# Operations Research
# 06. Convex Optimization Problems

# Optimization Problem: Standard Form

$$
\begin{array}{rll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leqslant 0, & i = 1, 2, \ldots, m \\
& h_i(x) = 0, & i = 1, 2, \ldots, p
\end{array}
$$

- $x \in \mathbb{R}^n$ is the optimization variable
- $f_0(x) : \mathbb{R}^n \to \mathbb{R}$ is the objective / cost
- $f_i(x) : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$ are the inequality constraints
- $h_i(x) : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, p$ are the equality constraints

# Feasible and Optimal Points

- $x \in \mathbb{R}^n$ is **feasible** if $x \in \mathrm{dom}\, f_0$ and satisfies the constraints
- **optimal value** $p^\star = \inf\{\, f_0(x) \mid x \text{ satisfies the constraints } \}$
- $p^\star = \infty$ if problem is **infeasible**
- $p^\star = -\infty$ if problem is **unbounded below**
- a feasible $x$ is **optimal** if $f_0(x) = p^\star$
- $X_{\mathrm{opt}}$ is the set of optimal points

# Locally Optimal Points

- $x$ is **locally optimal** if $\exists\, R > 0$ such that $x$ is optimal for

$$\begin{array}{rll} \text{minimize} & f_0(z) & \\ \text{subject to} & f_i(z) \leqslant 0, & i = 1, 2, \dots, m \\ & h_i(z) = 0, & i = 1, 2, \dots, p \\ & \|z - x\|_2 \leqslant R & \end{array}$$

- examples with $n = 1$, $m = p = 0$
  - $f_0(x) = \dfrac{1}{x}$, $\operatorname{dom} f_0 = \mathbb{R}_{++}$, $p^\star = 0$, no optimal point
  - $f_0(x) = -\log x$, $\operatorname{dom} f_0 = \mathbb{R}_{++}$, $p^\star = -\infty$
  - $f_0(x) = x \log x$, $\operatorname{dom} f_0 = \mathbb{R}_{++}$, $p^\star = -\dfrac{1}{e}$, $x = \dfrac{1}{e}$ is optimal
  - $f_0(x) = x^3 - 3x$, $p^\star = -\infty$, $x = 1$ is locally optimal

# Implicit and Explicit Constraints

- standard form optimization problem has **implicit constraints**

$$x \in \mathcal{D} \equiv \left( \bigcap_{i=0}^{m} \operatorname{dom} f_i \right) \bigcap \left( \bigcap_{i=1}^{p} \operatorname{dom} h_i \right)$$

- $\mathcal{D}$ is the **domain** of the optimization problem

- constraints $f_i(x) \leqslant 0$, $h_i(x) = 0$ are the **explicit constraints**

- a problem is unconstrained if it has no explicit constraints ($m = p = 0$)

- e.g.

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log\left(b_i - a_i^\top x\right)$$

  is an unconstrained problem with implicit constraints $a_i^\top x < b_i$

# Feasibility Problem

The feasibility problem

$$\begin{array}{rll} \text{find} & x & \\ \text{subject to} & f_i(x) \leqslant 0, & i = 1, 2, \ldots, m \\ & h_i(x) = 0, & i = 1, 2, \ldots, p \end{array}$$

can be consider as the standard optimization problem with $f_0(x) = 0$:

$$\begin{array}{rll} \text{minimize} & 0 & \\ \text{subject to} & f_i(x) \leqslant 0, & i = 1, 2, \ldots, m \\ & h_i(x) = 0, & i = 1, 2, \ldots, p \end{array}$$

- $p^\star = 0$ if constraints are feasible; any feasible $x$ is optimal
- $p^\star = \infty$ if constraints are infeasible

# Standard Form Convex Optimization Problem

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \quad i = 1, 2, \cdots, m \\ & a_i^\top x = b_i, \quad i = 1, 2, \ldots, p \end{array}$$

- objective and inequality constraints $f_0$, $f_1, \ldots, f_m$ are convex
- equality constraints are affine, often written as $A\,x = b$
- feasible and optimal sets of a convex optimization problem are convex

# An Example

- consider the following optimization problem

$$\begin{array}{rl} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = \dfrac{x_1}{1 + x_2^2} \leqslant 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

- $f_0$ is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leqslant 0\}$ is convex
- not a convex problem by our definition, for $f_1$ is not convex, $h_1$ is not affine
- equivalent, but not identical to the convex problem

$$\begin{array}{rl} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1 \leqslant 0 \\ & h_1(x) = x_1 + x_2 = 0 \end{array}$$

# Local and Global Optima

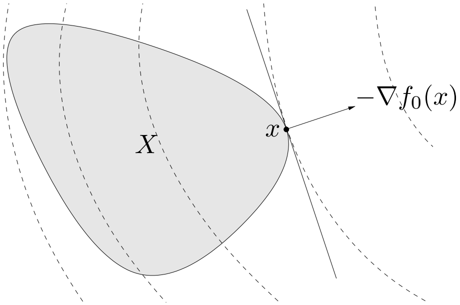
locally optimal point of a convex optimization problem is (globally) optimal

- suppose $x$ is locally optimal, but $\exists\, y$ with $f_0(y) < f_0(x)$
- $x$ locally optimal means $\exists\, R > 0$ such that if $x'$ is feasible and $\|x' - x\| \leqslant R$, then $f_0(x') \geqslant f_0(x)$
- set $z = \theta y + (1-\theta)x$ with $\theta = \dfrac{R}{2\|y - x\|_2}$
- $\|y - x\|_2 > R$, so $0 < \theta < \dfrac{1}{2}$
- $z$ is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = \dfrac{R}{2}$ and $f_0(z) \leqslant \theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$, which contradicts that $x$ is locally optimal

# Optimality Criterion for Differentiable $f_0$

- $x$ is optimal for a convex optimization problem
  $\iff$ $x$ is feasible and $\nabla f_0(x)^\top (y - x) \geqslant 0$ for all feasible $y$

- if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set $X$ at $x$



# Examples

- **unconstrained problem** $x$ minimizes $f_0(x) \iff$

$$\nabla f_0(x) = 0$$

- **equality constrained problem** $x$ minimizes $f_0(x)$ subject to $Ax = b \iff$

$$\exists\, v \quad \text{s.t.} \quad Ax = b, \quad \nabla f_0(x) + A^\top v = 0$$

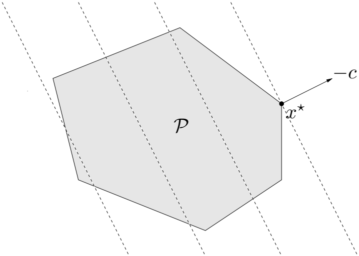
- **minimization over nonnegative orthant** $x$ minimizes $f_0(x)$ over $\mathbb{R}^n_+ \iff$

$$x \succcurlyeq 0, \quad \begin{cases} \nabla f_0(x)_i \geqslant 0, & x_i = 0 \\ \nabla f_0(x)_i = 0, & x_i > 0 \end{cases}$$

# Linear Program (LP)

$$\begin{array}{ll} \text{minimize} & c^\top x + d \\ \text{subject to} & G\,x \preceq h \\ & A\,x = b \end{array}$$

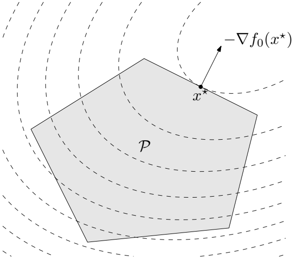
- convex problem with affine objective and constraints
- feasible set is a polyhedron



# Quadratic Program (QP)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\, x^\top P\, x + q^\top x + r \\ \text{subject to} & G\, x \preceq h \\ & A\, x = b \end{array}$$

- $P \in \mathsf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



# Quadratically Constrained Quadratic Program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}\, x^\top P_0\, x + q_0^\top x + r_0 \\
\text{subject to} & \frac{1}{2}\, x^\top P_i\, x + q_i^\top x + r_i \leqslant 0, \quad i = 1, 2, \dots, m \\
& A\, x = b
\end{array}
$$

- $P_i \in \mathsf{S}^n_+$; objective and constraints are convex quadratic
- if $P_1, P_2, \dots, P_m \in \mathsf{S}^n_+$, feasible region is intersection of $m$ ellipsoids and an affine set

# Second-Order Cone Programming (SOCP)

$$\begin{array}{ll} \text{minimize} & f^\top x \\ \text{subject to} & \|A_i\, x + b_i\|_2 \leqslant c_i^\top x + d_i, \quad i = 1, 2, \ldots, m \\ & F\, x = g \end{array}$$

where $A_i \in \mathbb{R}^{n_i \times n}$, $F \in \mathbb{R}^{p \times n}$

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i,\, c_i^\top x + d_i) \in \text{ second-order cone in } \mathbb{R}^{n_i+1}$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP
- more general than QCQP and LP

# Example: Robust Linear Programming

suppose constraints vectors $a_i$ are uncertain and LP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & a_i^\top x \leqslant b_i, \quad i = 1, 2, \ldots, m \end{array}$$

two common approaches to handle uncertainty

- **deterministic worst-case**: constraints must hold for all $a_i \in \mathcal{E}_i$ (uncertainty ellipsoids)

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & a_i^\top x \leqslant b_i \quad \forall\, a_i \in \mathcal{E}_i, \quad i = 1, 2, \ldots, m \end{array}$$

- **stochastic**: $a_i$ is random; constraints must hold with probability $\eta$

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \mathsf{P}\{a_i^\top x \leqslant b_i\} \geqslant \eta, \quad i = 1, 2, \ldots, m \end{array}$$

# Deterministic Worst-Case

- uncertainty ellipsoids are $\mathcal{E}_i = \{\overline{a}_i + P_i u \mid \|u\|_2 \leqslant 1\}$, where $\overline{a}_i \in \mathbb{R}^n$, $P_i \in \mathbb{R}^{n \times n}$
- center of $\mathcal{E}_i$ is $\overline{a}_i$; semi-axes determined by singular values / vectors of $P_i$
- robust LP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & a_i^\top x \leqslant b_i \quad \forall\, a_i \in \mathcal{E}_i, \quad i = 1, 2, \dots, m \end{array}$$

- equivalent to SOCP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \overline{a}_i^\top x + \|P_i^\top x\|_2 \leqslant b_i, \quad i = 1, 2, \dots, m \end{array}$$

which follows from $\sup_{\|u\|_2 \leqslant 1} (\overline{a}_i + P_i u)^\top x = \overline{a}_i^\top x + \|P_i^\top x\|_2$

# Stochastic Approach

- assume $a_i \sim \mathsf{N}(\overline{a}_i, \Sigma_i)$
- $a_i^\top x \sim \mathsf{N}(\overline{a}_i^\top x, x^\top \Sigma_i\, x)$, so

$$\mathsf{P}\{a_i^\top x \leqslant b_i\} = \Phi\left(\frac{b_i - \overline{a}_i^\top x}{\left\| \Sigma_i^{\frac{1}{2}} x \right\|_2}\right)$$

where $\Phi(u) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{u} e^{-\frac{t^2}{2}}\, \mathrm{d}t$ (cdf of $\mathsf{N}(0,1)$)

- $\mathsf{P}\{a_i^\top x \leqslant b_i\} \geqslant \eta$ can be expressed as $\overline{a}_i^\top x + \Phi^{-1}(\eta) \left\| \Sigma_i^{\frac{1}{2}} x \right\|_2 \leqslant b_i$
- for $\eta \geqslant \dfrac{1}{2}$, robust LP equivalent to SOCP

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \overline{a}_i^\top x + \Phi^{-1}(\eta) \left\| \Sigma_i^{\frac{1}{2}} x \right\|_2 \leqslant b_i, \quad i = 1, 2, \ldots, m \end{array}$$

# Semidefinite Program (SDP)

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & A x = b \end{array}$$

where $F_i,\, G \in \mathsf{S}^k$

- inequality constraint is called **linear matrix inequality (LMI)**
- includes problems with multiple LMI constraints: e.g.

  $$x_1 \widehat{F_1} + x_2 \widehat{F_2} + \cdots + x_n \widehat{F_n} + \widehat{G} \preceq 0, \quad x_1 \widetilde{F_1} + x_2 \widetilde{F_2} + \cdots + x_n \widetilde{F_n} + \widetilde{G} \preceq 0$$

  is equivalent to single LMI

  $$x_1 \begin{pmatrix} \widehat{F_1} & 0 \\ 0 & \widetilde{F_1} \end{pmatrix} + x_2 \begin{pmatrix} \widehat{F_2} & 0 \\ 0 & \widetilde{F_2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} \widehat{F_n} & 0 \\ 0 & \widetilde{F_n} \end{pmatrix} + \begin{pmatrix} \widehat{G} & 0 \\ 0 & \widetilde{G} \end{pmatrix} \preceq 0$$

# Example: Matrix Norm Minimization

$$\text{minimize} \quad \|A(x)\|_2 \equiv \left(\lambda_{\max}(A(x)^\top A(x)\right)^{\frac{1}{2}}$$

where $A(x) = A_0 + x_1 A_1 + x_2 A_2 + \cdot + x_n A_n$ with $A_i \in \mathbb{R}^{p \times q}$
equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{pmatrix} t\,I & A(x) \\ A(x)^\top & t\,I \end{pmatrix} \succcurlyeq 0 \end{array}$$

with variables $x \in \mathbb{R}^n$, $t \in \mathbb{R}$; constraint follows from

$$\|A(x)\|_2 \leqslant t \iff A(x)^\top A(x) \preccurlyeq t^2\,I,\ t > 0 \iff \begin{pmatrix} t\,I & A(x) \\ A(x)^\top & t\,I \end{pmatrix} \succcurlyeq 0$$

# Geometric Programming (GP)

- **monomial**

$$f(x) = c\, x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad \mathrm{dom}\, f = \mathbb{R}^n_{++}$$

- **posynomial**: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k\, x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \mathrm{dom}\, f = \mathbb{R}^n_{++}$$

- **geometric program (GP)**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m \\ & h_i(x) = 1, \quad i = 1, 2, \ldots, p \end{array}$$

with $f_i$ posynomial, $h_i$ monomial

# Geometric Program in Convex Form

- change variables to $y_i = \log x_i$, and take logarithm of cost, constraints
- monomial $f(x) = c\, x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, e^{y_2}, \ldots, e^{y_n}) = a^\top y + b \quad (b = \log c)$$

- posynomial $f(x) = \sum_{k=1}^{K} c_k\, x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, e^{y_2}, \ldots, e^{y_n}) = \log \left( \sum_{k=1}^{K} e^{a_k^\top y + b_k} \right)$$

- geometric program transforms to convex problem

$$
\begin{array}{ll}
\text{minimize} & \log \left( \displaystyle\sum_{k=1}^{K} e^{a_{0k}^\top y + b_{0k}} \right) \\
\text{subject to} & \log \left( \displaystyle\sum_{k=1}^{K} e^{a_{ik}^\top y + b_{ik}} \right) \leqslant 0, \quad i = 1, 2, \dots, m \\
& G\, y + d = 0
\end{array}
$$

# Example: Frobenius Norm Diagonal Scaling

- seek diagonal matrix $D = \mathrm{diag}(d)$, $d \succ 0$, to minimize

$$
\left\| DMD^{-1} \right\|_F^2 \equiv \sum_{i,j=1}^{n} \left( DMD^{-1} \right)_{ij}^2 = \sum_{i,j=1}^{n} M_{ij}^2 \, \frac{d_i^2}{d_j^2}
$$

- a posynomial in $d$ with exponents 0, 2, and $-2$
- in convex form, with $y = \log d$,

$$
\log \left\| DMD^{-1} \right\|_F^2 = \log \left( \sum_{i,j=1}^{n} e^{2\,(y_i - y_j + \log |M_{ij}|)} \right)
$$

# Change of Variables

- $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is injective with $\varphi(\mathbf{dom}\,\varphi) \supseteq \mathcal{D}$
- consider (possibly nonconvex) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m \\ & h_i(x) = 0, \quad i = 1, 2, \ldots, p \end{array}$$

- change variables to $z$ with $x = \varphi(z)$
- can solve equivalent problem

$$\begin{array}{ll} \text{minimize} & \widetilde{f_0}(z) \\ \text{subject to} & \widetilde{f_i}(z) \leqslant 0, \quad i = 1, 2, \ldots, m \\ & \widetilde{h_i}(z) = 0, \quad i = 1, 2, \ldots, p \end{array}$$

where $\widetilde{f_i}(z) = f_i(\varphi(z))$ and $\widetilde{h_i}(z) = h_i(\varphi(z))$

- recover original optimal point as $x^\star = \varphi(z^\star)$

## Example

- **nonconvex problem**

$$\begin{array}{ll} \text{minimize} & \dfrac{x_1}{x_2} + \dfrac{x_3}{x_1} \\ \text{subject to} & \dfrac{x_2}{x_3} + x_1 \leqslant 1 \end{array}$$

  with implicit constraint $x \succ 0$

- change variables using $x = \varphi(z) = e^z$ to get

$$\begin{array}{ll} \text{minimize} & e^{z_1 - z_2} + e^{z_3 - z_1} \\ \text{subject to} & e^{z_2 - z_3} + e^{z_1} \leqslant 1 \end{array}$$

  which is **convex**

# Transformation of Objective and Constraints

suppose

- $\varphi_0$ is monotone increasing
- $\psi_i(u) \leqslant 0 \iff u \leqslant 0,\ i = 1, 2, \dots, m$
- $\phi_i(u) = 0 \iff u = 0,\ i = 1, 2, \dots, p$

standard form optimization problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \varphi_0(f_0(x)) \\ \text{subject to} & \psi_i(f_i(x)) \leqslant 0, \quad i = 1, 2, \dots, m \\ & \phi_i(h_i(x)) = 0, \quad i = 1, 2, \dots, p \end{array}$$

example: minimizing $\|A\,x - b\|$ is equivalent to minimizing $\|A\,x - b\|^2$

# Converting Maximization to Minimization

suppose $\varphi_0$ is monotone decreasing, the maximization problem

$$\begin{array}{rll} \text{maximize} & f_0(x) & \\ \text{subject to} & f_i(x) \leqslant 0, & i = 1, 2, \ldots, m \\ & h_i(x) = 0, & i = 1, 2, \ldots, p \end{array}$$

is equivalent to the minimization problem

$$\begin{array}{rll} \text{minimize} & \varphi_0(f_0(x)) & \\ \text{subject to} & f_i(x) \leqslant 0, & i = 1, 2, \ldots, m \\ & h_i(x) = 0, & i = 1, 2, \ldots, p \end{array}$$

- $\varphi_0(u) = -u$ transforms maximizing a concave function to minimizing a convex function
- $\varphi_0(u) = \dfrac{1}{u}$ transforms maximizing a concave positive function to minimizing a convex function

# Eliminating Equality Constraints

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m \\ & A\,x = b \end{array}$$

is equivalent to

$$\begin{array}{rl} \text{minimize (over } z) & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leqslant 0, \quad i = 1, 2, \ldots, m \end{array}$$

where $F$ and $x_0$ are such that $A\,x = b \iff x = Fz + x_0$ for some $z$

# Introducing Equality Constraints

$$\begin{array}{rl} \text{minimize} & f_0(A_0 x + b_0) \\ \text{subject to} & f_i(A_i x + b_i) \leqslant 0, \quad i = 1, 2, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{rl} \text{minimize (over } x,\, y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leqslant 0, \quad i = 1, 2, \ldots, m \\ & y_i = A_i x + b_i, \quad i = 1, 2, \ldots, m \end{array}$$

# Introducing Slack Variables for Linear Ineqs

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^\top x \leqslant b_i, \quad i = 1, 2, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{rl} \text{minimize (over } x,\, s_i) & f_0(x) \\ \text{subject to} & a_i^\top + s_i = b_i, \quad i = 1, 2, \ldots, m \\ & s_i \geqslant 0, \quad i = 1, 2, \ldots, m \end{array}$$

# Epigraph Form

standard form convex problem is equivalent to

$$\begin{array}{rl} \text{minimize (over } x,\, t) & t \\ \text{subject to} & f_0(x) - t \leqslant 0 \\ & f_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m \\ & A\,x = b \end{array}$$

# Minimizing Over Some Variables

$$\begin{array}{rl} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leqslant 0, \quad i = 1, 2, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{rl} \text{minimize} & \widetilde{f_0}(x_1) \\ \text{subject to} & f_i(x_i) \leqslant 0, \quad i = 1, 2, \ldots, m \end{array}$$

where $\widetilde{f_0}(x_1) = \inf\limits_{x_2} f_0(x_1, x_2)$

# Convex Relaxation

- start with **nonconvex problem**: mimimize $h(x)$ s.t. $x \in C$
- find convex $\hat{h}$ with $\hat{h}(x) \leqslant h(x)$, $\forall\, x \in \operatorname{dom} h$
- find set $\widehat{C} \supseteq C$ (e.g. $\widehat{C} = \operatorname{conv} C$) described by

$$\widehat{C} = \{x \mid f_i(x) \leqslant 0,\ i = 1, 2, \dots, m;\ A\,x = b\}$$

- convex optimization problem

$$\begin{array}{ll} \text{minimize} & \hat{h}(x) \\ \text{subject to} & f_i(x) \leqslant 0, \quad i = 1, 2, \dots, m \\ & A\,x = b \end{array}$$

  is a **convex relaxation** of the original problem

- optimal value of relaxation is lower bound on optimal value of original problem

# Example: Boolean LP

- **mixed integer linear program (MILP)**

$$\begin{array}{ll} \text{minimize} & c^\top (x, z) \\ \text{subject to} & F(x, z) \preceq g, \quad A(x, z) = b \\ & x \in \mathbb{R}^n, \quad z \in \{0, 1\}^q \end{array}$$

- $z_i$ are called the **Boolean variables**
- this is hard to solve
- **LP relaxation**: replace $z \in \{0, 1\}^q$ with $z \in [0, 1]^q$
- optimal value of relaxation LP is lower bound on MILP
- can use as heuristic for approximately solveing MILP, e.g. **relax and round**

# Multicriterion Optimization

- **multicriterion** optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = (F_1(x), F_2(x), \ldots, F_q(x)) \\ \text{subject to} & f_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

- objective is the **vector** $f_0(x) \in \mathbb{R}^q$
- $q$ different objectives $F_1$, $F_2$, ..., $F_q$; hopefully all $F_i$'s to be small
- feasible $x^\star$ is **optimal** if $y$ feasible $\implies f_0(x^\star) \preccurlyeq f_0(y)$
- $x^\star$ simultaneously minimizes each $F_i$; the objectives are **noncompeting**
- not surprisingly, this doesn't happen very often

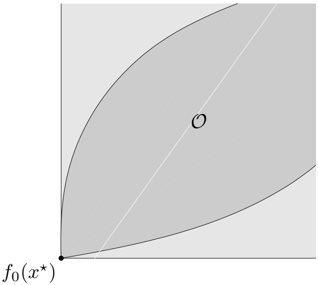
# Pareto Optimality

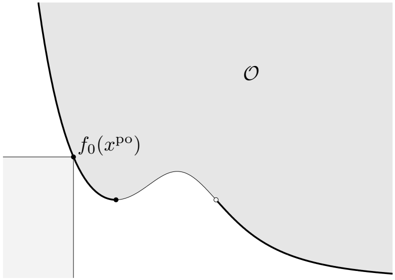
- feasible $x$ **dominates** another feasible $\tilde{x}$ if $f_0(x) \preccurlyeq f_0(\tilde{x})$ and for at least one $i$, $F_i(x) < F_i(\tilde{x})$
- i.e. $x$ meets $\tilde{x}$ on all objectives, and beats it on at least one
- feasible $x^{\mathrm{po}}$ is **Pareto optimal** if it is not dominated by any feasible point
- can be expressed as: $y$ feasible, $f_0(y) \preccurlyeq f_0(x^{\mathrm{po}}) \implies f_0(x^{\mathrm{po}}) = f_0(y)$
- there are typically many Pareto optimal points
- for $q = 2$, set of Pareto optimal objective values is the **optimal trade-off curve**
- for $q = 3$, set of Pareto optimal objective values is the **optimal trade-off surface**

# Optimal and Pareto Optimal Points

set of achievable objective values $\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$

- feasible $x$ is **optimal** if $f_0(x)$ is the minimum value of $\mathcal{O}$
- feasible $x$ is **Pareto optimal** if $f_0(x)$ is a minimal value of $\mathcal{O}$

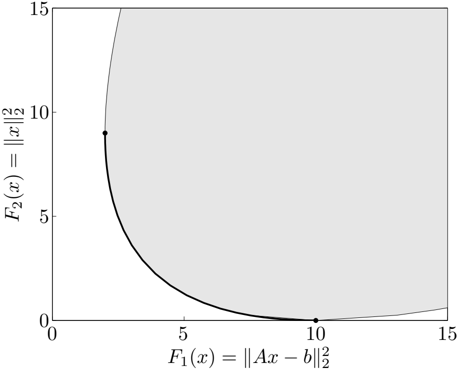


# Regularized Least-Squares

- minimize $(\|Ax - b\|_2^2, \|x\|_2^2)$ (first objective is loss, second is regularization)

- example with $A \in \mathbb{R}^{100\times10}$; heavy line shows Pareto optimal points



# Risk-Return Trade-off in Portfolio Optimization

- variable $x \in \mathbb{R}^n$ is investment portfolio, with $x_i$ fraction invested in asset $i$
- $\overline{p} \in \mathbb{R}^n$ is mean, $\Sigma$ is covariance of asset returns
- portfolio return has mean $\overline{p}^\top x$, variance $x^\top \Sigma\, x$
- minimize $(-\overline{p}^\top x, x^\top \Sigma\, x)$, subject to $\mathbf{1}^\top x = 1$, $x \succcurlyeq 0$
- Pareto optimal portfolios trace out optimal risk-return curve

# Scalarization

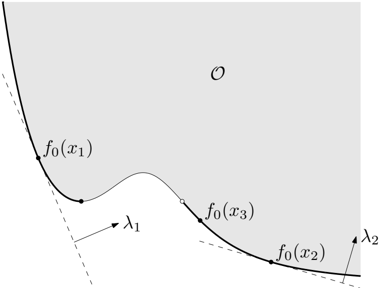
- **scalarization** combines the multiple objectives into one (scalar) objective
- a standard method for finding Pareto optimal points
- choose $\lambda \succ 0$ and solve scalar problem

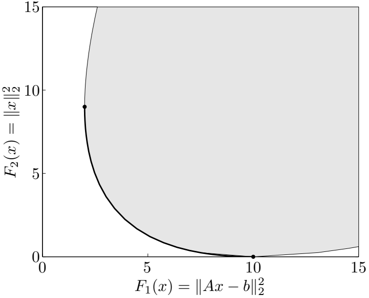
$$
\begin{array}{ll}
\text{minimize} & \lambda^\top f_0(x) \equiv \lambda_1 F_1(x) + \lambda_2 F_2(x) + \cdots + \lambda_q F_q(x) \\
\text{subject to} & f_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m \\
& h_i(x) = 0, \quad i = 1, 2, \ldots, p
\end{array}
$$

- $\lambda_i$ are relative weights on the objectives
- if $x$ is optimal for scalar problem, then it is Pareto-optimal for multicriterion problem
- for convex problems, can find (almost) all Pareto optimal points by varying $\lambda \succ 0$

# Example: Regularized Least-Squares

- minimize $(\|A\,x - b\|_2^2,\ \|x\|_2^2)$
- take $\lambda = (1, \gamma)$ with $\gamma > 1$, and minimize $\|A\,x - b\|_2^2 + \gamma \|x\|_2^2$



# Example: Risk-Return Trade-off in Portfolio Optimization

- risk-return trade-off: minimize $(-\overline{p}^\top x, x^\top \Sigma\, x)$, subject to $\mathbf{1}^\top x = 1$, $x \succeq 0$
- with $\lambda = (1, \gamma)$ we obtain the scalarized problem

$$\begin{array}{ll} \text{minimize} & -\,\overline{p}^\top x + \gamma\, x^\top \Sigma\, x \\ \text{subject to} & \mathbf{1}^\top x = 1, \quad x \succeq 0 \end{array}$$

- objective is negative **risk-adjusted return** $\overline{p}^\top x - \gamma\, x^\top \Sigma\, x$
- $\gamma$: **risk-aversion parameter**