Ising model for multivariate binary response

Some results from Joe (1997), Multivariate models and dependence concepts.

$Y_1, \ldots, Y_d$ dependent binary random variables in $\{0, 1\}$.

Exponential family model (analogous to multivariate normal) is

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \zeta^{-1} \exp\Big\{\sum_j \alpha_j y_j + \sum_{1\le j<k\le d} \gamma_{jk} y_j y_k\Big\}, \quad y_j \in \{0,1\} \forall j,$$

where ζ is a normalizing constant that depends on the α and γ parameters. This implies conditional distributions that are logistic in one variable given the remainder.

$$\frac{\mathbb{P}(Y_j = 1|\boldsymbol{Y}_{-j} = \boldsymbol{y}_{-j})}{\mathbb{P}(Y_j = 0|\boldsymbol{Y}_{-j} = \boldsymbol{y}_{-j})} = \frac{\exp\{\alpha_j + \sum_{k\ne j}[\alpha_k y_k + \gamma_{jk} y_k] + \sum_{k_1<k_2, k_1\ne j, k_2\ne j} \gamma_{k_1k_2} y_{k_1} y_{k_2}\}}{\exp\{\sum_{k\ne j} \alpha_k y_k + \sum_{k_1<k_2, k_1\ne j, k_2\ne j} \gamma_{k_1k_2} y_{k_1} y_{k_2}\}}$$

$$= \exp\Big\{\alpha_k + \sum_{k\ne j} \gamma_{jk} y_k\Big\}$$

or

$$\operatorname{logit} \mathbb{P}(Y_j = 1|\boldsymbol{Y}_{-j} = \boldsymbol{y}_{-j}) = \alpha_k + \sum_{k\ne j} \gamma_{jk} y_k.$$

The converse result [Joe and Liu, 1996, SAPL] is that if one has logistic regressions of any response variable on the remaining y's, then these logistic regressions if the coefficient γ_{jk} of y_j regressing on y_k matches the the coefficient γ_{kj} of y_k regressing on y_j. That is, $\gamma_{jk} = \gamma_{kj}$ for all $j \ne k$.

If $\gamma_{jk} = 0$, then it implies some conditional independence properties that are similar to multivariate Gaussian (it does not imply anything about conditional correlation).

Ising: $\gamma_{jk} = 0$ implies

$$\operatorname{logit} \mathbb{P}(Y_j = 1|\boldsymbol{Y}_{-j} = \boldsymbol{y}_{-j}) = \operatorname{logit} \mathbb{P}(Y_j = 1|Y_\ell = y_\ell, \ell \ne j, k),$$

and

$$\operatorname{logit} \mathbb{P}(Y_k = 1|\boldsymbol{Y}_{-k} = \boldsymbol{y}_{-k}) = \operatorname{logit} \mathbb{P}(Y_k = 1|Y_\ell = y_\ell, \ell \ne j, k).$$

That is, if $\gamma_{12} = 0$, then conditional distribution of Y_1 given $Y_2, \ldots, Y_d$ is the same as the conditional distribution of Y_1 given $Y_3, \ldots, Y_d$, etc.

This is analogous to a property of multivariate Gaussian: if $\boldsymbol{\Sigma}^{-1} = (\sigma^{jk})$, then $\sigma^{jk} = 0$ for $j \ne k$ implies

$$[Y_j|\boldsymbol{Y}_{-j} = \boldsymbol{y}_{-j}] = [Y_j|Y_\ell = y_\ell, \ell \ne j, k], \tag{2}$$

$$[Y_k|\boldsymbol{Y}_{-k} = \boldsymbol{y}_{-k}] = [Y_k|Y_\ell = y_\ell, \ell \ne j, k].$$

That is, there is a simplification in some Bayesian network representation.

For multivariate Gaussian, this results follows from: if Y_j is regressed on Y_ℓ, $\ell \ne j$, then the sign of the regression coefficient β_k for Y_k is the same as the sign of $\rho_{jk;rest}$. Hence $\beta_k = 0$ implies $\rho_{jk;rest} = \sigma^{jk} = 0$ and result (1).

MTP_2 properties (multivariate totally positive of order 2)

Multivariate density $f_{1:d}$ is MTP_2 if

$$f_{1:d}(\boldsymbol{y} \vee \boldsymbol{z})\, f_{1:d}(\boldsymbol{y} \wedge \boldsymbol{z}) \geq f_{1:d}(\boldsymbol{y})\, f_{1:d}(\boldsymbol{z}) \quad \forall \boldsymbol{y}, \boldsymbol{z}, \tag{1}$$

where $\vee$ and $\wedge$ denote coordinatewise max and min.

Multivariate normal density $f_{1:d}(\boldsymbol{y}) \propto \exp\{-Q(\boldsymbol{y})\}$ is MTP_2 if all of the coefficients of cross terms σ^{jk} in the quadratic form Q are negative or if $\rho_{jk;rest} \geq 0$ for all j, k.

Similarly, for the Ising exponential family multivariate binary density, $f_{1:d}(\boldsymbol{y})$ is MTP_2 if all $\gamma_{jk} \geq 0$.

If the inequality in (1) is reversed, this leads to MRR_2 (multivariate reverse rule of order 2), a weak concept of negative dependence.

Mathematical result: MTP_2 density implies all marginal densities of dimensions 2 or more are MTP_2. There is no analogous result for MRR_2.

Hence, multivariate normal density that is MTP_2 has correlations $\rho_{jk} \geq 0$ for all $j \neq k$.

Ising density that is MTP_2 has bivariate margins that are positive dependent.

trivariate Ising model :

α_1	α_2	α_3	γ_{12}	γ_{13}	γ_{23}	μ_1	μ_2	μ_3	ρ_{12}	ρ_{13}	ρ_{23}	OR12	OR13	OR23
0.5	0.4	0.6	0.2	0.3	0	0.696	0.631	0.691	0.0448	0.0650	0.0029	1.221	1.350	1.013
0.5	0.4	0.6	0.2	0.3	0.2	0.699	0.664	0.718	0.0466	0.0653	0.0459	1.237	1.362	1.237
0.5	0.4	0.6	0.2	0.3	0.4	0.702	0.697	0.746	0.0479	0.0650	0.0855	1.251	1.374	1.511
0.5	0.4	0.6	0.2	0.3	0.8	0.708	0.760	0.799	0.0485	0.0629	0.1535	1.276	1.395	2.255
0.5	0.4	0.6	0.2	0.3	2.0	0.722	0.903	0.918	0.0387	0.0469	0.2750	1.321	1.434	7.486
0.5	0.4	0.6	0.2	0.3	5.0	0.730	0.994	0.995	0.0104	0.0123	0.3359	1.344	1.455	150.4
0.5	0.4	0.6	0.2	0.3	-0.2	0.693	0.600	0.664	0.0424	0.0642	-0.0429	1.205	1.337	0.829
0.5	0.4	0.6	0.2	0.3	-1.0	0.683	0.491	0.573	0.0297	0.0581	-0.2392	1.136	1.285	0.373
0.5	0.4	0.6	0.2	0.3	-5.0	0.669	0.349	0.454	0.0001	0.0415	-0.6455	1.001	1.195	0.007
0.5	0.4	0.6	0.2	-0.3	0	0.609	0.627	0.602	0.0474	-0.0710	-0.0034	1.221	0.741	0.986
0.5	0.4	0.6	0.2	-0.3	0.2	0.608	0.656	0.633	0.0434	-0.0677	0.0428	1.205	0.748	1.204
0.5	0.4	0.6	0.2	-0.3	0.4	0.607	0.686	0.664	0.0395	-0.0643	0.0862	1.189	0.754	1.471
0.5	0.4	0.6	0.2	-0.3	0.8	0.605	0.745	0.728	0.0322	-0.0573	0.1624	1.162	0.765	2.194
0.5	0.4	0.6	0.2	-0.3	2.0	0.602	0.891	0.883	0.0165	-0.0368	0.3044	1.113	0.787	7.285
0.5	0.4	0.6	0.2	-0.3	5.0	0.599	0.993	0.993	0.0034	-0.0090	0.3787	1.088	0.798	146.3
0.5	0.4	0.6	0.2	-0.3	-0.2	0.609	0.600	0.573	0.0515	-0.0742	-0.0517	1.239	0.734	0.807
0.5	0.4	0.6	0.2	-0.3	-1.0	0.612	0.511	0.478	0.0665	-0.0848	-0.2480	1.314	0.705	0.363
0.5	0.4	0.6	-0.2	-0.3	0	0.550	0.572	0.606	-0.0491	-0.0725	0.0036	0.819	0.741	1.015
0.5	0.4	0.6	-0.2	-0.3	0.2	0.547	0.602	0.635	-0.0519	-0.0737	0.0508	0.807	0.734	1.240
0.5	0.4	0.6	-0.2	-0.3	0.4	0.543	0.634	0.664	-0.0542	-0.0744	0.0956	0.797	0.727	1.514
0.5	0.4	0.6	-0.2	-0.3	0.8	0.535	0.700	0.724	-0.0568	-0.0740	0.1752	0.779	0.715	2.259
0.5	0.4	0.6	-0.2	-0.3	2.0	0.516	0.867	0.878	-0.0493	-0.0593	0.3269	0.746	0.693	7.499
0.5	0.4	0.6	-0.2	-0.3	5.0	0.501	0.992	0.992	-0.0141	-0.0164	0.4079	0.729	0.681	150.6
0.5	0.4	0.6	-0.2	-0.3	-0.2	0.554	0.543	0.580	-0.0459	-0.0709	-0.0455	0.831	0.748	0.831
0.5	0.4	0.6	-0.2	-0.3	-1	0.564	0.451	0.496	-0.0313	-0.0623	-0.2400	0.881	0.778	0.373
0.5	0.4	0.6	-1	-1	-1	0.422	0.394	0.451	-0.1803	-0.1875	-0.1837	0.464	0.459	0.462
0.5	0.4	0.6	-4	-4	-4	0.286	0.260	0.315	-0.3376	-0.3866	-0.3617	0.048	0.043	0.046