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This is why we defined the conditional value functions the way that we did:

So that we could express the current-period choice probabilities conveniently

 $v_{jt}(X_{it}; \alpha, \beta, \gamma) = u_{jt}(X_{it}; \alpha) + \beta \int \mathbb{E}_{\epsilon} \left\{ \max_{k} v_{kt+1}(X_{it+1}; \alpha, \beta, \gamma) + \epsilon_{ikt+1} \right\} dF_{jt}(X_{it+1}|X_{it}; \gamma)$

Three sets of parameters to estimate:

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 α : Flow utility parameters

 $v_{jt}(X_{it}; \alpha, \beta, \gamma) = u_{jt}(X_{it}; \alpha) + \beta \int \mathbb{E}_{\epsilon} \left\{ \max_{k} v_{kt+1}(X_{it+1}; \alpha, \beta, \gamma) + \epsilon_{ikt+1} \right\} dF_{jt}(X_{it+1}|X_{it}; \gamma)$

 β : Discount factor

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 γ : Parameters governing state transitions

Putting it all together, assume the following:

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Then

$$p_{jt}(X_{it}; \alpha, \beta, \gamma) = \frac{\exp\left[X_{it}\alpha_j + \beta \int \log\left(\sum_k \exp\left(v_{kt+1}\left(X_{it+1}; \alpha, \beta, \gamma\right)\right)\right) dF_{jt}(X_{it+1}|X_{it}; \gamma) + \beta c\right]}{\sum_m \exp\left[X_{it}\alpha_m + \beta \int \log\left(\sum_{k'} \exp\left(v_{k't+1}\left(X_{it+1}; \alpha, \beta, \gamma\right)\right)\right) dF_{mt}(X_{it+1}|X_{it}; \gamma) + \beta c\right]}$$

When taking this model to data, there are two model objects we need to match:
1. Getting the p 's to match the d 's (choices)

2. Getting the mapping between X_t and X_{t+1}

The likelihood function thus incorporates both choice and state transition probabilities:

$$\mathcal{L}(\alpha, \beta, \gamma; X) = \prod_{i} \prod_{t} \prod_{j} \left[p_{jt}(X_{it}; \alpha, \beta, \gamma) f_{jt}(X_{it+1} | X_{it}; \gamma) \right]^{d_{it} = j}$$

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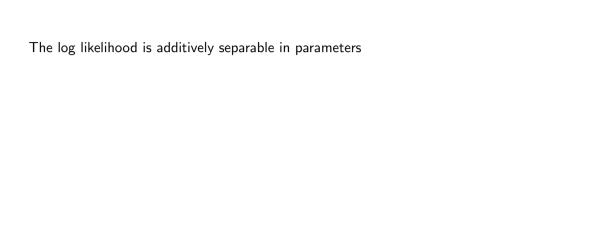
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Taking logs gives the log-likelihood:

$$\ell(\alpha, \beta, \gamma) = \sum_{i} \sum_{t} \sum_{j} (d_{it} = j) \left\{ \log[p_{jt}(X_{it}; \alpha, \beta, \gamma)] + \log[f_{jt}(X_{it+1}|X_{it}; \gamma)] \right\}$$



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Separability simplifies computational burden at cost of statistical efficiency

Bringing in the ba	ackwards recursion	ideas from last	video, algorithm	is as follows:

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- 5. Repeat steps 2–5 until convergence

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- The backwards recursion gives optimal behavior for any given parameter values
- MLE finds which parameter values make that optimal behavior consistent with observed data
- Must solve at all states—not just observed ones—because state transitions depend on choice probabilities at unvisited states