不定积分计算、基本公式推导课堂例题

(一) 线性运算即可直接得到

(1) $\int (x+\frac{1}{x})^2 dx = \int (x^2+2+\frac{1}{x^2}) dx = \frac{x^3}{3}+2x-\frac{1}{x}+C$

(2) $\int \sin^2\frac{x}{2}\, dx = \int \frac{1-\cos x}{2} dx = \frac{1}{2}x-\frac{1}{2}\sin x+C$

(3) $\int \tan^2 x\, dx = \int \sec^2 x-1\, dx = \tan x - x + C$

(4) $\int (2^x+3^x)^3 dx = \int 2^{3x}+3\cdot 2^{2x}3^x+3\cdot 2^x\cdot 3^{2x}+3^{3x}\, dx =$

$= \int 8^x+3\cdot 12^x+3\cdot 18^x+27^x\, dx$

$= \frac{8^x}{3\ln 2}+\frac{3\cdot 12^x}{\ln 12}+\frac{3\cdot 18^x}{\ln 18}+\frac{27^x}{3\ln 3}+C$

(5) $\int \frac{2x^2+3}{x^2+1} dx = \int 2+\frac{1}{x^2+1} dx = 2x+\arctan x+C$

(7) $\int x^3 e^{x^4} dx = \frac{1}{4}\int e^{x^4} dx^4 = \frac{1}{4}e^{x^4}+C$

(8) $\int \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}} dx = \int \sqrt{\arcsin x}\, d\arcsin x = \frac{2}{3}\arcsin^{\frac{3}{2}} x+C$

(9) $\int \frac{dx}{x\ln(\ln x)\ln x} = \int \frac{d\ln x}{\ln x\cdot \ln\ln x} = \frac{d\ln\ln x}{\ln\ln x}$

$= \ln|\ln\ln x|+C$

(10) $\int \tan x\, dx = -\int \frac{d\cos x}{\cos x} = -\ln|\cos x|+C$

(11) $\int \cot x\, dx = \int \frac{d\sin x}{\sin x} = \ln|\sin x|+C$

(12) $\int \sin^3 x\, dx = -\int \sin^2 x\, d\cos x = -\int (1-\cos^2 x) d\cos x$

$= -(\cos x - \frac{\cos^3 x}{3})+C$

$= -\cos x+\frac{\cos^3 x}{3}+C$

(13) $\int \sin^3 x\cos^3 x\, dx = \int \sin^3 x\cos^2 x\, d\sin x$

$= \int (1-\sin^2 x)\sin^3 x\, d\sin x$

$= \int \sin^3 x - \sin^5 x\, d\sin x$

$= \frac{\sin^4 x}{4}-\frac{\sin^6 x}{6}+C$

(14) $\int \sin^4 x\cos^2 x\, dx$

$= \int \frac{1+\cos 2x}{2}\cdot\left(\frac{1-\cos 2x}{2}\right)^2 dx$

$= \int \frac{1-\cos 2x-\cos^2 2x+\cos^3 2x}{8} dx$

$= \frac{1}{8}x-\frac{\sin 2x}{16}-\int \frac{\cos^2 2x}{8}dx+\int \frac{\cos^3 2x}{8}dx$

$= \frac{1}{8}x-\frac{\sin 2x}{16}-\int \frac{1+\cos 4x}{16}dx+\int \frac{1-\sin^2 2x}{2\cdot 8}d\sin 2x$

$= \frac{x}{16}-\frac{\sin 2x}{16}-\frac{1}{64}\sin 4x-\frac{\sin^3 2x}{48}+C$

(二) 第一换元法

(1) $\int (3x+2)^{100} dx = \frac{1}{3}\int (3x+2)^{100} d(3x+2) = \frac{(3x+2)^{101}}{3\cdot 101}+C$

(2) $\int \frac{1}{x^2+a^2} dx = \int \frac{\frac{1}{a}\cdot d(\frac{1}{a}x)}{(\frac{x}{a})^2+1} = \frac{\arctan(\frac{x}{a})}{a}+C$

(3) $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{d(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = \arcsin\frac{x}{a}+C$

$= -\arccos\frac{x}{a}+C$

(4) $\int \frac{dx}{4x^2+4x+9} = \int \frac{dx}{(2x+1)^2+8} = \frac{1}{4\sqrt{2}}\int \frac{d(\frac{x}{\sqrt{2}}+\frac{1}{2\sqrt{2}})}{1+(\frac{x}{\sqrt{2}}+\frac{1}{2\sqrt{2}})^2}$

$= \frac{1}{4\sqrt{2}}\arctan(\frac{x}{\sqrt{2}}+\frac{1}{2\sqrt{2}})+C$

(5) $\int \frac{\arctan x}{1+x^2} dx = \int \arctan x\cdot d\arctan x$

$= \frac{1}{2}\arctan^2 x+C$

(6) $\int \frac{x}{1+x^2} dx = \frac{1}{2}\int \frac{d(x^2+1)}{1+x^2} = \frac{1}{2}\ln(1+x^2)+C$

(15) $\int \frac{dx}{x^2-a^2} = \int \frac{dx}{(x-a)(x+a)} = \frac{1}{2a}\int \frac{1}{x-a} - \frac{1}{x+a}\, dx$

$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

(16) $\int \sec x\, dx = \int \frac{1}{\cos x} dx = \int \frac{d\sin x}{1-\sin^2 x}$

$= -\frac{1}{2}\ln\left|\frac{\sin x-1}{\sin x+1}\right| + C$

$= \ln\left|\frac{\sin x+1}{\sin x-1}\right|^{\frac{1}{2}} + C$

$= \ln\left|\frac{\sin x+1}{\cos x}\right| + C = \ln|\tan x+\sec x| + C$

(17) $\int \csc x\, dx = \int \frac{1}{\sin x} dx = -\int \frac{d\cos x}{1-\cos^2 x}$

$= \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C$

$= \ln\left|\frac{\cos x-1}{\sin x}\right| + C$

$= \ln|\cot x - \csc x| + C$

(18) $\int \tan^3 x \sec^2 x\, dx = \int \tan^3 x\, d\tan x$

$= \frac{\tan^4 x}{4} + C$

(19) $\int \tan^3 x \sec^3 x\, dx = \int \tan^2 x \sec^2 x\, d\sec x$

$= \int (\sec^2 - 1)\sec^2 x\, d\sec x$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

(20) $\int \frac{e^x+\cos x}{e^x+\sin x} dx = \int \frac{d(e^x+\sin x)}{e^x+\sin x}$

$= \ln|e^x+\sin x| + C$

(三)分部积分法

~~(1) $\int x\cdot 2^x dx = \int 2^x d\frac{x^2}{2} = \frac{x^2}{2}\cdot 2^x - \int \frac{x^2}{2}$~~

(1) $\int x\cdot 2^x dx = x\cdot\frac{2^x}{\ln 2} - \int \frac{2^x}{\ln 2} dx$

$= x\cdot\frac{2^x}{\ln 2} - \frac{2^x}{\ln^2 2} + C$

(2) $\int (x^2+x)e^x dx = (x^2+x)e^x - \int e^x(2x+1)dx$

$= (x^2+x)e^x - [(2x+1)e^x - \int e^x\cdot 2\, dx]$

$= (x^2+x)e^x - (2x+1)e^x + 2e^x + C$

$= e^x(x^2 - x + 1) + C$

(3) $\int x^2 \sin x\, dx = -x^2\cos x + \int \cos x\, 2x\, dx$

$= -x^2\cos x + 2x\sin x - \int \sin x\, 2dx$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$

(4) $\int \ln x\, dx = x\ln x - \int x\cdot\frac{1}{x} dx = x\ln x - x + C$

(5) $\int x^2 \ln^2 x\, dx = \frac{x^3}{3}\ln^2 x - \int \frac{x^3}{3}\cdot 2\ln x\cdot\frac{1}{x} dx$

$= \frac{x^3}{3}\ln^2 x - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}dx\right)$

$= \frac{x^3}{3}\ln^2 x - \frac{2x^3}{9}\ln x + \frac{2}{27}x^3 + C$

(6) $\int \frac{\ln x}{x^2} dx = -\frac{1}{x}\ln x - \frac{1}{x} + C$

(7) $\int e^{2x}\sin x\, dx$

解: $\int e^{2x}\sin x\, dx = \int e^{2x} d(-\cos x) = -\cos x\, e^{2x} + \int \cos x\cdot 2e^{2x} dx$

$= -\cos x\, e^{2x} + 2\sin x\, e^{2x} - 4\int \sin x\cdot e^{2x} dx$

故 $\int \sin x\cdot e^{2x} dx = -\frac{\cos x\, e^{2x}}{5} + \frac{2e^{2x}\sin x}{5} + C$

(8) $\int \sec^3 x\,dx$

解: $\cancel{\int\sec^3x\,dx = x\sec^3x - 3\int x\sec^3x\tan x\,dx}$

$\cancel{= x\sec^3x - 3(}$

~~其中~~, $\cancel{\int\sec^3x\tan x\,dx = \int\sec^2x\,d\sec x}$

$= \frac{\sec^3x}{3}$

~~代回得~~ $\cancel{\int\sec^3x\,dx = x\sec^3x - 3\left(\frac{x\sec^3x}{3} - \int\frac{\sec^3x}{3}dx\right)}$

$$\int\sec^3x\,dx = \int\sec x\,d\tan x$$

$$= \tan x\sec x - \int\tan^2x\sec x\,dx$$

$$= \tan x\sec x - \int(\sec^2x-1)\sec x\,dx$$

$$= \tan x\sec x - \int\sec^3x\,dx + \int\sec x\,dx$$

其中, $\int\sec x\,dx = \ln|\tan x+\sec x| + C$

故 $\int\sec^3x\,dx = \frac{\tan x\sec x + \ln|\tan x+\sec x|}{2} + C$

(9) $\int\sec^nx\,dx$ 的递推公式

设 $I_n = \int\sec^nx\,dx$. $I_1 = \ln|\sec x+\tan x| + C$

$I_2 = \tan x$

$n\geq 3$, $\int\sec^nx\,dx = \int\sec^{n-2}x\,d\tan x = \tan x\sec^{n-2}x$

$-\int\tan x\cdot(n-2)\sec^{n-2}\cancel{x\tan x}\,dx$

$= \tan x\sec^{n-2}x - (n-2)\int(\sec^2-1)\sec^{n-2}x\,dx$

$= \tan x\sec^{n-2}x - (n-2)I_n - 2I_{n-2}$

故 $(n-1)I_n = \tan x\sec^{n-2}x - 2)I_{n-2}$

$$I_n = \frac{\tan x\sec^{n-2}x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$

(10) $I_n = \int\sin^nx$ 的递推公式

$\int\sin^nx = \int\sin^{n-1}x\,d(\cos x) = -\cos x\sin^{n-1}x + \int\cos x\,d\sin^{n-1}x$

$= -\cos x\sin^{n-1}x + \int\cos^2x\,(n-1)\sin^{n-2}x\,dx$

$= -\cos x\sin^{n-1}x + (n-1)\int(1-\sin^2x)\sin^{n-2}x\,dx$

$= -\cos x\sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$

$$I_n = \frac{-\cos x\sin^{n-1}x}{n} + \frac{n-1}{n}I_{n-2}$$

同样, 求 $\int\cos^nx\,dx$ 看成 $\int\cos^{n-1}x\,d\sin x$

而 $\int\csc^nx\,dx$ 看成 $-\int\csc^{n-2}x\,d\cot x$

(11) $\int\arctan x\,dx = x\arctan x - \int x\cdot\frac{1}{1+x^2}dx$

$= x\arctan x - \frac{1}{2}\int\frac{d(x^2+1)}{1+x^2}$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

(12) $\int x\arctan x\,dx = \frac{x^2}{2}\arctan x - \int\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$

$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\arctan x - \frac{1}{2} + \frac{1}{2}\arctan x + C$

(13) $\int\arcsin x\,dx = x\arcsin x - \int x\frac{1}{\sqrt{1-x^2}}dx$

$= x\arcsin x + \frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{1-x^2}}$

$= x\arcsin x + \frac{\cancel{1}}{\sqrt{1-x^2}}\cancel{\frac{}{2}\arcsin(1-x^2)} + C$

(14) $\int(\arcsin x)^2dx = x\arcsin^2x - \int x\,2\arcsin x\frac{1}{\sqrt{1-x^2}}dx$

$= x\arcsin^2x - 2\int\frac{x}{\sqrt{1-x^2}}\arcsin x\,dx$

$= x\arcsin^2x - 2\left(\sqrt{1-x^2}\arcsin x - \int\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}}dx\right.$

$= x\arcsin^2x + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(15) $\int \sin(\ln x)\,dx$

解 $\int \sin(\ln x)\,dx = x\sin(\ln x) - \int x\cos(\ln x)\frac{1}{x}dx$

$= x\sin(\ln x) - (x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}dx)$

$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx$

故 $\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - x\cos(\ln x) + C$

(四) 第二换元法

(1) $\int (\arcsin^2 x)\,dx$

令 $t = \arcsin x$, $x = \sin t$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\int t^2 d\sin t = t^2 \sin t - \int \sin t \cdot 2t\,dt$

$= t^2\sin t - (-\cos t \cdot 2t + \int \cos t \cdot 2\,dt)$

$= t^2 \sin t + 2t\cos t - 2\sin t + C$

$= \arcsin^2 x \cdot x + 2\arcsin x\sqrt{1-x^2} - 2x + C$

(2) $\int \sin(\ln x)\,dx$

令 $t = \ln x$, $x = e^t$

$\int \sin t\,de^t = e^t \sin t - \int e^t \cos t\,dt$

$= e^t \sin t - (e^t \cos t + \int e^t \sin t\,dt)$

$\int \sin t\,de^t = \frac{1}{2}e^t(\sin t - \cos t) + C$

$= \frac{1}{2}x(\sin\ln x - \cos\ln x) + C$

(3) $\int \frac{\sqrt{x} - 1}{\sqrt[3]{x} + 1}\,dx$

令 $x = t^6$，则 $t = x^{\frac{1}{6}}$

$\int \frac{t^3 - 1}{t^2 + 1}\,dt^6 = 6\int \frac{t^8 - t^5}{t^2+1}\,dt$

$= 6\int \frac{(t^2+1)(t^6 - t^4 - t^3 + t^2 + t - 1) - t + 1}{t^2+1}\,dt$

$= 6\int t^6 - t^4 - t^3 + t^2 + t - 1 + \frac{-t+1}{t^2+1}\,dt$

$= 6(\frac{t^7}{7} - \frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} - t)$

$+ 6\int \frac{1}{1+t^2}\,dt - 3\int \frac{dt^2+1}{1+t^2}$

$= 6(\frac{t^7}{7} - \frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} - t + \arctan t$

$- \frac{1}{2}\ln(1+t^2)) + C$

把 $t = x^{\frac{1}{6}}$ 代入

(4) $\int e^{\sqrt[3]{x}}\,dx$

令 $t = x^{\frac{1}{3}}$, $x = t^3$

$\int e^t\,dt^3 = 3(e^t t^2 - \int e^t \cdot 2t\,dt)$

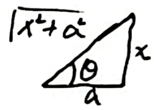
$= 3(e^t t^2 - (2e^t t - \int e^t \cdot 2\,dt))$

$= 3(e^t t^2 - 2e^t t + 2e^t) + C$

$= 3e^t(t^2 - 2t + 2) + C$

$= 3e^{\sqrt[3]{x}}(x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 2) + C$

(5) $\int \frac{dx}{\sqrt{x^2+a^2}}$



令 $x = a\tan\theta$,

则 $\sqrt{x^2+a^2} = a\sec\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\int \frac{a \cdot \sec^2\theta\,d\theta}{a\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$

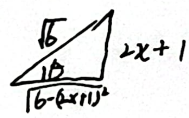
$= \ln\left|\left(\frac{a}{\sqrt{x^2+a^2}}\right)^{-1} + \frac{x}{a}\right| + C$

$= \ln|\sqrt{x^2+a^2} + x| + C$

(6) $\int \sqrt{5 - 4x - 4x^2}\,dx$

$\int \sqrt{5-4x-4x^2}\,dx = \int \sqrt{-(2x+1)^2 + 6}\,dx$

$= \int \sqrt{6 - (2x+1)^2}\,dx$，设



设 $2x+1 = \sqrt{6}\sin\theta$, $x = \frac{\sqrt{6}\sin\theta - 1}{2}$

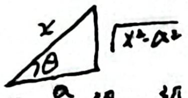
$\int \sqrt{6\cos^2\theta}\,dx = \int \sqrt{6}\cos\theta \cdot \frac{\sqrt{6}}{2} \cdot \cos\theta\,d\theta$

$= 3\int \cos^2\theta\,d\theta = 3(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}) + C$

解 θ 代入

(7) $\cancel{\int\sqrt{x^2+a^2}}$

$$\int\sqrt{x^2-a^2}\,dx$$



令 $x=a\sec\theta$，$\theta\in(0,\frac{\pi}{2})\cup(\pi,\frac{3\pi}{2})$

则 $\int a\tan\theta\cdot a\sec\theta\tan\theta\,d\theta$

$$=a^2\int\tan^2\theta\sec\theta\,d\theta$$

$$=a^2\left(\sec\theta\tan\theta-\int\sec^3\theta\,d\theta\right)$$

$$=a^2\left(\frac{\sec\theta\tan\theta-\ln|\tan\theta+\sec\theta|}{2}\right)+C$$