Exercises

- Ex. 1:

  Define $u(x,y)$ and $v(x,y)$ as

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^2} = \frac{x^2y^2}{x^2+y^2} + i\frac{xy^3}{x^2+y^2} \equiv u(x,y) + iv(x,y), \tag{1}$$

  where $z = x + iy$. By definition $z \neq 0$ and $f(0) = 0$. It follows that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{2xy^4}{(x^2+y^2)^2}, \\
\frac{\partial v}{\partial y} &= \frac{3x^3y^2 + xy^4}{(x^2+y^2)^2}, \\
\frac{\partial u}{\partial y} &= \frac{2x^4y}{(x^2+y^2)^4}, \\
\frac{\partial v}{\partial x} &= \frac{y^5 - x^2y^3}{(x^2+y^2)^2}.
\end{aligned}$$

  Note that since $z \neq 0$ (i.e. x and y cannot be zero simultaneously), the denominators of the partial derivatives are all well defined. In order to satisfy the Cauchy-Riemann conditions, we only need to examine the numerators.

  - Along the real-axis, $y = 0$: In this case, $\partial u/\partial x = \partial v/\partial y = 0$; $\partial u/\partial y = -\partial v/\partial x = 0$. As a result, $f(z)$ is analytic and differentiable along the real-axis except $z = 0$.
  - Along the imaginary-axis, $x = 0$: $\partial u/\partial x = \partial v/\partial y = 0$. However, $\partial u/\partial y = 0$, $-\partial v/\partial x = -y$. Therefore, $f(z)$ is not analytic nor differentiable on the imaginary-axis.
  - For any x, $y \neq 0$, in order to satisfy the Cauchy-Riemann condition, we need to have

$$2xy^4 = 3x^3y^2 + xy^4, \tag{2}$$
$$2x^4y = x^2y^3 - y^5. \tag{3}$$

    The first condition leads to $y^2 = 3x^2$. By substituting this identity into Eq. (3), we get:

$$x^4 = 0, \tag{4}$$

    which contradicts to the condition that x, $y \neq 0$. Thus, $f(z)$ is not analytic nor differentiable for any finite (x, y).

- Ex. 6:

  Given the definition $z = x + iy = re^{i\theta}$, we have $x = r\cos\theta$ and $y = r\sin\theta$. This transformation leads to

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \\
\frac{\partial y}{\partial r} &= \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.
\end{aligned}$$

  It follows that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{1}{\cos\theta}\frac{\partial u}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial u}{\partial \theta}, \\
\frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{1}{\sin\theta}\frac{\partial v}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial v}{\partial \theta}, \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{1}{\sin\theta}\frac{\partial u}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial u}{\partial \theta}, \\
\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{1}{\cos\theta}\frac{\partial v}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial v}{\partial \theta}.
\end{aligned}$$

With the above identities, the usual Cauchy-Riemann conditions in Cartesian coordinates

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y},\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y},\end{aligned}$$

translate to

$$\frac{1}{\cos\theta}\frac{\partial u}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial u}{\partial\theta} = \frac{1}{\sin\theta}\frac{\partial v}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial v}{\partial\theta}, \tag{5}$$

$$\frac{1}{\sin\theta}\frac{\partial u}{\partial r} + \frac{1}{r\cos\theta}\frac{\partial u}{\partial\theta} = -\frac{1}{\cos\theta}\frac{\partial v}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v}{\partial\theta}. \tag{6}$$

Now, Eq. (5)$\times\sin\theta$ $-$ Eq. (6)$\times\cos\theta$ gives

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}. \tag{7}$$

Similarly, Eq. (5)$\times\cos\theta$ $-$ Eq. (6)$\times\sin\theta$ leads to

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}. \tag{8}$$

- Ex. 7:
  Firstly, let's write $z = re^{i\theta}$.

  1. $\log z$:
     With the definition of z in polar form, we have

     $$\log z = \log r + i\theta \equiv u(r,\theta) + iv(r,\theta).$$

     It is straightforward to show that

     $$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial\theta},\\ \frac{\partial v}{\partial r} &= 0 = -\frac{1}{r}\frac{\partial u}{\partial\theta}.\end{aligned}$$

  2. $z^{1/2}$:

     $$z^{1/2} = r^{1/2}e^{i\theta/2} = r^{1/2}\cos\frac{\theta}{2} + ir^{1/2}\sin\frac{\theta}{2} = u(r,\theta) + iv(r,\theta)$$

     Thus, following the definition of polar Cauchy-Riemann conditions:

     $$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2} = \frac{1}{r}\frac{\partial v}{\partial\theta},\\ \frac{\partial v}{\partial r} &= \frac{1}{2}r^{-1/2}\sin\frac{\theta}{2} = -\frac{1}{r}\frac{\partial u}{\partial\theta}.\end{aligned}$$

  3. $z^{1/3}$:

     $$z^{1/3} = r^{1/3}e^{i\theta/3} = r^{1/3}\cos\frac{\theta}{3} + ir^{1/3}\sin\frac{\theta}{3} = u(r,\theta) + iv(r,\theta)$$

     Thus

     $$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{3}r^{-2/3}\cos\frac{\theta}{3} = \frac{1}{r}\frac{\partial v}{\partial\theta},\\ \frac{\partial v}{\partial r} &= \frac{1}{3}r^{-2/3}\sin\frac{\theta}{3} = -\frac{1}{r}\frac{\partial u}{\partial\theta}.\end{aligned}$$

In all the three cases, $z \neq 0$, i.e. $r \neq 0$.

- Ex. 13:

  1. $\tan(\tan^{-1} z) = z$
     Firstly, we have the following definitions:

     $$\tan z \equiv \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{1}{i}\frac{e^{2iz} - 1}{e^{2iz} + 1},$$
     $$\tan^{-1} z \equiv \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right).$$

     It follows that
     $$\exp\left(2i \cdot \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)\right) = \exp\left[-\ln\left(\frac{i+z}{i-z}\right)\right] = \frac{i-z}{i+z}.$$

     Therefore,
     $$\tan(\tan^{-1} z) = \frac{1}{i}\left(\frac{\frac{i-z}{i+z} - 1}{\frac{i-z}{i+z} + 1}\right) = z.$$

  2. $\log(e^z) = z + 2\pi n i$
     Write $z = re^{i\theta}$, we have

     $$e^z = e^{re^{i\theta}} = \underbrace{e^{r\cos\theta}}_{R} \cdot e^{i\overbrace{r\sin\theta}^{\phi}} \equiv R \cdot e^{i\phi}$$

     With this identity, we can proceed to show that

     $$\log(e^z) = \log(R \cdot e^{i\phi}) = \log R + i\phi + 2\pi n i = r\cos\theta + ir\sin\theta + 2\pi n i = z + 2\pi n i.$$

- Ex. 15: The complex function $\tanh z$ is defined as follows

  $$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}. \tag{9}$$

  To determine the singularities of $\tanh z$, we look at the second form. Namely, the function is singular if the denominator is zero or equivalently:
  $$e^{2z} = -1. \tag{10}$$

  This condition is satisified if $2z = \pm i(2n+1)\pi$ where $n = 0, 1, 2, \ldots$. As a result, $\tanh z$ is singular if z satisifies
  $$z = \pm i\left(n + \frac{1}{2}\right)\pi. \tag{11}$$

- Ex. 16:

  Between the following integrals

  $$\text{a)} \int_{-1}^{1} z^* dz, \qquad \text{b)} \int_0^i \sin 2z dz,$$

  the first integral does not make sense because z^* is not analytic anywhere in the complex plane. Thus the value of a) depends on the path of integration. In the textbook, it is already shown that along a unit circle C,
  $$\oint_C z^* dz = 2\pi i.$$

Using this result, a) can be viewed as going from $x = -1$ to $x = 1$ along either the *upper* unit circle or the *lower* unit circle, which gives the result of $-i\pi$ and $i\pi$ respectively. Moreover, if we just plug in the definition of $z = x + iy$ and recognize that along the real axis $y = 0$, $dy = 0$, we get

$$\int_{-1}^{i} z^* dz = \int_{-1}^{1} x\, dx = 0.$$

All these results just demonstrate that the integral in a) really is path-dependent.

It could be concluded that the integral in b) makes sense, and now we proceed to evaluate the integral. Using the definition $z = x + iy$ and the complex sine function, we have

$$\sin 2z\, dz = (\cosh 2y \sin 2x\, dx - \sinh 2y \cos 2x\, dy) + i(\cosh 2y \sin 2x\, dy + \sinh 2y \cos 2x\, dx) \tag{12}$$

Since $x = 0$ and $dx = 0$ along the y-axis, it follows that

$$\int_{0}^{i} \sin 2z\, dz = \int_{0}^{1} -\sinh 2y\, dy = \frac{1}{2}(1 - \cosh 2). \tag{13}$$

Instead of taking the path from the origin to $(x, y) = (0, 1)$ along the imaginary axis, we could compute the integral from the origin to $(x, y) = (1, 0)$ along the real axis (OA), then from $(1, 0)$ to $(1, 0)$ along the line $y = 1 - x$ (AB). Let's do this.

On the first path $y = 0$ and $dy = 0$. So the integrand Eq. (12) becomes $\sin 2x$, thus

$$\int_{OA} \sin 2z\, dz = \int_{0}^{1} \sin 2x\, dx = \frac{1}{2}(1 - \cos 2).$$

Going from $(1, 0)$ to $(0, 1)$, we have $y = 1 - x$. This leads to

$$\begin{aligned}
\int_{AB} \sin 2z\, dz &= \int_{AB} (\cosh 2y \sin 2x\, dx - \sinh 2y \cos 2x\, dy) + i \int_{AB} (\cosh 2y \sin 2x\, dy + \sinh 2y \cos 2x\, dx) \\
&= -\int_{0}^{1} (\cosh 2y \sin 2(1 - y)\, dy + \sinh 2y \cos 2(1 - y)\, dy) \\
&\quad + i \int_{0}^{1} (\cosh 2y \sin 2(1 - y)\, dy - \sinh 2y \cos 2(1 - y)\, dy) \\
&= -\frac{1}{2}\Big[\cosh 2y \cos 2(1 - y)\Big]_0^1 + \frac{i}{2}\Big[\sinh 2y \sin 2(1 - y)\Big]_0^1 \\
&= \frac{1}{2}(\cos 2 - \cosh 2).
\end{aligned}$$

Finally,

$$\int_{0}^{i} \sin 2z\, dz = \int_{OA} \sin 2z\, dz + \int_{AB} \sin 2z\, dz = \frac{1}{2}(1 - \cosh 2), \tag{14}$$

in agreement with Eq. (13) as expected.

- Ex. 18:

  1. Since C is a closed contour containing z_0, we write $f(z) = z^4 + 2z + 1$ and the integral can then be evaluated in the canonical form:

$$I = \oint_C \frac{f(z)}{(z - z_0)^4} dz = \frac{2\pi i}{3!} f^{(3)}(z_0) = \frac{2\pi i}{3!} \times 24 z_0 = 8\pi i z_0.$$

2. Because C is a closed contour containing the origin, we invoke the canonical formula:

$$I = \oint_C \frac{\cosh z}{z^{n+1}} dz = \frac{2\pi i}{n!} \frac{d^n}{dz^n} \cosh z \bigg|_{z=0}$$

The derivative on the right hand side is

$$\frac{d^n}{dz^n} \cosh z \bigg|_{z=0} = \begin{cases} \sinh z\big|_{z=0} = 0, & n = 1, 3, 5, \dots \\ \cosh z\big|_{z=0} = \frac{1}{2}, & n = 0, 2, 4, \dots \end{cases}$$

By combining the two cases, we arrive at

$$I = \frac{2\pi i}{n!} \frac{[1 + (-1)^n]}{2}.$$

- Ex. 19: Following the hint, we consider the integral

$$I = \frac{1}{2\pi i} \oint_C \frac{P(z) - P(a)}{z - a} dz, \tag{15}$$

where C is a contour enclosing the point a. Since the numerator is analytic, I can be evaluated by directly invoking Cauchy's integral formula. Namely:

$$I = [P(z) - P(a)]_{z=a} = 0.$$

However, we can also express the right hand side of Eq. (15) as follows:

$$I = \frac{1}{2\pi i} \oint_C \frac{P(z) - P(a)}{z - a} dz = \frac{1}{2\pi i} \oint_C \frac{P(z)}{z - a} dz - P(a) \cdot \frac{1}{2\pi i} \oint_C \frac{1}{z - a} dz.$$

The second integral is just $P(a)$ by Cauchy's integral formula. Thus we arrive at

$$\frac{1}{2\pi i} \oint_C \frac{P(z)}{z - a} dz = P(a). \tag{16}$$

The is a welcoming result which shows that Cauchy's integral formula can also be applied to complex polynomials.