

Exercises

- Ex. 1:

Define $u(x, y)$ and $v(x, y)$ as

$$f(z) = \frac{xy^2(x + iy)}{x^2 + y^2} = \frac{x^2y^2}{x^2 + y^2} + i\frac{xy^3}{x^2 + y^2} \equiv u(x, y) + iv(x, y), \quad (1)$$

where $z = x + iy$. By definition $z \neq 0$ and $f(0) = 0$. It follows that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{2xy^4}{(x^2 + y^2)^2}, \\ \frac{\partial v}{\partial y} &= \frac{3x^3y^2 + xy^4}{(x^2 + y^2)^2}, \\ \frac{\partial u}{\partial y} &= \frac{2x^4y}{(x^2 + y^2)^4}, \\ \frac{\partial v}{\partial x} &= \frac{y^5 - x^2y^3}{(x^2 + y^2)^2}. \end{aligned}$$

Note that since $z \neq 0$ (i.e. x and y cannot be zero simultaneously), the denominators of the partial derivatives are all well defined. In order to satisfy the Cauchy-Riemann conditions, we only need to examine the numerators.

- Along the real-axis, $y = 0$: In this case, $\partial u/\partial x = \partial v/\partial y = 0$; $\partial u/\partial y = -\partial v/\partial x = 0$. As a result, $f(z)$ is analytic and differentiable along the real-axis except $z = 0$.
- Along the imaginary-axis, $x = 0$: $\partial u/\partial x = \partial v/\partial y = 0$. However, $\partial u/\partial y = 0$, $-\partial v/\partial x = -y$. Therefore, $f(z)$ is not analytic nor differentiable on the imaginary-axis.
- For any $x, y \neq 0$, in order to satisfy the Cauchy-Riemann condition, we need to have

$$2xy^4 = 3x^3y^2 + xy^4, \quad (2)$$

$$2x^4y = x^2y^3 - y^5. \quad (3)$$

The first condition leads to $y^2 = 3x^2$. By substituting this identity into Eq. (3), we get:

$$x^4 = 0, \quad (4)$$

which contradicts to the condition that $x, y \neq 0$. Thus, $f(z)$ is not analytic nor differentiable for any finite (x, y) .

- Ex. 6:

Given the definition $z = x + iy = re^{i\theta}$, we have $x = r \cos \theta$ and $y = r \sin \theta$. This transformation leads to

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos \theta, & \frac{\partial x}{\partial \theta} &= -r \sin \theta, \\ \frac{\partial y}{\partial r} &= \sin \theta, & \frac{\partial y}{\partial \theta} &= r \cos \theta. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{1}{\cos \theta} \frac{\partial u}{\partial r} - \frac{1}{r \sin \theta} \frac{\partial u}{\partial \theta}, \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \frac{\partial \theta}{\partial y} = \frac{1}{\sin \theta} \frac{\partial v}{\partial r} + \frac{1}{r \cos \theta} \frac{\partial v}{\partial \theta}, \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y} = \frac{1}{\sin \theta} \frac{\partial u}{\partial r} + \frac{1}{r \cos \theta} \frac{\partial u}{\partial \theta}, \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{1}{\cos \theta} \frac{\partial v}{\partial r} - \frac{1}{r \sin \theta} \frac{\partial v}{\partial \theta}. \end{aligned}$$

With the above identities, the usual Cauchy-Riemann conditions in Cartesian coordinates

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y},\end{aligned}$$

translate to

$$\frac{1}{\cos \theta} \frac{\partial u}{\partial r} - \frac{1}{r \sin \theta} \frac{\partial u}{\partial \theta} = \frac{1}{\sin \theta} \frac{\partial v}{\partial r} + \frac{1}{r \cos \theta} \frac{\partial v}{\partial \theta}, \quad (5)$$

$$\frac{1}{\sin \theta} \frac{\partial u}{\partial r} + \frac{1}{r \cos \theta} \frac{\partial u}{\partial \theta} = -\frac{1}{\cos \theta} \frac{\partial v}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial v}{\partial \theta}. \quad (6)$$

Now, Eq. (5) $\times \sin \theta$ – Eq. (6) $\times \cos \theta$ gives

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}. \quad (7)$$

Similarly, Eq. (5) $\times \cos \theta$ – Eq. (6) $\times \sin \theta$ leads to

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \quad (8)$$

- Ex. 7:

Firstly, let's write $z = re^{i\theta}$.

1. $\log z$:

With the definition of z in polar form, we have

$$\log z = \log r + i\theta \equiv u(r, \theta) + iv(r, \theta).$$

It is straightforward to show that

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} &= 0 = -\frac{1}{r} \frac{\partial u}{\partial \theta}.\end{aligned}$$

2. $z^{1/2}$:

$$z^{1/2} = r^{1/2} e^{i\theta/2} = r^{1/2} \cos \frac{\theta}{2} + ir^{1/2} \sin \frac{\theta}{2} = u(r, \theta) + iv(r, \theta)$$

Thus, following the definition of polar Cauchy-Riemann conditions:

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{2} r^{-1/2} \cos \frac{\theta}{2} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} &= \frac{1}{2} r^{-1/2} \sin \frac{\theta}{2} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.\end{aligned}$$

3. $z^{1/3}$:

$$z^{1/3} = r^{1/3} e^{i\theta/3} = r^{1/3} \cos \frac{\theta}{3} + ir^{1/3} \sin \frac{\theta}{3} = u(r, \theta) + iv(r, \theta)$$

Thus

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{3} r^{-2/3} \cos \frac{\theta}{3} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} &= \frac{1}{3} r^{-2/3} \sin \frac{\theta}{3} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.\end{aligned}$$

In all the three cases, $z \neq 0$, i.e. $r \neq 0$.

• Ex. 13:

1. $\tan(\tan^{-1} z) = z$

Firstly, we have the following definitions:

$$\tan z \equiv \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{1}{i} \frac{e^{2iz} - 1}{e^{2iz} + 1},$$

$$\tan^{-1} z \equiv \frac{i}{2} \ln \left(\frac{i+z}{i-z} \right).$$

It follows that

$$\exp \left(2i \cdot \frac{i}{2} \ln \left(\frac{i+z}{i-z} \right) \right) = \exp \left[-\ln \left(\frac{i+z}{i-z} \right) \right] = \frac{i-z}{i+z}.$$

Therefore,

$$\tan(\tan^{-1} z) = \frac{1}{i} \left(\frac{\frac{i-z}{i+z} - 1}{\frac{i-z}{i+z} + 1} \right) = z.$$

2. $\log(e^z) = z + 2\pi ni$

Write $z = re^{i\theta}$, we have

$$e^z = e^{re^{i\theta}} = \underbrace{e^{r \cos \theta}}_R \cdot e^{i \overbrace{r \sin \theta}^{\phi}} \equiv R \cdot e^{i\phi}$$

With this identity, we can proceed to show that

$$\log(e^z) = \log(R \cdot e^{i\phi}) = \log R + i\phi + 2\pi ni = r \cos \theta + ir \sin \theta + 2\pi ni = z + 2\pi ni.$$

• Ex. 16:

Between the following integrals

$$\text{a) } \int_{-1}^1 z^* dz, \quad \text{b) } \int_0^i \sin 2z dz,$$

the first integral does not make sense because z^* is not analytic anywhere in the complex plane. Thus the value of a) depends on the path of integration. In the textbook, it is already shown that along a unit circle C ,

$$\oint_C z^* dz = 2\pi i.$$

Using this result, a) can be viewed as going from $x = -1$ to $x = 1$ along either the *upper* unit circle or the *lower* unit circle, which gives the result of $-i\pi$ and $i\pi$ respectively. Moreover, if we just plug in the definition of $z = x + iy$ and recognize that along the real axis $y = 0$, $dy = 0$, we get

$$\int_{-1}^i z^* dz = \int_{-1}^1 x dx = 0.$$

All these results just demonstrate that the integral in a) really is path-dependent.

It could be concluded that the integral in b) makes sense, and now we proceed to evaluate the integral. Using the definition $z = x + iy$ and the complex sine function, we have

$$\sin 2z dz = (\cosh 2y \sin 2x dx - \sinh 2y \cos 2x dy) + i(\cosh 2y \sin 2x dy + \sinh 2y \cos 2x dx) \quad (9)$$

Since $x = 0$ and $dx = 0$ along the y -axis, it follows that

$$\int_0^i \sin 2z \, dz = \int_0^1 -\sinh 2y \, dy = \frac{1}{2}(1 - \cosh 2). \quad (10)$$

Instead of taking the path from the origin to $(x, y) = (0, 1)$ along the imaginary axis, we could compute the integral from the origin to $(x, y) = (1, 0)$ along the real axis (OA), then from $(1, 0)$ to $(1, 1)$ along the line $y = 1 - x$ (AB). Let's do this.

On the first path $y = 0$ and $dy = 0$. So the integrand Eq. (9) becomes $\sin 2x$, thus

$$\int_{OA} \sin 2z \, dz = \int_0^1 \sin 2x \, dx = \frac{1}{2}(1 - \cos 2).$$

Going from $(1, 0)$ to $(0, 1)$, we have $y = 1 - x$. This leads to

$$\begin{aligned} \int_{AB} \sin 2z \, dz &= \int_{AB} (\cosh 2y \sin 2x \, dx - \sinh 2y \cos 2x \, dy) + i \int_{AB} (\cosh 2y \sin 2x \, dy + \sinh 2y \cos 2x \, dx) \\ &= - \int_0^1 (\cosh 2y \sin 2(1 - y) \, dy + \sinh 2y \cos 2(1 - y) \, dy) \\ &\quad + i \int_0^1 (\cosh 2y \sin 2(1 - y) \, dy - \sinh 2y \cos 2(1 - y) \, dy) \\ &= - \frac{1}{2} \left[\cosh 2y \cos 2(1 - y) \right]_0^1 + \frac{i}{2} \left[\sinh 2y \sin 2(1 - y) \right]_0^1 \\ &= \frac{1}{2}(\cos 2 - \cosh 2). \end{aligned}$$

Finally,

$$\int_0^i \sin 2z \, dz = \int_{OA} \sin 2z \, dz + \int_{AB} \sin 2z \, dz = \frac{1}{2}(1 - \cosh 2), \quad (11)$$

in agreement with Eq. (10) as expected.