Linear System

1. Superposition principle

$$\mathcal{L}\{a_1u_1(t)+a_2u_2(t)\} = a_1\mathcal{L}\{u_1(t)\}+a_2\mathcal{L}\{u_2(t)\}$$

2. Table of laplace Transforms.

$f(t)$	$\delta(t)$	$1(t)$	t	$\frac{1}{2}t^2$	$\frac{t^k}{k!}$ $k>0$	e^{-at}	$\frac{1}{n!}t^n e^{-at}$	$\sin\omega t$	$\cos\omega t$	$e^{-at}\sin\omega t$	$e^{-at}\cos\omega t$
$F(s)$	1	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s^3}$	$\frac{1}{s^{k+1}}$	$\frac{1}{s+a}$	$\frac{1}{(s+a)^{n+1}}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{s+a}{(s+a)^2+\omega^2}$

3. Property of Laplace

① $F(s)=\int_0^{+\infty} f(t)e^{-st}dt$ ② $\mathcal{L}[f'(t)] = sF(s) - f(0)$ ③ $\mathcal{L}[f^{(n)}(t)] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^-)$

④ $\mathcal{L}[af(t) \pm bg(t)] = a\mathcal{L}[f(t)] \pm b\mathcal{L}[g(t)]$ ⑤ $\mathcal{L}[e^{at}f(t)] = F(s-a)$ ⑥ $\mathcal{L}[f(t-a)] = e^{-as}F(s)$

⑦ $\mathcal{L}[\int_0^t f(\tau)d\tau] = \frac{1}{s}F(s)$ ⑧ $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

4. Calculate Pole

exp. $\frac{2s+7}{s(s+4)(s+1)}$ Pole $= 0, -4, -1$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+4} + \frac{C_3}{s+1}$$

$$C_1 = \frac{2s+7}{s(s+4)(s+1)} \times s \Big|_{s=0} = \frac{7}{4}$$

$$C_2 = \frac{2s+7}{s(s+4)(s+1)} \times (s+4) \Big|_{s=-4} = -\frac{1}{12}$$

$$C_3 = \frac{2s+7}{s(s+4)(s+1)} \times (s+1) \Big|_{s=-1} = -\frac{5}{3}$$

$$X(s) = \frac{\frac{7}{4}}{s} - \frac{\frac{1}{12}}{(s+4)} - \frac{\frac{5}{3}}{(s+1)}$$

$\Downarrow \mathcal{L}^{-1}$

$$x(t) = \frac{7}{4} - \frac{1}{12}e^{-4t} - \frac{5}{3}e^{-t}$$

$$\frac{5s+7}{s(s+2)(s+3)^3} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_{31}}{(s+3)^3} + \frac{C_{32}}{(s+3)^2} + \frac{C_{33}}{(s+3)}$$

$$C_1 = \frac{5s+7}{s(s+2)(s+3)^3} \times s \Big|_{s=0}$$

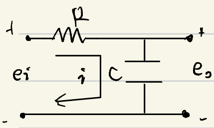
$$C_2 = \frac{5s+7}{s(s+2)(s+3)^3} \times (s+2) \Big|_{s=-2}$$

$$C_{31} = \frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3 \Big|_{s=-3}$$

$$C_{32} = \left[\frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3\right]' \Big|_{s=-3}$$

$$C_{33} = \left[\frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3\right]'' \Big|_{s=-3}$$

5. I/O Discription



$$\begin{cases} e_i = Ri + \frac{1}{C}\int i\,dt \\ e_o = \frac{1}{C}\int i\,dt \end{cases} \Rightarrow \frac{de_o}{dt} = \frac{i}{C} \Rightarrow i = C\frac{de_o}{dt}$$

$$e_i = RC\frac{de_o}{dt} + e_o \Rightarrow e_o(\tau) = \frac{1}{RC}\int_0^{\tau} e^{(t-\tau)/RC} e_i\,dt$$

① $e_i=0$ $e_o(0)\neq 0$ → zero input response : $e_o(t) = e_o(0)e^{-\frac{t}{RC}}$

② $e_i\neq 0$ $e_o(0)=0$ → zero state response : $e_o(t) = A(1-e^{-t/RC})$

③ $e_i\neq 0$ $e_o(0)\neq 0$ → general response : $e_o(t) = e_o(0)e^{-\frac{t}{RC}} + A(1-e^{-\frac{t}{RC}})$

6. show linear:

exp: $y(t) = \int_0^t u(\tau)\,d\tau$

Step 1: $u_1 \to y_1$; $u_2 \to y_2 \Rightarrow y_1 = \int_0^t u_1 d\tau \quad y_2 = \int_0^t u_2 d\tau$

Step 2: $u = a_1 u_1 + a_2 u_2 \to y$ Superposition principle

Step 3: $y = \int_0^t (a_1 u_1 + a_2 u_2)\,d\tau = a_1 \int_0^t u_1 d\tau + a_2 \int_0^t u_2 d\tau = a_1 y_1 + a_2 y_2$

7. Causality:

The system is said to be causal if the output at time t does not depend on the input at time after t

8. Time variance

$u(t) \to u(t-t_0) \to \boxed{L} = y_1(t)$

$y(t) \to y(t-t_0) \qquad y_1(t) = y(t-t_0)$

9. Types of signal.

Step: $f(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$ Ramp: $f(t) = \begin{cases} t & t \geq 0 \\ 0 & t<0 \end{cases}$ impulse: $\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t \neq 0 \end{cases}$ and $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$

10. Impulse Response $u(t) = \int_{-\infty}^{+\infty} u(\tau)\,\delta(t-\tau)\,d\tau$

$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau$ then $y(t) = L\{\int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau\} = \int_{-\infty}^{\infty} u(\tau)\,h(t,\tau)\,d\tau$

LTI: $\delta(t-\tau) \to h(t-\tau, 0) \equiv h(t-\tau) \Rightarrow y(t) = \int_0^t u(\tau)\,h(t-\tau)\,dt$

exp: $h(t,\tau) = e^{-(t-\tau)^2}$, $\sin(t-\tau)$ $e^{\sin(t-\tau)}$

LTV. $y(t) = \int_{-\infty}^{+\infty} u(\tau)\, h(t,\tau)\, d\tau$

exp: $h(t,\tau) = e^{-\frac{t-\tau}{t}}$, $\sin t$, $e^{\sin t}$

II. Computation Impulse response

exp: use laplace

$\ddot{y} + 4\dot{y} + 3y = 4\dot{u} + 5u \quad \mathcal{L} \Rightarrow \quad s^2 Y(s) + 4sY(s) + 3Y(s) = 4sU(s) + 5U(s)$

①

$H(s) = \frac{Y(s)}{U(s)} = \frac{4s+5}{s^2+4s+3} \Rightarrow h(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{\frac{7}{2}}{s+3} + \frac{\frac{1}{2}}{s+1}\right)$

$= \frac{7}{2}e^{-3t} + \frac{1}{2}e^{-t} \quad t \geq 0$

②. step input

$y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau$

$\begin{cases} h(t) = \frac{7}{2}e^{-3t} + \frac{1}{2}e^{-t}, \ t \geq 0 \\ u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \end{cases}$

$y(t) = \int_0^t \left[\frac{7}{2}e^{-3(t-\tau)} + \frac{1}{2}e^{-(t-\tau)}\right] 1 \cdot d\tau$

$= \frac{5}{3} - \frac{7}{6}e^{-3t} \frac{1}{2}e^{-t}$

③ use T.F.

$Y(s) = H(s)U(s) \qquad H(s) = \frac{4s+5}{s^2+4s+3} \qquad U(s) = \frac{1}{s}$

$\therefore Y(s) = \frac{4s+5}{s(s^2+4s+3)} \Rightarrow y(t) = \frac{5}{3} - \frac{7}{6}e^{-3t} - \frac{1}{2}e^{-t}$

④. If $y(0) = 1 \quad \dot{y}(0) = 0$

$\mathcal{L}\{\ddot{y} + 4\dot{y} + 3y = 4\dot{u} + 5u\}$

$\mathcal{L}\{\ddot{y}\} = s^2 Y(s) - sY(0^-) - \dot{y}(0^-)$

$\mathcal{L}\{\dot{y}\} = sY(s) - y(0^-)$

$\Downarrow$

$s^2 Y(s) - s y(0) - \dot{y}(0) + 4[sY(s) - y(0)] + 3Y(s) = 4sU(s) + 5U(s)$

$Y(s) = \frac{4s+5}{s^2+4s+3} \times U(s) + \frac{s+4}{s^2+4s+3}$

$y(t) = \mathcal{L}^{-1} Y(s) = \frac{5}{3} - \frac{5}{3}e^{-3t} + e^{-t}$

State Diagram

exp

Find a state space representation (A, B, C, D) for the system described by

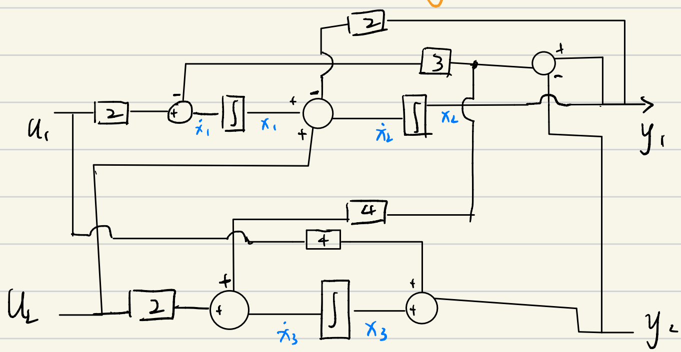
$$\ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + \dot{u}_2$$

$$\dot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4\dot{u}_1$$

Hint: Use the state diagram approach.

$$\begin{cases} \ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + \dot{u}_2 \\ \dot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4\dot{u}_1 \end{cases} \Rightarrow \begin{cases} \ddot{y}_1 = 2u_1 + \dot{u}_2 - 2\dot{y}_1 - 3(y_1 - y_2) \\ \dot{y}_2 = 4(y_1 - y_2) + 2u_2 + 4\dot{u}_1 \end{cases} \xRightarrow{\text{integra}} \begin{cases} y_1 = \iint [2u_1 - 3(y_1 - y_2)] + \int [u_2 - 2y_1] \\ y_2 = \int [4(y_1 - y_2) + 2u_2] + 4u_1 \end{cases}$$

state Diagram ↓



$$\begin{cases} \dot{x}_1 = 2u_1 - 3(y_1 - y_2) \\ \dot{x}_2 = x_1 - 2y_1 + u_2 \\ \dot{x}_3 = 2u_2 + 4(y_1 - y_2) \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = 2u_1 - 3(x_2 - x_3 - 4u_1) \\ \dot{x}_2 = x_1 - 2x_2 + u_2 \\ \dot{x}_3 = 2u_2 + 4(x_2 - x_3 - 4u_1) \end{cases}$$

$$\begin{cases} y_1 = x_2 \\ y_2 = x_3 + 4u_1 \end{cases}$$

$$\boxed{\therefore A = \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 0 \\ 0 & 4 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 14 & 0 \\ 0 & 1 \\ -16 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}}$$

LTI example

(1) The response of a LTI causal system to the input $u_1(t) = u_s(t)\,u_s(2-t)$ is $y_1(t) = (1-e^{-t})\,u_s(t)\,u_s(2-t) + e^{-(t-2)}\,u_s(t-2)$. Find the response of the system to a new input $u_2(t) = (t-2)\,u_s(t-2)\,u_s(4-t)$ for $t \le 4$. Note that $u_s(t)$ designates a unit step function.

$$u_1(t-2) = u_s(t-2)\,u_s(4-t) \Rightarrow y_1(t-2) = (1-e^{-(t-2)})\,u_s(t-2)\,u_s(4-t) + e^{-(t-4)}\,u_s(t-4)$$

$$(t-2)u_1(t-2) = (t-2)\,u_s(t-2)\,u_s(4-t) = u_2(t) \Rightarrow (t-2)y_1(t-2) = (t-2)(1-e^{-(t-2)})\,u_s(t-2)\,u_s(4-t) + (t-2)\,e^{-(t-4)}\,u_s(t-4)$$

$\boxed{\text{for } t \le 4}$ $\quad u_s(t-4) = 0$

$$\therefore\ y_2(t) = (t-2)(1-e^{-(t-2)})\,u_s(4-t)$$

(2) Consider a LTI system $\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$

Let $u(t) = \begin{bmatrix} -3e^{-t} + 5e^{-2t} \\ 3e^{-2t} \end{bmatrix}$ for $t \ge 0$, and suppose that $y(t) = -2e^{-t} + 6e^{-2t}$. If $\lim_{t\to\infty} x(t) = 0$, find $x(0)$.

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + u_1 - u_2 \\ \dot{x}_2 = u_1 + u_2 \\ y = x_1 - x_2 \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = -x_1 + x_2 + (-3e^{-t} + 5e^{-2t}) - 3e^{-2t} \\ \dot{x}_2 = -3e^{-t} + 5e^{-2t} + 3e^{-2t} = -3e^{-t} + 8e^{-2t} \Rightarrow x_2 = 3e^{-t} - 4e^{-2t} + C \\ \dot{y} = \dot{x}_1 - \dot{x}_2 = -x_1 + x_2 - 3e^{-t} + 2e^{-2t} + 3e^{-t} - 8e^{-2t} \\ \quad = 2e^{-t} - 12e^{-2t} \end{cases}$$

Since $\lim_{t\to\infty} x(t) = 0$ $\therefore C = 0$

$$\begin{cases} -x_1 + x_2 - 3e^{-t} + 2e^{-2t} + 3e^{-t} - 8e^{-2t} = 2e^{-t} - 12e^{-2t} \\ x_2 = 3e^{-t} - 4e^{-2t} \end{cases} \Rightarrow \begin{cases} x_1 = \cancel{3e^{-t}} - 4e^{-2t} - \cancel{3e^{-t}} + 2e^{-2t} + 3e^{-t} - 8e^{-2t} - 2e^{-t} + 12e^{-2t} \\ \quad = e^{-t} + 2e^{-2t} \\ x_2 = 3e^{-t} - 4e^{-2t} \end{cases}$$

$$\therefore\ x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

3) Given $\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} x_1(t-1)+|u(t)| \\ x_1(t)\end{bmatrix}$; $y(t)=x_1(t+1)+x_2(t)-u(t)$ $t\to\infty$ noncausal

Find the appropriate description of the system among:

Linear	Time-Invariant	Causal	Lumped	Continuous-time
Nonlinear	Time-Variant	Noncausal	Distributed	Discrete-time

$$\begin{cases} \dot{x}_1(t) = x_1(t-1) + |u(t)| \\ \dot{x}_2(t) = x_1(t) \end{cases} \quad \therefore \begin{cases} \dot{x} = f(x(t), u(t)) \\ y = g(x(t), u(t)) \end{cases}$$

$$y(t) = x_1(t+1) + x_2(t) - u(t)$$

Nonlinear TI

Distinction Lumped and Distributed

① ODE is lumped (ordinary differential equation)
PDE is distributed (partial differential equation)

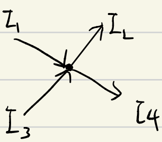
② has delay signal is distributed

exp $\dot{x}_1(t) = x_1(t-1) + u(t)$

State Models

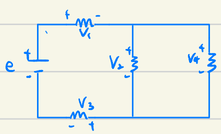
①. RLC network the voltage across the capacitor and the current through an inductor are the states.

② KCL



$\Rightarrow I_1 + I_3 = I_2 + I_4$

③ KVL =



$\Rightarrow$ $e = V_1 + V_2 + V_3$ $\quad V_2 = V_4$

$e = V_1 + V_4 + V_3$

12. General State Space Representation

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) & \leftarrow \text{state equation} \\ y(t) = g(x(t), u(t), t) & \leftarrow \text{Output equation} \end{cases}$$

nonlinear time varying causal system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}$$

Nonlinear time invariant causal

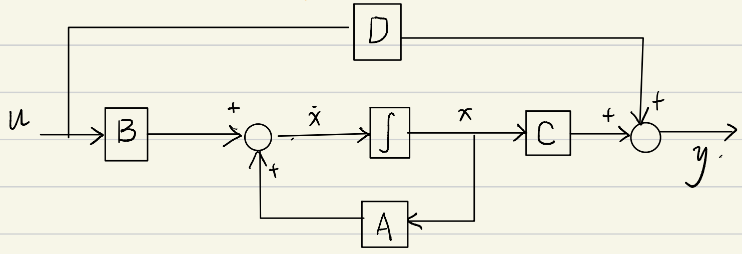
$$\begin{cases} \dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) \\ y(t) = C(t)\, x(t) + D(t)\, u(t) \end{cases}$$

Time varying, Linear

$$\begin{cases} \dot{x}(t) = A\, x(t) + B\, u(t) \\ y(t) = C\, x(t) + D\, u(t) \end{cases}$$

Linear Time Invariant

$\Downarrow$



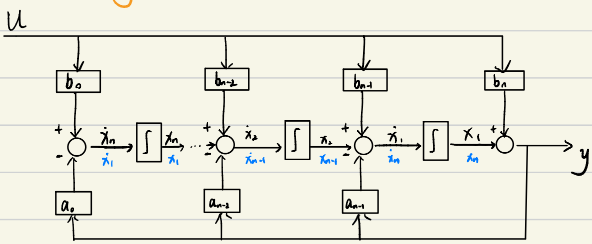
13. Observable Canonical Form (O.C.F)

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_n \frac{d^n u}{dt^n} + b_{n-1}\frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_0 u$$

define $D \equiv \frac{d}{dt} \Rightarrow D^n y + a_{n-1} D^{n-1} y + \cdots a_0 y = b_n D^n u + b_{n-1} D^{n-1} u + \cdots b_0 u$

$$y = b_n u + \frac{1}{D}[b_{n-1}u - a_{n-1}y] + \cdots \frac{1}{D^n}[b_0 u - a_0 y]$$

State Diagram:



$$\begin{cases} \dot{x}_1 = -a_{n-1}x_1 + x_2 + (b_{n-1} - a_{n-1}b_n)u \\ \dot{x}_2 = -a_{n-2}x_1 + x_3 + (b_{n-2} - a_{n-2}b_n)u \\ \vdots \\ \dot{x}_n = -a_0 x_1 + (b_0 - a_0 b_n)u \end{cases} \qquad y = b_0 u + x_1$$

$\boxed{\text{pair } (A, C) \text{ is observeable canonical forms}}$

$$A = \left[\begin{array}{c|cccc} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ \hline -a_0 & 0 & 0 & \cdots & 0 \end{array}\right] \quad B = \begin{bmatrix} b_{n-1} - a_{n-1}b_n \\ b_{n-2} - a_{n-2}b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{bmatrix}$$

$$C = \left[\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \end{array}\right] \quad D = [b_n]$$

usually write

$$A = \left[\begin{array}{cccccc|c} 0 & 0 & \cdots & 0 & 0 & & -a_0 \\ \hline 1 & 0 & \cdots & 0 & 0 & & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & & -a_{n-1} \end{array}\right] \quad B = \begin{bmatrix} b_0 - a_0 b_n \\ b_1 - a_1 b_n \\ b_2 - a_2 b_n \\ \vdots \\ b_{n-2} - a_{n-2} b \\ b_{n-2} - a_{n-1} b \end{bmatrix}$$

$$C = \left[\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right] \quad D = [b_n]$$

13.1 Observability

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad O_x = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n \times n} \qquad \text{rank}\{O_x\} = n. \quad \text{if is observable}$$

column rank

13.2 Transform method

Let $\begin{cases} \dot{x}_o = A_o x_o + B_o u \\ y = C_o x_o + D_o u \end{cases}$

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \qquad D_o = D$$

$$\begin{cases} A_o = T_o^{-1} A T_o \\ B_o = T_o^{-1} B \\ C_o = C T_o \\ D_o = D \end{cases} \qquad C_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$O_{x_o} = \begin{bmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n-1} \end{bmatrix} \qquad O_{x_o} = O_x T_o$$

$\therefore\ T_o^{-1} = O_{x_o}^{-1} O_x$ or $T_o = O_x^{-1} O_{x_o}$

Step: ① check observability use O_x

② Determine the coefficients of the $\boxed{\text{C.P.}}$ of matrix A $\quad \det(\lambda I - A) = 0$

Characteristic polynomial $\pi_A(\lambda)$ $\quad a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0$

③ Form A_o C_o

④ Calculate O_x and O_{x_o} using A, C. $A_o\, C_o$

⑤ Find the transformation $\boxed{T_o^{-1} = O_{x_o}^{-1} O_x}$

⑥ Use $T_o^{-1}B$ to get B_o

⑦ Check $A_o = T_o^{-1} A T_o$ $\quad C_o = C T_o$

⑧ $D_o = D$

14. Controllable Canonical Form (C.C.F)

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots b_1 s^1 + b_0}{s^n + a_{n-1} s^{n-1} + \cdots a_1 s^1 + a_0}$$

$$\Downarrow$$

$$U \rightarrow \underbrace{\frac{1}{s^n + a_{n-1} s^{n-1} + \cdots a_1 s^1 + a_0}}_{\text{all pole system}} \xrightarrow{Z} \underbrace{b_n s^n + b_{n-1} s^{n-1} + \cdots b_1 s^1 + b_0}_{\text{all zero system}} \rightarrow Y$$

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0} \xrightarrow{\mathcal{L}^{-1}} \boxed{z^{(n)} + a_{n-1} z^{n-1} + \cdots a_0 z = u}$$

$$\text{Let} \begin{cases} x_1 = z \\ x_2 = \dot{z} \\ \vdots \\ x_{n-1} = z^{(n-2)} \\ x_n = z^{(n-1)} \end{cases} \Rightarrow \boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = z^{(n)} \end{cases} \quad z^{(n)} = u - a_0 x_1 - a_1 x_2 - \cdots a_{n-1} x_n}$$

$$\frac{Y(s)}{Z(s)} = b_n s^n + b_{n-1} s^{n-1} + \cdots b_1 s^1 + b_0 \xrightarrow{\mathcal{L}^{-1}} y = b_n z^{(n)} + b_{n-1} z^{(n-1)} + \cdots b_1 z' + b_0 z$$

$$\boxed{y = (b_0 - b_n a_0) x_1 + (b_1 - b_n a_1) x_2 + \cdots (b_{n-1} - b_n a_{n-1}) x_n + b_n u}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [(b_0 - a_0 b_n) \ (b_1 - a_1 b_n) \ \cdots (b_{n-1} - a_{n-1} b_n)] \quad D = [b_n]$$

pair (A, B) is in the Controllable Canonical Form

$A.CCF = A.OCF^T$

$B.CCF = C.OCF^T$

$C.CCF = B.OCF^T$

$D.CCF = D.OCF$

14.1 controllability

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$C_x = [B : AB : A^2B \cdots : A^{n-1}B]_{n\times n}$

$Rank\{C_x\} = n$ it is controllable

row rank

14.2. Transform method

find a nonsingular transformation $x_c = T_c^{-1} x$

then we have
$$\begin{cases} \dot{x}_c = A_c x_c + B_c u \\ y = C_c x_c + D_c u \end{cases}$$

$$A_c = \begin{bmatrix} 0 & & & & \\ \vdots & & I_{(n-1)\times(n-1)} & & \\ 0 & & & & \\ 0 & & & & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1} \quad D_c = D$$

$$\begin{cases} A_c = T_c^{-1} A T_c \\ B_c = T_c^{-1} B \\ C_c = C T_c \\ D_c = D \end{cases}$$

$$C_{x_c} = [B_c : A_c B_c : \cdots A_c^{n-1} B_c] = [T_c^{-1}B : (T_c^{-1} A \underbrace{\boxed{T_c)(T_c^{-1}}}_{I} B) : \cdots (T_c^{-1} A T_c)^{n-1} (T_c^{-1} B)]$$

$$= T_c^{-1} C_x$$

$$\therefore T_c = C_x C_{x_c}^{-1} \qquad T_c^{-1} = C_{x_c} C_x^{-1}$$

Step: ① Check controllability use C_x

② Determine the coefficients of the $\boxed{C.P.}$ of matrix A $\det(\lambda I - A) = 0$

Characteristic polynomial $\Pi_A(\lambda)$

$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0$

③ Form A_c B_c

$$A_c = \begin{bmatrix} 0 & & & & \\ \vdots & & I_{(n-1)\times(n-1)} & & \\ 0 & & & & \\ 0 & & & & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$$

④ Calculate C_x and C_{x_c} using A, B. A_c, B_c

⑤ Find the transformation $T_c = C_x C_{x_c}^{-1}$

⑥ Use $C_c = C T_c$ to get C_c

⑦ Check $A_c = T_c^{-1} A T_c$ $B_c = T_c^{-1} B$

⑧ $D_c = D$

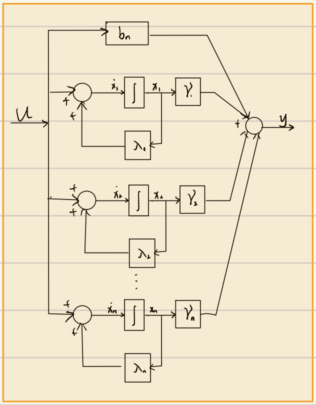
15. Jordan Canonical Form (J.C.F)

$$Y(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0} U(s) = b_n U(s) + \frac{N(s)}{D(s)} U(s)$$

$$N(s) = (b_{n-1} - a_{n-1} b_n) s^{n-1} + (b_{n-2} - a_{n-2} b_n) s^{n-2} + \cdots (b_0 - a_0 b_n)$$
$$D(s) = s^n + a_{n-1} s^{n-1} + \cdots a_0$$

①. Distinct Poles

$$Y(s) = b_n U(s) + \left[\frac{\gamma_1}{s-\lambda_1} + \frac{\gamma_2}{s-\lambda_2} + \cdots \frac{\gamma_n}{s-\lambda_n}\right] U(s)$$



$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + u \\ \dot{x}_2 = \lambda_2 x_2 + u \\ \vdots \\ \dot{x}_n = \lambda_n x_n + u \end{cases}$$

$$y = b_n u + \gamma_1 x_1 + \cdots \gamma_n x_n$$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\gamma_1 \ \ \gamma_2 \ \ \gamma_3 \ \cdots \ \gamma_n] \quad D = [b_n]$$

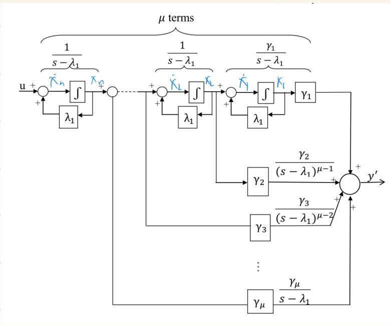
②. Multiple Repeated Poles

$$D(s) = (s-\lambda_1)^{\mu} \times (s-\lambda_2)^{\rho} \times (s-\lambda_3)(s-\lambda_4)\cdots(s-\lambda_{n-\mu-\rho})$$

$$Y = b_0 U + \left[\frac{\gamma_1}{(s-\lambda_1)^{\mu}} + \frac{\gamma_2}{(s-\lambda_1)^{\mu-1}} + \cdots \frac{\gamma_{\mu}}{s-\lambda_1}\right] U$$

$$+ \left[\frac{\gamma_{\mu+1}}{(s-\lambda_2)^{\rho}} + \cdots \frac{\gamma_{\mu+\rho}}{(s-\lambda_2)}\right] U$$

$$+ \left[\frac{\gamma_{\mu+\rho+1}}{(s-\lambda_3)} + \frac{\gamma_{\mu+\rho+2}}{(s-\lambda_4)} + \cdots \frac{\gamma_n}{s-\lambda_{n-\mu-\rho}}\right] U$$



$$A = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-\mu-\rho} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\gamma_1 \cdots \gamma_{\mu} \; \gamma_{\mu+1} \cdots \gamma_{\mu+\rho} \cdots \gamma_n] \quad D = [b_0]$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix} \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

15.1 Transform method

①. Distinct Eigenvalues

we have $\hat{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ $\hat{A} = M^{-1}AM$ $M = [x_1\ x_2 \cdots x_n]$

$$\begin{cases} \dot{x}_J = M^{-1}AMx_J + M^{-1}Bu \\ y = CMx_J + Du \end{cases} \therefore$$

$$\begin{cases} \dot{x}_J = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} x_J + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \\ y = [c_1 \quad c_2 \quad \cdots \quad c_n] x_J + Du \end{cases}$$

Controllable: $b_i \neq 0$ $i = 1, 2, \cdots n$

Observable: $c_i \neq 0$ $i = 1, 2, \cdots, n$

Given a matrix A and eigenvalue λ, the eigenvalue and eigenvector v relations are given by: $Av = \lambda v$ v is eigenvector

v need to be normalized (归一化)

$M = [v_1\ v_2\ v_3 \cdots]$

②. Repeated Eigenvalues

nullity : $\boxed{n - \text{rank}(A - \lambda_i I)}$

if nullity = multiplicity of the eigenvalues

Generalized Eigenvector

non zero vector x $(A-\lambda I)^{k-1} x \neq 0$ $(A-\lambda I)^k x = 0$

define: $x_k = x$

$x_{k-1} = (A-\lambda I)x = (A-\lambda I)x_k$

$x_{k-2} = (A-\lambda I)^2 x = (A-\lambda I)x_{k-1}$

$\vdots$

$x_1 = (A-\lambda I)^{k-1} x = (A-\lambda I)x_2$

$\boxed{\text{rank}(A-\lambda I)^k = n - m_i}$

multiplicity of the eigenvalues

exp: $(A-\lambda I)^3 x_3 = 0$
$x_2 = (A-\lambda I)x_3$
$x_1 = (A-\lambda I)x_2$ $\Rightarrow M = [x_1 x_2 x_3]$

nullity equal the number of linearly independent eigenvectors for matrix A with its associated eigenvalue λ_i

Also means how many Jordan block does the eigenvalue λ_i have

$n-r$. equal how many "1" in this Jordan block

r means total number of block

exp:
$$\left[\begin{array}{cc|c|c} \lambda & 1 & & \\ 0 & \lambda & & \\ \hline & & \lambda & \\ \hline & & & \lambda_2 \end{array}\right] \quad r=3$$

$$\left[\begin{array}{ccc|c} \lambda_1 & 1 & 0 & \\ 0 & \lambda_1 & 1 & \\ 0 & 0 & \lambda_1 & \\ \hline & & & \lambda_2 \end{array}\right] \quad r=2$$

$n-r=1$ only one "1" $\qquad$ $n-r=2$ only 2 "1"

15.2 Transform Method

Step ①: Compute eigen value $\det(A-\lambda I)=0$

② compute number of linearly independent eigenvector for λ_i

use $n-\text{rank}(A-\lambda_i I)$

③ use ①.② Form each Jordan block

④ Determine the linearly independent vector for λ_i

⑤ generate Modal matrix M by placing the eigen vectors as columns

15.3. controllability and Observability

① each last row of each Jordan block B_i are linearly independent

$$B_i^l = \begin{bmatrix} b_{li1} \\ b_{li2} \\ \vdots \\ b_{lir(i)} \end{bmatrix}$$

the number of columns and rows need to be equal to C's rows and columns.

② each first column of each Jordan block C_i are linearly independent $C_i^l = [c_{1i1} \; c_{1i2} \; c_{1i3} \cdots c_{1ir(i)}]$

exp:

$$\dot{x} = \left[\begin{array}{cccccccccc}
\lambda_1 & 1 & \vdots & & \vdots & & \vdots & & & \\
0 & \lambda_1 & \vdots & & \vdots & & \vdots & & & \\
\cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
 & & \vdots & \lambda_1 & \vdots & & \vdots & & & \\
\cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
 & & \vdots & & \vdots & \lambda_1 & \vdots & & & \\
\cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
 & & \vdots & & \vdots & & \vdots & \lambda_2 & 1 & 0 \\
 & & \vdots & & \vdots & & \vdots & 0 & \lambda_2 & 1 \\
 & & \vdots & & \vdots & & \vdots & 0 & 0 & \lambda_2
\end{array}\right] x + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 1 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u$$

$$y = \left[\begin{array}{cccccccccc}
1 & 1 & \vdots & 2 & \vdots & 0 & \vdots & 0 & 2 & 0 \\
1 & 0 & \vdots & 1 & \vdots & 2 & \vdots & 0 & 1 & 1 \\
1 & 0 & \vdots & 2 & \vdots & 3 & \vdots & 0 & 2 & 2
\end{array}\right] x$$

$B_1^l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ full rank

$B_2^l = [0 \; 0 \; 1]$ linearly independent

$C_1^l = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ full rank

$C_2^l = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ rank=0 ∴ unobservable

15.4 JCF with complex conjugate eigenvalues

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & \bar{A}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ \bar{b}_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & \bar{C}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Introduce $\bar{x} = P x$

$$\boxed{P = \begin{bmatrix} I & I \\ jI & -jI \end{bmatrix} \text{ and } P^{-1} = \frac{1}{2} \begin{bmatrix} I & -jI \\ I & jI \end{bmatrix}}$$

then:
$$\begin{cases} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\,A_1 & \mathrm{Im}\,A_1 \\ -\mathrm{Im}\,A_1 & \mathrm{Re}\,A_1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{Re}\,b_1 \\ -2\,\mathrm{Im}\,b_1 \end{bmatrix} u \\ y = \begin{bmatrix} \mathrm{Re}\,C_1 & \mathrm{Im}\,C_1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{cases}$$

exp: $A \to 2 \quad 1\pm 2j \quad 1\pm 2j$

$$\dot{x} = \left[\begin{array}{cc|cc|c} 1+2j & 1 & & & \\ 0 & 1+2j & & & \\ \hline & & 1-2j & 1 & \\ & & 0 & 1-2j & \\ \hline & & & & 2 \end{array}\right] x + \left[\begin{array}{c} 2-3j \\ 1 \\ \hline 2+3j \\ 1 \\ \hline 2 \end{array}\right] u$$

$$y = \begin{bmatrix} 1 & -j & 1 & j & 2 \end{bmatrix} x$$

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ j & 0 & -j & 0 & 0 \\ 0 & j & 0 & -j & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \dot{\bar{x}} = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \bar{x} + \begin{bmatrix} 4 \\ 2 \\ 6 \\ 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \end{bmatrix} \bar{x}$$

16. Transfer Function (LTI System)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

17. Solution of state Equation

$$x(t) = \underbrace{e^{\int_{t_0}^{t} a(\tau)d\tau} x(t_0)}_{\text{zero input response}} + \underbrace{\int_{t_0}^{t} e^{\int_{\tau}^{t} a(\tau')d\tau'} b(\tau)u(\tau)\, d\tau}_{\text{zero state response}}$$

if $a(t) = a$ (constant):

$$x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} b(\tau) u \, d\tau$$

if $t_0 = 0$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} b(\tau) u \, d\tau$$

prove: $\mathcal{L}\{\dot{x} = ax + b(t)u\}$ $t=0$

$$sX(s) - x(0) = aX(s) + \mathcal{L}\{b(t)u\}$$

$$X(s) = \frac{1}{s-a}x(0) + \frac{1}{s-a}\mathcal{L}\{b(t)u\}$$

$$\Downarrow \mathcal{L}^{-1}$$

$$x(t) = e^{at}x(0) + \mathcal{L}^{-1}\{\underbrace{\frac{1}{s-a}}_{G_1}\underbrace{\mathcal{L}\{b(t)u\}}_{G_2}\}$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} b(\tau) u \, d\tau$$

$$\boxed{\mathcal{L}^{-1}\{G_1(s)G_2(s)\} = \int_0^t g_1(t-\tau)g_2(\tau)d\tau = g_1(t) * g_2(t)}$$

18. Fundamental Matrix $(M(t))$ $\dot{x} = A(t)x$ for Time variant System

① n columns of $M(t)$ are n linearly independent solutions of $\dot{x}=A(t)x$.

$$\dot{M}(t) = A(t)M(t)$$

each columns are $\boxed{\text{linearly independent}}$

② $x(t) = e^{\int A(\tau)d\tau} x(0)$

$M(t) = e^{\int A\tau d\tau} M(0)$

19. State Transition Matrix $[\Phi(t)]$

$$\dot{x} = A(t)x$$

$$\underbrace{\Phi(t,t_0)}_{\text{unique}} = \underbrace{M(t)}_{\text{not unique}} M^{-1}(t_0) \qquad x(t) = \Phi(t,t_0)\,x(t_0)$$

Properties:

① $\dfrac{\partial \Phi(t,t_0)}{\partial t} = A(t)\Phi(t,t_0)$

② $\Phi(t_0,t_0) = I$

③ $\left.\dfrac{\partial \Phi(t,t_0)}{\partial t}\right|_{t=t_0} = A(t_0)\cdot I$

④ $\Phi(t_2,t_0) = \Phi(t_2,t_1)\,\Phi(t_1,t_0)$

⑤ $\Phi^{-1}(t_1,t_0) = \Phi(t_0,t_1)$

20. State Transition Matrix for $\boxed{\text{LTI}}$ System

Exp: Assuming $\dot{x} = Ax \Rightarrow x(t) = e^{A(t-t_0)}x(t_0)$

$\dot{x}(t) = A\,e^{A(t-t_0)}x(t_0)$

Since $x(t) = \Phi(t,t_0)x(t_0)$

$\therefore\ \Phi(t,t_0) = e^{A(t-t_0)}$

$\Phi(t,t_0) = \Phi(t-t_0)$

Let $t_0 = 0$

$\Phi(t,t_0) = \Phi(t) \qquad \Phi^{-1}(t) = \Phi(-t)$

20.1 Approaches to Compute $\Phi(t)$ $\boxed{LTI}$

①. $\Phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

① compute $(sI-A)^{-1}$ using Leverrier's algorithm:

$$(sI-A)^{-1} = \frac{\operatorname{adj}(sI-A)}{\det(sI-A)} = \frac{P_{n-1}s^{n-1} + P_{n-2}s^{n-2} + \cdots + P_1 s + P_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$P_{n-1} = I_{n\times n}$

$a_{n-1} = -tr(A)$

$P_{n-2} = P_{n-1}A + a_{n-1}I$

$a_{n-2} = -\frac{1}{2}tr(P_{n-2}A)$

$\vdots$

$$\boxed{\begin{aligned} P_k &= P_{k+1}A + a_{k+1}I \\ a_k &= \frac{-1}{n-k}tr(P_kA) \end{aligned}}$$

$tr = trace = sum(a_{ii})\ i = 1, 2, \cdots n$

$P_0A + a_0I = 0$

② $\left[sI-A \;\vdots\; I_{(n\times n)} \right] \Rightarrow \left[I_{(n\times n)} \;\vdots\; (sI-A)^{-1} \right]$

②. Cayley-Hamilton Technique

1. $\Pi_A(\lambda) = \det(\lambda I - A) = 0$

$\Pi_A(A) = 0$

$\Downarrow$

2. $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$

$A^n + a_1A^{n-1} + \cdots a_{n-1}A + a_nI = 0$

$P(A) = R(A)$ $\quad R(A) = \alpha_1A^{n-1} + \alpha_2A^{n-2} + \cdots \alpha_{n-1}A + \alpha_0 I$

if λ repeat m times

$$\left.\frac{dp}{d\lambda}\right|_{\lambda=\lambda_1} = \left.\frac{dR}{d\lambda}\right|_{\lambda=\lambda_1}$$

$$\left.\frac{d^2p}{d\lambda^2}\right|_{\lambda=\lambda_1} = \left.\frac{d^2R}{d\lambda^2}\right|_{\lambda=a_1}$$

$$\left.\frac{d^{m-1}p}{d\lambda^{m-1}}\right|_{\lambda=\lambda_1} = \left.\frac{d^{m-1}p}{d\lambda^{m-1}}\right|_{\lambda=\lambda_1}$$

exp 1: $A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$ Find e^{At}

$n=2$ $\quad P(A)=e^{At}$ $\quad P(\lambda)=e^{\lambda t}$ $\quad R(\lambda)=\alpha\lambda+\beta$

$\pi_A(\lambda)=\det\begin{bmatrix}\lambda & -1\\ 1 & \lambda+2\end{bmatrix}=0$ $\qquad \lambda(\lambda+2)+1=0$

$(\lambda+1)^2=0$

$\lambda=-1,-1$ (Repeated)

$\therefore \lambda=-1 \Rightarrow \begin{cases} e^{-t}=-\alpha+\beta \\ \dfrac{de^{\lambda t}}{d\lambda}=\dfrac{d\alpha\lambda+\beta}{d\lambda}\end{cases} \Rightarrow \begin{cases} e^{-t}=-\alpha+\beta \\ te^{-t}=\alpha\end{cases} \Rightarrow \begin{cases}\beta=e^{-t}+te^{-t} \\ \alpha=te^{-t}\end{cases}$

Since $P(A)=R(A)$

$\therefore e^{At}=te^{-t}A+(e^{-t}+te^{-t})\times I$

---

exp 2: $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 27 & -27 & 9\end{bmatrix}$ Find e^{At}

$n=3$ $\quad P(A)=e^{At}$ $\quad P(\lambda)=e^{\lambda t}$ $\quad R\lambda=\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0\cdot I$

$\pi_A(\lambda)=\det\begin{bmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ -27 & 27 & \lambda-9\end{bmatrix}=\lambda[\lambda(\lambda-9)+27]-(-1)(0-27)$

$=\lambda(\lambda^2-9\lambda+27)-27$

$=\lambda^3-9\lambda^2+27\lambda-27$

$=(\lambda-3)^3$

$\lambda=3$ repeated

$\begin{cases} e^{3t}=\alpha_2\times 9+\alpha_1\times 3+\alpha_0 \\ \dfrac{de^{\lambda t}}{d\lambda}=2\alpha_2\lambda+\alpha_1 \\ \dfrac{d^2e^{\lambda t}}{d\lambda^2}=2\alpha_2\end{cases} \Rightarrow \lambda=3 \qquad \begin{cases}\alpha_0=e^{3t}(1-3t+\frac{9}{2}t^2) \\ \alpha_1=(t-3t^2)e^{3t} \\ \alpha_2=\frac{1}{2}t^2e^{3t}\end{cases}$

$e^{At}=\alpha_0 I+\alpha_1 A+\alpha_2 A^2$

③. $\boxed{e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}}$

If A is $\boxed{\text{sparse matrix}}$

$A^n = 0$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots$$

Properties :

① $e^0 = I$

② $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$ (t_1, t_2) scalars

③. $e^{(A+B)t} = e^{At}e^{Bt}$ $AB=BA$ (A,B commute)

④ $\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$ (A, e^{At} commute)

④ Jordan Form

$$A = \begin{bmatrix} J_1 & \cdots & & & 0 \\ \vdots & J_2 & \cdots & & 0 \\ \vdots & \vdots & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{J_1 t} & \cdots & & & 0 \\ \vdots & e^{J_2 t} & \cdots & & 0 \\ \vdots & \vdots & e^{\lambda_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

$$e^{J_1 t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_1 t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & \cdots & \frac{t^{n-3}}{(n-3)!}e^{\lambda_1 t} & \frac{t^{n-2}}{(n-2)!}e^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} & \cdots & \frac{t^{n-4}}{(n-4)!}e^{\lambda_1 t} & \frac{t^{n-3}}{(n-3)!}e^{\lambda_1 t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & \cdots & 0 & e^{\lambda_1 t} \end{bmatrix}$$

21. special formula

$$e^{jx} = \cos x + j\sin x \qquad \sin A = \frac{e^{jA} - e^{-jA}}{2j} \qquad \cosh^2 A - \sinh^2 A = I \qquad \cosh A = \frac{e^{A} + e^{-A}}{2}$$

$$1 \pm j = \sqrt{2}\, e^{\frac{-j\pi}{4}} \qquad \cos A = \frac{e^{jA} + e^{-jA}}{2} \qquad \sinh A = \frac{e^{A} - e^{-A}}{2}$$

22. $\Phi(t,t_0)$ for $\boxed{LTV}$ System can't use Laplace

① $\Phi(t,t_0) = M(t)\, M^{-1}(t_0)$

② Direct solution $\dot{x} = A(t)\, x$ to get $x(t) = \Phi(t,t_0)\, x(t_0)$

③ if $A(t)$ $A(t_0)$ commute $\quad \Phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)\, d\tau}$

④ if $A(t) = \sum_{i=1}^{k} \alpha_i(t)\, M_i$ $\quad$ a_i is scalar $\quad \Phi(t,t_0) = \prod_{i=1}^{k} e^{M_i \int_{t_0}^{t} a_i(\tau)\, d\tau}$

constant

$M_i\ M_j$ commute

Special formula

$$e^{\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$$

23. Solution of the state Equation and Output Equation

LTI System

$$x(t) = \Phi(t,t_0)\, x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau$$

$$\frac{\partial \Phi(t,t_0)}{\partial t} x(t_0) + \frac{\partial}{\partial t} \int_{t_0}^{t} \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau = A(t)\Phi(t,t_0)\, x(t_0) + A(t)\int_{t_0}^{t} \Phi(t,\tau)\, B(\tau)\, u(\tau)\, d\tau + B(t)\, u(t)$$

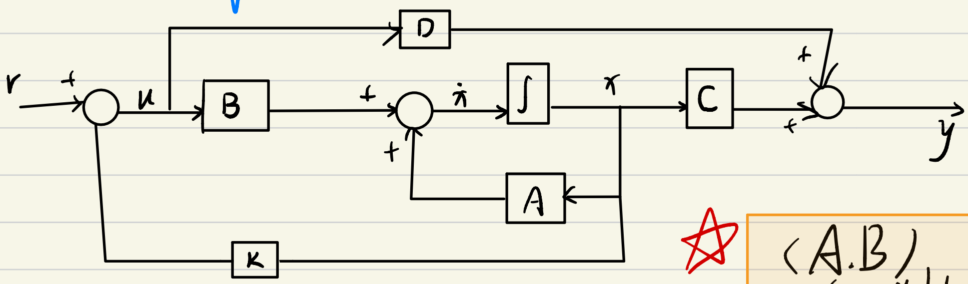
Leibnitz's Rule

$$\boxed{\frac{\partial}{\partial t} \int_{u(t)}^{v(t)} f(x,t)\, dx = \int_{u(t)}^{v(t)} \frac{\partial f(x,t)}{\partial t}\, dx + f(v,t)\frac{\partial v}{\partial t} - f(u,t)\frac{\partial u}{\partial t}}$$

$$y(t) = C(t)\underbrace{\boxed{\Phi(t,t_0)\,x(t_0)} + C(t)\boxed{\int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau}}_{x(t)} + D(t)\,u(t)$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

24. State feed back



$\boxed{(A,B)\ \text{controllable}}$

open loop system $\quad \dot{x} = Ax + Bu$

let $u = \underset{\substack{\uparrow \\ r=0}}{r} - Kx = -Kx$

$$\begin{cases} \dot{x} = (A-BK)x = A_f x \\ A_f = A - BK \leftarrow \text{close loop matrix} \end{cases}$$

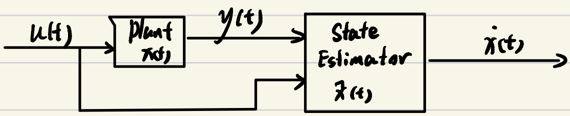
Characteristic polynomial for the close loop system

$$\boxed{\begin{aligned} \det(sI - A_f) &= \det(sI - A + BK) \\ &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \end{aligned}}$$

Ackerman's Formula

$$\begin{cases} K = [0\ 0 \cdots 0\ 1]\,[B\ \underbrace{AB \cdots A^{n-2}B\ \ A^{n-1}}_{Cx}B]^{-1}\,\alpha_c(A) \\ \alpha_c(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I \end{cases}$$

25. Full-Order Oberver



☆ A.C is observable

in this system

$$\boxed{\begin{aligned} &\dot{\hat{x}} = A\hat{x} + Bu + Gy - GC\hat{x} \\ &\dot{\hat{x}} = (A-GC)\hat{x} + Bu + Gy \qquad \hat{y} = C\hat{x} \end{aligned}}$$

state estimation error

$$\boxed{\begin{aligned} &e(t) = x(t) - \hat{x}(t) \\ &\dot{e} = (A-GC)e \end{aligned}}$$

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + G(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

char. ploy. is

$$\boxed{\det(sI - A + GC) = \alpha_e(s)}$$

Use Ackermann's formula

$$\begin{cases} G_{n\times 1} = \alpha_e(A)\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ \alpha_e(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I \end{cases}$$

estimator equations:

$$\begin{cases} \dot{\hat{x}} = (A-GC)\hat{x} + B(-K)\hat{x} + Gy \\ \quad = (A-GC-BK)\hat{x} + Gy \\ u = -Kx \end{cases}$$

transfer function of controller-estimator

$$\boxed{G_{ec} = K(SI-A+BK+GC)^{-1}G}$$

26. Closed-loop system

$$1 + G_{ec}(s)\,G_p(s) = 0$$

27. Mason's gain formula

$$G = \frac{\sum_{k=1}^{N} G_k \Delta_k}{\Delta}$$

$\Delta = 1 - \Sigma L_i + \Sigma L_i L_j - \Sigma L_i L_j L_k + \cdots + (-1)^m \Sigma \cdots + \cdots$

L_i = loop gain of each closed loop

L_{ij} = product of the loop gains of any two non-touching loops

Δ_k = the cofactor value of Δ for the k^{th} forward path with the loops touching the k^{th} forward path removed

G_k = path gain of the k^{th} forward path between y_{in} and y_{out}

28. Reduced-Order state Estimator

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad A_{(n\times n)} \quad B_{(n\times p)} \quad C_{(q\times n)}$$

$$P = \begin{bmatrix} C \\ R \end{bmatrix}_{n\times n} \rightarrow \text{full rank} \qquad \Rightarrow \begin{cases} \dot{\bar{x}} = PAP^{-1}\bar{x} + PBu \\ y = CP^{-1}\bar{x} = [I_q \ \ 0]\bar{x} \end{cases}$$

$$Q = P^{-1}$$

$$\bar{x} = Px$$

$$\begin{cases} \begin{bmatrix} \dot{\bar{x}}_{1\,(q\times 1)} \\ \dot{\bar{x}}_{2\,[(n-q)\times 1]} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11\,(q\times q)} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22\,[(n-q)(n-q)]} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u \\ y = [I_q \ \ 0]\,\bar{x} = \bar{x}_1 \end{cases}$$

$$\Downarrow$$

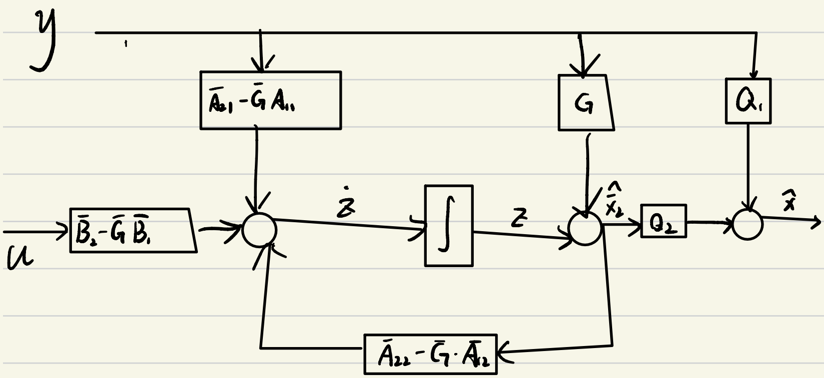
$$\begin{cases} A \leftarrow \bar{A}_{22} \\ Bu \leftarrow (\bar{A}_{21}\bar{x}_1 + \bar{B}_2 u) \\ C \leftarrow \bar{A}_{12} \\ y \leftarrow \dot{\bar{x}}_1 - A_{11}\bar{x}_1 - \bar{B}_1 u \end{cases}$$

$$\bar{G} = \alpha_e(\bar{A}_{22}) \begin{bmatrix} \bar{A}_{12} \\ \bar{A}_{12}\bar{A}_{22} \\ \vdots \\ \bar{A}_{12}\bar{A}_{22}^{\,n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\dot{\hat{\bar{x}}}_2 = (\bar{A}_{22} - \bar{G}\cdot\bar{A}_{12})\hat{\bar{x}}_2 + \bar{G}(\dot{y} - \bar{A}_{11}y - \bar{B}_1 u) + (\bar{A}_{21}y + \bar{B}_2 u)$$

$$\dot{z} = (\bar{A}_{22} - \bar{G}\bar{A}_{12})z + [(\bar{A}_{22} - \bar{G}\cdot\bar{A}_{12})\bar{G} + (\bar{A}_{21} - \bar{G}\cdot\bar{A}_{11})]y + (\bar{B}_2 - \bar{G}\bar{B}_1)u$$

$$\hat{\bar{x}} = \begin{bmatrix} \hat{\bar{x}}_1 \\ \hat{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} y \\ \bar{G}y + z \end{bmatrix} \qquad \hat{x} = P^{-1}\hat{\bar{x}} = Q\hat{\bar{x}} = Q\begin{bmatrix} I_q & 0 \\ \bar{G} & I_{n-q} \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$



29. tracking problem

1. $y = x_1$ $\quad k_r = k_1$

2. $y \neq x_1$

$$u = -Kx + K_r r = K_a(r - y) - K_b x$$

$$= K_a r - (K_a C + K_b)x$$

$$K_r = K_a \qquad K = (K_a C + K_b)$$

30. closed-loop

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H_{eq}(s)}$$

$G(s)$ open loop plant TF

H_{eq} feedback TF

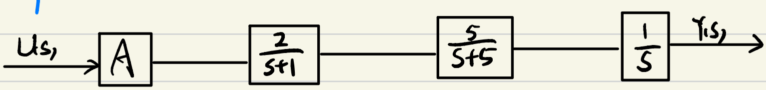
$$G(s) = \frac{Y(s)}{U(s)} = \frac{K(s^m + C_{m-1}s^{m-1} + \cdots C_1 s + C_0)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \quad m<n$$

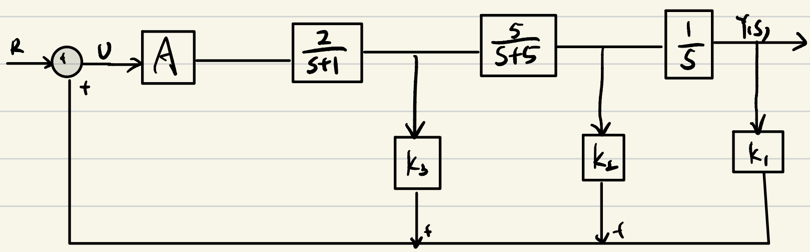
$$H_{eq}(s) = \frac{k_n s^{n-1} + k_{n-1}s^{n-2} + \cdots k_2 s + k_1}{s^n + C_{m-1}s^{m-1} + \cdots + C_1 s + C_0}$$

$$GH_{eq}(s) = \frac{K(k_n s^{n-1} + k_{n-1}s^{n-2} + \cdots + k_2 s + k_1)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$\frac{Y(s)}{R(s)} = \frac{K(s^m + C_{m-1}s^{m-1} + \cdots + C_0)}{s^n + (a_{n-1} + Kk_n)s^{n-1} + \cdots (a_0 + Kk_1)}$$

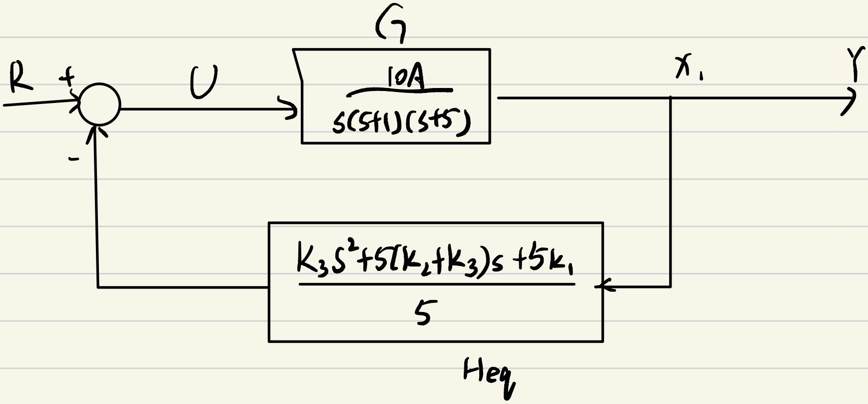
Example





Mason's gain formula

$$\frac{Y(s)}{R(s)} = \frac{10A}{s^3+(6+2Ak_3)s^2+[5+(k_2+k_3)\cdot 10A]s+10Ak_1}$$



$\%OS = 4.3 \Rightarrow S_{1,2} = -0.708 \pm j0.704$

To get $e_{ss}|_{\text{unit step}} = 0$ $\quad k_1 = 1$

$T_s = 5.65,\ \zeta\omega_n = 0.708$

place the 3rd pole at $S_3 = -100$

Therefore desired T.F.

$$\frac{Y(s)}{R(s)} = \frac{10A}{(s+100)(s+0.708\pm j0.7064)} = \frac{10A}{s^3+101.4s^2+142.6s+100}$$

$$\begin{cases} 6+2Ak_3 = 101.4 \\ 5+(k_2+k_3)10A = 142.6 \\ 10A = 100 \end{cases} \Rightarrow \begin{cases} A = 10 \\ k_1 = 1 \\ k_2 = -3.393 \\ k_3 = 4.77 \end{cases}$$

$$\frac{G}{1+G\,H_{eq}} = \frac{Y(s)}{R(S)}$$

$$G H_{eq} = G\frac{R(S)}{Y(S)} - 1$$

31. Root formula

1. $\xi = 0$ $\quad S_{1,2} = \pm j\omega_n$

2. $0 < \xi < 1$ $\quad S_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

3. $\xi = 1$ $\quad S_{1,2} = -\omega_n$

4. $\xi > 1$ $\quad S_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

$$T_s = \frac{4}{\xi\omega_n} \qquad \%OS = 100 \times e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$