Lecture 13

Part 3 Estimation and Hypothesis Test

Continue our talk about interval estimation...

90% Confidence Interval for μ

$$P\left(\bar{X} - 1.645\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.645\frac{\sigma}{\sqrt{n}}\right) = 0.90$$

- We get a 90% confidence interval by using 1.645:

$$\bar{X} \pm 1.645\frac{\sigma}{\sqrt{n}} = \left(\bar{X} - 1.645\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.645\frac{\sigma}{\sqrt{n}}\right)$$

99% Confidence Interval for μ

$$P\left(\bar{X} - 2.575\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 2.575\frac{\sigma}{\sqrt{n}}\right) = 0.99$$

- We get a 99% confidence interval by using 2.575:

$$\bar{X} \pm 2.575\frac{\sigma}{\sqrt{n}} = \left(\bar{X} - 2.575\frac{\sigma}{\sqrt{n}}, \bar{X} + 2.575\frac{\sigma}{\sqrt{n}}\right)$$

100(1 - α)% Confidence Interval for μ

$$P\left(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

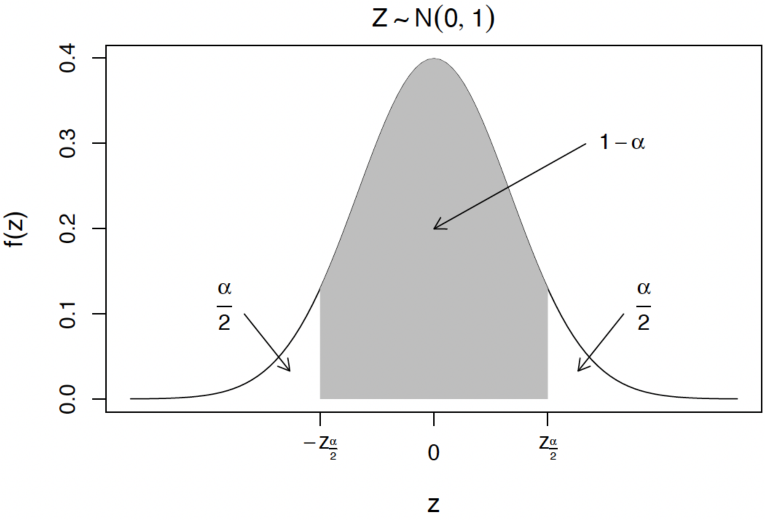
- A 100(1 - α)% confidence interval for μ when σ^2 is known is given by:

$$\bar{X} \pm z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} = \left(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

100(1 - α)% Confidence Interval for μ

- $\bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ is called the **lower confidence limit**.
- $\bar{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ is called the **upper confidence limit**.
- 1 - α is called the **confidence level** and is equal to the proportion of intervals under repeated sampling that contain the population mean.
- $\pm z_{\frac{\alpha}{2}}$ are the points which cut off an area of $\frac{\alpha}{2}$ in the tails of the standard normal PDF and leave an area of 1 - α in the middle.

100(1 - α)% Confidence Interval for μ



Factors Affecting the Confidence Interval

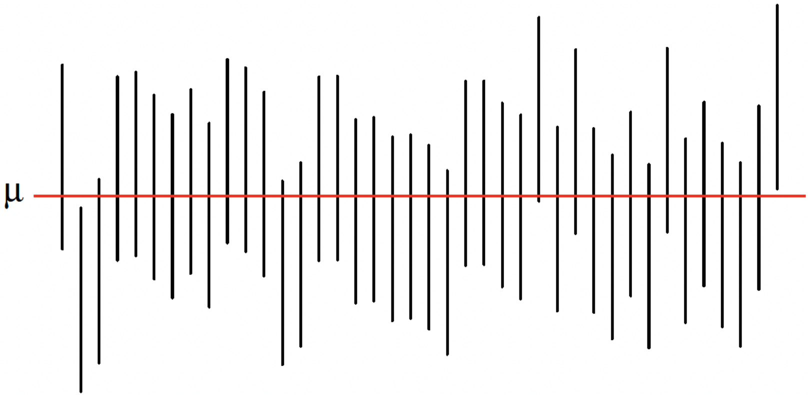
$$\left(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

- Population variance: Larger variation in the random variable widens the interval.
- Sample size: As n gets bigger, the interval gets narrower.
- Confidence level: Increasing confidence level will make the interval wider. For example, to go from 95% to 99%, we change 1.96 to 2.575, which widens the interval.

Interpreting a Confidence Interval

- Remember that it is the *interval* that is random and therefore changes from sample to sample.
- The population mean μ is a fixed and constant value - it is either within the interval or not.
- You should interpret a 100(1 - α)% confidence interval as saying "in repeated sampling, 100(1 - α)% of such intervals created would contain the true population mean".

Interpreting a Confidence Interval



Example 1

- The average height of a sample of 25 men is found to be 178cm. Assume that the standard deviation of male heights is known to be 10cm, and that heights follow a normal distribution.
  (a) Find a 95% confidence interval for the population mean height.
  (b) To what confidence level does an interval of $(174.71, 181.29)$ correspond?

Solution - Part (a)

- For a 95% confidence interval, we know that $z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$. Therefore:

$$\begin{aligned} \bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} &= 178 \pm 1.96 \times \frac{10}{\sqrt{25}} \\ &= (174.08, 181.92) \end{aligned}$$

- So, in repeated sampling, we would expect 95% of the intervals created this way to contain μ.

Solution - Part (b)

- From the lower confidence limit, we get:

$$\begin{aligned}
\bar{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} &= 174.71 \\
178 - z_{\frac{\alpha}{2}} \frac{10}{\sqrt{25}} &= 174.71 \\
z_{\frac{\alpha}{2}} &= \frac{\sqrt{25}}{10} \times (178 - 174.71) \\
z_{\frac{\alpha}{2}} &= 1.645
\end{aligned}$$

- From the z-tables, we know that $\frac{\alpha}{2} = 0.05$ so this corresponds to a $100(1-\alpha) = 90\%$ confidence interval.

Backward - How large should sample size be?

- Suppose that before we gather data, we know that we want to get an estimate within a certain distance of the true population value.
- We can use the CLT to find the minimum sample size required to meet this condition, if the population standard deviation is known.

Example 2

- I time my morning bus trips to work, and get an average of 35 minutes. Assuming that the standard deviation of times is known to be 5 minutes, I want to estimate the true population mean length to within 3 minutes, with **99% certainty**. How many bus trips should I time for calculating my average?

Solution

- Step 1: Set up the required equation, then standardize:

$$
\begin{aligned}
P(|\bar{X} - \mu| < 3) &= 0.99 \\
P(-3 < \bar{X} - \mu < 3) &= 0.99 \\
P\left(-\frac{3}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{3}{\frac{\sigma}{\sqrt{n}}}\right) &= 0.99 \\
P\left(-\frac{3}{\frac{5}{\sqrt{n}}} < Z < \frac{3}{\frac{5}{\sqrt{n}}}\right) &= 0.99
\end{aligned}
$$

Solution

- Step 2: We know $P(-2.575 < Z < 2.575) = 0.99$. Therefore solve for n:

$$\frac{3}{\frac{5}{\sqrt{n}}} = 2.575$$

$$\sqrt{n} = 2.575 \times \frac{5}{3}$$

$$n = 18.42 \approx 19$$

- I need to time at least 19 (round up!) bus trips in order to derive a 99% CI that estimates μ to within 3 minutes.

Hypothesis Test

Statistical Inference

- Estimation (last time):
  - Draw inferences about a population by selecting a sample and *estimating* population parameters (point and interval estimators).
- Hypothesis testing (this time):
  - Draw inferences about a population by making a *claim* or *hypothesis* about a population parameter and testing whether the hypothesis is supported by the sample.
- They are different!

Let's Go Bowling!

- Suppose we all go bowling.
- Before the game, I claim that my average bowling score is 150 (but you think it's lower).
- Over 10 games, my average score turns out to be 80.
- Do you believe my claim?

Let's Go Bowling!

- Assume bowling scores are normally distributed.
- The population:

  An infinite number of my bowling scores.
- The claim/hypothesis:

  That the population mean is 150 ($\mu = 150$).
- The sample and sample statistic:

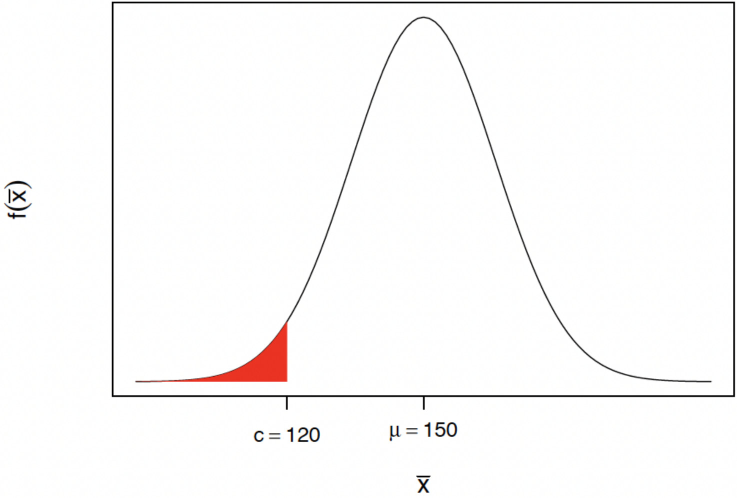
  Bowling scores from 10 games and $\bar{X} = 80$.
- Does the sample and sample statistic support my hypothesis?
- How can we decide?

One Possible Approach

- Assume that the hypothesis is true (i.e, my true bowling average is 150).
- Choose some (lower) cut-off score. (Chosen by person)
- Compare my sample mean bowling score from 10 games to this cut-off.
- If my sample mean is below the cut-off, reject the hypothesis.

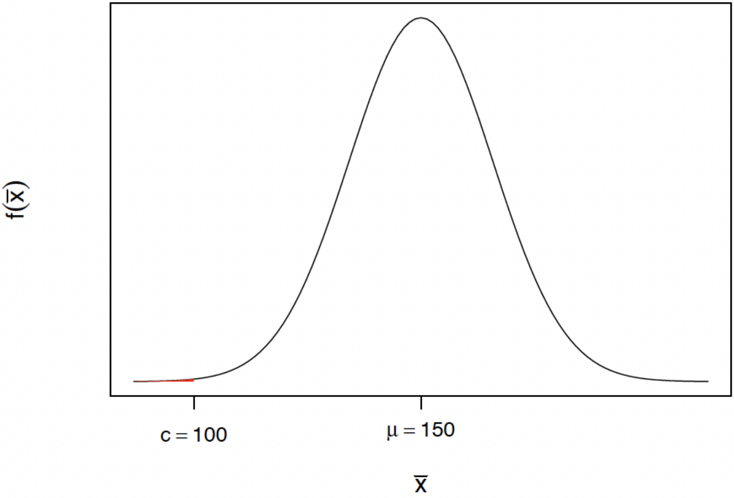
One Possible Approach

- For example, we might choose a cut-off value of $c = 120$:



One Possible Approach

- To be even more confident in our decision to reject the hypothesis, we could lower our cut-off:



1. Hypotheses

- We always test between two hypotheses, the *null hypothesis* and the *alternative hypothesis*.
- We start by assuming the null hypothesis is true.
- Goal is to determine if there is enough evidence to conclude that the alternative hypothesis is true.
- So, based on our sample data, we make one of two decisions:
  (a) *Reject* the null hypothesis (enough evidence in favour of the alternative hypothesis). Or,
  (b) *Fail to reject* the null hypothesis (not enough evidence in favour of the alternative hypothesis).

Null Hypothesis

- The **null hypothesis** (H_0) usually corresponds to a default claim about a population parameter.
- For hypothesis tests concerning a *single* population parameter, almost always involves an "$=$" sign.
- Bowling example:
  - $H_0 : \mu = 150$, i.e., mean bowling score is 150.

Alternative Hypothesis

- The **alternative hypothesis** (H_1) usually represents a claim about the population parameter that we are trying to prove.
- Generally involves a "$<$", "$>$" or "$\neq$" sign.
- Bowling example:

  $H_1 : \mu < 150$, i.e., mean bowling score is less than 150.