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This is why we defined the conditional value functions the way that we did:

So that we could express the current-period choice probabilities conveniently

Before we get to the likelihood, let's remind ourselves what we're trying to estimate:

$$v_{jt}(X_{it};\alpha,\beta,\gamma) = u_{jt}(X_{it};\alpha) + \beta \int \mathbb{E}_{\epsilon}\left\{\max_{k} v_{kt+1}(X_{it+1};\alpha,\beta,\gamma) + \epsilon_{ikt+1}\right\} dF_{jt}(X_{it+1}|X_{it};\gamma)$$

Three sets of parameters to estimate:

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α: Flow utility parameters

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β: Discount factor

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γ: Parameters governing state transitions

Putting it all together, assume the following:

- $\epsilon_{ijt} \overset{iid}{\sim}$ T1EV
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Then

$$p_{jt}(X_{it};\alpha,\beta,\gamma) = \frac{\exp\left[X_{it}\alpha_j + \beta\int\log\left(\sum_k \exp\left(v_{kt+1}\left(X_{it+1};\alpha,\beta,\gamma\right)\right)\right) dF_{jt}(X_{it+1}|X_{it};\gamma) + \beta c\right]}{\sum_m \exp\left[X_{it}\alpha_m + \beta\int\log\left(\sum_{k'} \exp\left(v_{k'\,t+1}\left(X_{it+1};\alpha,\beta,\gamma\right)\right)\right) dF_{mt}(X_{it+1}|X_{it};\gamma) + \beta c\right]}$$

When taking this model to data, there are two model objects we need to match:

1. Getting the p's to match the d's (choices)

2. Getting the mapping between X_t and X_{t+1}

The likelihood function thus incorporates both choice and state transition probabilities:

$$\mathcal{L}(\alpha, \beta, \gamma; X) = \prod_i \prod_t \prod_j \left[p_{jt}(X_{it}; \alpha, \beta, \gamma) f_{jt}(X_{it+1} | X_{it}; \gamma)\right]^{d_{it}=j}$$

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Taking logs gives the log-likelihood:

$$\ell(\alpha,\beta,\gamma)=\sum_i\sum_t\sum_j(d_{it}=j)\left\{\log[p_{jt}(X_{it};\alpha,\beta,\gamma)]+\log[f_{jt}(X_{it+1}|X_{it};\gamma)]\right\}$$

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Separability simplifies computational burden at cost of statistical efficiency

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4. Evaluate log likelihood $\ell(\alpha, \beta, \gamma)$ and update parameter guesses
5. Repeat steps 2–5 until convergence

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- MLE finds which parameter values make that optimal behavior consistent with observed data
- Must solve at *all* states—not just observed ones—because state transitions depend on choice probabilities at unvisited states