_2 + x_3 \leqslant -5 \\ & 2x_1 - x_2 + 2x_3 \leqslant 4 \\ & x_1,\ x_2,\ x_3 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 \leqslant 4 \\ & 2x_1 + 3x_3 \leqslant 5 \\ & 2x_1 + x_2 + 3x_3 \leqslant 7 \\ & x_1,\ x_2,\ x_3 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 6x_2 + 9x_3 + 8x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_4 \leqslant 5 \\ & x_1 + x_2 + 2x_3 + 3x_4 \leqslant 3 \\ & x_1,\ x_2,\ x_3,\ x_4 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 5x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + 3x_2 + x_3 \leqslant 3 \\ & -x_1 + 3x_3 \leqslant 2 \\ & 2x_1 - x_2 + 2x_3 \leqslant 4 \\ & 2x_1 + 3x_2 - x_3 \leqslant 2 \\ & x_1,\ x_2,\ x_3 \geqslant 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & x_1 + 8x_2 + 9x_3 + 2x_4 + 7x_5 + 3x_6 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \geqslant 1 \\ & -x_1 + x_4 + x_5 = 0 \\ & -x_2 - x_4 + x_6 = 0 \\ & x_3 + x_5 + x_6 \leqslant 1 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geqslant 0 \end{aligned}$$

LP in standard form:

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 + x_3 + x_4 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 \leqslant 2 && (1) \\
& x_2 + x_4 \leqslant 1 && (2) \\
& x_1 + 2x_3 \leqslant 1 && (3) \\
& x_j \geqslant 0, \quad j = 1, \ldots, 4
\end{aligned}
$$

Suppose the LP-solver find an "optimal" solution $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = 0$, $x_4 = \frac{1}{2}$ with maximum $\frac{5}{2}$, how can we check this?

Scale (1) by $\frac{1}{2}$, add (2), and add (3) scaled by $\frac{1}{2}$, we get for every feasible x_j, $j = 1, \ldots, 4$

$$x_1 + 2x_2 + \frac{3}{2}x_3 + x_4 \leqslant \frac{5}{2}$$

The objective

$$x_1 + 2x_2 + x_3 + x_4 \leqslant x_1 + 2x_2 + \frac{3}{2}x_3 + x_4 \leqslant \frac{5}{2}$$

so $\frac{5}{2}$ is indeed optimal.

LP in standard form:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{1j} x_j \leqslant b_1 \\
& \sum_{j=1}^{n} a_{2j} x_j \leqslant b_2 \\
& \dots\dots\dots \\
& \sum_{j=1}^{n} a_{mj} x_j \leqslant b_m \\
& x_j \geqslant 0, \quad j = 1, 2, \dots, n
\end{aligned}
$$

For every choice of the "scaling factors" $y_i \geqslant 0$, $i = 1, 2, \dots, m$,

$$
y_1 \sum_{j=1}^{n} a_{1j} x_j + y_2 \sum_{j=1}^{n} a_{2j} x_j + \cdots + y_m \sum_{j=1}^{n} a_{mj} x_j \leqslant y_1 b_1 + y_2 b_2 + \cdots + y_m b_m
$$

Rearrange the inequality as

$$
x_1 \sum_{i=1}^{m} a_{i1} y_i + x_2 \sum_{i=1}^{m} a_{i2} y_i + \cdots + x_n \sum_{i=1}^{m} a_{in} y_i \leqslant y_1 b_1 + y_2 b_2 + \cdots + y_m b_m
$$

The Dual of Linear Program (Cont'd)

Choose y_i, $i = 1, 2, \ldots, m$ such that

$$c_1 \leqslant \sum_{i=1}^{m} a_{i1} y_i, \quad c_2 \leqslant \sum_{i=1}^{m} a_{i2} y_i, \quad \ldots, \quad c_n \leqslant \sum_{i=1}^{m} a_{in} y_i$$

Then

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \leqslant x_1 \sum_{i=1}^{m} a_{i1} y_i + x_2 \sum_{i=1}^{m} a_{i2} y_i + \cdots + x_n \sum_{i=1}^{m} a_{in} y_i \leqslant y_1 b_1 + y_2 b_2 + \cdots + y_m b_m$$

To make the upper bound tighter, we have the LP in dual form:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} b_i y_i \\
\text{subject to} \quad & \sum_{i=1}^{n} a_{i1} y_i \geqslant c_1 \\
& \sum_{i=1}^{n} a_{i2} y_i \geqslant c_2 \\
& \cdots\cdots\cdots \\
& \sum_{i=1}^{n} a_{in} y_i \geqslant c_m \\
& y_i \geqslant 0, \quad i = 1, 2, \ldots, m
\end{aligned}$$

The Primal

$$\begin{aligned} \text{maximize} \quad & c^\top x \\ \text{subject to} \quad & A\,x \preccurlyeq b \\ & x \succcurlyeq 0 \end{aligned}$$

The Dual

$$\begin{aligned} \text{minimize} \quad & b^\top y \\ \text{subject to} \quad & A^\top\,y \succcurlyeq c \\ & y \succcurlyeq 0 \end{aligned}$$

The Primal

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & A\,x \succcurlyeq b \\ & x \succcurlyeq 0 \end{aligned}$$

The Dual

$$\begin{aligned} \text{maximize} \quad & b^\top y \\ \text{subject to} \quad & A^\top\,y \preccurlyeq c \\ & y \succcurlyeq 0 \end{aligned}$$

Primal (max)	Dual (min)
$\geqslant$ constraint	$\leqslant 0$ variable
$\leqslant$ constraint	$\geqslant 0$ variable
$=$ constraint	free variable
$\geqslant 0$ variable	$\leqslant$ constraint
$\leqslant 0$ variable	$\geqslant$ constraint
free variable	$=$ constraint