Part I: Theory

1. (a) $M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (b) $M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (c) $M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (d) $M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & RT \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_1 t_x \\ r_{21} & r_{22} & r_{23} & r_2 t_y \\ r_{31} & r_{32} & r_{33} & r_3 t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

   where r_1, r_2, r_3 are the row vectors of R: $R = \begin{pmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{pmatrix}$.

2. Let $M := (M_1 - M_2) =: \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$.

   "$\Rightarrow$":
   We show that M is skew-symmetric by distinguishing diagonal and off-diagonal elements of M:

   (a) $\forall i \colon 0 = e_i^\top M e_i = m_{ii}$ where $e_i =$ i-th unit vector

   (b) $\forall i \neq j \colon 0 = (e_i + e_j)^\top M (e_i + e_j)$ where $e_j =$ j-th unit vector
   $= m_{ii} + m_{jj} + m_{ij} + m_{ji} \Rightarrow m_{ij} = -m_{ij}$

   hence, $m_{ii} = 0$ and $m_{ij} = -m_{ij}$, i.e. M is skew-symmetric.

"$\Leftarrow$":
using $M = -M^\top$, we directly calculate

$$\forall x \colon x^\top M x = (x^\top M x)^\top = x^\top M^\top x = -(x^\top M x)$$
$$\Rightarrow x^\top M x = 0$$

3. We know: $\omega = (\omega_1\ \omega_2\ \omega_3)^\top$ with $||\omega|| = 1$ and $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

(a)

$$
\hat{\omega}^2 = \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix}
$$

$$
= \begin{pmatrix} \omega_1^2 - \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{1} & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 - \underbrace{(\omega_2^2 + \omega_1^2 + \omega_3^2)}_{1} & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 - \underbrace{(\omega_3^2 + \omega_1^2 + \omega_2^2)}_{1} \end{pmatrix}
$$

$$
= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
= \omega\omega^\top - I
$$

$$
\begin{aligned}
\hat{\omega}^3 &= \hat{\omega}\,\hat{\omega}^2 \\
&= \hat{\omega}\,(\omega\omega^\top - I) \\
&= \hat{\omega}\,\omega\,(\omega^\top) - \hat{\omega}I \\
&= (\omega \times \omega)\,\omega^\top - \hat{\omega} \\
&= -\hat{\omega} \qquad\qquad (\text{as } \omega \times \omega = 0)
\end{aligned}
$$

Alternative solution for $\hat{\omega}^3$:

$$
\hat{\omega}^3 = \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_2^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & \omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{1} & -\omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{1} \\ -\omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{1} & 0 & \omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{1} \\ \omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{1} & -\omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_{1} & 0 \end{pmatrix}
$$

$$
= -\hat{\omega}
$$

(b) The formulas for n even and odd can be found by writing down the solutions for $n = 1, \ldots, 6$:

$$\hat{\omega}$$
$$\hat{\omega}^2$$
$$\hat{\omega}^3 = -\hat{\omega}$$
$$\hat{\omega}^4 = -\hat{\omega}^2 \qquad \text{as: } \hat{\omega}^4 = \hat{\omega}^3\hat{\omega} = -\hat{\omega}\hat{\omega} = -\hat{\omega}^2$$
$$\hat{\omega}^5 = \hat{\omega} \qquad \text{as: } \hat{\omega}^5 = \hat{\omega}^4\hat{\omega} = -\hat{\omega}^2\hat{\omega} = -\hat{\omega}^3 = -(-\hat{\omega}) = \hat{\omega}$$
$$\hat{\omega}^6 = \hat{\omega}^2 \qquad \text{as: } \hat{\omega}^6 = \hat{\omega}^5\hat{\omega} = \hat{\omega}\hat{\omega} = \hat{\omega}^2$$

For even numbers:

$$\hat{\omega}^2$$
$$\hat{\omega}^4 = -\hat{\omega}^2$$
$$\hat{\omega}^6 = \hat{\omega}^2$$

For odd numbers:

$$\hat{\omega}$$
$$\hat{\omega}^3 = -\hat{\omega}$$
$$\hat{\omega}^5 = \hat{\omega}$$

$$\text{even:} \quad \hat{\omega}^{2n} = (-1)^{n+1}\,\hat{\omega}^2 \quad \text{for } n \geq 1$$
$$\text{odd:} \quad \hat{\omega}^{2n+1} = (-1)^n\,\hat{\omega} \qquad \text{for } n \geq 0$$

Proof via complete induction:

i. For even numbers $2n$ where $n \geq 1$:
   - $n = 1: \quad \hat{\omega}^2 = (-1)^2\hat{\omega}^2$
   - Induction step $n \rightarrow n + 1$:

$$
\begin{aligned}
\hat{\omega}^{2(n+1)} &= \hat{\omega}^{2n} \cdot \hat{\omega}^2 \\
&= (-1)^{n+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 \qquad \text{(assumption)} \\
&= (-1)^{n+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\
&\overset{(a)}{=} (-1)^{(n+1)+1} \cdot \hat{\omega}^2
\end{aligned}
$$

ii. For odd numbers $2n + 1$ where $n \geq 0$:
   - $n = 0: \quad \hat{\omega}^1 = (-1)^0\hat{\omega}$
   - Induction step $n \rightarrow n + 1$:

$$
\begin{aligned}
\hat{\omega}^{2(n+1)+1} &= \hat{\omega}^{2n+1} \cdot \hat{\omega}^2 \\
&= (-1)^n \cdot \hat{\omega} \cdot \hat{\omega}^2 \qquad \text{(assumption)} \\
&= (-1)^n \cdot \hat{\omega}^3 \\
&\overset{(a)}{=} (-1)^{n+1} \cdot \hat{\omega}
\end{aligned}
$$

(c) We know: $\omega \in \mathbb{R}^3$. Let $\nu = \frac{\omega}{\|\omega\|}$ and $t = \|\omega\|$. Hence, $w = \nu t$, $\hat{\omega} = \hat{\nu} t$.

$$
\begin{aligned}
e^{\hat{\omega}} &= e^{\hat{\nu} t} \\
&= \sum_{n=0}^{\infty} \frac{(\hat{\nu} t)^n}{n!} \\
&= I + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \hat{\nu}^{2n} + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \hat{\nu}^{2n+1} \\
&\overset{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}}_{1-\cos(t)} \hat{\nu}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)} \hat{\nu} \\
&\overset{(\text{def.})}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|)
\end{aligned}
$$