16. 设 $X_1, X_2, \ldots, X_n$ 是自参数为 λ 的 Poisson 分布的样本，试证明

$$\hat{\sigma} = \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sqrt{\sum_{i=1}^{n} (\mathrm{X}_i - \overline{\mathrm{X}})^2} \tag{1}$$

是 σ 无偏估计。

证明. 要证 $\hat{\sigma}$ 是 σ 的无偏估计，只需证明 $E(\hat{\sigma}) = \sigma$ 即可。

$$\begin{aligned} E(\hat{\sigma}) =& E(\frac{1}{\sqrt{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sqrt{\sum_{i=1}^{n} (\mathrm{X}_i - \overline{\mathrm{X}})^2}) \\ =& \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} E(\sqrt{\sum_{i=1}^{n} (\mathrm{X}_i - \overline{\mathrm{X}})^2}) \end{aligned}$$

由定理 3.3 知 $\frac{1}{\sigma^2} \sum\limits_{i=1}^{n} (\mathrm{X}_i - \overline{\mathrm{X}})^2 \sim \chi^2(n-1)$, 所以令 $Y = \sum\limits_{i=1}^{n} (\mathrm{X}_i - \overline{\mathrm{X}})^2$，则

$$\begin{aligned} E(\hat{\sigma}) =& \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} E(\sqrt{\sigma^2 Y}) \\ =& \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sigma \int_0^{+\infty} \sqrt{x} \frac{2^{\frac{-n+1}{2}}}{\Gamma(\frac{n-1}{2})} e^{-\frac{x}{2}} x^{\frac{n-1}{2}-1} dx \end{aligned}$$

再令 $x = 2y$，则

$$\begin{aligned} E(\hat{\sigma}) =& \frac{1}{\sqrt{2}} \frac{1}{\Gamma(\frac{n}{2})} \sigma \int_0^{+\infty} 2^{\frac{-n+1}{2}} e^{-y} (2y)^{\frac{n}{2}-1} 2dy \\ =& \frac{1}{\sqrt{2}} \frac{1}{\Gamma(\frac{n}{2})} \sigma \int_0^{+\infty} 2^{\frac{1}{2}} e^{-y} (y)^{\frac{n}{2}-1} dy \end{aligned}$$

由欧拉第二积分定义得：$\int_0^{+\infty} e^{-y} (y)^{\frac{n}{2}-1} dy = \Gamma(\frac{n}{2})$，即

$$E(\hat{\sigma}) = \frac{1}{\sqrt{2}} \frac{1}{\Gamma(\frac{n}{2})} \sigma 2^{\frac{1}{2}} \Gamma(\frac{n}{2}) = \sigma$$

□