Consider a scenario with only two alternatives and ϵ's that are Type I extreme value

$\epsilon_{ij} \overset{\text{iid}}{\sim}$ Standard T1EV with $F(\epsilon) = e^{-e^{-\epsilon}}$ (CDF) and $f(\epsilon) = e^{-\epsilon}e^{-e^{-\epsilon}}$ (PDF)

Consider a scenario with only two alternatives and ϵ's that are Type I extreme value

$\epsilon_{ij} \overset{\text{iid}}{\sim}$ Standard T1EV with $F(\epsilon) = e^{-e^{-\epsilon}}$ (CDF) and $f(\epsilon) = e^{-\epsilon}e^{-e^{-\epsilon}}$ (PDF)

We want to know the formula for the probability of choosing option 1:

$$P_{i1} = \Pr(u_{i1} + \epsilon_{i1} > u_{i2} + \epsilon_{i2})$$

$$= \Pr(\epsilon_{i2} - \epsilon_{i1} < u_{i1} - u_{i2})$$

$$= \Pr(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2})$$

Start with the probability as a double integral:

$$P_{i1} = \iint I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2}) f(\epsilon_{i1}, \epsilon_{i2}) d\epsilon_{i1} d\epsilon_{i2}$$

Start with the probability as a double integral:

$$P_{i1} = \iint I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2}) \underbrace{f(\epsilon_{i1}, \epsilon_{i2})}_{f(\epsilon_{i1})f(\epsilon_{i2})} d\epsilon_{i1} d\epsilon_{i2}$$

Start with the probability as a double integral:

$$P_{i1} = \iint I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2}) \underbrace{f(\epsilon_{i1}, \epsilon_{i2})}_{f(\epsilon_{i1})f(\epsilon_{i2})} d\epsilon_{i1} d\epsilon_{i2}$$

So we can separate the integrals:

$$P_{i1} = \int f(\epsilon_{i1}) \left[\int I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2}) f(\epsilon_{i2}) d\epsilon_{i2} \right] d\epsilon_{i1}$$

Start with the probability as a double integral:

$$P_{i1} = \iint I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2})\,\underbrace{f(\epsilon_{i1}, \epsilon_{i2})}_{f(\epsilon_{i1})f(\epsilon_{i2})}\,d\epsilon_{i1} d\epsilon_{i2}$$

So we can separate the integrals:

$$P_{i1} = \int f(\epsilon_{i1})\underbrace{\left[\int I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2}) f(\epsilon_{i2}) d\epsilon_{i2}\right]}_{\text{CDF of } \epsilon_{i2}} d\epsilon_{i1}$$

Start with the probability as a double integral:

$$P_{i1} = \iint I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2}) \underbrace{f(\epsilon_{i1}, \epsilon_{i2})}_{f(\epsilon_{i1})f(\epsilon_{i2})} d\epsilon_{i1} d\epsilon_{i2}$$

So we can separate the integrals:

$$P_{i1} = \int f(\epsilon_{i1}) \underbrace{\left[\int I(\epsilon_{i2} < \epsilon_{i1} + u_{i1} - u_{i2}) f(\epsilon_{i2}) d\epsilon_{i2} \right]}_{\text{CDF of } \epsilon_{i2}} d\epsilon_{i1}$$

$$= \int_{-\infty}^{\infty} F(\epsilon_{i1} + u_{i1} - u_{i2}) f(\epsilon_{i1}) d\epsilon_{i1}$$

Substituting in the formulas for $F(\cdot)$ and $f(\cdot)$:

$$P_{i1} = \int_{-\infty}^{\infty} \underbrace{e^{-e^{-(\epsilon_{i1}+u_{i1}-u_{i2})}}}_{\text{T1EV CDF}} \underbrace{e^{-e^{-\epsilon_{i1}}} e^{-\epsilon_{i1}}}_{\text{T1EV PDF}} d\epsilon_{i1}$$

Substituting in the formulas for $F(\cdot)$ and $f(\cdot)$:

$$P_{i1} = \int_{-\infty}^{\infty} e^{-e^{-(\epsilon_{i1}+u_{i1}-u_{i2})}} e^{-e^{-\epsilon_{i1}}} e^{-\epsilon_{i1}} d\epsilon_{i1}$$

Substituting in the formulas for $F(\cdot)$ and $f(\cdot)$:

$$P_{i1} = \int_{-\infty}^{\infty} e^{-e^{-(\epsilon_{i1}+u_{i1}-u_{i2})}} e^{-e^{-\epsilon_{i1}}} e^{-\epsilon_{i1}} d\epsilon_{i1}$$

$$= \int_{-\infty}^{\infty} \exp\left(-e^{-\epsilon_{i1}}[e^{u_{i2}-u_{i1}} + 1]\right) e^{-\epsilon_{i1}} d\epsilon_{i1}$$

Substituting in the formulas for $F(\cdot)$ and $f(\cdot)$:

$$P_{i1} = \int_{-\infty}^{\infty} e^{-e^{-(\epsilon_{i1}+u_{i1}-u_{i2})}} e^{-e^{-\epsilon_{i1}}} e^{-\epsilon_{i1}} d\epsilon_{i1}$$

$$= \int_{-\infty}^{\infty} \exp\left(-e^{-\epsilon_{i1}}[e^{u_{i2}-u_{i1}} + 1]\right) e^{-\epsilon_{i1}} d\epsilon_{i1}$$

Substitution: $t = e^{-\epsilon_{i1}}$, so $dt = -e^{-\epsilon_{i1}} d\epsilon_{i1}$

Bounds: $\epsilon_{i1} : -\infty \to \infty$ becomes $t : \infty \to 0$

After substitution:

$$P_{i1} = \int_{\infty}^{0} \exp\left(-t[e^{u_{i2}-u_{i1}} + 1]\right)(-dt)$$

After substitution:

$$
\begin{aligned}
P_{i1} &= \int_{\infty}^{0} \exp\left(-t[e^{u_{i2}-u_{i1}} + 1]\right)(-dt) \\
&= \int_{0}^{\infty} \exp\left(-t[e^{u_{i2}-u_{i1}} + 1]\right) dt
\end{aligned}
$$

After substitution:

$$\begin{aligned}
P_{i1} &= \int_{\infty}^{0} \exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right)(-dt) \\
&= \int_{0}^{\infty} \exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right) dt \\
&= \left.\frac{\exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right)}{-[e^{u_{i2}-u_{i1}}+1]}\right|_{0}^{\infty}
\end{aligned}$$

After substitution:

$$\begin{aligned}
P_{i1} &= \int_{\infty}^{0} \exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right)(-dt) \\
&= \int_{0}^{\infty} \exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right) dt \\
&= \left.\frac{\exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right)}{-[e^{u_{i2}-u_{i1}}+1]}\right|_{0}^{\infty} \\
&= 0 - \frac{1}{-[e^{u_{i2}-u_{i1}}+1]}
\end{aligned}$$

After substitution:

$$
\begin{aligned}
P_{i1} &= \int_{\infty}^{0} \exp\left(-t[e^{u_{i2}-u_{i1}} + 1]\right)(-dt) \\
&= \int_{0}^{\infty} \exp\left(-t[e^{u_{i2}-u_{i1}} + 1]\right) dt \\
&= \left. \frac{\exp\left(-t[e^{u_{i2}-u_{i1}} + 1]\right)}{-[e^{u_{i2}-u_{i1}} + 1]} \right|_{0}^{\infty} \\
&= 0 - \frac{1}{-[e^{u_{i2}-u_{i1}} + 1]} \\
&= \frac{1}{e^{u_{i2}-u_{i1}} + 1} = \frac{e^{u_{i1}}}{e^{u_{i1}} + e^{u_{i2}}}
\end{aligned}
$$

After substitution:

$$
\begin{aligned}
P_{i1} &= \int_{\infty}^{0} \exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right)(-dt) \\
&= \int_{0}^{\infty} \exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right) dt \\
&= \left.\frac{\exp\left(-t[e^{u_{i2}-u_{i1}}+1]\right)}{-[e^{u_{i2}-u_{i1}}+1]}\right|_{0}^{\infty} \\
&= 0 - \frac{1}{-[e^{u_{i2}-u_{i1}}+1]} \\
&= \frac{1}{e^{u_{i2}-u_{i1}}+1} = \frac{e^{u_{i1}}}{e^{u_{i1}}+e^{u_{i2}}}
\end{aligned}
$$

This generalizes to J alternatives: $P_{ij} = \frac{e^{u_{ij}}}{\sum_k e^{u_{ik}}}$ (see Train 2009, sec. 3.10)