CSE 559A: Computer Vision

[credit: danjodon.deviantart.com]

Fall 2017: T-R: 11:30-1pm @ Lopata 101

Instructor: Ayan Chakrabarti (ayan@wustl.edu).
Staff: Abby Stylianou (abby@wustl.edu), Jarett Gross (jarett@wustl.edu)

http://www.cse.wustl.edu/~ayan/courses/cse559a/

Sep 7, 2017

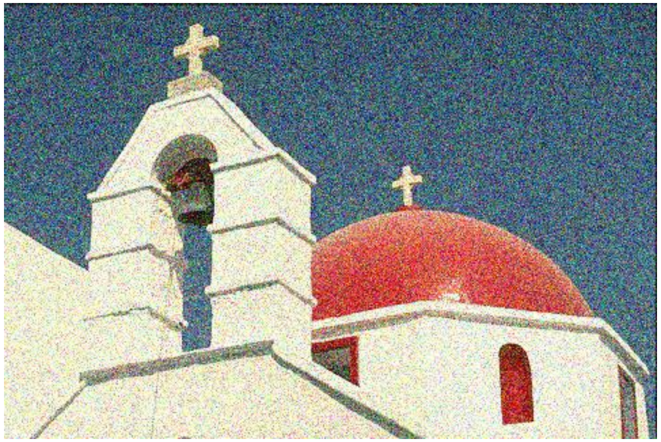
OFFICE HOURS

Jarett Gross	Mon	5:40pm-6:30pm	Jolley 431
Ayan Chakrabarti	Wed	9:30am-10:30am	Jolley 205
Abby Stylianou*	Fri	10:00am-11:00am	**9/[8,15]: Jolley 420** **9/22- : Jolley 309**

BILATERAL FILTERING

Denoising by Smoothing (with a Gaussian filter):

$$X$$



$$Y = X * G$$

$$Y[n] = \sum_{n'} G[n']X[n-n']$$

$$G'[n_1, n_2] = G[n_1 - n_2] \propto \exp\left(-\frac{|n_1 - n_2|^2}{2\sigma^2}\right)$$

$$Y[n_1] = \sum_{n_2} G'[n_1, n_2]X[n_2]$$

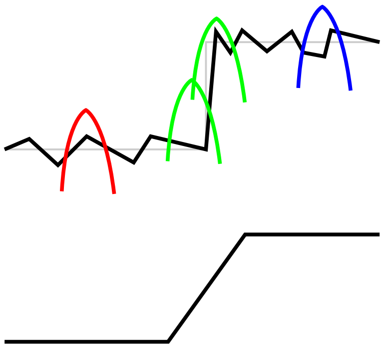
$$\sum_{n_2} G'[n_1, n_2] = 1$$

BILATERAL FILTERING

Denoising by Smoothing (with a Gaussian filter):

X

$Y = X * G$



$$G'[n_1, n_2] = G[n_1 - n_2] \propto \exp\left(-\frac{|n_1 - n_2|^2}{2\sigma^2}\right)$$

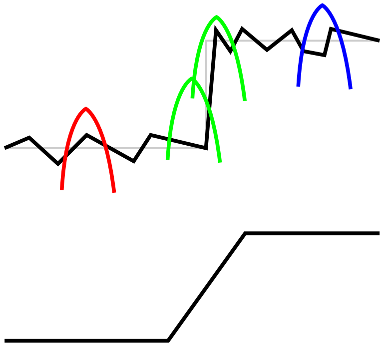
$$\sum_{n_2} G'[n_1, n_2] = 1$$

BILATERAL FILTERING

Denoising by Smoothing (with a Gaussian filter):

X

Make the filter weights data dependent !



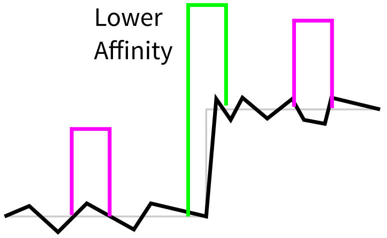
$$B[n_1, n_2] \propto \exp\left(-\frac{|n_1 - n_2|^2}{2\sigma^2} - \frac{|X[n_1] - X[n_2]|^2}{2\sigma_I^2}\right)$$

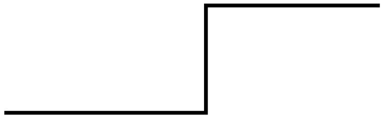
$$\sum_{n_2} B[n_1, n_2] = 1$$

BILATERAL FILTERING

Denoising by Smoothing (with a Gaussian filter):

X





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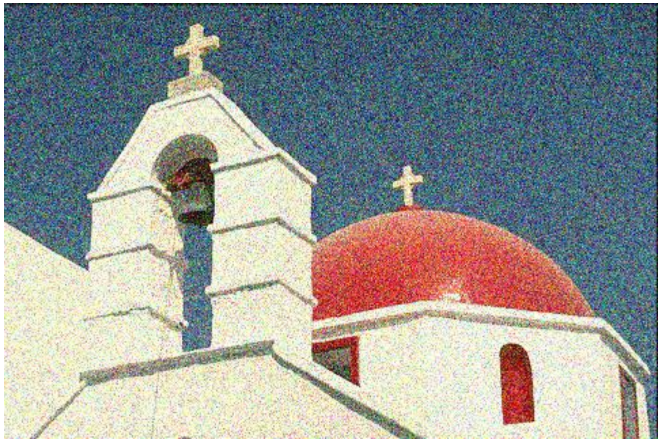
$$\sum_{n_2} B[n_1, n_2] = 1$$

BILATERAL FILTERING

Denoising with a Bilateral Filter



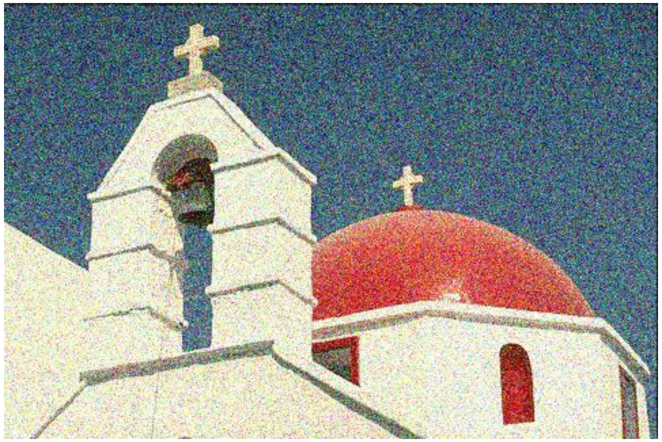
Gaussian Filter Result

σ_I High

$$B[n_1, n_2] \propto \exp\left(-\frac{|n_1 - n_2|^2}{2\sigma^2} - \frac{|X[n_1] - X[n_2]|^2}{2\sigma_I^2}\right)$$

BILATERAL FILTERING

Denoising with a Bilateral Filter



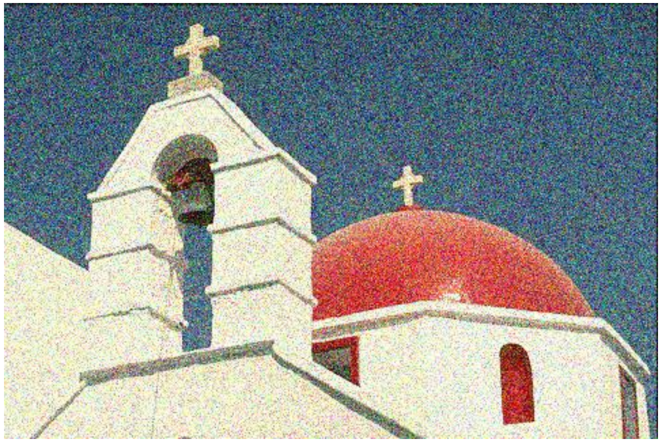
Gaussian Filter Result

σ_I Medium

$$B[n_1, n_2] \propto \exp\left(-\frac{|n_1 - n_2|^2}{2\sigma^2} - \frac{|X[n_1] - X[n_2]|^2}{2\sigma_I^2}\right)$$

BILATERAL FILTERING

Denoising with a Bilateral Filter



Gaussian Filter Result

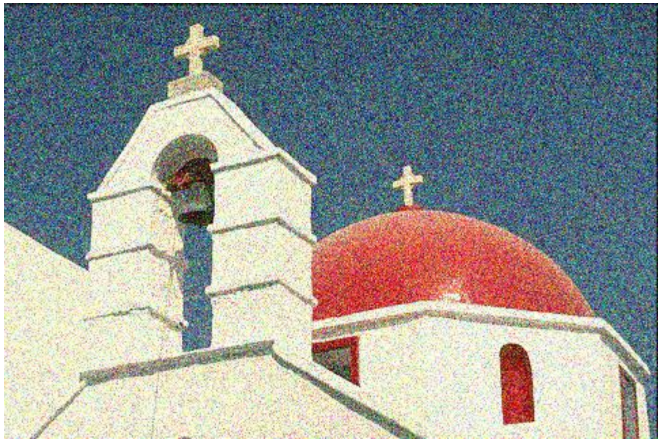
σ_I Low

$$B[n_1, n_2] \propto \exp\left(-\frac{|n_1 - n_2|^2}{2\sigma^2} - \frac{|X[n_1] - X[n_2]|^2}{2\sigma_I^2}\right)$$

BILATERAL FILTERING

Denoising with a Bilateral Filter

σ_I Low Repeated



Gaussian Filter Result

$$B[n_1, n_2] \propto \exp\left(-\frac{|n_1 - n_2|^2}{2\sigma^2} - \frac{|X[n_1] - X[n_2]|^2}{2\sigma_I^2}\right)$$

BILATERAL FILTERING

- *Guided Bilateral Filter*: $B[n_1, n_2]$ based on a separate image $Z[n]$: depth, infra-red, etc.
- Far less efficient than convolution
  - Filter also has to be computed, normalized, at each output location.
  - Efficient Datastructures Possible
- Further Reading:
  - Paris et al., SIGGRAPH/CVPR Course on Bilateral Filtering
  - Recent work on using this for inference, best paper runner up at ECCV 2016
    Barron & Poole, The Fast Bilateral Solver, ECCV 2016.

FOURIER TRANSFORM

The Discrete 2D Fourier Transform

$$\mathcal{F}[X] = F[u,v] = \frac{1}{WH} \sum_{n_x=0}^{W-1} \sum_{n_y=0}^{H-1} X[n_x, n_y] \exp\left(-j\, 2\pi \left(\frac{u\, n_x}{W} + \frac{v\, n_y}{H}\right)\right)$$

$$\exp(j\,\theta) = \cos\theta + j\sin\theta$$

We follow EE convention and use $j = \sqrt{-1}$ instead of i.

- Defined for a single-channel / grayscale image X.
- F is a "complex valued" array indexed by integers u, v.
- Note that $F[u,v] = F[u+W, v] = F[u, v+H]$because of periodicity.
- Therefore, we typically store $F[u,v]$ for $u \in \{0, \ldots, W-1\}, v \in \{0, \ldots, H-1\}$.
- Can think of $F[u,v]$ as a complex-valued "image" with the same number of pixels as X.

Can be implemented fairly efficiently using the FFT algorithm
(often, FFT is used to refer to the operation itself).

FOURIER TRANSFORM

The Discrete 2D Fourier Transform Pair

$$\mathcal{F}[X] = F[u,v] = \frac{1}{WH} \sum_{n_x=0}^{W-1} \sum_{n_y=0}^{H-1} X[n_x, n_y] \, \exp\left(-j\, 2\pi \left(\frac{u\, n_x}{W} + \frac{v\, n_y}{H}\right)\right)$$

$$\mathcal{F}^{-1}[F] = X[n_x, n_y] = \sum_{u=0}^{W-1} \sum_{v=0}^{H-1} F[u,v] \, \exp\left(j\, 2\pi \left(\frac{u\, n_x}{W} + \frac{v\, n_y}{H}\right)\right)$$

- If X is real-valued, $F[-u,-v] = F[W-u, H-v] = \bar{F}[u,v]$, where $\bar{F}$ implies complex conjugate.
- $F[0,0]$ is often called the DC component. It is the average intensity of X. It is real if X is real.
- Only WH independent "numbers" in $F[u,v]$ (counting real and imaginary separately) if X is real.
- Parseval's Theorem: (energy preserving upto constant factor)

$$\sum_{u,v} \|F[u,v]\|^2 = \sum_{u,v} F[u,v]\bar{F}[u,v] = \frac{1}{WH} \sum_{n_x,n_y} \|X[n_x,n_y]\|^2$$

FOURIER TRANSFORM

DFT as a Co-ordinate Transform

$$F[u,v] = \frac{1}{\sqrt{WH}} \left\langle \; S_{uv}, \; X \; \right\rangle$$

(Remember for $u, v \in \mathbb{C}^n$, $\langle u, v \rangle = u^* v$).

where each S_{uv} can be thought of as a different (complex-valued) image:

$$S_{uv}[n_x, n_y] = \frac{1}{\sqrt{WH}} \exp\left(j\, 2\pi \left(\frac{u\, n_x}{W} + \frac{v\, n_y}{H}\right)\right)$$

$F[u,v]$ is the inner-product between X and S_{uv}. (scaled by $\sqrt{WH}$)

Suv =

+ j

FOURIER TRANSFORM

DFT as a Co-ordinate Transform

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$F[u,v]$ is the inner-product between X and S_{uv}. (scaled by $\sqrt{WH}$)

Property: $\langle S_{uv}, S_{u'v'} \rangle = 1 \text{ if } u' = u \;\&\; v' = v, \text{ and } 0 \text{ otherwise.}$

Inverse-DFT:

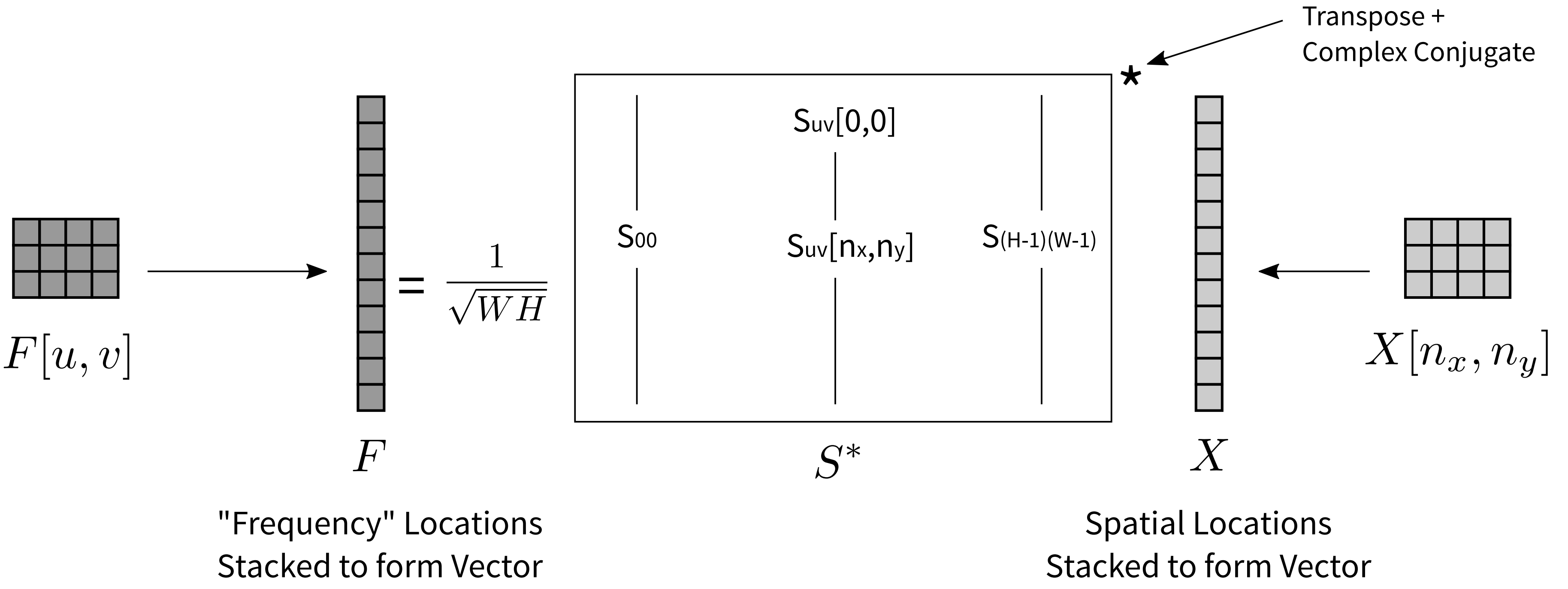
$$X = \sqrt{WH} \; \sum_{u=0}^{W-1} \sum_{v=0}^{H-1} \; F[u,v] \; S_{uv}$$

FOURIER TRANSFORM

DFT as a Co-ordinate Transform

$$F[u,v] = \frac{1}{\sqrt{WH}} \left\langle\ S_{uv},\ X\ \right\rangle, \qquad X = \sqrt{WH} \sum_{u=0}^{W-1} \sum_{v=0}^{H-1} F[u,v]\ S_{uv}$$

$$\langle S_{uv}, S_{u'v'} \rangle = 1 \text{ if } u' = u \ \&\ v' = v, \text{ and 0 otherwise.}$$

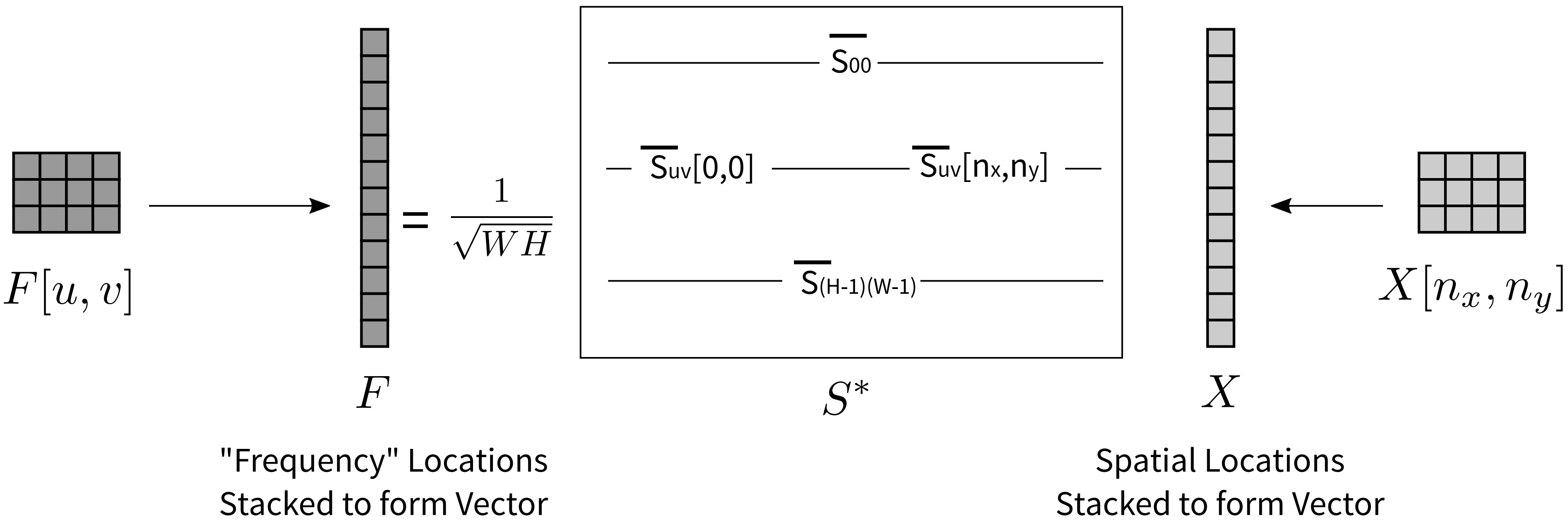


FOURIER TRANSFORM

DFT as a Co-ordinate Transform

$$F[u,v] = \frac{1}{\sqrt{WH}} \left\langle \; S_{uv}, \; X \; \right\rangle, \qquad X = \sqrt{WH} \sum_{u=0}^{W-1} \sum_{v=0}^{H-1} F[u,v] \; S_{uv}$$

$$\langle S_{uv}, S_{u'v'} \rangle = 1 \text{ if } u' = u \;\&\; v' = v, \text{ and 0 otherwise.}$$



FOURIER TRANSFORM

DFT as a Co-ordinate Transform

$$F = \frac{1}{\sqrt{WH}} S^* X, \qquad X = \sqrt{WH}\, S\, F$$

S is a $WH \times WH$ matrix with each column a different S_{uv}.

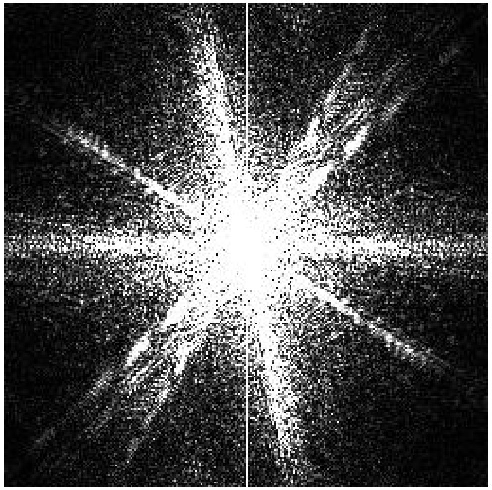
So, $SS^* = S^* S = I \Rightarrow S^{-1} = S^*$.

- This means S is a unitary matrix.
- Multiplication by S is a co-ordinate transform:
    - X are the co-ordinates of a point in a WH dimensional space.
    - Multiplication by S^* changes the 'co-ordinate system'.
    - In the new co-ordinate system, each 'dimension' now corresponds to frequency rather than location.
    - S is a length-preserving matrix ($\|S^* X\|^2 = \|X\|^2$).
    - It does rotations or reflections (in WH dimensional space).

FOURIER TRANSFORM

X

$|F|^2$



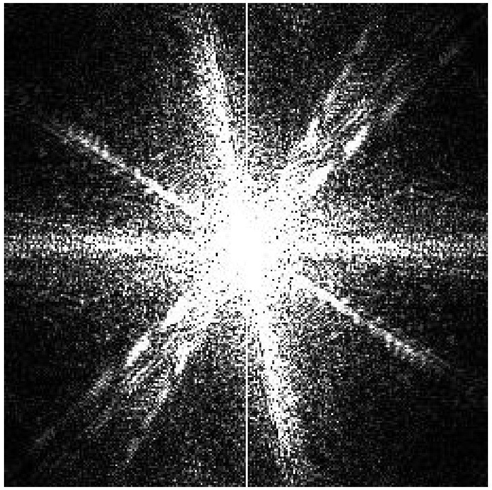
(W-u) 0 u

Zero-centered Co-ordinates
for frequencies [u,v]

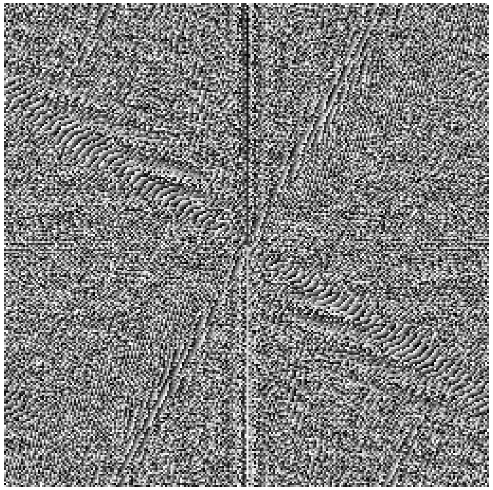
FOURIER TRANSFORM

X

$|F|^2$



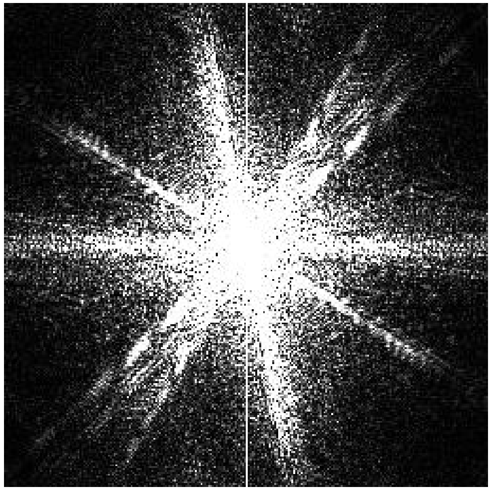
$\angle F$



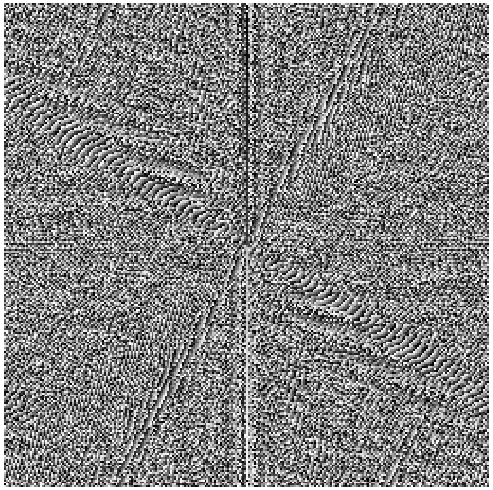
FOURIER TRANSFORM

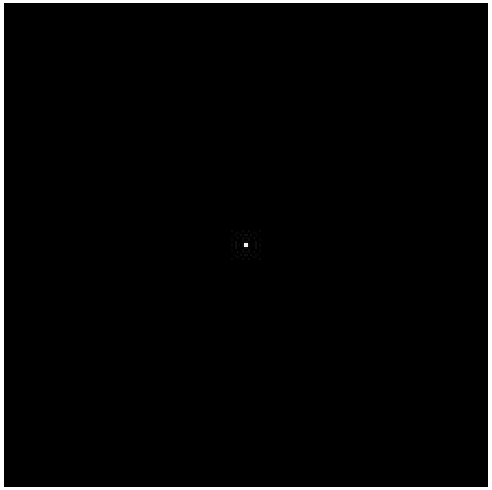
X

$|F|^2$



$\angle F$

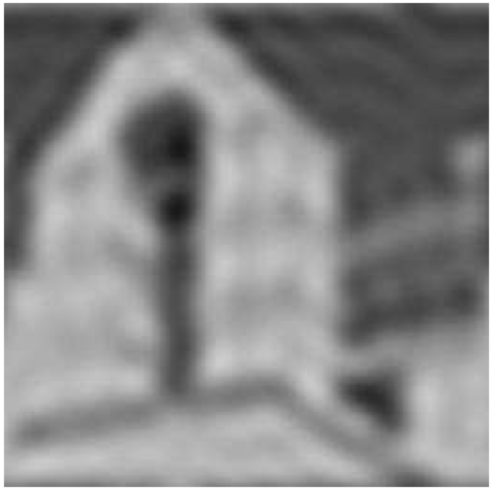




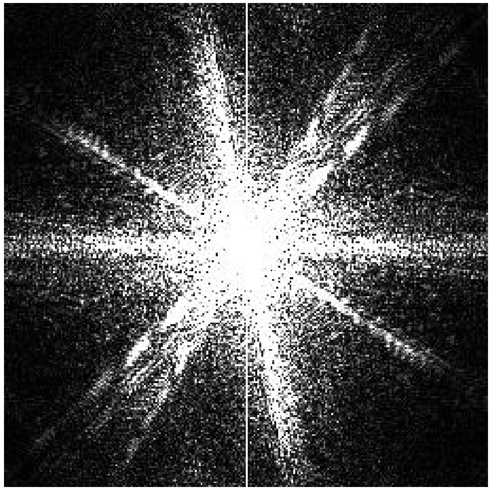
Reconstruct with only
these frequency components

FOURIER TRANSFORM

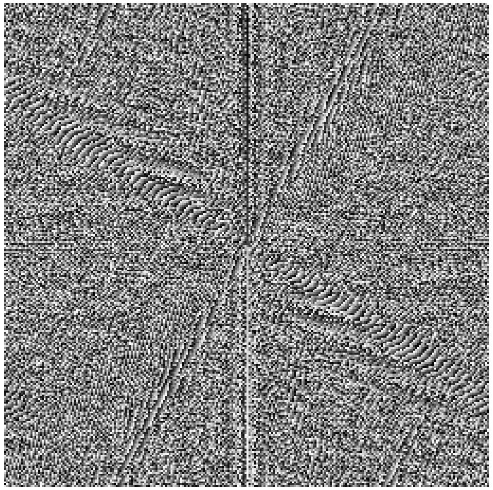
X

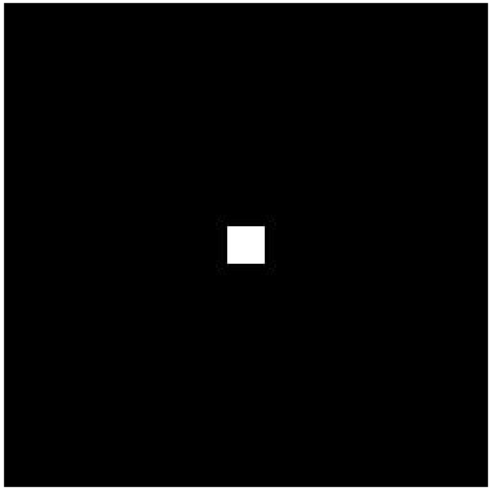


$|F|^2$



$\angle F$



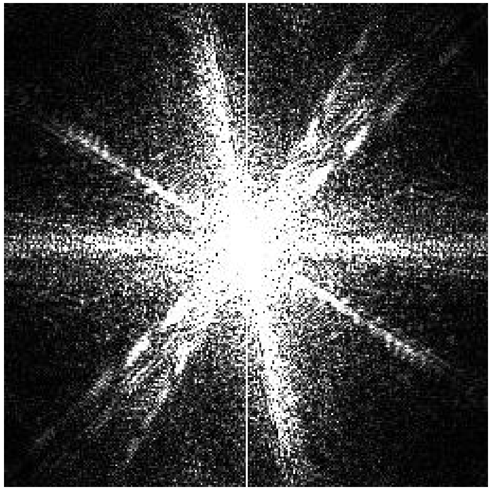


Reconstruct with only
these frequency components

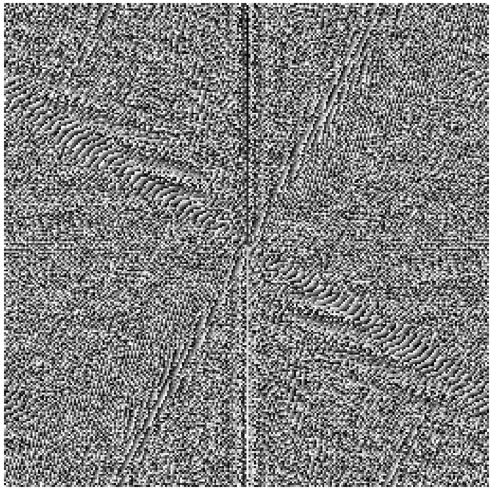
FOURIER TRANSFORM

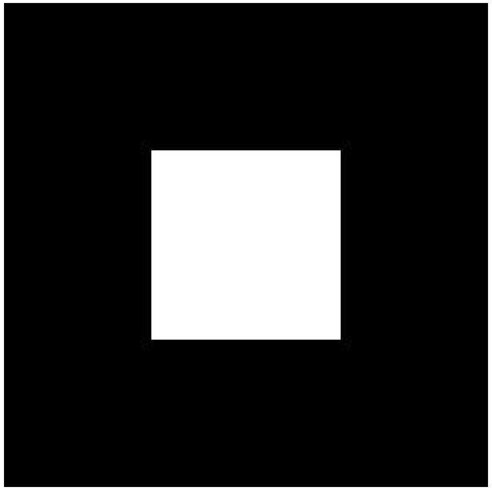
X

$|F|^2$



$\angle F$



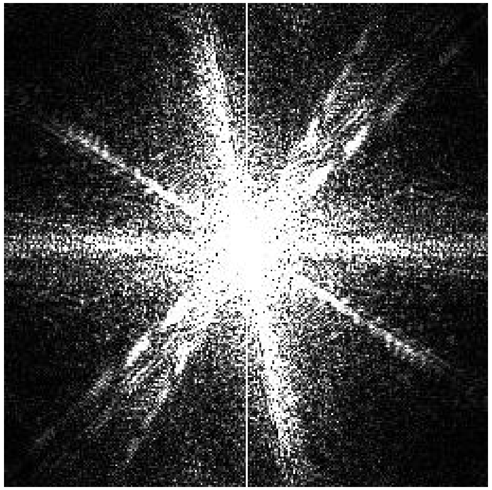


Reconstruct with only
these frequency components

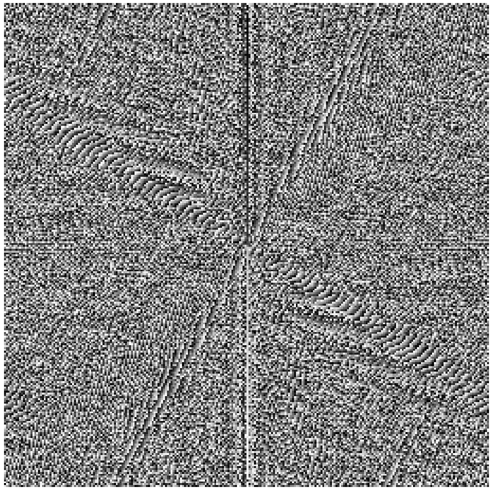
FOURIER TRANSFORM

X

$|F|^2$



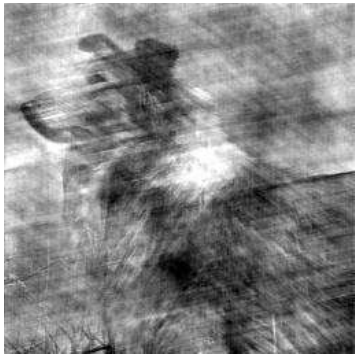
$\angle F$



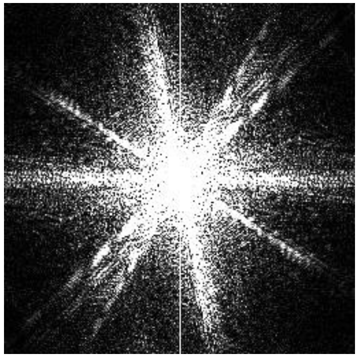
Reconstruct with only
these frequency components

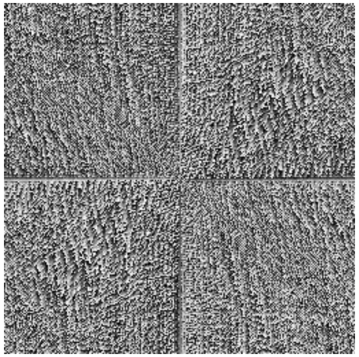
FOURIER TRANSFORM

A



Magnitude A
Phase B

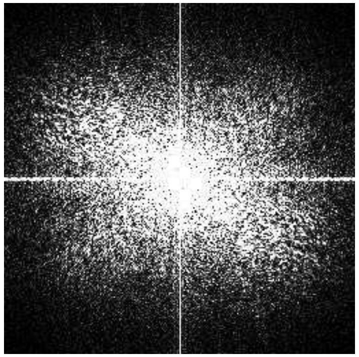


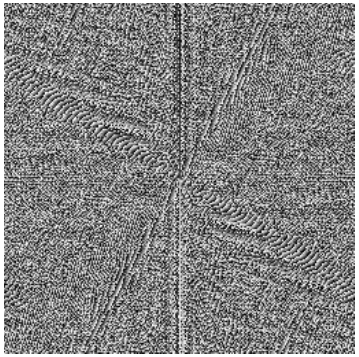


Location of edges / structure,
defined by phase more than magnitude.

B

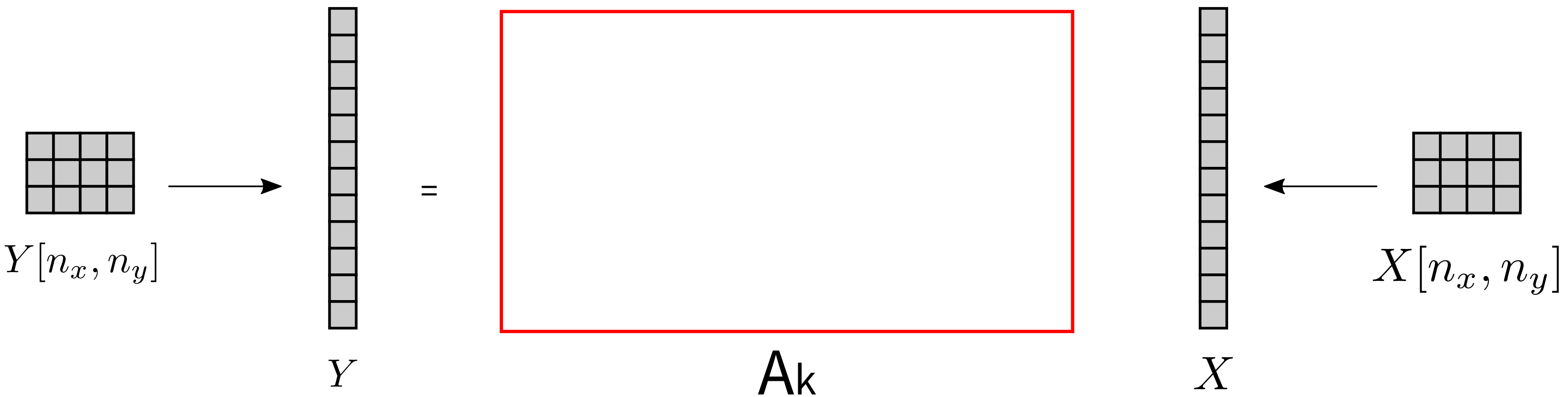
Magnitude B
Phase A





CONVOLUTION THEOREM

Convolution in "matrix" form



Spatial Locations
Stacked to form Vector

- Mostly 0 (sparse)
- Has $W_k H_k$ non-zero entries per row.
- Same set of values, but at different places in each row

Spatial Locations
Stacked to form Vector

$$Y[n_x, n_y] = \sum_{n'_x} \sum_{n'_y} k[n'_x, n'_y] \;\; X[n_x - n'_x, n_y - n'_y]$$

CONVOLUTION THEOREM

$$Y = X * k \Rightarrow Y = A_k X$$

A_k is not square for valid / long convolution.

Question:

Let $Y = A_k\ X$ correspond to $Y = X *_{\text{valid}}\ k$. Now, let $X' = A_k^T Y$. How is X' related to Y by convolution ? What operation does A_k^T represent ?

A: Full convolution with $k[-n_x, -n_y]$ (flipped version of k)

CONVOLUTION THEOREM

$$Y = X * k \Rightarrow Y = A_k X$$

Now if we consider the square A_k matrix corresponding to 'same' convolution with circular padding, i.e. padding as $X[W + n_x, n_y] = X[n_x, n_y], X[n_x, -n_y] = X[n_x, H - n_y]$, etc.

Then, A_k is *diagonalized* by the Fourier Transform !

$$A_k = S\, D_k\, S^*$$

- Here, D_k is a diagonal matrix.
- The above equation holds for every A_k
  - You get different diagonal matrices D_k.
  - But S is the diagonalizing basis for all kernels.
- In the Fourier co-ordinate system, convolution is a 'point-wise' operation !

$$Y = A_k X = S\ \ D_k\ \ S^*\ X \Rightarrow (S^* Y) = D_k (S^* X)$$

CONVOLUTION THEOREM

Why does this happen ?

- $X = \sqrt{WH} \ \sum_{u,v} F[u,v] S_{uv}$
- $Y = X * k = \sqrt{WH} \ \sum_{u,v} F[u,v] S_{uv} * k$(by linearity / distributivity)
- $(S_{uv} * k)[n] = \sum_{n'} k[n'] S_{uv}[n - n']$
- $S_{uv}[n - n']$, assuming circular padding, is also a sinusoid with the same frequency (u, v) and magnitude, but different phase.
- Multiplying by $k[n']$ changes the magnitude, but frequency still the same.
- Adding different sinusoids of the same frequency gives you another sinusoid of the same frequency. $\alpha \cos \theta + \beta \sin \theta$.
- $(S_{uv} * k)[n_x, n_y] = d_{uv:k} \ S_{uv}[n_x, n_y]$, where $d_{uv:k}$ is some complex scalar.

Sinusoids are eigen-functions of convolution

$$Y = X * k = \sqrt{WH} \ \sum_{u,v} F[u,v] S_{uv} * k = \sqrt{WH} \ \sum_{u,v} \left(F[u,v] \ d_{uv:k} \right) \ S_{uv}$$

CONVOLUTION THEOREM

$$A_k = S\, D_k\, S^*$$

- What's more, the diagonal elements of D_k are the $(W_x \times W_y)$ Fourier transform of k.

$$D_k = \mathrm{diag}\left(\frac{1}{\sqrt{WH}} S^*\, k \right)$$

- This is the convolution theorem.
  - Computational advantage for performing (and inverting!) convolution, albeit under circular padding.
  - Good way of analyzing what a kernel is doing by looking at its Fourier transform.
- Why did we use complex numbers ? Like quaternions in Graphics, for convenience!
  - If we used real number co-ordinate transform, convolution would convert to several 2×2 transforms on pairs of co-ordinates.
  - Complex numbers are just a way of grouping these pairs into a single 'number'.

CONVOLUTION THEOREM

Doing Convolutions in the Fourier Domain:

- DFT, Point-wise multiply with FT of kernel, Inverse DFT
- Need to keep in mind some padding / size issues.

CONVOLUTION THEOREM

X

k

Kernel has to be the same size as the image.

CONVOLUTION THEOREM

X

k

1. Zero-pad
2. Circularly shift
   to center at (0,0)

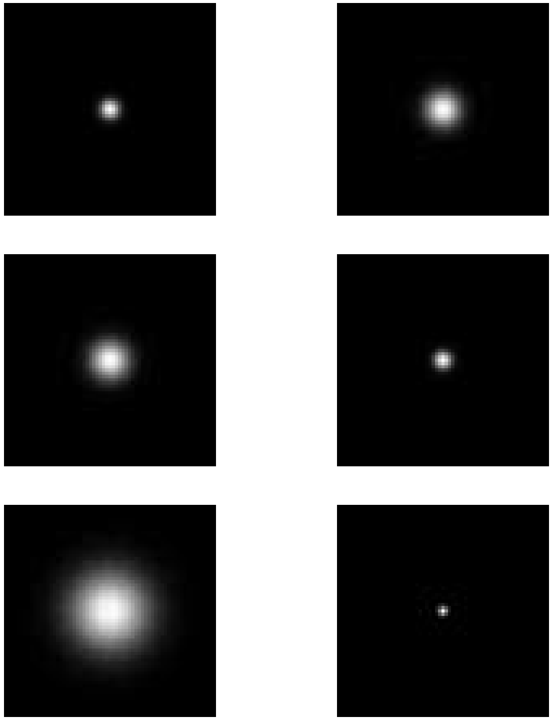
Kernel has to be the same size as the image.

- From same circular, you can always get 'valid' by cropping.
- To get full / same with zero-padding, pad your original image first.

CONVOLUTION THEOREM

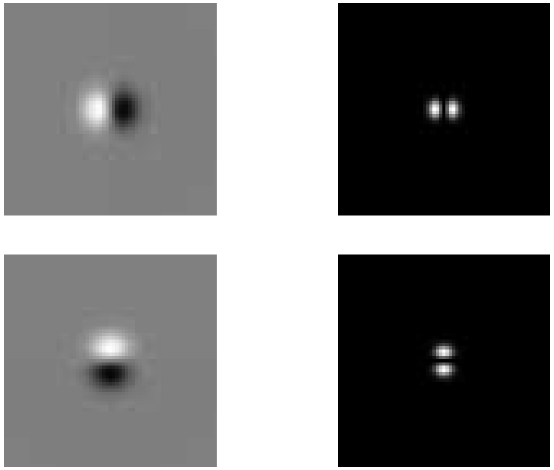
Kernel / Fourier Transform (magnitude) Pairs

Kernel Magnitude



Gaussian Kernels: Low Pass (attenuate higher frequencies)
Larger spatial support: smaller Fourier support.

Kernel Magnitude



Gaussian Derivatives: Band-pass

For more indepth coverage:
Szeliski Sec 3.4