CSE 559A: Computer Vision

[credit: danjodon.deviantart.com]

Fall 2017: T-R: 11:30-1pm @ Lopata 101

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http://www.cse.wustl.edu/~ayan/courses/cse559a/

Nov 7, 2017

GENERAL

- PSET 4 due next Tuesday.

- Recitation this Friday November 10.

MACHINE LEARNING

- Given inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, we want to learn a function $y = f(x)$, i.e., $f : \mathcal{X} \to \mathcal{Y}$
- Function should be a "good" predictor of y, as measured in terms of a risk or loss function: $L(y, \hat{y})$.
- Ideally, we want to find the best $f \in \mathcal{H}$, among some class or space of functions $\mathcal{H}$ (called the hypothesis space), which minimizes the expected loss under the joint distribution $P_{XY}(x, y)$:

$$f = \arg\min_{f \in \mathcal{H}} \int_{\mathcal{X}} \int_{\mathcal{Y}} L(y, f(x)) \; P_{XY(x,y)} \; dy \, dx$$

- But we don't know this joint distribution, but we have a **training set** $(x_1, y_1), (x_2, y_2), \ldots (x_T, y_T)$, which (we hope!) are samples from P_{XY}.
- So we approximate the integral over the P_{XY} with an average over the training set (x_t, y_t),

$$f = \arg\min_{f \in \mathcal{H}} \frac{1}{T} \sum_{t} L(y_t, f(x_t))$$

You're given data. Choose a loss function that matches your task, a hypothesis space $\mathcal{H}$, and minimize.

MACHINE LEARNING

$$f = \arg\min_{f \in \mathcal{H}} \frac{1}{T} \sum_t L(y_t, f(x_t))$$

Consider:

- $x \in \mathcal{X} = \mathbb{R}^d$
- $y \in \mathcal{Y} = \mathbb{R}$
- $\mathcal{H}$ is the space of all "linear functions" of $\mathcal{X}$.
  - $f(x; w, b) = w^T x + b, \quad w \in \mathbb{R}^d, b \in \mathbb{R}$
  - Minimization of $f \in \mathcal{H}$ becomes a minimization of w, b
- Consider $L(y, \hat{y}) = (y - \hat{y})^2$

And then we have our familiar least-squares regression!

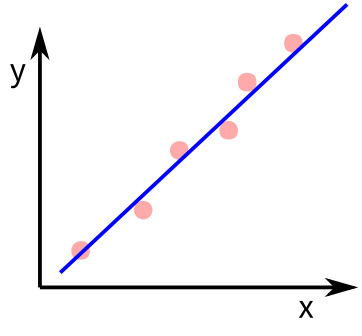
MACHINE LEARNING

$$f = \arg\min_{f \in \mathcal{H}} \frac{1}{T} \sum_t L(y_t, f(x_t))$$

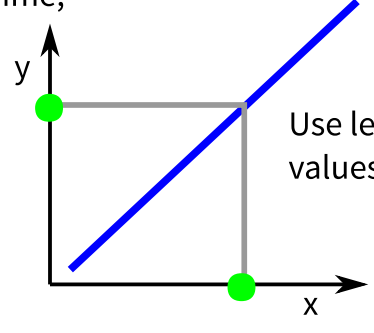
$$w, b = \arg\min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \frac{1}{T} \sum_t (y_t - w^T x_t - b)^2$$

So we know how to solve this: take derivative and set to 0.

Training



At Test Time,



Use learned function to predict values of new inputs

Not just for fitting "lines". Imagine x is a vector corresponding to a patch of intensities in a noisy image. y is corresponding clean intensity of the center pixel. You could use this to "learn" a linear "denoising filter", by fitting to many examples of pairs of noisy and noise-free intensities.

MACHINE LEARNING

$$w, b = \arg \min_{w \in \mathbb{R}^d, b \in \mathbb{R}} \frac{1}{T} \sum_t (y_t - w^T x_t - b)^2$$

Define $\tilde{x}_t = [x_t^T, 1]^T$

$$w = \arg \min_{w \in \mathbb{R}^{d+1}} \frac{1}{T} \sum_t (y_t - w^T \tilde{x}_t)^2$$

$$w = \left(\sum_t \tilde{x}_t \tilde{x}_t^T \right)^{-1} \left(\sum_t \tilde{x}_t y_t \right)$$

Now, let's say we wanted to fit a polynomial instead of a linear function.

For $x \in \mathbb{R}$,

$f(x; w_0, w_1, w_2, \ldots, w_n) = w_0 + w_1 x + w_2 x^2 + \ldots w_n x^n.$

This is our hypothesis space. Same loss function $L(y, \hat{y}) = (y - \hat{y})^2$.

MACHINE LEARNING

$$w = \arg \min_{w \in \mathbb{R}^{n+1}} \frac{1}{T} \sum_t (y_t - w_0 - w_1 x_t - w_2 x_t^2 \ldots - w_n x_t^n)^2$$

Set $\tilde{x}_t = \left[1, x_t, x_t^2, x_t^3, \ldots x_t^n\right]^T$.

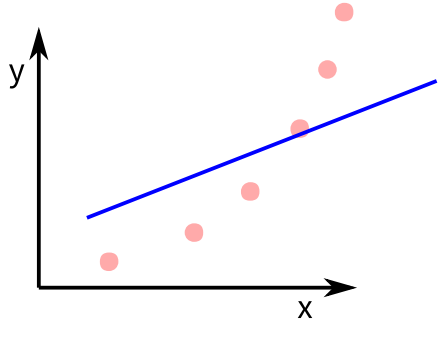
And you get exactly the same equation !

$$w = \left(\sum_t \tilde{x}_t \tilde{x}_t^T \right)^{-1} \left(\sum_t \tilde{x}_t y_t \right)$$

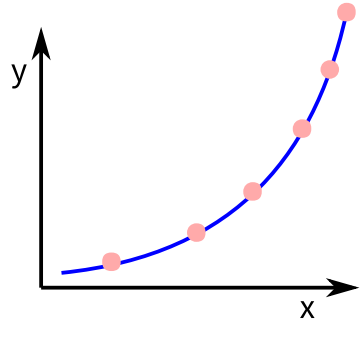
- But now, inverting a larger matrix.
- Can apply the same idea to polynomials of multi-dimensional x.
- Can apply least-squares fitting to any task with an $L2$ loss, and where the hypothesis space is linear in the parameters (not necessarily in the input).

MACHINE LEARNING

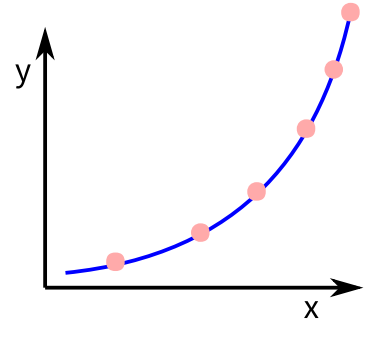
Why not just fit the more complex model ?



Linear Fit



Quadratic Fit



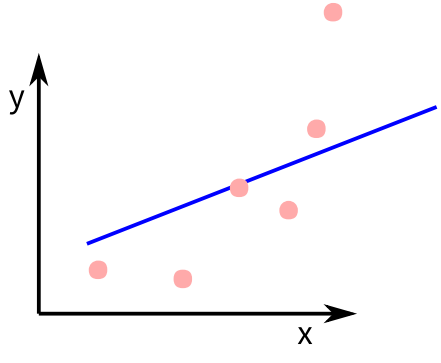
12th Order Polynomial

nth Order Polynomials can fit all orders upto n.

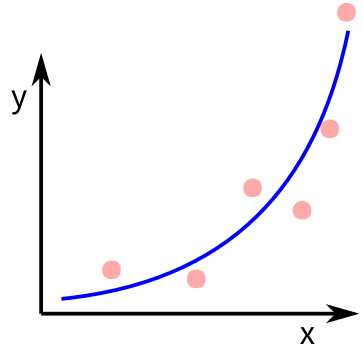
Why not just choose the most complex inclusive hypothesis space ?

MACHINE LEARNING

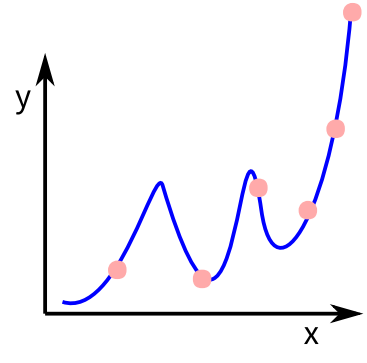
Why not just fit the more complex model ?



Linear Fit



Quadratic Fit



12th Order Polynomial

Given enough capacity, f will hallucinate structure in the noise

nth Order Polynomials can fit all orders upto n.

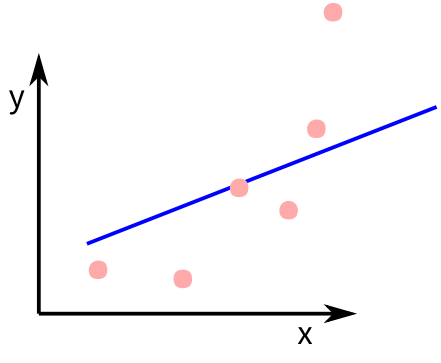
Why not just choose the most complex inclusive hypothesis space ?

Because, y may have noise (P(y|x) is not deterministic).

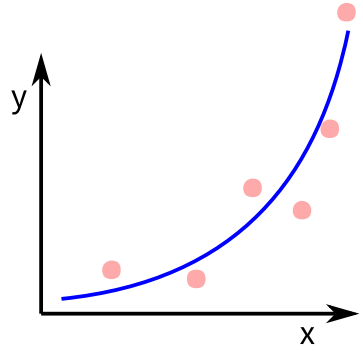
MACHINE LEARNING

Why not just fit the more complex model ?

Too simple

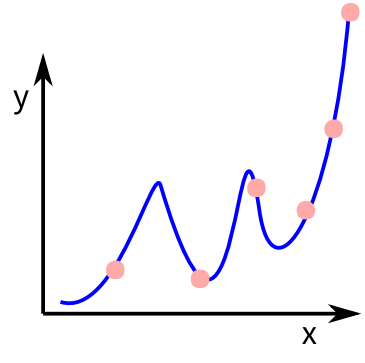


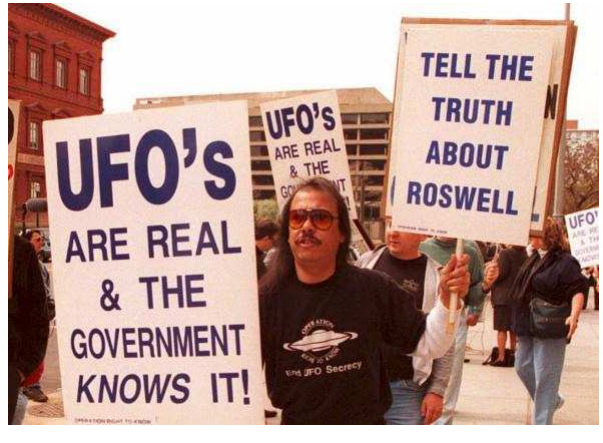
Just Right



Quadratic Fit

Too complex



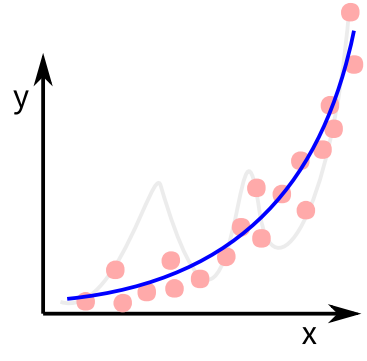


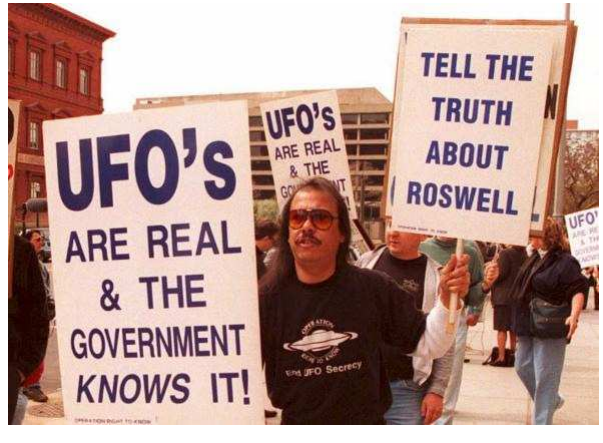
MACHINE LEARNING

Why not just fit the more complex model ?

Too complex

Can be fixed if we had a lot more data.





MACHINE LEARNING

While we train on empirical loss,

$$\frac{1}{T}\sum_{t} L(y_t, f(x_t))$$

we care about the actual expected loss:

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} L(y, f(x))\ \ P_{XY(x,y)}\ \ dy\ dx$$

Why ? Because we don't want to explain the training set. We want to do well on new inputs. We want to "generalize" from the training set.

MACHINE LEARNING

Error = Bayes Error + Approximation Error + Estimation Error

- Bayes Error: This is the error due to the uncertainty in $p(y|x)$. This is the error you would have even if you knew the exact distribution and could craft a function f with infinite complexity.

$$\text{Bayes Error} = \int \left(\int \min_{\hat{y}} L(y, \hat{y}) P_{XY}(x, y) dy \right) dx$$

- Approximation Error: This is the error due to the limited capacity of our hypothesis space $\mathcal{H}$. It is the error of the true best function $f \in \mathcal{H}$, minus the Bayes error, assuming we had perfect knowledge of P_{XY}.
  Also called the "Bias"^{2}.
- Estimation Error: This is the remaining component of error, caused by the fact that we don't know the true P_{XY}, but only have a limited number of samples.
  This depends on the size of our training set (and also, how well we are able to do the minimization). Called "variance".

MACHINE LEARNING

Error = Bayes Error + Approximation Error + Estimation Error

Bias-Variance Tradeoff

Choosing a simple function class: higher approximation error, lower estimation error.

Choosing a complex function class: lower approximation error, higher estimation error.

How do I decrease Bayes Error ? By getting better inputs!

MACHINE LEARNING

Overfitting

- Definitions of complexity of a function or hypothesis space $\mathcal{H}$: VC-dimension, Rademacher complexity
- Try to capture that one function or function space provides a "simpler" explanation than another
- Useful as intuition. But often don't "work" for very complex functions and tasks.
- But the idea is:
  - Given two functions with the same error, you want to choose one that's simpler.
  - You never want to consider a class of functions that can fit 'random' T pairs of (x, y), where T is the size of your training set.
- Choose hypothesis space based on size of training set.
- Add a "regularizer" (for example, a squared penalty on higher order polynomial coefficients).

MACHINE LEARNING

Overfitting: Good ML "Hygiene"

Remember, you can overfit not just the parameters, but your design choices !

For any given task:

- Have train, dev, val, and test set.
- Train your estimators on the train set.
- Choose hyperparameters based on dev set.
  - Function class
  - Regularization weight
  - Number of iterations to train, etc.
- Keep periodically checking to see if it generalizes to val set.
- Look at the test set only at the "end" of the project.

MACHINE LEARNING

Classification

Consider the case when y is binary, i.e., $\mathcal{Y} = \{0, 1\}$.

How do you define the loss function then ?

- Ideally, $L(y, \hat{y})$ is 0 if they are equal, 1 otherwise.

But don't know how to solve that. What if we solved by regression ?

$$w = \arg\min_{w} \frac{1}{T} \sum_{t} (y_t - w^T \tilde{x}_t)^2$$

And at test time, we can output $y = 1$ if $w^T \tilde{x} > 0.5$ and 0 otherwise.

The problem is the loss function will penalize $w^T \tilde{x}_t > 1$ when $y_t = 1$. While at test time, this would give us exactly the right answer !

MACHINE LEARNING

Logistic regression

- Learn a function $f(x) = P(y == 1)$ which regresses to the probability y is 1.
- We have to choose f such that the domain of $f(x)$ lies between $[0, 1]$.

$$f(x; w) = \sigma\left(w^T \tilde{x}\right), \quad \sigma(p) = \frac{\exp(p)}{1 + \exp(p)}$$

- This ensures that the output of f is between $[0, 1]$
- $w^T \tilde{x}$ can be interpreted as the log of the odds, or log of ratio between $P(y == 1)$ to $P(y == 0)$
- Again, can choose any way of constructing augmented vector $\tilde{x}$ from x. This means that then the log-odds will be linear / polynomial / etc. function of x.

What about the loss ?

MACHINE LEARNING

Logistic regression

$$P(y == 1) = f(x) = \sigma\left(w^T \tilde{x}\right)$$

Cross-Entropy Loss

If true y is 1, we want $f(x)$ to be high, and if it is 0, we want it to be low.

$$L(y, f(x)) = -y \log f(x) - (1 - y) \log(1 - f(x))$$

There's a minus because this is the loss.

Minimizing $\sum_t L(y_t, f(x_t))$ can be viewed as maximizing the product of the probabilities of the labels y_t.

But now, how do we minimize this function in terms of w ? No longer least-squares.

NEXT TIME

- Gradient Descent & Stochastic Gradient Descent
- Back-propagation & Neural Networks