Reviewing from the previous video:

When s is unobserved, the log likelihood function is not additively separable:

$$\ell = \sum_{s} \log \left( \sum_{s} \pi_{s} \prod_{t} \mathcal{L}(Y_{1t}|X_{1t}, \alpha_{1}, s) \mathcal{L}(Y_{2t}|X_{2t}, \alpha_{2}, s) \right)$$

where  $\mathcal{L}$  is a likelihood function

### FINITE MIXTURE DISTRIBUTIONS, SEQUENTIAL LIKELIHOOD

### By Peter Arcidiacono and John Bailey Jones<sup>1</sup>

A popular way to account for unobserved heterogeneity is to assume that the data are driven from a finite instruct edistribution. Abstrair to using finite instruct models is that parameters that could previously be estimated in stages must now be estimated pitally, which is the stage of the

KEYWORDS: Unobserved heterogeneity, mixture distributions, EM algorithm, dynamic discrete choice.

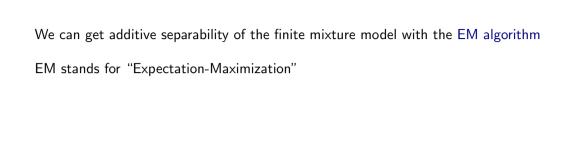
#### 1. INTRODUCTION

ONE WAY TO ACCOUNT for unobserved heterogeneity in data, and the related problem of self-selection, is to assume that the data are drawn from a finite mixture distribution. Under this approach, each observation is assumed to belong to one of several different "types," each of which has its own distribution. While the connentrician does not observe each observation's type, if her model is sufficiently structured she can infer it by applying Bawes' Thocrem.

Models with finite mixtures have appeared in numerous applications. In labor comonies, Kaena and Wolpin (1997) and Eckstein and Wolpin (1999) use mixtures to control for person-specific differences in models of dynamic discrete choice. Finite mixture models from the basis of Hamilton's (1989, 1999) influential regime-switching model of economic time series. A particularly important application has been to use finite mixture models an onoparametric approximations to more general maxime models. Important papers in this vein include Laird (1978), Lindsuy (1983), and Heckman and Singer (1984). More creenly, Cameron and Heckman (1989, 2001) use this sort of nonparametrie maximum likelihood estimation to study the effect of family background on colocational achievement of the control of the contr

<sup>&</sup>lt;sup>1</sup> We thank Donald Andrews, Arie Beresteanu, Mark Coppejans, Michael McCracken, Tom Mroz, Barbara Rossi, Wilbert van der Klaauw, and two anonymous referees for valuable comments.

Although we focus on economic applications, finite mixture models have been used widely in other fields as well. Titterington, Smith, and Makov (1985) and McLachlan and Peel (2000) provide lists.



We can get additive separability of the finite mixture model with the EM algorithm

EM stands for "Expectation-Maximization"

The algorithm iterates on two steps:

**E-step:** estimate parameters of the mixing distribution (the  $\pi$ 's)

**M-step:** pretend you observe the unobserved variable and estimate the  $\alpha$ 's

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The EM algorithm is used in other applications to fill in missing data

In this case, the missing data is the permanent unobserved heterogeneity

With the EM algorithm, the non-separable likelihood function

$$\ell = \sum_{i} \log \left( \sum_{s} \pi_{s} \prod_{t} \mathcal{L}(Y_{1t}|X_{1t}, \alpha_{1}, s) \mathcal{L}(Y_{2t}|X_{2t}, \alpha_{2}, s) \right)$$

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can be written in a form that is separable:

$$\ell = \sum_i \sum_s q_{is} \sum_t \left\{ \ell_1(Y_{1t}|X_{1t},\alpha_1,s) + \ell_2(Y_{2t}|X_{2t},\alpha_2,s) \right\}$$

where  $q_{is}$  is the (posterior) probability that i belongs to group s

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**Stage 2 of M-step:** estimate  $\ell(Y_{2t}|X_{2t},\alpha_2,s)$  weighting by the q's

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( 21/ 22, 32, 3)

**E-step:** update the q's by calculating

$$q_{is} = rac{\pi_s \prod_t \mathcal{L}(Y_{1t}|X_{1t},lpha_1,s)\mathcal{L}(Y_{2t}|X_{2t},lpha_2,s)}{\sum_t \pi_m \prod_t \mathcal{L}(Y_{1t}|X_{1t},lpha_1,m)\mathcal{L}(Y_{2t}|X_{2t},lpha_2,m)}$$

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Iterate on E and M steps until the q's converge

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Both issues (local optima and estimation error) are problem-specific

You need to understand your specific case