Lecture 4

Part 2 Probability and Distributions

Let's start from something about gambling...

A casino offer you a game of chance: a fair coin is tossed at every stage.

- The initial stake begins at \$2 and is *doubled* every time tails appears.
- The first time heads appears, the game ends and the player wins whatever is the current stake.
- Thus the player wins \$2 if heads appears on the first toss, \$4 if tails appears on the first toss and heads on the second, \$8 if tails appears on the first two tosses and heads on the third, and so on.

Question: How much are you willing to pay for playing this game?

St. Petersburg paradox

Your expected return for playing this game:

$$
\begin{aligned}
\text{Expected Return} &= 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + \cdots \\
&= 2 \times \frac{1}{2} + 2^2 \times \frac{1}{2^2} + 2^3 \times \frac{1}{2^3} + \cdots \\
&= \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty
\end{aligned}
$$

Question: Are you willing to pay an infinite number for this game?
Let me ask you again, how much are you willing to pay?

Random Experiment

- A **random experiment** is a process that results in one of several possible outcomes, none of which can be predicted with certainty.
  For example:
  - Rolling a conventional die. Possible outcomes are 1, 2, 3, 4, 5 or 6.
  - Flipping a coin. Possible outcomes are heads or tails.
- Outcomes are denoted by O_i.

Sample Space

- The sample space of a random experiment is a list of all the possible outcomes and is denoted by

$$S = \{O_1, O_2, O_3, \cdots\}$$

  For example, when flipping a coin, $S = \{H, T\}$
- The outcomes in a sample space must be both *mutually exclusive* and *exhaustive*.
  - **Mutually exclusive**: No two outcomes can both occur at the same time on any single trial of the experiment.
  - **Exhaustive**: The sample space must include all possible outcomes that can occur.

Probabilities of Outcomes

- The probability of an outcome occurring on a single trial is written as $P(O_i)$.
- Probabilities associated with the outcomes in a sample space must satisfy two important requirements:
  - $0 \leq P(O_i) \leq 1$ for all i.
  - $\sum_i P(O_i) = 1$.

Classical Approach

The *classical approach* is based on the assumption that the outcomes of an experiment are equally likely to happen.

- Suppose we roll a fair six-sided dice.
- What is the probability of rolling a 5?
- The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
- Assuming all six outcomes are equally likely, the probability of rolling a 5 is $P(5) = \frac{1}{6}$.
- Note that in more complicated experiments, we often need to use mathematical rules to count the total number of outcomes.

Relative Frequency Approach

The relative frequency approach defines probability as the long-run relative frequency with which an outcomes occurs.

- Suppose we roll a loaded six-sided dice.
  For example:

Relative Frequency Approach

The *relative frequency approach* defines probability as the long-run relative frequency with which an outcomes occurs.

- Suppose we roll a loaded six-sided dice.
- What is the probability of rolling a 5?
- We can't assume equally likely outcomes.
- But suppose we know that in the past 1000 rolls, a 5 came up 190 times.
- Based on this past history, we assign the probability of rolling a 5 to be the long-run relative frequency, i.e., $\frac{19}{100} = 0.19$.

Subjective Approach

The *subjective approach* is based on personal judgment, accumulation of knowledge, and experience.

- Suppose we roll a loaded six-sided dice.
- What is the probability of rolling a 5?
- We can't assume equally likely outcomes.
- Suppose I have no historical data on the die.
- So, based on what I read about loaded dice on Wikipedia and also based on how this particular die feels in my hand (weight distribution, size, etc.), I decide that the probability of rolling a 5 is 0.2.

Events

- A simple event is an individual outcome from the sample space.
- An event is a collection of one or more simple events (or outcomes).

For example, when rolling a die, let A denote the event that an odd number comes up. Then $A = \{1, 3, 5\}$.

Probability of an Event

- The probability of an event is equal to the sum of the probabilities of the simple events that make up the event:

$$P(A) = P(1) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

- If we assume all simple events have equal probability, we can also determine the probability of an event by counting the number of simple events that make up the event:

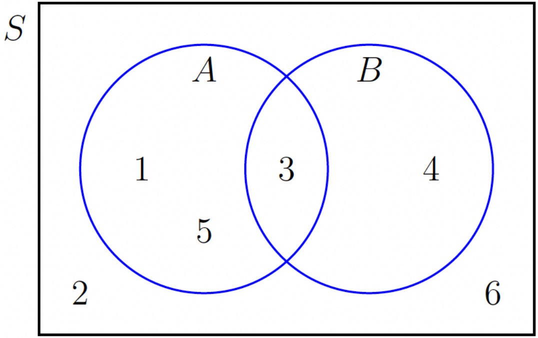
$$P(A) = \frac{\text{number of simple events in } A}{\text{number of simple events in } S} = \frac{3}{6} = \frac{1}{2}$$

Combining Events

- There are some important ways in which events cane combined that we will encounter repeatedly throughout this course.
- Suppose we have two events, A and B.
  - For example, suppose we roll a conventional die and let $A = \{1, 3, 5\}$ denote the event of rolling an odd number and let $B = \{3, 4\}$ denote the event of rolling either a 3 or a 4.

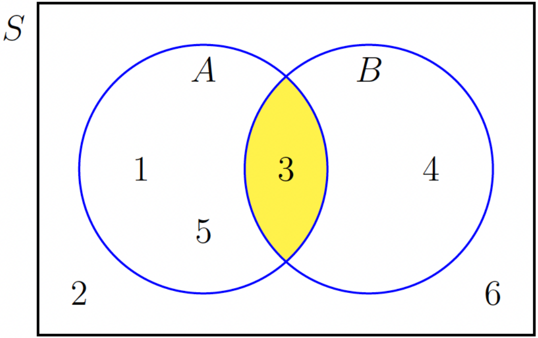
Venn Diagrams

- We can use Venn diagrams to visually describe the events.



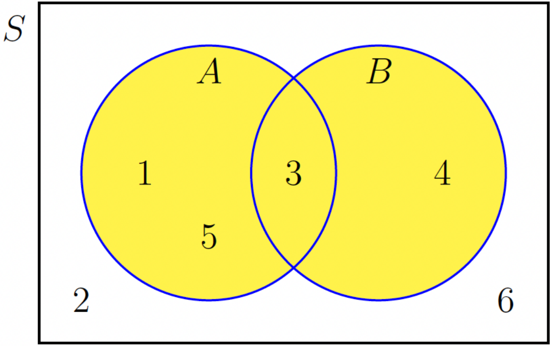
Intersection

- The **intersection** of A and B, denoted by $A \cap B$, is the event that happens when both A and B occur.
  - For example, $A \cap B = \{3\}$.



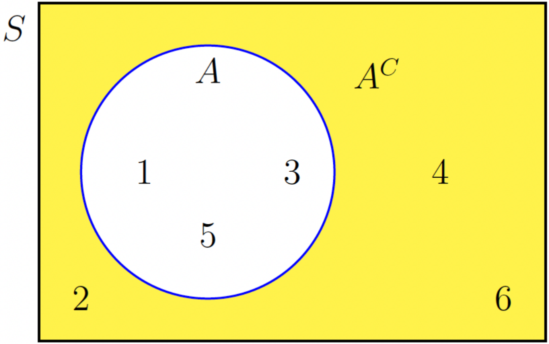
Union

- The **union** of A and B, denoted by $A \cup B$, is the event that happens when either A and B occur.
  - For example, $A \cup B = \{1, 3, 4, 5\}$.



Complement

- The **complement** of A, denoted A^C, is the event that happens when A does not occur.
  - For example, $A^C = \{2, 4, 6\}$.



Intersection and Union

- Both intersection and union are commutative operators.
- That is, for any events A and B, the following are true:

$$A \cap B = B \cap A$$

  and

$$A \cup B = B \cup A$$

- The order of the events does not matter.

Important Facts

- For the sample space S,

$$P(S) = 1$$

- For any event A,

$$0 \leq P(A) \leq 1$$

- If A and B are mutually exclusive (i.e., $A \cap B = \emptyset$), then

$$P(A \cap B) = 0$$

Joint Probability

- A joint probability is the probability of the intersection of two or more events.
  - For example, $P(A \cap B)$ or $P(A \cap B \cap C)$.
- Suppose we are investigating the relationship between how well a mutual fund performs and where the fund manager earned their MBA.

Joint Probability

Let's define the following events:

- Let event A_1 be the event that the fund manager graduated from a program ranked in the top 10.
- Let A_2 be the event that the fund manager graduated from a program ranked between 10 and 20.
- Let A_3 be the event that the fund manager graduated from a program ranked 20 or below.
- Let B_1 be the event that the fund outperforms the market.
- Let B_2 be the event that the fund does not outperform the market.

Joint Probability Table

	B_1: Fund outperforms	B_2: Fund doesn't outperform	Totals
A_1: Rank ≤ 10	0.06	0.16	
A_2: $10 < \text{Rank} < 20$	0.05	0.13	
A_3: Rank ≥ 20	0.06	0.54	
Totals			1

$$P(A_1 \cap B_1) = 0.06 \qquad P(A_1 \cap B_2) = 0.16$$

$$P(A_2 \cap B_1) = 0.05 \qquad P(A_2 \cap B_2) = 0.13$$

$$P(A_3 \cap B_1) = 0.06 \qquad P(A_3 \cap B_2) = 0.54$$

Marginal Probability

- A **marginal probability** is the unconditional probability of an event, irrespective of all other events.
- Given a properly specified joint probability table,marginal probabilities are calculated by adding across the rows of the table, or adding down the columns of the table.
- They are so called because they are calculated in the *margins* of the table.

Marginal Probability

	B_1: Fund outperforms	B_2: Fund doesn't outperform	Totals
A_1: Rank ≤ 10	0.06	0.16	0.22
A_2: $10 <$ Rank < 20	0.05	0.13	0.18
A_3: Rank ≥ 20	0.06	0.54	0.60
Totals	0.17	0.83	1

For example:

$$\begin{aligned} P(A_1) &= P(A_1 \cap B_1) + P(A_1 \cap B_2) \\ &= 0.06 + 0.16 \\ &= 0.22 \end{aligned}$$

Marginal Probability

- Why can we find $P(A_1)$ by adding across the first row of the table, and similarly for $P(A_2)$ and $P(A_3)$?
- Recall that B_2 was the complement of B_1, i.e., $B_2 = B_1^C$.
- So another way of writing what we did when calculating the marginal probability of A_1 is:

$$P(A_1) = P(A_1 \cap B_1) + P(A_1 \cap B_1^C)$$

Marginal Probability

- The previous statement is always true because of two important properties regarding an event and its complement:
  - $B_1 \cap B_1^C = \emptyset$, i.e., an event and its complement are mutually exclusive (they can't both occur at the same time).
  - $B_1 \cup B_1^C = S$, i,e., the union of an event and its complement is the entire sample space.

Marginal Probability