1 The Cauchy-Riemann Conditions

A single-valued function $f(z)$ of complex variable $z = x + iy$ is analytic if it has derivatives throughout a region of the complex plane. Or equivalently, a single-valued $f(z)$ is said to be analytic at $z = z_0$ if it has derivatives at z_0 and all points in some neighborhoods of z_0.

If we write $f(z) = u(x, y) + iv(x, y)$, then the **necessary** condition for $f(z)$ to be differentiable at a point is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{1}$$

which is known as the Cauchy-Riemann conditions. The **sufficient** conditions for the differentiability of $f(z)$ at z_0 are

1. The Cauchy-Riemann condition holds at z_0.
2. The first partial derivatives of $u(x, y)$ and $v(x, y)$ exists and be continuous at z_0.

It can be also shown that $\partial f/\partial z^* = 0$ if and only if the Cahchy-Riemann conditions holds. This implies that, loosely speaking, analytic functions are independent of z^*, they are functions of z alone.

2 Complex Integration

Theorem 1 Cauchy's Theorem: *If a function $f(z)$ is analytic within and on a closed contour C, and $f'(z)$ is continuous throughout this region, then*

$$\oint_C f(z)dz = 0. \tag{2}$$

The theorem can be proved without assuming the continuity of $f'(z)$ because any function which is analytic in a region necessarily has a continuous derivative (cf. the Cauchy-Riemann conditions). If fact, it can be proved that an analytic function has derivatives of all orders, and therfore its derivatives are continuous. In this sense, we have:

Theorem 2 Cauchy-Goursat Theorem: *If a function $f(z)$ is analytic within and on a closed contour C, then*

$$\oint_C f(z)dz = 0. \tag{3}$$

Following the Cauchy-Goursat theorem, one can prove the following result which is extremely useful in all mathematical physics:

Theorem 3 *If $f(z)$ is analytic within and on a closed contour C, then for any point z_0 interior to C,*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz. \tag{4}$$

The theorem shows the amazingly strong inner structure of analytic functions. It means that if a function is analytic within and on a contour C, its value at every point inside C is determined by its value on the bounding curve C.

If we differentiate both sides of Eq. (4) with respect to z_0 (since z_0 is any point inside C, it can be treated as a variable), one obtains

$$f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz. \tag{5}$$

In general, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz. \tag{6}$$

Thus $f(z)$ have derivatives of all orders inside C. The kth derivative is continuous within C because the $k + 1$ derivative exists.

A function that is analytic in the entire complex plane is called an entire function. Thus:

Theorem 4 Liouville's Theorem *If $f(z)$ is entire and $|f(z)|$ is bounded for all values of z, then $f(z)$ is a constant.*

3 The Expansion of an Analytic Function

An important application of the Cauchy-Goursat theorem concerns about the possibility of expanding an analytic function in a power series. The results are stated as the following theorem.

Theorem 5 Laurent's Theorem *Let $f(z)$ be analytic throughout the closed annular region between the two circles C_1 and C_2 with common center z_0. Then at each point in the annulus*

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n, \tag{7}$$

with the series converging uniformly in any closed region, R, lying wholly within the annulus. Here

$$A_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz', \tag{8}$$

for $n = 0, \pm 1, \pm 2, \ldots$, and C is any closed contour in the annulus which encloses z_0

A particularly important consequence of Laurent's theorem arises when the function is analytic and has no singularities within the inner circle C_2. This is summarized as the following theorem

Theorem 6 Taylor's Theorem *If $f(z)$ is analytic at all points interior to a circle C centered about z_0, then in any closed region contained wholly inside C*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n, \tag{9}$$

and this series converges uniformly.

4 Residue Theory

This is really just an application of the above theorems

Theorem 7 The Residue Theorem *The integral of $f(z)$ around a closed contour C containing a finite number n of singular points of $f(z)$ equals the sum of n integrals of $f(z)$ about n circles, each enclosing one and only one of the n singular points. In other words,*

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^{n} R_j, \tag{10}$$

where R_j is called the residue at the point z_j:

$$R_j = \frac{1}{2\pi i} \oint_{C_j} f(z) dz. \tag{11}$$

In addition to the definition Eq. (11), there is another way of computing residues which is sometimes useful. If we expand $f(z)$ in a Laurent series about the singular point z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_n)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}. \tag{12}$$

It can be shown that

$$b_1 = \frac{1}{2\pi i} \oint_{C_0} f(z) dz \equiv R_0. \tag{13}$$

In other words, the residue around a point z_0 can be found by expanding the function around z_0 in a Laurent series and picking out the coefficient of the term in $(z - z_0)^{-1}$.

5 Evaluation of definite integrals

5.1 $\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta$

Here $R(\cos\theta, \sin\theta)$ is a rational function

$$R(\cos\theta, \sin\theta) = \frac{a_1 \cos\theta + a_2 \sin\theta + a_3 \cos^2\theta + \dots}{b_1 \cos\theta + b_s \sin\theta + b_3 \cos^2\theta + b_4 \sin^2\theta + \dots} \tag{14}$$

To evaluate the integral, we let $z = e^{i\theta}$ and proceed with the substitution

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2}\left(z - \frac{1}{z}\right), \tag{15}$$

and $d\theta = -i(dz/z)$. The integral becomes

$$-i \oint_C R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2}\left(z - \frac{1}{z}\right)\right] \frac{dz}{z}, \tag{16}$$

where C is the unit circle.

5.2 $\int_{-\infty}^{\infty} R(x) dx$

In this case, $R(x)$ is a rational function without poles on the real axis. Furthermore, suppose the degree of the denominator of $R(x)$ is at least two unit higher than the degree of the numerator, meaning $|R(z)| \to 1/|z^2|$ as $|z| \to \infty$, and all the singularities of $R(z)$ are in the upper half-plane, then

$$\int_{-\infty}^{\infty} R(x) dx = \oint_C R(z) dz = 2\pi i \sum_{y>0} \operatorname{Res} R(z). \tag{17}$$

where C encloses all singularities in the upper half-plane.

5.3 $\int_{-\infty}^{\infty} R(x) e^{ix} dx$

Again, assuming there are no poles on the real axis, and if the raional function $R(z)$ has a zero of at least order two at infinity, then

$$\int_{-\infty}^{\infty} R(x) e^{ix} dx = \oint_C R(z) e^{iz} dz = 2\pi i \sum_{y>0} \operatorname{Res}\left[R(z) e^{iz}\right] \tag{18}$$

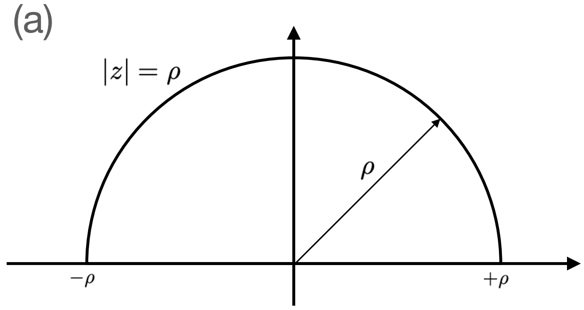

Figure 1: (a) Contour for the integrals considered in 5.4. (b) Contour for the integrals considered in 5.5

5.4 $\int_{-\infty}^{\infty} R(x)e^{i\alpha x}dx$

Here $R(x)$ has a zero of order one at infinity, and again, it is assumed that there are no poles on the real axis. To evaluate the integral, we consider a contour consisting of a semi-circle in the upper half-plane and the real axis and write:

$$\oint_C R(z)e^{i\alpha z}dz = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} R(x)e^{i\alpha x}dx + \int_{semi,|z|=\rho} R(z)e^{i\alpha z}dz. \tag{19}$$

For $\alpha > 0$, the Jordan's Lemma tells us that the integral over the semi-circle vanishes. The integral on the left hand side is given by the residue theory. Thus

$$\int_{-\infty}^{\infty} R(x)e^{i\alpha x}dx = 2\pi i\sum_{y>0}\text{Res}\left[R(z)e^{i\alpha z}\right] \tag{20}$$

5.5 $\int_{-\infty}^{\infty} Q(x)dx$

We consider the function $Q(z)$ and assume that it is meromorphic (i.e. no essential singularities) in the upper half-plane, and has poles of order one (i.e. simple poles) on the real axis. If $Q(z)$ behaves at infinity like any of the integrands discussed in 5.2, 5.3, or 5.4, then we can extend those techniques and, using the contour shown in Fig. 1(b), evaluate the integral as follows

$$P\int_{-\infty}^{\infty} Q(x)dx = 2\pi i\sum_{y>0}\text{Res}\,Q(z) + \pi i\sum_{y=0}\text{Res}\,Q(z), \tag{21}$$

where the second term denotes the sum of the residues of $Q(z)$ at each of its simple poles on the real axis.

5.6 $\int_0^{\infty} x^{\lambda-1}R(x)dx$

In this last case, we assume that λ is not an integer. $R(z)$ is rational and analytic at $z = 0$, and has no poles on the positive real axis, and $|z^\lambda R(z)| \to 0$ uniformly as $|z| \to 0$, and as $|z| \to \infty$. This problem invloces branch points and branch cuts because the power function $z^{\lambda-1}$ is in general not a single-valued function. In the current case, $z^{\lambda-1}$ has a branch point at the origin. We can consider the following branch of the function ($z = re^{i\theta}$)

$$z^{\lambda-1} = \exp\left[(\lambda-1)\log z\right] = \exp\left[(\lambda-1)(\log r + i\theta)\right], \tag{22}$$

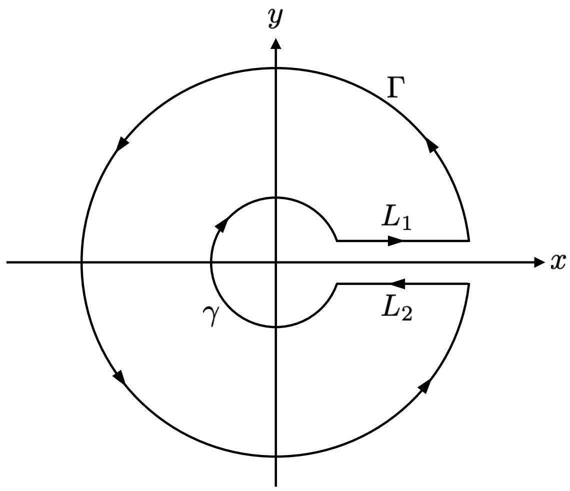

Figure 2: Contour for the integrals considered in 5.6.

where $0 < \theta < 2\pi$ and $r > 0$.

Now consider the contour shown in Fig. 2. Because of the branch cut along the positive real axis:

$$\oint z^{\lambda-1} R(z) dz = \int_\Gamma + \int_\gamma + \int_{L_1} + \int_{L_2} . \tag{23}$$

Since the cut is along the positve real axis, the line integrals above and below the cut will not cancel. In fact, along the line L_1, we have $z = x$, the usual convention. For the line just below the cut, $z = xe^{2\pi i}$ since it picks up a 2π phase. The integrals over Γ and γ vanish, because $|z^\lambda R(z)| \to 0$ as $|z| \to 0$, and as $|z| \to \infty$. Thus

$$\begin{aligned}\oint z^{\lambda-1} R(z) dz &= \int_\infty^0 e^{2\pi i(\lambda-1)} x^{\lambda-1} R(x) dx + \int_0^\infty x^{\lambda-1} R(x) dx. \\ &= \frac{-2i \sin \pi\lambda}{e^{-\pi i\lambda}} \int_0^\infty x^{\lambda-1} R(x) dx \end{aligned} \tag{24}$$

The left hand side of the above equation is nothing but the sum of residues inside the contour:

$$\oint z^{\lambda-1} R(z) dz = 2\pi i \sum_{interior} \mathrm{Res}\left[z^{\lambda-1} R(z)\right] \tag{25}$$

Thus it follows that

$$\int_0^\infty x^{\lambda-1} R(x) dx = \frac{\pi(-1)^{\lambda-1}}{\sin \pi\lambda} \sum_{interior} \mathrm{Res}\left[z^{\lambda-1} R(z)\right] . \tag{26}$$

Theorem 8 *Let $f(z)$ be a meromorphic function (i.e. no essential singularities) and let C be a contour which encloses the zeros of $\sin \pi z$, located at $z = \rho$, $\rho + 1$, ..., n. If we assume the poles of $f(z)$ and $\sin \pi z$ are distinct, then*

$$\sum_{m=\rho}^{n} f(m) = \frac{1}{2\pi i} \oint_C \pi \cot \pi z \, f(z) dz - \sum Res \left[\pi \cot(\pi z) \, f(z)\right] \tag{27}$$

where the summation on the right hand side is over poles of $f(z)$ inside C.

6 Reference

1. Mathematics of Classical and Quantum Physics, Frederick W. Byron Jr. and Robert W. Fuller.