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Period $T-2$:

$$v_{ijT-2} = u_{ijT-2} + \beta \mathbb{E} \max_k \{v_{ikT-1} + \epsilon_{ikT-1} | d_{iT-2} = j\}$$

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General form:

$$v_{ijt} = u_{ijt} + \beta \mathbb{E} \max_k \{v_{ikt+1} + \epsilon_{ikt+1} | d_{it} = j\}$$

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- Choice set: $\{$Physics, Literature, Other$\}$
- X_{it}: GPA, completed courses, job market conditions
- If you choose physics $\rightarrow$ $GPA_{i,t+1}$ may go down (due to harsh grading)
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Key insight: Today's major choice changes tomorrow's opportunities

Past choices create path dependence through skill accumulation

Conditional value function:

$$v_{jt}(X_{it}) = u_{jt}(X_{it}) + \beta \int \mathbb{E}_{\epsilon} \left\{ \max_{k} v_{kt+1}(X_{it+1}) + \epsilon_{ikt+1} \right\} dF_{jt}(X_{it+1}|X_{it})$$

Let's break down each component:

Conditional value function:

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$v_{jt}(X_{it})$: Conditional value of choosing j today given state X_{it}

Conditional value function:

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$u_{jt}(X_{it})$: Flow utility today from choosing j

Conditional value function:

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β: Discount factor

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$\int$: Integral over all possible future states X_{it+1}

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$\mathbb{E}_\epsilon \left\{ \max_k v_{kt+1}(X_{it+1}) + \epsilon_{ikt+1} \right\}$: Expected future value (integrated over pref. shocks)

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$dF_{jt}(X_{it+1}|X_{it})$: State transition probability (depends on today's choice j)

Also assumes Markov property: future states depend only on current state and choice, not entire past history

Notation: $dF_{jt}(X_{it+1}|X_{it}) = f_{jt}(X_{it+1}|X_{it})dX_{it+1}$ where $f(\cdot)$ is the PDF

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Note that the integral could turn into a summation if we discretize $F_{jt}(\cdot)$

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$$\mathbb{E}_\epsilon \left\{ \max_k v_{kt+1}(X_{it+1}) + \epsilon_{ikt+1} \right\} = \log \left(\sum_k \exp\left(v_{kt+1}(X_{it+1})\right) \right) + \underbrace{c}_{\text{Euler's constant}}$$

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Computational challenge: State space grows with T

Various simplifying assumptions and workarounds can sidestep this challenge