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THE DIRECT KINEMATICS OF PLANAR PARALLEL MANIPULATORS: SPECIAL ARCHITECTURES AND NUMBER OF SOLUTIONS

CLÉMENT M. GOSSELIN

Département de Génie Mécanique, Université Laval, Ste-Foy, Québec, Canada G1K 7P4

JEAN-PIERRE MERLET

INRIA, Unité de Sophia-Antipolis, 2004 Route des Lucioles, 06565 Valbonne cedex, France

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Abstract—This paper presents new results on the direct kinematic problem of planar three-degree-of-freedom parallel manipulators. This subject has been addressed in the past. Indeed, the latter problem has been reduced to the solution of a minimal polynomial of degree 6 by several researchers working independently. This paper focuses on the direct kinematic problem associated with particular architectures of planar parallel manipulators. For some special geometries, namely, manipulators for which all revolute joints on the platform and on the base are respectively collinear, it has been conjectured that only four solutions are possible, as opposed to six in the general case. However, this fact has never been shown and the polynomial solution derived for the general case still gives six solutions for the special geometry, two of which are spurious and unfeasible. In this paper, a formal proof of the aforementioned conjecture is derived using Sturm's theorem. Then, alternative derivations of the polynomial solutions are pursued and a robust computational scheme is given for the direct kinematics. The scheme accounts for special cases that would invalidate the previous derivations. Finally, possible simplifications of the general polynomial are discussed and related to particular geometries of the manipulator. It is first shown that it is not possible to find an architecture that would lead to a vanishing coefficient for the term of degree 6 in the polynomial. Then, a special geometry different from the one mentioned above and leading to closed-form solutions is introduced. A simplified planar three-degree-of-freedom parallel manipulator can be of great interest, especially for applications in which the manipulator is working on a vertical plane, i.e., when gravity is in the plane of motion.

INTRODUCTION

The theoretical and practical problems associated with parallel manipulators have been addressed by many authors since the first parallel robotic architectures have been proposed by Hunt [1] and MacCallion [2]. However, fewer authors have studied planar parallel manipulators (see for instance [3–6]). In the latter references, several properties of a planar three-degree-of-freedom parallel manipulator with either prismatic or revolute actuators are investigated. Closed-form solutions are given for the inverse kinematic problem and issues related to workspace analysis and optimization as well as kinematic accuracy and conditioning are discussed. Potential applications for planar parallel robotic manipulators include metal cutting, deburring, pick-and-place operations over a plane surface and mobile bases for spatial manipulators.

One of the recent trends in the study of parallel manipulators is the derivation of polynomial solutions to the direct kinematic problem. Indeed, it is well known that this problem leads to complex nonlinear coupled algebraic equations which are, in general, very difficult to solve. A polynomial solution is an interesting result since it provides an upper bound for the number of solutions to the direct kinematic problem. In the case of a planar three-degree-of-freedom parallel manipulator, the direct kinematic problem admits a maximum of 6 real solutions. A geometric proof of this result was given by Hunt [3]. A polynomial of degree 12—therefore not minimal—was first proposed by Merlet [6] for the solution of this problem. Later, a minimal polynomial—of degree 6—has been derived independent by several researchers [7–10]. In [9], particular architectures have also been studied. It has been conjectured that the manipulators for which the revolute joints on the platform are aligned lead to only four real solutions of the direct kinematic problem.

In this paper, particular geometries of the planar three-degree-of-freedom parallel manipulator are studied and the aforementioned conjecture is formally proven using Sturm's theorem. Then, alternative derivations of the polynomial solution are given and it is shown that a minimal polynomial in any of the three Cartesian variables can be obtained. The polynomial in y is studied in detail and a second proof of the number of solutions in the simplified case is given. A robust computational scheme based on the latter polynomial is given, accounting for special cases that can arise. Finally, possible simplifications of the general polynomial are discussed and related to particular geometries of the manipulator. It is first shown that it is not possible to find an architecture that would lead to a vanishing coefficient of the term in degree 6 in the polynomial. Then, a special geometry different from the one mentioned above and leading to closed-form solutions is introduced.

These results complete the study on the direct kinematics of planar three-degree-of-freedom manipulators undertaken in [7–10]. They are particularly relevant in the context of design engineering since special architectures of planar parallel manipulators are of practical interest. Indeed, as shown in [11] using stiffness plots, a simplified planar three-degree-of-freedom parallel manipulator would be a very good candidate for applications in which the manipulator is working on a vertical plane, i.e., when gravity is in the plane of motion.

DIRECT KINEMATICS OF THE GENERAL PLANAR THREE-DEGREE-OF-FREEDOM PARALLEL MANIPULATOR

A general planar three-degree-of-freedom parallel manipulator is represented in Fig. 1. Three actuated prismatic joints are mounted on fixed passive revolute joints (A_1, A_2, A_3) and are connected to a common platform which plays the role of the end effector of common serial manipulators. The revolute joints connecting the legs to the platform (B_1, B_2, B_3) are also passive. The actuation of the prismatic joints allows one to adjust the length of each of the legs and therefore to position and orient the platform (B_1, B_2, B_3) on the plane. As shown in [5], an equivalent manipulator can be designed with three fixed revolute actuators (the mathematical formulation of the direct kinematic problem is the same in both cases, as demonstrated in [9]). A minimal polynomial solution—of degree 6—is derived in [7–10] for the direct kinematic problem associated

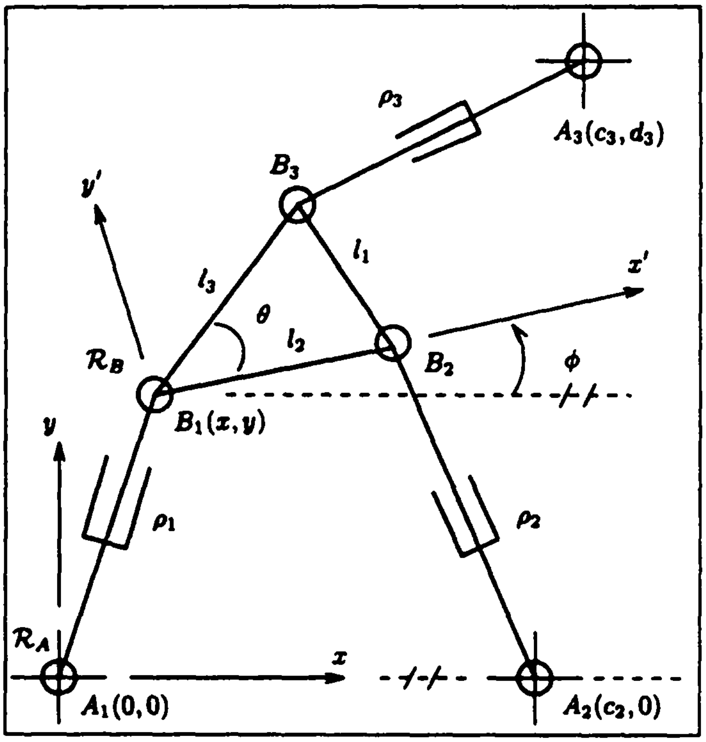

Fig. 1. General three-degree-of-freedom planar parallel manipulator.

with this manipulator. The derivation presented in [9] is now briefly outlined. To begin with, a fixed coordinate frame, noted $\mathcal{R}_A$, is attached to the base and a moving frame, noted $\mathcal{R}_B$, to the platform (Fig. 1). For purposes of simplification and without loss of generality, these frames are located at points A_1 and B_1, respectively, and are oriented in such a way that the X axes respectively intersect points A_2 and B_2. Hence, the Cartesian coordinates of the manipulator are defined as the position of point B_1, noted (x, y), on the plane and the orientation of the platform, given by angle ϕ (Fig. 1). Moreover, the joint coordinates are given by the length of the legs, noted ρ_1, ρ_2 and ρ_3. The equations associated with the inverse kinematic problem can then be written by considering the distance between the three pairs of points (A_i, B_i) for a given position and orientation of the platform. One gets:

$$\rho_1^2 = x^2 + y^2 \tag{1}$$

$$\rho_2^2 = (x + l_2 \cos\phi - c_2)^2 + (y + l_2 \sin\phi)^2 \tag{2}$$

$$\rho_3^2 = (x + l_3 \cos(\theta + \phi) - c_3)^2 + (y + l_3 \sin(\theta + \phi) - d_3)^2 \tag{3}$$

where c_2, c_3, d_3, l_2, l_3 and θ and the geometric parameters of the manipulator (Fig. 1).

Now, subtracting equation (1) from equations (2) and (3), respectively, leads to a new system of equations which can be written as

$$\rho_1^2 = x^2 + y^2 \tag{4}$$

$$\rho_2^2 - \rho_1^2 = Rx + Sy + Q \tag{5}$$

$$\rho_3^2 - \rho_1^2 = Ux + Vy + W \tag{6}$$

where the coefficients, R, S, Q, U, V and W are functions of the geometric parameters of the robot and of the angle of orientation of the platform ϕ which are written as

$$R = 2l_2 \cos\phi - 2c_2 \tag{7}$$

$$S = 2l_2 \sin\phi \tag{8}$$

$$Q = -2c_2 l_2 \cos\phi + l_2^2 + c_2^2 \tag{9}$$

$$U = 2l_3 \cos(\phi + \theta) - 2c_3 \tag{10}$$

$$V = 2l_3 \sin(\phi + \theta) - 2d_3 \tag{11}$$

$$W = -2l_3 d_3 \sin(\phi + \theta) - 2l_3 c_3 \cos(\phi + \theta) + l_3^2 + c_3^2 + d_3^2 \tag{12}$$

Equations (5) and (6) form a linear system of equations in x and y which can be readily solved. The expressions obtained for x and y are then substituted into equation (4) which leads to an equation in ϕ only. Finally, the following substitutions are used in the latter equation

$$\sin\phi = \frac{2T}{1 + T^2}, \quad \cos\phi = \frac{1 - T^2}{1 + T^2} \tag{13}$$

and a polynomial of degree 6 in T is obtained, i.e.,

$$\sum_{i=0}^{6} C_i T^i = 0 \tag{14}$$

where

$$T = \tan\left(\frac{\phi}{2}\right) \tag{15}$$

and where the coefficients, C_i, $i = 0, \ldots, 6$, are functions of the actuator lengths and of the geometric parameters. For each of the real roots of this polynomial, a unique solution for x and y—and hence a unique configuration of the platform—can be found, using the linear system consisting of equations (5) and (6). Since the maximum number of solutions to the problem is 6, the aforementioned polynomial is minimal. This result was reported in [7–10].

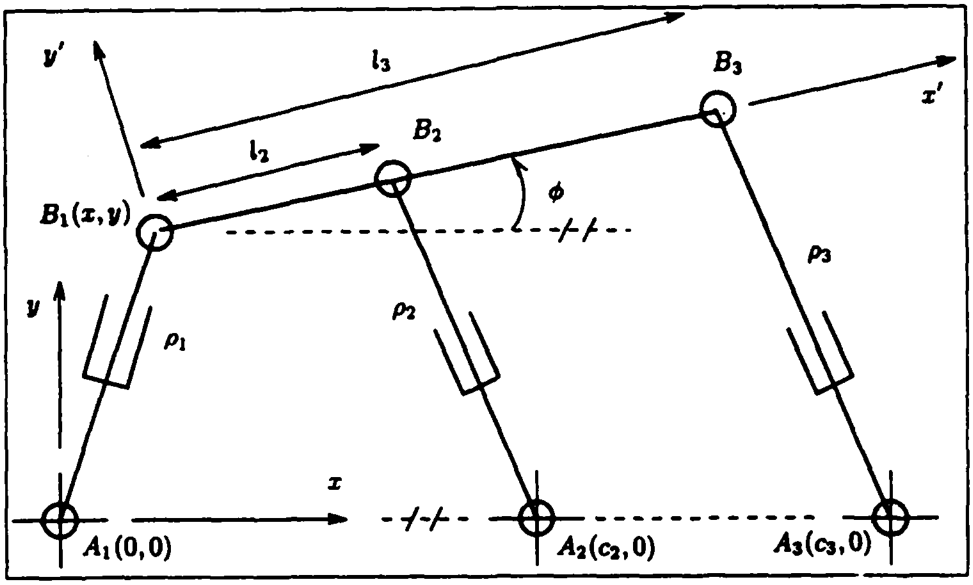

Fig. 2. Three-degree-of-freedom planar parallel manipulator of simplified architecture.

SIMPLIFIED MANIPULATOR AND NUMBER OF SOLUTIONS

A simplified version of the manipulator of the preceding section is represented in Fig. 2. In this particular case, the three revolute joints on the base and on the platform are respectively aligned. This architecture is obtained by setting angle θ and dimension d_3 to 0. Equations (1)–(3) are then simplified to:

$$\rho_1^2 = x^2 + y^2 \tag{16}$$

$$\rho_2^2 = (x + l_2 \cos\phi - c_2)^2 + (y + l_2 \sin\phi)^2 \tag{17}$$

$$\rho_3^2 = (x + l_3 \cos\phi - c_3)^2 + (y + l_3 \sin\phi)^2 \tag{18}$$

Using the procedure described above, an equation in ϕ only is obtained. In fact, in this particular case, the equation will be a cubic in $\cos\phi$, i.e., an equation of the form

$$a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0 \tag{19}$$

where $z = \cos\phi$ and where the coefficients a_3 to a_0 are functions of the geometric parameters and the joint coordinates which are given in the Appendix. Hence, the solution is cascaded in a cubic [equation (19)] and a quadratic (to uniquely define angle ϕ from the value of $\cos\phi$). A closed-form solution is therefore possible. In [9], it has been conjectured—no proof was given—that, in this particular case, only four feasible solutions are possible. This will now be shown.

To begin with, it should be noted that the variable in the cubic of equation (19) is in fact $\cos\phi$ and hence should be comprised between -1 and 1, i.e.,

$$-1 \leqslant z \leqslant 1 \tag{20}$$

Therefore, in order to determine the maximum number of real solutions, it suffices to determine only the number of solutions which lie in the aforementioned interval. To this end, Sturm's method [12] will be used. This method is briefly explained in the next subsection.

Sturm's method for the determination of the number of roots of a polynomial

Let $f_0(x)$ be a polynomial of degree n in x. The roots of $f_0(x)$ are given as the roots of the following equation:

$$f_0(x) = \sum_{i=0}^{i=n} a_i x^i = 0 \tag{21}$$

Sturm's theorem allows the determination of the number of real roots of a polynomial in a given interval $[x_1, x_2]$, without actually computing the roots [12]. To begin with, $f_1(x)$ is defined as the first derivative of $f_0(x)$ with respect to x, i.e.,

$$f_1(x) = f_0'(x) \tag{22}$$

Then, polynomial $f_0(x)$ is divided by $f_1(x)$, which leads to

$$f_0(x) = f_1(x) * d(x) + r_2(x) \tag{23}$$

where $d(x)$ is the result of the polynomial division and $r_2(x)$ is the remainder. At this point, $f_2(x)$ is defined as:

$$f_2(x) = -r_2(x) \tag{24}$$

This procedure is repeated iteratively, i.e., polynomial $f_{i-1}(x)$ is divided by $f_i(x)$, the remainder of this polynomial division is noted r_{i+1} and, finally, $f_{i+1}(x)$ is defined as $-r_{i+1}$. When f_{i+1} no longer contains any term in x—which occurs when $i = n - 1$—the procedure is stopped. In other words, the algorithm for the derivation of a Sturm sequence can be written as

For $i = 1$ to $n - 1$, do

$$f_{i-1}(x) = f_i(x) d_i(x) + r_{i+1}(x)$$
$$f_{i+1}(x) = -r_{i+1}(x)$$

where $f_1(x)$ is defined as in equation (22). Upon completion of the above operations, the expressions obtained for $f_0, f_1, \ldots f_n$ constitute the Sturm sequence. Now, let x_1 and x_2 be respectively the lower and the upper limit of interval of interest of $f_0(x)$. Sturm's theorem states that the number of real roots of $f_0(x)$ in this closed interval is equal to the number of sign changes between $f_i(x_1)$ and $f_{i+1}(x_1)$ for $i \in [0, n-1]$ minus the number of sign changes between $f_i(x_2)$ and $f_{i+1}(x_2)$ for $i \in [0, n-1]$. In other words, Sturm's theorem allows us to write

$$n_r = n_{c1} - n_{c2} \tag{25}$$

where n_r is the number of real roots in the interval of interest, n_{c1} is the number of sign changes between $f_i(x_1)$ and $f_{i+1}(x_1)$ for $i \in [0, n-1]$ and n_{c2} is the number of sign changes between $f_i(x_2)$ and $f_{i+1}(x_2)$ for $i \in [0, n-1]$.

Number of solutions of the direct kinematics of the simplified planar parallel manipulator

Since the polynomial of equation (19) is of degree 3, the application of Sturm's method will lead to four functions, f_0, f_1, f_2 and f_3, where f_3 is a constant. Moreover, the interval of interest of x—which is equal to $\cos\phi$—is $[-1, 1]$. One has

$$\begin{aligned} f_0(-1) = (&c_2^2 l_3 - c_2 l_3^2 + 2c_2 l_2 l_3 - l_3^2 l_2 + l_3 l_2^2 + c_2^2 c_3 - 2c_2 c_3 l_3 \\ &+ 2c_2 c_3 l_2 - 2c_3 l_2 l_3 + c_3 l_2^2 - c_2 c_3^2 - c_3^2 l_2 - c_2 \rho_1^2 + l_3 \rho_1^2 \\ &- l_2 \rho_1^2 + c_2 \rho_1^2 - l_3 \rho_2^2 - c_3 \rho_2^2 + l_2 \rho_3^2 + c_2 \rho_3^2)^2 \end{aligned} \tag{26}$$

$$\begin{aligned} f_0(1) = (&c_2^2 l_3 + c_2 l_3^2 - 2c_2 l_2 l_3 \\ &- l_3^2 l_2 + l_3 l_2^2 - c_2^2 c_3 - 2c_2 c_3 l_3 + 2c_2 c_3 l_2 + 2c_3 l_2 l_3 \\ &- c_3 l_2^2 - c_3^2 l_2 + c_2 c_3^2 + c_2 \rho_1^2 + l_3 \rho_1^2 - l_2 \rho_1^2 \\ &- c_3 \rho_1^2 - l_3 \rho_2^2 + c_3 \rho_2^2 - c_2 \rho_3^2 + l_2 \rho_3^2)^2 \end{aligned} \tag{27}$$

From equations (26) and (27), it is clear that quantities $f_0(-1)$ and $f_0(1)$ are both positive definite. Furthermore, f_3 being a constant, the two possibilities arising from its sign can easily be investigated.

The case for which f_3 is positive is first considered and the potential sequences of signs of f_1 and f_2 that would maximize the number of real roots in the interval of interest will now be determined for this case. Referring to equation (25), it is clear that, in order to maximize the number of real roots, the number of sign changes of $f_i(-1)$ has to be maximized and the number of sign changes

Table 1. First case maximizing the number of roots in the interval $[-1, 1]$ for a positive value of f_3

	f_0	f_1	f_2	f_3	No. of sign changes
$x = -1$	+	−	+	+	2
$x = 1$	+	+	+	+	0

of $f_i(1)$ minimized. The three possible cases that arise are illustrated in Tables 1–3. In the first case, it is assumed that $f_1(-1)$ is negative and that $f_2(-1)$ is positive while in the second one it is assumed that $f_1(-1)$ is positive and $f_2(-1)$ is negative. Finally, in the third case it is assumed that both $f_1(-1)$ and $f_2(-1)$ are negative. In all cases, $f_1(1)$ and $f_2(1)$ are assumed to be positive, in order to minimize the number of sign changes in $f_i(1)$. All three cases lead to two real roots comprised in the interval $[-1, 1]$, which is the maximum possible in this case.

Let us now consider the case for which f_3 is negative. Again three possible cases that would maximize the number of real roots arise. They are illustrated in Tables 4–6. In the first case, it is assumed that $f_1(1)$ and $f_2(1)$ are both positive while in the second case it is assumed that they are both negative. Finally, in the last case, it is assumed that $f_1(1)$ is positive while $f_2(1)$ is negative. In all three cases, $f_1(-1)$ is assumed to be negative while $f_2(-1)$ is assumed to be positive since this is the only way to maximize the number of sign change in the first line of the table. All cases lead to two real roots which is therefore the maximum number if f_3 is negative.

Hence, in any case, the maximum number of real solutions of equation (19) in the interval $[-1, 1]$ is 2. This leads to a maximum of four real solutions for the direct kinematics of the simplified parallel manipulator, which confirms the conjecture stated in [9]. The above derivation using Sturm's theorem therefore constitutes a valid proof of this statement.

Additionally, it is also possible to show that, except for special cases, the number of solutions to the direct kinematic problem will always be 4. Indeed, let us first assume that f_3 is positive. In this case, each line of the table will always contain an even number of sign changes, equal to 0 or 2. Therefore, the difference between these two values will always be even and, in general, equal to 2 which leads to four solutions of the direct kinematic problem. (Notice that if the difference is equal to zero then the mechanism cannot be assembled.) Similarly, if f_3 is assumed to be negative, the number of sign changes in each of the lines of the table will always be an odd number, equal to 1 or 3. Therefore, the difference between these numbers will always be an even number, in general equal to 2, which leads to four solutions for the direct kinematics.

ALTERNATIVE DERIVATION OF THE DIRECT KINEMATICS OF THE SIMPLIFIED PLANAR PARALLEL MANIPULATOR

The derivation presented in the preceding section for the simplified planar parallel manipulator had led to a polynomial solution of the direct kinematic problem in the form of a cascade of a cubic and a quadratic. Moreover, it has been used to show that, for this special manipulator, the direct kinematic problem leads to a maximum of four solutions. The proof was based on Sturm's theorem for polynomials. However, although the proof on the number of solutions is clear, it has not been possible to identify the spurious roots from the outset. Therefore, the three solutions of the cubic must be computed—even though it is known that only two are valid—and subsequently checked for validity.

The purpose of this section is to investigate an alternative method for the derivation of the polynomial solution of the direct kinematics in order to try to obtain a solution with no spurious roots. Other objectives of this new derivation are: (i) a further proof of the number of solutions obtained to confirm the previous approach and (ii) the development of an alternative computational scheme which could be used in special situations for which the previous derivation would

Table 2. Second case maximizing the number of roots in the interval $[-1, 1]$ for a positive value of f_3

	f_0	f_1	f_2	f_3	No. of sign changes
$x = -1$	+	+	−	+	2
$x = 1$	+	+	+	+	0

Table 3. Third case maximizing the number of roots in the interval $[-1, 1]$ for a positive value of f_3

	f_0	f_1	f_2	f_3	No. of sign changes
$x = -1$	+	−	−	+	2
$x = 1$	+	+	+	+	0

not be valid. Indeed, the previous derivation was based on the elimination of variables x and y from the equations through the solution of a linear system. This approach is valid as long as the latter linear system is of full rank and alternative schemes are needed if the system becomes singular.

The derivation now introduced is also based on equations (16)–(18). As in the previous procedure, an equivalent system of equations is obtained by subtracting the first equation from the second and the third and by using the first equation together with the two new equations thereby obtained. The substitutions of equation (13) are then used in the above equations, which leads to three polynomial equations in x, y and T, where T is defined as in equation (15).

Since T does not appear in the first equation, the resultant—using Bézout's theorem—of the last two equations can be used to eliminate T and obtain a new polynomial equation in x and y. Finally, the resultant of the latter equation and the first one is obtained, which leads to a polynomial of degree 6 in y, i.e.,

$$P_y(y) = \sum_{i=0}^{i=6} h_i y^i = 0 \tag{28}$$

where the coefficients, h_i $i = 1, \ldots, 6$ are functions of the geometry of the manipulator and of the joint variables and where

$$h_1 = h_3 = h_5 = 0 \tag{29}$$

The detailed expressions of the other coefficients are not given here because of space limitation but they can be obtained from the authors, in machine-readable form. Since the coefficients of the terms of odd degrees of this polynomial are equal to zero, it can be expressed as a polynomial of degree 3 in Y, with

$$Y = y^2 \tag{30}$$

One obtains,

$$P_Y(Y) = h_6 Y^3 + h_4 Y^2 + h_2 Y + h_0 \tag{31}$$

Again, the solution of the direct kinematic problem leads to a cascade of one cubic and one quadratic and the spurious solutions cannot be eliminated from the outset. However, coefficient h_6 has a simple form and can be written as

$$h_6 = 16384 c_2^2 c_3^2 (l_2 - l_3)^2 (c_3 l_2 - c_2 l_3)^2 \tag{32}$$

which is a positive definite quantity. This property of the polynomial will now be used in the determination of the maximum number of real solutions. The polynomial of equation (31) can be used in instances where the polynomial in T derived in the preceding section does not apply.

Determination of the maximum number of real solutions

In order to obtain an additional proof for the number of solutions of the direct kinematics of the simplified manipulator, the number of real roots of equation (31) will now be investigated. In fact, only real positive roots of this polynomial are valid since Y is defined as y^2. Hence, Sturm's theorem will be used on the interval given by $Y \in [0, \infty[$. Since h_6 is a positive definite quantity, one has, following the notation of the preceding section,

$$P_Y(\infty) = f_0(\infty) > 0 \tag{33}$$

Moreover, using the same notation, one can write

$$f_1(Y) = 3h_6 Y^2 + 2h_4 Y + h_2 \tag{34}$$

Table 4. First case maximizing the number of roots in the interval $[-1, 1]$ for a negative value of f_3

	f_0	f_1	f_2	f_3	No. of sign changes
$x=-1$	+	−	+	−	3
$x=1$	+	+	+	−	1

which leads to

$$f_1(\infty) > 0 \tag{35}$$

Therefore, for $Y = \infty$, there will be two positive elements in the Sturm sequence and hence a maximum of two sign changes. Since the number of sign changes obtained from the sequence derived for $Y = 0$ will be subtracted from that number, it can be readily concluded that the polynomial of equation (31) will never have more than two positive real roots which again shows that the direct kinematic problem has a maximum of four solutions.

It is pointed out that the above derivation can be slightly modified in order to obtain a polynomial of degree 6 in x. However, in this case, none of the coefficients of the polynomial obtained vanish.

Special cases

In the above derivation, one important special case arises when $P_Y(0) = 0$, i.e., when $Y = 0$ is a root of the polynomial. In this case, $y = 0$ is a solution of equations (16)–(18). Again, an equivalent system of equations is obtained by subtracting equation (16) from equations (17) and (18) and by using equation (16) as the third equation. The first of these equations is linear in x and can be solved as

$$x = -\frac{c_2^2 - 2l_2c_2\cos\phi + l_2^2 - \rho_2^2 + \rho_1^2}{2l_2\cos\phi - 2c_2} \tag{36}$$

This solution is then substituted into the other two equations, which leads to

$$A_1\cos^2\phi + A_2\cos\phi + A_3 = 0 \tag{37}$$

and

$$A_4\cos^2\phi + A_5\cos\phi + A_6 = 0 \tag{38}$$

where

$$A_1 = 4l_2^2c_2^2 - 4\rho_1^2l_2^2 \tag{39}$$

$$A_2 = (4l_2c_2 - 4l_2c_2^3 + 4l_2c_2\rho_1^2 - 4l_2^3c_2) \tag{40}$$

$$A_3 = (2c_2^2l_2^2 - 2c_2^2\rho_2^2 - 2c_2^2\rho_1^2 - 2l_2^2\rho_2^2 + 2l_2^2\rho_1^2 + \rho_2^4 - 2\rho_2^2\rho_1^2 + \rho_1^4 + c_2^4 + l_2^4 + \rho_2^2) \tag{41}$$

$$A_4 = 2l_3l_2c_2 - 2c_3l_2l_3 \tag{42}$$

$$A_5 = (-l_3c_2^2 - l_3l_2^2 + l_3\rho_2^2 - l_3\rho_1^2 - 2c_2c_3l_2 + 2c_3c_2l_3 + c_3^2l_2 + l_3^2l_2 - l_2\rho_3^2 + l_2\rho_1^2) \tag{43}$$

$$A_6 = c_3c_2^2 + c_3l_2^2 - c_3\rho_2^3 + c_3\rho_1^2 - c_3^2c_2 - l_3^2c_2 + c_2\rho_3^2 - c_2\rho_1^2 \tag{44}$$

Equations (37) and (38) are quadratic equations in $\cos\phi$ and will therefore lead to a maximum of four solutions for angle ϕ. Additionally, the consistency equation given in the Appendix must be satisfied.

Table 5. Second case maximizing the number of roots in the interval $[-1, 1]$ for a negative value of f_3

	f_0	f_1	f_2	f_3	No. of sign changes
$x=-1$	+	−	+	−	3
$x=1$	+	−	−	−	1

Table 6. Third case maximizing the number of roots in the interval $[-1, 1]$ for a negative value of f_3

	f_0	f_1	f_2	f_3	No. of sign changes
$x = -1$	+	−	+	−	3
$x = 1$	+	+	−	−	1

In the above derivation, the solution obtained for variable x assumed that the following condition was verified

$$c_2 \neq l_2 \cos \phi \tag{45}$$

If this is not the case for one of the solutions obtained, then the procedure is not valid. Alternatively, the second equation of the system can be used to solve for x. This leads to

$$x = -\frac{c_3^2 - 2l_3 c_3 \cos \phi + l_3^2 - \rho_3^2 + \rho_1^2}{2l_3 \cos \phi - 2c_3} \tag{46}$$

Substituting this result into the other two equations then gives

$$B_1 \cos^2 \phi + B_2 \cos \phi + B_3 = 0 \tag{47}$$

and

$$B_4 \cos^2 \phi + B_5 \cos \phi + B_6 = 0 \tag{48}$$

where

$$B_1 = 4l_3^2 c_3^2 - 4l_3^2 \rho_1^2 \tag{49}$$

$$B_2 = -4l_3 c_3^3 - 4l_3^3 c_3 + 4l_3 c_3 \rho_3^2 + 4l_3 c_3 \rho_1^2 \tag{50}$$

$$B_3 = (2c_3^2 l_3^2 - 2c_3^2 \rho_3^2 - 2c_3^2 \rho_1^2 - 2l_3^2 \rho_3^2 + 2l_3^2 \rho_1^2 + \rho_3^4 - 2\rho_3^2 \rho_1^2 + \rho_1^4 + c_3^4 + l_3^4) \tag{51}$$

$$B_4 = 2l_2 l_3 c_3 - 2l_2 l_3 c_2 \tag{52}$$

$$B_5 = (-l_2 c_3^2 - l_2 l_3^2 + l_2 \rho_3^2 - l_2 \rho_1^2 - 2c_2 c_3 l_3 + 2l_2 c_2 c_3 + c_2^3 l_3 + l_2^2 l_3 - l_3 \rho_2^2 + l_3 \rho_1^2) \tag{53}$$

$$B_6 = c_2 c_3^2 + c_2 l_3^2 - c_2 \rho_3^2 + c_2 \rho_1^2 - c_2^2 c_3 - l_2^2 c_3 + c_3 \rho_2^2 - c_3 \rho_1^2 \tag{54}$$

Again, two quadratic equations in $\cos \phi$ are obtained which leads to a maximum of four real solutions for ϕ. The corresponding consistency equation given in the Appendix must be satisfied.

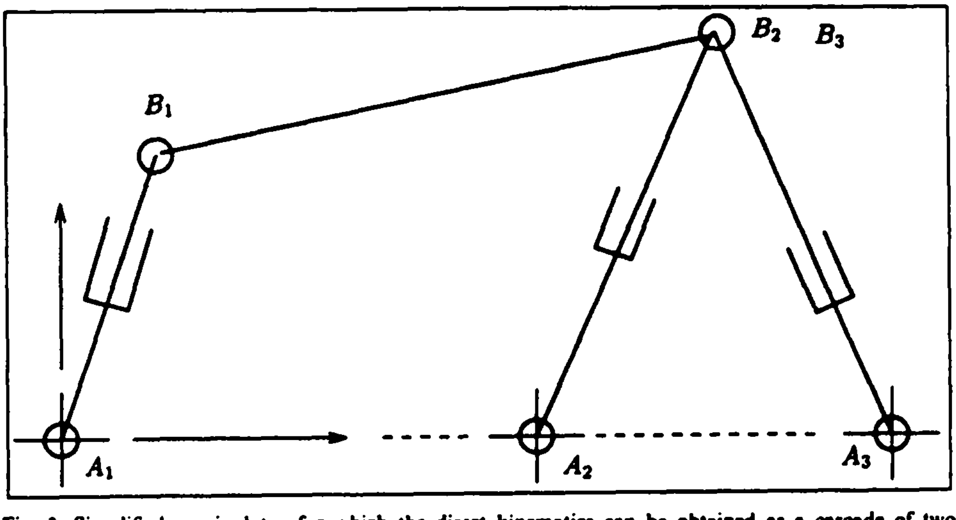

Fig. 3. Simplified manipulator for which the direct kinematics can be obtained as a cascade of two quadratics.

In this case, the derivation is not valid if the following condition is satisfied

$$c_3 = l_3 \cos\phi \tag{55}$$

Therefore, the case for which the following conditions are satisfied must be considered:

$$c_2 = l_2 \cos\phi \quad \text{and} \quad c_3 = l_3 \cos\phi \tag{56}$$

When subjected to these conditions, the original system of equation becomes

$$x^2 - \rho_1^2 = 0 \tag{57}$$

$$l_2^2 \cos^2\phi - l_2^2 + \rho_2^2 - \rho_1^2 = 0 \tag{58}$$

$$l_3^2 \cos^2\phi - l_3^2 + \rho_3^2 - \rho_1^2 = 0 \tag{59}$$

Hence, two values of opposite sign are obtained for $\cos\phi$. However, the consistency equation will invalidate one of these solutions and only two solutions are obtained for ϕ. Equation (57) gives two values of x and the direct kinematics leads to a maximum of four solutions. Equations (58) and (59) lead to the following consistency condition

$$l_3^2\rho_1^2 - l_3^2\rho_2^2 + l_2^2\rho_3^2 - l_2^2\rho_1^2 = 0 \tag{60}$$

With the solution scheme derived above, all special cases can be solved and the proof for the number of solution holds in all cases.

POLYNOMIAL SIMPLIFICATIONS AND SPECIAL ARCHITECTURES

The polynomial of degree 6 obtained for the solution of the direct kinematic problem of general three-degree-of-freedom planar parallel manipulators does not allow for closed-form solutions. Indeed, as is well known, expressions for the roots of general polynomials of degree greater than 4 cannot be obtained in closed-form. However, since the coefficients of the polynomial obtained are functions of the geometric parameters of the manipulator and of the joint coordinates, it could be interesting to address the problem of finding a manipulator architecture that would lead to a reduction in the degree of the polynomial for any value of the joint coordinates. Such cases are studied in the next subsections.

Possible vanishing of the coefficient of the term of degree 6 in the general polynomial

Conditions under which the coefficient of the term of degree 6 in the original polynomial—equation (14)—would vanish would lead to simplified direct kinematics since the degree of the polynomial would be reduced. In order to study this possibility, the term in $\rho_2^2\rho_3^2$ of this coefficient is first examined. Indeed, for the coefficient of the term of degree 6 to vanish over the whole workspace of the manipulator, it has to be identically equal to zero for any value of the joint variables. The latter term, noted r_{23}, can be written as

$$r_{23} = 2l_2(l_2 + c_2)(c_3 + l_3\cos\theta)\rho_2^2\rho_3^2 \tag{61}$$

Since l_2 and c_2 are positive definite quantities, the vanishing of this term requires that the following condition be verified:

$$c_3 = -l_3\cos\theta \tag{62}$$

When condition (62) is imposed, the term in $\rho_1^2\rho_3^2$ in the coefficient of degree 6 of the polynomial, noted r_{13}, then becomes

$$r_{13} = 2(l_2 + c_2)^2\rho_1^2\rho_3^2 \tag{63}$$

which cannot be equal to zero for arbitrary values of ρ_1 and ρ_2. Therefore, it is not possible to find a manipulator with the general architecture of Fig. 1 for which the coefficient of degree 6 of equation (14)—the polynomial solution of the direct kinematic problem—would vanish.

It is worth mentioning, however, that this approach can lead to interesting special architectures if the simplified manipulator studied in the preceding sections is considered, i.e., the manipulator for which the revolute joints are aligned on the base and on the platform (Fig. 2). In this case, the coefficient of degree 3 of the cubic of equation (19) can be investigated, an expression of which is given in the Appendix as

$$a_3 = -8l_2c_2l_3c_3(c_2 - c_3)(l_2 - l_3) \tag{64}$$

It is clear, from this expression, that if all quantities are positive definite, then this coefficient can vanish if c_2 is equal to c_3 or if l_2 is equal to l_3. In other words, the cubic equation becomes a quadratic if two of the revolute joints on the base or on the platform coincide. An example of such an architecture is given in Fig. 3. The real gain of simplicity in the solution of the direct kinematics is not very important, however, because the simplified aligned architecture already leads to a closed-form solution.

Possible vanishing of the coefficients of the terms of odd degrees in the general polynomial

If the terms of odd degrees of the polynomial of equation (14) vanish, then it is possible to reduce this polynomial to a polynomial of degree 3 in T^2. This would allow for a closed-form solution through the cascade of one cubic and one quadratic, just as in the case of the simplified manipulator discussed in the preceding sections. The problem to be addressed now is the identification of special architectures—different from the one of the simplified manipulator presented above—which would also lead to a simplification of the polynomial through the elimination of the terms of odd degrees. From equation (13), it is clear that the terms of odd degrees in the polynomial arise from the terms in $\sin\phi$ in the original equation, i.e., the equation obtained before the substitutions of equation (13) are used. Hence, the condition for the elimination of the terms in odd degrees is the vanishing of the term in $\sin\phi$ in the original equation. This term is written as

$$(u_{11}\cos\phi + u_{12})\sin\phi \tag{65}$$

where u_{11} and u_{12} are given in the Appendix.

The expression for u_{12} is now examined and the terms in $\rho_2^2\rho_3^2$ and ρ_2^4 are collected. One has

$$u_{12} = (2l_2d_3 - 2c_2l_3\sin\theta)\rho_2^2\rho_3^2 + (2c_3l_3\sin\theta - 2l_3d_3\cos\theta)\rho_2^4 + \cdots \tag{66}$$

Imposing the vanishing of these terms leads to

$$2l_2d_3 - 2c_2l_3\sin\theta = 0 \tag{67}$$

$$2c_3l_3\sin\theta - 2l_3d_3\cos\theta = 0 \tag{68}$$

From these equations, it is clear that if $\sin\theta$ is equal to zero, d_3 must also be equal to zero, which corresponds to the case of the simplified manipulator introduced in the preceding sections. It is now assumed that $\sin\theta$ is not equal to zero in order to try to identify other special architectures. Solving equation (67) for d_3 leads to

$$d_3 = \frac{c_2l_3\sin\theta}{l_2} \tag{69}$$

Substituting this result into equation (68) and solving for l_2, one has, finally

$$d_3 = \frac{c_3\sin\theta}{\cos\theta} \tag{70}$$

$$l_2 = \frac{l_3c_2\cos\theta}{c_3} \tag{71}$$

If these conditions on the geometry of the manipulator are satisfied, the polynomial solution of the direct kinematic problem will contain only terms of even degrees. This is easily verified by substituting equations (70) and (71) back into the expressions of u_{11} and u_{12}, which leads to

$$u_{11} = 0, \quad u_{12} = 0 \tag{72}$$

Moreover, it can be easily verified that the geometric interpretation of conditions (70) and (71) is simply that the triangle formed by the three points of attachment of the revolute joints on the base and the triangle formed by the three points of attachment of the revolute joints on the platform are similar triangles. In other words, if the base and platform triangle are a scaled version of one another, the direct kinematics will be cascaded and will hence lead to a closed-form solution.

In this special case, the polynomial of degree 6 contains only terms of even degree in T and can therefore be expressed as a polynomial of degree 3 in T^2. Futhermore, it is possible to show that the latter polynomial can be factored as a polynomial of degree 1 and a polynomial of degree 2. Indeed, if z is defined as $\cos\phi$, the resulting polynomial can be expressed as a polynomial of degree 3 in z which can be factored as

$$P(z) = P_1(z)P_2(z) = 0 \tag{73}$$

with

$$P_1(z) = l_3^2\cos^2\theta + c_3^2 - (2c_3l_3\cos\theta)z = 0 \tag{74}$$

and where $P_2(z)$ is a polynomial of degree 2 in z. The first root for z, noted z_0, can be obtained from $P_1(z)$ as

$$z_0 = \frac{l_3^2\cos^2\theta + c_3^2}{2c_3l_3\cos\theta} \tag{75}$$

which can be rewritten as

$$z_0 = 1 + \frac{(l_3\cos\theta - c_3)^2}{2c_3l_3\cos\theta} \tag{76}$$

Since one of the conditions on the geometry of the base and the platform triangles—equation (71)—implies that c_3 and $\cos\theta$ always have the same sign and since l_3 is a positive definite quantity, z_0 will always be greater than 1 and cannot be a solution for z (which is equal to $\cos\phi$). Hence there will be only two solutions for z—given by the roots of $P_2(z)$—which means that the direct kinematic problem will have only four solutions in this case. The only exception occurs when $c_3 = l_3\cos\theta$, i.e., when the base and platform triangles are identical. In this case, one has

$$z_0 = 1 \tag{77}$$

which is within the range of the cosine function.

CONCLUSION

This paper has presented several results on the direct kinematics of planar three-degree-of-freedom parallel manipulators. First of all it was shown, using Sturm's theorem, that the direct kinematic problem of the simplified manipulator—for which the revolute joints on the base and on the platform are respectively aligned—leads to a maximum of four solutions. Moreover, alternative derivations of the direct kinematics of this manipulator have been given. It was shown that polynomials of degree 6 in x, y or $T = \tan(\phi/2)$ can be derived. In the latter two cases, a cascaded form of the direct kinematics allowing for closed-form solutions is obtained. The solution based on the polynomial in y was studied in detail and a robust computational scheme accounting for all special cases was given. Special architectures different from the simplified manipulator with aligned revolute joints and leading to simplified direct kinematics were then investigated. It was shown that if the base and platform triangles are similar, the direct kinematics simplifies in a cascaded sequence and can be solved in closed-form. Furthermore, the sequence obtained involves two quadratics which means that spurious roots are eliminated from the outset in this case. The results introduced in this paper are of interest in the context of analysis and design of planar parallel manipulators, which may find several applications in robotics as well as in motion systems in general.

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APPENDIX

In all the expressions given below, the following notation is used:

$$\rho_{i2} = \rho_i^2, \quad i = 1\text{–}3 \tag{A1}$$

Coefficients a_i of equation (19)

$$a_3 = -8l_2c_2l_3c_3(c_2 - c_3)(l_2 - l_3) \tag{A2}$$

$$\begin{aligned} a_2 = {} & 4l_3^2l_2^2c_3^2 - 16c_2l_3^2l_2^2c_3 \\ & + 4l_3^2l_2^2c_2^2 - 4l_3^3l_2c_2^2 + 8l_2l_3^3c_2c_3 \\ & + 8l_2l_3c_3^3c_2 + 8\rho_{12}c_3l_2l_3c_2 - 4c_3^2l_2^3l_3 \\ & - 16c_3^2l_2l_3c_2^2 - 4c_3^2l_2l_3\rho_{12} - 4c_3^2l_2^2\rho_{12} \\ & - 4c_2c_3^3l_2^2 - 4\rho_{12}l_2l_3c_2^2 - 4c_2\rho_{12}l_2^2c_3 \\ & + 8l_2^3l_3c_2c_3 + 8l_2l_3c_2^3c_3 + 4l_3^2c_2^2\rho_{12} \\ & - 4c_2^3l_3^2c_3 - 4c_2l_3^2\rho_{12}c_3 + 4c_2^2l_3^2c_3^2 + 4l_2^2c_3^2c_2^2 \end{aligned} \tag{A3}$$

$$\begin{aligned} a_1 = {} & 6l_3\rho_{12}c_2c_3^2 - 2l_3\rho_{12}^2c_3 - 8l_3c_2^2\rho_{12}c_3 \\ & - 4l_3^3l_2^2c_3 + 2l_3c_2^3\rho_{12} + 2l_3^3\rho_{12}c_2 \\ & + 2l_3\rho_{12}^2c_2 - 2l_3c_2^4c_3 + 2l_2^3\rho_{12}c_3 \\ & - 2l_2^4l_1c_3 + 6l_3c_2^3c_3^2 + 2l_2\rho_{12}^2c_3 \\ & + 6l_2c_3^3c_2^2 + 2l_2c_3^3\rho_{12} - 2l_2\rho_{12}^2c_2 \\ & + 6l_2\rho_{12}c_2^2c_3 - 4c_2^2l_3^3c_3 - 4c_2^2l_3c_3^3 \\ & - 4l_2^3c_3^2c_2 - 4l_2c_2^2c_3^3 - 4l_3^2l_2c_2^3 \\ & - 4l_3^2l_2^3c_2 - 4l_3c_3^3l_2^2 - 8l_3^2l_2c_2\rho_{12} \\ & + 6l_3^2c_3l_2\rho_{12} + 6\rho_{12}l_2^2l_3c_2 - 8l_2c_3^2c_2\rho_{12} \\ & + 6l_3^3l_2^2c_2 + 6l_3^2c_3l_2^3 - 2l_2l_3^4c_2 \\ & - 8l_3c_3l_2^2\rho_{12} - 2l_2c_3^4c_2 + 2l_3^3c_2^3 + 2l_2^3c_3^3 + 10l_2l_3^2c_3c_2^2 \\ & - 4l_2l_3^2c_3^2c_2 + 10l_2^2l_3c_3^2c_2 - 4l_2^2l_3c_2^2c_3 \end{aligned} \tag{A4}$$

$$\begin{aligned} a_0 = {} & (l_2^2 + l_3^2 + c_2^2 + c_3^2 - 2l_2l_3 - 2c_2c_3) \\ & \times (l_3^2l_2^2 + c_3^2l_2^2 - 2l_3\rho_{12}l_2 + \rho_{12}^2 - 2c_2\rho_{12}c_3 + c_3^2c_3^2 + l_3^2c_2^2) \end{aligned} \tag{A5}$$

Condition for the consistency of equation (37) and (38)

$$\begin{aligned} & -4l_2^2(\rho_{12} - \rho_{22} + l_2^2 - c_2^2)^2(2\rho_{12}l_2^3c_3^2l_3 - l_2^2c_2^2\rho_{12}^2 \\ & + 2\rho_{12}\rho_{32}c_2^2l_2^2 - 2\rho_{12}l_2\rho_{32}c_2^2l_3 - 2\rho_{12}l_2\rho_{32}l_3\rho_{22} - 2\rho_{12}l_2c_3c_2^3l_3 \\ & - 2\rho_{12}l_2l_3^3c_2c_3 - 2\rho_{12}l_2^3c_3l_3c_2 + 2\rho_{12}l_2l_3^3c_2^2 + l_3^2\rho_{12}^3 - c_3^2\rho_{12}^2l_3^2 \\ & - c_2^2\rho_{32}^2l_2^2 + l_2^2\rho_{12}^3 - l_2^2c_3^4c_2^2 - l_2^2l_3^4c_2^2 - l_2^4l_3^2c_3^2 - c_2^4l_3^2c_3^2 \\ & - \rho_{22}^2l_3^2c_3^2 - 4l_2^2c_3^2c_2^2l_3^2 + 2l_2l_3^3c_2^3c_3 + 2l_2c_3^3c_2^3l_3 \\ & - 2l_2^3\rho_{32}c_2l_3c_3 + 2l_2\rho_{32}c_2\rho_{22}l_3c_3 - 2l_2l_3^3c_2\rho_{22}c_3 + 2c_2^2\rho_{22}l_3^2c_3^2 \\ & + 2l_2^2\rho_{22}l_3^2c_3^2 - 2l_2\rho_{32}c_2^3l_3c_3 + 2l_2^2c_3^2c_2^2\rho_{32} + 2l_2^3c_3^3c_2l_3 \\ & - 2l_2c_2^3c_2\rho_{22}l_3 + 2l_2^2l_3^2c_2^2\rho_{32} + 2l_2^3l_3^3c_2c_3 - 2\rho_{12}^2l_2l_3 - 2\rho_{12}^2l_2^3l_3 \\ & - 2\rho_{12}^2l_2l_3^3 + 4\rho_{12}^2l_2^2l_3^2 - 2\rho_{12}^2\rho_{32}l_2^2 - 2\rho_{12}^2l_2^2c_3^2 - 2\rho_{12}^2l_3^2c_2^2 \\ & - 2\rho_{12}^2\rho_{22}l_3^2 + \rho_{12}l_2^4l_3^2 - 2\rho_{12}l_2^3l_3^3 + \rho_{12}l_2^2l_3^4 + \rho_{12}\rho_{32}^2l_2^2 \\ & + 2\rho_{12}^2l_2c_3^2l_3 + 2\rho_{12}^2l_2\rho_{22}l_3 + 2\rho_{12}^2l_2\rho_{32}l_3 + 2\rho_{12}l_2^2c_2^2c_3^2 \\ & + 2\rho_{12}^2l_2c_2^2l_3 + 2\rho_{12}^2l_2c_3l_3c_2 + 2\rho_{12}c_2^2l_3^2c_3^2 - 2\rho_{12}\rho_{22}l_3^2c_2^2 \end{aligned}$$

$$
\begin{aligned}
&+ 2\rho_{12}\rho_{22}l_1^2c_3^2 + 2\rho_{12}l_2^3\rho_{32}l_3 - 2\rho_{12}l_2^2l_3^3\rho_{22} + \rho_{12}l_2^2c_3^4 + \rho_{12}\rho_{22}^2l_3^2 \\
&+ \rho_{12}c_2^4l_3^2 - 2\rho_{12}l_2c_3^2c_2^2l_3 + 2\rho_{12}l_2c_3\rho_{22}l_3c_2 - 2\rho_{12}l_2c_3^2\rho_{22}l_3 \\
&+ 2\rho_{12}l_2\rho_{32}c_2l_3c_3 + 2\rho_{12}l_2l_3^3\rho_{22} - 2\rho_{12}l_2^2c_3^2\rho_{32} \\
&- 2\rho_{12}l_2^2l_3^2\rho_{32} - 2\rho_{12}l_2c_3^3c_2l_3) = 0
\end{aligned} \tag{A6}
$$

Condition for the consistency of equations (47) and (48)

$$
\begin{aligned}
&- 4l_3^2(\rho_{12} - \rho_{32} + l_3^2 - c_3^2)^2(\rho_{12}l_3^2c_2^4 - 2\rho_{12}l_3^3l_2^3 + l_3^2\rho_{12}^3 \\
&+ \rho_{12}l_3^2\rho_{22}^2 - l_3^2c_3^2\rho_{12}^2 + \rho_{12}l_3^4l_2^2 - 2l_3^3\rho_{22}c_3l_2c_2 + 2l_3\rho_{22}c_3\rho_{32}l_2c_2 \\
&- l_3^2c_2^4c_3^2 - l_3^2l_2^4c_3^2 - l_3^4l_2^2c_2^2 - l_3^2\rho_{22}^2c_3^2 - c_3^4l_2^2c_2^2 \\
&- \rho_{32}^2l_2^2c_2^2 - 4l_3^2c_2^2c_3^2l_2^2 - 2l_3l_2^3c_3\rho_{32}c_2 + 2c_3^2\rho_{32}l_2^2c_2^2 \\
&+ 2l_3^2\rho_{32}l_2^2c_2^2 + 2l_3c_2^3c_3^3l_2 + 2l_3l_2^3c_3^3c_2 - 2l_3\rho_{22}c_3^3l_2c_2 \\
&+ 2l_3^2c_2^2c_3^2\rho_{22} + 2l_3^3c_2^3c_3l_2 - 2l_3c_2^3c_3\rho_{32}l_2 + 2l_3^2l_2^2c_3^2\rho_{22} \\
&+ 2l_3^3l_2^3c_3c_2 + \rho_{12}^3l_2^2 - 2\rho_{12}^3l_3l_2 - 2\rho_{12}^2l_3l_2^3 - 2\rho_{12}^2l_3^3l_2 - \rho_{12}^2c_2^2l_2^2 \\
&+ 4\rho_{12}^2l_3^2l_2^2 - 2\rho_{12}^2l_3^2\rho_{22} - 2\rho_{12}^2l_3^2c_2^2 - 2\rho_{12}^2\rho_{32}l_2^2 - 2\rho_{12}^2l_2^2c_3^2 \\
&+ \rho_{12}c_3^4l_2^2 + \rho_{12}\rho_{32}^2l_2^2 + 2\rho_{12}^2l_3c_2l_2c_1 + \rho_{12}l_3^2l_2^4 + 2\rho_{12}^2l_3c_2^2l_2 \\
&+ 2\rho_{12}^2l_3c_3^2l_2 + 2\rho_{12}^2l_3\rho_{32}l_2 + 2\rho_{12}^2l_3\rho_{22}l_2 + 2\rho_{12}l_3^2c_3^2c_2^2 + 2\rho_{12}l_3^3c_2^3l_2 \\
&+ 2\rho_{12}l_3^2c_3^2\rho_{22} + 2\rho_{12}l_3l_2^3c_3^2 + 2\rho_{12}c_3^2l_2^2c_2^2 - 2\rho_{12}\rho_{32}l_2^2c_3^2 \\
&+ 2\rho_{12}\rho_{32}l_2^2c_2^2 + 2\rho_{12}l_3^3\rho_{22}l_2 - 2\rho_{12}l_3^2l_2^2\rho_{32} + 2\rho_{12}l_3l_2^3\rho_{32} \\
&- 2\rho_{12}l_1^2c_2^2\rho_{22} - 2\rho_{12}l_3^2l_2^2\rho_{22} - 2\rho_{12}l_3^2l_2^2\rho_{22} + 2\rho_{12}l_1c_2\rho_{32}l_2c_3 \\
&- 2\rho_{12}l_3c_2^2\rho_{32}l_2 - 2\rho_{12}l_3\rho_{22}c_3^2l_2 + 2\rho_{12}l_3\rho_{22}c_3l_2c_2 - 2\rho_{12}l_3^3c_2l_2c_3 \\
&- 2\rho_{12}l_3c_2^2c_3^2l_2 - 2\rho_{12}l_1l_2^3c_3c_2 - 2\rho_{12}l_3c_2^3c_3l_2 \\
&- 2\rho_{12}l_1c_2c_2^3l_2 - 2\rho_{12}l_1\rho_{22}l_2\rho_{32} = 0
\end{aligned} \tag{A7}
$$

Coefficients u_{11}, u_{12} of equation (65)

$$
\begin{aligned}
u_{11} = {} & 4c_2^3l_3^2d_3 - 4d_3^3l_2^2c_2 + 16c_3^2l_2c_2^2l_3\sin\theta + 8c_2^2l_3^2d_3^2\sin\theta\cos\theta \\
&- 16l_2^2l_3^2c_2d_3\cos^2\theta + 16l_2^2l_3^2\cos\theta c_2c_3\sin\theta - 8l_2^3c_3c_2l_1\sin\theta + 16c_2^2l_3^2d_3c_3\cos^2\theta \\
&- 8l_2c_2^3c_3l_3\sin\theta + 8l_2c_2c_3\rho_{22}l_3\sin\theta - 8c_2l_3^2d_3\rho_{12}\cos^2\theta - 8c_2^3l_3^2d_3\cos^2\theta \\
&+ 8c_2l_3^2d_3\rho_{22}\cos^2\theta + 8c_2l_3^2c_3\rho_{12}\sin\theta\cos\theta + 8\rho_{12}c_3l_2^2d_3 - 8c_2^2l_3^2c_3^2\cos\theta\sin\theta \\
&+ 8c_2^3l_3^2c_3\cos\theta\sin\theta - 8c_2l_3^2c_3\rho_{22}\cos\theta\sin\theta - 8c_2^2l_3^2d_1c_3 - 4c_2l_3^2d_3\rho_{22} \\
&+ 8l_3^2d_3^2l_2^2\sin\theta\cos\theta + 16l_3^2c_3l_2^2d_3\cos^2\theta - 8l_3^2c_3^2l_2^2\cos\theta\sin\theta + 4c_2l_3^2d_3\rho_{12} \\
&- 4d_3^2l_2\rho_{12}l_3\sin\theta - 16l_3c_3l_2d_3c_2^2\cos\theta + 8l_3c_3l_2d_3\rho_{22}\cos\theta - 8l^3c_3l_2^3d_3\cos\theta \\
&+ 4d_3^2l_2\rho_{22}l_3\sin\theta - 4d_3^2l_2^3l_3\sin\theta - 4\rho_{32}l_2c_2^2l_3\sin\theta + 4\rho_{32}l_2^2d_3c_2 \\
&+ 8l_2c_2^3l_3d_3\cos\theta - 8l_2c_2l_3d_3\rho_{22}\cos\theta + 8l_2c_2l_3d_3\rho_{12}\cos\theta \\
&- 8c_2^2l_3^2\rho_{12}\cos\theta\sin\theta \\
&+ 4l_3^3l_2c_2^2\sin\theta - 8l_3c_3l_2d_3\rho_{12}\cos\theta + 4c_3^2l_2^3l_3\sin\theta - 4c_2^3l_2\rho_{22}l_3\sin\theta \\
&+ 4c_3^2l_2\rho_{12}l_3\sin\theta - 4c_3^2l_2^2d_3c_2 + 8l_2^3l_3d_3c_2\cos\theta + 4\rho_{12}l_2c_2^2l_3\sin\theta \\
&- 4\rho_{12}l_2^2d_3c_2 + 8l_2d_3^2c_2l_3\cos\theta - 8l_2d_3^2c_2l_3c_3\sin\theta - 8l_2\rho_{32}c_2l_3d_3\cos\theta \\
&+ 8l_2\rho_{32}c_2l_3c_3\sin\theta + 8l_2c_3^2c_2l_3d_3\cos\theta - 8l_2c_3^3c_2l_3\sin\theta + 8l_2l_3^3c_2d_3\cos\theta \\
&- 8l_2l_3^3c_2c_3\sin\theta - 8l_2\rho_{12}c_2l_3c_3\sin\theta - 8l_3^2c_3l_2^2d_3
\end{aligned} \tag{A8}
$$

$$
\begin{aligned}
u_{12} = {} & 2\rho_{12}l_2^3d_3 - 2d_3^3l_2\rho_{22} - 2c_2^3l_3^3\sin\theta + 2l_3^2l_2^3d_3 \\
&+ 2d_3^3l_2\rho_{12} + 2c_3^2l_2^3d_3 + 2\rho_{12}^2l_2d_3 - 2\rho_{32}l_2^3d_3 \\
&+ 2d_3^3l_2c_2^2 + 2d_3^3l_2^3 - 2\rho_{22}^3l_3d_3\cos\theta + 4\rho_{22}l_3d_3\rho_{12}\cos\theta - 2\rho_{12}^2l_3d_3\cos\theta \\
&+ 4l_2^2c_3c_2l_3d_3\cos\theta - 6l_2^2c_3^2c_2l_3\sin\theta + 2l_2^4c_3l_3\sin\theta + 4l_2^2c_3c_2^2l_3\sin\theta \\
&- 4l_2^2c_3\rho_{22}l_3\sin\theta - 2c_2l_3^3\rho_{12}\sin\theta + 8c_2^2\rho_{12}l_3c_3\sin\theta - 2c_2\rho_{12}l_2^2l_3\sin\theta \\
&- 2c_2^3\rho_{12}l_3\sin\theta + 2c_2\rho_{12}\rho_{22}l_3\sin\theta - 2c_2\rho_{12}^2l_3\sin\theta + 4c_2^3c_3l_3d_3\cos\theta \\
&+ 2c_2^4c_3l_3\sin\theta - 4c_2^2c_3\rho_{22}l_3\sin\theta - 4\rho_{22}c_3c_2l_3d_3\cos\theta + 2\rho_{22}^2c_3l_3\sin\theta \\
&- 4\rho_{22}c_3\rho_{12}l_3\sin\theta + 4\rho_{12}c_3c_2l_3d_3\cos\theta + 2\rho_{12}^2c_3l_3\sin\theta - 2c_2d_3^2l_2^2l_3\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
&-2c_2^3d_3^2l_3\sin\theta+2c_2d_3^2\rho_{22}l_3\sin\theta-2c_2d_3^2\rho_{12}l_3\sin\theta+4c_2^2\rho_{32}l_3d_3\cos\theta\\
&-4c_2^2\rho_{32}l_3c_3\sin\theta+2c_2d_3^2\rho_{32}l_2^2l_3\sin\theta+2c_2^3\rho_{32}l_3\sin\theta-2c_2\rho_2^3\rho_{22}l_3\sin\theta\\
&+2c_2\rho_{32}\rho_{12}l_3\sin\theta-4c_2^2c_3^2l_3d_3\cos\theta+4c_2^2c_3^3l_3\sin\theta-6c_2^3c_3^2l_3\sin\theta\\
&+6c_2c_3^2\rho_{22}l_3\sin\theta-6c_2c_3^2\rho_{12}l_3\sin\theta-4c_2^2l_3^3d_3\cos\theta+4c_2^2l_3^3c_3\sin\theta\\
&-2c_2l_3^3l_2^2\sin\theta+2c_2l_3^3\rho_{22}\sin\theta-8\rho_{12}c_3l_2d_3c_2+4l_3c_3^3l_2^2\sin\theta\\
&+4\rho_{32}l_2^2l_3d_3\cos\theta-4\rho_{32}l_2^2l_3c_3\sin\theta+4l_3^2d_3l_2\rho_{12}\cos^2\theta-4d_3^3l_2^2l_3\cos\theta\\
&+4d_3^2l_2^2l_3c_3\sin\theta+8\rho_{12}l_3^2l_2c_2\sin\theta\cos\theta-4l_3^2c_3l_2\rho_{12}\sin\theta\cos\theta+4l_3^2c_3l_2\rho_{22}\sin\theta\cos\theta\\
&-4c_3^2l_2^2l_3d_3\cos\theta-4l_3^2c_3l_2c_2^2\sin\theta\cos\theta-4l_3^2d_3l_2\rho_{22}\cos^2\theta+4l_3^2d_3l_2c_2^2\cos^2\theta\\
&-2\rho_{32}l_2d_3c_2^2+2\rho_{32}l_2d_3\rho_{22}-2\rho_{32}l_2d_3\rho_{12}+2c_3^2l_2d_3c_2^2\\
&-2c_3^2l_2d_3\rho_{22}+2c_3^2l_2d_3\rho_{12}+2l_3^2l_2d_3c_2^2-2l_3^2l_2d_3\rho_{22}\\
&+2l_3^2l_2d_3\rho_{12}+4l_2^2l_3d_3\rho_{22}\cos\theta-8l_2^2l_3d_3\rho_{12}\cos\theta-2c_2^4l_3d_3\cos\theta\\
&-4c_2^2d_3^3l_3\cos\theta+4c_2^2d_3^2l_3c_3\sin\theta-4l_2^2l_3d_3c_2^2\cos\theta+4l_3^2d_3l_2^3\cos^2\theta\\
&+4c_2^2l_3d_3\rho_{22}\cos\theta-4l_3^2c_3l_2^3\sin\theta\cos\theta-4l_3^3l_2^2d_3\cos\theta+4l_3^3c_3l_2^2\sin\theta\\
&+2\rho_{12}l_2d_3c_2^2-2\rho_{12}l_2d_3\rho_{22}-2l_2^4l_3d_3\cos\theta
\end{aligned}
\qquad \text{(A9)}
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ARCHITECTURES SPÉCIALES ET NOMBRE DE SOLUTIONS AU PROBLÈME GÉOMÉTRIQUE DIRECT DES MANIPULATEURS PARALLÈLES PLANS

Résumé—Cet article traite du problème géométrique direct des manipulateurs parallèles plans à trois degrés de liberté. Ce problème a déjà fait l'objet de travaux dans le passé et il a été montré que le problème géométrique direct de tels manipulateurs pouvait en général être ramené à la solution d'un polynôme de degré 6. De plus, pour des manipulateurs ayant une géométrie simplifiée, c'est-à-dire lorsque les liaisons rotoïdes sur la base et sur la plate-forme sont respectivement alignées, il a été conjecturé que le nombre maximum de solutions était alors réduit à 4. Ce résultat est démontré ici pour la première fois. La preuve repose sur le théorème de Sturm, qui permet de déterminer le nombre de solutions réelles d'un polynôme sur un intervalle donné par l'étude, aux bornes de l'intervalle, de polynômes obtenus par la division polynomiale de l'expression de départ et de sa dérivée. Par ailleurs, une nouvelle formulation des équations est également donnée, ce qui permet d'obtenir un polynôme de degré 6 en x, en y on en $T = \tan(\phi/2)$, selon le choix. Le polynôme obtenu en y est analysé en détail et une procédure de calcul robuste est obtenue en considérant les cas particuliers qui pourraient invalider les formulations précédentes. Cette procédure conduit à des solutions explicites robustes qui pourraient être directement utilisées pour l'analyse ou la commande d'un manipulateur. Finalement, des architectures conduisant à une simplification des équations du problème géométrique direct sont investiguées. Il est d'abord montré qu'il n'est pas possible de trouver une architecture qui annulerait le coefficient de degré 6 du polynôme. Ensuite, il est démontré que si le triangle formé par la position des liaisons rotoïdes sur la base et le triangle formé par les liaisons rotoïdes sur la plate-forme sont des triangles semblables, alors les termes de puissances impaires du polynôme s'annulent et la solution se simplifie en une cascasde d'une cubique et d'une quadratique. Des solutions explicites sont alors possibles.

Les résultats présentés dans cet article sont particulièrement intéressants pour la conception et la commande de manipulateurs parallèles plans. Ceux-ci peuvent trouver des applications dans plusieurs domaines comme la fabrication mécanique, la manipulation ou la génération de mouvements pour la simulation. Les architectures spéciales présentent l'avantage de permettre une solution explicite du problème géométrique direct et sont parfois très appropriées, spécialement si la gravité agit dans le plan de mouvement.