SCHOOL OF COMPUTER TECHNOLOGY

AASD 4001
Mathematical Concepts for Machine Learning
Lecture 5

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Session 5

- Fourier Transforms in Signal and Image processing
  - Introduction
  - 1D and 2D Fourier transforms
  - Filtering in the frequency domain

Intro to Fourier transform and frequency domain

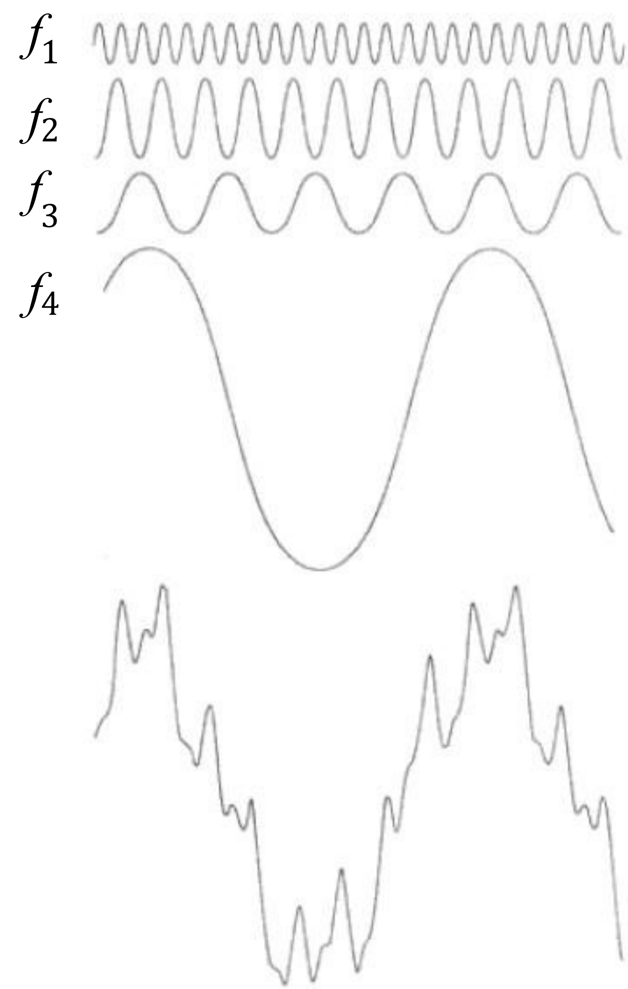
Fourier's contribution (the analytic Theory of Heat, 1822):

> Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies, each multiplied by a different coefficient.

The concept that complicated functions could be represented as a sum of simple sines and cosines was not at all intuitive. When it first appeared, it was a revolutionary concept which took mathematicians over a century to "adjust".

Even functions that are not periodic but whose area under the curve is finite can be expressed as the integral of sines and cosines multiplied by a weighing function. The formulation in this case is ***the Fourier transform***, and its utility is even greater than the Fourier series in most practical applications.

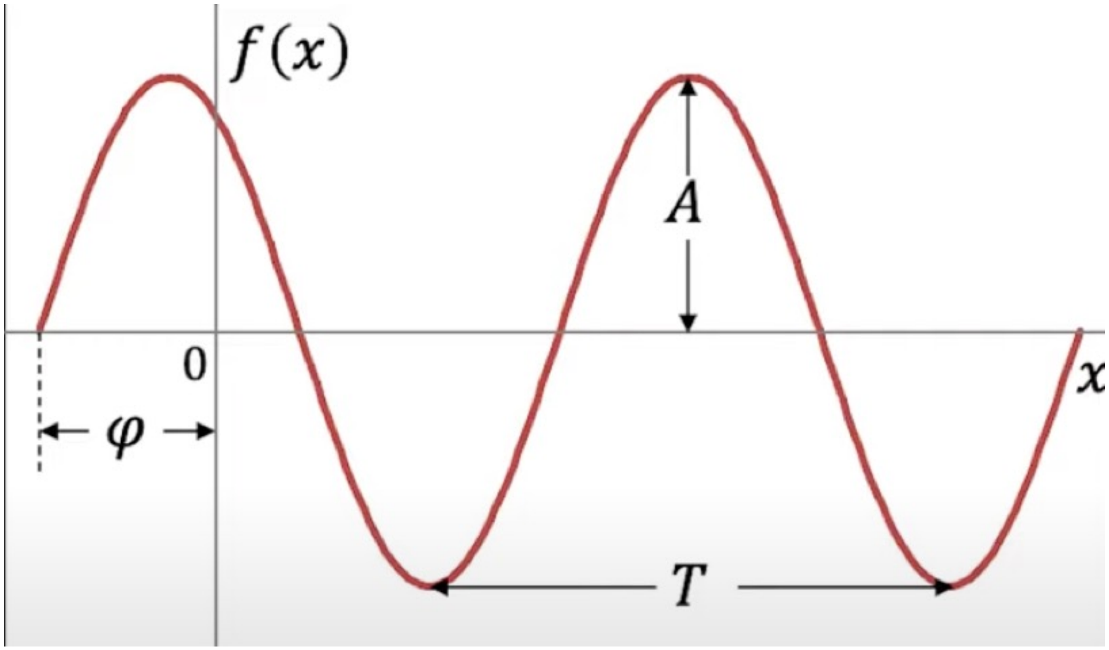
This core technology allowed for the first time practical processing and meaningful interpretation of a wide range of signals, from medical monitors and scanners to modern electronic communication.



$$a \cdot f_1 + b \cdot f_2 + c \cdot f_3 + d \cdot f_4$$

Sinusoid

$$f(x) = A\sin(2\pi u x + \varphi)$$



A: amplitude
u: ordinary frequency, the number of cycles per unit of time (Hertz) or distance
$\omega = 2\pi u$: angular frequency, radians per unit of time/distance
$T = 2\pi/(2\pi u) = 1/u$: period
φ: phase

The 1D Fourier Transform

The Fourier transform of a single variable, continuous function $f(t)$ is defined by the equation

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt \qquad (1)$$

where $j = \sqrt{-1}$. Conversely, given $F(u)$, we can obtain $f(t)$ by the inverse Fourier transform

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ut} du \qquad (2)$$

These two equations comprise the Fourier transform pair.

- The function can be recovered from its Fourier transform.
- The domain of the Fourier transform is the frequency domain. If t is in seconds, u is Hertz $(1/second)$.

The Fourier transform of a discrete function of one variable $f(t)$, $t = 0, 1, 2, \ldots, M-1$, where M is the number of samples, is given by

$$F(u) = \frac{1}{M} \sum_{t=0}^{M-1} f(t) e^{-j2\pi ut/M} \qquad \text{for } u = 0, 1, 2, \ldots, M-1$$

We can obtain the original function back using the inverse DFT

$$f(t) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ut/M} \qquad \text{for } t = 0, 1, 2, \ldots, M-1$$

1D Discrete Fourier transform

How do we compute $F(u) = \frac{1}{M}\sum_{t=0}^{M-1} f(t)e^{-\frac{j2\pi ut}{M}}$?

Firstly, we substitute $u=0$ and sum for all values of $f(t)$. We then substitute $u=1$ in the exponential and repeat the summation over all values of t. We repeat this process for all values of u in order to obtain the complete Fourier transform. It takes approximately M^2 summations and multiplications to compute DFT.

Is the transform $F(u)$ a discrete quantity? Does it have the same number of components as $f(t)$?

The concept of the frequency domain follows directly from Euler's formula

$$e^{j\theta} = cos\theta + j \cdot sin\theta$$

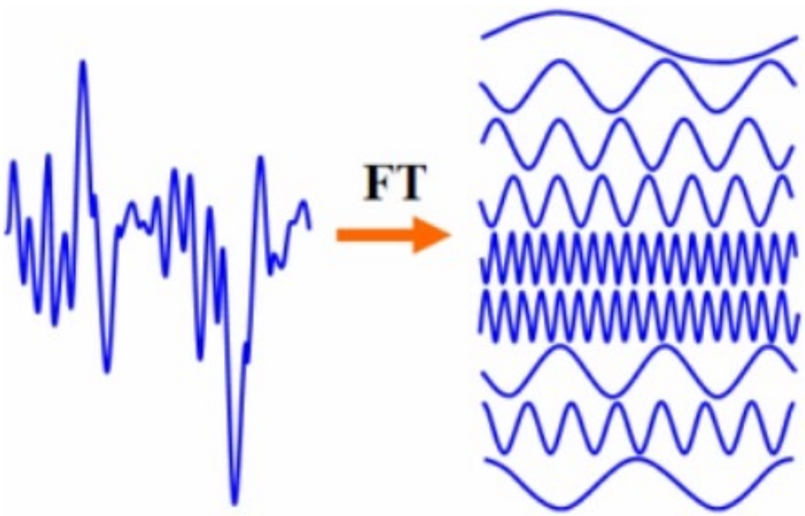
Substituting this formula into the equation for $F(u)$ and knowing that $cos(-\theta) = cos\theta$ gives us

$$F(u) = \frac{1}{M}\sum_{t=0}^{M-1} f(t)(\cos\frac{2\pi ut}{M} - j\sin\frac{2\pi ut}{M}) \qquad \text{for } u = 0, 1, 2, ..., M-1$$

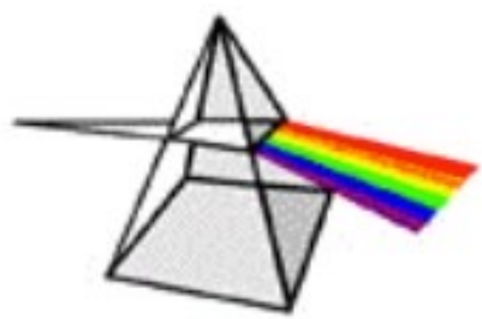
Each term of the Fourier transform is composed of the sum of all values of the function $f(t)$. The values of $f(t)$ are multiplied by sines and cosines of various frequencies. The domain (values of u) over which the values of u range is called **the frequency domain**, because u determines the frequency of the components of the transform. Each of the M terms of $F(u)$ is called a **frequency component** of the transform.

1D Discrete Fourier transform

The Fourier transform is analogous to a glass prism. The prism is a physical device that separates light into various colour components each depending on its wavelength content. When we consider light, we talk about its spectrum or frequency content. Likewise, the Fourier transform acts like a "mathematical prism" that separates a function into different components, each based on frequency. The Fourier transform characterizes a function by its frequency content.



Mathematical prism



Glass prism

Important DFT quantities

From equation $F(u) = \frac{1}{M}\sum_{t=0}^{M-1} f(t)(\cos\frac{2\pi ut}{M} - j\sin\frac{2\pi ut}{M})$ it follows that the components of the Fourier transform are complex quantities. Let's express $F(u)$ in exponential coordinates

$$F(u) = |F(u)|e^{-j\phi(u)}$$

where $|F(u)| = [R^2(u) + I^2(u)]^{1/2}$ is called the magnitude or **spectrum** of the Fourier transform. $R(u)$ and $I(u)$ are real and imaginary parts of $F(u)$, respectively. The properties of the spectrum are used for image enhancement.

Another quantity that we will rely on is the **power spectrum** also referred to as **spectral density is** defined as the square of the Fourier spectrum.

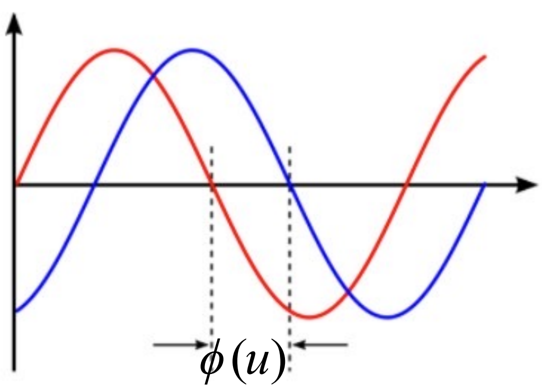
$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

and

$$\phi(u) = tan^{-1}\frac{I(u)}{R(u)}$$

Is phase angle or **phase spectrum** of the transform.

It determines how the sinusoids line up relative to one another to form $f(t)$.

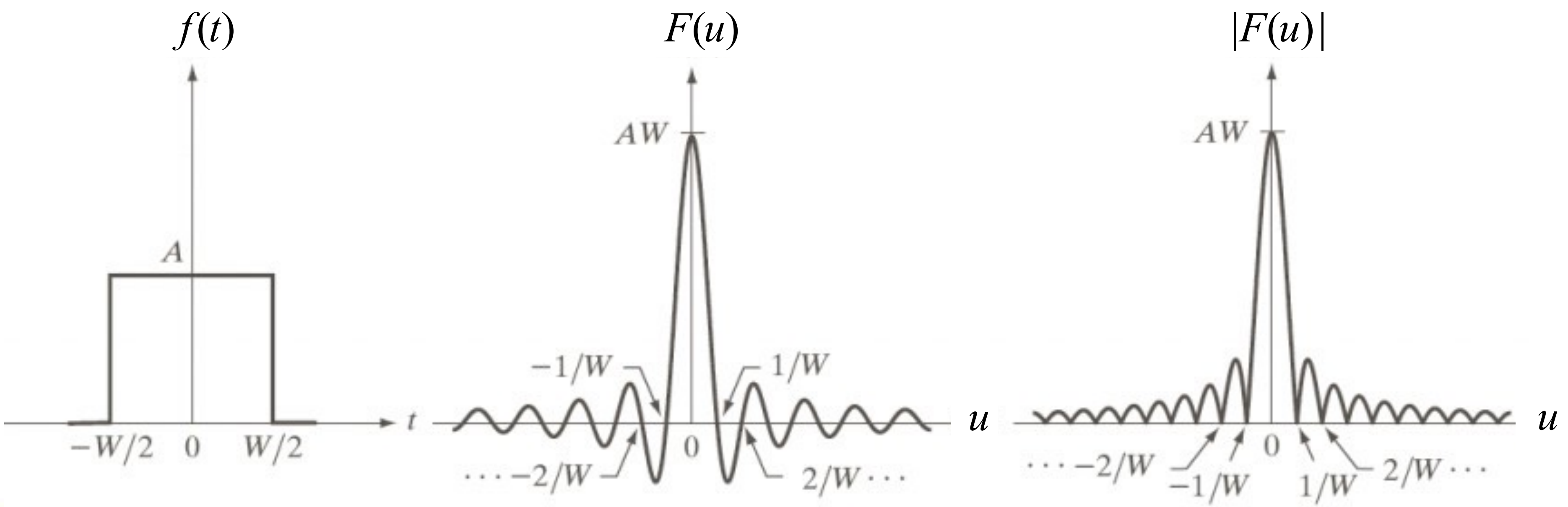


Example of 1D Fourier transform

Fourier transform of the box function is the sinc function. In this case the Fourier transform is a real-valued function. Both $f(t)$ and $F(u)$ could be discrete functions (the points on the plots are linked).

It is important to keep in mind that samples $f(t)$ are not necessarily taken at integer values of t in a finite interval. These can be equally spaced floating-point numbers.

$$F(u) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ut}dt = Aw\,[\mathrm{sinc}(uw)] \quad ; \quad \mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



2D DFT and its Inverse

Extension of 1D Fourier transform and its inverse 2 dimensions is straightforward. The discrete Fourier transform of $f(x, y)$ of size M x N is given by equation

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \qquad (1)$$

The expression must be computed for values $u = 0, 1, 2, \ldots, M-1$ and also for $v = 0, 1, 2, \ldots, N-1$.

Given $F(u, v)$, we obtain $f(x, y)$ via the inverse Fourier transform, given by the expression

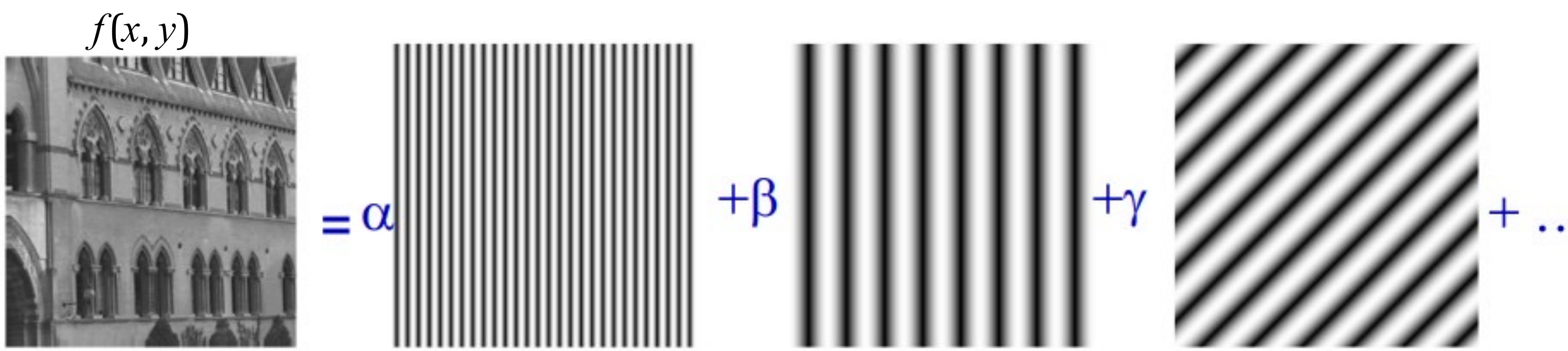
$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})} \qquad (2)$$

for $x = 0, 1, 2, \ldots, M-1$ and $y = 0, 1, \ldots, N-1$

Equations (1)-(2) comprise the 2D discrete Fourier transform pair. The variables u and v are frequency variables and x and y are spatial or image variables.

Visual interpretation of 2D DFT

$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$ is decomposed into a weighted sum of 2D functions using scalar products.



2D Discrete Fourier Transform

We define the 2D Fourier spectrum

$$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

phase angle

$$\phi(u,v) = tan^{-1} \frac{I(u,v)}{R(u,v)}$$

and power spectrum as

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

where $R(u,v)$ and $I(u,v)$ are the real and imaginary parts of $F(u,v)$.

It is a common practice to multiply the input image function by $(-1)^{x+y}$ prior to computing the Fourier transform. It has been shown mathematically that

$$F(f(x,y)(-1)^{x+y}) = F(u-M/2\,,\, v-N/2)$$

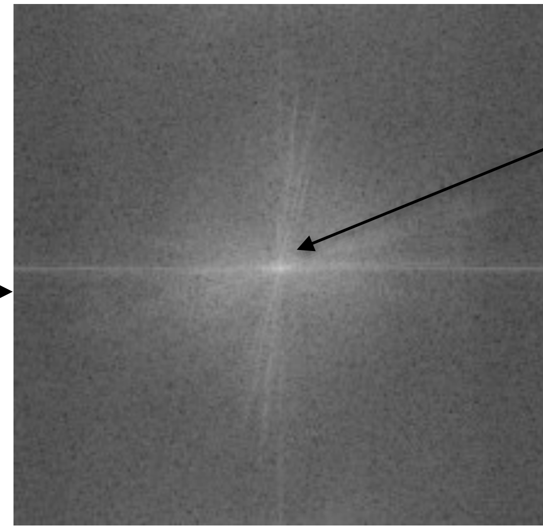
The equation states that the origin of the Fourier transform of $f(x,y)(-1)^{x+y}$ is located at $u=M/2$ and $v=N/2$, which is the centre of the M x N area occupied by the 2D DFT. We refer to this area as the **frequency domain.**

2D DFT Magnitude and Phase

y

x

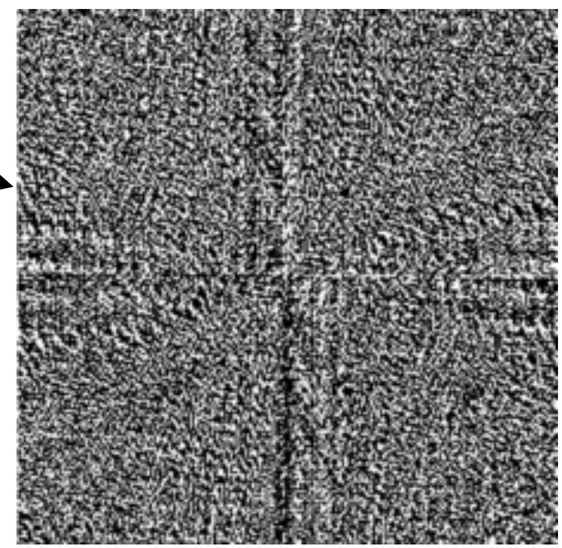
$f(x, y)$



$(u, v) = (0, 0)$

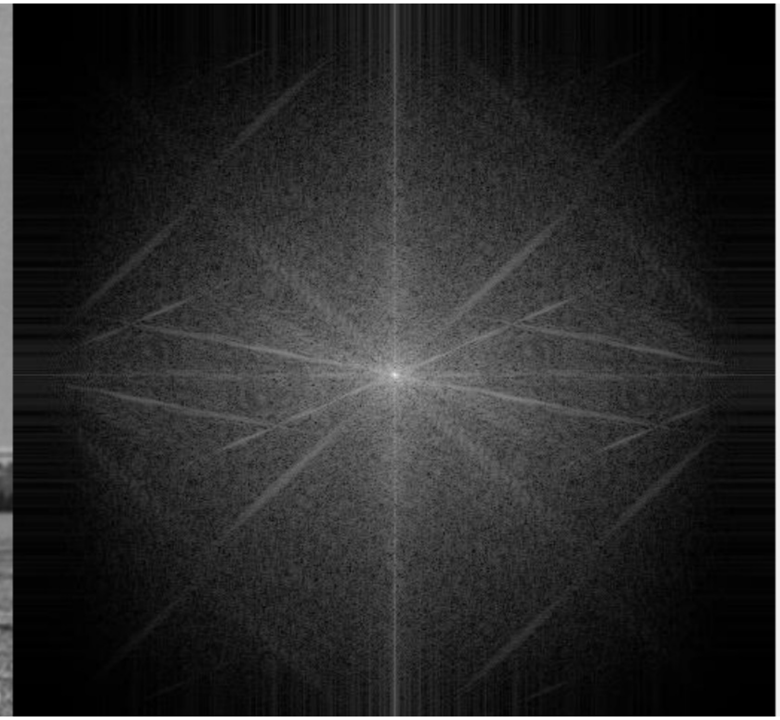
$|F(u, v)|$

Phase $F(u, v)$



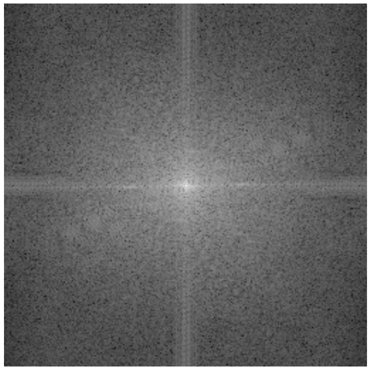
Does phase appear informative?
$|F(u, v)|$ decreases with higher frequencies. Hence, low frequencies contain more image information than the higher ones. Higher frequencies correspond to faster gray level changes in the image.

2D DFT Magnitude and Phase

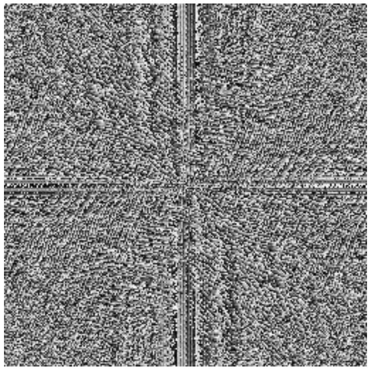


2D DFT Magnitude and Phase

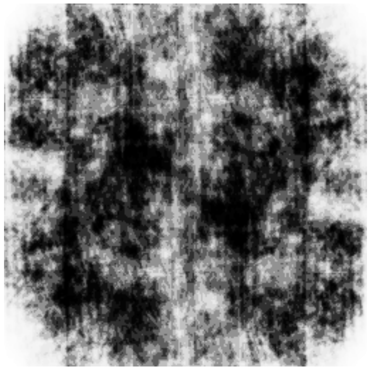
Image



FT-Magnitude



FT-Phase



Inverse FT
Ignoring the phase

Lowpass filter

Highpass filter

2D DFT important properties

1. The value of the Fourier Transform $F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$ at $(u, v) = (0,0)$

is $F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$

which is the average of $f(x, y)$. If $f(x, y)$ is an image, then the value of its Fourier transform at the origin is equal to the average gray level of the image. Because both frequencies are 0 at the origin, $F(0,0)$ sometimes is called the *dc* component of the spectrum. This terminology is from electrical engineering where "dc" refers to direct current.

2. The spectrum of a Fourier transform is symmetric about origin

$$|F(u, v)| = |F(-u, -v)|$$

Filtering in the frequency domain

Filtering in the frequency domain is straightforward and it can be illustrated by the following diagram.

Fourier transform → Filter function $H(u, v)$ → Inverse Fourier transform

$F(u, v)$

$H(u, v)F(u, v)$

Pre-processing

Post-processing

$H(u, v)$ is called a filter since it suppresses certain frequencies in the transform while Leaving others unchanged.

The FT of the output (filtered) image is given by

$$G(u, v) = H(u, v)F(u, v)$$

The multiplication is defined on element-by element basis. Each component of H multiplies both, real and imaginary parts of the corresponding component F. The filtered image is obtained by taking the inverse FT of $G(u, v)$:

$$g(x, y) = F^{-1}\{G(u, v)\}$$

$f(x, y)$

Input image

$g(x, y)$

Enhanced image

Basic filters and their properties

Notch filter

Used to remove repetitive "Spectral" noise from an image

A notch filter is a filter that contains nulls in its frequency response.

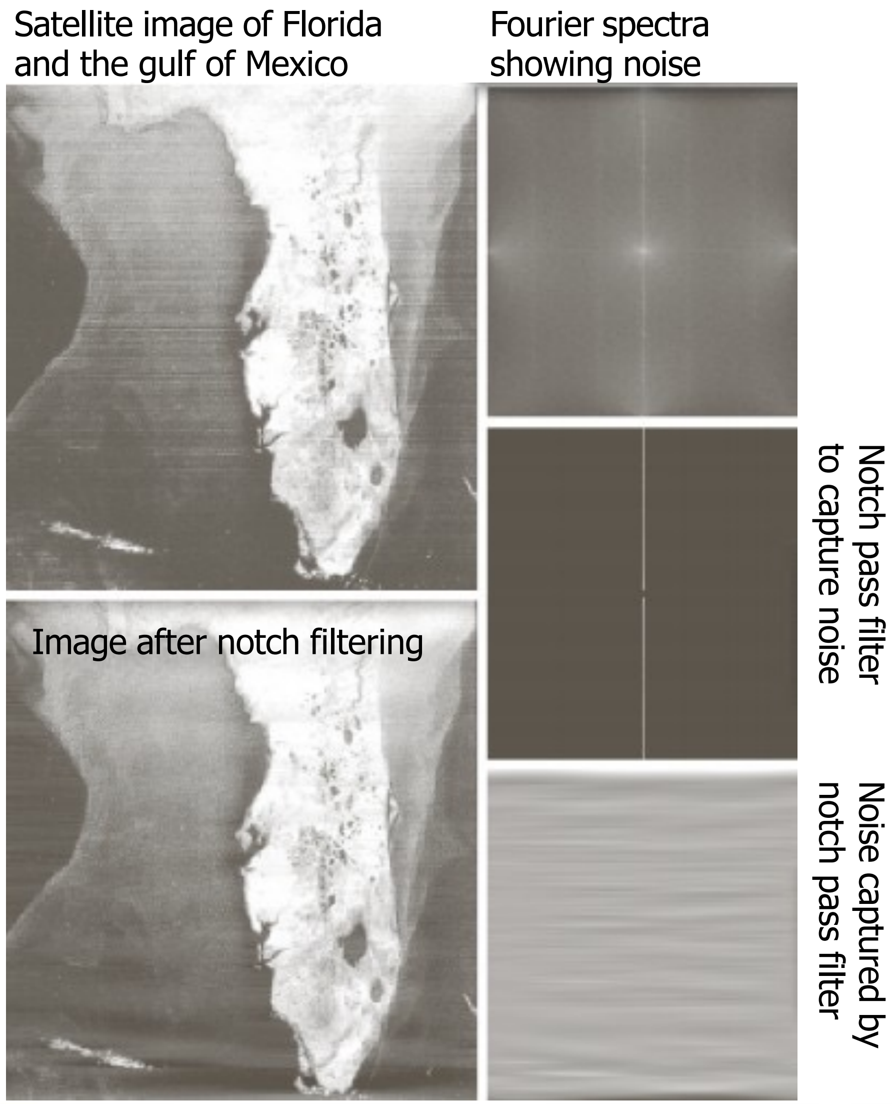
They are used in many applications where specific frequency components must be eliminated.

Assuming that the FT has been centered, we can mathematically define the Notch filter for an illustration on the right- hand side as

$$H = \begin{cases} 0 & \text{if } v = 0, \\ 1 & \text{otherwise} \end{cases}$$

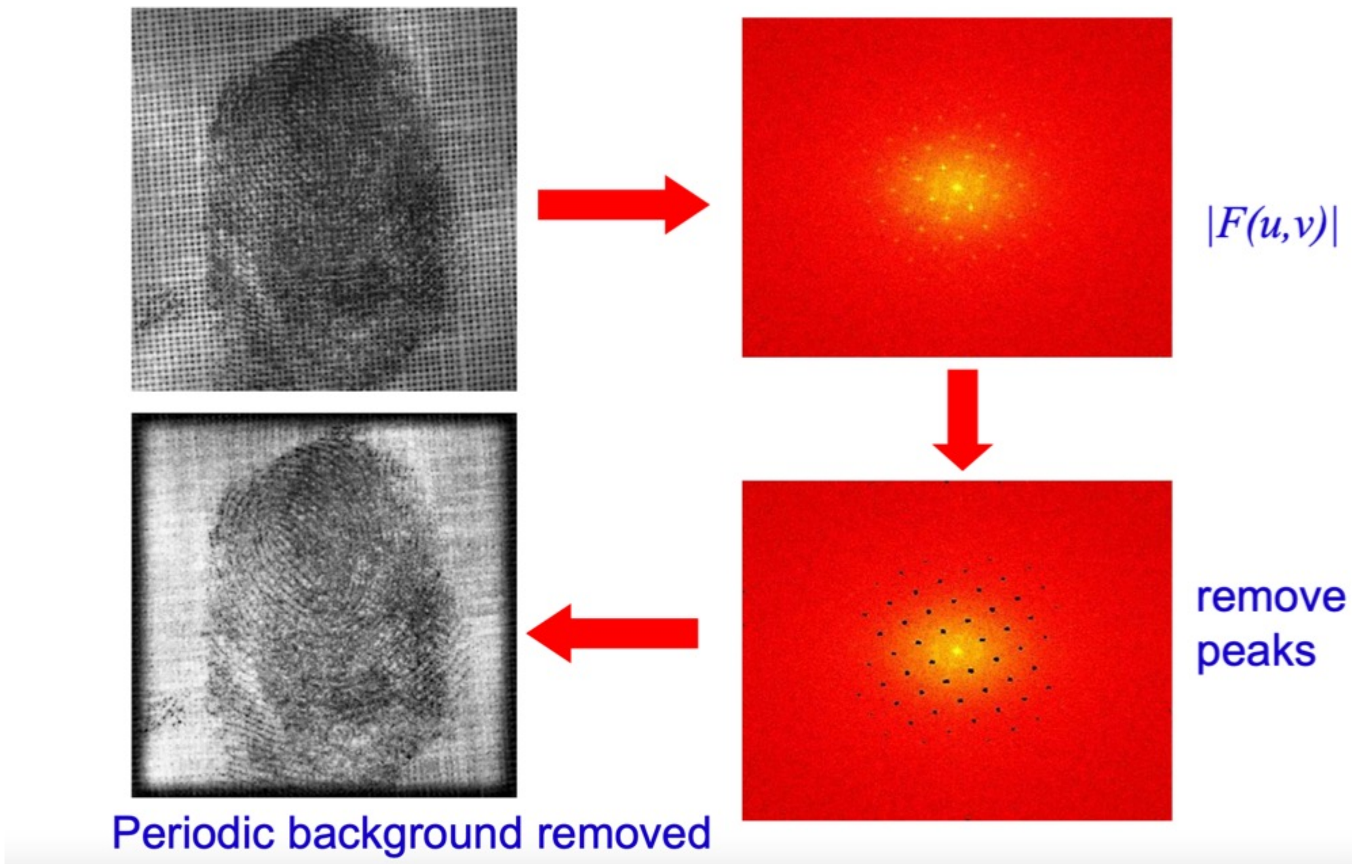
The filter would set the central vertical line to 0 and leave all other frequency components untouched.

How can we obtain the processed image?



Applications: image processing

Forensic Application

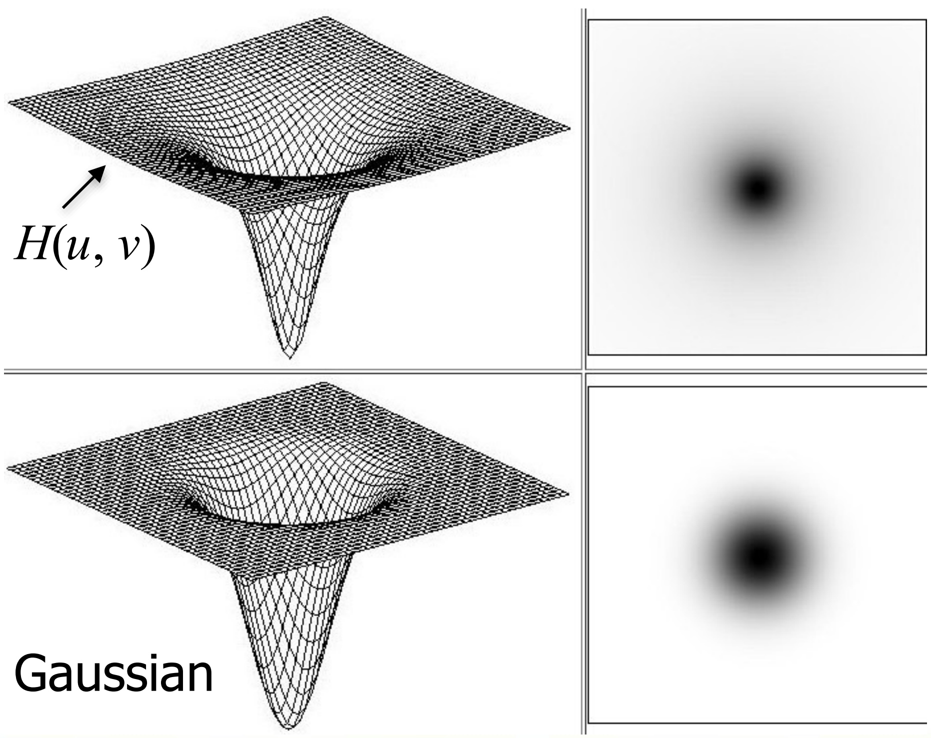


Low-pass and high-pass 2D filters

These filters can be created directly in the frequency domain. Low-pass filters are known as smoothing filters. High-pass filters are known as sharpening filters.

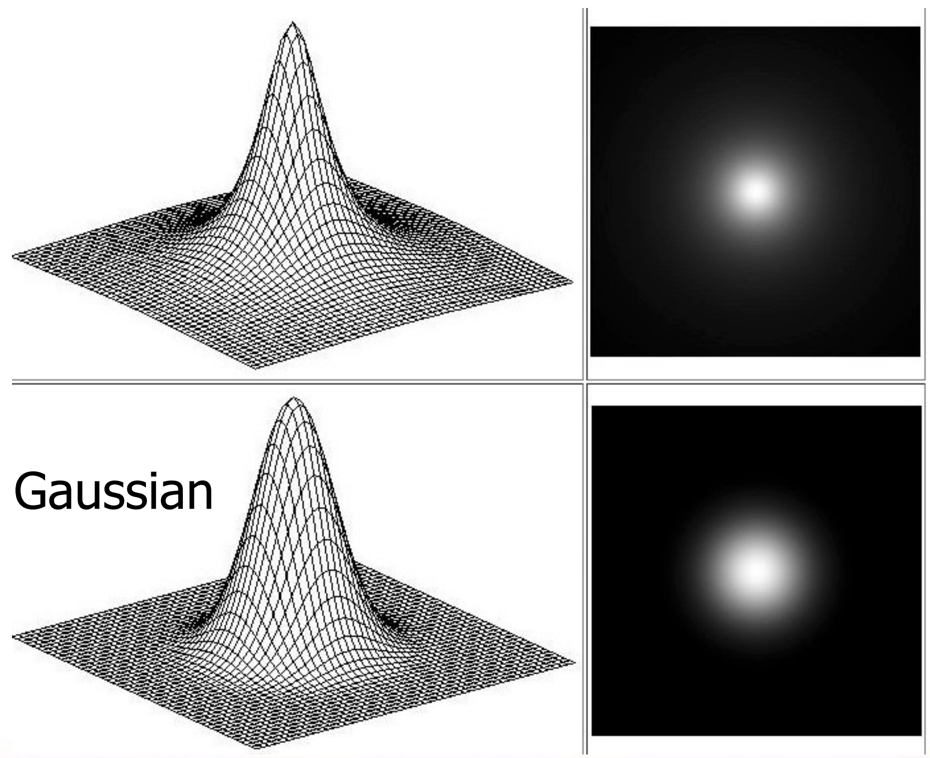
High-pass filter examples

Butterworth $|H(u,v)|$

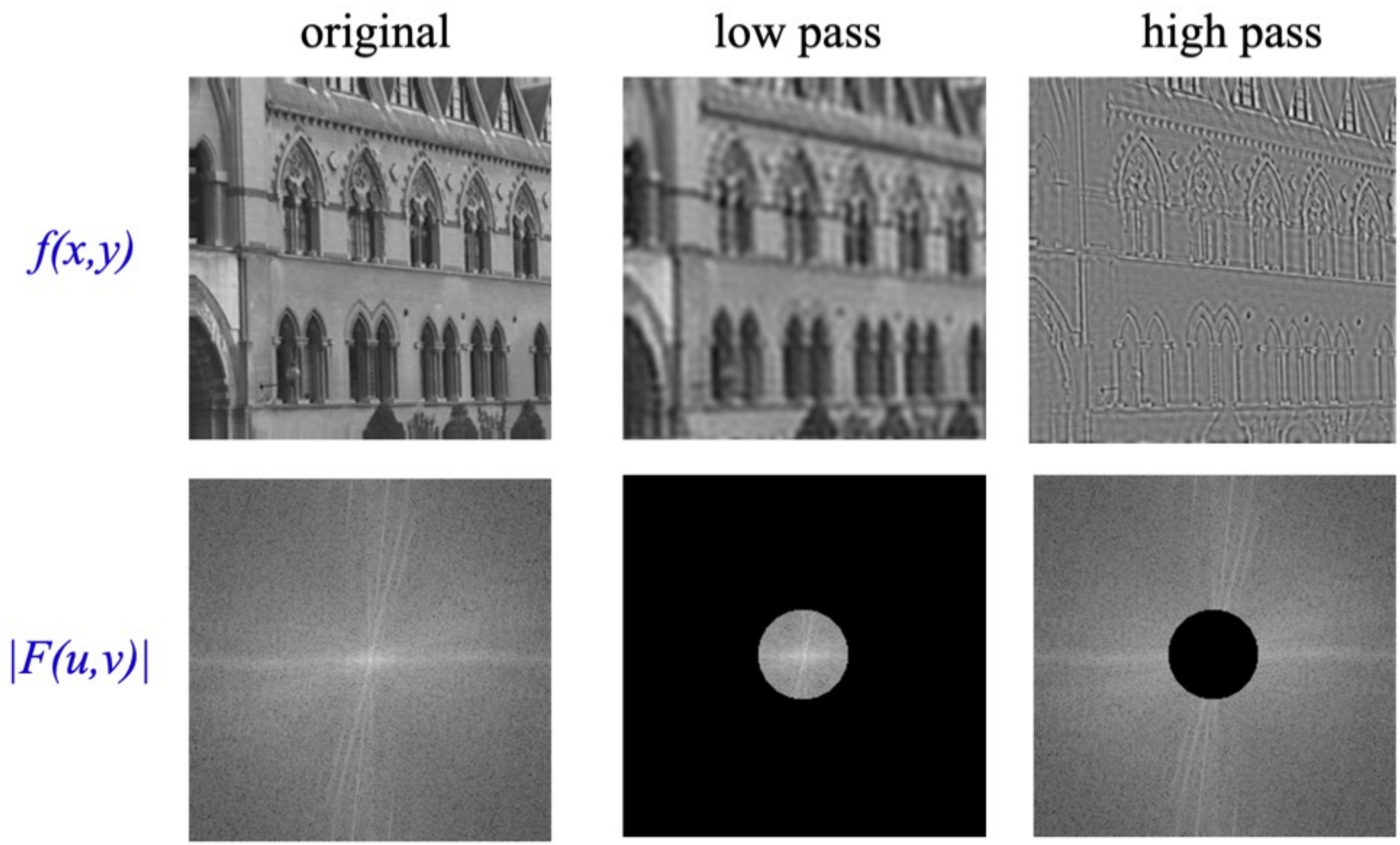


Low-pass filter examples

Butterworth $|H(u,v)|$



Low-pass and high-pass 2D filters

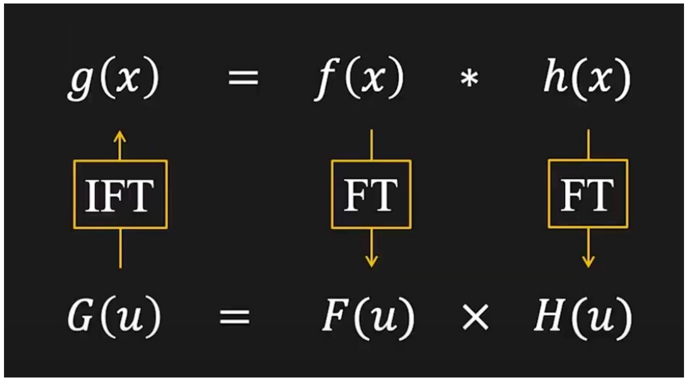


The effect of high-pass and low-pass filtering

Let $F(u, v)$ and $H(u, v)$ (filter) denote the Fourier transforms of $f(x, y)$ and $h(x, y)$. Then the convolution theorem states that

$$f(x, y) * h(x, y) \Leftrightarrow F(u, v)\, H(u, v)$$

Spatial Domain		Frequency Domain
$g(x) = f(x) * h(x)$ Convolution	$\longleftrightarrow$	$G(u) = F(u)\, H(u)$ Multiplication
$g(x) = f(x)\, h(x)$ Multiplication	$\longleftrightarrow$	$G(u) = F(u) * H(u)$ Convolution



The effect of high-pass and low-pass filtering

$$f(x, y) * h(x, y) \Leftrightarrow F(u, v)\,H(u, v)$$

The expression on the left (spatial convolution) can be obtained by taking the inverse Fourier transform of the expression on the right. Conversely, the expression on the right can be obtained by taking the forward Fourier transform of the expression on the left.

The operation of the **discrete convolution of two functions** is defined by

Function h is mirrored about the origin

$$f(x,y) * h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$$