A Simplified Proof of Brouwer's Fixed Point Theorem

Directed Reading Program

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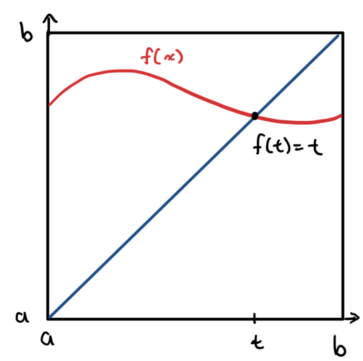
In 1912, Luitzen Brouwer published his fixed point theorem, a famous topological result.

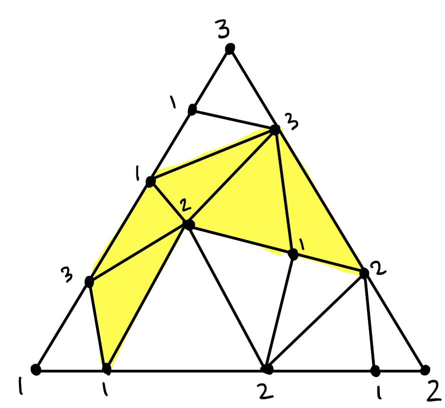
Theorem (Brouwer's Fixed Point): Every continuous function $f : B^n \to B^n$ of an n-dimensional ball to itself has a fixed point. That is, there exists $x \in B^n$ such that $f(x) = x$.

For the $n = 1$ case, the proof is simple. Suppose f is a continuous function that maps $[a, b]$ to $[a, b]$. Define $g(x) = f(x) - x$. Notice that $g(a) = f(a) - a \geq 0$ and that $g(b) = f(b) - b \leq 0$. By the Intermediate Value Theorem, there exists $t \in [a, b]$ for which $g(t) = f(t) - t = 0$, implying $f(t) = t$.



It turns out that the proof of higher-dimensional cases gets much more complicated, and that we can prove it using Sperner's lemma, a combinatorial result that does not seem at all related to fixed points or n-dimensional balls.

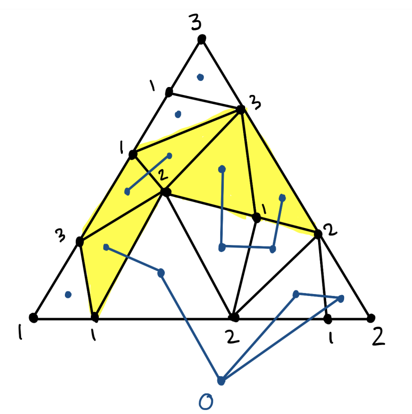
Lemma (Sperner): Take a triangle with vertices V_1, V_2, V_3. Triangulate it (or decompose it into a finite number of smaller triangles, and only triangles). Give each vertex a "color" from the set $1, 2, 3$ such that V_i gets the color i for each i, vertices along the edge V_i, V_j get only the colors i, j with $i \neq j$, and the interior vertices are colored arbitrarily.



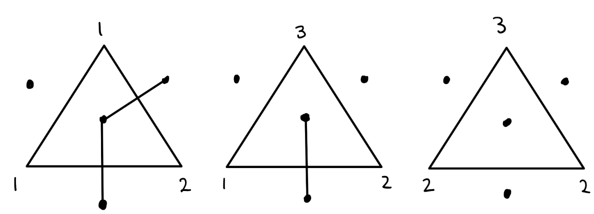
Then there exists in the triangulation some triangle with vertices colored $1, 2, 3$.

Proof (Sperner): We will prove not only the existence of such a triangle, but that there must be an odd number of such triangles given a triangulation.

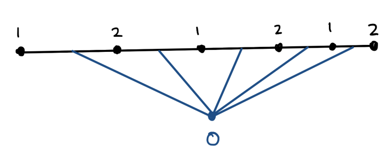
Suppose we have a triangulation that satisfies the conditions of the lemma. Place a point within each of the small triangles, and place a point O underneath the edge containing the colors 1 and 2. Connect two points if the small edge between them are a 1-2 edge - that is, if the edge is made up of vertices with colors 1 and 2. We define the neighbor of some point P as the a point immediately connected to P. Note that the number of neighbors of P and the degree of P are the same.



For a point P in the interior of the large triangle, we see that it has three possible number of neighbors. If P is within a triangle with vertices colored 1 and 2, it will have two neighbors. If P is within a triangle with vertices lacking either 1 or 2, it will have no neighbors. If P is within a 1-2-3 triangle (or a triangle with vertices of all three colors), it will have one neighbor. Note that having three neighbors is impossible, so one neighbor is the only odd-degree possibility.



For the point O, we see that it must have an odd number of neighbors. To see why, consider a line segment, labeled 1 and 2 at its endpoints. Suppose it has some number of points between the endpoints, with each consecutive number different from the point before it. Since the line segment starts at 1 and ends at 2, there must be the same number of points labeled 1 and points labeled 2, meaning that our line segment is composed of an even number of points, or that it is divided into an odd number of segments. Since each segment "corresponds" to a connection between O and a neighbor, O has an odd number of neighbors.



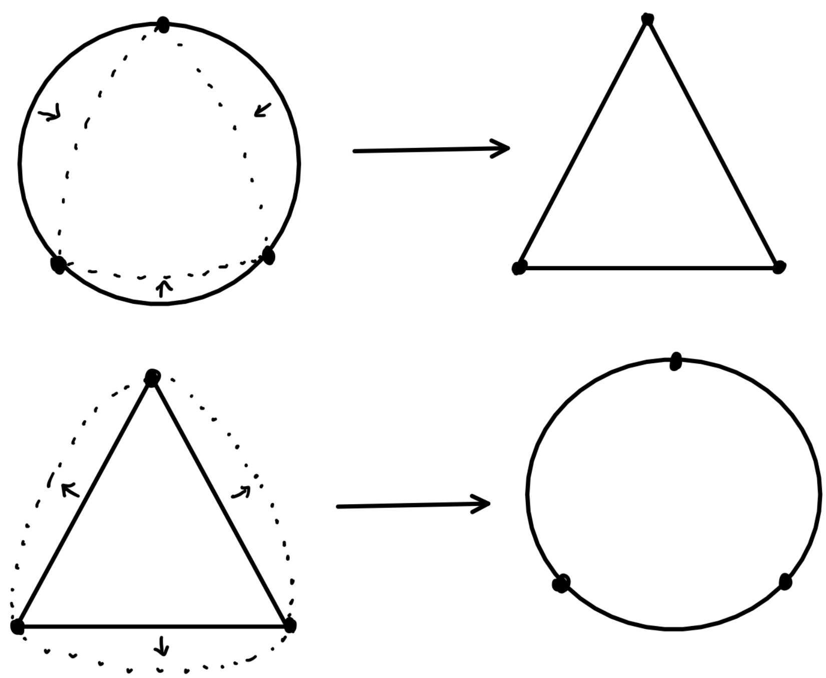
Finally, we need the fact that the number of odd-degree points is even. Note that the sum of all degrees is just twice the number of connections (since each connection will add two degrees to the total). So, the sum of degrees must be even. It is also obvious that the sum of degrees of even-degree points is even. Thus, the sum of degrees of odd-degree points is even. Therefore, there must be an even number of odd-degree points.

So, removing point O from the set of odd-degree points gives us an odd number of odd-degree points within the triangle, which is equivalent to saying that there are an odd number of 1-2-3 triangles.

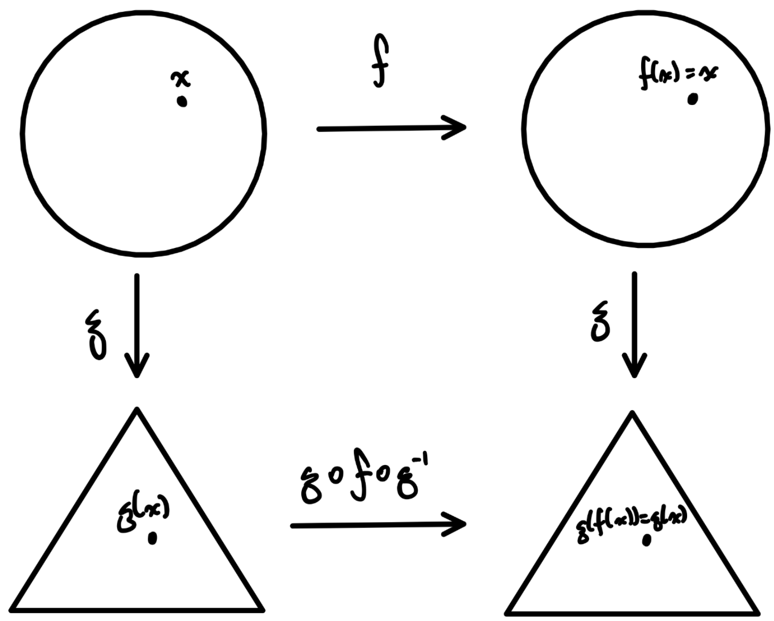
$\square$

With Sperner's lemma now proved, we can turn our attention back to our theorem at hand. Specifically, we will be proving the $n = 2$ case for simplicity, although higher-dimensional cases can be proved using induction and more indices.

Consider a continuous function $f : B^2 \to B^2$. There exists a continuous, invertible function $g : B^2 \to \Delta$ with $g^{-1} : \Delta \to B^2$. To give a rough idea of how to transform from a disk to a triangle, take three distinct points on the boundary of the disk, and "squeeze in" the sides until it resembles a triangle. The reverse direction works by "stretching out" the sides back into a disk.



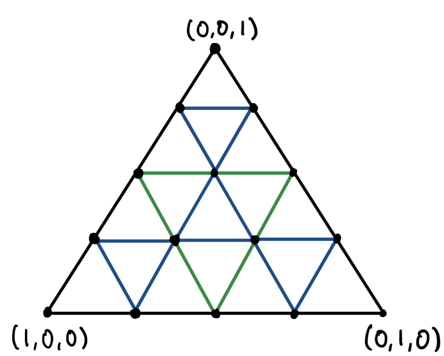
Suppose that there was a fixed point x in the disk. Then g would send x and $f(x)$ to the same point, because $f(x) = x$. But then we see that when we apply $g \circ f \circ g^{-1}$ to $g(x)$, we must get $g(x)$. So, $f : B^2 \to B^2$ has a fixed point if and only if $g \circ f \circ g^{-1} : \Delta \to \Delta$ has a fixed point (in this case, that's $g(x)$). Essentially, we need to show that a continuous transformation from a triangle to itself has a fixed point. Furthermore, since any triangle can be "stretched" into any other triangle, we need only show that this is true for one triangle.



This works because of the topological fact that a triangle and a two-dimensional ball are homeomorphic (and so are triangles to other triangles). For intuition, think of this as an isomorphism, but for shapes. You can even think of our goal as finding eigenvalues of f with value 1!

Proof (Brouwer): Suppose we have a triangle Δ with vertices $(1,0,0),(0,1,0),(0,0,1)$. This triangle sits in the plane $x+y+z=1$. Let $f:\Delta\to\Delta$ be a continuous function. Assume to the contrary that f does not have a fixed point.

We will use a controlled method of triangulating Δ to simplify the proof. We can view Δ as an equilateral triangle (by homeomorphism). Take the midpoints of each edge, and connect the midpoints in such a way that the new triangles formed are also equilateral. Repeat this process n times for the n-th triangulation of Δ.



2nd triangulation of Δ

Define $\text{color}(\mathbf{v}) = \min\{i : f(\mathbf{v})_i < v_i\}$, or the smallest index i such that the i-th coordinate of $f(\mathbf{v})$ is less than the i-th coordinate of $\mathbf{v}$. Denote $\mathbf{v} = (v_1, v_2, v_3)$ and $f(\mathbf{v}) = (f(\mathbf{v})_1, f(\mathbf{v})_2, f(\mathbf{v})_3)$.

If the smallest index i does not exist for some $\mathbf{v}$, then we have found a fixed point. Suppose there exists $\mathbf{w}$ such that color($\mathbf{w}$) is undefined. Then $f(\mathbf{w})_i \geq w_i$. But since $\mathbf{w} \in \Delta$ and $f(\mathbf{w}) \in \Delta$, $w_1 + w_2 + w_3 = 1$ and $f(\mathbf{w})_1 + f(\mathbf{w})_2 + f(\mathbf{w})_3 = 1$. If $f(\mathbf{w})_1 > w_1$, then $f(\mathbf{w})_2 + f(\mathbf{w})_3 < w_2 + w_3$, contradicting our initial inequality assumption. So, we must have $f(\mathbf{w})_1 = w_1$. It follows that $f(\mathbf{w})_2 = w_2$ and $f(\mathbf{w})_3 = w_3$. But then $\mathbf{w}$ is a fixed point, contradicting our assumption that f does not have a fixed point. So, color($\mathbf{v}$) is well-defined on our region.

It turns out that this coloring actually satisfies Sperner's lemma. For $i \in \{1, 2, 3\}$, color($\mathbf{e}_i$) $= i$ because i is the only index for which $f(\mathbf{e}_i)_i < e_i$. For each $\mathbf{v}$ on the side from $(1, 0, 0)$ to $(0, 1, 0)$, we have $v_3 = 0$ and so $f(\mathbf{v}_3) \geq v_3$, which implies that color($\mathbf{v}$) $\neq 3$ (and similarly for the other sides of Δ). Interior vertices are colored arbitrarily. Thus, by Sperner's lemma, there exists a triangle with vertices of colors 1, 2, and 3 for any triangulation with edges of finite length.

We want to construct a triangulation such that the maximal length of an edge in the triangulation gets really small. We can construct a sequence of triangulations such that the sequence of maximal lengths of an edge approaches zero. We can use our triangulation example from before, where the maximal length of an edge is $\frac{1}{2^n}$. As the number of triangulations approaches infinity, the maximal length approaches $\lim_{x\to\infty} \frac{1}{2^n} = 0$.

We will use (without proof) the fact that every bounded sequence always contains a convergent subsequence. We want to show that the sequence of 1-2-3 triangles $\Delta_1, \Delta_2, \Delta_3, \ldots$ in each consecutive triangulation converges to a triangle in Δ. Each triangle has three vertices, which each have three coordinates. So, we can write

$$\Delta_n = (\mathbf{a}_n, \mathbf{b}_n, \mathbf{c}_n) = ((a_{1,n}, a_{2,n}, a_{3,n}), (b_{1,n}, b_{2,n}, b_{3,n}), (c_{1,n}, c_{2,n}, c_{3,n}))$$

where (for example) $\mathbf{a}_n$ denotes a vertex with color 1, $\mathbf{b}_n$ denotes a vertex with color 2, and $\mathbf{c}_n$ denotes a vertex with color 3.

Since $a_{1,1}, a_{1,2}, a_{1,3}, \ldots$ is a bounded sequence, there exists a convergent subsequence $a_{1,n_1}, a_{1,n_2}, a_{1,n_3}, \ldots$. So, the x-coordinate of say the left-most vertex of the subsequence $\Delta_{n_1}, \Delta_{n_2}, \Delta_{n_3}, \ldots$ converges. We can repeat this argument using this subsequence to obtain a subsequence $\Delta_{n_{k_1}}, \Delta_{n_{k_2}}, \Delta_{n_{k_3}}, \ldots$ of our subsequence in which both x- and y-coordinates of the left-most vertex of each triangle in the sequence converges. We go through this process a total of nine times, one for each vertex coordinate of a triangle. Thus, we can find a subsequence (indexed by l_i) such that the x, y, z coordinates of each vertex of $\Delta_{l_1}, \Delta_{l_2}, \Delta_{l_3}, \ldots$ converge.

By our construction method of getting smaller edge lengths with each consecutive triangulation, the converging vertices get closer to each other the further along the sequence. Since the maximal length approaches 0 as the triangulation count approaches infinity, the three vertices of our converging sequence of 1-2-3 triangles will actually collapse to a single point, which we call $\mathbf{x} = (x_1, x_2, x_3)$. To simplify the argument, suppose we have arrived at $\mathbf{x}$ by the example process shown above.

To find $f(\mathbf{x})$, we note that $f(\mathbf{a}_n)_1 < a_{1,n}$ for all n and that $\lim_{n\to\infty} a_{1,n} = x_1$, so $\lim_{n\to\infty} f(\mathbf{a}_n)_1 \leq \lim_{n\to\infty} a_{1,n}$. We can give a similar argument for the other two coordinates and derive that $f(\mathbf{x})_i \leq \mathbf{x}_i$ for $i \in \{1, 2, 3\}$. But since the sum of the coordinates of $\mathbf{x}$ and $f(\mathbf{x})$ are both equal to 1, these inequalities are actually equalities as shown previously, contradicting our assumption that there is no fixed point.

$\square$