Exercises

- Ex. 1:

  Define $u(x,y)$ and $v(x,y)$ as

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^2} = \frac{x^2y^2}{x^2+y^2} + i\frac{xy^3}{x^2+y^2} \equiv u(x,y) + iv(x,y), \tag{1}$$

  where $z = x + iy$. By definition $z \neq 0$ and $f(0) = 0$. It follows that

$$\frac{\partial u}{\partial x} = \frac{2xy^4}{(x^2+y^2)^2},$$
$$\frac{\partial v}{\partial y} = \frac{3x^3y^2 + xy^4}{(x^2+y^2)^2},$$
$$\frac{\partial u}{\partial y} = \frac{2x^4y}{(x^2+y^2)^4},$$
$$\frac{\partial v}{\partial x} = \frac{y^5 - x^2y^3}{(x^2+y^2)^2}.$$

  Note that since $z \neq 0$ (i.e. x and y cannot be zero simultaneously), the denominators of the partial derivatives are all well defined. In order to satisfy the Cauchy-Riemann conditions, we only need to examine the numerators.

  - Along the real-axis, $y = 0$: In this case, $\partial u/\partial x = \partial v/\partial y = 0$; $\partial u/\partial y = -\partial v/\partial x = 0$. As a result, $f(z)$ is analytic and differentiable along the real-axis except $z = 0$.
  - Along the imaginary-axis, $x = 0$: $\partial u/\partial x = \partial v/\partial y = 0$. However, $\partial u/\partial y = 0$, $-\partial v/\partial x = -y$. Therefore, $f(z)$ is not analytic nor differentiable on the imaginary-axis.
  - For any x, $y \neq 0$, in order to satisfy the Cauchy-Riemann condition, we need to have

$$2xy^4 = 3x^3y^2 + xy^4, \tag{2}$$
$$2x^4y = x^2y^3 - y^5. \tag{3}$$

    The first condition leads to $y^2 = 3x^2$. By substituting this identity into Eq. (3), we get:

$$x^4 = 0, \tag{4}$$

    which contradicts to the condition that x, $y \neq 0$. Thus, $f(z)$ is not analytic nor differentiable for any finite (x,y).

- Ex. 6:

  Given the definition $z = x + iy = re^{i\theta}$, we have $x = r\cos\theta$ and $y = r\sin\theta$. This transformation leads to

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta,$$
$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

  It follows that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{1}{\cos\theta}\frac{\partial u}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial u}{\partial \theta},$$
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{1}{\sin\theta}\frac{\partial v}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial v}{\partial \theta},$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{1}{\sin\theta}\frac{\partial u}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial u}{\partial \theta},$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{1}{\cos\theta}\frac{\partial v}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial v}{\partial \theta}.$$

With the above identities, the usual Cauchy-Riemann conditions in Cartesian coordinates

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

translate to

$$\frac{1}{\cos\theta}\frac{\partial u}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial u}{\partial\theta} = \frac{1}{\sin\theta}\frac{\partial v}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial v}{\partial\theta}, \tag{5}$$
$$\frac{1}{\sin\theta}\frac{\partial u}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial u}{\partial\theta} = -\frac{1}{\cos\theta}\frac{\partial v}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v}{\partial\theta}. \tag{6}$$

Now, Eq. (5)$\times \sin\theta$ $-$ Eq. (6)$\times\cos\theta$ gives

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}. \tag{7}$$

Similarly, Eq. (5)$\times\cos\theta$ $-$ Eq. (6)$\times\sin\theta$ leads to

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}. \tag{8}$$

- Ex. 7:
  Firstly, let's write $z = re^{i\theta}$.

  1. $\log z$:
     With the definition of z in polar form, we have

     $$\log z = \log r + i\theta \equiv u(r,\theta) + iv(r,\theta).$$

     It is straightforward to show that

     $$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial\theta},$$
     $$\frac{\partial v}{\partial r} = 0 = -\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

  2. $z^{1/2}$:

     $$z^{1/2} = r^{1/2}e^{i\theta/2} = r^{1/2}\cos\frac{\theta}{2} + ir^{1/2}\sin\frac{\theta}{2} = u(r,\theta) + iv(r,\theta)$$

     Thus, following the definition of polar Cauchy-Riemann conditions:

     $$\frac{\partial u}{\partial r} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2} = \frac{1}{r}\frac{\partial v}{\partial\theta},$$
     $$\frac{\partial v}{\partial r} = \frac{1}{2}r^{-1/2}\sin\frac{\theta}{2} = -\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

  3. $z^{1/3}$:

     $$z^{1/3} = r^{1/3}e^{i\theta/3} = r^{1/3}\cos\frac{\theta}{3} + ir^{1/3}\sin\frac{\theta}{3} = u(r,\theta) + iv(r,\theta)$$

     Thus

     $$\frac{\partial u}{\partial r} = \frac{1}{3}r^{-2/3}\cos\frac{\theta}{3} = \frac{1}{r}\frac{\partial v}{\partial\theta},$$
     $$\frac{\partial v}{\partial r} = \frac{1}{3}r^{-2/3}\sin\frac{\theta}{3} = -\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

In all the three cases, $z \neq 0$, i.e. $r \neq 0$.

- Ex. 13:

  1. $\tan(\tan^{-1} z) = z$
     Firstly, we have the following definitions:

     $$\tan z \equiv \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{1}{i}\frac{e^{2iz} - 1}{e^{2iz} + 1},$$
     $$\tan^{-1} z \equiv \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right).$$

     It follows that
     $$\exp\left(2i \cdot \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)\right) = \exp\left[-\ln\left(\frac{i+z}{i-z}\right)\right] = \frac{i-z}{i+z}.$$

     Therefore,
     $$\tan(\tan^{-1} z) = \frac{1}{i}\left(\frac{\frac{i-z}{i+z} - 1}{\frac{i-z}{i+z} + 1}\right) = z.$$

  2. $\log(e^z) = z + 2\pi n i$
     Write $z = re^{i\theta}$, we have

     $$e^z = e^{re^{i\theta}} = \underbrace{e^{r\cos\theta}}_{R} \cdot e^{i\overbrace{r\sin\theta}^{\phi}} \equiv R \cdot e^{i\phi}$$

     With this identity, we can proceed to show that

     $$\log(e^z) = \log(R \cdot e^{i\phi}) = \log R + i\phi + 2\pi n i = r\cos\theta + ir\sin\theta + 2\pi n i = z + 2\pi n i.$$

- Ex. 15: The complex function $\tanh z$ is defined as follows

  $$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}. \tag{9}$$

  To determine the singularities of $\tanh z$, we look at the second form. Namely, the function is singular if the denominator is zero or equivalently:
  $$e^{2z} = -1. \tag{10}$$
  This condition is satisified if $2z = \pm i(2n+1)\pi$ where $n = 0, 1, 2, \ldots$. As a result, $\tanh z$ is singular if z satisifies
  $$z = \pm i\left(n + \frac{1}{2}\right)\pi. \tag{11}$$

- Ex. 16:

  Between the following integrals

  $$\text{a)}\ \int_{-1}^{1} z^* dz, \qquad \text{b)}\ \int_{0}^{i} \sin 2z dz,$$

  the first integral does not make sense because z^* is not analytic anywhere in the complex plane. Thus the value of a) depends on the path of integration. In the textbook, it is already shown that along a unit circle C,
  $$\oint_C z^* dz = 2\pi i.$$

Using this result, a) can be viewed as going from $x = -1$ to $x = 1$ along either the *upper* unit circle or the *lower* unit circle, which gives the result of $-i\pi$ and $i\pi$ respectively. Moreover, if we just plug in the definition of $z = x + iy$ and recognize that along the real axis $y = 0$, $dy = 0$, we get

$$\int_{-1}^{i} z^* dz = \int_{-1}^{1} x\,dx = 0.$$

All these results just demonstrate that the integral in a) really is path-dependent.

It could be concluded that the integral in b) makes sense, and now we proceed to evaluate the integral. Using the definition $z = x + iy$ and the complex sine function, we have

$$\sin 2z\,dz = (\cosh 2y \sin 2x\,dx - \sinh 2y \cos 2x\,dy) + i(\cosh 2y \sin 2x\,dy + \sinh 2y \cos 2x\,dx) \tag{12}$$

Since $x = 0$ and $dx = 0$ along the y-axis, it follows that

$$\int_0^i \sin 2z\,dz = \int_0^1 -\sinh 2y\,dy = \frac{1}{2}(1 - \cosh 2). \tag{13}$$

Instead of taking the path from the origin to $(x, y) = (0, 1)$ along the imaginary axis, we could compute the integral from the origin to $(x, y) = (1, 0)$ along the real axis (OA), then from $(1, 0)$ to $(1, 0)$ along the line $y = 1 - x$ (AB). Let's do this.

On the first path $y = 0$ and $dy = 0$. So the integrand Eq. (12) becomes $\sin 2x$, thus

$$\int_{OA} \sin 2z\,dz = \int_0^1 \sin 2x\,dx = \frac{1}{2}(1 - \cos 2).$$

Going from $(1, 0)$ to $(0, 1)$, we have $y = 1 - x$. This leads to

$$\begin{aligned}
\int_{AB} \sin 2z\,dz = &\int_{AB} (\cosh 2y \sin 2x\,dx - \sinh 2y \cos 2x\,dy) + i \int_{AB} (\cosh 2y \sin 2x\,dy + \sinh 2y \cos 2x\,dx) \\
= &-\int_0^1 (\cosh 2y \sin 2(1-y)\,dy + \sinh 2y \cos 2(1-y)\,dy) \\
&+ i \int_0^1 (\cosh 2y \sin 2(1-y)\,dy - \sinh 2y \cos 2(1-y)\,dy) \\
= &-\frac{1}{2}\Big[\cosh 2y \cos 2(1-y)\Big]_0^1 + \frac{i}{2}\Big[\sinh 2y \sin 2(1-y)\Big]_0^1 \\
= &\frac{1}{2}(\cos 2 - \cosh 2).
\end{aligned}$$

Finally,

$$\int_0^i \sin 2z\,dz = \int_{OA} \sin 2z\,dz + \int_{AB} \sin 2z\,dz = \frac{1}{2}(1 - \cosh 2), \tag{14}$$

in agreement with Eq. (13) as expected.

- Ex. 18:

  1. Since C is a closed contour containing z_0, we write $f(z) = z^4 + 2z + 1$ and the integral can then be evaluated in the canonical form:

  $$I = \oint_C \frac{f(z)}{(z - z_0)^4} dz = \frac{2\pi i}{3!} f^{(3)}(z_0) = \frac{2\pi i}{3!} \times 24 z_0 = 8\pi i z_0.$$

2. Because C is a closed contour containing the origin, we invoke the canonical formula:

$$I = \oint_C \frac{\cosh z}{z^{n+1}} dz = \frac{2\pi i}{n!} \frac{d^n}{dz^n} \cosh z \bigg|_{z=0}$$

The derivative on the right hand side is

$$\frac{d^n}{dz^n} \cosh z \bigg|_{z=0} = \begin{cases} \sinh z|_{z=0} = 0, & n = 1, 3, 5, \ldots \\ \cosh z|_{z=0} = \frac{1}{2}, & n = 0, 2, 4, \ldots \end{cases}$$

By combining the two cases, we arrive at

$$I = \frac{2\pi i}{n!} \frac{[1 + (-1)^n]}{2}.$$

- Ex. 19: Following the hint, we consider the integral

$$I = \frac{1}{2\pi i} \oint_C \frac{P(z) - P(a)}{z - a} dz, \tag{15}$$

where C is a contour enclosing the point a. Since the numerator is analytic, I can be evaluated by directly invoking Cauchy's integral formula. Namely:

$$I = [P(z) - P(a)]_{z=a} = 0.$$

However, we can also express the right hand side of Eq. (15) as follows:

$$I = \frac{1}{2\pi i} \oint_C \frac{P(z) - P(a)}{z - a} dz = \frac{1}{2\pi i} \oint_C \frac{P(z)}{z - a} dz - P(a) \cdot \frac{1}{2\pi i} \oint_C \frac{1}{z - a} dz.$$

The second integral is just $P(a)$ by Cauchy's integral formula. Thus we arrive at

$$\frac{1}{2\pi i} \oint_C \frac{P(z)}{z - a} dz = P(a). \tag{16}$$

The is a welcoming result which shows that Cauchy's integral formula can also be applied to complex polynomials.

- Ex. 20: We'd like to evaluate the following contour integral for an analytic function $f(z)$:

$$I = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - \alpha)(z - \beta)} dz,$$

where α and β are two distinct arbitrary points inside the contour C. We can choose C to be a circle such that α is the center of C and $z = r_0 e^{i\theta}$, as shown in Fig. 1. In the figure, we choose the axis so that α and β lie on the real axis. However, this arrangement is really not necessary.

In any case, we can add two circles C_α and C_β around α and β respectively, along with two sets of opposite-traversing line segments joining the inner circles with C. Let's call the contour as C'. Then we can write

$$\frac{1}{2\pi i} \oint_{C'} \frac{f(z)}{(z - \alpha)(z - \beta)} dz = I + \frac{1}{2\pi i} \oint_{C_\alpha} \frac{f(z)}{(z - \alpha)(z - \beta)} dz + \frac{1}{2\pi i} \oint_{C_\beta} \frac{f(z)}{(z - \alpha)(z - \beta)} dz.$$

The integral along C' vanishes because now the integrand is analytic within the contour C' For the second term on the right-hand side, we notice that the term

$$\frac{f(z)}{z - \beta}$$

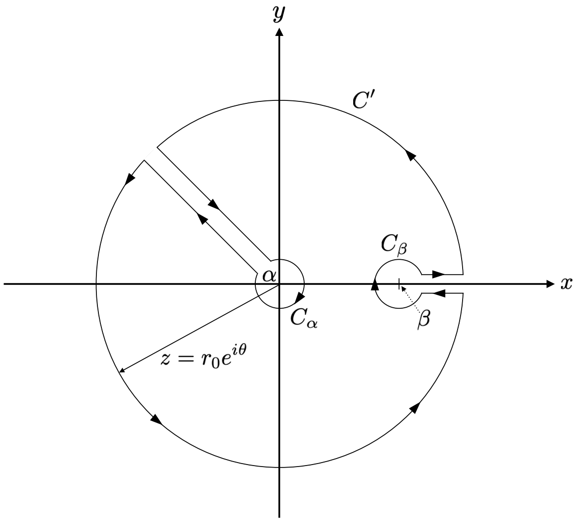

Figure 1: Contour for Exercise 20.

is analytic within C_α, so the integral is given immediately by Cauchy's integral formula:

$$\frac{1}{2\pi i}\oint_{C_\alpha}\frac{f(z)}{(z-\alpha)(z-\beta)}dz = -\frac{f(\alpha)}{\alpha-\beta}.$$

Similarly for the last term, we have

$$\frac{1}{2\pi i}\oint_{C_\beta}\frac{f(z)}{(z-\alpha)(z-\beta)}dz = -\frac{f(\beta)}{\beta-\alpha}.$$

Therefore, it follows that

$$\frac{1}{2\pi i}\oint_{C}\frac{f(z)}{(z-\alpha)(z-\beta)}dz = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}. \tag{17}$$

In order to deduce the Liouville's Theorem, we take Eq. (17) and rewrite it as:

$$f(\alpha)-f(\beta) = \frac{\alpha-\beta}{2\pi i}\oint_{C}\frac{f(z)}{(z-\alpha)(z-\beta)}dz.$$

Now consider the norm of the left-hand side:

$$\begin{aligned}|f(\alpha)-f(\beta)| &= \frac{|\alpha-\beta|}{2\pi}\left|\oint_{C}\frac{f(z)}{(z-\alpha)(z-\beta)}dz\right| \\ &\le \frac{|\alpha-\beta|}{2\pi}\oint_{C}\frac{|f(z)|}{|z-\alpha||z-\beta|}|dz|.\end{aligned}$$

By construction, $|z-\alpha| = r_0$, and $|dz| = r_0 d\theta$. By assumption $f(z)$ is bounded, so $|f(z)| \le M$. We can also choose the radius of C such that $|z-\beta| \ge r_0/2$, for example. Then it follows that

$$|f(\alpha)-f(\beta)| \le \frac{|\alpha-\beta|}{2\pi r_0}\frac{2M}{r_0}2\pi r_0 = \frac{2|\alpha-\beta|}{r_0}M.$$

We can choose the radius of C arbitarrily large, the right hand side of the above inequality tends to be zero as $r_0 \to \infty$. Thus

$$|f(\alpha)-f(\beta)| = 0. \tag{18}$$

This result implies that $f(\alpha) = f(\beta)$ for arbitrary α and β. Thus $f(z) = \text{const}$.