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We can mathematically express this choice in two different ways:

- d_i is integer-valued, $d_i \in \{1, \ldots, J\}$; ($J = 2$ in this case)
- d_{ij} is binary, $d_{ij} \equiv 1[d_i = j]$

Assuming $\epsilon_1, \epsilon_2 \overset{iid}{\sim} T1EV$ gives the likelihood of choosing 1 and 2:

$$P_{i1} = \frac{\exp(u_{i1} - u_{i2})}{1 + \exp(u_{i1} - u_{i2})}$$
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We can view the event $d_i = j$ as a weighted coin flip

This gives us a random variable that follows the Bernoulli distribution

Supposing our sample is of size N, the likelihood function is:

$$\mathcal{L}\left(X, Z, d; \beta, \gamma\right) = \prod_{i=1}^{N} P_{i1}^{d_{i1}} P_{i2}^{d_{i2}}$$

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$$= \prod_{i=1}^{N} \left(\frac{\exp((\beta_1 - \beta_2)X_i + \gamma(Z_1 - Z_2))}{1 + \exp((\beta_1 - \beta_2)X_i + \gamma(Z_1 - Z_2))} \right)^{d_{i1}} \times \left(\frac{1}{1 + \exp((\beta_1 - \beta_2)X_i + \gamma(Z_1 - Z_2))} \right)^{(1-d_{i1})}$$

For many reasons, it's better to maximize the log likelihood function

Taking the log gives:

$$\ell\left(X, Z, d; \beta, \gamma\right) = \sum_{i=1}^{N} d_{i1} \log P_{i1} + \left(1 - d_{i1}\right) \log\left(1 - P_{i1}\right)$$

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$$= \sum_{i=1}^{N} \sum_{j=1}^{2} d_{ij} \log P_{ij}$$

Expanding the log likelihood:

$$\ell(X, Z, d; \beta, \gamma) = \sum_{i=1}^{N} \{ d_{i1} \left[\log\left(\exp(u_{i1} - u_{i2})\right) - \log\left(1 + \exp(u_{i1} - u_{i2})\right)\right] + (1 - d_{i1}) \left[\log(1) - \log\left(1 + \exp(u_{i1} - u_{i2})\right)\right] \}$$

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&\qquad \left. (1 - d_{i1}) \left[\log(1) - \log\left(1 + \exp(u_{i1} - u_{i2})\right) \right] \right\} \\
&= \sum_{i=1}^{N} \left\{ d_{i1} \left[u_{i1} - u_{i2} \right] - \log\left(1 + \exp(u_{i1} - u_{i2})\right) \right\} \\
&= \sum_{i=1}^{N} \left\{ d_{i1} \left[(\beta_1 - \beta_2) X_i + \gamma(Z_1 - Z_2) \right] - \right. \\
&\qquad \left. \log\left(1 + \exp((\beta_1 - \beta_2) X_i + \gamma(Z_1 - Z_2))\right) \right\}
\end{aligned}
$$

We can then estimate $(\beta_1 - \beta_2)$ and γ using any nonlinear optimizer