Lecture #21

The Rendering Equation

Computer Graphics

Winter Term 2016/17

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Up to now

- Very simple lighting model
  - Phong Lighting (or maybe Torrance-Sparrow or similar)
  - Lighting from point lights (or directional lights = point lights at infinity)
- Distribution Ray Tracing
  - supports area lights → how related to point lights ?
  - could support indirect illumination → how defined correctly ?
  - …
- We need a physically correct description of lighting !

→ Rendering Equation

Content

- Radiometric Concepts
- BRDFs
- The Rendering Equation

Radiometry

- Physical measurement of light
- “How bright is a light source” ?
- “How bright is a surface point ?”
- “How much light does a surface point emit towards the camera ?”

- For now, we ignore color
- In fact, all following magnitudes are wavelength dependent
- In practice, this means we simply use RGB-triples

Radiant Flux

- Light source emits photons
  - varying frequency, each photon has some small energy

- Radiant Flux: emitted light energy per time
  = power (energy per time)
  = flux (term for radiant power)
  - $\Phi = \frac{dE}{dt}$
  - unit: Watts = W

- Simple example: 100 W light bulb, 5% efficiency → radiant flux = 5W

Radiant Flux

- Flux can also be used to describe incident light
- Example:
  - Each pixel on a camera chip counts incident photons
  - Camera chip counts n incident photons while shutter is open
  - assume: photon frequency ν → energy of one photon is $h\nu$
  - assume: shutter time t → incident radiant flux $\Phi = n\frac{h\nu}{t}$

Radiosity

- Flux is light emitted by **entire** light source
- Does not describe emission of a single point
  → points on light can have different brightness
  → “brightness” is probably “emitted light per surface area”

- Area light source
  - Flux Φ emitted over area *A*
  - „Brightness“ is emitted light per area = *Radiosity B*

$$B = \frac{\Phi}{A} \quad \left[\frac{W}{m^2}\right]$$

→ assumes constant light over light source

  - or spatially varying emission:

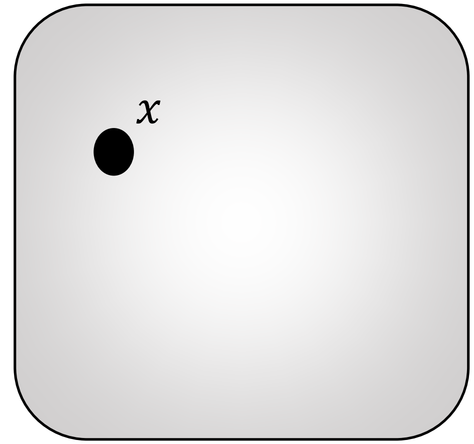
$$B(x) = \frac{d\Phi(\mathrm{x})}{dA_x}$$

Radiosity

- Examples
  - a light panel emits 25W of light on an area of 10cm x 10cm
    → Radiosity is $2500 \frac{W}{m^2}$
  - another light panel emits the same flux on an area of 1m x 1m
    → Radiosity is $25 \frac{W}{m^2}$
    → same flux, different radiosity

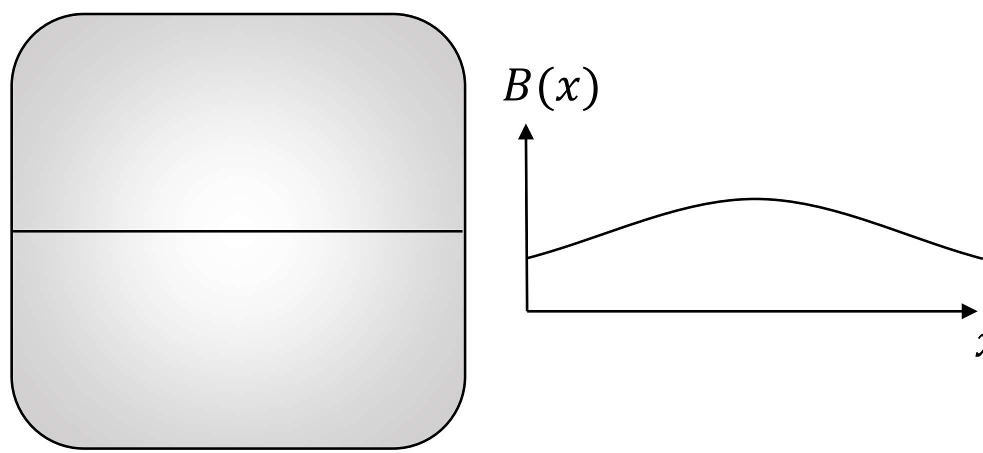
Radiosity

- another example:
  - Brightness not uniform over panel
- how to measure brightness at x?
  - consider small area around x, measure emitted flux → Φ
  - $B(x) = \Phi/A$
  - make area smaller → derivative



$$B(x) = \frac{d\Phi}{dA_x}$$

  - $B(x)$ varies over panel (see right)



Irradiance

- also radiosity can be incident = *Irradiance E*
- Example: one pixel on a camera chips has surface area A and counts n photons of frequency ν

$$E = n\frac{h\nu}{At}$$

- or spatially varying incident light:

$$E(x) = \frac{d\Phi(x)}{dA_x}$$

Intensity

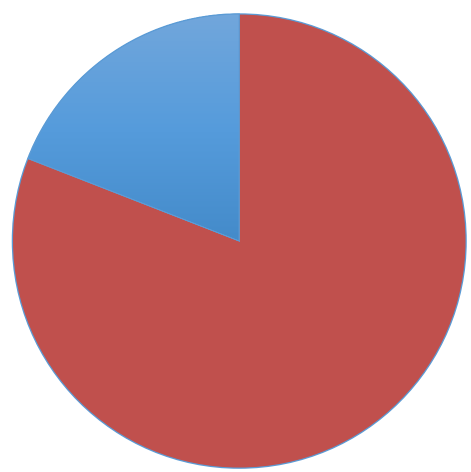
- Instead of spatial dependence we can also consider angular dependence = Intensity I

$$I(\omega) = \frac{d\Phi(\omega)}{d\omega}$$

- ???

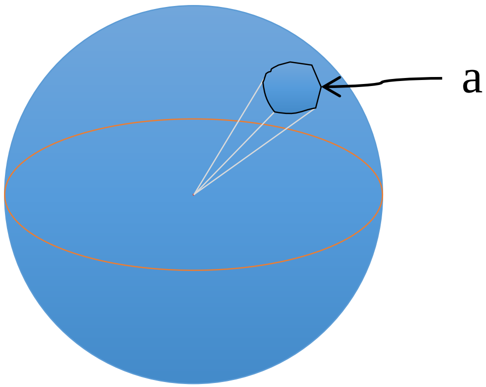
Solid Angle

- Angle – span of directions in 2D
  - Angle θ of circular arc of length l is equal to l/r [radians $\equiv$ rad]
  - Example of circle: Circumference: $l = 2\pi r$; $\theta = \frac{l}{r} = 2\pi[rad]$
    $\rightarrow$ total angle of a circle is 2π



Solid Angle

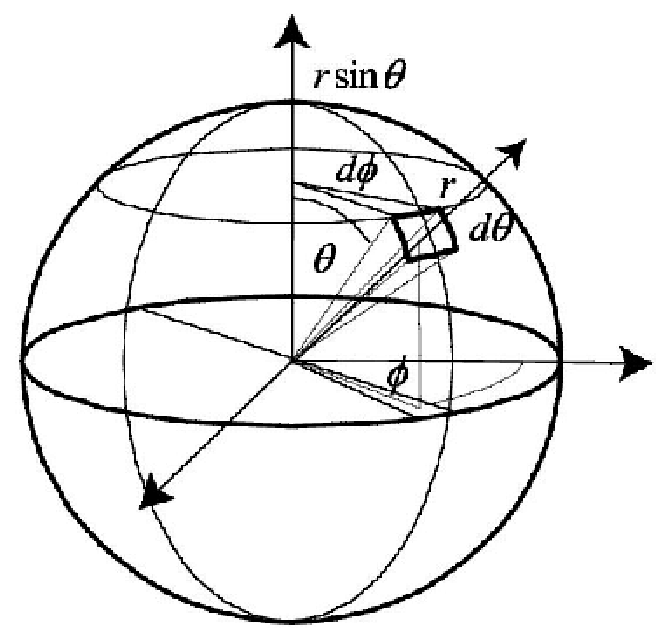
- Solid angle – equivalent value in 3D
  - Solid angle Ω of spherical area a is equal to a/r^2 [steradians $\equiv$ sr], "radians squared"
  - Example of sphere
    - Surface area: $a = 4\pi r^2$,
    - therefore $\Omega = a/r^2 = 4\pi r^2/r^2$ [sr]
    - The solid angle of the sphere is 4π [sr]



Solid Angle

- Differential solid angle $d\Omega$ on sphere using polar coordinates (θ, φ)
  - Length of an arc at $[\theta, \theta + d\theta]$: $r\, d\theta$
  - Length of an arc at $[\varphi, \varphi + d\varphi]$: $r \sin\theta\, d\varphi$
  - Differential area on sphere:

$$d\Omega = (r d\theta)(r \sin\theta\, d\varphi) = r^2 \sin\theta d\theta d\varphi$$



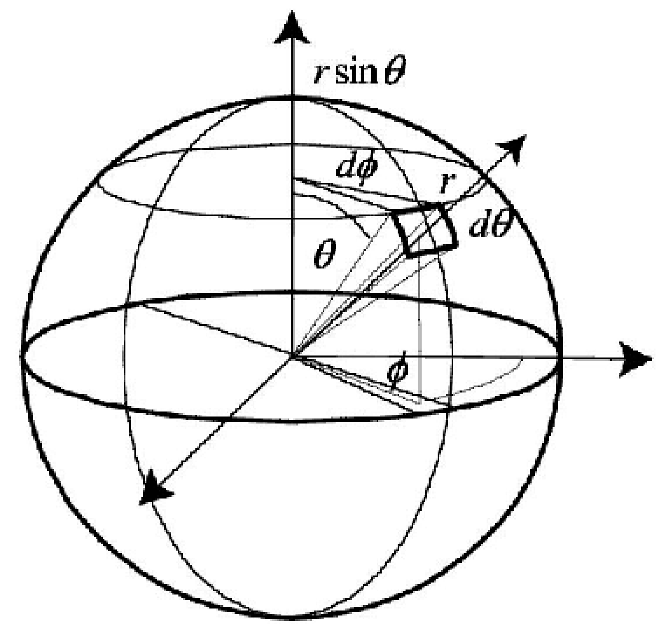
Solid Angle

- Differential solid angle $d\omega$ on sphere

$$d\omega = \frac{d\Omega}{r^2} = \sin\theta d\theta d\phi$$

- Example: integrate "1" over sphere -> surface area

$$\int_0^{\pi}\int_0^{2\pi} 1 \cdot \sin\theta d\theta \; d\varphi = 4\pi$$

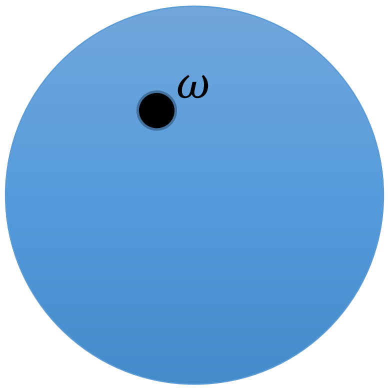


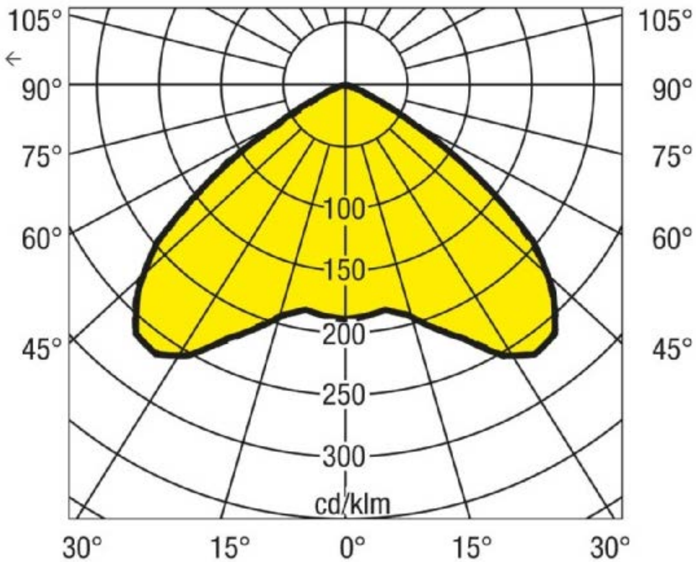
Intensity

- Now back to intensity in direction ω:
  - consider solid angle Ω around ω
  - measure emitted flux Φ
  - $I = \frac{\Phi}{\Omega}$
- make solid angle infinitely small:
  $\rightarrow I(\omega) = \frac{d\Phi(\omega)}{d\omega}$

  = light emitted into direction ω

- example:
  - radial intensity plot
    taken from www.osram.de



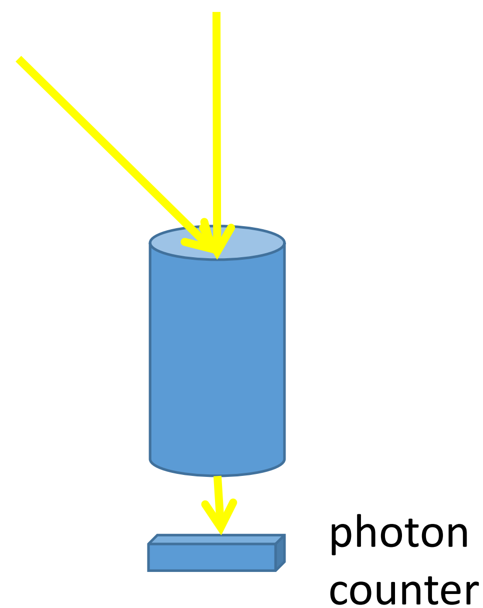


Strahlungsverteilung

Intensity

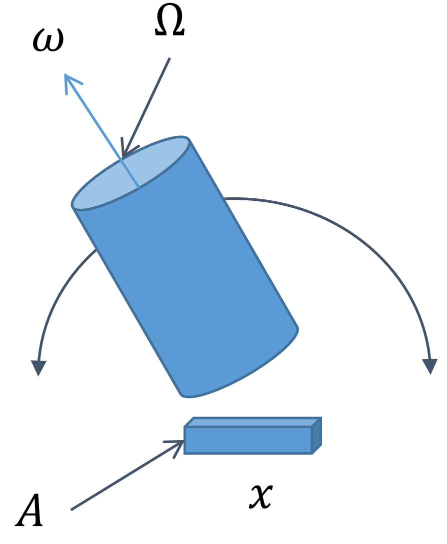
- Incident intensity:
  think of special measuring device:
  - lengthy tube only lets light pass coming from a restricted range of directions
    → solid angle
  - only counts photons coming from certain directions with some solid angle Ω
  - incident intensity in direction of cylinder:

$$I = \frac{nh\nu}{t\Omega}$$



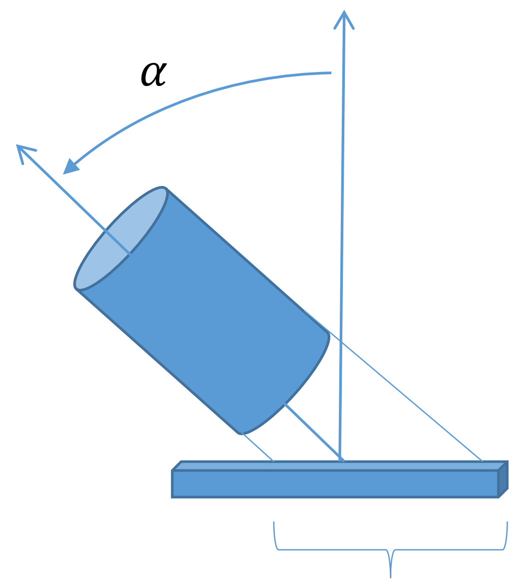
Radiance

- Finally: what we want is
  incident or exitant light, depending on *position and direction*
  → **Radiance**
- let's start with incident radiance of a surface *(not the same as irradiance!)*
- again, we restrict the solid angle using a tube with some solid angle Ω
- what area should we use ?



Radiance

- we have to consider the projected area
- for steep angles, the area that emits light increases by cosα
  - $L(x,\omega) = \frac{\Phi}{A\,\Omega} = \frac{\Phi}{cos\alpha\, A'\,\Omega}$
- in terms of flux:
  - $L(x,\omega) = \frac{d^2\Phi}{cos\alpha\,\mathrm{d}A_\mathrm{x}\,\mathrm{d}\omega} = \frac{d^2\Phi}{<N_x,\omega> dA_x\, d\omega}$



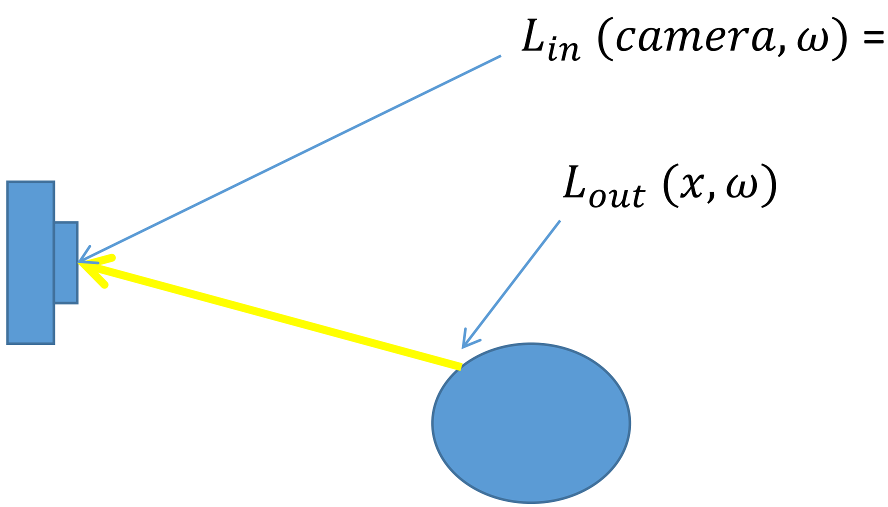
$$A' = \frac{A}{cos\alpha}$$

Radiance

- Same for exitant flux
- Unit is $\frac{W}{m^2 sr}$
- Radiance is what is measured by camera

Radiance

- In vacuum, radiance does not change along a ray
  - basis for ray tracing



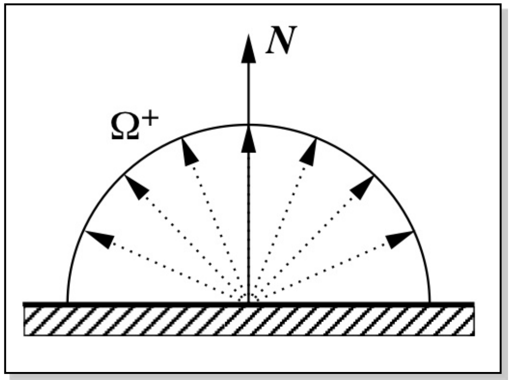
Radiometry

- Radiant flux or radiant power: $\Phi = \frac{dE}{dt}$

- Radiant intensity: $I(\omega) = \frac{d\Phi}{d\omega}$

- Flux density or radiosity: $B(x) = \frac{d\Phi}{dx}$

- Radiance: $L(x, \omega) = \frac{d^2\Phi}{<N_x, \omega> d\omega dA_x}$



Radiometry

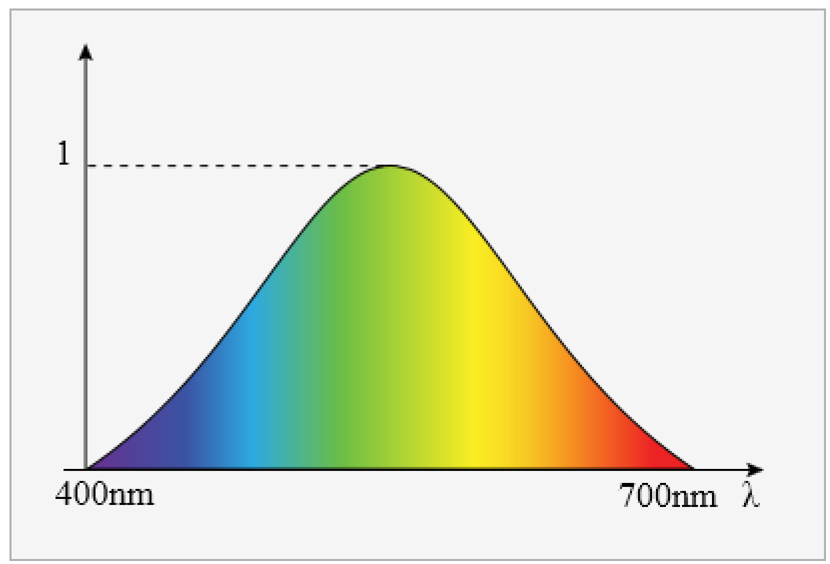
- Integration
  - $B(x) = \int_{\Omega^+} L(x, \omega) < N_x, \omega > d\omega$
  - $I(\omega) = \int_S L(x, \omega) dA_x$
  - $\Phi = \int_S B(x) dA_x = \int_\Omega I(\omega) d\omega = \int_S \int_\Omega L(x, \omega) < N_x, \omega > d\omega \, dA_x$

Radiometry

Quantity	Definition	Units
Radiant Energy	E	J
radiant flux (power)	$\phi = \frac{dE}{dt}$	W
Radiant Intensity	$I = \frac{d\phi}{d\omega}$	W sr^{-1}
Radiosity	$B = \frac{d\phi^{out}}{dA}$	W m^{-1}
Irradiance	$E = \frac{d\phi^{in}}{dA}$	W m^{-1}
Radiance	$L = \frac{dI}{\cos \alpha dA}$	W m^{-2} sr^{-1}

Photometry

- Radiometric quantities are wave length dependent
- Photometry: measure brightness *as perceived by human observer*
- Luminosity function $V(\lambda)$ measures perceived brightness over wave length



Photometry

- set of quantities as in radiometry, but wave lengths weighted according to $V(\lambda)$:

$$F = 683 \frac{lm}{W} \int \Phi(\lambda) V(\lambda) d\lambda$$

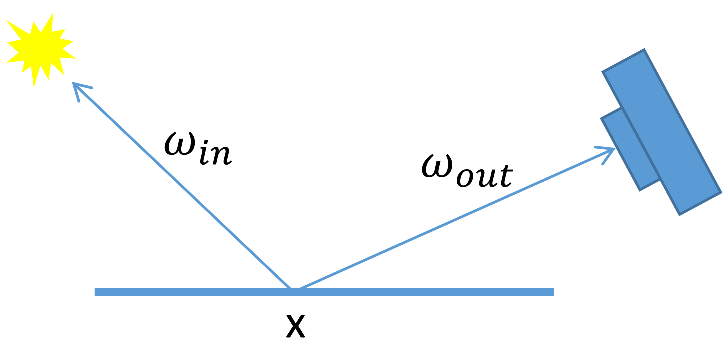
- *F* = photometric equivalent to flux
  = *luminous flux*

Photometry

Radiometry		Photometry	
Energy Energie	Joule [J]	Luminous Energy Lichtmenge	Talbot, Lumen-sekunde [lm s]
Flux Strahlungsfluss	Watts [W= J/s]	Luminous Flux Lichtstrom	Lumen [lm]
Irradiance Flussdichte	W/m^2	Illuminance Beleuchtungsstärke	Lux [lx]
Radiosity Radiometrisches Emmissionsvermögen	W/m^2	Luminosity Photometrisches Emissionsvermögen	Lx / m^2
Intensity Intensität	W / sr	Luminous Intensity Lichtstärke	Candela [cd]
Radiance Strahlungsdichte	$W / sr\ m^2$	Luminance Leuchtdichte	cd / m^2

Reflection

- how can we describe reflection off a surface?
  - e.g. Phong-Model in radiometric context?
- How is light incident at surface point x from direction ω_{in} reflected in direction ω_{out}?



Reflection

reflected exitant light:
radiance in direction ω_{out}

- more precisely:

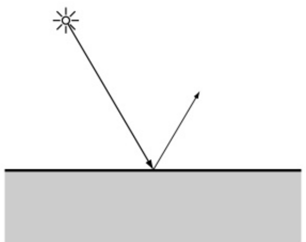
$$f(x, \omega_{in}, \omega_{out}) = \frac{dL(x, \omega_{out})}{dE(x, \omega_{in}) < N_x, \omega_{in} >}$$

incident light:
irradiance from direction ω_{in}

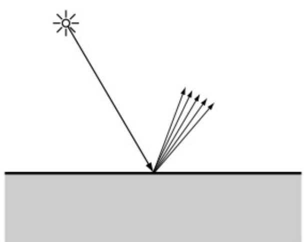
- cosine ?
  - similar argument as for radiance definition
  - diffuse BRDF → $f = const$

Reflection

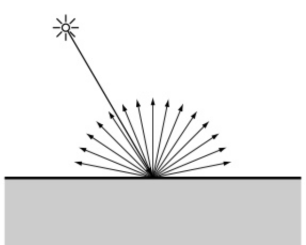
- Major types of BRDFs



mirror reflection



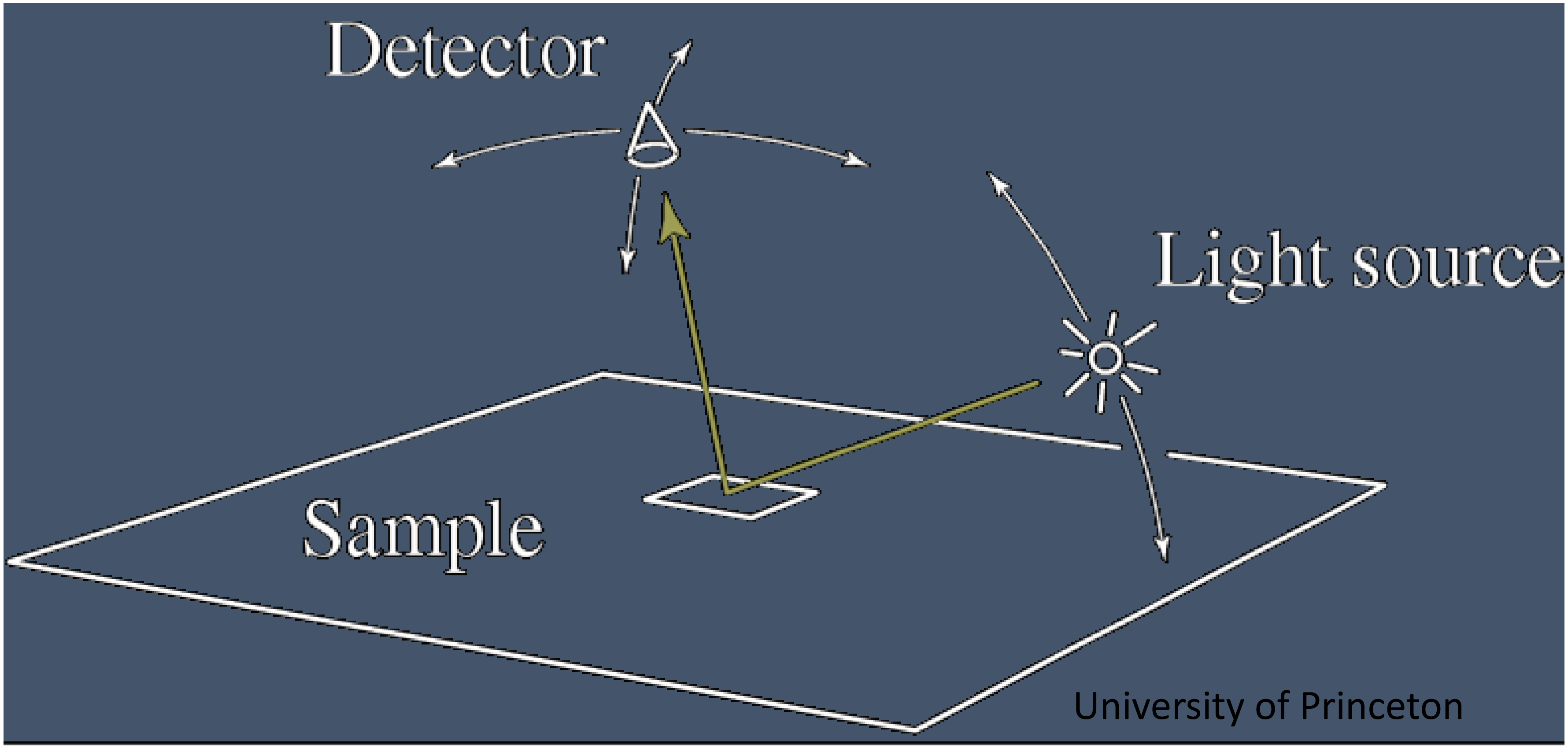
glossy reflection



diffuse (lambert) reflection

Reflection

- Much work on measuring realistic BRDFs



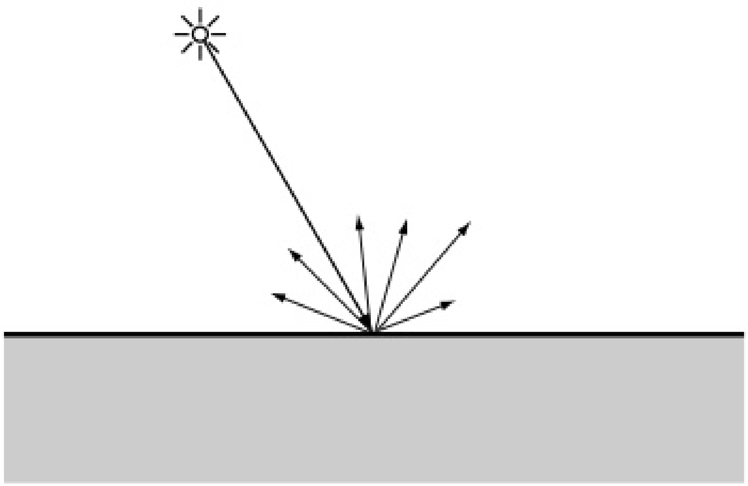
Reflection

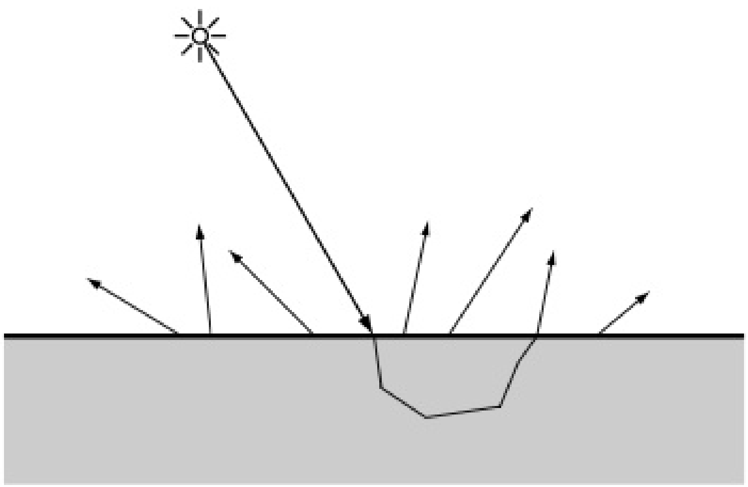
- Gonioreflectometer

Reflection

- Measuring BRDF of a face

Side remark: BRDF / BSSRDF





BRDF: light leaves surface
where it entered

BSSRDF: light leaves surface
at a different position. It describes
subsurface scattering

Side remark: BRDF / BSSRDF

BRDF

BSSRDF

Reflection

- Back to BRDFs: Phong Lighting Model as BRDF
- $f_{Phong}(x, \omega_{in}, \omega_{out}) =$

$$\underbrace{k_d < N_x, \omega_{in} >}_{\text{diffuse}} + \underbrace{k_s \, (\text{reflect}(N_x, \omega_{in}) \circ \omega_{out})^n}_{\text{specular}}$$

- no ambient term

Reflection

- given:
  - incident light $L_{in}(x, \omega_{in})$
  - BRDF $f(x, \omega_{in}, \omega_{out})$

- to compute: exitant light

$$L_{out}(x, \omega_{out}) = \int f(x, \omega_{in}, \omega_{out}) < \mathrm{N_x}, \omega_{in} > L_{in}(x, \omega_{in}) d\omega_{in}$$

Point Lights, Parallel Lights

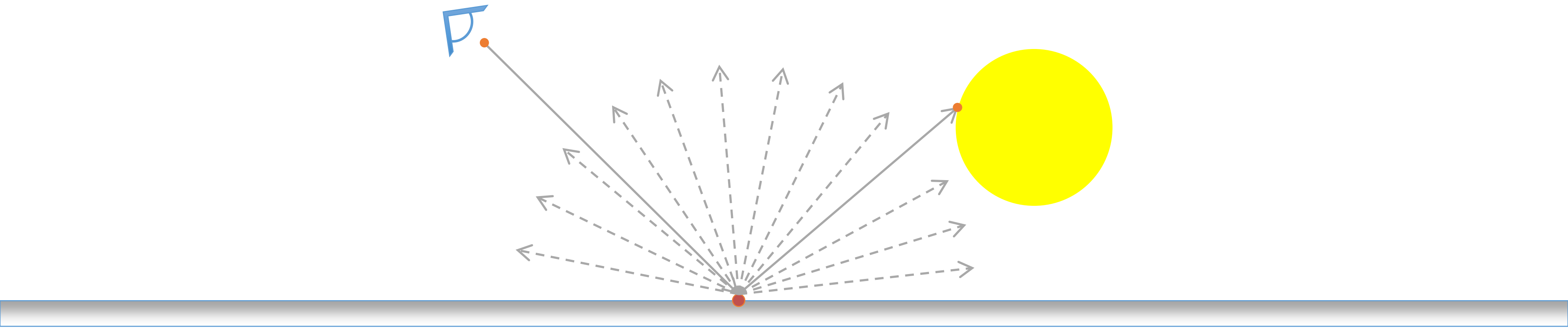
- Problem:
  - point lights and parallel lights have zero area
  - thus their radiance is infinite
  - these sorts of light do not fit in this context
  - We can squeeze them in using Dirac-functions
  - Or replacing them by small sphere lights (point lights) or by very distant area lights (parallel lights)
- But:
  - In the Phong model, reflected light decreases with the squared distance to the light source. Radiance along a ray is constant. !?!?

Illumination Computation

- Illumination Integral:

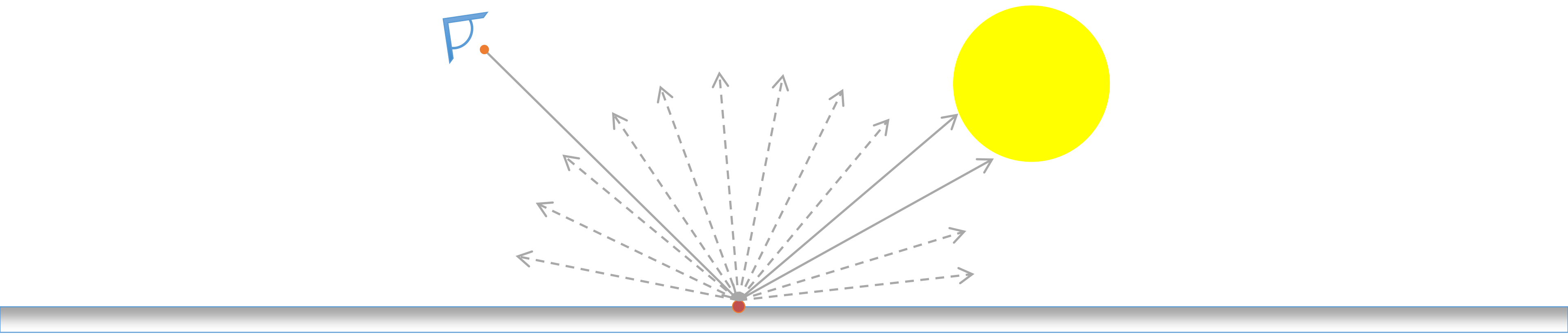
$$L_{out}(x, \omega_{out}) = \int f(x, \omega_{in}, \omega_{out}) < \mathrm{N_x}, \omega_{in} > L_{in}(x, \omega_{in}) d\omega_{in}$$

- simple integration method for area light sources:
  - sample incident light using n sample directions ω_i
  - if ray hits light source → $L_{in} = L_e$ otherwise $L_{in} = 0$
  - $L_{out}(x, \omega) = \int \ldots d\omega_{in} \approx \frac{2\pi}{n} \sum_i L_{in(\omega_i)} f(x, \omega_i, w_{out}) < N_x, \omega_i >$



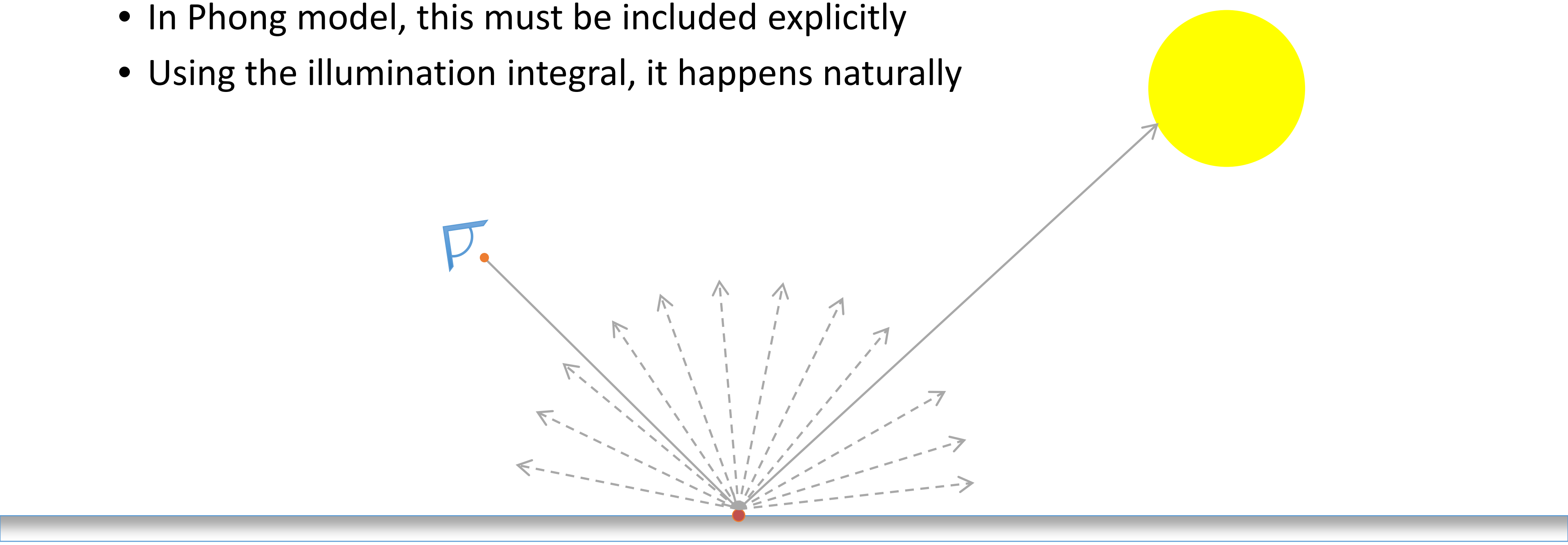
Illumination Computation

- Observation: two rays hit sphere



Illumination Computation

- If we move light away, only one ray hits sphere
  → reflection decreases
- This is how the distance to the light source comes in !
- The solid angle of the light decreases with its squared distance!
  - In Phong model, this must be included explicitly
  - Using the illumination integral, it happens naturally

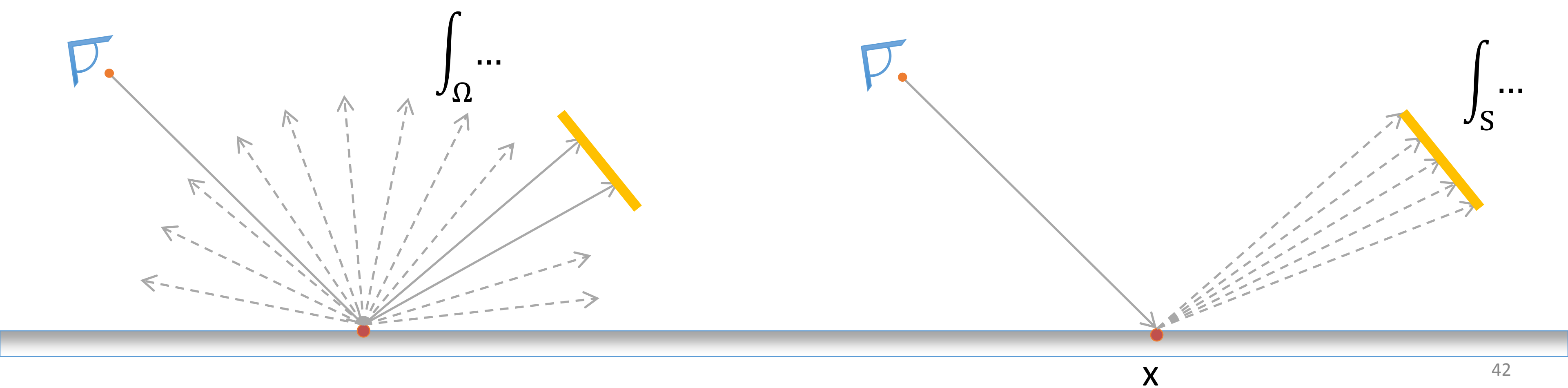


Illumination from Area Lights

- Up to now, we gather incident light over the entire hemisphere Ω_x

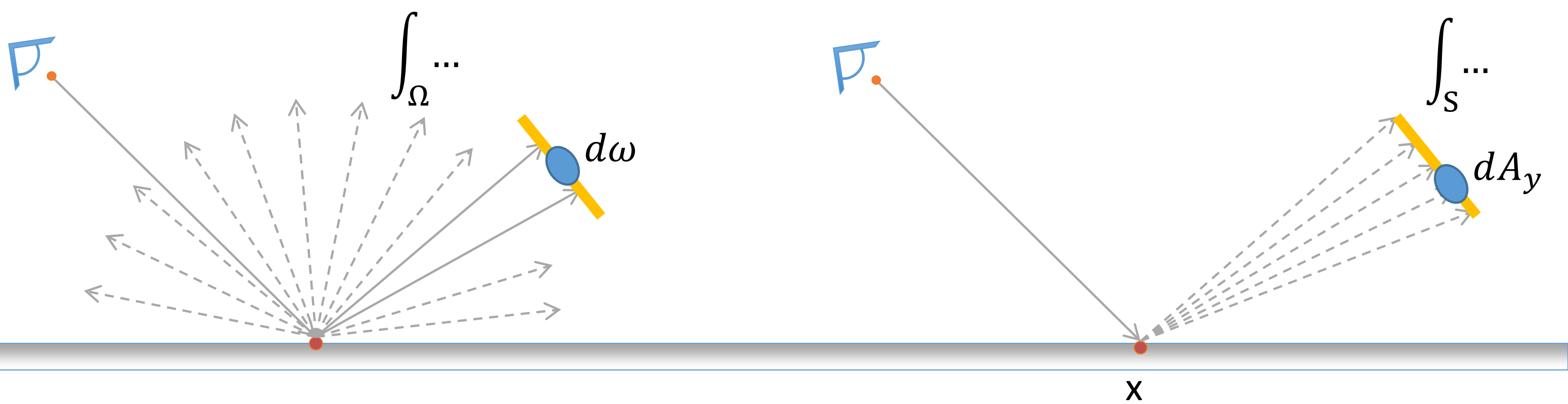
$$L_{out}(x, \omega_{out}) = \int_{\Omega_{\mathrm{x}}} f(x, \omega_{in}, \omega_{out}) < \mathrm{N_x}, \omega_{in} > L_{in}(x, \omega_{in}) d\omega_{in}$$

- If we want to compute the illumination from an area light source, the integrand is zero in most cases
$\rightarrow$ instead of gathering over all incident directions, we should gather over points on the light source:



Illumination from Area Lights

- → **Reparameterize** integral over surface of light source S !
- Instead of integrating over directions ($d\omega$), we integrate over light source (dA_y)
- Each little dA_y on light source corresponds to a small solid angle $d\omega$
- simple geometry: $d\omega = \frac{<N_y,\omega>}{\|x-y\|^2} dA_y$



Illumination from Area Lights

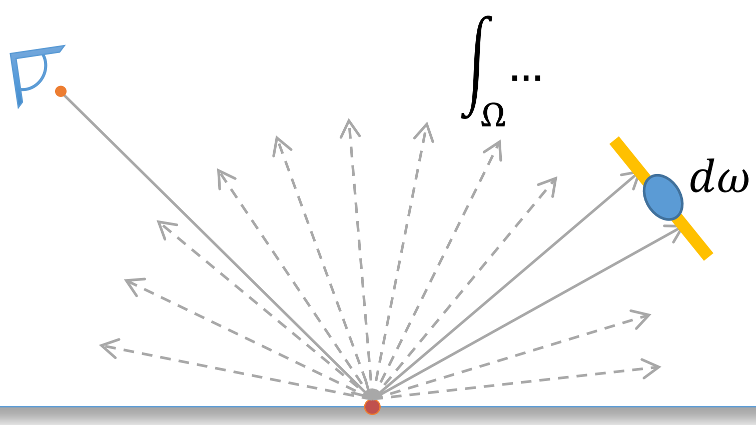
- So our integral

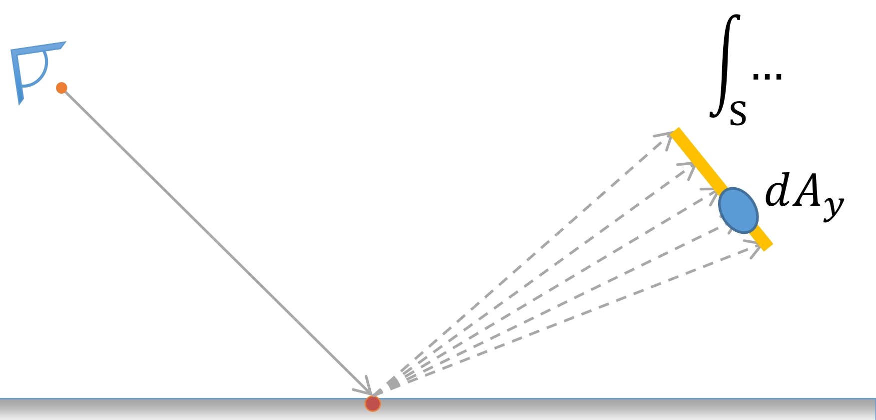
$$L(x,\omega_{out}) = \int_{\Omega_{\mathrm{x}}} f(x,\omega_{in},\omega_{out}) < \mathrm{N_x},\omega_{in} > L_{in}(x,\omega_{in})d\omega_{in}$$

- becomes

$$L(x,\omega_{out}) = \int_{S} G(x,y) f\left(x,\omega_{x,y},w_{out}\right) L_{\mathrm{e}}\left(y,\omega_{y,x}\right) G(x,y) d\mathrm{A_y}$$

- with $G(x,y) = \frac{<N_x,\omega_{x,y}><N_y,\omega_{y,x}>}{\|x-y\|^2}$



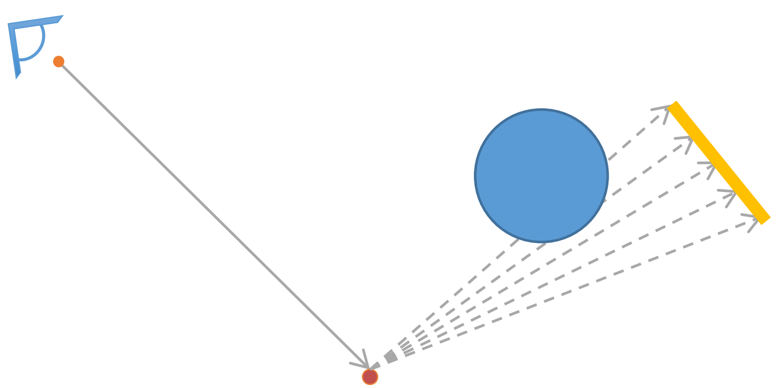


Illumination from Area Lights

- Additionally, we have to consider visibility now:

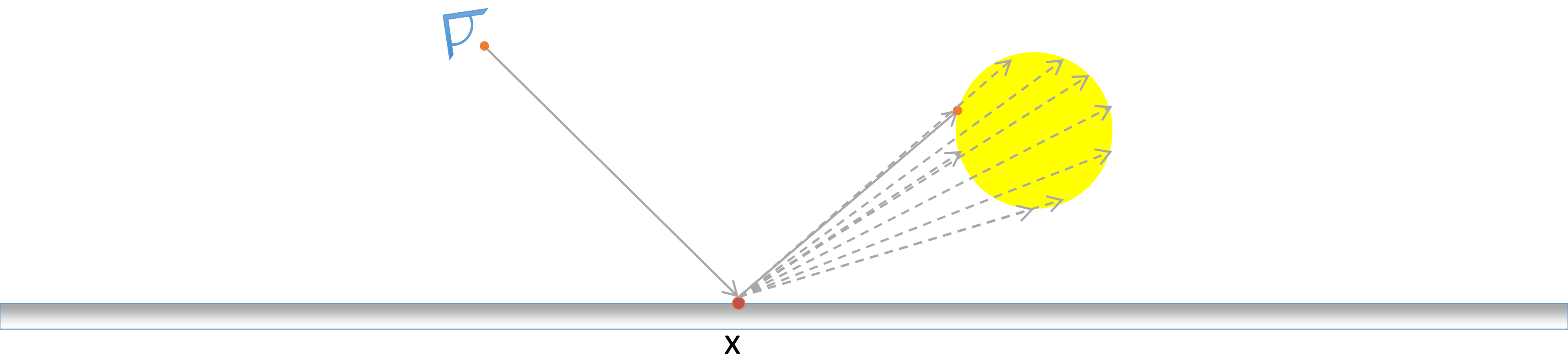
$$L(x, \omega_{out}) = \int_S f(x, \omega_{x,y}, w_{out}) L_{\mathrm{e}}(y, \omega_{y,x}) G(x, y) V(x, y) d\mathrm{A}_{\mathrm{y}}$$

Visibility:
does x „see“ y ?



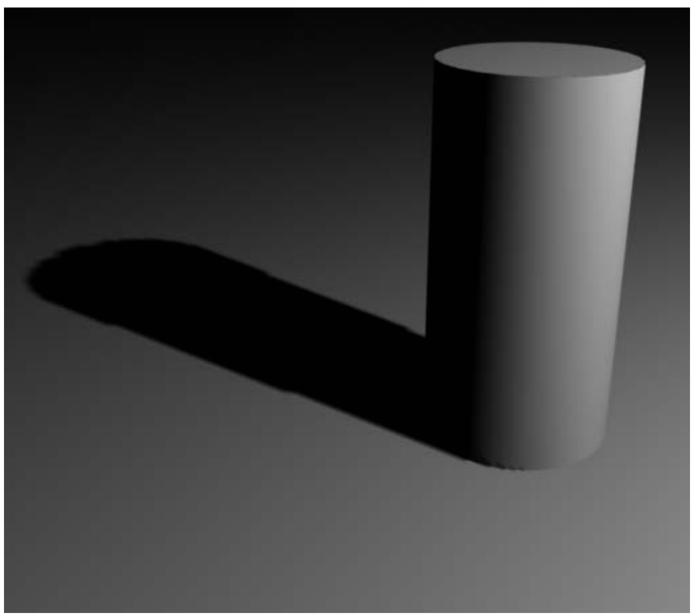
Illumination from Area Lights

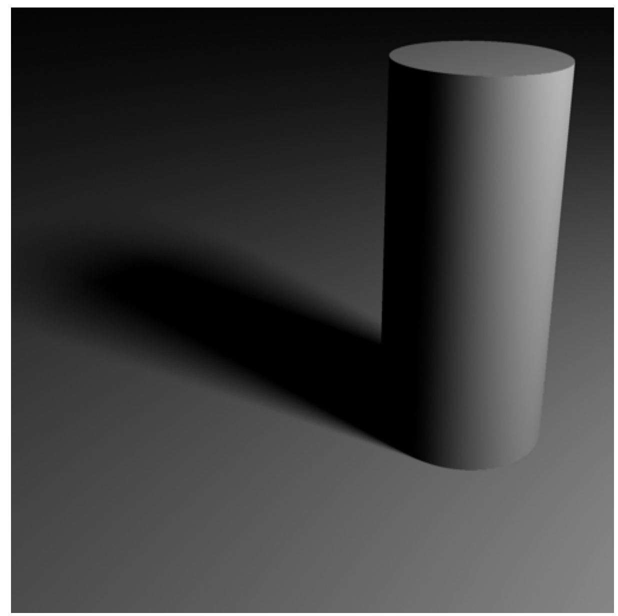
- area light source with emitted radiance L_e
  - sample incident light using n samples on light source y_i
  - $L_{out}(x,\omega) \approx \frac{1}{n}\sum_i L_e f\left(x, \omega_{x,y_i}, w_{out}\right) G(x,y_i) V(x,y_i)$



Integration Techniques

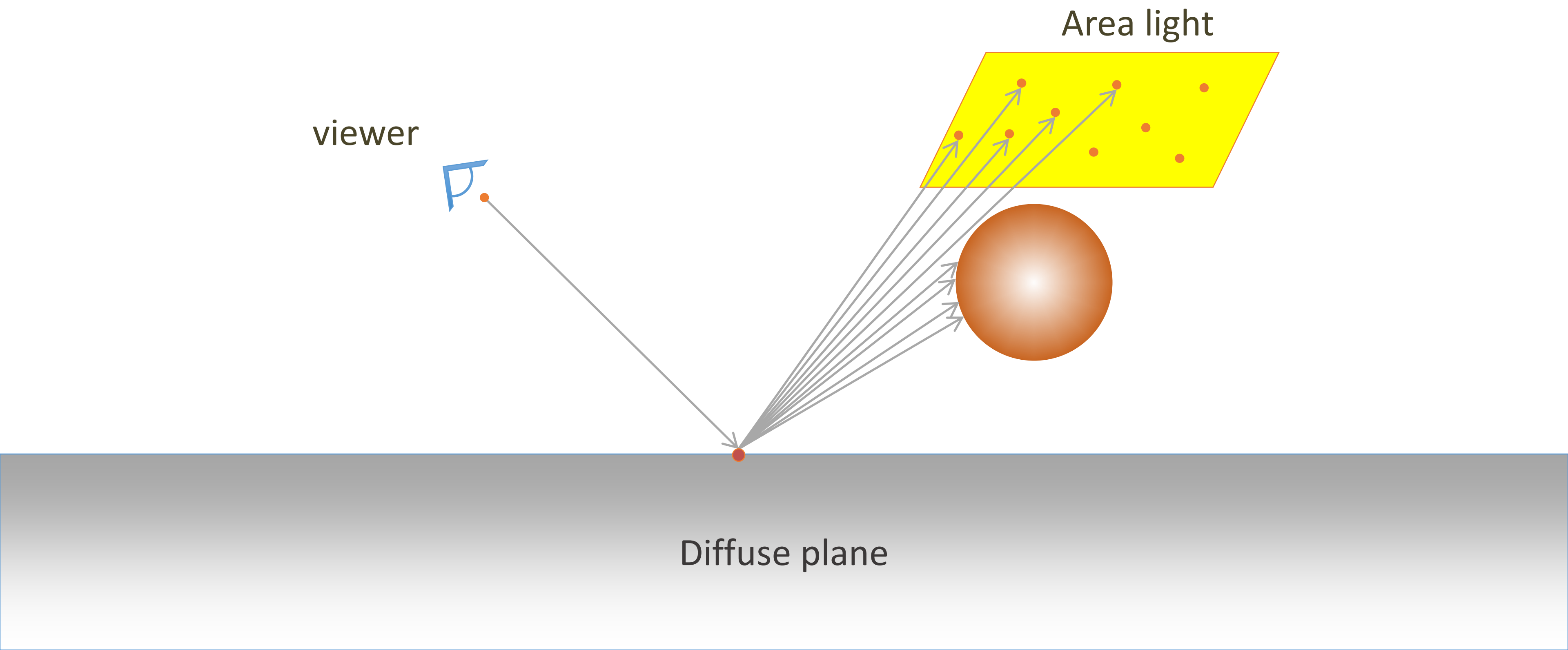
- Remember: Illumination from Area Light Sources
  - Area lights cast soft shadows. These become softer with increasing distance to the occluder





Integration Techniques

- Area Lights
  - Shoot multiple rays to the area light. Choose adequate positions on the light source (sampling!)
  - Samples must be weighted by $G(x, y)$!
  - → more distant samples get lower weight



Rendering Equation

- Up to now:
  Illumination from area lights:
  $$L_{\text{refl}}(x, \omega_{out}) = \int_S f\left(x, \omega_{x,y}, \omega_{out}\right) L_e\left(y, \omega_{y,x}\right) G(x, y) V(x, y) dy$$
- L_e is self-emittance, making a normal surface a light source
- L_{refl} is the reflected light of a surface
- The final radiance of a surface is the sum of both:
  $$L(x, \omega_{out}) = L_e(x, \omega_{out}) + L_{refl}(x, \omega_{out})$$
- But also normal surfaces get indirect light sources when being illuminated
  → replace L_e under integral by L
  → recursive rendering equation

$$L(\mathrm{x}, \omega_{out}) = L_e(x, \omega_{out}) + \int_S f\left(x, \omega_{x,y}, \omega_{out}\right) L\left(y, \omega_{y,x}\right) G(x, y) V(x, y) dy$$

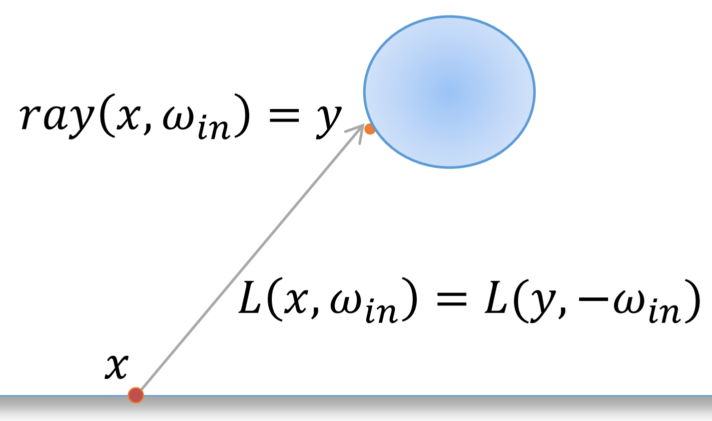
Rendering Equation

- Rendering Equation as surface integral

$$L(\mathrm{x}, \omega_{out}) = L_e(x, \omega_{out}) + \int_S f\left(x, \omega_{x,y}, \omega_{out}\right) L\left(y, \omega_{y,x}\right) G(x, y) V(x, y) dy$$

- As directional integral

$$L(\mathrm{x}, \omega_{out})$$
$$= L_e(x, \omega_{out}) + \int_{\Omega_x} f(x, \omega_{\mathrm{in}}, \omega_{out}) L(\mathrm{ray}(\mathrm{x}, \omega_{in}), -\omega_{\mathrm{in}}) \langle N_x | \omega_{in} \rangle d\omega_{in}$$



That’s it

- So much on the rendering equation

- How to solve this equation efficiently:

  Lecture “Global Illumination” during summer term

  2h lecture + 2h programming assignments

- Next week: preparation for exam