Little notes on Dynamic Programming

The problem can be written as

$$v(k) = \max_x u(f(k) - x) + \beta v(x) \tag{1}$$

Where k is a variable and x is a picked value depend on k such that $v(k)$ is maximized.

1) To solve the problem, firstly do the derivative on both side of (1)

$$v'(k) = u'(f(k) - x)f'(k) \tag{2}$$

The derivative of $\beta v(x)$ is ruled out by the envelope theorem.

To understand this step, recall what we will do in maximize a function about k, that is simply do the derivative.

Then set the policy function

$$x = g(k) \tag{3}$$

as the characterized x of (1). Then

$$v'(k) = u'(f(k) - g(k))f'(k) \tag{4}$$

2) The second step is to do the derivative on the RHS of (1) with respect to x, that is to find the best x

$$-u'(f(k) - x) + \beta v'(x) = 0 \tag{5}$$

Which yields

$$u'(f(k) - x) = \beta v'(x) \tag{6}$$

The economics interpretation of this step is that given endowment k, the best x that maximize the $v(k)$ is obtained through smoothing consumption.

Plug the policy function and get

$$u'(f(k) - g(k)) = \beta v'(g(k)) \tag{7}$$

3) Iterate (4) forward to the next period by substituting $g(k)$ into k in (4), get

$$v'(g(k)) = u'(f(g(k)) - g(g(k)))f'(g(k)) \tag{8}$$

You may feel confused on why we need to substitute $g(k)$ into k, as $x = g(k)$. Mathematically this is to help get the RHS of (7), while analytically it is to help iterate toward a fixed point where $k = g(k)$.

4) Combine (7) and (8) by replacing $v'(g(k))$, get the inter-temporal Euler equation on marginal utility.

$$u'(f(k) - g(k)) = \beta u'(f(g(k)) - g(g(k)))f'(g(k)) \tag{9}$$

Q6 from OU Fall 2022 Second midterm Prove Shephard's lemma using the envelope theorem

My solution

$$\min_x c = w \cdot x$$

$$\text{s.t. } y = f(x)$$

The Lagrangian $L(x, \lambda) = w \cdot x + \lambda \cdot (y - f(x))$, The FOC yields

$$\frac{\partial L}{\partial x} = w - \lambda \cdot f'(x) = 0$$

$$y - f(x) = 0$$

Solve and get $\lambda^* = \lambda^*(w, y)$ and $x^* = x^*(w, y)$, where $w - \lambda^* \cdot f'(x^*) = 0$ and $y - f(x^*) = 0$. Then the Lagrangian can be rewritten as

$$L(w, y) = w \cdot x^*(w, y) + \lambda^*(w, y)[y - f(x^*(w, y))]$$

The FOC with respect to w is

$$\begin{aligned}\frac{\partial L}{\partial w} &= (w)' \cdot x^*(w, y) + w \cdot \frac{\partial x^*(w, y)}{\partial w} + \frac{\partial \lambda^*(w, y)}{\partial w}[y - y - f(x^*(w, y))] + \lambda^*(w, y)[-\frac{\partial f(x^*)}{\partial x} \cdot \frac{\partial x^*(w, y)}{\partial w}] \\ &= x^*(w, y) + [w - \lambda^*(w, y)\frac{\partial f(x^*)}{\partial x}] \cdot \frac{\partial x^*(w, y)}{\partial w} + [y - f(x^*(w, y))]\frac{\partial \lambda^*(w, y)}{\partial w}\end{aligned}$$

Since $w - \lambda^* \cdot f'(x) = 0$ and $y - f(x^*) = 0$ by the FOC of the first Lagrangian

$$\frac{\partial L}{\partial w} = x^*(w, y)$$

Therefore,

$$\frac{\partial c(w, y)}{\partial w} = \frac{\partial L}{\partial w} = x^*(w, y).$$