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$$\sum_{X_{t+1}} f_k(X_{t+2}|X_{t+1})f_j(X_{t+1}|X_t) = \sum_{X_{t+1}} f_k(X_{t+2}|X_{t+1})f_{j'}(X_{t+1}|X_t) \quad \forall j, j'$$

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We can use renewal, along with clever normalizations, to eliminate recursion

... Even when our choice set does not contain a terminal option

Why? Because CCPs don't have to correspond to optimal decisions

Normalize future value term relative to renewal action for choice j:

$$v_{jt}(X_t) = u_j(X_t) + \beta \sum_{X_{t+1}} \left[v_{kt+1}(X_{t+1}) - \log(p_{kt+1}(X_{t+1}))\right] f_j(X_{t+1}|X_t) + \beta c$$

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Expanding $v_{kt+1}(X_{t+1})$:

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Substituting in yields:

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For choice j', normalize FV term relative to choice k:

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Renewal property: $V_{t+2}(X_{t+2})$ terms cancel out in differences

$$\begin{aligned} v_{jt}(X_t) - v_{j't}(X_t) = {} & u_j(X_t) - u_{j'}(X_t) \\ & + \beta \sum_{X_{t+1}} \left[u_k(X_{t+1}) - \log(p_{kt+1}(X_{t+1}))\right] f_j(X_{t+1}|X_t) \\ & - \beta \sum_{X_{t+1}} \left[u_k(X_{t+1}) - \log(p_{kt+1}(X_{t+1}))\right] f_{j'}(X_{t+1}|X_t) \end{aligned}$$

Let's revisit the Rust (1987) bus engine model—two choices with flow payoffs:

$$u(X_t, d_t, \theta) = \begin{cases} -c(X_t, \theta) & \text{if } d_t = 0 \\ -[\overline{P} - \underline{P} + c(0, \theta)] & \text{if } d_t = 1 \end{cases}$$

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Replacing the engine sets $X_{t+1} = 0$ regardless of $X_t \implies j = 1$ is renewal

Only need one-period-ahead replacement probability for future value

Value functions (choice 1 is renewal, $k = 1$):

$$v_1(X) = u_1(X) + \beta \left[v_1(0) - \log(p_1(0))\right] + \beta c \qquad \text{(replace)}$$

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Together: both current flow utility and all future values are independent of X

Taking differences:

$$\begin{aligned}
v_1(X) - v_0(X) = {} & u_1(X) - u_0(X) \\
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Since $v_1(X') = v_1(0)$ for all X':

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Estimation: logit with adjustment term, calculate p_1 and $f(X'|X)$ in first stage