Linear System

1. Superposition principle

$$\mathcal{L}\{a_1u_1(t) + a_2u_2(t)\} = a_1\mathcal{L}\{u_1(t)\} + a_2\mathcal{L}\{u_2(t)\}$$

2. Table of laplace Transforms.

$f(t)$	$\delta(t)$	$1(t)$	t	$\frac{1}{2}t^2$	$\frac{t^k}{k!}$ $k>0$	e^{-at}	$\frac{1}{n!}t^n e^{-at}$	$\sin\omega t$	$\cos\omega t$	$e^{-at}\sin\omega t$	$e^{-at}\cos\omega t$
$F(s)$	1	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s^3}$	$\frac{1}{s^{k+1}}$	$\frac{1}{s+a}$	$\frac{1}{(s+a)^{n+1}}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{s+a}{(s+a)^2+\omega^2}$

3. Property of Laplace

$F(s) = \int_0^{+\infty} f(t)\, e^{-st}dt$ $\qquad \mathcal{L}[f'(t)] = sF(s) - f(0)$ $\qquad \mathcal{L}[f^{(n)}(t)] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^-)$

$\mathcal{L}[af(t) \pm bg(t)] = a\mathcal{L}[f(t)] \pm b\mathcal{L}[g(t)]$ $\qquad \mathcal{L}[e^{at}f(t)] = F(s-a)$ $\qquad \mathcal{L}[f(t-a)] = e^{-as}F(s)$

$\mathcal{L}[\int_0^t f(\tau)d\tau] = \frac{1}{s}F(s)$ $\qquad \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

4. Calculate Pole

exp. $\frac{2s+7}{s(s+4)(s+1)}$ $\qquad$ Pole $= 0, -4, -1$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+4} + \frac{C_3}{s+1}$$

$$C_1 = \frac{2s+7}{s(s+4)(s+1)} \times s \Big|_{s=0} = \frac{7}{4}$$

$$C_2 = \frac{2s+7}{s(s+4)(s+1)} \times (s+4) \Big|_{s=-4} = -\frac{1}{12}$$

$$C_3 = \frac{2s+7}{s(s+4)(s+1)} \times (s+1) \Big|_{s=-1} = -\frac{5}{3}$$

$$X(s) = \frac{\frac{7}{4}}{s} - \frac{\frac{1}{12}}{(s+4)} - \frac{\frac{5}{3}}{(s+1)}$$

$$\Downarrow \mathcal{L}^{-1}$$

$$x(t) = \frac{7}{4} - \frac{1}{12}e^{-4t} - \frac{5}{3}e^{-t}$$

$$\frac{5s+7}{s(s+2)(s+3)^3} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_{31}}{(s+3)^3} + \frac{C_{32}}{(s+3)^2} + \frac{C_{33}}{(s+3)}$$

$$C_1 = \frac{5s+7}{s(s+2)(s+3)^3} \times s \Big|_{s=0}$$

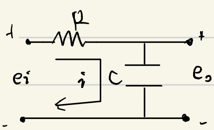
$$C_2 = \frac{5s+7}{s(s+2)(s+3)^3} \times (s+2) \Big|_{s=-2}$$

$$C_{31} = \frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3 \Big|_{s=-3}$$

$$C_{32} = \left[\frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3\right]' \Big|_{s=-3}$$

$$C_{33} = \left[\frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3\right]'' \Big|_{s=-3}$$

5. I/O Discription



$$\begin{cases} e_i = Ri + \frac{1}{C}\int i\,dt \\ e_o = \frac{1}{C}\int i\,dt \end{cases} \Rightarrow \frac{de_o}{dt} = \frac{i}{C} \Rightarrow i = C\frac{de_o}{dt}$$

$$e_i = RC\frac{de_o}{dt} + e_o \quad \Longrightarrow \quad e_o(\tau) = \frac{1}{RC}\int_0^{\tau} e^{(t-\tau)/RC} e_i dt$$

① $e_i = 0$ $\quad e_o(0) \neq 0$ $\to$ zero input response : $e_o(t) = e_o(0)\, e^{-\frac{t}{RC}}$

② $e_i \neq 0$ $\quad e_o(0) = 0$ $\to$ zero state response : $e_o(t) = A(1 - e^{-t/RC})$

③ $e_i \neq 0$ $\quad e_o(0) \neq 0$ $\to$ general response : $e_o(t) = e_o(0)\, e^{-\frac{t}{RC}} + A(1 - e^{-\frac{t}{RC}})$

6. show linear:

exp: $y(t) = \int_0^t u(\tau)\,d\tau$

Step 1: $u_1 \to y_1$; $u_2 \to y_2 \Rightarrow y_1 = \int_0^t u_1 d\tau \quad y_2 = \int_0^t u_2 d\tau$

Step 2: $u = a_1 u_1 + a_2 u_2 \to y$

Step 3: $y = \int_0^t (a_1 u_1 + a_2 u_2)\,d\tau = a_1 \int_0^t u_1 d\tau + a_2 \int_0^t u_2 d\tau = a_1 y_1 + a_2 y_2$

7. Causality:

The system is said to be causal if the output at time t does not depend on the input at time after t

8. Time variance

$u(t) \to u(t-t_0) \to \boxed{L} = y_1(t)$

$y(t) \to y(t-t_0) \qquad y_1(t) = y(t-t_0$

9. Types of signal.

Step: $f(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$ Ramp: $f(t) = \begin{cases} t & t \geq 0 \\ 0 & t<0 \end{cases}$ impulse: $\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t \neq 0 \end{cases}$ and $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$

10. Impulse Response

$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau$ then $y(t) = L\{\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau\} = \int_{-\infty}^{\infty} u(\tau)\,h(t,\tau)\,d\tau$

LTI: $\delta(t-\tau) \to h(t-\tau, 0) \equiv h(t-\tau) \Rightarrow y(t) = \int_0^t u(\tau)\,h(t-\tau)\,dt$

LTV. $y(t)=\int_{-\infty}^{+\infty} u(\tau)\,h(t,\tau)\,d\tau$

II. Computation Impulse response

exp: use laplace

$$\ddot{y}+4\dot{y}+3y=4\dot{u}+5u \quad \overset{\mathcal{L}}{\Rightarrow} \quad s^2Y(s)+4sY(s)+3Y(s)=4sU(s)+5U(s)$$

①

$$H(s)=\frac{Y(s)}{U(s)}=\frac{4s+5}{s^2+4s+3} \Rightarrow h(t)=\mathcal{L}^{-1}(H(s))=\mathcal{L}^{-1}\left(\frac{\frac{7}{2}}{s+3}+\frac{\frac{1}{2}}{s+1}\right)$$

$$=\frac{7}{2}e^{-3t}+\frac{1}{2}e^{-t} \quad t>0$$

②. step input

$$y(t)=\int_0^t h(t-\tau)\,u(\tau)\,d\tau$$

$$\begin{cases} h(t)=\frac{7}{2}e^{-3t}+\frac{1}{2}e^{-t}, & t\geq 0 \\ u(t)=\begin{cases} 1 & t\geq 0 \\ 0 & t<0 \end{cases} \end{cases}$$

$$y(t)=\int_0^t\left[\frac{7}{2}e^{-3(t-\tau)}+\frac{1}{2}e^{-(t-\tau)}\right]1\cdot d\tau$$

$$=\frac{5}{3}-\frac{7}{6}e^{-3t}\ \frac{1}{2}e^{-t}$$

③ use T.F.

$$Y(s)=H(s)U(s) \qquad H(s)=\frac{4s+5}{s^2+4s+3} \qquad U(s)=\frac{1}{s}$$

$$\therefore Y(s)=\frac{4s+5}{s(s^2+4s+3)} \Rightarrow y(t)=\frac{5}{3}-\frac{7}{6}e^{-3t}-\frac{1}{2}e^{-t}$$

④. If $y(0)=1$ $\dot{y}(0)=0$

$$\mathcal{L}\{\ddot{y}+4\dot{y}+3y=4\dot{u}+5u\}$$

$$\mathcal{L}\{\ddot{y}\}=s^2Y(s)-sY(0^-)-\dot{y}(0^-)$$

$$\mathcal{L}\{\dot{y}\}=sY(s)-y(0^-)$$

$$\Downarrow$$

$$s^2Y(s)-sy(0)-\dot{y}(0)+4[sY(s)-y(0)]+3Y(s)=4sU(s)+5U(s)$$

$$Y(s)=\frac{4s+5}{s^2+4s+3}\times U(s)+\frac{s+4}{s^2+4s+3}$$

$$y(t)=\mathcal{L}^{-1}Y(s)=\frac{5}{3}-\frac{5}{3}e^{-3t}+e^{-t}$$

State Diagram

exp

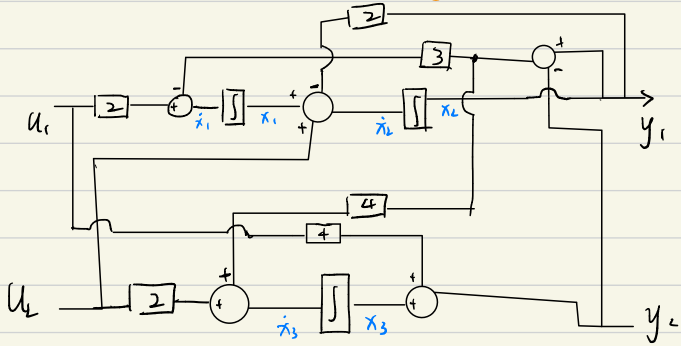
Find a state space representation (A, B, C, D) for the system described by

$$\ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + \dot{u}_2$$
$$\dot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4\dot{u}_1$$

Hint: Use the state diagram approach.

$$\begin{cases} \ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + \dot{u}_2 \\ \dot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4\dot{u}_1 \end{cases} \Rightarrow \begin{cases} \ddot{y}_1 = 2u_1 + \dot{u}_2 - 2\dot{y}_1 - 3(y_1 - y_2) \\ \dot{y}_2 = 4(y_1 - y_2) + 2u_2 + 4\dot{u}_1 \end{cases} \xRightarrow{\text{integra}} \begin{cases} y_1 = \iint 2u_1 - 3(y_1 - y_2) + \int u_2 - 2y_1 \\ y_2 = \int [4(y_1 - y_2) + 2u_2] + 4u_1 \end{cases}$$

State Diagram ↓



$$\begin{cases} \dot{x}_1 = 2u_1 - 3(y_1 - y_2) \\ \dot{x}_2 = x_1 - 2y_1 + u_2 \\ \dot{x}_3 = 2u_2 + 4(y_1 - y_2) \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = 2u_1 - 3(x_2 - x_3 - 4u_1) \\ \dot{x}_2 = x_1 - 2x_2 + u_2 \\ \dot{x}_3 = 2u_2 + 4(x_2 - x_3 - 4u_1) \end{cases}$$

$$\begin{cases} y_1 = x_2 \\ y_2 = x_3 + 4u_1 \end{cases}$$

$$\therefore \boxed{A = \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 0 \\ 0 & 4 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 14 & 0 \\ 0 & 1 \\ -16 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}}$$

12. General State Space Representation

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) & \leftarrow \text{state equation} \\ y(t) = g(x(t), u(t), t) & \leftarrow \text{Output equation} \end{cases}$$

nonlinear time varying causal system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}$$

Nonlinear time invariant causal

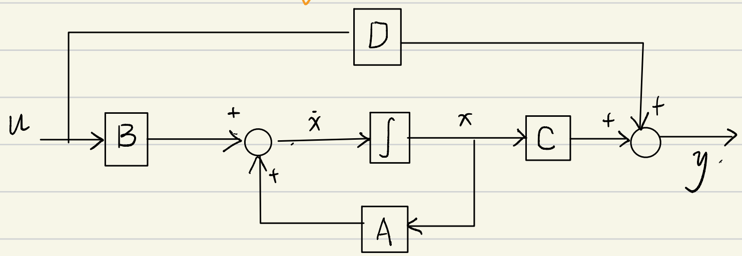
$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \end{cases}$$

Time varying, Linear

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

Linear Time Invariant

$\Downarrow$



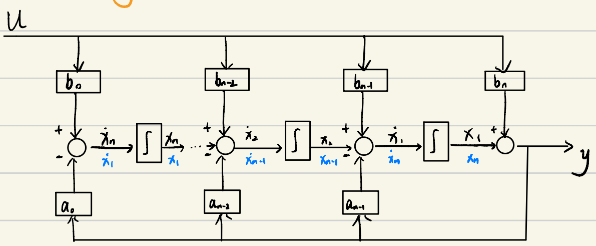
13. Observable Canonical Form (O.C.F)

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_n \frac{d^n u}{dt^n} + b_{n-1}\frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_0 u$$

define $D \equiv \frac{d}{dt} \Rightarrow D^n y + a_{n-1} D^{n-1} y + \cdots a_0 y = b_n D^n u + b_{n-1} D^{n-1} u + \cdots b_0 u$

$$y = b_n u + \frac{1}{D}[b_{n-1}u - a_{n-1}y] + \cdots \frac{1}{D^n}[b_0 u - a_0 y]$$

State Diagram:



$$\begin{cases} \dot{x}_1 = -a_{n-1}x_1 + x_2 + (b_{n-1} - a_{n-1}b_n)u \\ \dot{x}_2 = -a_{n-2}x_1 + x_3 + (b_{n-2} - a_{n-2}b_n)u \\ \vdots \\ \dot{x}_n = -a_0 x_1 + (b_0 - a_0 b_n)u \end{cases} \qquad y = b_0 u + x_1$$

$$\boxed{\text{pair } (A, C) \text{ is observeable canonical forms}}$$

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_{n-1} - a_{n-1}b_n \\ b_{n-2} - a_{n-2}b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad D = [b_n]$$

usually write

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} b_0 - a_0 b_n \\ b_1 - a_1 b_n \\ b_2 - a_2 b_n \\ \vdots \\ b_{n-2} - a_{n-2} b \\ b_{n-2} - a_{n-1} b \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \quad D = [b_n]$$

13.1 Observability

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$O_x = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n\times n}$$

$\text{rank}\{O_x\} = n.$ if is observable

column rank

13.2 Transform method

Let $\begin{cases} \dot{x}_o = A_o x_o + B_o u \\ y = C_o x_o + D_o u \end{cases}$

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \qquad D_o = D$$

$$\begin{cases} A_o = T_o^{-1} A T_o \\ B_o = T_o^{-1} B \\ C_o = C T_o \\ D_o = D \end{cases}$$

$$C_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$O_{x_o} = \begin{bmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n-1} \end{bmatrix} \qquad O_{x_o} = O_x T_o$$

$\therefore T_o^{-1} = O_{x_o}^{-1} O_x$ or $T_o = O_x^{-1} O_{x_o}$

Step: ① check observability use O_x

② Determine the coefficients of the $\boxed{\text{C.P.}}$ of matrix A $\quad \det(\lambda I - A) = 0$

Characteristic polynomial $\Pi_A(\lambda)$ $\quad a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0$

③ Form A_o C_o

④ Calculate O_x and O_{x_o} using $A, C.$ $A_o C_o$

⑤ Find the transformation $\boxed{T_o^{-1} = O_{x_o}^{-1} O_x}$

⑥ Use $T_o^{-1}B$ to get B_o

⑦ Check $A_o = T_o^{-1} A T_o$ $\quad C_o = C T_o$

⑧ $D_o = D$

14. Controllable Canonical Form (C.C.F)

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s^1 + b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s^1 + a_0}$$

$$\Downarrow$$

$$U \rightarrow \underbrace{\frac{1}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s^1 + a_0}}_{\text{all pole system}} \xrightarrow{Z} \underbrace{b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s^1 + b_0}_{\text{all zero system}} \rightarrow Y$$

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} \xrightarrow{\mathcal{L}^{-1}} \boxed{z^{(n)} + a_{n-1}z^{n-1} + \cdots a_0 z = u}$$

$$\text{Let} \begin{cases} x_1 = z \\ x_2 = \dot{z} \\ \vdots \\ x_{n-1} = z^{(n-2)} \\ x_n = z^{(n-1)} \end{cases} \Rightarrow \boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = z^{(n)} \end{cases} \quad z^{(n)} = u - a_0 x_1 - a_1 x_2 - \cdots a_{n-1} x_n}$$

$$\frac{Y(s)}{Z(s)} = b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s^1 + b_0 \xrightarrow{\mathcal{L}^{-1}} y = b_n z^{(n)} + b_{n-1} z^{(n-1)} + \cdots b_1 \dot{z} + b_0 z$$

$$\boxed{y = (b_0 - b_n a_0)x_1 + (b_1 - b_n a_1)x_2 + \cdots (b_{n-1} - b_n a_{n-1})x_n + b_n u}$$

$$A = \left[\begin{array}{c|ccccc} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \hline -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{array}\right] \quad B = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hline 1 \end{array}\right]$$

$$C = [(b_0 - a_0 b_n) \; (b_1 - a_1 b_n) \; \cdots (b_{n-1} - a_{n-1} b_n)] \quad D = [b_n]$$

$\boxed{\text{pair } (A, B) \text{ is in the Controllable Canonical Form}}$

$A.CCF = A.OCF^T$
$B.CCF = C.OCF^T$
$C.CCF = B.OCF^T$
$D.CCF = D.OCF$

14.1 controllability

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$C_x = [B \vdots AB \vdots A^2B \cdots \vdots A^{n-1}B]_{n\times n}$

$Rank\{C_x\} = n$ it is controllable

row rank

14.2. Transform method

find a nonsingular transformation $x_c = T_c^{-1}x$

then we have
$$\begin{cases} \dot{x}_c = A_c x_c + B_c u \\ y = C_c x_c + D_c u \end{cases}$$

$$A_c = \left[\begin{array}{c|cccc} 0 & & & & \\ \vdots & & I_{(n-1)\times(n-1)} & & \\ 0 & & & & \\ 0 & & & & \\ \hline -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{array}\right] \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1} \quad D_c = D$$

$$\begin{cases} A_c = T_c^{-1} A T_c \\ B_c = T_c^{-1} B \\ C_c = C T_c \\ D_c = D \end{cases}$$

$$C_{x_c} = [B_c \vdots A_c B_c \vdots \cdots A_c^{n-1} B_c] = [T_c^{-1}B \vdots (T_c^{-1} A \underbrace{\boxed{T_c)(T_c^{-1}}}_{I} B) \vdots \cdots (T_c^{-1}AT_c)^{n-1}(T_c^{-1}B)]$$
$$= T_c^{-1} C_x$$

$\therefore T_c = C_x C_{x_c}^{-1} \qquad T_c^{-1} = C_{x_c} C_x^{-1}$

Step: ① check controllability use C_x

② Determine the coefficients of the $\boxed{C.P.}$ of matrix A $\quad \det(\lambda I - A) = 0$

Characteristic polynomial $\Pi_A(\lambda)$ $\quad a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0$

③. Form A_c B_c

$$A_c = \left[\begin{array}{c|cccc} 0 & & & & \\ \vdots & & I_{(n-1)\times(n-1)} & & \\ 0 & & & & \\ 0 & & & & \\ \hline -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{array}\right] \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$$

④ Calculate C_x and C_{x_c} using $A, B.$ A_c, B_c

⑤ Find the transformation $T_c = C_x C_{x_c}^{-1}$

⑥ Use $C_c = CT_c$ to get C_c

⑦ Check $A_c = T_c^{-1} A T_c \quad B_c = T_c^{-1} B$

⑧ $D_c = D$

15. Jordan Canonical Form (J.C.F)

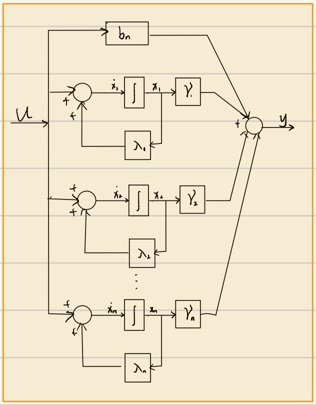
$$Y(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0} U(s) = b_n U(s) + \frac{N(s)}{D(s)} U(s)$$

$$N(s) = (b_{n-1} - a_{n-1} b_n) s^{n-1} + (b_{n-2} - a_{n-2} b_n) s^{n-2} + \cdots (b_0 - a_0 b_n)$$

$$D(s) = s^n + a_{n-1} s^{n-1} + \cdots a_0$$

①. Distinct Poles

$$Y(s) = b_n U(s) + \left[\frac{\gamma_1}{s-\lambda_1} + \frac{\gamma_2}{s-\lambda_2} + \cdots \frac{\gamma_n}{s-\lambda_n} \right] U(s)$$



$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + u \\ \dot{x}_2 = \lambda_2 x_2 + u \\ \vdots \\ \dot{x}_n = \lambda_n x_n + u \end{cases}$$

$$y = b_n u + \gamma_1 x_1 + \cdots \gamma_n x_n$$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \cdots \quad \gamma_n] \quad D = [b_n]$$

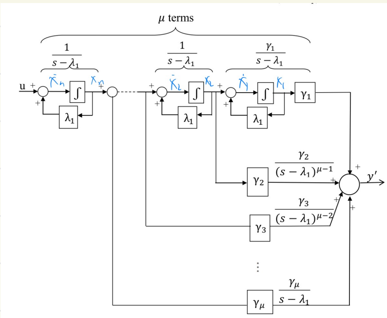
②. Multiple Repeated Poles

$$D(s) = (s-\lambda_1)^{\mu} \times (s-\lambda_2)^{\rho} \times (s-\lambda_3)(s-\lambda_4)\cdots(s-\lambda_{n-\mu-\rho})$$

$$Y = b_0 U + \left[\frac{\gamma_1}{(s-\lambda_1)^{\mu}} + \frac{\gamma_2}{(s-\lambda_1)^{\mu-1}} + \cdots \frac{\gamma_{\mu}}{s-\lambda_1}\right] U$$

$$+ \left[\frac{\gamma_{\mu+1}}{(s-\lambda_2)^{\rho}} + \cdots \frac{\gamma_{\mu+\rho}}{(s-\lambda_2)}\right] U$$

$$+ \left[\frac{\gamma_{\mu+\rho+1}}{(s-\lambda_3)} + \frac{\gamma_{\mu+\rho+2}}{(s-\lambda_4)} + \cdots \frac{\gamma_n}{s-\lambda_{n-\mu-\rho}}\right] U$$



$$A = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-\mu-\rho} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\gamma_1 \cdots \gamma_{\mu} \; \gamma_{\mu+1} \cdots \gamma_{\mu+\rho} \cdots \gamma_n] \quad D = [b_0]$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix} \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

15.1 Transform method

①. Distinct Eigenvalues

we have $\hat{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ $\hat{A} = M^{-1} A M$ $M = [x_1 \; x_2 \cdots x_n]$

$$\begin{cases} \dot{x}_J = M^{-1} A M x_J + M^{-1} B u \\ y = C M x_J + D u \end{cases} \therefore$$

$$\begin{cases} \dot{x}_J = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} x_J + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \\ y = [c_1 \quad c_2 \quad \cdots \quad c_n] x_J + Du \end{cases}$$

Controllable: $b_i \neq 0 \quad i = 1, 2, \cdots n$

Observable: $c_i \neq 0 \quad i = 1, 2, \cdots, n$

Given a matrix A and eigenvalue λ, the eigenvalue and eigenvector v relations are given by: $Av = \lambda v$ v is eigenvector

v need to be normalized

②. Repeated Eigenvalues

nullity : $\boxed{n - \text{rank}(A - \lambda_i I)}$

if nullity = multiplicity of the eigenvalues

Generalized Eigenvector

non zero vector x $(A - \lambda I)^{k-1} x \neq 0$ $(A - \lambda I)^k x = 0$

define : $x_k = x$

$x_{k-1} = (A - \lambda I) x = (A - \lambda I) x_k$

$x_{k-2} = (A - \lambda I)^2 x = (A - \lambda I) x_{k-1}$

$\vdots$

$x_1 = (A - \lambda I)^{k-1} x = (A - \lambda I) x_2$

$\boxed{\text{rank}(A - \lambda_i I)^k = n - m_i}$

multiplicity of the eigenvalues

nullity equal the number of linearly independent eigenvectors for matrix A with its associated eigenvalue λ_i

Also means how many Jordan block does the eigenvalue λ_i have

$n-r$. equal how many "1" in this Jordan block

r means total number of block

exp: $\begin{bmatrix} \lambda & 1 & \vdots & \vdots \\ 0 & \lambda & \vdots & \\ \cdots & & \lambda & \cdots \\ & & \vdots & \lambda_2 \end{bmatrix}$ $r=3$ $\qquad$ $\begin{bmatrix} \lambda_1 & 1 & 0 & \vdots \\ 0 & \lambda_1 & 1 & \vdots \\ 0 & 0 & \lambda_1 & \vdots \\ & \cdots & & \lambda_2 \end{bmatrix}$ $r=2$

$n-r=1$ only one "1" $\qquad$ $n-r=2$ only 2 "1"

15.2 Transform Method

Step ①: compute eigen value $\det(A-\lambda I)=0$

② compute number of linearly independent eigenvector for λ_i

use $n-\text{rank}(A-\lambda_i I)$

③ use ①.② Form each Jordan block

④ Determine the linearly independent vector for λ_i

⑤ generate Modal matrix M by placing the eigen vectors as columns

15.3. controllability and Observability

① each last row of each Jordan block B_i are linearly independent

$$B_i^l = \begin{bmatrix} b_{li1} \\ b_{li2} \\ \vdots \\ b_{lir(i)} \end{bmatrix}$$

② each first column of each Jordan block C_i' are linearly independent $C_i' = [C_{1i1} \; C_{1i2} \; C_{1i3} \cdots C_{1ir(i)}]$

exp:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ 0 & \lambda_1 & & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_1 & & & \\ & & & & \lambda_2 & 1 & 0 \\ & & & & 0 & \lambda_2 & 1 \\ & & & & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 0 & 2 & 2 \end{bmatrix} x$$

$B_1^l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ full rank

$B_2^l = [0 \; 0 \; 1]$ linearly independent

$C_1' = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ full rank

$C_2' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ rank $= 0$ $\therefore$ unobservable

15.4 JCF with complex conjugate eigenvalues

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & \bar{A}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ \bar{b}_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & \bar{C}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Introduce $\bar{x} = Px$

$$\boxed{P = \begin{bmatrix} I & I \\ jI & -jI \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{2} \begin{bmatrix} I & -jI \\ I & jI \end{bmatrix}}$$

then:
$$\begin{cases} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \operatorname{Re} A_1 & \operatorname{Im} A_1 \\ -\operatorname{Im} A_1 & \operatorname{Re} A_1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 2\operatorname{Re} b_1 \\ -2\operatorname{Im} b_1 \end{bmatrix} u \\ y = \begin{bmatrix} \operatorname{Re} C_1 & \operatorname{Im} C_1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{cases}$$

16. Transfer Function (LTI System)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

17. Solution of State Equation

$$x(t) = \underbrace{e^{\int_{t_0}^{t} a(\tau)d\tau} x(t_0)}_{\text{zero input response}} + \underbrace{\int_{t_0}^{t} e^{\int_{\tau}^{t} a(\tau')d\tau'} b(\tau)u(\tau)\,d\tau}_{\text{zero state response}}$$

if $a(t) = a$ (constant):

$$x(t) = e^{a(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} b(\tau) u \, d\tau$$

if $t_0 = 0$

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-\tau)} b(\tau) u \, d\tau$$

prove: $\mathcal{L}\{\dot{x} = ax + b(t)u\}$ $\quad t=0$

$$sX(s) - x(0) = aX(s) + \mathcal{L}\{b(t)u\}$$

$$X(s) = \frac{1}{s-a} x(0) + \frac{1}{s-a}\mathcal{L}\{b(t)u\}$$

$\Downarrow \mathcal{L}^{-1}$

$$x(t) = e^{at}x(0) + \mathcal{L}^{-1}\{\underbrace{\frac{1}{s-a}}_{G_1}\underbrace{\mathcal{L}\{b(t)u\}}_{G_2}\}$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} b(\tau)u\,d\tau$$

$$\boxed{\mathcal{L}^{-1}\{G_1(s)G_2(s)\} = \int_0^t g_1(t-\tau)g_2(\tau)d\tau = g_1(t) * g_2(t)}$$

18. Fundamental Matrix ($M(t)$)

n columns of $M(t)$ are n linearly independent solutions of $\dot{x} = A(t)x$.

$\dot{M}(t) = A(t)M(t)$

each columns are $\boxed{\text{linearly independent}}$

19. State Transition Matrix [$\Phi(t)$]

$$\dot{x} = A(t)x$$

$$\underbrace{\Phi(t,t_0)}_{\text{unique}} = \underbrace{M(t)}_{\text{not unique}} M^{-1}(t_0) \qquad x(t) = \Phi(t,t_0)\,x(t_0)$$

Properties :

① $\dfrac{\partial \Phi(t,t_0)}{\partial t} = A(t)\Phi(t,t_0)$

② $\Phi(t_0,t_0) = I$

③ $\left.\dfrac{\partial \Phi(t,t_0)}{\partial t}\right|_{t=t_0} = A(t_0)\cdot I$

④ $\Phi(t_2,t_0) = \Phi(t_2,t_1)\,\Phi(t_1,t_0)$

⑤ $\Phi^{-1}(t_1,t_0) = \Phi(t_0,t_1)$

20. State Transition Matrix for $\boxed{\text{LTI}}$ System

Exp: Assuming $\dot{x} = Ax \Rightarrow x(t) = e^{A(t-t_0)}x(t_0)$

$$\dot{x}(t) = A\,e^{A(t-t_0)}x(t_0)$$

Since $x(t) = \Phi(t,t_0)x(t_0)$

$\therefore\ \Phi(t,t_0) = e^{A(t-t_0)}$

$\Phi(t,t_0) = \Phi(t-t_0)$

Let $t_0 = 0$

$\Phi(t,t_0) = \Phi(t) \qquad \Phi^{-1}(t) = \Phi(-t)$

20.1 Approaches to Compute $\Phi(t)$ $\boxed{LTI}$

① $\Phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

① compute $(sI-A)^{-1}$ using Leverrier's algorithm:

$$(sI-A)^{-1} = \frac{adj(sI-A)}{det(sI-A)} = \frac{P_{n-1}s^{n-1} + P_{n-2}s^{n-2} + \cdots + P_1 s + P_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$P_{n-1} = I_{n\times n}$

$a_{n-1} = -tr(A)$ $\qquad$ $tr = \text{trace} = \text{sum}(a_{ii})\ i=1,2,\cdots n$

$P_{n-2} = P_{n-1}A + a_{n-1}I$

$a_{n-2} = -\frac{1}{2}tr(P_{n-2}A)$

$\vdots$

$$\boxed{\begin{aligned} P_k &= P_{k+1}A + a_{k+1}I \\ a_k &= \frac{-1}{n-k}tr(P_kA) \end{aligned}}$$

$P_0 A + a_0 I = 0$

② $\left[\, sI-A \;\vdots\; I_{(n\times n)} \,\right] \Rightarrow \left[\, I_{(n\times n)} \;\vdots\; (sI-A)^{-1} \,\right]$

② Cayley-Hamilton Technique

1. $\Pi_A(\lambda) = det(\lambda I - A) = 0$

$\Pi_A(A) = 0$

$\Downarrow$

2. $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$

$A^n + a_1 A^{n-1} + \cdots\ a_{n-1}A + a_n I = 0$

$P(A) = R(A)$ $\qquad$ $R(A) = \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots \alpha_{n-1}A + \alpha_0 I$

if λ repeat m times

$$\left.\frac{dp}{d\lambda}\right|_{\lambda=\lambda_1} = \left.\frac{dR}{d\lambda}\right|_{\lambda=\lambda_1}$$

$$\left.\frac{d^2p}{d\lambda^2}\right|_{\lambda=\lambda_1} = \left.\frac{d^2R}{d\lambda^2}\right|_{\lambda=a_1}$$

$$\left.\frac{d^{m-1}p}{d\lambda^{m-1}}\right|_{\lambda=\lambda_1} = \left.\frac{d^{m-1}p}{d\lambda^{m-1}}\right|_{\lambda=\lambda_1}$$

exp 1: $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ Find e^{At}

$n=2$ $P(A) = e^{At}$ $P(\lambda) = e^{\lambda t}$ $R(\lambda) = \alpha\lambda + \beta$

$\pi_A(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix} = 0$ $\lambda(\lambda+2)+1=0$

$(\lambda+1)^2 = 0$

$\lambda = -1, -1$ (Repeated)

$\therefore \lambda = -1 \Rightarrow \begin{cases} e^{-t} = -\alpha+\beta \\ \dfrac{de^{\lambda t}}{d\lambda} = \dfrac{d\,\alpha\lambda+\beta}{d\lambda} \end{cases} \Rightarrow \begin{cases} e^{-t} = -\alpha+\beta \\ te^{-t} = \alpha \end{cases} \Rightarrow \begin{cases} \beta = e^{-t} + te^{-t} \\ \alpha = te^{-t} \end{cases}$

Since $P(A) = R(A)$

$\therefore e^{At} = te^{-t}A + (e^{-t} + te^{-t}) \times I$

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exp 2: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & -27 & 9 \end{bmatrix}$ Find e^{At}

$n=3$ $P(A) = e^{At}$ $P(\lambda) = e^{\lambda t}$ $R(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 \cdot I$

$\pi_A(\lambda) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -27 & 27 & \lambda-9 \end{bmatrix} = \lambda[\lambda(\lambda-9)+27] - (-1)(0-27)$

$= \lambda(\lambda^2 - 9\lambda + 27) - 27$

$= \lambda^3 - 9\lambda^2 + 27\lambda - 27$

$= (\lambda-3)^3$

$\lambda = 3$ repeated

$\begin{cases} e^{3t} = \alpha_2 \times 9 + \alpha_1 \times 3 + \alpha_0 \\ \dfrac{de^{\lambda t}}{d\lambda} = 2\alpha_2\lambda + \alpha_1 \\ \dfrac{d^2 e^{\lambda t}}{d\lambda^2} = 2\alpha_2 \end{cases} \Rightarrow \lambda = 3$ $\begin{cases} \alpha_0 = e^{3t}(1 - 3t + \frac{9}{2}t^2) \\ \alpha_1 = (t - 3t^2)e^{3t} \\ \alpha_2 = \frac{1}{2}t^2 e^{3t} \end{cases}$

$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$

③. $\boxed{e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}}$

If A is $\boxed{\text{sparse matrix}}$

$A^n = 0$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots$$

Properties :

① $e^0 = I$

② $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$ (t_1, t_2) scalars

③. $e^{(A+B)t} = e^{At}e^{Bt}$ AB=BA (A,B commute)

④ $\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$ (A , e^{At} commute)

④ Jordan Form

$$A = \begin{bmatrix} J_1 & \cdots & & & 0 \\ \vdots & J_2 & \cdots & & 0 \\ \vdots & \vdots & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{J_1 t} & \cdots & & & 0 \\ \vdots & e^{J_2 t} & \cdots & & 0 \\ \vdots & \vdots & e^{\lambda_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

$$e^{J_1 t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_1 t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & \cdots & \frac{t^{n-3}}{(n-3)!}e^{\lambda_1 t} & \frac{t^{n-2}}{(n-2)!}e^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} & \cdots & \frac{t^{n-4}}{(n-4)!}e^{\lambda_1 t} & \frac{t^{n-3}}{(n-3)!}e^{\lambda_1 t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & \cdots & 0 & e^{\lambda_1 t} \end{bmatrix}$$

21. special formula

$$e^{jx} = \cos x + j\sin x \qquad \sin A = \frac{e^{jA} - e^{-jA}}{2j} \qquad \cosh^2 A - \sinh^2 A = I \qquad \cosh A = \frac{e^{A} + e^{-A}}{2}$$

$$1 \pm j = \sqrt{2}\, e^{-j\frac{\pi}{4}} \qquad \cos A = \frac{e^{jA} + e^{-jA}}{2} \qquad \sinh A = \frac{e^{A} - e^{-A}}{2}$$

22. $\Phi(t,t_0)$ for $\boxed{\text{LTV}}$ system can't use Laplace

① $\Phi(t,t_0) = M(t)\,M^{-1}(t_0)$

② Direct solution $\dot{x} = A(t)\,x$ to get $x(t) = \Phi(t,t_0)\,x(t_0)$

③ if $A(t)$, $A(t_0)$ commute $\quad \Phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)\,d\tau}$

④ if $A(t) = \sum_{i=1}^{k} \alpha_i(t)\, M_i$ $\quad$ a_i is scalar $\quad \Phi(t,t_0) = \prod_{i=1}^{k} e^{M_i \int_{t_0}^{t} a_i(\tau)\,d\tau}$

constant

$M_i\, M_j$ commute

Special formula

$$e^{\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$$

23. Solution of the state Equation and Output Equation

LTI System

$$x(t) = \Phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

$$\frac{\partial \Phi(t,t_0)}{\partial t}\,x(t_0) + \frac{\partial}{\partial t}\int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau = A(t)\,\Phi(t,t_0)\,x(t_0) + A(t)\int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau + B(t)\,u(t)$$

Leibniz's Rule

$$\boxed{\frac{\partial}{\partial t}\int_{u(t)}^{v(t)} f(x,t)\,dx = \int_{u(t)}^{v(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f(v,t)\frac{\partial v}{\partial t} - f(u,t)\frac{\partial u}{\partial t}}$$

$$y(t) = C(t)\boxed{\Phi(t,t_0)x(t_0)} + C(t)\boxed{\int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau} + D(t)u(t)$$

$x(t)$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$