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OPTIMIZATION OF PLANAR AND SPHERICAL FUNCTION GENERATORS AS MINIMUM-DEFECT LINKAGES

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Abstract—In this paper, minimum-defect linkages are defined as linkages having a fully rotatable input and a maximum transmission quality. The transmission quality, introduced elsewhere, is defined as the integral, over a full revolution of the input link, of the square of the sine of the transmission angle. The *complement* of this quantity is defined here as the *transmission defect*. It is shown here that it can be expressed as a positive-definite quadratic form, which readily leads to a nonlinear least-square formulation of the associated linkage-optimization problem. Moreover, this quadratic form is shown to equal the square of the mean value plus the variance of the cosine of the transmission angle. This leads to the study of a particular class of linkages for which the mean value of the cosine of the transmission angle vanishes. Linkages pertaining to this class are defined as *zero-mean linkages*. Theorems concerning the mobility of zero-mean linkages are stated and proven. Moreover, guidelines for the design of this class of linkages are given. Transmission-defect minimization is then applied to the design of quick-return planar and spherical mechanisms.

1. INTRODUCTION

The linkage-optimization problem aimed at maximizing the quality of transmission has been given due attention by many a researcher. A crucial development in this context is the concept of *transmission index*, introduced first by Sutherland and Roth[1], which allows to extend the concept of *transmission angle* to any spatial linkage. On the other hand, Gupta[2] introduced a method of planar-linkage synthesis with an input crank, whose transmission angle is constrained to lie between 45° and 135°. This method was then extended to the *exact* synthesis of *RSSR* linkages[3]. Furthermore, Tinubu and Gupta[4] showed that a linkage optimization based on minimizing the structural error, rather than the *design error*, leads to branching-defect elimination. Moreover, the optimization of planar, spherical, and spatial linkages *having a quadratic input–output equation*, with a minimum design error and a maximum transmission quality, was presented in [5]. In this reference, the method used by the author is based on the Newton–Gauss algorithm for nonlinear least squares—see, for instance [6, 7]. On the other hand, the concept of *linkage discriminant*, first introduced in [8], and applied to the determination of the linkage mobility region, was applied to the optimization of linkages with maximum transmission quality and prescribed mobility characteristics[9].

In this paper, results concerning the transmission quality of planar and spherical linkages are derived. A particular class of linkages, called here *zero-mean linkages*, is defined and analyzed in detail. Their mobility regions are introduced as a particular case of the ones presented in [10] for general planar and spherical 4-bar linkages. Some important theorems governing their mobility characteristics are also stated and guidelines for their design are given.

A more general class of linkages, called minimum-defect linkages, is also defined and this concept is applied to the design of quick-return mechanisms using the orthogonal-decomposition method, presented in [11].

2. TRANSMISSION QUALITY

The transmission quality of a four-bar linkage, which is to be maximized, was defined in [9] as the square root of the following positive definite quantity:

$$z = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 \mu \, d\psi, \tag{1a}$$

where μ represents the *transmission angle* of the function-generating linkage under study. For brevity, the transmission quality is defined in what follows as z itself, rather than its square root. The *complement* of the transmission quality, which is to be minimized, is thus defined as:

$$z' = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 \mu \, d\psi, \tag{1b}$$

and is termed the *transmission defect*. Hence,

$$z + z' = 1, \tag{1c}$$

and

$$0 < z' < 1. \tag{1d}$$

Of course, in these definitions, the input link is assumed to be of the crank type, for the associated integrals are not defined for input links of the rocker type. In the particular cases of planar and spherical linkages, the cosine of the transmission angle can be written as follows:

$$\cos \mu = c_1 + c_2 \cos \psi, \tag{2}$$

where c_1 and c_2 are constants depending only upon the linkage parameters, expressions for which will be given presently. Thus, for planar and spherical linkages, z' becomes

$$z' = c_1^2 + \tfrac{1}{2}c_2^2, \tag{3}$$

where c_1^2 and c_2^2 are *positive semidefinite* and *positive definite* quantities, respectively, as shown in Sections 2.1 and 2.2, i.e.

$$c_1^2 \geqslant 0, \quad c_2^2 > 0. \tag{4}$$

From the foregoing discussion, it is apparent that the transmission quality is maximized if the transmission defect is minimized. The practical application of this fact is that linkages with maximum transmission quality can be found using least-square based optimization algorithms, which aim intrinsically at minimizing a positive semidefinite performance index, rather than at its maximization.

2.1. Planar linkages

A general planar linkage is shown in Fig. 1, where a_i, for $i = 1, \ldots, 4$, denote the link lengths. Next, a vector $\mathbf{k}$ of *linkage parameters* is defined as follows:

$$\mathbf{k} = [k_1, k_2, k_3]^T, \tag{5a}$$

where

$$k_1 = \frac{a_1^2 + a_2^2 - a_3^2 + a_4^2}{2a_2 a_4}, \quad k_2 = \frac{a_1}{a_2}, \quad k_3 = \frac{a_1}{a_4}, \tag{5b}$$

with the following inverse relations:

$$a_1 = 1, \quad a_2 = \frac{1}{k_2}, \quad a_3 = \frac{\sqrt{k_2^2 + k_3^2 + k_2^2 k_3^2 - 2k_1 k_2 k_3}}{|k_2 k_3|}, \quad a_4 = \frac{1}{k_3}. \tag{6}$$

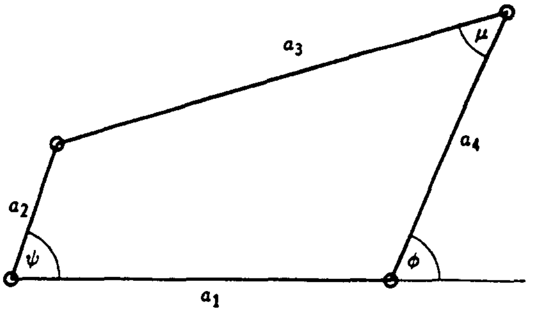

Fig. 1. Planar four-bar linkage.

The cosine of the transmission angle is given as[12]:

$$\cos\mu = \frac{a_3^2 + a_4^2 - a_1^2 - a_2^2 + 2a_1 a_2 \cos\psi}{2a_3 a_4}, \tag{7a}$$

or, in terms of **k**, as

$$\cos\mu = \frac{\operatorname{sgn}(k_2 k_2)(k_2 - k_1 k_3 + k_3^2 \cos\psi)}{\sqrt{k_2^2 + k_3^2 + k_2^2 k_3^2 - 2k_1 k_2 k_3}}. \tag{7b}$$

Constants c_1 and c_2 appearing in (2) and (3) are, then,

$$c_1 = \frac{\pm(k_2 - k_1 k_3)}{\sqrt{k_2^2 + k_3^2 + k_2^2 k_3^2 - 2k_1 k_2 k_3}}, \quad c_2 = \frac{\pm k_3^2}{\sqrt{k_2^2 + k_3^2 + k_2^2 k_3^2 - 2k_1 k_2 k_3}}, \tag{8}$$

from which it is apparent that c_1^2 positive semidefinite, whereas c_2^2 is positive definite.

2.2. Spherical linkages

A general spherical linkage is shown in Fig. 2, where α_i, for $i = 1, \ldots, 4$, denote the linkage dimensions. Again, a vector **k** of linkage parameters is defined as:

$$\mathbf{k} = [k_1, k_2, k_3, k_4], \tag{9a}$$

where

$$k_1 = \frac{\cos\alpha_1 \cos\alpha_2 \cos\alpha_4 - \cos\alpha_3}{\sin\alpha_1 \cos\alpha_2 \sin\alpha_4}, \quad k_2 = \frac{\tan\alpha_2}{\tan\alpha_4}, \quad k_3 = \frac{\tan\alpha_2}{\sin\alpha_1}, \quad k_4 = \frac{\tan\alpha_2}{\tan\alpha_1}; \tag{9b}$$

a possible inversion of which is the following:

$$\cos\alpha_1 = \frac{k_4}{k_3}, \quad \sin\alpha_1 = \frac{\sqrt{k_3^2 - k_4^2}}{|k_3|},$$

$$\cos\alpha_2 = \frac{\operatorname{sgn}(k_3)}{\sqrt{1 + k_3^2 - k_4^2}}, \quad \sin\alpha_2 = \sqrt{\frac{k_3^2 - k_4^2}{1 + k_3^2 - k_4^2}},$$

$$\cos\alpha_3 = \frac{k_2 k_4 - k_1(k_3^2 - k_4^2)}{|k_3|\sqrt{(1 + k_3^2 - k_4^2)(k_2^2 + k_3^2 - k_4^2)}},$$

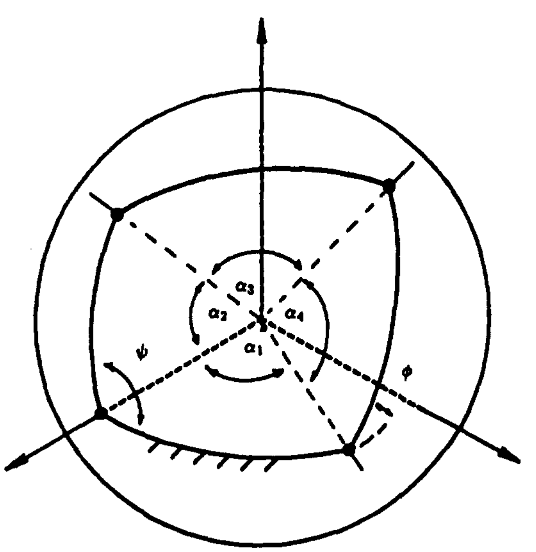

Fig. 2. Spherical four-bar linkage.

$$\sin \alpha_3 = \frac{1}{k_3}\sqrt{\frac{(k_3^2-k_4^2)[k_2^2+k_3^2(1+k_2^2)+2k_1k_2k_4+(k_3^2-k_1^2)(k_3^2-k_4^2)]}{(1+k_3^2-k_4^2)(k_2^2+k_3^2-k_4^2)}},$$

$$\cos \alpha_4 = \frac{k_2}{\sqrt{k_2^2+k_3^2-k_4^2}}, \quad \sin \alpha_4 = \sqrt{\frac{k_3^2-k_4^2}{k_2^2+k_3^2-k_4^2}}\,\mathrm{sgn}(k_3). \tag{10a}$$

This inversion requires, of course, that

$$k_4^2 \leqslant k_3^2. \tag{10b}$$

The cosine of the transmission angle, given in [13], is multiplied by factor $\sqrt{(1-\cos\alpha_3)/2}$, in order to render it compatible with the general definition of transmission index given in [14]. This produces the following:

$$\cos\mu = \sqrt{\frac{1-\cos\alpha_3}{2}}\,\frac{\cos\alpha_1\cos\alpha_2-\cos\alpha_3\cos\alpha_4+\sin\alpha_1\sin\alpha_2\cos\psi}{\sin\alpha_3\sin\alpha_4}. \tag{11}$$

Constants c_1 and c_2 of equations (2) and (3) are now defined by:

$$c_1 = \sqrt{\frac{1-\cos\alpha_3}{2}}\,\frac{\cos\alpha_1\cos\alpha_2-\cos\alpha_3\cos\alpha_4}{\sin\alpha_3\sin\alpha_4}. \tag{12a}$$

$$c_2 = \sqrt{\frac{1-\cos\alpha_3}{2}}\,\frac{\sin\alpha_1\sin\alpha_2}{\sin\alpha_3\sin\alpha_4}, \tag{12b}$$

or, in terms of vector **k**,

$$c_1 = F\gamma_1, \quad c_2 = F\gamma_2, \tag{12c}$$

where

$$F = \sqrt{\frac{|k_3|\sqrt{(1+k_3^2-k_4^2)(k_2^2+k_3^2-k_4^2)}-k_2k_4+k_1(k_3^2-k_4^2)}{2|k_3|\sqrt{(1+k_3^2-k_4^2)(k_2^2+k_3^2-k_4^2)}}}, \tag{12d}$$

and

$$\gamma_1 = \frac{k_1k_2+k_4}{\sqrt{k_2^2+k_3^2(1+k_2^2)+2k_1k_2k_4+(k_3^2-k_1^2)(k_3^2-k_4^2)}}; \tag{12e}$$

$$\gamma_2 = \frac{k_2^2+k_3^2-k_4^2}{\sqrt{k_2^2+k_3^2(1+k_2^2)+2k_1k_2k_4+(k_3^2-k_1^2)(k_3^2-k_4^2)}}. \tag{12f}$$

If none of angles α_i, for $i = 1, 2, 3, 4$, is allowed to vanish, an issue that is given due attention in Section 4.2, it is clear that c_1^2 is positive semidefinite, and c_2^2 is positive definite—the positive definiteness of c_2^2 can also be readily realized from condition (10b). From expression (12d), neither c_1 nor c_2, and not even their squares, are smooth functions of the linkage-parameter vector **k**. This would prevent us from minimizing z' using nonlinear least-square techniques, which rely on such smoothness. This is readily overcome by formulating the problem in the linkage-dimension space of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, in which, from equations (12a, b), c_1 and c_2 are smooth functions of the said linkage dimensions.

3. ZERO-MEAN LINKAGES AND THEIR PROPERTIES

Minimum-defect linkages are defined as linkages having an input crank, for which the transmission defect, as given by equation (3), is a minimum. It has been shown in Section 2 that the second term of the right-hand side of equation (3) cannot vanish, whereas the first one can. This leads to the definition of a specific class of linkages, called *zero-mean linkages*, for which the value of c_1 is equal to zero. From equation (2) it is apparent that c_1 and $\frac{1}{2}c_2^2$ are, in fact, the expected value and the variance of the cosine of the transmission angle i.e.

$$c_1 = E(\cos\mu); \tag{13a}$$

$$\tfrac{1}{2}c_2^2 = E[(\cos\mu - c_1)^2] \equiv \mathrm{Var}(\cos\mu). \tag{13b}$$

and hence the zero-mean adjective for linkages having a vanishing c_1.

3.1. Planar linkages

The zero-mean condition, in this case, leads to:

$$k_2 = k_1 k_3, \tag{14}$$

Thus, the transmission defect can be expressed as:

$$z' = \tfrac{1}{2}c_2^2 = \frac{k_3^2}{2(1 - k_1^2 + k_1^2 k_3^2)}, \tag{15}$$

and the mobility conditions for an input crank, derived in [10], reduce to the following:

$$(k_1 + k_3)^2 \leqslant (1 + k_1 k_3)^2. \tag{16a}$$

$$(k_1 - k_3)^2 \leqslant (1 - k_1 k_3)^2. \tag{16b}$$

The two foregoing inequalities can be readily reduced to a single one, namely.

$$(k_1^2 - 1)(1 - k_3^2) \leqslant 0, \tag{16c}$$

which represents the dashed region of the k_1–k_3 plane shown in Fig. 3. This region represents the domain of definition of zero-mean linkages, i.e. linkages having an input crank and for which $c_1 = 0$. Moreover, one can readily prove the following:

Theorem 1—*Zero-mean planar linkages are of the drag-link type when they correspond to points located in the inner square of their region of definition and of the crank-rocker type when they correspond to points located elsewhere within the said region.*

Proof: The conditions under which a planar four-bar linkage has a fully-rotatable output link are given in [15] as:

$$(k_1 + k_2)^2 \leqslant (1 + k_3)^2; \tag{17a}$$

$$(k_1 - k_2)^2 \leqslant (1 - k_3)^2. \tag{17b}$$

Substitution of the zero-mean condition in these expressions leads to:

$$(k_1^2 - 1)(1 + k_3)^2 \leqslant 0; \tag{18a}$$

$$(k_1^2 - 1)(1 - k_3)^2 \leqslant 0, \tag{18b}$$

which can be reduced to:

$$k_1^2 \leqslant 1. \tag{19}$$

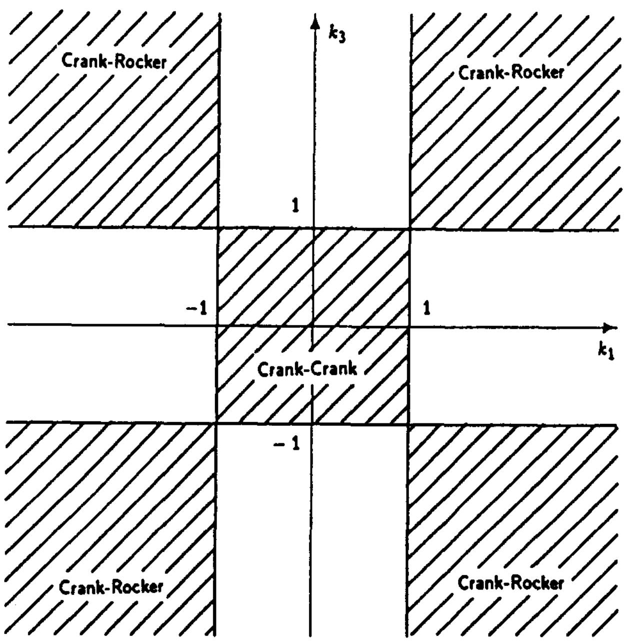

Fig. 3. Domain of definition of planar zero-mean linkages.

Therefore, only the zero-mean linkages corresponding to points located in the inner square have a fully-rotatable output link, i.e. they are of the drag-link type. The other subregions represent zero-mean linkages of the crank-rocker type since their input link is a crank but their output link is not. This completes the proof.

A function-generation problem that arises rather frequently in applications calls for *quick-return mechanisms*. In this case, one is rather interested in linkages of the crank-rocker type. The motion of such linkages is defined by the time ratio of its two phases. If the first phase takes place as the input link sweeps an angle $\pi + \Delta\psi$, whereas the second phase—the return—as the input link sweeps an angle $\pi - \Delta\psi$, the time ratio T_R is defined as:

$$T_R = \frac{\pi + \Delta\psi}{\pi - \Delta\psi}. \tag{20}$$

It was mentioned in Theorem 1 that planar zero-mean linkages can be of the crank-rocker type. The following theorem is now proven:

Theorem 2—*Planar zero-mean linkages which are of the crank-rocker type have a time ratio of one.*

Proof: Consider the two geometric constructions of Fig. 5 where a planar crank-rocker linkage is shown in its two extreme positions. Moreover, the angle $\Delta\psi$, as defined in equation (20), is given by:

$$\Delta\psi = \psi_2 - \psi_1, \tag{21a}$$

where ψ_1 and ψ_2 are assumed to be bounded as follows:

$$0 \leqslant \psi_1, \quad \psi_2 \leqslant \pi. \tag{21b}$$

Using the law of cosines, we can write

$$\cos\psi_1 = \frac{a_1^2 - a_4^2 + (a_3 + a_2)^2}{2a_1(a_3 + a_2)} \tag{22a}$$

and

$$\cos\psi_2 = \frac{a_1^2 - a_4^2 + (a_3 - a_2)^2}{2a_1(a_3 - a_2)}. \tag{22b}$$

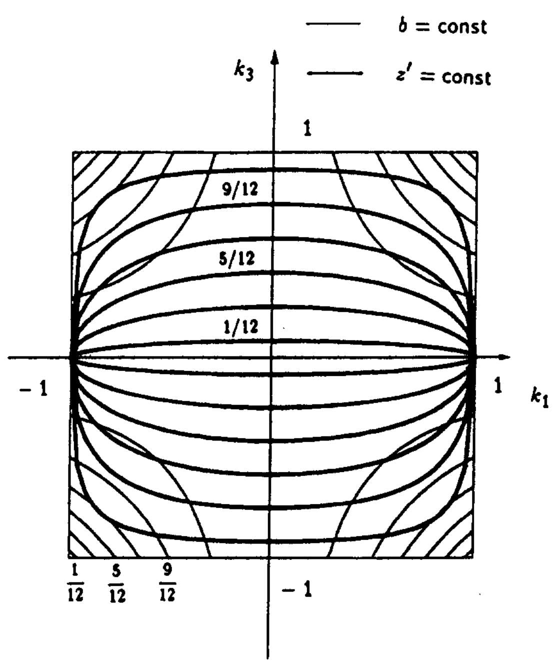

Fig. 4. Lines of constant balance and of constant transmission defect for planar zero-mean linkages.

If we now impose the zero-mean condition, i.e., if we substitute $a_1^2 + a_2^2$ by $a_3^2 + a_4^2$, we obtain

$$\cos \alpha_1 = \cos \alpha_2 = \frac{a_3}{a_1} \tag{23}$$

which, by virtue of relations (21a & b), leads to $\Delta\psi = 0$ and, from equation (20), we have $T_R = 1$ which completes the proof.

Therefore, planar zero-mean linkages of the crank-rocker type cannot be candidates for quick-return mechanisms.

Planar zero-mean linkages which are of the drag-link type can be optimized by finding the minimum transmission defect for a given minimum value of the mechanism's *dimensional balance*. This is defined as the following real number:

$$b = \left(\frac{a_4}{a_2}\right)^2 + \left(\frac{a_3}{a_2}\right)^2 - 1 \tag{24a}$$

which turns out to be positive definite, for

$$b = k_1^2 k_3^2. \tag{24b}$$

It can be readily shown that $0 < b < 1$ for zero-mean linkages of the drag-link type since for these we have $|k_1| \leqslant 1$ and $|k_3| \leqslant 1$. Lines of constant balance and of constant transmission defect are plotted in Fig. 4. The optimum drag-link mechanism, for a given minimum balance b_m, is found at the point of tangency of the contour $b = b_m$ with a contour of constant transmission defect. This point can be readily determined in closed form. Indeed, linkages with a dimensional balance b_m verify the following equation:

$$b_m - k_1^2 k_3^2 = 0, \tag{25a}$$

whereas zero-mean linkages with a constant transmission defect z_0' verify

$$(\tfrac{1}{2} - k_1^2 z_0')k_3^2 - z_0'(1 - k_1^2) = 0. \tag{25b}$$

The solution of the nonlinear system of equations obtained when minimizing z_0' in equation (25b) subjected to the constraint (25a) is the following:

$$k_3^2 = \frac{2b_m}{1 + b_m} \tag{26a}$$

and

$$k_1^2 = \frac{b_m}{k_3^2}. \tag{26b}$$

Alternatively, the foregoing solution can be determined visually with the aid of a graphics terminal and a digitizer or a joystick.

3.2. Spherical linkages

In this case, the zero-mean condition leads to:

$$k_4 = -k_1 k_2 \tag{27}$$

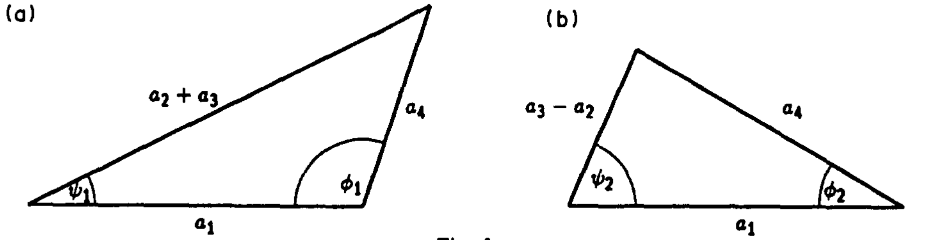

Fig. 5.

and the transmission defect can be written as:

$$z' = \tfrac{1}{2}c_2^2 = \tfrac{1}{4}AB, \tag{28a}$$

where

$$A = \frac{|k_3|\sqrt{(1 + k_3^2 - k_1^2 k_2^2)(k_2^2 + k_3^2 - k_1^2 k_2^2)} + k_1 k_2^2 + k_1(k_3^2 - k_1^2 k_2^2)}{|k_3|\sqrt{1 + k_3^2 - k_1^2 k_2^2}} \tag{28b}$$

and

$$B = \frac{(k_2^2 + k_3^2 - k_1^2 k_2^2)^{3/2}}{k_2^2 + k_3^2(1 + k_2^2) - 2k_1^2 k_2^2 + (k_3^2 - k_1^2)(k_3^2 - k_1^2 k_2^2)}. \tag{28c}$$

The mobility conditions leading to an input crank, and introduced in [10], take on the form that follows, under condition (27):

$$(k_2 - k_1)^2 \leqslant (1 - k_1 k_2)^2; \tag{29a}$$

$$(k_2 + k_1)^2 \leqslant (1 + k_1 k_2)^2, \tag{29b}$$

which are equivalent to the following single inequality:

$$(k_1^2 - 1)(1 - k_2^2) \leqslant 0. \tag{29c}$$

The region of the k_1–k_2 plane defined by the foregoing inequality is represented in Fig. 6. This is the domain of definition of spherical zero-mean linkages. One now can prove the following:

Theorem 3—*Zero-mean spherical linkages are of the crank-rocker type when they correspond to points inside the inner square of their region of definition and of the drag-link type when they correspond to points located elsewhere within the said region.*

Proof: The conditions under which the spherical linkage has a fully rotatable output link were derived in [15]. They can be written as:

$$(k_2 + k_4)^2 \geqslant (1 - k_1)^2 \tag{30a}$$

and

$$(k_2 - k_4)^2 \geqslant (1 + k_1)^2. \tag{30b}$$

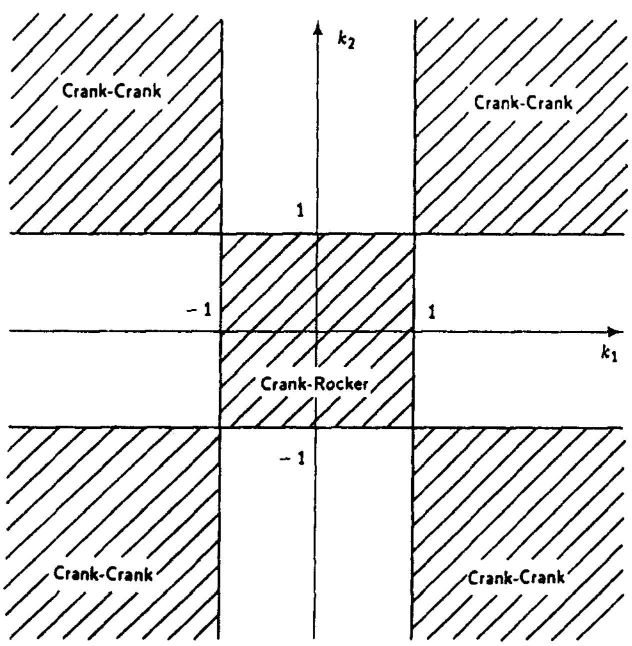

Fig. 6. Domain of definition of spherical zero-mean linkages.

Upon substitution of the zero-mean condition, we obtain

$$(k_2^2 - 1)(1 - k_1)^2 \geq 0 \tag{31a}$$

and

$$(k_2^2 - 1)(1 + k_1)^2 \geq 0, \tag{31b}$$

which can be reduced to

$$k_2^2 \geq 1. \tag{32}$$

Therefore, the linkages associated with points located in the peripheral sections of the mobility region are of the drag-link type and the ones corresponding to points inside the inner square are of the crank-rocker type. The proof is then completed.

Moreover, one has the following:

Theorem 4—*Zero-mean spherical linkages that are of the crank-rocker type have a time ratio of one.*

Proof: Consider now the two extreme configurations of the spherical linkage shown in Fig. 7. In this case, the angle $\Delta\psi$, defined in equation (20), can be expressed as:

$$\Delta\psi = \psi_1 - \psi_2 \tag{33a}$$

where ψ_1 and ψ_2 are constrained as follows:

$$0 \leq \psi_1, \psi_2 \leq \pi \tag{33b}$$

Using the law of cosines for spherical triangles, one can also write:

$$\cos\psi_1 = \frac{\cos\alpha_4 - \cos\alpha_1 \cos(\alpha_3 + \alpha_2)}{\sin\alpha_1 \sin(\alpha_3 + \alpha_2)}. \tag{34a}$$

and

$$\cos\psi_2 = \frac{\cos\alpha_4 - \cos\alpha_1 \cos(\alpha_3 - \alpha_2)}{\sin\alpha_1 \sin(\alpha_3 - \alpha_2)} \tag{34b}$$

If we now introduce the zero-mean condition, i.e., if we substitute $\cos\alpha_4$ by $(\cos\alpha_1 \cos\alpha_2/\cos\alpha_3)$, the following is derived:

$$\cos\psi_1 = \cos\psi_2 = \cot\alpha_1 \tan\alpha_3 \tag{35}$$

which, similarly to the planar case, leads to $\Delta\psi = 0$ and therefore $T_R = 1$, thereby completing the proof.

The fact that zero-mean crank-rocker linkages have a time ratio of unity disables them from being candidates for quick-return mechanisms. Hence, the optimization of quick-return mechanisms should be tackled with an alternate approach, which is done in the following section.

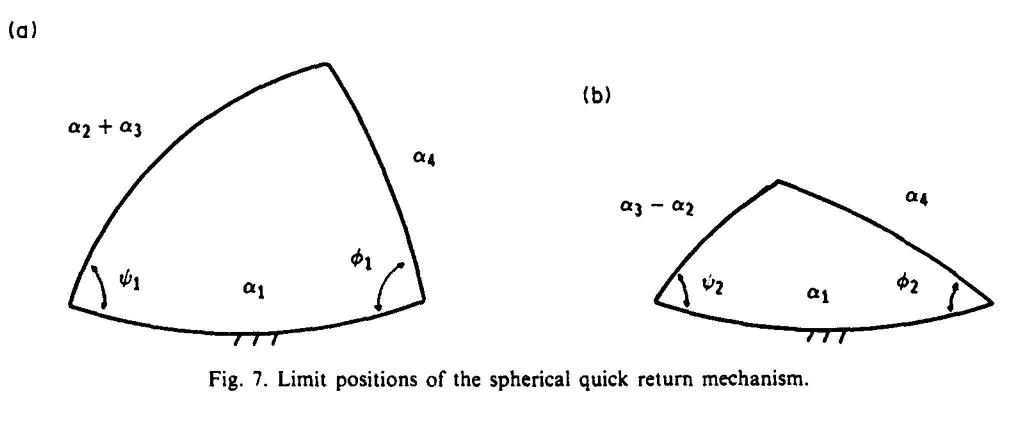

Fig. 7. Limit positions of the spherical quick return mechanism.

4. OPTIMIZATION OF QUICK-RETURN MECHANISMS AS MINIMUM-DEFECT LINKAGES

Quick-return mechanism, as defined in Section 3.1, will now be designed using the concept of minimum-defect linkages. The problem then consists of minimizing the quadratic form of equation (3) subjected to constraints on the time ratio and output swing angle of the linkage.

4.1. Planar linkages

4.1.1. Problem formulation. This is based on the approach introduced by Cleghorn and Fenton[16]. In order to set up the constraint equations of the problem at hand, the following transformations of the link lengths are introduced:

$$r_2 = \frac{a_2}{a_1}, \quad r_3 = \frac{a_3}{a_1}, \quad r_4 = \frac{a_4}{a_1} \tag{36}$$

and

$$\begin{aligned} q_1 &= r_3 - r_2; \\ q_2 &= r_3 + r_2. \end{aligned} \tag{37}$$

The two extreme positions of the output link give rise to the geometric constructions of Fig. 5. Application of the *cosines law* to these triangles gives the following constraints[16]:

$$g_1 = r_4^2 - 1 - q_2^2 + 2q_2 \cos\psi_1 = 0; \tag{38a}$$

$$g_2 = r_4^2 - 1 - q_1^2 + 2q_1 \cos\psi_2 = 0; \tag{38b}$$

$$g_3 = q_2^2 - 1 - r_4^2 + 2r_4 \cos\phi_1 = 0; \tag{38c}$$

$$g_4 = q_1^2 - 1 - r_4^2 + 2r_4 \cos\phi_2 = 0; \tag{38d}$$

$$g_5 = \psi_2 - \psi_1 - \Delta\psi = 0; \tag{38e}$$

$$g_6 = \phi_1 - \phi_2 - \Delta\phi = 0; \tag{38f}$$

or, in vector form,

$$\mathbf{g} = \mathbf{0}, \tag{38g}$$

where g_i denotes the ith component of the 6-dimensional vector $\mathbf{g}$, the time ratio being defined as

$$T_R = \frac{\pi - \Delta\psi}{\pi + \Delta\psi}. \tag{39}$$

In the foregoing discussion, $\Delta\phi$ is the prescribed output swing angle, and $\Delta\psi$ is as defined in equation (21a). Moreover, ψ_1, ψ_2, ϕ_1 and ϕ_2 are defined in Fig. 5.

The vector of design variables $\mathbf{x}$ will therefore be defined as:

$$\mathbf{x} = [r_4, q_1, q_2, \psi_1, \psi_2, \phi_1, \phi_2]^T. \tag{40}$$

The objective function to be minimized is defined as the linkage defect, i.e. as z', which can be readily expressed as the following quadratic form:

$$z' = \tfrac{1}{2}\mathbf{f}^T\mathbf{W}\mathbf{f}, \tag{41a}$$

where $\mathbf{f} = [c_1\ c_2]^T$, with c_1 and c_2 defined as follows:

$$c_1 = \frac{q_1 q_2 + r_4^2 - 1}{r_4(q_1 + q_2)}; \tag{41b}$$

$$c_2 = \frac{q_2 - q_1}{r_4(q_1 + q_2)} \tag{41c}$$

and

$$\mathbf{W} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{41d}$$

The Jacobians of $\mathbf{f}$ and $\mathbf{g}$ with respect to $\mathbf{x}$, $\mathbf{F}$ and $\mathbf{G}$, respectively, are then readily derived as the following 2×7 and 6×7 matrices:

$$\mathbf{F} = \frac{1}{r_4^2(q_1+q_2)^2}\begin{bmatrix} (q_1+q_2)(2r_4^2 - N) & [q_2 r_4(q_1+q_2) - N] & 0 & 0 & 0 & 0 & 0 \\ (q_1^2 - q_2^2) & -2r_4 q_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{42a}$$

where

$$N = q_1 q_2 + r_4^2 - 1 \tag{42b}$$

and

$$\mathbf{G} = [\mathbf{G}_1 \quad \mathbf{G}_2]; \tag{43a}$$

where

$$\mathbf{G}_1 = \begin{bmatrix} 2r_4 & 0 & -2q_2 + 2\cos\psi_1 \\ 2r_4 & -2q_1 + 2\cos\psi_2 & 0 \\ -2r_4 + 2\cos\phi_1 & 0 & 2q_2 \\ -2r_4 + 2\cos\phi_1 & 2q_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{43b}$$

and

$$\mathbf{G}_2 = \begin{bmatrix} -2q_2\sin\psi_1 & 0 & 0 & 0 \\ 0 & -2q_1\sin\psi_2 & 0 & 0 \\ 0 & 0 & -2r_4\sin\phi_1 & 0 \\ 0 & 0 & 0 & -2r_4\sin\phi_2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \tag{43c}$$

which completes the formulation of the problem. This problem was solved numerically using the orthogonal-decompositon method[11]. This method is meant to minimize an objective function which is an m-dimensional quadratic form of n variables subjected to p constraints. In this case, we have $m = 2$ with $n = 7$ variables subjected to $p = 6$ constraints.

4.1.2. Examples. Two examples are presented here. They are taken from [12] for purposes of comparison. the results obtained using the aforementioned procedure are given in Table 1 and they are in full agreement with the results reported in that reference.

Several tests performed with this formulation of the problem for the design of planar linkages show that the procedure usually converges within ten iterations.

4.2. Spherical linkages

4.2.1. Problem formulation. The formulation of this problem is similar to the one used in the planar case. The constraints are established using the extreme positions, which are shown in Fig. 7. The *cosines law* for spherical triangles is applied on these two configurations. Moreover, since the link dimensions are now angles, the design variables will be chosen as the sines and the cosines of these, rather than as the angles themselves. This will simplify the formulation and will enhance

Table 1. Optimum planar four-bar linkages

	Case 1		Case 2	
	$\Delta\phi = 40^\circ$	$\Delta\psi = -20^\circ$	$\Delta\phi = 64^\circ$	$\Delta\psi = 28^\circ$
a_1	1.3420		1.0410	
a_2	0.3230		0.4940	
a_3	0.7290		0.9360	
a_4	1.0000		1.0000	
c_1	−0.2564		0.2929	
c_2	0.5946		0.5494	
z'	0.2425		0.2367	

the numerical stability of the problem, but will require additional constraints. The global set of constraints will then be:

$$g_1 = u_4 - u_1(u_2u_3 - v_2v_3) - v_1(u_2v_3 + v_2u_3)\cos\psi_1 = 0; \tag{44a}$$

$$g_2 = u_4 - u_1(u_2u_3 + v_2v_3) - v_1(u_2v_3 - v_2u_3)\cos\psi_2 = 0; \tag{44b}$$

$$g_3 = u_2u_3 - v_2v_3 - u_1u_4 - v_1v_4\cos\phi_1 = 0; \tag{44c}$$

$$g_4 = u_2u_3 + v_2v_3 - u_1u_4 - v_1v_4\cos\phi_2 = 0; \tag{44d}$$

$$g_5 = \psi_1 - \psi_2 - \Delta\psi = 0; \tag{44e}$$

$$g_6 = \phi_1 - \phi_2 - \Delta\phi = 0; \tag{44f}$$

$$g_7 = u_1^2 + v_1^2 - 1 = 0; \tag{44g}$$

$$g_8 = u_2^2 + v_2^2 - 1 = 0; \tag{44h}$$

$$g_9 = u_3^2 + v_3^2 - 1 = 0; \tag{44i}$$

$$g_{10} = u_4^2 + v_4^2 - 1 = 0, \tag{44j}$$

where

$$u_i = \cos\alpha_i, \quad v_i = \sin\alpha_i, \quad i = 1, \ldots, 4, \tag{44k}$$

or, in vector form,

$$\mathbf{g} = \mathbf{0}, \tag{44l}$$

where g_i is the ith component of the 10-dimensional vector $\mathbf{g}$. The output swing angle is given by $\Delta\phi$ and $\Delta\psi$ is related to the time ratio by equation (39). Angles ψ_1, ψ_2, ϕ_1 and ϕ_2 are defined in Fig. 7.

The objective function to be minimized will be the transmission defect. However, tests run with the program implementing the orthogonal-decomposition algorithm showed that the procedure is deadly attracted by the degenerate case for which $\alpha_1 = \alpha_3 = \alpha_4 = 0$. One can easily verify that in this case, all the constraints are satisfied—providing $\psi_1 - \psi_2 = \Delta\psi$ and $\phi_1 - \phi_2 = \Delta\phi$—and that the objective function goes to zero. To overcome this problem, we augment the objective function with the squares of the cosines of the link angles. This will force the angles of the mechanism to be as close as possible to 90°, which will lead to dimensionally well-balanced mechanisms. The objective function then becomes:

$$z' = \tfrac{1}{2}\mathbf{f}^T\mathbf{W}\mathbf{f}, \tag{45}$$

where

$$\mathbf{f} = [\sqrt{2}c_1, c_2, \cos\alpha_1, \cos\alpha_3, \cos\alpha_3, \cos\alpha_4]^T. \tag{46a}$$

Matrix $\mathbf{W}$ allows one to introduce some weights in the quadratic form. For example, if one assigns less importance to the dimensional balance of the mechanism—and gets closer to the original problem—, then $\mathbf{W}$ can be defined as

$$\mathbf{W} = \mathrm{diag}(1, 1, w, w, w, w), \tag{46b}$$

where w is a positive quantity smaller than 1.

Notice that, in this case, the vector of design variables will be defined as:

$$\mathbf{x} = [u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4, \psi_1, \psi_2, \phi_1, \phi_2]^T. \tag{47}$$

Therefore, the Jacobians of $\mathbf{f}$ and $\mathbf{g}$ with respect to $\mathbf{x}$, denoted by $\mathbf{F}$ and $\mathbf{G}$, respectively, are written as the following 6×12 and 10×12 matrices:

$$\mathbf{F} = (\mathbf{F}_1 \quad \mathbf{F}_2 \quad \mathbf{F}_3), \tag{48a}$$

where

$$\mathbf{F}_1 = \begin{bmatrix} \frac{\sqrt{2}Qu_2}{v_3v_4} & 0 & \frac{\sqrt{2}Qu_1}{v_3v_4} & 0 & \left[\frac{-\sqrt{2}Qu_4}{v_3v_4} + \frac{\sqrt{2}(u_3u_4 - u_1u_2)}{4Qv_3v_4}\right] \\ 0 & \frac{Qv_2}{v_3v_4} & 0 & \frac{Qv_1}{v_3v_4} & \frac{-v_1v_2}{4Qv_3v_4} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \tag{48b}$$

$$\mathbf{F}_2 = \begin{bmatrix} \frac{\sqrt{2}Q(u_3u_4 - u_1u_2)}{v_3^2v_4} & \frac{-\sqrt{2}u_3Q}{v_3v_4} & \frac{\sqrt{2}Q(u_3u_4 - u_1u_2)}{v_3v_4^2} \\ \frac{-Qv_1v_2}{v_4v_3^2} & 0 & \frac{-Qv_1v_2}{v_3v_4^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{48c}$$

$$\mathbf{F}_3 = \mathbf{0}_{6\times 4}, \tag{48d}$$

in which $\mathbf{0}_{6\times 4}$ denotes the 6×4 zero matrix, and Q is defined as

$$Q = \sqrt{\frac{1-u_3}{2}}, \tag{48e}$$

with

$$\mathbf{G} = (\mathbf{G}_1 \quad \mathbf{G}_2 \quad \mathbf{G}_3), \tag{49a}$$

where

$$\mathbf{G}_1 = \begin{bmatrix} (v_2v_3 - u_2u_3) & -(u_2v_3 + v_2u_3)\cos\psi_1 & -(u_1u_3 + v_1v_3\cos\psi_1) & (u_1v_3 - v_1u_3\cos\psi_1) \\ -(u_2u_3 + v_2v_3) & (v_2u_3 - u_2v_3)\cos\psi_2 & -(u_1u_3 + v_1v_3\cos\psi_2) & (-u_1v_3 + v_1u_3\cos\psi_2) \\ -u_4 & -v_4\cos\phi_1 & u_3 & -v_3 \\ -u_4 & -v_4\cos\phi_2 & u_3 & v_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2u_1 & 2v_1 & 0 & 0 \\ 0 & 0 & 2u_2 & 2v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{49b}$$

$$
\mathbf{G}_2 = \begin{bmatrix}
-(u_1u_2 + v_1v_2\cos\psi_1) & (u_1v_2 - v_1u_2\cos\psi_1) & 1 & 0 \\
(-u_1u_2 + v_1v_2\cos\psi_2) & -(u_1v_2 + v_1u_2\cos\psi_2) & 1 & 0 \\
u_2 & -v_2 & -u_1 & -v_1\cos\phi_1 \\
u_2 & v_2 & -u_1 & -v_1\cos\phi_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2u_3 & 2v_3 & 0 & 0 \\
0 & 0 & 2u_4 & 2v_4
\end{bmatrix} \tag{49c}
$$

$$
\mathbf{G}_3 = \begin{bmatrix}
v_1(u_2v_3 + v_2u_3)\sin\psi_1 & 0 & 0 & 0 \\
0 & u_1(u_2v_3 - v_2u_3)\sin\psi_2 & 0 & 0 \\
0 & 0 & v_1v_4\sin\phi_1 & 0 \\
0 & 0 & 0 & v_1v_4\sin\phi_2 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \tag{49d}
$$

which completes the formulation of this problem. Notice that, in this case, we have $n = 12$ variables subjected to $p = 10$ constraints and that we are aiming at minimizing a performance index for which $m = 6$.

4.2.2. Examples. Three examples are presented for this case, the results appearing in Table 2. Notice that in the first two examples given for this problem, we specified the same time ratio and output swing angle. However, in the second one, we have used some weights to give less importance to the terms $\cos^2\alpha_i$ in the objective function. The optimum linkage obtained, then, has a better transmission quality, but is dimensionally less balanced.

In the case of spherical linkages, convergence usually occurs within about 15 iterations.

5. CONCLUSIONS

Minimum-defect linkages are defined in this paper as linkages having a fully rotatable input and a maximum transmission quality. This criterion is used to design a particular class of function generators, namely, quick-return mechanisms. This is accomplished using an orthogonal-decomposition method, which produces results within a few iterations, and hence appears suitable for this type of problems. Whereas closed-form solutions for optimum planar quick-return mechanisms exist, no such solutions are available for spherical and spatial linkages, which justifies the numerical

Table 2. Optimum spherical four-bar linkages

	Case 1	Case 2	Case 3
$\Delta\phi$	70°	70°	90°
$\Delta\psi$	20°	20°	30°
Weights	1.0	0.1	0.1
α_1 (deg)	104.1	97.6	80.2
α_2 (deg)	33.7	34.3	152.4
α_3 (deg)	83.4	56.0	46.9
α_4 (deg)	88.7	89.8	88.7
c_1	−0.13749	−0.06312	−0.09061
c_2	0.36078	0.31653	0.24891
z'	0.08399	0.05408	0.03919

approach introduced here. This allows, in turn, the formulation of the problem of optimization of quick-return mechanisms, whether planar or spherical, within a unified framework. Another class of linkages, called zero-mean linkages, is defined, and general results pertaining to these are derived. Finally, the results presented here should be applicable to spatial linkages, the extension to which requires further research.

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OPTIMISATION DES MÉCANISMES PLANS ET SPHÉRIQUES À DÉFAUT MINIMUM POUR LA SYNTHÈSE DE FONCTIONS

Résumé—Dans cet article, les mécanismes *à défaut minimum* sont définis comme étant les mécanismes pour lesquels le membre entraînant a une mobilité complète et ayant une qualité de transmission maximale. La qualité de transmission est à son tour définie comme étant l'intégrale, sur une rotation complète du membre entraînant, du carré du sinus de l'angle de transmission. Le complément de cette quantité, appelé *défaut de transmission* peut être exprimé sous une forme quadratique que l'on cherchera à minimiser. La moyenne et la variance du cosinus de l'angle de transmission sont alors exprimées en fonction des paramètres du mécanisme ce qui conduit à l'étude d'une classe particulière de mécanismes pour lesquels cette moyenne est nulle et que l'on nomme *mécanismes à valeur moyenne nulle*. Des théorèmes décrivant la mobilité de ces mécanismes sont démontrés et des critères utiles pour leur conception sont donnés. La minimisation du *défaut de transmission* est alors appliquée à la conception de mécanismes à retour rapide plans et sphériques.