

Linear

System

## 1. Superposition principle

$$L\{f_1 u_1(t) + f_2 u_2(t)\} = a_1 L\{u_1(t)\} + a_2 L\{u_2(t)\}$$

## 2. Table of Laplace Transforms.

$f(t)$	$F(s)$	$1(t)$	$t$	$\frac{1}{2}t^2$	$\frac{t^k}{k!} \quad k > 0$	$e^{-at}$	$\frac{1}{n!}t^n e^{-at}$	$\sin wt$	$\cos wt$	$e^{-at} \sin wt$	$e^{-at} \cos wt$
$f(t)$	$F(s)$	1	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s^3}$	$\frac{1}{s^{k+1}}$	$\frac{1}{s+a}$	$\frac{1}{(s+a)^n}$	$\frac{w}{s^2+w^2}$	$\frac{s}{s^2+w^2}$	$\frac{sw}{(s+a)^2+w^2}$

## 3. Property of Laplace

$$F(s) = \int_0^{+\infty} f(t) e^{-st} dt \quad L[f(t)] = sF(s) - f(0) \quad L[f^{(n)}(t)] = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$$

$$L[a f(t) \pm b g(t)] = a L[f(t)] \pm b L[g(t)] \quad L[e^{at} f(t)] = F(s-a) \quad L[f(t-a)] = e^{-as} F(s)$$

$$\text{L}[\int_0^t f(t) dt] = \frac{1}{s} F(s) \quad \lim_{s \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s)$$

## 4. Calculate Pole

exp.  $\frac{2s+7}{s(s+4)(s+1)}$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+4} + \frac{C_3}{s+1}$$

$$C_1 = \frac{2s+7}{s(s+4)(s+1)} \times s \Big|_{s=0} = \frac{7}{4}$$

$$C_2 = \frac{2s+7}{s(s+4)(s+1)} \times (s+4) \Big|_{s=-4} = -1 \quad X(t) = \frac{7}{4} - \frac{1}{12} e^{-4t} - \frac{5}{3} e^t$$

$$C_3 = \frac{2s+7}{s(s+4)(s+1)} \times (s+1) \Big|_{s=-1} = -\frac{5}{3}$$

$$\text{Pole} = 0, -4, -1$$

$$\frac{5s+7}{s(s+2)(s+3)} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+3} + \frac{C_{33}}{(s+3)^2}$$

$$C_1 = \frac{5s+7}{s(s+2)(s+3)^2} \times s \Big|_{s=0}$$

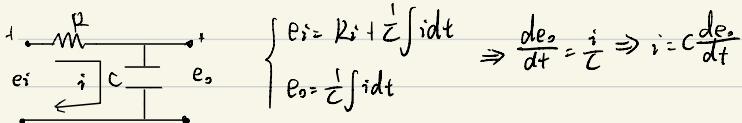
$$C_2 = \frac{5s+7}{s(s+2)(s+3)^2} \times (s+2) \Big|_{s=-2}$$

$$C_{33} = \frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3 \Big|_{s=-3}$$

$$C_{33} = \left[ \frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3 \right]' \Big|_{s=-3}$$

$$C_{33} = \left[ \frac{5s+7}{s(s+2)(s+3)^3} \times (s+3)^3 \right] \Big|_{s=-3}$$

## 5. I/O Description



$$\begin{cases} L \dot{i} = R \dot{i} + \frac{1}{C} \int i dt \\ e_i = \frac{1}{C} \int i dt \end{cases} \Rightarrow \frac{de_i}{dt} = \frac{i}{C} \Rightarrow i = C \frac{de_i}{dt}$$

$$e_i = RC \frac{de_i}{dt} + e_o \Rightarrow E_o(t) = \frac{1}{RC} \int_0^t e^{(t-\tau)/RC} e_i d\tau$$

$$\textcircled{1} \quad e_i = 0, e_o(0) \neq 0 \rightarrow \text{zero input response} : \quad E_o(t) = E_o(0) e^{-\frac{t}{RC}}$$

$$\textcircled{2} \quad e_i \neq 0, e_o(0) = 0 \rightarrow \text{zero state response} : \quad E_o(t) = A(1 - e^{-\frac{t}{RC}})$$

$$\textcircled{3} \quad e_i \neq 0, e_o(0) \neq 0 \rightarrow \text{general response} : \quad E_o(t) = e_o(0) e^{-\frac{t}{RC}} + A(1 - e^{-\frac{t}{RC}})$$

## 6. Show linear:

$$\text{exp: } y(t) = \int_0^t u(\tau) d\tau$$

$$\text{Step 1: } u_1 \rightarrow y_1; u_2 \rightarrow y_2 \Rightarrow y_1 = \int_0^t u_1 d\tau \quad y_2 = \int_0^t u_2 d\tau$$

$$\text{Step 2: } u = a_1 u_1 + a_2 u_2 \rightarrow y \quad \text{Superposition principle}$$

$$\text{Step 3: } y = \int_0^t (a_1 u_1 + a_2 u_2) d\tau = a_1 \int_0^t u_1 d\tau + a_2 \int_0^t u_2 d\tau = a_1 y_1 + a_2 y_2$$

## 7. Causality:

The system is said to be causal if the output at time  $t$  does not depend on the input after  $t$

## 8. Time variance

$$u(t) \rightarrow u(t-t_0) \rightarrow \boxed{y_1(t)} = y_1(t)$$

$$y(t) \rightarrow y(t-t_0) \quad y_1(t) = y(t-t_0)$$

## 9. Types of signal.

$$\text{Step: } f(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \quad \text{Ramp: } f(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \text{impulse: } \delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t) dt = 1$$

## 10. Impulse Response $u(t) = \int_{-\infty}^{+\infty} u(\tau) \delta(t-\tau) d\tau$

$$u(t) = \int_{-\infty}^{\infty} u(\tau) \delta(t-\tau) d\tau \quad \text{then} \quad y(t) = L \left\{ \int_{-\infty}^{\infty} u(\tau) \delta(t-\tau) d\tau \right\} = \int_{-\infty}^{\infty} u(\tau) h(t, \tau) d\tau$$

$$\text{LTI: } \delta(t-\tau) \rightarrow h(t-\tau, 0) = h(t-\tau) \Rightarrow y(t) = \int_0^t u(\tau) h(t-\tau) d\tau$$

$$\text{Exp: } h(t, \tau) = e^{-\frac{(t-\tau)^2}{2}}, \sin(t-\tau), e^{5\sin(t-\tau)}$$

$$LTV: y(t) = \int_{-\infty}^{+\infty} u(\tau) h(t, \tau) d\tau$$

exp:  $h(t, \tau) = e^{-\frac{t-\tau}{T}} \cdot \sin t \cdot e^{\sin t}$

## 11. Computation Impulse response

exp, use laplace

$$\ddot{y} + 4\dot{y} + 3y = 4u + 5v \quad L \Rightarrow s^2 Y(s) + 4s Y(s) + 3Y(s) = 4s U(s) + 5V(s)$$

$$① H(s) = \frac{Y(s)}{V(s)} = \frac{4s+5}{s^2+4s+3} \Rightarrow h(t) = L^{-1}(H(s)) = L^{-1}\left(\frac{\frac{1}{2}}{s+3} + \frac{\frac{1}{2}}{s+1}\right)$$

② step input

$$y(t) = \int_0^t h(t-\tau) u(\tau) d\tau$$

$$\begin{cases} h(t) = \frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t}, t \geq 0 \\ u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \end{cases}$$

$$y(t) = \int_0^t \left[ \frac{1}{2}e^{-3(t-\tau)} + \frac{1}{2}e^{-(t-\tau)} \right] 1 \cdot d\tau$$

$$= \frac{5}{3} - \frac{7}{6}e^{-3t} - \frac{1}{2}e^{-t}$$

③ use T.F.

$$Y(s) = H(s)U(s) \quad H(s) = \frac{4s+5}{s^2+4s+3} \quad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{4s+5}{s(s^2+4s+3)} \Rightarrow y(t) = \frac{5}{3} - \frac{7}{6}e^{-3t} - \frac{1}{2}e^{-t}$$

④ If  $y(0)=1$   $\dot{y}(0)=0$

$$L\{\ddot{y} + 4\dot{y} + 3y = 4u + 5v\}$$

$$L\{\ddot{y}\} = s^2 Y(s) - s y(0) - \dot{y}(0)$$

$$L\{\dot{y}\} = s Y(s) - y(0)$$

$$\downarrow$$

$$s^2 Y(s) - s Y(0) - \dot{y}(0) + 4[s Y(s) - y(0)] + 3Y(s) = 4s U(s) + 5V(s)$$

$$Y(s) = \frac{4s+5}{s^2+4s+3} \times U(s) + \frac{s+4}{s^2+4s+3}$$

$$y(t) = L^{-1} Y(s) = \frac{5}{3} - \frac{5}{3}e^{-3t} + e^{-t}$$

# State Diagram

Ex)

Find a state space representation  $(A, B, C, D)$  for the system described by

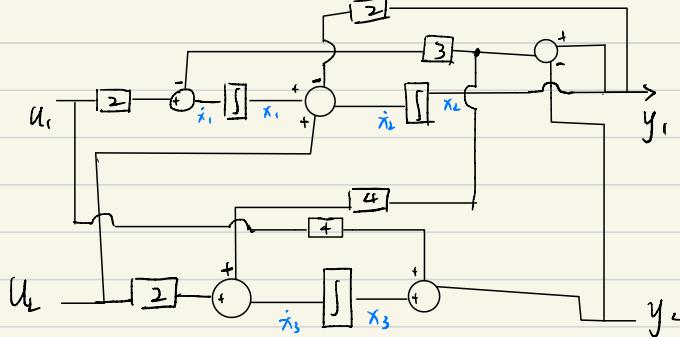
$$\ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + u_2$$

$$\ddot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4u_1$$

Hint: Use the state diagram approach.

$$\begin{cases} \ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + u_2 \\ \ddot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4u_1 \end{cases} \Rightarrow \begin{cases} \dot{y}_1 = 2u_1 + u_2 - 2\dot{y}_1 - 3(y_1 - y_2) \\ \dot{y}_2 = 4(y_1 - y_2) + 2u_2 + 4u_1 \end{cases} \xrightarrow{\text{integrate}} \begin{cases} y_1 = \int [2u_1 - 3(y_1 - y_2)] + \int u_2 - 2y_1 \\ y_2 = \int [4(y_1 - y_2) + 2u_2] + 4u_1 \end{cases}$$

## State Diagram ↴



$$\begin{cases} \dot{x}_1 = 2u_1 - 3(y_1 - y_2) \\ \dot{x}_2 = x_1 - 2y_1 + u_2 \\ \dot{x}_3 = 2u_2 + 4(y_1 - y_2) \end{cases}$$

⇒

$$\begin{cases} \dot{x}_1 = 2u_1 - 3(x_2 - x_3 - 4u_1) \\ \dot{x}_2 = x_1 - 2x_2 + u_2 \\ \dot{x}_3 = 2u_2 + 4(x_2 - x_3 - 4u_1) \end{cases}$$

$$\begin{cases} y_1 = x_2 \\ y_2 = x_3 + 4u_1 \end{cases}$$

$$\therefore A = \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 0 \\ 0 & 4 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 14 & 0 \\ 0 & 1 \\ -16 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$$

## LTI example

- (1) The response of a LTI causal system to the input  $u_1(t) = u_s(t) u_s(2-t)$  is  $y_1(t) = (1 - e^{-t}) u_s(t) u_s(2-t) + e^{-(t-2)} u_s(t-2)$ . Find the response of the system to a new input  $u_2(t) = (t-2) u_s(t-2) u_s(4-t)$  for  $t \leq 4$ . Note that  $u_s(t)$  designates a unit step function.

$$u_1(t-2) = u_s(t-2) u_s(4-t) \Rightarrow y_1(t-2) = (1 - e^{-(t-2)}) u_s(t-2) u_s(4-t) + e^{-(t-4)} u_s(t-4)$$

$$(t-2)u_1(t-2) = (t-2) u_s(t-2) u_s(4-t) = u_s(t) \Rightarrow (t-2)y_1(t-2) = (t-2)(1 - e^{-(t-2)}) u_s(t-2) u_s(4-t) + (t-2) e^{-(t-4)} u_s(t-4)$$

$$\text{for } t \leq 4 \quad u_s(t-4) = 0$$

$$\therefore y_2(t) = (t-2)(1 - e^{-(t-2)}) u_s(4-t)$$

$$(2) \text{ Consider a LTI system} \quad \dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} u$$

$$\text{Let } u(t) = \begin{bmatrix} -3e^{-t} + 5e^{-2t} \\ 3e^{-2t} \end{bmatrix} \text{ for } t \geq 0, \text{ and suppose that}$$

$$y(t) = -2e^{-t} + 6e^{-2t}. \text{ If } \lim_{t \rightarrow \infty} x(t) = 0, \text{ find } x(0).$$

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + u_1 - u_2 \\ \dot{x}_2 = u_1 + u_2 \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = -x_1 + x_2 + (-3e^{-t} + 5e^{-2t}) - 3e^{-2t} \\ \dot{x}_2 = -3e^{-t} + 5e^{-2t} + 3e^{-2t} = -3e^{-t} + 8e^{-2t} \Rightarrow x_2 = 3e^{-t} - 4e^{-2t} + C \end{cases} \text{ since } \lim_{t \rightarrow \infty} x(t) = 0 \Leftrightarrow C = 0$$

$$y = x_1 - x_2 = -x_1 + x_2 - 3e^{-t} + 2e^{-2t} + 3e^{-t} - 8e^{-2t} = 2e^{-t} - 12e^{-2t}$$

$$\begin{cases} -x_1 + x_2 - 3e^{-t} + 2e^{-2t} + 3e^{-t} - 8e^{-2t} = 2e^{-t} - 12e^{-2t} \\ x_2 = 3e^{-t} - 4e^{-2t} \end{cases} \Rightarrow \begin{cases} x_1 = 3e^{-t} - 4e^{-2t} - 3e^{-t} + 2e^{-2t} + 3e^{-t} - 8e^{-2t} - 2e^{-t} + 12e^{-2t} \\ x_2 = 3e^{-t} - 4e^{-2t} \end{cases}$$

$$\therefore x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

3) Given  $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t-1) + |u(t)| \\ x_2(t) \end{bmatrix}; y(t) = x_1(t+) + x_2(t-) - u(t)$  noncausal

Find the appropriate description of the system among:

Linear	Time-Invariant	Causal	Lumped	Continuous time
Nonlinear	Time-Variant	Noncausal	Distributed	Discrete-time

$$\begin{cases} \dot{x}_1(t) = x_1(t-1) + |u(t)| \\ \dot{x}_2(t) = x_2(t) \\ y(t) = x_1(t+1) + x_2(t) - u(t) \end{cases} \quad \therefore \quad \begin{cases} \dot{x} = f(x, t), u(t) \\ y = g(x, t) \end{cases}$$

Nonlinear    TI

Distinction      Lumped      and      Distributed

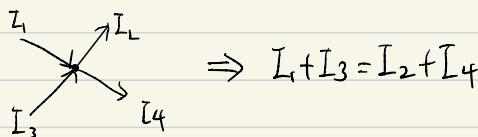
- ① ODE is lumped      ordinary differential equation  
PDE is distributed      partial differential equation
- ② has delay signal is distributed

exp       $\dot{x}_1(t) = x_1(t-1) + u(t)$

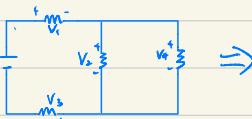
## State Models

- ① RLC network the voltage across the capacitor and the current through an inductor are the states.

② KCL



③ KVL



$$e = V_1 + V_2 + V_3 \quad V_2 = V_4 \\ e = V_1 + V_4 + V_3$$

## 12. General State Space Representation

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \end{cases} \leftarrow \text{state equation}$$

$$\begin{cases} y(t) = g(x(t), u(t), t) \end{cases} \leftarrow \text{Output equation}$$

nonlinear time varying causal system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \end{cases}$$

Nonlinear time invariant causal

$$\begin{cases} y(t) = g(x(t), u(t)) \end{cases}$$

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \end{cases}$$

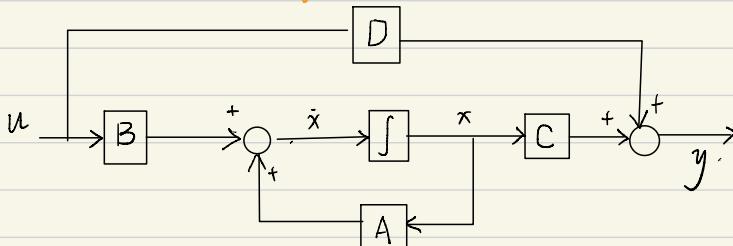
Time varying, Linear

$$\begin{cases} y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \end{cases}$$

Linear Time Invariant

$$\begin{cases} y(t) = Cx(t) + Du(t) \end{cases}$$



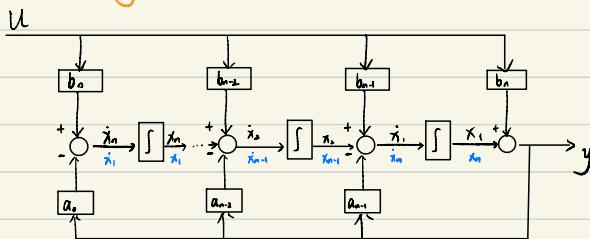
### 13. Observable Canonical Form (O.C.F)

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = b_n \frac{d^n u}{dt^n} + b_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_0 u$$

$$\text{define } D \equiv \frac{d}{dt} \Rightarrow D^n y + a_{n-1} D^{n-1} y + \dots + a_0 y = b_n D^n u + b_{n-1} D^{n-1} u + \dots + b_0 u$$

$$y = b_0 u + \frac{1}{D} [b_{n-1} u - a_{n-1} y] + \dots + \frac{1}{D^n} [b_0 u - a_0 y]$$

State Diagram:



$$\begin{cases} \dot{x}_1 = -a_{n-1}x_1 + x_2 + (b_{n-1} - a_{n-1}b_0)u \\ \dot{x}_2 = -a_{n-2}x_2 + x_3 + (b_{n-2} - a_{n-2}b_1)u \\ \vdots \\ \dot{x}_{n-1} = -a_{n-2}x_{n-1} + x_n + (b_{n-2} - a_{n-2}b_{n-1})u \\ \dot{x}_n = -a_0x_n + (b_0 - a_0b_{n-1})u \end{cases} \quad y = b_0 u + x_1$$

pair (A, C) is observable canonical forms

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \dots & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_n + a_{n-1}b_n \\ b_{n-1} - a_{n-1}b_n \\ \vdots \\ b_1 - a_0b_n \\ b_0 - a_0b_n \end{bmatrix}$$

usually write

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} b_n - a_0b_n \\ b_{n-1} - a_1b_n \\ b_{n-2} - a_2b_n \\ \vdots \\ b_1 - a_{n-1}b_n \\ b_0 - a_nb_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \quad D = [b_n]$$

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad D = [b_0]$$



### 13.1 Observability

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$O_x = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n \times n}$$

rank  $\{O_x\} = n$ . if it is observable  
column rank

### 13.2 Transform method

$$\begin{cases} \dot{x}_o = A_o x_o + B_o u \\ y = C_o x_o + D_o u \end{cases}$$

$$\begin{cases} A_o = T_o^{-1} A T_o \\ B_o = T_o^{-1} B \\ C_o = C T_o \\ D_o = D \end{cases}$$

$$O_{x_o} = \begin{bmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n-1} \end{bmatrix}$$

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix}$$

$$C_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$O_{x_o} = O_x T_o$$

$$\therefore T_o^{-1} = O_{x_o}^{-1} O_x \text{ or } T_o = O_x^{-1} O_{x_o}$$

Step: ① Check observability use  $O_x$

② Determine the coefficients of the [C.P] of matrix A

characteristic polynomial  $T_A(\lambda)$

$$\det(\lambda I - A) = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

③ Form  $A_o$ ,  $C_o$

$A_o$	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	-a_n

④ Calculate  $O_x$  and  $O_{x_o}$  using A.C. A.C.

⑤ Find the transformation  $T_o^{-1} = O_{x_o}^{-1} O_x$

⑥ Use  $T_o^{-1} B$  to get  $B_o$

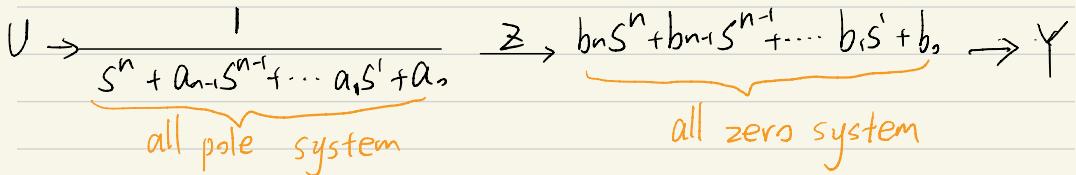
⑦ Check  $A_o = T_o^{-1} A T_o$      $C_o = C T_o$

⑧  $D_o = D$

## 14. Controllable Canonical Form (C.C.F)

$$\frac{Y(S)}{U(S)} = \frac{b_n S^n + b_{n-1} S^{n-1} + \dots + b_1 S + b_0}{S^n + a_{n-1} S^{n-1} + \dots + a_1 S + a_0}$$

↓



$$\frac{Z(S)}{U(S)} \xrightarrow{\cancel{U(S)}} \frac{1}{S^n + a_{n-1} S^{n-1} + \dots + a_1 S + a_0} \xrightarrow{\mathcal{L}^{-1}} Z^{(n)} + a_{n-1} Z^{n-1} + \dots + a_0 Z = u$$

Let  $\begin{cases} x_1 = z \\ x_2 = \dot{z} \\ \vdots \\ x_{n-1} = z^{(n-2)} \\ x_n = z^{(n-1)} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = z^{(n)} \end{cases}$

$$z^{(n)} = u - a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n$$

$$\frac{Y(S)}{Z(S)} = \frac{b_n S^n + b_{n-1} S^{n-1} + \dots + b_1 S + b_0}{S^n + a_{n-1} S^{n-1} + \dots + a_1 S + a_0} \xrightarrow{\mathcal{L}^{-1}} y = b_n z^{(n)} + b_{n-1} z^{(n-1)} + \dots + b_1 z + b_0 z$$

$y = (b_0 - b_0 a_0) x_1 + (b_1 - b_1 a_1) x_2 + \dots + (b_{n-1} - b_{n-1} a_{n-1}) x_n + b_n u$

$$A = \left[ \begin{array}{c|cccccc} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \hline -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{array} \right] \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

pair  $(A, B)$  is in the  
Controllable Canonical Form

$$A \cdot CCF = A \cdot OCF^T$$

$$B \cdot CCF = C \cdot OCF^T$$

$$C \cdot CCF = B \cdot OCF^T$$

$$D \cdot CCF = D \cdot OCF$$

$$C = [ (b_0 - a_0 b_0) \quad (b_1 - a_1 b_1) \quad \dots \quad (b_{n-1} - a_{n-1} b_{n-1}) ] \quad D = [b_n]$$

## 14.1 controllability

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$C_x = [B : AB : A^2B : \dots : A^{n-1}B]_{n \times n}$$

$\text{Rank}[C_x] = n$  it is controllable  
Row rank

## 14.2. Transform method

find a nonsingular transformation  $x_c = T_c^{-1}x$

then we have

$$\begin{cases} \dot{x}_c = A_c x_c + B_c u \\ y = C_c x_c + D_c u \end{cases}$$

$$A_c = \left[ \begin{array}{c|ccccc} 0 & & & & & \\ \vdots & & & & & \\ 0 & & I_{(n-1) \times (n-1)} & & & \\ 0 & & & & & \\ \hline -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} & \end{array} \right] \quad B_c = \left[ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array} \right]_{n \times 1} \quad D_c = 0$$

$$\begin{cases} A_c = T_c^{-1}AT_c \\ B_c = T_c^{-1}B \\ C_c = CT_c \\ D_c = 0 \end{cases} \quad C_{x_c} = [B_c : A_c B_c : \dots : A_c^{n-1} B_c] = [T_c^{-1}B : (T_c^{-1}A)(T_c^{-1}B) : \dots : (T_c^{-1}A)^{n-1}(T_c^{-1}B)] \\ = T_c^{-1}C_x \quad \therefore T_c = C_x C_{x_c}^{-1} \quad T_c^{-1} = C_{x_c} C_x^{-1}$$

Step: ① Check controllability use  $C_x$

② Determine the coefficients of the C.P. of matrix A  
Characteristic polynomial  $T_A(\lambda)$

$$\det(\lambda I - A) = 0$$

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$$

③ Form  $A_c$   $B_c$

$$A_c = \left[ \begin{array}{c|ccccc} 0 & & & & & \\ \vdots & & & & & \\ 0 & & I_{(n-1) \times (n-1)} & & & \\ \hline -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} & \end{array} \right] \quad B_c = \left[ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array} \right]_{n \times 1}$$

④ Calculate  $C_x$  and  $C_{x_c}$  using  $A, B, A_c, B_c$

⑤ Find the transformation  $T_c = C_x C_{x_c}^{-1}$

⑥ Use  $C_c = C T_c$  to get  $C_c$

⑦ Check  $A_c = T_c^{-1}AT_c \quad B_c = T_c^{-1}B$

⑧  $D_c = 0$

## 15. Jordan Canonical Form (J.C.F)

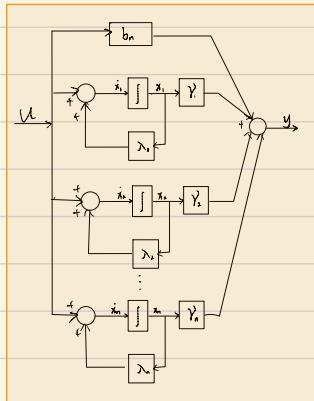
$$Y(S) = \frac{b_n S^n + b_{n-1} S^{n-1} + \dots + b_1 S + b_0}{S^n + a_{n-1} S^{n-1} + \dots + a_1 S + a_0} U(S) = b_n U(S) + \frac{N(S)}{D(S)} U(S)$$

$$\boxed{N(S) = (b_{n-1} - a_{n-1} b_n) S^{n-1} + (b_{n-2} - a_{n-2} b_n) S^{n-2} + \dots + (b_1 - a_0 b_n)}$$

$$D(S) = S^n + a_{n-1} S^{n-1} + \dots + a_0$$

①. Distinct Poles

$$Y(S) = b_n U(S) + \left[ \frac{\gamma_1}{S-\lambda_1} + \frac{\gamma_2}{S-\lambda_2} + \dots + \frac{\gamma_n}{S-\lambda_n} \right] U(S)$$



$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + u \\ \dot{x}_2 = \lambda_2 x_2 + u \\ \vdots \\ \dot{x}_n = \lambda_n x_n + u \end{cases}$$

$$y = b_n u + \gamma_1 x_1 + \dots + \gamma_n x_n$$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\gamma_1 \ \gamma_2 \ \gamma_3 \ \dots \ \gamma_n] \quad D = [b_n]$$

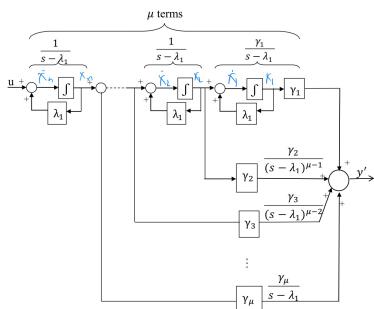
## ② Multiple Repeated Poles

$$D(s) = (s - \lambda_1)^{\mu} \times (s - \lambda_2)^{\rho} \times (s - \lambda_3)(s - \lambda_4) \cdots (s - \lambda_n)^{\mu - \rho}$$

$$Y = b_0 V + \left[ \frac{Y_1}{(s - \lambda_1)^{\mu}} + \frac{Y_2}{(s - \lambda_1)^{\mu-1}} + \cdots + \frac{Y_{\mu}}{s - \lambda_1} \right] V$$

$$+ \left[ \frac{Y_{\mu+1}}{(s - \lambda_2)^{\rho}} + \cdots + \frac{Y_{\mu+\rho}}{(s - \lambda_2)} \right] V$$

$$+ \left[ \frac{Y_{\mu+\rho+1}}{(s - \lambda_3)} + \frac{Y_{\mu+\rho+2}}{(s - \lambda_4)} + \cdots + \frac{Y_n}{s - \lambda_{n-\mu}} \right] V$$



$$A = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-\mu} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [Y_1 \ Y_2 \ Y_3 \ \dots \ Y_{\mu+1} \ \dots \ Y_{\mu+\rho} \ \dots \ Y_n] \quad D = [b_0]$$

$$J_i = \begin{bmatrix} \lambda_i & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

$$B_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

## 15.1 Transform method

### ①. Distinct Eigenvalues

We have

$$\hat{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\hat{A} = M^{-1} A M \quad M = [x_1 \ x_2 \ \cdots \ x_n]$$

$$\begin{cases} \dot{x}_j = M^{-1} A M x_j + M^{-1} B u \\ y = C M x_j + D u \end{cases} \therefore$$

Controllable:  $b_i \neq 0 \quad i=1,2,\dots,n$

Observable:  $c_i \neq 0 \quad i=1,2,\dots,n$

$$\begin{cases} \dot{x}_J = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} x_J + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \\ y = [c_1 \ c_2 \ \cdots \ c_n] x_J + D u \end{cases}$$

Given a matrix  $A$  and eigenvalue  $\lambda$ , the eigenvalue and eigenvector  $v$  relations are given by:  $A v = \lambda v$   $v$  is eigenvector

$v$  need to be normalized ( $\|v\| = 1$ )

$$M = [v_1 \ v_2 \ v_3 \ \cdots]$$

### ②. Repeated Eigenvalues

nullity :  $n - \text{rank}(A - \lambda I)$

if nullity = multiplicity of the eigenvalues

#### Generalized Eigenvector

non zero vector  $x$   $(A - \lambda I)^{k-1} x \neq 0 \quad (A - \lambda I)^k x = 0$

define:  $x_k = x$

$$x_{k-1} = (A - \lambda I)x = (A - \lambda I)x_k$$

$$x_{k-2} = (A - \lambda I)^2 x = (A - \lambda I)x_{k-1}$$

$$\text{rank}(A - \lambda I)^k = n - m$$

multiplicity of the eigenvalues

$$\text{exp: } (A - \lambda I)^3 x_3 = 0 \quad x_1 = (A - \lambda I)^{k-1} x = (A - \lambda I) x_2$$

$$\begin{aligned} x_2 &= (A - \lambda I)x_3 \\ x_1 &= (A - \lambda I)x_2 \end{aligned} \Rightarrow M = [x_1 \ x_2 \ x_3]$$

nullity equal the number of linearly independent eigenvectors for matrix A with its associated eigenvalue  $\lambda_i$ .  
 Also means how many Jordan block does the eigenvalue  $\lambda_i$  have

$n-r$  equal how many "1" in this Jordan block

$r$  means total number of block

$$\text{ex: } \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \quad r=3$$

$n-r = 1$  only one "1"

$$\begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \quad r=2$$

$n-r = 2$  only 2 "1"

## 15.2 Transform Method

Step ①: compute eigen value  $\det(A - \lambda I) = 0$

②. compute number of linearly independent eigenvector for  $\lambda_i$

use  $n - \text{rank}(A - \lambda_i I)$

③ use ①, ② Form each Jordan block

④ Determine the linearly independent vector for  $\lambda_i$

⑤ generate Modal matrix  $M$  by placing the eigenvectors as columns

## 15.3. Controllability and Observability

- ① each last row of each Jordan block  $B_i^l$  are linearly independent

$$B_i^l = \begin{bmatrix} b_{11}^{l,i} \\ b_{22}^{l,i} \\ \vdots \\ b_{rr}^{l,i} \end{bmatrix}$$

the number of columns and rows need to be equal to  $C$ 's rows and columns.

- ② each first column of each Jordan block  $C_i^l$  are linearly independent  $C_i^l = [C_{11}^l \ C_{12}^l \ C_{13}^l \ \dots \ C_{1r(i)}^l]$

exp:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ 0 & \lambda_1 & & & & & \\ \vdots & \vdots & \ddots & & & & \\ & \ddots & & \ddots & & & \\ & & & \lambda_1 & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & \ddots \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & \vdots & 2 & \vdots & 0 & \vdots & 0 & 2 & 0 \\ 1 & 0 & \vdots & 1 & \vdots & 2 & \vdots & 0 & 1 & 1 \\ 1 & 0 & \vdots & 2 & \vdots & 3 & \vdots & 0 & 2 & 2 \end{bmatrix} x$$

$$B_i^l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{bmatrix} \quad \text{full rank}$$

$$C_i^l = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{full rank}$$

$$C_2^l = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{rank} = 0 \quad \therefore \text{unobservable}$$

## 15.4 JCF with complex conjugate eigenvalues

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & \bar{A}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ \bar{b}_1 \end{bmatrix} u$$

$$y = [C_1 \quad \bar{C}_1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Introduce

$$\bar{x} = P x$$

$$P = \begin{bmatrix} I & I \\ jI & -jI \end{bmatrix} \text{ and } P^{-1} = \frac{1}{2} \begin{bmatrix} I & -jI \\ I & jI \end{bmatrix}$$

then:  $\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \operatorname{Re} A_1 & \operatorname{Im} A_1 \\ -\operatorname{Im} A_1 & \operatorname{Re} A_1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 2\operatorname{Re} b_1 \\ 2\operatorname{Im} b_1 \end{bmatrix} u$

$$y = [\operatorname{Re} C_1 \quad \operatorname{Im} C_1] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Ex:  $A \rightarrow 2 \begin{pmatrix} 1+2j & 1 \\ 0 & 1+2j \end{pmatrix}$

$$\dot{x} = \left[ \begin{array}{cc|ccc} 1+2j & 1 & & & & \\ 0 & 1+2j & & & & \\ \hline & & 1-2j & -1 & & \\ & & 0 & 1-2j & & \\ \hline & & & & 2 & \\ & & & & & 2 \end{array} \right] x + \begin{bmatrix} 2-3j \\ 1 \\ 2+3j \\ 1 \\ 2 \end{bmatrix} u$$

$$y = [1 \quad -j \quad 1 \quad j \quad 2] x$$

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ j & 0 & -j & 0 & 0 \\ 0 & j & 0 & -j & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \dot{\bar{x}} = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} 4 \\ 2 \\ 6 \\ 0 \\ 2 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0 \quad -1 \quad 2] \bar{x}$$

## 16. Transfer Function (LTI System)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} sX(s) = Ax(s) + Bu(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$\frac{Y(s)}{U(s)} = C(SI - A)^{-1}B + D$$

## 17. Solution of State Equation

$$x(t) = e^{\int_{t_0}^t a(z) dz} x(t_0) + \int_{t_0}^t e^{\int_z^t a(z) dz} b(z) u(z) dz$$

zero input response      zero state response

if  $a(z) = a$  (constant) :

$$x(t) = e^{at-t_0} x(t_0) + \int_{t_0}^t e^{a(t-z)} b(z) u(z) dz$$

if  $t_0 = 0$

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-z)} b(z) u(z) dz$$

prove :  $\mathcal{L}\{ \dot{x} = ax + b(t)u \} \quad t=0$

$$sX(s) - x(0) = aX(s) + \mathcal{L}\{ b(t)u \}$$

$$X(s) = \frac{1}{s-a} x(0) + \frac{1}{s-a} \mathcal{L}\{ b(t)u \}$$

$$x(t) = e^{at} x(0) + \mathcal{L}^{-1}\left\{\frac{1}{s-a} \mathcal{L}\{ b(t)u \}\right\}$$

$$\begin{aligned} \mathcal{L}^{-1}\{G_1(s)G_2(s)\} &= \int_0^t g_1(t-z)g_2(z)dz \\ &= g_1(t) * g_2(t) \end{aligned}$$

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-z)} b(z) u(z) dz$$

18. Fundamental Matrix. ( $M(t)$ )  $\dot{x} = A(t)x$  for Time Variant System

①  $n$  columns of  $M(t)$  are  $n$  linearly independent solutions of  $\dot{x} = A(t)x$ .  
 $\dot{M}(t) = A(t)M(t)$

each columns are linearly independent

②  $x(t) = e^{\int_{t_0}^t A(\tau)d\tau} x(t_0)$   
 $M(t) = e^{\int_{t_0}^t A(\tau)d\tau} M(t_0)$

19. State Transition Matrix  $[\underline{\Phi}(t)]$

$$\dot{x} = A(t)x$$

$$\underbrace{\underline{\Phi}(t, t_0)}_{\text{unique}} = \underbrace{M(t)}_{\text{not unique}} M^{-1}(t_0)$$

$$x(t) = \underline{\Phi}(t, t_0)x(t_0)$$

Properties :

①  $\frac{\partial \underline{\Phi}(t, t_0)}{\partial t} = A(t) \underline{\Phi}(t, t_0)$

③  $\left. \frac{\partial \underline{\Phi}(t, t_0)}{\partial t} \right|_{t=t_0} = A(t_0) \cdot I$

②  $\underline{\Phi}(t_0, t_0) = I$

④  $\underline{\Phi}(t_2, t_0) = \underline{\Phi}(t_2, t_1) \underline{\Phi}(t_1, t_0)$

⑤  $\underline{\Phi}^{-1}(t_1, t_0) = \underline{\Phi}(t_0, t_1)$

20. State Transition Matrix for LTI System

Exp: Assuming  $\dot{x} = A x \Rightarrow x(t) = e^{A(t-t_0)} x(t_0)$   
 $\dot{x}(t) = A e^{A(t-t_0)} x(t_0)$

Since  $x(t) = \underline{\Phi}(t, t_0)x(t_0)$

$$\therefore \underline{\Phi}(t, t_0) = e^{A(t-t_0)}$$

$$\underline{\Phi}(t, t_0) = \underline{\Phi}(t-t_0)$$

Let  $t_0 = 0$

$$\underline{\Phi}(t, t_0) = \underline{\Phi}(t) \quad \underline{\Phi}^{-1}(t) = \underline{\Phi}(-t)$$

## 20.1 Approaches to Compute $\Phi(t)$

LTI

$$① \quad \Phi(t) = e^{At} = L^{-1}\{(sI - A)^{-1}\}$$

② compute  $(sI - A)^{-1}$  using Leverrier's algorithm:

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} = \frac{P_{n-1}s^{n-1} + P_{n-2}s^{n-2} + \dots + P_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0 s + a_0}$$

$$P_{n-1} = I_{n \times n}$$

$$a_{n-1} = -\text{tr}(A)$$

$$P_{n-2} = P_{n-1}A + a_{n-1}I$$

$$a_{n-2} = \frac{1}{2} \text{tr}(P_{n-2}A)$$

$$\vdots$$

$$P_0 = P_{n-1}A + a_{n-1}I$$

$$a_{n-0} = \frac{1}{n!} \text{tr}(P_0A)$$

$$P_0A + a_0I = 0$$

②

$$\begin{bmatrix} sI - A & | & I_{n \times n} \end{bmatrix} \Rightarrow \begin{bmatrix} I_{n \times n} & | & (sI - A)^{-1} \end{bmatrix}$$

## ② Cayley - Hamilton Technique

$$1. \quad \text{Tr}_A(\lambda) = \det(\lambda I - A) = 0$$

$$\text{Tr}_A(A) = 0$$

↓

$$2. \quad \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$$

$$A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I = 0$$

$$PA = RA$$

$$RA = a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_{n-1} A + a_n I$$

if  $\lambda$  repeat m times

$$\left. \frac{dp}{d\lambda} \right|_{\lambda=\lambda_1} = \left. \frac{dR}{d\lambda} \right|_{\lambda=\lambda_1}$$

$$\left. \frac{d^2p}{d\lambda^2} \right|_{\lambda=\lambda_1} = \left. \frac{d^2R}{d\lambda^2} \right|_{\lambda=\lambda_1}$$

$$\left. \frac{d^{m-1}p}{d\lambda^{m-1}} \right|_{\lambda=\lambda_1} = \left. \frac{d^{m-1}R}{d\lambda^{m-1}} \right|_{\lambda=\lambda_1}$$

$$\text{Exp 1: } A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{Find } e^{At}$$

$$n=2 \quad P(A) = e^{At} \quad P(\lambda) = e^{\lambda t} \quad R(\lambda) = \alpha\lambda + \beta$$

$$\pi_A(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda+2 \end{bmatrix} = 0 \quad \lambda(\lambda+2) + 1 = 0$$

$$(\lambda+1)^2 = 0$$

$\lambda = -1, -1$  (Repeated)

$$\therefore \lambda = -1 \Rightarrow e^{-t} = -\alpha + \beta$$

$$\left\{ \frac{de^{-t}}{d\lambda} = \frac{d\alpha\lambda + \beta}{d\lambda} \right\} \Rightarrow \left\{ \begin{array}{l} e^{-t} = -\alpha + \beta \\ te^{-t} = \alpha \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \beta = e^{-t} + te^{-t} \\ \alpha = te^{-t} \end{array} \right.$$

$$\text{Since } P(A) = R(A)$$

$$\therefore e^{At} = te^{-t}A + (e^{-t} + te^{-t})I$$

$$\text{Exp 2: } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & 27 & 9 \end{bmatrix} \quad \text{Find } e^{At}$$

$$n=3 \quad P(A) = e^{At} \quad P(\lambda) = e^{\lambda t} \quad R(\lambda) = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 I$$

$$\pi_A(\lambda) = \det \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & -1 \\ 27 & 27 & \lambda-9 \end{bmatrix} = \lambda(\lambda(\lambda-9)+27) - (-1)(0-27)$$

$$= \lambda(\lambda^2 - 9\lambda + 27) - 27$$

$$= \lambda^3 - 9\lambda^2 + 27\lambda - 27$$

$$= (\lambda-3)^3$$

$\lambda = 3$  repeated

$$\left\{ \begin{array}{l} e^{3t} = \alpha_0 \times 9 + \alpha_1 \times 3 + \alpha_2 \\ \frac{de^{3t}}{d\lambda} = 2\alpha_2\lambda + \alpha_1 \Rightarrow \lambda = 3 \\ \frac{d^2e^{3t}}{d\lambda^2} = 2\alpha_2 \end{array} \right.$$

$$\left\{ \begin{array}{l} \alpha_0 = e^{3t}(1-3t+\frac{9}{2}t^2) \\ \alpha_1 = (t-3t^2)e^{3t} \\ \alpha_2 = \frac{1}{2}t^2e^{3t} \end{array} \right.$$

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$③. e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

If  $A$  is sparse matrix  
 $A^n = 0$

$$e^{At} = I + tAt + \frac{t^2}{2!} A^2 + \dots$$

Properties :

$$\textcircled{1} \quad e^0 = I$$

$$\textcircled{2} \quad e^{A(t_1+t_2)} = e^{At_1} e^{At_2} \quad (t_1, t_2) \text{ scalars}$$

$$\textcircled{3} \quad e^{(A+B)t} = e^{At} e^{Bt} \quad AB = BA \quad (A, B \text{ commute})$$

$$\textcircled{4} \quad \frac{de^{At}}{dt} = Ae^{At} = e^{At} A \quad (A, e^{At} \text{ commute})$$

## ④ Jordan Form

$$A = \begin{bmatrix} J_1 & & & & & 0 \\ & J_2 & & & & 0 \\ & & J_3 & & & 0 \\ & & & \ddots & & \vdots \\ & & & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \quad e^{At} = \begin{bmatrix} e^{J_1 t} & & & & & 0 \\ & e^{J_2 t} & & & & 0 \\ & & e^{J_3 t} & & & 0 \\ & & & e^{\lambda_4 t} & & 0 \\ & & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}$$

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2!} e^{\lambda_i t} & \cdots & \frac{t^{n-2}}{(n-2)!} e^{\lambda_i t} & \frac{t^{n-1}}{(n-1)!} e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n-3}}{(n-3)!} e^{\lambda_i t} & \frac{t^{n-2}}{(n-2)!} e^{\lambda_i t} \\ 0 & 0 & e^{\lambda_i t} & \cdots & \frac{t^{n-4}}{(n-4)!} e^{\lambda_i t} & \frac{t^{n-3}}{(n-3)!} e^{\lambda_i t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & 0 & 0 & \cdots & 0 & e^{\lambda_i t} \end{bmatrix}$$

## 21. special formula

$$e^{ix} = \cos x + j \sin x \quad \sin A = \frac{e^{jA} - e^{-jA}}{2j} \quad \cosh^2 A - \sinh^2 A = 1 \quad \cosh A = \frac{e^A + e^{-A}}{2}$$

$$1 \pm j = \sqrt{2} e^{\pm j\frac{\pi}{4}}$$

$$\cos A = \frac{e^{jA} + e^{-jA}}{2}$$

22.  $\Phi(t, t_0)$  for LT V System can't use Laplace

①  $\Phi(t, t_0) = M_{(t)} M^{-1}(t_0)$

② Direct solution  $\dot{x} = A(t)x$  to get  $x(t) = \Phi(t, t_0)x(t_0)$

③ if  $A(t), A(t_0)$  commute  $\Phi(t, t_0) = e^{\int_{t_0}^t A(z) dz}$

④ if  $A(t) = \sum_{i=1}^k a_i(t) M_i$    
  $a_i$  is scalar  
 $M_i, M_j$  commute  $\Phi(t, t_0) = \prod_{i=1}^k e^{\int_{t_0}^t a_i(z) dz}$

Special formula

$$e^{\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$$

23. Solution of the state Equation and Output Equation  
LTI System

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau) d\tau$$

$$\frac{\partial \Phi(t, t_0)}{\partial t} x(t_0) + \frac{\partial}{\partial t} \int_{t_0}^t \Phi(t, \tau) B(\tau) u(\tau) d\tau = A(t)\Phi(t, t_0)x(t_0) + A(t) \int_{t_0}^t \Phi(t, \tau) B(\tau) u(\tau) d\tau + B(t)u(t)$$

Leibnitz's Rule

$$\frac{\partial}{\partial t} \int_{u(t)}^{v(t)} f(x, t) dx = \int_{u(t)}^{v(t)} \frac{\partial f(x, t)}{\partial t} dx + f(v, t) \frac{\partial v}{\partial t} - f(u, t) \frac{\partial u}{\partial t}$$

$$y(t) = C(t) \boxed{\tilde{\Phi}(t, t_0) x(t_0)} + C(t) \boxed{\int_{t_0}^t \tilde{\Phi}(t, \tau) B(\tau) u(\tau) d\tau} + D(t) v(t)$$

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau$$