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Question: What remains identifiable?

With terminal or renewal choices, can identify f_{jt}'s and u_{jt}'s until period $\mathcal{T} - 1$

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Extreme value case:

$$v_j(X_t) = u_{jt}(X_t) + \beta \sum_{X_{t+1}} \log\left(\sum_k \exp[v_{kt+1}(X_{t+1})]\right) f_{jt}(X_{t+1}|X_t) + \beta c$$

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Last term differences out or is constant

Expression holds *regardless* of expectations at $t+2$

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- Reason: future flow payoffs appear in expectation of future utility, but we lack corresponding data to recover them
- Implication: possible to estimate non-stationary games

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If ϵ's are Normal:

- No closed-form expression for V_{t+1}
- Need simulation to compute $\mathbb{E}\max$ integral

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Probability of choosing j in nest J_r:

$$p_{jt}(X_t) = \frac{\left(\sum_{j' \in J_r} \exp\left(\frac{v_{j'}(X_t)}{\lambda_r} \right) \right)^{\lambda_r - 1} \exp\left(\frac{v_j(X_t)}{\lambda_r} \right)}{G_t}$$

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Probability of choosing nest r:

$$p_{rt}(X_t) = \frac{\left(\sum_{j' \in J_r} \exp\left(\frac{v_{j'}(X_t)}{\lambda_r} \right) \right)^{\lambda_r}}{G_t}$$

From nest probability:

$$\left(G_t p_{rt}(X_t)\right)^{1/\lambda_r} = \sum_{j' \in J_r} \exp\left(\frac{v_{j'}(X_t)}{\lambda_r}\right)$$

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Simplify:

$$p_{jt}(X_t) = G_t^{\frac{-1}{\lambda_r}} p_{rt}(X_t)^{\frac{\lambda_r - 1}{\lambda_r}} \exp\left(\frac{v_j(X_t)}{\lambda_r}\right)$$

Take logs and rearrange:

$$(1/\lambda_r)\log(G_t) = -\log(p_{jt}(X_t)) + ((\lambda_r - 1)/\lambda_r)\log(p_{rt}(X_t)) + (1/\lambda_r)v_j(X_t)$$

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Note: when $\lambda_r = 1$, this reduces to multinomial logit

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One equation redundant $\Rightarrow J-1$ system solving for $J-1$ differences in v's