Optimal Recommender System Design

Changhwa Lee*

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Abstract

Intermediaries such as Amazon and Google recommend products and services to consumers for which they receive compensation from the recommended sellers. Consumers will find these recommendations useful only if they are informative about the quality of the match between the sellers' offerings and the consumer's needs. The intermediary would like the consumer to purchase the product from the recommended seller, but is constrained because consumers need not follow the recommendation. I frame the intermediary's problem as a mechanism design problem in which the mechanism designer cannot directly choose the outcome, but must encourage the consumer to choose the desired outcome. I show that in the optimal mechanism, the recommended seller has the largest non-negative virtual willingness to pay adjusted for the cost of persuasion. The optimal mechanism can be implemented via a handicap auction.

I use this model to provide insights for current policy debates. First, to examine the impact of the intermediary's use of seller data, I identify types of seller data that lead to benefit or harm to the consumer and sellers. Second, I find that the optimal direct mechanism protects consumer privacy, but consumer data is leaked to sellers under other implementations. Lastly, I show that the welfare-maximizing mechanism increases the consumer surplus, but reduces the joint profit of the intermediary and sellers relative to the revenue-maximizing mechanism. An alternative interpretation of the model as a search engine is discussed.

*University of Pennsylvania. Email: changhwa@sas.upenn.edu. I am deeply indebted to Rakesh Vohra, George Mailath and Andrew Postlewaite for their guidance and support at every stage of this paper. I thank Kevin He, Aislinn Bohren, Hanming Fang, Pedro Solti, Nawaaz Khalfan, Cuimin Ba, Alice Gindin, Andelyn Russell and several seminar participants for helpful comments and suggestions.

1 Introduction

As consumers increasingly use websites and digital services for shopping, online platforms play a larger role in choosing products and services. Many platforms make personalized recommendations based on past data about consumers, providing them with greater insights into which products and services best fit their needs. For example, more than 75% of Netflix selections arise from personalized recommendations derived from past viewership and stated preference. Likewise, more than 35% of Amazon's sales result from the platform's recommendations to consumers.[1]

While some platforms like Netflix focus solely on providing the best matches for users, others monetize the recommendations by collecting payments from sellers in exchange for recommending their products and services. For instance, Amazon recommends sponsored products by displaying the products at the top of search lists. Google and Facebook recommend products by displaying targeted advertisements. A unique feature of these platforms is that sellers pay for their products to be recommended, yet the platform fully designs how to recommend and the payment structure. I call this pair, of a recommendations rule and a payment rule, a *recommender system*.

In this paper, I consider a monopolistic intermediary designing a recommender system to maximize the revenue collected from sellers. There are three types of players: a representative consumer, N representative sellers and an intermediary. The consumer may choose from one of the N products or an outside option. While the consumer does not know the match values of the products, the intermediary does and monetizes this knowledge by collecting payments from sellers in exchange for recommending their products. Sellers are willing to pay for recommendations to increase their sales. The (ex-post) willingness to pay is drawn from two sources: the seller's private information, such as profit margin, and the match values, which only the intermediary knows.

I frame the intermediary's problem as a revenue-maximizing mechanism design problem of allocating one unit of sales to one of multiple sellers, but with a constraint. Unlike a standard optimal auction (Myerson (1981)), the intermediary cannot directly choose an outcome of the mechanism, the sales, but must rely on the consumer to choose an outcome. The only way to influence the consumer's choice is by recommending products that are a good match so that the consumer will find it optimal to choose the recommended option. That is, the intermediary is constrained to persuade the consumer to choose the desired outcome.

The intermediary's objective of raising revenue from sellers and constraint of persuading

[1] *McKinsey & Company*, "How retailers can keep up with consumers," https://www.mckinsey.com/industries/retail/our-insights/how-retailers-can-keep-up-with-consumers

the consumer interact in a non-trivial way. To raise revenue from sellers, the intermediary has to persuade the consumer to purchase the product from the recommended seller and the outside option if no seller is recommended. Otherwise, the consumer ignores the recommendations, and sellers would not pay for a recommendation. Persuading the consumer to take the recommended option requires recommending a product with a high match value even if its seller does not necessarily have the highest expected willingness to pay.

The presence of the consumer's outside option is important. Without it, the constraint of persuading the consumer is trivially satisfied. In the symmetric environment where products are ex-ante identical, the consumer is indifferent among all options and follows recommendations as long as they contain some information about match values. If the intermediary runs an optimal auction (with no reserves) with sellers and recommends the product of the seller with the highest virtual willingness to pay, then such recommendations are informative because sellers' virtual willingness to pay partially depends on match values. In other words, the revenue-maximizing mechanism designed ignoring the persuasion constraint trivially satisfies the constraint.

With the outside option, the constraint of persuading the consumer bites. Suppose the intermediary first runs an optimal auction with sellers and recommends the product of the seller with the highest non-negative virtual willingness to pay (Myerson (1981)). If the consumer is nearly ex-ante indifferent between the outside option and products, the consumer follows the recommendations. However, when the consumer strongly prefers his outside option over products or vice versa, the recommendations are not informative enough about match values, so the recommendations are ignored. To make recommendations informative, the intermediary adjusts the virtual willingness to pay by match values, and recommends according to the adjusted virtual willingness to pay (Theorem 1.a and Theorem 1.b). The adjusted virtual willingness to pay is larger when the product is a good match. The precise size of the adjustment is shadow price of the persuasion constraint that I call the *cost of persuasion*.

In solving the intermediary's mechanism design problem, I reformulate the problem as a Bayesian persuasion problem in which the intermediary persuades the consumer to take the recommended option by strategically releasing information about match values as well as sellers' willingness to pay. In this Bayesian persuasion problem, the intermediary has state-dependent preferences over recommendations, the state space is multidimensional and possibly infinite, and the consumer has multiple options to choose from. These features combined make the three popular approaches - concavification (Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Dworczak and Kolotilin (2019)) - hard to apply tractably. Instead, I use a guess and verify approach by focusing on a class of recommendations rules that I call *value-switching monotone*. As it tractably characterizes the binding obedience constraints and the structure of the optimal recommendations rule, I expect this approach to be useful in similar Bayesian persuasion problems.

In the second part of this paper, I use the model to examine policy questions on the regulation of platforms. The first question is whether the intermediary should be allowed to collect and use data that reflects sellers' private information, which I call *additional information*. For example, Amazon sometimes demands receipts from third-party sellers to prove their products' authenticity. The receipts may contain sensitive information such as from where and at what prices the products are purchased, which would enable Amazon to directly purchase and sell the identical products without leaving any margin to third-party sellers.[2] Google is accused of using past bidding data to estimate bids advertisers are likely to submit.[3] Regulators have initiated a series of antitrust investigations on intermediaries' use of additional information about sellers on the basis of the potential harm to consumers and sellers.[4]

I find that the intermediary's use of additional information does not necessarily harm consumers and sellers. Additional information changes the revenue gains the intermediary makes by recommending products of sellers with higher willingness to pay, and hence, the optimal recommender system (Theorem 2.a and Theorem 2.b). In particular, additional information that decreases (increases) the revenue gains benefits (harms) consumers by making the intermediary more (less) likely to recommend products based on match values (Theorem 3 and Theorem 6). The same property provides sufficient conditions under which additional information harms sellers (Theorem 4, Corollary 1, Theorem 7 and Corollary 2).

The second question is whether consumer data is protected or leaked to sellers through the recommender system.[5] I show that the intermediary cannot earn higher revenue by sharing

[2] *U.S. House Judiciary Committee's Subcommittee on Antitrust, Commercial, and Administrative Law*, "Investigation of Competition in Digital Markets," https://judiciary.house.gov/uploadedfiles/competition_in_digital_markets.pdf?utm_campaign=4493-519

[3] *The Wall Street Journal*, "Google's Secret 'Project Bernanke' Revealed in Texas Antitrust Case," https://www.wsj.com/articles/googles-secret-project-bernanke-revealed-in-texas-antitrust-case-11618097760

[4] The European Commission has launched an antitrust investigation against Amazon (https://ec.europa.eu/commission/presscorner/detail/en/ip_20_2077). Ten states led by Texas have sued Google for anti-competitive policies in online advertisement markets, including Project Bernanke. (https://www.wsj.com/articles/states-sue-google-over-digital-ad-practices-11608146817?mod=article_inline).

[5] Some intermediaries such as Facebook allow sellers to define target audience using attributes including date of birth, gender and location before they bid. Korolova (2010) demonstrates that sellers can select attributes so that they are satisfied only by a single user, effectively revealing the target consumer's demographic information that was supposed to be private. See Korolova (2010) and Venkatadri, Andreou, Liu, Mislove, Gummadi, Loiseau, and Goga (2018) for more details. This has sparked concerns about consumer data leakage through targeted advertisements, and served as one of the motivations for data protection

consumer data with sellers. In the optimal direct mechanism, the intermediary can always extract the benefit sellers would have from receiving consumer data by providing the sellers with better matches and charging more (Theorem 8). Consumer data is protected in that sellers do not learn about consumers' match values until the auction ends. However, data leakage is a feature of some indirect mechanims that implement the optimal recommender system (Theorem 9 and Theorem 10).

Lastly, I show that the welfare-maximizing mechanism increases consumer surplus but reduces the joint profit of the intermediary and sellers relative to the revenue-maximizing mechanism when the consumer's outside option is so undesirable that he always prefers products over the outside option (Theorem 11). Under the welfare maximization regime, the welfare gains by recommending products of sellers with higher willingness to pay is lower, and that by recommending better-matched products is higher, relative to the revenue gains under the revenue maximization regime. This change in gains leads the social planner to recommend products based on match values more often, increasing consumer surplus and decreasing the joint profit.

The remainder of this paper proceeds as follows. The next subsection discusses related literature. Section 2 provides an example to demonstrate the key properties of the revenue-maximizing recommender system. Section 3 describes the model. Section 4 characterizes the revenue-maximizing recommender system. Section 5 characterizes how additional information changes the optimal recommender system and the payoffs of the consumer, sellers and intermediary. Section 6 explores whether consumer data is protected or leaked to sellers through the recommender system. Section 7 discusses several extensions and relaxation of assumptions, including an alternative interpretation of the model as a search engine. Section 8 concludes. All proofs are collected in the Appendix.

1.1 Related Literature

Sales of Information

This paper contributes to the emerging literature on the sale of information by a monopolistic information seller. Starting with Admati and Pfleiderer (1986, 1990), several papers focus on how to sell information to an information buyer who directly receives the information to make better decisions. Recent works study a monopolistic information seller selling experiments to a decision maker who has private information about the states of the world (Bergemann, Bonatti, and Smolin (2018)), statistics to a decision maker who has private information about what kinds of information it needs (Segura-Rodriguez (2021)) and consumer segments

regulations such as the European Union's General Data Protection Regulation (https://gdpr.eu/).

to a producer who uses it to better price-discriminate (Yang (2021)).

In contrast, in this paper, the information buyers are product sellers, and the information seller is an intermediary. Product sellers pay the intermediary in order to influence the information provided to the consumer, instead of directly receiving information. The closest to my paper is Yang (2019), which studies an intermediary who designs a recommendations rule, a transfer rule, and a pricing rule over a single product and seller. Instead, I study an intermediary who designs a recommendations rule and a transfer rule over multiple products with exogenously given prices. The consumer benefits from recommendations because the intermediary can better distinguish between ex-ante identical products. This source of consumer surplus plays a crucial role in analyzing the impact of additional information on the consumer surplus. Inderst and Ottaviani (2012), Mitchell (2021), and Aridor and Gonçalves (2021) also analyze problems of information buyers paying to influence information others receive, but the information buyers are non-strategic or do not have private information.

Regulation of Platforms

This paper is closely related to a series of papers on the use of data by platforms and their regulation. de Cornière and Taylor (2019), Hagiu, Teh, and Wright (2020) and Aridor and Gonçalves (2021) study how an intermediary uses consumer data to promote its own product when it competes with a third-party seller on prices and qualities. Madsen and Vellodi (2021) studies how the intermediary uses seller data to launch its own private-label product. Fang and Kim (2021) examines how the intermediary shares consumer data with a third-party seller when the intermediary's private-label product competes with the seller's. Hagiu and Wright (2015), Hagiu and Wright (2019) and Kang and Muir (2021) focus on how different market structures, instead of platforms' use of data, affect outcomes. While the prior literature studies how platforms and sellers interact through the downstream market competition, I focus on how platforms use data to give informative recommendations and how their regulations change the recommendations and players' welfare.

Mechanism Design

This paper combines mechanism design with Bayesian persuasion. The intermediary solves a revenue-maximizing mechanism design problem, but with a constraint that it has to persuade the consumer to take the recommended options. If the intermediary could force the consumer to take the recommended options, then the intermediary's problem reduces to a standard revenue-maximizing auction design problem (Myerson (1981)).

There are several papers that study mechanism design problems in which the mechanism designer cannot fully control the outcome. In Myerson (1982), agents choose outcomes after communicating with the mechanism designer, and it is without loss of generality to restrict the mechanism designer's attention to honest and obedient mechanisms. Myerson (1983) studies an incentive compatible communications mechanism in a Bayesian game where the outcome relies on agents' private information. Dworczak (2020) studies a problem of allocating an object to one of several agents that is followed by a black-box aftermarket that the mechanism designer cannot control. By contrast, in this paper, the intermediary (i) has private information, (ii) elicits information from one party (sellers) and recommends outcomes to the other party (the consumer), and (iii) directly interacts with the consumer who solely chooses the outcome and does not care about sellers' private information per se.

Bayesian Persuasion

This paper contributes to the Bayesian persuasion literature (Rayo and Segal (2010), Kamenica and Gentzkow (2011), Bergemann and Morris (2019)). In this literature, the persuader's preference is often given exogenously and is simplified to be independent of the states (Gentzkow and Kamenica (2016)), depend only on the posterior mean (Dworczak and Martini (2019)) or semi uppercontinuous in beliefs (Dworczak and Kolotilin (2019), Dizdar and Kováč (2020)). The state space or action space are often simplified to be finite (Kamenica and Gentzkow (2011)) or even binary (Rayo and Segal (2010), Alonso and Câmara (2016), Kolotilin (2018), Aridor and Gonçalves (2021)). Without these assumptions, the three popular tools in Bayesian persuasion are not always tractable: concavification (Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Kolotilin (2018), Galperti and Perego (2018), Dworczak and Kolotilin (2019), Dworczak and Martini (2019), Dizdar and Kováč (2020)). In this paper, I demonstrate that even without the above assumptions, a Bayesian persuasion problem can still be tractably analyzed by focusing on value-switching monotone recommendations rules.

Online Targeted Advertisements

Intermediaries often sell online targeted advertisements by auctioning positions of products in search results. Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) study the generalized second-price position auction and find that it has a unique perfect Bayesian equilibrium that is outcome equivalent to that under Vickery-Clark-Groves mechanism (Vickrey (1961), Clarke (1971), Groves (1973)). Positions of products in their models, however, do not convey any information about match values for the consumer. Athey and Ellison (2011) studies a position auction under which sellers that are a better match for the consumer make higher bids, so that the higher positions convey higher match values, and emphasizes the informational role of search engines. Complementary to these papers, I focus on the intermediary's role as an information provider and allow the intermediary to use a fully flexible set of mechanisms instead of the specific protocols of position auctions.

2 Example

Consider a situation where a potential consumer searches *bug spray* on Amazon in a world where there are only two bug spray products on Amazon, *chemical* and *natural*. Amazon has one spot for a *sponsored product* that appears at the top of the search list.

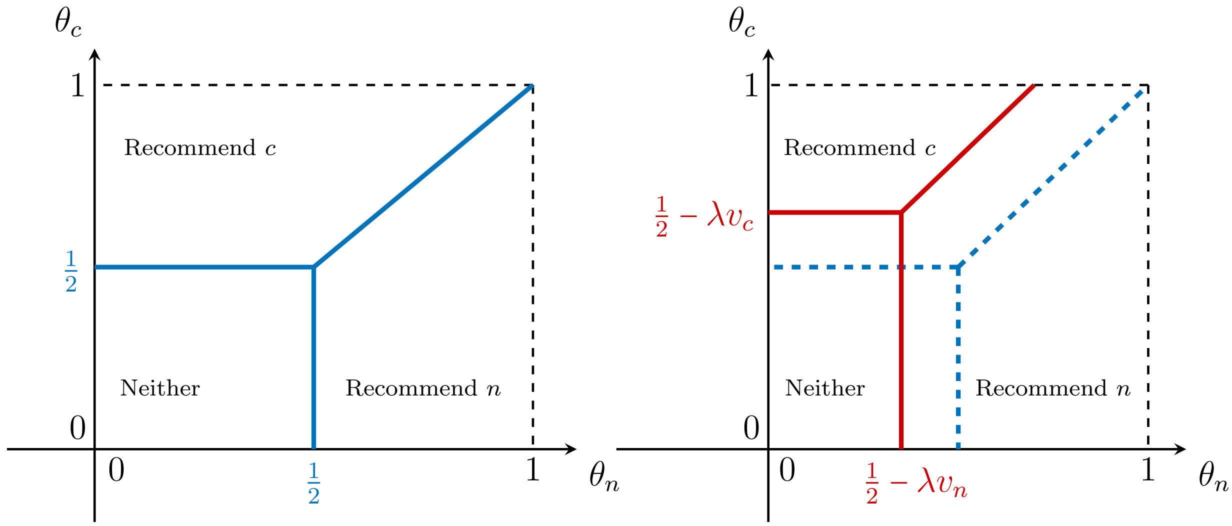
The consumer wants to purchase a bug spray only if the product is good match for him, but without recommendations, thinks that both are unlikely to be a good match. Amazon, on the other hand, has better information about whether each product would be a good match for the consumer.[6] Formally, each product may be a good match with a probability $0 < q < \frac{1}{2}$ or a bad match with a probability $1 - q$. If the consumer chooses a product, he gets $v > 0$ if it is a good match and $-v < 0$ if a bad match. If the consumer chooses neither product, then the consumer gets 0. In the absence of additional information, the consumer does not purchase any of the products. Amazon, on the other hand, privately observes the match value $v_i \in \{v, -v\}$ for each product $i \in \{c, n\}$ where c stands for *chemical* and n for *natural*.

Each seller $i \in \{c, n\}$ makes a marginal profit θ_i whenever the consumer purchases seller i's product. The marginal profit θ_i is each seller's private information and is drawn from a uniform distribution over $[0, 1]$ independently of the other seller's marginal profit as well as the consumer's match values. Sellers are risk-neutral - they try to maximize their expected profits.

How should Amazon choose a sponsored product to maximize the revenue it can raise from sellers? A sensible guess is to run a second-price auction with sellers and recommend the winner's product with a reserve $\frac{1}{2}$. If sellers bid their marginal profits, the resulting recommendations rule is depicted in Figure 1a. The problem, however, is that the consumer does not purchase the recommended product because there is no information about match values in the sponsored products. For example, when the consumer sees *natural* as the sponsored product, the only information that the consumer learns is that the seller of *natural* has paid more money to the intermediary. Hence, the consumer ignores the sponsored products, and sellers do not participate in the auction for sponsored products.

One way to make recommendations informative about match values is to give *discounts* to sellers based on how well their products match consumer's needs. Consider a variant of the second-price auction where the highest bidder wins the auction but is required to pay the second highest bid discounted by λv, where $\lambda > 0$. For example, if the seller *natural* bids

[6] For example, Amazon can infer how much the consumer will be satisfied with each product by looking up other consumers who have similar purchase histories as this particular consumer and seeing how much they are satisfied with each product.



(a) Second-price auction.

(b) Second-price auction with discounts.

Figure 1: Two auction rules for the sponsored product recommendations when $v_n > v_c$.

b_n and *chemical* bids b_c with $b_n > b_c$, then the seller *natural* wins, but is required to pay

$$\max\left(b_c, \frac{1}{2}\right) - \lambda v_n.$$

The parameter λ governs how informative the sponsored products are about match values. The larger λ is, the greater the discount is for products that are a better match, and the more likely the sponsored product is a good match for the consumer. This encourages sellers of better products to bid higher and thus win more often. Furthermore, the discount is negative if the product is a bad match. The intermediary charges additional money when sellers of poorly matched products win the auction in order to discourage them from winning. Figure 1b depicts the resulting recommendations rule when sellers bid according to $b_i = \theta_i + \lambda v_i$ and when *natural* is a good match but *chemical* is a bad match for the consumer.

When sellers bid below the reserve, the intermediary needs to induce the consumer not to purchase any of the products. This is achieved when λ is large enough by not displaying any sponsored products. Because discounts imply that products are recommended less often when they are a worse match, when the consumer sees *no sponsored products* on his search list, the consumer understands that this is partially because the sellers did not bid high enough, but also because the products are not a good match. In the specific case in which the consumer does not buy products without recommendations, $q < \frac{1}{2}$, any $\lambda \geq 0$ successfully persuades the consumer not to purchase. When the consumer would have purchased products even without recommendations, $q \geq \frac{1}{2}$, a sufficiently high λ would persuade.

When $\lambda = 0$, displaying a sponsored product does not update the consumer's belief on

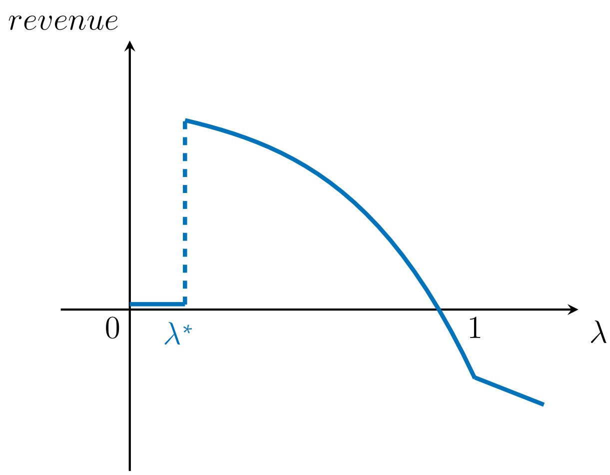


Figure 2: Amazon's revenue under the second price auction with discounts as a function of λ. When $\lambda < \lambda^*$, the consumer ignores recommendations and the sellers do not pay for recommendations. Under the optimal auction, the intermediary provides information just enough to induce the consumer to purchase recommneded products.

match values, and the auction reduces to a standard second-price auction. When λ is positive but very small, displaying a sponsored product updates the consumer's belief about the product's match value positively, but not enough to convince him to purchase the product. There is a lowest number $\lambda^* > 0$ at which the informativeness of the sponsored product is just enough so that the consumer is indifferent between taking the sponsored product and not purchasing any of products.[7]. For any $\lambda \geq \lambda^*$, the consumer purchases the sponsored product, sellers are willing to pay for sponsorship and Amazon raises positive revenue from sellers. The informativeness that maximizes Amazon's revenue is precisely λ^* that leaves the consumer indifferent between purchasing and not purchasing the sponsored product. Figure 2 depicts Amazon's revenue as a function of λ.

Note that even the lowest type seller $\theta = 0$ can win the auction if his product is a good match, and the seller receives a positive profit. Thus, Amazon can further raise its revenue by collecting a *participation fee* P^* amounting to the expected profit of the lowest type $\theta = 0$ from each seller, and still induce all sellers to participate to the auction. As it will be shown in Theorem 10, the second-price auction with *discounts* $\lambda^* v$ and *participation fees* P^* as above is a revenue-maximizing mechanism in this particular setup.

[7]As it will be shown in Section 4, at such λ^*, the consumer prefers the displayed sponsored product over the other non-sponsored product.

3 Model

3.1 Setup

There is a consumer (he), N sellers (she) and an intermediary (it). Each seller sells one product. Each product $i \in \{1, \cdots, N\} = \mathcal{N}$ has a *match value* $v_i \in \mathbb{R}$ that is independently drawn from a common distribution F that has a bounded support $\mathcal{V}$ with $-\infty < \inf \mathcal{V} = \underline{v} \leq \bar{v} = \sup \mathcal{V} < \infty$. Only the intermediary knows the match values $\boldsymbol{v} = (v_1, \cdots, v_N) \in \boldsymbol{\mathcal{V}} = \mathcal{V}^N$ of the products; the consumer and sellers do not.

The consumer may choose from one of N products or his outside option. If the seller i's product is purchased, the consumer receives utility v_i. If the consumer does not purchase any of the products and chooses the outside option, the consumer receives utility v_0, a value commonly known to all players. The consumer's expected payoff of choosing $i \in \mathcal{N} \cup \{0\}$ with probability r_i when match values are $\boldsymbol{v}$ is

$$\sum_{i \in \mathcal{N} \cup \{0\}} v_i r_i.$$

Each seller $i \in \mathcal{N}$ has an ex-post profit

$$(\theta_i + w(v_i))r_i - t_i$$

where $\theta_i + w(v_i)$ is the seller i's (ex-post) *willingness to pay*, r_i is the probability of the consumer purchasing the product i and t_i is a transfer that the seller i pays to the intermediary. The willingness to pay consists of two parts.

The first part θ_i is *private willingness to pay*. This is derived from the seller- or product-specific information that seller i privately knows such as its marginal cost,[8] and is independently drawn from a common distribution G that has a support $[\underline{\theta}, \bar{\theta}] = \Theta$ where $-\infty < \underline{\theta} < \bar{\theta} < \infty$. The distribution G has a probability distribution function g and satisfies Myerson's regularity, that is, $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}$ is strictly increasing.

The second part $w(v_i)$ is *value-dependent willingness to pay*, which is a part of the seller's profit that is increasing in the match value v_i. This reflects that the sellers prefer consumers who are a better match for their products. For example, consumers who are a better match would be more likely to repurchase the product, which increases the sellers' willingness to pay for a recommendation.[9] The value-dependent willingness to pay is a reduced form way

[8] For example, if each product's price p_i is public knowledge and marginal cost c_i is each seller's private knowledge, then private willingness to pay is $\theta_i = p_i - c_i$.

[9] Consumers who are a better match would be less likely to return products, which leads to higher profits and hence willingness to pay. Similarly, better matched consumers are more likely to purchase products after

to capture such interactions between the consumer and sellers.

3.2 Recommender Systems

The intermediary knows $\boldsymbol{v}$, but does not know $\boldsymbol{\theta}$. Before learning $\boldsymbol{v}$, the intermediary designs and commits to a recommendations rule $\boldsymbol{r}$ and a transfer $\boldsymbol{t}$. I call this pair a *recommender system*. Formally, a recommender system is

$$(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$$

such that $\sum_{i \in \mathcal{N} \cup \{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. The recommender system specifies with what probability to recommend option i, $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ and how much each seller i pays the intermediary, $t_i(\boldsymbol{v}, \boldsymbol{\theta})$, when the intermediary observes $\boldsymbol{v}$ and sellers report as $\boldsymbol{\theta}$.

Given a recommender system $(\boldsymbol{r}, \boldsymbol{t})$, when recommended with $i \in \mathcal{N} \cup \{0\}$, the consumer updates his beliefs on the expected value of each option $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_j \mid r_i(\boldsymbol{v}, \boldsymbol{\theta}) > 0)$ and chooses the option with the highest expected value. The constraint for the consumer to optimally take the recommended option i over another option j is called an *obedience constraint from i to j*, which formally is written as

$$OB_{ij} : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_j r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}). \tag{1}$$

Note that OB_{ij} is trivially satisfied if the intermediary does not recommend i almost surely. The recommender system $(\boldsymbol{r}, \boldsymbol{t})$ is *obedient* if all OB_{ij} are satisfied for all $i, j \in \mathcal{N} \cup \{0\}$. Since the transfer $\boldsymbol{t}$ is irrelevant for obedience, I interchangeably use the obedience of a recommender system and of the corresponding recommendations rule $\boldsymbol{r}$. Define

$$\Pi_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \Big((\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}) \Big) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i})$$

to be the expected profit of seller i when her private willingness to pay is θ_i, and

$$Q_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i})$$

to be the probability of recommending seller i's product. The recommender system is incentive compatible if for all $i \in \mathcal{N}$ and $\theta_i, \theta_i' \in \Theta$

$$IC_i : \Pi_i(\theta_i) \geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \Big[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}) - t_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}) \Big] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}), \tag{2}$$

clicking the advertisements.

and individually rational if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta$

$$IR_i : \Pi_i(\theta_i) \geq 0. \tag{3}$$

By the revelation principle from mechanism design (Myerson (1981)) and information design (Bergemann and Morris (2019)),[10] the intermediary can restrict attention to obedient, incentive compatible and individually rational recommender systems without loss of generality. Using such recommender systems, the intermediary maximizes the expected revenue

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} t_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}).$$

A defining feature of my model is that the intermediary is solving a revenue-maximizing mechanism design problem, but with a constraint. Instead of the intermediary choosing the outcome, i.e. which option to choose, the consumer chooses the best outcome for himself given the information provided by the intermediary. The intermediary designs recommendations informative so that the consumer chooses the outcome that intermediary wants him to choose. If the intermediary were able to choose the outcome by itself, then its problem is a standard optimal auction design problem (Myerson (1981)).

Timing of the Game

1. Intermediary offers and commits to a recommender system $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ where $\sum_{i \in \{0\} \cup \mathcal{N}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.
2. Sellers report their private information $\boldsymbol{\theta}$.
3. Intermediary observes the consumer's match values $\boldsymbol{v}$.
4. Intermediary recommends an action and collects transfers according to $(\boldsymbol{r}(\boldsymbol{v}, \boldsymbol{\theta}), \boldsymbol{t}(\boldsymbol{v}, \boldsymbol{\theta}))$.
5. Consumer gets a recommendation and takes an action.

[10] The intermediary, in principle, may attempt to provide information in more flexible ways than recommendations. That is, the intermediary can design and commit to a pair of an information structure $(\sigma, \mathcal{S})$ where

$$\sigma : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to \Delta \mathcal{S}.$$

and a transfer $\boldsymbol{t} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to \mathbb{R}^N$, instead of a recommender system. By revelation principle, an outcome of such an indirect mechanism can always be represented as an outcome of an obedient, incentive compatible and individually rational recommender system, so the intermediary can restrict attention to such recommender systems without loss of generality.

4 Optimal Recommender System

In this section, I characterize the optimal recommender system using a class of recommendations that I call value-switching monotone.

4.1 Intermediary as a Bayesian Persuader

Notice that the transfer $\boldsymbol{t}$ is irrelevant for the consumer's obedience constraints, so that the standard characterization of incentive compatible and individually rational recommender system (Myerson (1981)) applies for any obedient recommendations rule $\boldsymbol{r}$.

Lemma 1. *An obedient* $(\boldsymbol{r}, \boldsymbol{t})$ *recommender system is incentive compatible and individually rational if and only if for all* $i \in \mathcal{N}$ *and* $\theta_i \in \Theta$,

$$Q_i(\theta_i) \text{ is increasing in } \theta_i, \tag{4}$$

$$\Pi_i(\theta) = \Pi_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i, \tag{5}$$

$$\Pi_i(\underline{\theta}) \geq 0. \tag{6}$$

The standard arguments of substituting the expected revenue with virtual willingness to pay $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$, dropping incentive compatibility and individual rationality, and setting the lowest type's expected profit to zero apply as well.

Lemma 2. *Suppose that a recommendations rule* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^N$ *maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \tag{7}$$

subject to obedience constraints (1) *and monotonicity constraints* (4)*. Suppose also that*

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}) = (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}) d\tilde{\theta}_i. \tag{8}$$

Then, $(\boldsymbol{r}, \boldsymbol{t})$ *is an optimal recommender system.*

Ignoring the monotonicity constraints, Lemma 2 recasts the intermediary's revenue maximization problem as a Bayesian persuasion problem that only uses a recommendations rule $\boldsymbol{r}$. In this Bayesian persuasion problem, the intermediary persuades the consumer to take recommended option by strategically releasing information about $(\boldsymbol{v}, \boldsymbol{\theta})$. The problem has the following features: The intermediary's state-dependent preference over recommendations is given by its virtual willingness to pay; the consumer can choose from $N+1$ options; the state space problem is multi-dimensional and possibly infinite.

Each feature of the problem brings a difficulty in applying the three popular approaches in Bayesian persuasion literature: concavification (Aumann and Maschler (1995), Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Kolotilin (2018), Galperti and Perego (2018), Dworczak and Kolotilin (2019), Dworczak and Martini (2019)).[11] Whenever departing away from the three approaches, the immediate challenge lies in identifying which of the obedience constraints bind and not bind at the optimal recommendations rule. With $N+1$ options, there are $\frac{N(N+1)}{2}$ obedience constraints to check, a seemingly daunting task. I overcome this challenge by applying the guess and verify method using value-switching monotone recommendations rules.

4.2 Value-Switching Monotone Recommendations Rule

Definition 1. A recommendations rule $\boldsymbol{r}$ is *value-switching monotone* if

1. $r_0(\boldsymbol{v}, \boldsymbol{\theta})$ decreases in (v_i, θ_i) for all $i \in \mathcal{N}$.

2. $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ increases in (v_i, θ_i) for all $i \in \mathcal{N}$.

3. $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ decreases whenever v_j is switched with a larger v_i for all $i, j \in \mathcal{N}$, i.e. for all $i, j \in \mathcal{N}$, $(\boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}}_{-ij} \times \boldsymbol{\Theta}$ and $v > v'$,

$$r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \geq r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}).$$

Value-switching monotonicity is weaker than monotonicity which would replace the third condition with $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ decreasing in v_j for all $j \in \mathcal{N} \setminus \{i\}$. The optimal recommendations rule in Theorem 1.a is value-switching monotone, but not monotone. Value-switching monotonicity requires $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ to be increasing in θ_i to ensure the monotonicity constraints (4) satisfied, but does not require any particular behavior with $\boldsymbol{\theta}_{-i}$.

The following lemma states that the intermediary can ignore the obedience constraints between products as long as the intermediary uses a value-switching monotone recommendations rule.

[11] Concavification has limited applicability when state space is large (Gentzkow and Kamenica (2016)). Convex function characterization necessarily assumes the sender's payoff to depend only on the expected value of the states (Gentzkow and Kamenica (2016)). Duality approach often assumes state space to be either an interval or discrete (Kolotilin (2018), Galperti and Perego (2018)), and the sender's payoff to be semi upper-continuous in beliefs (Dworczak and Kolotilin (2019), Dworczak and Martini (2019), Dizdar and Kováč (2020)), which are not necessarily satisfied in this environment.

Lemma 3. *Any value-switching monotone recommendations rule* $\boldsymbol{r}$ *satisfies obedience constraints between products, i.e.* OB_{ij} *for all* $i, j \in \mathcal{N}$.

Lemma 3 reduces the number of possibly binding constraints to check from $\frac{N(N+1)}{2}$ to $2N$. The remaining obedience constraints are one of the two types of obedience constraints: obedience constraints from outside option to products,

$$OB_{0i} : \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) r_0(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0, \tag{9}$$

and those from products to outside option.

$$OB_{i0} : \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_i - v_0) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{10}$$

The following lemma states that whether the remaining obedience constraints are satisfied for a given value-switching recommendations rule depends on two thresholds.

Lemma 4. *For any value-switching monotone recommendations rule* $\boldsymbol{r}$, *for each* $i \in \mathcal{N}$, *there are* $-\infty \leq \underline{v}_i \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}_i \leq \infty$ *such that*

1. OB_{i0} *is satisfied if and only if* $v_0 \leq \bar{v}_i$,

2. OB_{0i} *is satisfied if and only if* $v_0 \geq \underline{v}_i$,

where

$$\bar{v}_i = \begin{cases} \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))} & \text{if } \Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}) > 0) > 0 \\ \infty & \text{if } \Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}) > 0) = 0 \end{cases}$$

and

$$\underline{v}_i = \begin{cases} \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_0(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta}))} & \text{if } \Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta}) > 0) > 0 \\ -\infty & \text{if } \Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta}) > 0) = 0 \end{cases}.$$

The first part of Lemma 4 states that OB_{i0} is satisfied if and only if the outside option value is below the threshold $\bar{v}_i$. A lower outside option value provides less incentive for the consumer to take the outside option over the recommended product i, and hence, is easier to satisfy OB_{i0}. If the intermediary recommends i with positive probability, then OB_{i0} is equivalent to

$$v_0 \leq \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i(\boldsymbol{v},\boldsymbol{\theta})) = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))},$$

which implies the threshold $\bar{v}_i = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))}$. If the intermediary does not recommend i almost surely, the intermediary does not worry about keeping the consumer incentivized to purchase the recommended product i over the outside option regardless of the

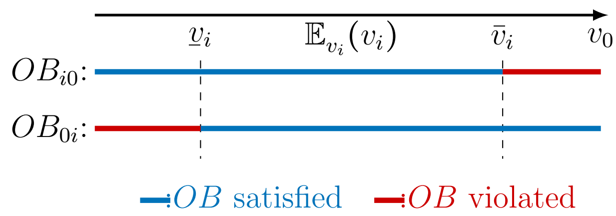


Figure 3: A graphical illustration of regions on which OB_{i0} and OB_{0i} are satisfied and violated.

outside option value. In other words, OB_{i0} is trivially satisfied for all v_0, which implies the threshold $\bar{v}_i = \infty$. The second part about the other threshold $\underline{v}_i$ for OB_{0i} may be explained in a similar manner.

Lemma 4 also states that a product's ex-ante expected value $\mathbb{E}_{v_i}(v_i)$ has to be in-between the two thresholds, that is, $\underline{v}_i \le \mathbb{E}_{v_i}(v_i) \le \bar{v}_i$. To see why this has to be the case, consider $v_0 = \mathbb{E}_{v_i}(v_i)$. The consumer is ex-ante indifferent between all products and the outside option, so that he follows any recommendations as long as recommendations contain some (or no) information about match values, i.e. $Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta})) \ge 0$ and $Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_0(\boldsymbol{v},\boldsymbol{\theta})) \le 0$. The value-switching monotonicity requires $r_i(\boldsymbol{v},\boldsymbol{\theta})$ to be increasing and $r_0(\boldsymbol{v},\boldsymbol{\theta})$ to be decreasing in v_i, which ensures that recommendations are informative about the match values. Therefore, when $v_0 = \mathbb{E}_{v_i}(v_i)$, any value-switching monotone recommendations rule satisfies all obedience constraints. All obedience constraints continue to hold as long as v_0 is close enough to $\mathbb{E}_{v_i}(v_i)$, i.e. $v_0 \in [\underline{v}_i, \bar{v}_i]$. See Figure 3 for a graphical illustration of Lemma 4.

In particular, if the intermediary runs an optimal auction with sellers ignoring the obedience constraints, the resulting recommendations rule is value-switching monotone and satisfies all obedience constraints as long as v_0 is close enough to $\mathbb{E}_{v_i}(v_i)$. Let $\boldsymbol{\rho}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ denote the resulting recommendations rule that I call by the *unconstrained optimal recommendations rule* and is given by

$$\rho_i^*(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \mathcal{M} \text{ and } \theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) \ge 0 \\ 0 & \text{otherwise} \end{cases}, \tag{11}$$

where $\mathcal{M} = \{i \in \mathcal{M} \mid \arg\max_{j \in \mathcal{N}} \{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\}\}$. That is, $\boldsymbol{\rho}^*$ is characterized by recommending the product of the seller with the highest non-negative virtual willingness to pay, and the outside option if all sellers' virtual willingness to pay is negative (Myerson (1981)). Note that $\boldsymbol{\rho}^*$ is symmetric,[12] so that the thresholds $\bar{v}_i$ and $\underline{v}_i$ are identical across

[12] A recommendations rule $\boldsymbol{r}$ is symmetric if for any $i \in \mathcal{N}$, any bijective function $\iota : \mathcal{N} \to \mathcal{N}$ and any $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = r_{\iota(i)}(\boldsymbol{v}^\iota, \boldsymbol{\theta}^\iota)$$

all products $i \in \mathcal{N}$. Let $\bar{v}^*$ and $\underline{v}^*$ be the respective common thresholds.

4.3 Optimal Recommender System

The intermediary's problem is linear in $\boldsymbol{r}$, so that the method of Lagrangean is both necessary and sufficient for an optimal solution. The following theorem characterizes an optimal recommender system when $v_0 \in (\underline{v}, \bar{v})$.

Theorem 1.a. *Let $v_0 \in (\underline{v}, \bar{v})$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for each $i \in \mathcal{N}$,*

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \textit{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1 - G(\theta_j)}{g(\theta_j)} + w(v_j)}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^*(\boldsymbol{v})}_{\substack{\textit{cost of}\\ \textit{persuasion}}}, 0 \right\} \\ 0 & \textit{otherwise} \end{cases} \tag{12}$$

where $\mathcal{M} = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v}) \right\}$, and

$$\ell_i^*(\boldsymbol{v}) = \begin{cases} 0 & \textit{if } v_0 \in [\underline{v}^*, \bar{v}^*] \\ \lambda_1^*(v_0) \cdot (v_0 - v_i) & \textit{if } v_0 > \bar{v}^* \\ \lambda_2^*(v_0) \cdot \sum_{k \in \mathcal{N}} (v_0 - v_k) & \textit{if } v_0 < \underline{v}^* \end{cases} \tag{13}$$

where $\lambda_1^*(v_0)$ and $\lambda_2^*(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively. Let $\boldsymbol{t}$ be as in (8). Then, $\boldsymbol{r}^*$ is value-switching monotone, and $(\boldsymbol{r}^*, \boldsymbol{t})$ is an optimal recommender system.

The optimal recommendations rule $\boldsymbol{r}^*$ is characterized by recommending a product with the highest non-negative virtual willingness to pay adjusted for the cost of persuasion, $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) - \ell_i^*(\boldsymbol{v})$, and the outside option if the adjusted virtual willingness to pay is negative for all sellers. The cost of persuasion is the shadow price of each of the binding constraints.

To gain intuition for Theorem 1.a, consider first the intermediary running an optimal auction with sellers and recommending the winner's product ignoring the obedience constraints, i.e. $\boldsymbol{\rho}^*$ that recommends only based on the virtual willingness to pay. If the consumer always follows the recommendations, $\boldsymbol{\rho}^*$ is the revenue-maximizing recommendations rule. By

where $(\boldsymbol{v}^\iota, \boldsymbol{\theta}^\iota)$ is such that $v^\iota_{\iota(i)} = v_i$ and $\theta^\iota_{\iota(i)} = \theta_i$ for all $i \in \mathcal{N}$.

Lemma 4, the consumer optimally follows recommendations from $\boldsymbol{\rho}^*$ if $v_0 \in [\underline{v}^*, \bar{v}^*]$. Since none of the obedience constraints bind, the cost of persuasion ℓ_i^* is zero.

When $v_0 > \bar{v}^*$ or $v_0 < \underline{v}^*$, the unconstrained optimal recommendations rule $\boldsymbol{\rho}^*$ fails in persuading the consumer to take recommended options. To provide an incentive for the consumer to take recommended options, the intermediary needs to recommend products more often when match values are high and less often otherwise, so that the recommendations would be more informative about match values. To the extent that the intermediary cannot recommend based on virtual willingness to pay, there is a loss of revenue associated with keeping the recommendations informative. The optimal way to improve the informativness is to adjust the virtual willingness to pay with the cost of persuasion, the shadow price of the obedience constraints.

For outside option values that are always above or below the value of the products, the optimal recommender system is characterized in the following theorem.

Theorem 1.b. *1. Let $v_0 > \bar{v}$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that*

$$r_0^*(\boldsymbol{v}, \boldsymbol{\theta}) = 1$$

for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. Let $\boldsymbol{t}$ be as in (8). Then, $\boldsymbol{r}^*$ is value-switching and $r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $i \in \mathcal{N} \cup \{0\}$ almost surely, and $(\boldsymbol{r}^*, \boldsymbol{t})$ is an optimal recommender system.

2. Let $v_0 < \underline{v}$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for each $i \in \mathcal{N}$,

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^*|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) \right\} \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $\mathcal{M}^* = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) \right\}$ and

$$\ell_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = 0 \text{ for all } i \in \mathcal{N} \text{ and } (\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \tag{15}$$

Let $\boldsymbol{t}$ be as in (8). Then, $\boldsymbol{r}^*$ is value-switching monotone and $r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $i \in \mathcal{N} \cup \{0\}$ almost surely, and $(\boldsymbol{r}, \boldsymbol{t})$ is an optimal recommender system.

When $v_0 > \bar{v}$, the consumer always prefers the outside option over the products. For such a consumer, the only obedient recommendations rule is to always recommend the outside option. When $v_0 < \underline{v}$, the consumer always prefers products over the outside option, but does not know which product the consumer prefers the most. The intermediary is restricted to recommend products only, but not the outside option. Consequently, the intermediary always recommends the product with the highest virtual willingness to pay, even though it may be negative.

When $v_0 = \bar{v}$, the intermediary is restricted to recommending outside option except for when $v_i = \bar{v}$ for some $i \in \mathcal{N}$. Conditioning on such $\boldsymbol{v}$, the intermediary recommends a product with the highest non-negative virtual willingness to pay among the products that have the valuation of $\bar{v}$. When $v_0 = \underline{v}$, the intermediary is restricted to recommending products except for when $v_i = \underline{v}$ for all $i \in \mathcal{N}$. Conditioning on such $\boldsymbol{v}$, the intermediary recommends a product with the highest non-negative virtual willingness to pay.

5 Additional Information

This section analyzes how the intermediary's use of additional information about sellers' private willingness to pay affects on the consumer surplus, intermediary's revenue and sellers' profits. I reformulate the additional information as a change in the intermediary's preference over recommendations to provide sufficient conditions under which the additional information benefits the consumer and sellers. Interpreting a class of additional information as the intermediary's monopolization of products market with private-label products, I show that the monopolization is beneficial to the consumer under mild conditions.

5.1 Optimal Recommender System with Additional Information

I begin with extending the baseline model of Section 3 to incorporate the additional information. The intermediary observes *additional signals* $\boldsymbol{z} = (z_1, \cdots, z_N)$ about sellers' private information $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_N)$. Each $z_i \in \mathcal{Z} \subset \mathbb{R}$ is independently drawn from a common distribution $H(\cdot \mid \theta_i)$ conditioning on each θ_i, and is common knowledge between a seller i and the intermediary, but not known to others.[13] Let $\mathcal{H} = \{H(\cdot \mid \theta)\}_{\theta \in \Theta}$ be *additional information*, a collection of distribution functions conditioning on each $\theta \in \Theta$. Let $\mathcal{Z}(\theta)$ be a support of $H(\cdot \mid \theta)$, and $\Theta(z)$ be a set of states at which z is generated with a positive probability. I present two examples of additional information below.

Example 1 (Perfectly revealing additional information)**.** Additional information $\mathcal{H}$ is *per-*

[13]More generally, it may be assumed that each i observes a signal ζ_i about additional signals about others $\boldsymbol{z}_{-i}$ without affecting any of the results. The signal ζ_i may be uninformative about $\boldsymbol{z}_{-i}$ as in here, may be completely revealing or may be related with $\boldsymbol{z}_{-i}$ in any arbitrary way.

fectly revealing if $\mathcal{Z} = [\underline{\theta}, \bar{\theta}]$ and

$$H(z \mid \theta) = \begin{cases} 1 & \text{if } z \geq \theta \\ 0 & \text{if } z < \theta \end{cases}$$

with $\mathcal{Z}(\theta) = \{\theta\}$ and $\Theta(z) = \{z\}$. □

Example 2 (Lower censorship additional information)**.** Additional information $\mathcal{H}$ is *lower censorship* if it reveals θ if $\theta \geq \theta^*$, but does not reveal otherwise. Formally, $\mathcal{Z} = \{z_0\} \cup [\theta^*, \bar{\theta}]$ where $z_0 < \theta^*$, and

$$H(z \mid \theta) = \begin{cases} 1 & \text{if } \theta \geq \theta^* \text{ and } z \geq \theta, \text{ or } \theta < \theta^* \text{ and } z \geq z_0 \\ 0 & \text{otherwise} \end{cases}$$

with $\mathcal{Z}(\theta) = \{\theta\}$ when $\theta \geq \theta^*$ and $\mathcal{Z}(\theta) = \{z_0\}$ when $\theta < \theta^*$, and $\Theta(z) = \{z\}$ when $z \geq \theta^*$ and $\Theta(z) = [\underline{\theta}, \theta^*]$ and $z = z_0$ □

The state space is $\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. The intermediay's recommender system is

$$(\mathbf{r}, \mathbf{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$$

such that $\sum_{i \in \mathcal{N} \cup \{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. The obedience, incentive compatibility and individual rationality are defined in the standard manner. Define

$$Q_i(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) \boldsymbol{H}(d\boldsymbol{z}_{-i})$$

to be the probability of recommending the seller i's product when her private willingness to pay is θ_i and additional signal is z_i. Applying the standard arguments gives the following lemma.

Lemma 5. *Suppose that a recommendations rule* $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^N$ *maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{16}$$

subject to obedience constraints

$$OB_{ij} : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_j r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{17}$$

and monotonicity constraints, i.e. for all $i \in \mathcal{N}$, $\theta_i \in \Theta$ *and* $z_i \in \mathcal{Z}$, $Q_i(\theta_i, z_i)$ *increases in* θ_i. *Suppose also that*

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = (\theta_i + w(v_i))r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}, \boldsymbol{z}) d\tilde{\theta}_i. \tag{18}$$

Then, $(\boldsymbol{r}, \boldsymbol{t})$ *is an optimal recommender system.*

Similar arguments of using value-switching recommendations rules from Section 4 may be applied to characterize the optimal recommender system.

Theorem 2.a. *Let* $v_0 \in (\underline{v}, \bar{v})$. *Let* $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ *be a recommendations rule such that for each* $i \in \mathcal{N}$,

$$r_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \begin{cases} \frac{1}{|\mathcal{M}^A|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1 - G(\theta_j \mid z_j)}{g(\theta_j \mid z_j)} + w(v_j)}_{\text{virtual willingness to pay}} - \underbrace{\ell_j^A(\boldsymbol{v})}_{\substack{\text{cost of} \\ \text{persuasion}}}, 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

where $\mathcal{M}^A = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j) - \ell_j^A(\boldsymbol{v}) \right\}$, *and*

$$\ell_i^A(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^A, \bar{v}^A] \\ \lambda_1^A(v_0) \cdot (v_0 - v_i) & \text{if } v_0 > \bar{v}^A \\ \lambda_2^A(v_0) \cdot \sum_{k \in \mathcal{N}} (v_0 - v_k) & \text{if } v_0 < \underline{v}^A \end{cases} \tag{20}$$

where $\lambda_1^A(v_0)$ *and* $\lambda_2^A(v_0)$ *are Lagrangian multipliers for* OB_{i0} *and* OB_{0i} *that may vary depending on* v_0, *respectively, and* $\bar{v}^A$ *and* $\underline{v}^A$ *are the thresholds from the unconstrained optimal recommendations rule. Let* $\boldsymbol{t}$ *be as in* (18). *Then,* $\boldsymbol{r}^A$ *is value-switching monotone, and* $(\boldsymbol{r}^A, \boldsymbol{t})$ *is an optimal recommender system.*

For outside option values that are always above or below the value of the products, the optimal recommender system is characterized in the following theorem.

Theorem 2.b. *1. Let* $v_0 > \bar{v}$. *Let* $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ *be a recommendations rule such that*

$$r_0^A(\boldsymbol{v}, \boldsymbol{\theta}) = 1$$

for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. *Let* $\boldsymbol{t}$ *be as in* (8). *Then,* $(\boldsymbol{r}^A, \boldsymbol{t})$ *is an optimal recommender system.*

2. Let $v_0 < \underline{v}$. Let $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for each $i \in \mathcal{N}$,

$$r_i^A(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^A|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j) \right\} \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

where $\mathcal{M}^A = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) \right\}$ and

$$\ell_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 0 \text{ for all } i \in \mathcal{N} \text{ and } (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \tag{22}$$

Let $\boldsymbol{t}$ be as in (18). Then, $\boldsymbol{r}^A$ is value-switching monotone and $r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $i \in \mathcal{N} \cup \{0\}$ almost surely, and $(\boldsymbol{r}^A, \boldsymbol{t})$ is an optimal recommender system.

It remains to analyze how an optimal recommendations rule with additional information $\boldsymbol{r}^A$ is different from that without additional information $\boldsymbol{r}^*$, and how does the difference impact on consumer surplus, intermediary's revenue and sellers' profits. I begin the analysis with recasting the additional information as a change in the intermediary's preference.

5.2 Additional Information as Change in Intermediary's Preference

A key observation is that additional information changes the intermediary's state-dependent preference over recommendations, but nothing else. The intermediary's persuasion problem without additional information, i.e. maximizing (7) subject to (1), can be reformulated as maximizing

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{23}$$

subject to obedience constraints (17). Although a recommendations rule is allowed to vary depending on additional signals $\boldsymbol{z}$, the optimal solution ignores $\boldsymbol{z}$ because the integrands of both the objective function (23) and the constraints (17) do not depend on $\boldsymbol{z}$, resulting in the same solution as maximizing (7) subject to (1).

Comparing the intermediary's persuasion problem without and with additional information, the only difference is the inverse hazard rates in the intermediary's preference. Without additional information, the intermediary's preference is given by the virtual willingness to pay,

$$\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i).$$

With additional information, the inverse hazard rate is conditioned on each additional signal,

$$\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i).$$

That is, additional information changes the intermediary's preference through inverse hazard rates, but nothing else.

The following definition is useful in capturing the change in the intermediary's preference caused by additional information.

Definition 2. Let $\mathcal{H}$ be additional information. A *θ-revenue difference* for $\theta > \theta'$ without additional information is

$$\Delta(\theta, \theta') = \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')}\right)$$

A θ-revenue difference with additional signals $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$ is

$$\Delta^{z,z'}(\theta, \theta') = \left(\theta - \frac{1 - G(\theta \mid z)}{g(\theta \mid z)}\right) - \left(\theta' - \frac{1 - G(\theta' \mid z')}{g(\theta' \mid z')}\right).$$

The θ-revenue difference without additional information measures an increase in virtual willingness to pay by recommending a product with higher θ over that with lower θ' holding others fixed. In other words, this measures how much the revenue increases as θ increases. By Myerson's regularity,

$$\Delta(\theta, \theta') > 0.$$

The θ-revenue difference with additional information measures the same except that the additional signals z and z', each corresponding to θ and θ', may be different from each other.

Example 3. To understand why θ-revenue difference is useful, consider an environment with 2 products $\{i, j\}$ where $\mathcal{V} = [\underline{v}, \bar{v}]$ and $\Theta = [\underline{\theta}, \bar{\theta}]$. Suppose that $w(v)$ strictly increases in v. Each product i is characterized by a pair (v, θ). The area inside the dashed square in Figure 4 is the space of all possible pairs for the product i. Let (v_j, θ_j) be the pair for the product j. An iso-revenue curve at (v_j, θ_j) is a set of points (v_i, θ_i) that gives the same revenue, the virtual willingness to pay, as (v_j, θ_j), and is drawn as a blue curve in Figure 4. Since the virtual willingness to pay increases in v and θ, the increasing direction of the iso-revenue curve is northeast. If the product i's pair (v_i, θ_i) is above the indifference curve, then the intermediary gets more revenue by recommending i over j; if below, then otherwise.

Assume, for simplicity, that the intermediary always recommends products based on the revenue and the consumer always follows the recommendations.

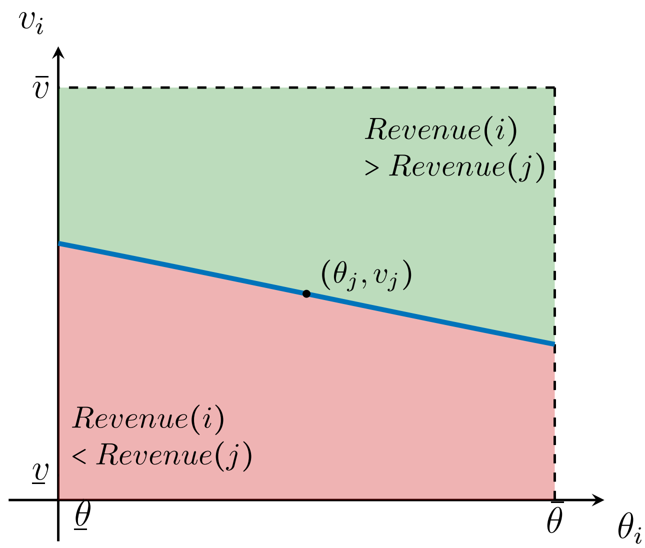


Figure 4: Iso-revenue Curve

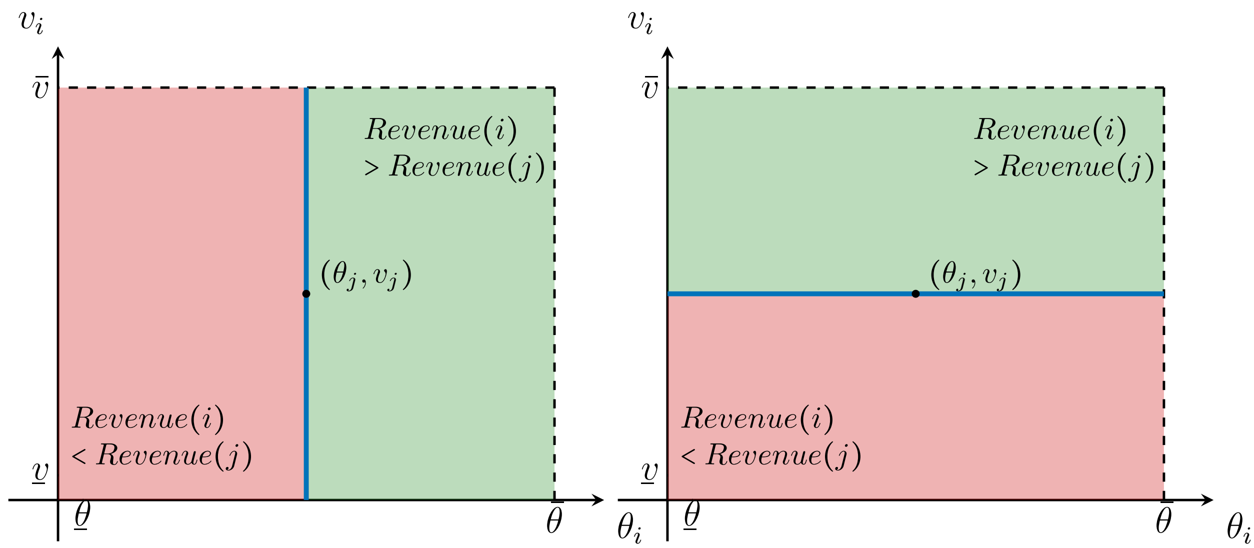
The higher the slope of the iso-revenue curve is, the more likely to recommend a product with higher θ, the lower consumer surplus is. One extreme case is in Figure 5a where the slope is so high that the iso-revenue is a vertical line. Under this iso-revenue curve, the intermediary recommends a product whichever has higher θ. If the consumer follows the recommendations, the consumer is low because the recommendations do not reflect match values at all. Another extreme case is in Figure 5b where the slope is so low that the iso-revenue is a horizontal line. The intermediary recommends a product whiehver has higher v. If the consumer follows the recommendations, the consumer is low because the recommendations are made only based match values.

Whether additional information benefits the consumer depends on whether additional information decreases the slope of the iso-revenue curve. It can be shown that the additional information decreases the slope for every point $(\boldsymbol{v}, \boldsymbol{\theta})$ if it decreases θ-revenue difference, i.e. for all $\theta, \theta' \in \Theta$, $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$,

$$\Delta^{z,z'}(\theta, \theta') \geq \Delta(\theta, \theta'),$$

and hence, benefits the consumer. The opposite holds as well: Additional information increases the slope if it increases θ-revenue difference, and hence, harms the consumer. $\square$

The above example illustrates the main intuitions behind how additional information changes the intermediary's preference and the consumer surplus through θ-revenue difference. However, there are two caveats. First, the graphical analysis only applies to how recommendations change between products, not between a product and the outside option.



(a) Vertical Iso-revenue Curve (b) Horizontal Iso-revenue Curve

Figure 5: Extreme Iso-revenue Curves.

Second, the consumer in this example is assumed to always follow the recommendations. These caveats motivate a class of additional information and environment under which the intuition well-applies.

5.3 Consumer Surplus under Small Inverse Hazard Rates Environment

Definition 3. Additional information $\mathcal{H}$ is *well-behaving* if it

1. satisfies *generalized Myerson's regularity* if for all $\theta > \theta'$, $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$,

$$\Delta^{z,z'}(\theta, \theta') > 0.$$

2. *increases (decreases) θ-revenue differences* if for all $\theta > \theta'$, $z \in \mathcal{Z}(\theta)$ and $z' \in \mathcal{Z}(\theta')$,

$$\Delta^{z,z'}(\theta, \theta') \geq (\leq) \Delta(\theta, \theta').$$

The first condition requires the virtual willingness to pay to be strictly increasing in θ no matter the additional signals. The second condition requires θ-revenue to be uniformly increasing or decreasing for all pairs of θ and z. Together, it increases or decreases the downward sloping iso-revenue curve as in Example 3.

Example 1, cont. Let $\mathcal{H}$ be a perfectly revealing additional information. The perfectly revealing additional information is well-behaving, and decreases (increases) θ-revenue difference if and only if $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.

□

Example 2, cont. Let $\mathcal{H}$ be a lower censorship additional information with θ^*. Let G be a distribution that has a decreasing inverse hazard rate $\frac{1-G(\theta)}{g(\theta)}$ on $[\underline{\theta}, \bar{\theta}]$ and has a density function such that for some neighborhood $B(\underline{\theta})$ of $\underline{\theta}$, $\inf_{\theta \in B(\underline{\theta})} g(\theta) > 0$ and $\sup_{\theta \in B(\underline{\theta})} g'(\theta) < \infty$. This nests a rich class of distributions including uniform distribution, linear virtual valuation distribution, (truncated) normal distribution, (truncated) exponential distribution and unimodal distribution with appropriate restrictions.

For sufficiently small θ^*, the lower censorship additional information is well-behaving, and always decreases θ-revenue difference. □

Definition 4. A triple $(G, \mathcal{H}, w)$ is said to have *small inverse hazard rates* if for all $i \in \mathcal{N}$

$$\inf_{v_i \in \mathcal{V}, \theta_i \in \Theta} \theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) > 0$$

and

$$\inf_{v_i \in \mathcal{V}, \theta_i \in \Theta, z_i \in \mathcal{Z}} \theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) > 0.$$

A small inverse hazard rates environment is likely to arise when sellers' marginal profit through recommender systems are high relative to their costs. For example, when online targeted advertisements often have better returns than other media (Hu, Shin, and Tang (2016)) or generate more revenue per ad and higher conversion rates than non-targeted ads (Howard (2010)), the environment is likely to have small inverse hazard rates.

In a small inverse hazard rates environment, the intermediary always prefer recommending products over the outside option. That is, the intermediary does not recommend the outside option unless doing so is necessary for the persuasion. Recommending the outside option is required only when the outside option value is very high, under which OB_{i0} binds and the consumer surplus is zero with and without additional information. When the outside option value is lower, the intermediary always recommends products over the outside option, and the graphical analysis from Example 3 applies.

The consumer surplus under a recommendations rule $\boldsymbol{r}$ at v_0 is

$$CS(v_0; \boldsymbol{r}) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \left[\sum_{i \in \{0\} \cup \mathcal{N}} (v_i - u^*) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})$$

where $u^* = \max(v_0, \mathbb{E}_{v_i}(v_i))$ is the consumer's optimal payoff without the recommendations.

Theorem 3. *Consider a small inverse hazard rates environment. Let $\mathcal{H}$ be any well-behaving additional information. Additional information increases (decreases) consumer surplus for all v_0 if it decreases (increase) θ-revenue difference.*

Example 1, cont. By Theorem 3, the perfectly revealing additional information increases (decreases) consumer surplus for all v_0 if $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.

$\square$

Recall that many 'natural' distributions (uniform, normal, exponential, log-concave, etc.) have decreasing $\frac{1-G(\theta)}{g(\theta)}$. Therefore, for natural distributions, the perfectly revealing additional information *increases* the consumer surplus for all consumers. This is a surprising result, as one of the grounds for restricting platforms from collecting seller data is the potential for consumer harm.[14] Instead, restricting the intermediary from collecting the most precise seller data harms all consumers by adding an information friction between the intermediary and sellers.

Example 2, cont. By Theorem 3, lower censorship additional information always increases consumer surplus for all v_0. $\square$

5.4 Sellers' Profits in Small Inverse Hazard Rates Environment

Additional information about sellers does not necessarily harm sellers' profits. Additional information reduces information rents, which in turn reduces sellers' profits conditioning on recommending products. However, the reduced information rent also allows the intermediary to recommend products when information rents restrained it from doing so, increasing the chance of recommending products, and hence, sellers' profits.

Example 4. Consider an environment where there is only one seller whose private willingness to pay θ is drawn from a uniform distribution over $[0,1]$. Match value v is drawn from a uniform distribution over $\{\underline{v}, \bar{v}\}$. The consumer's outside option value is $v_0 = \frac{1}{2}(\underline{v} + \bar{v})$, so that the consumer follows the intermediary's recommendations as long is it is value-switching monotone. There is no value-dependent willingness to pay $w(v_i) = 0$, so that the seller's virtual willingness to pay is

$$\theta - \frac{1 - G(\theta)}{g(\theta)} + w(v) = 2\theta - 1.$$

[14] *European Commission*, "Antitrust: Commission sends Statement of Objections to Amazon for the use of non-public independent seller data and opens second investigation into its e-commerce business practices," https://ec.europa.eu/commission/presscorner/detail/en/ip_20_2077

Without additional information, the intermediary recommends the product if and only if $\theta \leq \frac{1}{2}$ and the consumer follows the recommendations.

Consider *partitional* additional information that informs whether θ is above or below $\frac{1}{2}$, i.e. $\mathcal{Z} = \{z^L, z^H\} \subset \mathbb{R}^1$ with $z^L < z^H$ such that

$$H(z_i \mid \theta_i) = \begin{cases} 1 & \theta < \frac{1}{2} \text{ and } z \geq z^L \text{ and } \theta \geq \frac{1}{2} \text{ and } z \geq z^H \\ 0 & \text{otherwise} \end{cases}.$$

Conditioning on $z = z^H$, the seller's virtual willingness to pay $2\theta_i - 1$ is the same as before, so the intermediary recommends in the same manner and the seller gets the same profit. Conditioning on $z = z^L$, the intermediary learns that the seller has $\theta \leq \frac{1}{2}$, which reduces the inverse hazard rates to $\frac{1}{2} - \theta$ and increases the virtual willingness to pay to

$$\theta - \left(\frac{1}{2} - \theta\right) = 2\theta - \frac{1}{2}.$$

With the increased virtual willingness to pay, the intermediary recommends the product for θ that it used to recommend the outside option, $\frac{1}{4} \leq \theta \leq \frac{1}{2}$, increasing the seller's profit. Since every type of seller earns the same or more profit than before, the additional information increases the seller's ex-ante profit.[15] $\square$

The θ-revenue difference continues to play an important role determining whether additional information harms sellers. A seller i's ex-ante expected profit without additional information is

$$\Pi_i^* = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \frac{1 - G(\theta_i)}{g(\theta_i)} r_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}). \tag{24}$$

and with additional information is

$$\Pi_i^A = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} r_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}). \tag{25}$$

Notice that two objects change from (24) to (25): The recommendations rule from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$ and inverse hazard rates from $\frac{1-G(\theta_i)}{g(\theta_i)}$ to $\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)}$. It is helpful to separate the change in total profit by each of the changes. To this end, define a fictitious expected profit function obtained by fixing the inverse hazard rates at $\frac{1-G(\theta_i)}{g(\theta_i)}$ but changing the recommendations rule

[15] Note that the additional information is Pareto-improving in this example.

changes from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$

$$\Pi_i^F = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \frac{1 - G(\theta_i)}{g(\theta_i)} r_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}). \tag{26}$$

The total change in the seller's profit $\Pi_i^A - \Pi_i^*$ can be decomposed into two terms,

$$\underbrace{\Pi_i^A - \Pi_i^*}_{\text{total change}} = \underbrace{(\Pi_i^A - \Pi_i^F)}_{\text{inverse hazard rates effect}} + \underbrace{(\Pi_i^F - \Pi_i^*)}_{\substack{\text{recommendations} \\ \text{rule effect}}}$$

where *recommendations rule effect* $\Pi_i^F - \Pi_i^*$ captures the change in profit caused by a change in recommendations rule from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$, and *inverse hazard rates effect* $\Pi_i^A - \Pi_i^F$ captures the change in profit caused by a change in inverse hazard rates with and without additional information. I say the recommendations rule effect increases (decreases) all sellers' profits if $\Pi_i^F - \Pi_i^* \geq (\leq)\ 0$. The following theorem characterizes how each effect changes the profit.

Theorem 4. *Consider a small inverse hazard rates environment. Let $\mathcal{H}$ be any well-behaving additional information and $v_0 \leq \mathbb{E}_{v_i}(v_i)$.*

1. *Recommendations rule effect increases (decreases) all sellers' profits if one of the following conditions is satisfied:*

   (a) *Additional information increases (decreases) θ-revenue difference and inverse hazard rates $\frac{1-G(\theta)}{g(\theta_i)}$ increases (decreases) in θ_i.*

   (b) *Additional information decreases (increases) θ-revenue difference and inverse hazard rates $\frac{1-G(\theta)}{g(\theta_i)}$ decreases (increases) in θ_i.*

2. *Inverse hazard rates effect decreases all sellers' profits if for all $z_i \in \mathcal{Z}$ and $\theta_i \in \Theta(z_i)$,*

$$\frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} \leq \frac{1 - G(\theta_i)}{g(\theta_i)}.$$

A similar intuition from Example 3 applies. An increased θ-revenue difference increases the slope of the iso-revenue curve, so that the intermediary is more likely to recommend products with higher private willingness to pay θ instead of those with higher match values v. This change increases sellers' profits if $\frac{1-G(\theta_i)}{g(\theta_i)}$ is increasing in θ_i, but decreases if $\frac{1-G(\theta_i)}{g(\theta_i)}$ is decreasing in θ_i. Consequently, one sufficient condition for additional information to increase sellers' profits is for it to increase θ-revenue difference and $\frac{1-G(\theta_i)}{g(\theta_i)}$ to be increasing in θ_i. By a similar argument, if additional information decreases θ-revenue difference, then it increases sellers' profits if $\frac{1-G(\theta_i)}{g(\theta_i)}$ is decreasing in θ_i.

Let us say additional information *reduces inverse hazard rates* if for all $z_i \in \mathcal{Z}$ and $\theta_i \in \Theta(z_i)$,

$$\frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} \leq \frac{1 - G(\theta_i)}{g(\theta_i)}.$$

This is a sufficient condition for sellers' profits to be decreased through the inverse hazard rates effect. Both perfectly revealing and lower censorship additional information reduces inverse hazard rates, but not all well-behaving additional information does so.

Sufficient conditions under which additional information decreases sellers' profits are provided below.

Corollary 1. *Consider a small inverse hazard rates environment. Let $\mathcal{H}$ be any well-behaving additional information. Let $v_0 \leq \mathbb{E}_{v_i}(v_i)$. Additional information decreases sellers' profits if it reduces inverse hazard rates and one of the following conditions is satisfied:*

1. *Additional information increases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ.*
2. *Additional information decreases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ increases in θ.*

Example 1, cont. Perfectly revealing additional information always decreases sellers' profits to 0. □

Example 2, cont. By Corollary 1, lower censorship additional information decreases sellers' profits if $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. □

5.5 General Environment

This section examines the impact of additional information without small inverse hazard rates assumption.

Additional information always increases the intermediary's revenue, because the intermediary can always choose to ignore additional information.

Theorem 5. *Let $\mathcal{H}$ be any additional information and $v_0 \in \mathcal{R}^1$. Additional information always increases the intermediary's revenue.*

For consumers with outside option values lower than $\underline{v}$, the intermediary is restricted to recommend products, so the same analysis from the small inverse hazard rates environment applies for the consumer surplus. For consumers with outside options values higher than $\bar{v}$, the intermediary is restricted to recommend the outside option, so additional information is irrelevant.

Theorem 6. *Let $\mathcal{H}$ be any well-behaving additional information.*

1. *Let $v_0 < \underline{v}$. Additional information increases (decreases) consumer surplus for all v_0 if it decreases (increase) θ-revenue difference.*

2. *Let $v_0 > \bar{v}$. Additional information does not change the consumer surplus.*

Example 1, cont. By Theorem 3, perfectly revealing additional information *increases* (decreases) consumer surplus for all v_0 if $\frac{1-G(\theta)}{g(\theta)}$ *decreases* (increases) in θ. $\square$

Example 2, cont. Since lower censorship additional information always decreases θ-revenue difference, by Theorem 6, the additional information increases consumer surplus for $v_0 \leq \underline{v}$, but does not change consumer surplus for $v_0 \geq \bar{v}$. $\square$

For consumers with outside option values lower than $\underline{v}$, the intermediary is restricted to recommend products, so the same analysis from the small inverse hazard rates environment applies for the sellers' profits.

Theorem 7. *Let $\mathcal{H}$ be any well-behaving additional information and $v_0 < \underline{v}$.*

1. *Recommendations rule effect increases (decreases) all sellers' profits if one of the following conditions is satisfied:*

   (a) Additional information increases (decreases) θ-revenue difference and inverse hazard rates $\frac{1-G(\theta)}{g(\theta_i)}$ increases (decreases) in θ_i.

   (b) Additional information decreases (increases) θ-revenue difference and inverse hazard rates $\frac{1-G(\theta)}{g(\theta_i)}$ decreases (increases) in θ_i.

2. *Inverse hazard rates effect decreases all sellers' profits if for all $z_i \in \mathcal{Z}$ and $\theta_i \in \Theta(z_i)$,*

$$\frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} \leq \frac{1 - G(\theta_i)}{g(\theta_i)}.$$

Corollary 2. *Let $\mathcal{H}$ be any well-behaving additional information. Let $v_0 < \underline{v}$. Additional information decreases sellers' profits if it reduces inverse hazard rates and one of the following conditions is satisfied:*

1. *Additional information increases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ.*

2. *Additional information decreases θ-revenue and $\frac{1-G(\theta)}{g(\theta)}$ increases in θ.*

6 Consumer Data Protection

This section explores whether consumer data is protected or leaked to sellers through the recommender system. I find that the intermediary does not earn a higher revenue by sharing the consumer data with sellers under the optimal direct mechanism. However, there are indirect mechanisms that implement the optimal recommender system and leak consumer data to sellers.

For concreteness, I consider the environment from Section 3 with $v_0 \in (\underline{v}, \bar{v})$, but all results extend to any other environments from this paper.

6.1 Sharing Consumer Data

The analysis so far has assumed that that the intermediary cannot directly communicate any information about the consumer's match values to sellers. Consumer data is protected in that sellers do not learn about the consumer's match value $\boldsymbol{v}$ until the game ends. The intermediary can potentially earn a higher revenue by sharing some consumer data with sellers.

A data sharing policy is a pair $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$ where $\boldsymbol{\mathcal{Y}} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N$ and $\boldsymbol{Y} : \boldsymbol{\mathcal{V}} \to \Delta \boldsymbol{\mathcal{Y}}$ that privately sends $y_i \in \mathcal{Y}_i$ to each seller i before reporting θ_i. The distribution $\boldsymbol{Y}$ can potentially be asymmetric across sellers. The intermediary's problem is to choose a pair of a data sharing policy $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$ and a recommender system $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Y}} \to [0,1]^{N+1} \times \mathbb{R}^N$.

Fix a data sharing policy $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$. For each seller i with (θ_i, y_i), her expected profit is

$$\Pi_i^Y(\theta_i, y_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Y}}_{-i}} \Big((\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \Big) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{\mathcal{Y}}_{-i}(d\boldsymbol{y}_{-i} \mid y_i) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}),$$

where $\boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) = \frac{\boldsymbol{Y}(d\boldsymbol{y}|\boldsymbol{v})\boldsymbol{F}(d\boldsymbol{v})}{\int_{\boldsymbol{\mathcal{V}}} \boldsymbol{Y}(d\boldsymbol{y}|\boldsymbol{v})\boldsymbol{F}(d\boldsymbol{v})}$, and probability of getting recommended is

$$Q_i^Y(\theta_i, y_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Y}}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{\mathcal{Y}}_{-i}(d\boldsymbol{y}_{-i} \mid y_i) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}).$$

Data sharing signals $\boldsymbol{y}$ do not affect on sellers' incentive to report $\boldsymbol{\theta}$ truthfully. For each given $\boldsymbol{y}$, the incentive compatibility and individual rationality may be characterized in the standard way.

Lemma 6. *Let $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$ be a data sharing policy. An obedient $(\boldsymbol{r}, \boldsymbol{t})$ recommender system is* *incentive compatible and individually rational if and only if for all $i \in \mathcal{N}$, $\theta_i \in \Theta$ and $y_i \in \mathcal{Y}_i$,*

$$\begin{aligned}
&Q_i^Y(\theta_i, y_i) \text{ is increasing in } \theta_i,\\
&\Pi_i^Y(\theta_i, y_i) = \Pi_i^Y(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i^Y(\tilde{\theta}_i, y_i) d\tilde{\theta}_i,\\
&\Pi_i^Y(\underline{\theta}, y_i) \geq 0.
\end{aligned}$$

Applying the standard arguments, for a given data sharing policy $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$, the intermediary's problem is to maximize

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Y}}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{Y}(d\boldsymbol{y}) \boldsymbol{G}(d\boldsymbol{\theta}) \tag{27}$$

subject to obedience constraints, for all $i, j \in \mathcal{N} \cup \{0\}$,

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Y}}} (v_i - v_j) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{y}) \boldsymbol{F}(d\boldsymbol{v} \mid \boldsymbol{y}) \boldsymbol{Y}(d\boldsymbol{y}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{28}$$

Note that the integrands of both the objective function (27) and the constraints (28) do not depend on $\boldsymbol{y}$, so that the optimal recommendations rule ignores $\boldsymbol{y}$. Therefore, the optimal recommender system remains the same regardless of the data sharing policy.

Even in the absence of data sharing, the intermediary already extracts all of sellers' potential benefit from having a better estimate about value-dependent willingness to pay $w(v_i)$ by recommending better based on $\boldsymbol{v}$ but charging more accordingly. Another potential incentive for strategic data sharing is to affect sellers' incentive to report their private information. This channel is muted by the additive separability between θ_i and $w(v_i)$, and would have been important if sellers' profits were not additively separable, $\theta_i w(v_i)$, for example. How the non-separability would affect on the optimal data sharing policy is yet an open question.

Let $(\boldsymbol{r}^*, \boldsymbol{t}^*)$ be the optimal recommender system without data sharing. The intermediary does not share consumer data with sellers if $\boldsymbol{\mathcal{Y}} = \varnothing$. When data is not shared, the consumer data is protected.

Theorem 8. *When data sharing is allowed, the intermediary's optimal recommender system is the same regardless of the data sharing policy $(\boldsymbol{Y}, \boldsymbol{\mathcal{Y}})$. In particular, the recommender system $(\boldsymbol{r}^*, \boldsymbol{t}^*)$ without data sharing is optimal.*

6.2 Implementation

Under a mild condition, a variant of handicap auction (Eső and Szentes (2007)) implements the optimal recommender system. In a special environment with linear private virtual willingness to pay and no value-dependent willingness to pay, a second-price auction with discounts and participation fees implement implements. This includes the example in Section 2 as a special case. For concreteness, I consider the environment from Section 3 with $v_0 \in (\underline{v}, \bar{v})$. Both versions of the implementation extend to any other environments that I consider in this paper.

6.2.1 Handicap Auction

Let $v_0 \in (\underline{v}, \bar{v})$.[16] The handicap auction consists of two rounds. In the first round, each seller i with θ_i chooses a price premium $p_i \in \mathbb{R}^1$ at a fee $C_i(p_i)$ from the menu of price premia $(p, C_i(p))_{p\in\mathbb{R}^1}$ proposed by the intermediary. The price premium chosen by each seller is known only to the seller and the intermediary, but not to other sellers and the consumer. After the first round and before the second, the intermediary discloses $\boldsymbol{v}$ to sellers and announces the cost of persuasion $\ell_i : \boldsymbol{\mathcal{V}} \to \mathbb{R}^1$ for each $i \in \mathcal{N}$. In the second round, a second-price auction with zero reservation price, price premia and costs of persuasion follows. The seller with the highest bid wins the auction, but is required to pay the second highest pay plus the price premium and the cost of persuasion. That is, if others bid $(b_j)_{j\in\mathcal{N}\setminus\{i\}}$, the price premium is p_i and the cost of persuasion is $\ell_i(\boldsymbol{v})$ for each i, then if the seller i bids $b_i > \max_{j\in\mathcal{N}\setminus\{i\}}(b_j, 0)$, then she wins the auction and is required to pay

$$\max_{j\in\mathcal{N}\setminus\{i\}}(b_j, 0) + p_i + \ell_i(\boldsymbol{v}).$$

If she bids $b_i < \max_{j\in\mathcal{N}\setminus\{i\}}(b_j, 0)$, then she loses and pays nothing.

Arguments below closely follow Eső and Szentes (2007). For the completeness, I present the full arguments here. I begin with characterizing an equilibrium in the second round auction: it is weakly dominant strategy for each seller i to bid his willingness to pay minus the price premium.

Lemma 7. *Suppose each seller i is informed with $\boldsymbol{v}$ and is charged with a price premium p_i. In the second round of the handicap auction, it is a weakly dominant strategy for seller i to bid $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$.*

From here on, sellers are assumed to play according to their weakly dominant strategy in the second round. The handicap auction is represented by a triple of of functions $p_i : \Theta \to \mathbb{R}^1$,

[16]When $v_0 > \bar{v}$, the optimal recommender system from Theorem 1.b is implemented by a handicap auction with positive infinite reservation prices and $\ell_i^*(\boldsymbol{v}) = 0$ for all i and $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$. When $v_0 < \underline{v}$, the optimal recommender system from the same theorem is implemented by a handicap auction with negative infinite reservation prices and $\ell_i^*(\boldsymbol{v}) = 0$ for all i and $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$ as in (15).

$c_i : \Theta \to \mathbb{R}^1$ and $\ell_i : \boldsymbol{\mathcal{V}} \to \mathbb{R}^1$ for each $i \in \mathcal{N}$ where $p_i(\theta_i)$ is the price premium that θ_i chooses at the fee of $c_i(\theta_i) = C_i(p_i(\theta_i))$.

Given a handicap auction $(p_i, c_i, \ell_i)_{i\in\mathcal{N}}$, by Lemma 7, if the seller θ_i reports non-truthfully as $\hat{\theta}_i$ in the first round, the seller bids $\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v})$ in the second round. Denote each seller's equilibrium bid assuming that she bids truthfully in each stage by

$$b_i^*(\boldsymbol{v}, \theta_i) = \theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}).$$

Let $b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) = \max_{j\in\mathcal{N}\setminus\{i\}}(b_j^*(\boldsymbol{v}, \theta_j), 0)$ be the equilibrium highest bid and reservation price excluding i's bid in the second round at each given state $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.

The handicap auction $(p_i, c_i, \ell_i)_{i\in\mathcal{N}}$ is incentive compatible if for each seller i seller optimally reports its true type in the first round. The seller θ_i's expected profit after reporting as $\hat{\theta}_i$ assuming others report truthfully is

$$\pi_i^H(\theta_i, \hat{\theta}_i) = \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\left[(\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i+w(v_i)-p_i(\theta_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] - c(\hat{\theta}_i),$$

and wins the auction with a probability of

$$Q_i^H(\theta_i, \hat{\theta}_i) = \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\left[\mathbf{1}_{\{\theta_i+w(v_i)-p_i(\theta_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right].$$

The incentive compatibility of handicap auctions is characterized in the following lemma.

Lemma 8. *A handicap auction $(p_i, c_i, \ell_i)_{i\in\mathcal{N}}$ is incentive compatible if and only if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta$,*

$$\pi_i^H(\theta_i, \theta_i) = \pi_i^H(\underline{\theta}, \underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) d\tilde{\theta}_i \tag{29}$$

and for all $\theta_i', \theta_i'' \in \Theta$ such that $\theta_i' < \theta_i < \theta_i''$,

$$Q_i^H(\theta_i, \theta_i') \leq Q_i^H(\theta_i, \theta_i) \leq Q_i^H(\theta_i, \theta_i''). \tag{30}$$

The inequality (30) states that for each seller with a given type, if he reports his type to be higher (lower) in the first round, then he is more (less) likely to win the auction in the second round. Since misreporting in the first round only changes the price premium, (30) is satisfied if the price premium $p_i(\boldsymbol{v}, \theta_i)$ decreases in θ_i.

Theorem 9. *Let $v_0 \in (\underline{v}, \bar{v})$. Suppose that $\frac{1-G(\theta_i)}{g_i(\theta_i)}$ decreases in θ_i. The intermediary can implement the optimal recommendations rule (12) and attain the same revenue via a handicap auction $(p_i^*, c_i^*, \ell_i^*)_{i\in\mathcal{N}}$ where*

$$p_i^*(\boldsymbol{v}, \theta_i) = \frac{1 - G(\theta_i)}{g(\theta_i)}, \tag{31}$$

$$c_i^*(\theta_i) = \mathbb{E}_{\boldsymbol{v},\theta_{-i}}\left[(\theta_i + w(v_i) - p_i^*(\theta_i) - \ell_i^*(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i+w(v_i)-p_i^*(\theta_i)-\ell_i^*(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] - \int_{\underline{\theta}}^{\theta_i} Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)d\tilde{\theta}_i, \tag{32}$$

and the cost of persuasion $(\ell_i^*)_{i\in\mathcal{N}}$ *is as in* (13).

Note that the premium $p_i^*(\boldsymbol{v},\theta_i)$ decreases in θ_i, so that (30) is satisfied. The first term in the fee schedule c_i^* is the seller i's expected profit from the second round. The second term is the information rent. By paying the fee, the seller's expected profit conditioning on θ_i is exactly the information rent, so that (29) is satisfied with $\pi_i^H(\underline{\theta},\underline{\theta}) = 0$. By Lemma 8, the handicap auction $(p_i^*, c_i^*, \ell_i^*)_{i\in\mathcal{N}}$ is incentive compatible. The handicap auction is individually rational and attains attains the same revenue as the optimal recommender system because $\pi_i^H(\underline{\theta},\underline{\theta}) = 0$ and the second round auction implements the optimal recommendations rule (12).

6.2.2 Second-Price Auction with Discounts and Participation Fees

Consider an environment where the virtual private willingness to pay is linear

$$\alpha\theta_i - \beta \text{ for some } \alpha > 1, \beta > 0$$

with support $\left[\underline{\theta}, \frac{\beta}{\alpha-1}\right]$ where $0 \leq \underline{\theta} < \frac{\beta}{\alpha-1}$, and the value-dependent willingness to pay is zero, i.e. $w(v) = 0$. The class of distributions with linear virtual private willingness to pay includes includes uniform, exponential distribution, Pareto distribution and log-logistic distribution.

A second-price auction with discounts and participation fees is represented by a pair of a discount function $d_i : \boldsymbol{\mathcal{V}} \to \mathbb{R}^1$ and a participation fee $P_i \in \mathbb{R}^1$ for each $i \in \mathcal{N}$. Each seller first decides whether to participate by paying the fee P_i. Once participated, each seller is informed of the discount $d_i(\boldsymbol{v})$, but not $\boldsymbol{v}$ directly. The seller with the highest bid wins, but is required to pay the second-highest bid minus a discount. With appropriately chosen discounts, participation fees and reserve prices, the auction implements the optimal recommender system.

Theorem 10. *Let* $v_0 \in (\underline{v}, \bar{v})$*. Suppose that* $\theta_i - \frac{1-G(\theta_i)}{g_i(\theta_i)} = \alpha\theta_i - \beta$ *with* $\alpha, \beta > 0$ *and* $w(v_i) = 0$ *for all* $v_i \in \mathcal{V}$*. The intermediary can implement the optimal recommendations rule* (12) *and attain the same revenue via a second price auction with discounts and participation fees* $(d_i^*, P_i^*)_{i\in\mathcal{N}}$ *where*

$$d_i^*(\boldsymbol{v}) = -\frac{1}{\alpha}\ell_i^*(\boldsymbol{v}), \tag{33}$$

and

$$P_i^* = \mathbb{E}_{\boldsymbol{v},\theta_{-i}} \left(\theta_i - \frac{1}{\alpha}\ell_i^*(\boldsymbol{v}) - \max_{j\in\mathcal{N}\setminus\{i\}} \left(\theta_j - \frac{1}{\alpha}\ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha} \right) \right) \mathbf{1}_{\{\theta_i - \frac{1}{\alpha}\ell_i^*(\boldsymbol{v}) > \max_{j\in\mathcal{N}\setminus\{i\}}(\theta_j - \frac{1}{\alpha}\ell_j^*(\boldsymbol{v}), \frac{\beta}{\alpha})\}} \tag{34}$$

with reservation price $\frac{\beta}{\alpha}$.

Conditioning on a participation, each seller's optimal bid is

$$\theta_i + d_i(\boldsymbol{v}) = \theta_i - \frac{1}{\alpha}\ell_i^*(\boldsymbol{v}).$$

If all sellers of every type participates, this auction implements the optimal recommendations rule (12).

It remains to make sure that the lowest type's expected profit is 0. Note that the lowest private willingness to pay type $\underline{\theta}$ may have a positive probability of winning the auction when he sells a product that is high match. When he wins the auction, he always gets a non-negative profit. The participation fee P_i^* is exactly the expected profit of the lowest type without the fee, making the expected profit of $\underline{\theta}$ to be is 0. Consequently, all sellers of all types participate; the intermediary attains the same revenue as under the optimal recommender system.

6.3 Discussions on Consumer Data Protection

Under the direct mechanism, consumer data is protected in that sellers do not learn about the consumer's match values $\boldsymbol{v}$ until the game ends. Furthermore, the intermediary does not earn higher revenue by sharing information about match values with sellers. However, the handicap auction involves disclosing $\boldsymbol{v}$ to sellers when they bid. Under the second-price auction with discounts and participation fees, sellers infer some of $\boldsymbol{v}$ from the discounts even though they are not directly informed of it. For example, if the seller i is informed of a discount $\lambda(v_i - v_0)$, then the seller can infer that the consumer's match value is v_i. In both cases, the consumer data is leaked in that sellers learn about $\boldsymbol{v}$ before the game ends.

7 Discussion and Additional Results

7.1 Constrained Welfare Maximization

Define α-welfare, a weighted sum of the consumer welfare and the joint profit of the intermediary and sellers,

$$\alpha \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) + (1-\alpha)\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}),$$

where $\alpha \in (0,1)$. A recommender system $(\boldsymbol{r}^\alpha, \boldsymbol{t}^\alpha)$ is a *constrained α-welfare maximizing* recommender system if it maximizes the α-welfare subject to obedience constraints (1), incentive compatibility (2) and individual rationality (3). In this section, I characterize α-welfare maximizing recommender system and its implication on the consumer surplus. Throughout this section, I assume that $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ and $v_0 < \underline{v}$.

Note that transfer $\boldsymbol{t}$ is irrelevant for the constrained α-welfare maximization problem as long as it makes a given recommendations rule incentive compatible and individually rational. In particular, in characterizing the constrained α-welfare maximizing recommendations rule and α-welfare, it is without loss of generality to assume $\boldsymbol{t}$ to be (8) and drop incentive compatibility and individual rationality constraints. Subtracting a constant αv_0 from the α-welfare gives the following lemma.

Lemma 9. *Suppose that a recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta} \to [0,1]^N$ maximizes*

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i + w(v_i) + \frac{\alpha}{1-\alpha}(v_i - v_0)\right) r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \tag{35}$$

subject to obedience constraints (1) and monotonicity constraints (4). Suppose also that

$$t_i(\boldsymbol{v},\boldsymbol{\theta}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i})d\tilde{\theta}_i. \tag{36}$$

Then, $(\boldsymbol{r},\boldsymbol{t})$ is a constrained α-welfare recommender system.

Comparing the α-welfare maximization problem in Lemma 9 to the intermediary's revenue maximization problem in Lemma 5, the only difference is the integrand in each objective function is changed from

$$\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) \tag{37}$$

to

$$\theta_i + w(v_i) + \frac{\alpha}{1-\alpha}(v_i - v_0). \tag{38}$$

The identical graphical analysis from Example 3 applies. Unlike in Section 5, however, the transition from (37) to (38) changes not only the term related to θ_i, but also the term related to v_i as well. The difference terms are defined for both θ as well as v.

Definition 5. A *θ-revenue difference* for $\theta > \theta'$ is

$$\Delta(\theta, \theta') = \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) - \left(\theta' - \frac{1 - G(\theta')}{g(\theta')}\right)$$

and *v-revenue difference* for $v > v'$ is

$$D(v, v') = w(v) - w(v')$$

Similarly, a *θ-welfare difference* for $\theta > \theta'$ is

$$\Delta^\alpha(\theta, \theta') = \theta - \theta'$$

and *v-welfare difference* for $v > v'$ is

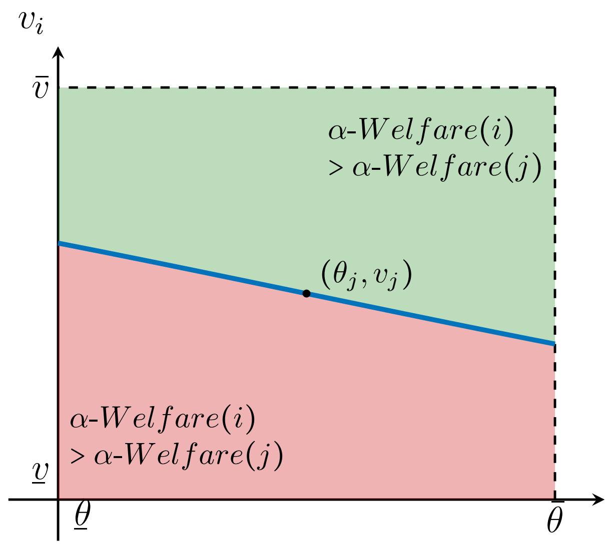
$$D^\alpha(v, v') = \left(w(v) + \frac{\alpha}{1-\alpha}v\right) - \left(w(v') + \frac{\alpha}{1-\alpha}v'\right).$$

Note that $\Delta^\alpha(\theta, \theta') \leq \Delta(\theta, \theta')$ for all $\theta > \theta'$ and $D^\alpha(v, v') \geq D(v, v')$ for all $v > v'$. Relative to the revenue-maximization regime, the additional gain from recommending a product with a higher θ decreases and that with a higher v increases under the α-maximization regime. In other words, the iso-welfare curve has a lower slope than the iso-revenue curve as in Figure 6. Consequently, α-welfare-maximizing recommendations rule recommends products with higher match values more often, which increases the consumer surplus and decreases the joint profit.

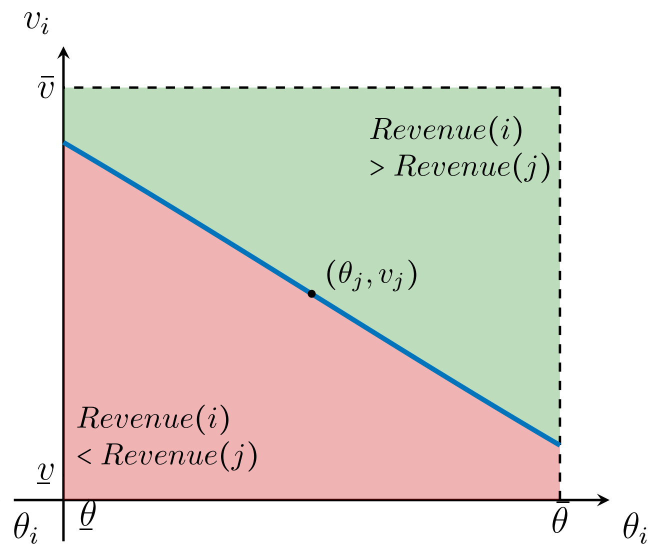
Theorem 11. *Suppose that $\frac{1-G(\theta_i)}{g(\theta_i)}$ decreases in θ_i and $v_0 < \underline{v}$. Relative to the revenue maximizing recommender system $(\boldsymbol{r}^*, \boldsymbol{t}^*)$, under the α-welfare maximizing recommender system $(\boldsymbol{r}^\alpha, \boldsymbol{t}^\alpha)$,*

1. *Consumer surplus is higher.*

2. *Joint profit of the intermediary and sellers is lower.*



(a) Iso-welfare Curve



(b) Iso-revenue Curve

Figure 6: Extreme Iso-revenue Curves.

7.2 Search Engine Interpretation

One can interpret a recommender system with some modifications as a search engine. Suppose there are $N_p + N_o$ search items. For each of the first N_p search items, there is an advertiser (seller) $i \in \mathcal{N}_p = \{1, \ldots, N_p\}$ who is willing to pay for a paid search for his item (recommendation). For the later N_o items, owners of the item $i \in \mathcal{N}_o = \{N_p + 1, \ldots, N_p + N_o\}$ are not willing to pay. Items of the first type are called *paid search* while the second are *organic search* items. As in the baseline case, each seller has private willingness to pay θ_i and value-dependent willingness to pay $w_p(v_i)$ that are additively separable. If an advertiser i pays t_i for his item to be searched with probability r_i, the advertiser's profit is

$$(\theta_i + w_p(v_i))r_i - t_i$$

where $w_p(.)$ is any strictly increasing function. The private willingness to pay θ_i is independently drawn from a common distribution G that is absolutely continuous and has support $[\underline{\theta}, \bar{\theta}]$ with $-\infty < \underline{\theta} < \bar{\theta} < \infty$.

The user (the consumer) does not know the valuation of each search item, but only knows that v_i is independently drawn from a common distribution F_p if $i \in \mathcal{N}_p$, and from F_o if $i \in \mathcal{N}_o$. For simplicity, assume that $F_p(v) = \tilde{F}(v - \mu_p)$ and $F_o(v) = \tilde{F}(v - \mu_o)$ where $\mu_p, \mu_o \in \mathbb{R}^1$ are mean shifters of F_p and F_o, respectively. Let $\boldsymbol{v}_p = (v_1, \ldots, v_{N_p}) \in \boldsymbol{\mathcal{V}}_o = \mathcal{V}^{N_p}$, $\boldsymbol{v}_o = (v_{N_p+1}, \ldots, v_{N_p+N_o}) \in \boldsymbol{\mathcal{V}}_o = \mathcal{V}^{N_o}$, $\boldsymbol{v} = (\boldsymbol{v}_p, \boldsymbol{v}_o) \in \boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o$, $\boldsymbol{F} = \boldsymbol{F}_p \times \boldsymbol{F}_o$ and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_{N_p}) \in \boldsymbol{\Theta} = \Theta^{N_p}$.

The intermediary knows about the value of the options better than the user himself. The intermediary privately observes $\boldsymbol{v}$ and designs how to provide information to the user while raising revenue from advertisers, which I call a search engine (a recommender system). Since a search engine always returns some output, I assume that there is no outside option. By the revelation principle, it is without loss to assume that the intermediary's search engine is a direct mechanism $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o \times \boldsymbol{\Theta} \to [0,1]^{N_p+N_o} \times \mathbb{R}^{N_p}$ with $\sum_{i \in \mathcal{N}_p \cup \mathcal{N}_o} r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) = 1$. The intermediary collects revenue from advertisers, but also has reputation concerns that incentivizes the intermediary to search for the right item for the user. Formally, the intermediary's objective function is a weighted sum of revenue and value of the recommendations

$$\int_{\boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o \times \boldsymbol{\Theta}} \left(\sum_{i \in \mathcal{N}_p} t_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) + \beta \sum_{i \in \mathcal{N}_p \cup \mathcal{N}_o} w_o(v_i) r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) \right) \boldsymbol{F}_p(d\boldsymbol{v}_p) \boldsymbol{F}_o(d\boldsymbol{v}_o) \boldsymbol{G}(d\boldsymbol{\theta})$$

where $\beta \in (0,1)$ is a relative weight between the revenue and value of the recommendations, and $w_o(.)$ is any strictly increasing function. After applying the usual arguments, the intermediary's problem is reduced to a Bayesian persuasion problem of maximizing

$$\int_{\boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}_p \cup \mathcal{N}_o} \psi_i^E(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) \boldsymbol{F}_p(d\boldsymbol{v}_p) \boldsymbol{F}_o(d\boldsymbol{v}_o) \boldsymbol{G}(d\boldsymbol{\theta})$$

where

$$\psi_i^E(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) = \begin{cases} \theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w_p(v_i) + w_o(v_i) & \text{if } i \in \mathcal{N}_p \\ w_o(v_i) & \text{if } i \in \mathcal{N}_o \end{cases}$$

subject to obedience constraints, i.e. OB_{ij} for any $i, j \in \mathcal{N}_p \cup \mathcal{N}_o$

$$\int_{\boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o \times \boldsymbol{\Theta}} (v_i - v_j)\, r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) \boldsymbol{F}_p(d\boldsymbol{v}_p) \boldsymbol{F}_o(d\boldsymbol{v}_o) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0$$

and monotonicity constraints, that is, for all $\theta_i > \theta_i'$

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \theta_i, \boldsymbol{\theta}_{-i}) \boldsymbol{F}_p(d\boldsymbol{v}_p) \boldsymbol{F}_o(d\boldsymbol{v}_o) \boldsymbol{G}(d(\theta_i, \boldsymbol{\theta}_{-i})) \\ \geq &\int_{\boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \theta_i', \boldsymbol{\theta}_{-i}) \boldsymbol{F}_p(d\boldsymbol{v}_p) \boldsymbol{F}_o(d\boldsymbol{v}_o) \boldsymbol{G}(d(\theta_i', \boldsymbol{\theta}_{-i})). \end{aligned} \tag{39}$$

Define a value-switching monotone recommendations rule as in the following.

Definition 6. A recommendations rule $\boldsymbol{r}$ is value-switching monotone if

1. $r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta})$ increases in (v_i, θ_i) for all $i \in \mathcal{N}_p \cup \mathcal{N}_o$.

2. $r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta})$ decreases in v_j whenever $i \in \mathcal{N}_p$ and $j \in \mathcal{N}_o$, or, $i \in \mathcal{N}_o$ and $j \in \mathcal{N}_p$.

3. $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ decreases whenever v_j is switched with a larger v_i for all $i, j \in \mathcal{N}_p$ or $i, j \in \mathcal{N}_o$, i.e. $(\boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}}_{-ij} \times \boldsymbol{\Theta}$ and $v > v'$,

$$r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \geq r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}).$$

We can apply the same logic as before. Any value-switching monotone recommendations rule always satisfy OB_{ij} for $i, j \in \mathcal{N}_p$ and $i, j \in \mathcal{N}_o$; satisfy all of the obedience constraints if μ_p and μ_o are similar to each other; violates OB_{op} if μ_p is significantly larger than μ_o; violates OB_{po} if μ_p is significantly smaller than μ_o, where $p \in \mathcal{N}_p$ and $o \in \mathcal{N}_o$. An optimal search engine is characterized by two numbers $\underline{\mu}^E \leq 0 \leq \bar{\mu}^E$

$$r_i^E(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^E|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \psi_j^E(\boldsymbol{v}, \boldsymbol{\theta}) - \ell_j^E(\boldsymbol{v}), 0 \right\} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{M}^E = \arg\max_{j \in \mathcal{N}} \left\{ \psi_j^E(\boldsymbol{v}, \boldsymbol{\theta}) + w_j - \ell_j^E(\boldsymbol{v}) \right\}$, for $i \in \mathcal{N}_p$

$$\ell_i^E(\boldsymbol{v}) = \begin{cases} 0 & \text{if } \mu_p - \mu_o \geq \underline{\mu}^E \\ \lambda_2^E(v_0) \cdot \sum_{o \in \mathcal{N}_o} (v_o - v_i) & \text{if } \mu_p - \mu_o < \underline{\mu}^E \end{cases}$$

and for $i \in \mathcal{N}_o$

$$\ell_i^E(\boldsymbol{v}) = \begin{cases} \lambda_1^E(v_0) \cdot \sum_{p \in \mathcal{N}_p} (v_p - v_i) & \text{if } \mu_p - \mu_o > \bar{\mu}^E \\ 0 & \text{if } \mu_p - \mu_o \leq \bar{\mu}^E \end{cases}$$

where $\lambda_1^E(v_0)$ and $\lambda_2^E(v_0)$ are Lagrangian multipliers for OB_{po} and OB_{op} that may vary depending on v_0, respectively. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}) = (\theta_i + w_p(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}) d\tilde{\theta}_i$$

for each i.

7.3 Relaxing v_0 as Common Knowledge

The value of the outside option v_0 has been assumed to be a constant that is commonly known to all players. One way to relax this assumption is to assume that v_0 is common knowledge between the intermediary and consumer, but not to the sellers. Sellers instead believe that v_0 is drawn from a distribution F_0. All results continue to hold identically under this assumption. Below I explore two different ways to relax the symmetric knowledge of v_0 between the intermediary and the consumer: One under which v_0 is the intermediary's private information; the other under which v_0 is the consumer's private information.

7.3.1 Value of Outside Option as Private Knowledge of Intermediary

Suppose that only the intermediary observes v_0 that is drawn from a distribution F_0. For simplicity,[17] assume that $F_0(v_0) = F(v_0 - \mu_0)$ where $\mu_0 \in \mathbb{R}^1$ is a mean shifter of F_0. If $\mu_0 = 0$, then the distribution for the outside option F_0 is identical to those of other products F; if $\mu_0 > 0$, then F_0 first-order stochastically dominates F; if $\mu_0 < 0$, then F_0 is first-order stochastically dominated by F. All of the results in Section 4 and Section 5 continues to hold after replacing v_0 with μ_0.

7.3.2 Value of Outside Option as Private Knowledge of Consumer

Suppose that only the consumer observes v_0 that is drawn from a distribution F_0 that has a full support on the real line. Suppose that the intermediary offers the same recommender system to the consumer of all types, and the consumer of each type decides whether to follow the recommendation. This is equivalent to the intermediary designing the set $\mathcal{V}_0^{NR} \subset \mathbb{R}^1$ of the consumer types who will obey recommendations on top of designing a recommender system itself. For any given recommender system, a consumer with high v_0 disobeys when recommended with a product; a consumer with low v_0 disobeys when recommended with the outside option. Consequently, the intermediary's problem reduces to designing a recommender system and picking up two thresholds $\underline{v}^{NR} \leq \bar{v}^{NR}$ such that the consumer obeys if and only if $v_0 \in [\underline{v}^{NR}, \bar{v}^{NR}]$.

The intermediary faces another layer of trade-offs, setting target population $\mathcal{V}_0^{NR} = [\underline{v}^{NR}, \bar{v}^{NR}]$, on top of the trade-off between raising revenue and keeping the consumer incentivized to obey for each consumer $v_0 \in \mathcal{V}_0^{NR}$. Which population to target depends on whether the intermediary chooses to recommend the outside option with positive probability or not.

If the intermediary does not recommend the outside option almost surely, then any consumer with bad enough outside option is obedient to the recommender system. In particular, an optimal recommender system $(\boldsymbol{r}^{NR}, \boldsymbol{t}^{NR})$ is characterized by $\bar{v}^{NR} \geq \mathbb{E}_{v_i}(v_i)$ such that the consumer is obedient to $\boldsymbol{r}^{NR}$ if and only if $v_0 \in \mathcal{V}_0^{NR} = (-\infty, \bar{v}^{NR}]$, and

$$r_i^{NR}(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{1}{|\mathcal{M}^{NR}|} \text{ if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1 - G(\theta_j)}{g(\theta_j)} + w_j \right\} \tag{40}$$

[17] All results here can be generalized to any family distributions $F_0(v_0; \mu_0)$ where $\mu_0 \in \mathbb{R}^1$ is an index such that $F_0(v_0; \mu_0)$ first-order stochastically dominates $F_0(v_0; \mu_0')$ whenever $\mu_0 > \mu_0'$.

where $\mathcal{M}^{NR} = \arg\max_{j\in\mathcal{N}}\left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j\right\}$. Note that there is no cost of persuasion because none of the obedience constraints bind. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i^{NR}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{w})d\tilde{\theta}_i. \tag{41}$$

If the intermediary recommends the outside option with positive probability, then the optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ is characterized by $\underline{v} \leq \underline{v}^{NR} = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1) \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}^{NR} = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1) \leq \bar{v}^{NR} \leq \bar{v}$such that the consumer is obedient to $\boldsymbol{r}^{NR}$ if and only if $v_0 \in \mathcal{V}_0^{NR} = [\underline{v}^{NR},\bar{v}^{NR}]$. Furthermore, OB_{0i} binds at $\underline{v}^{NR}$ and OB_{i0} binds at $\bar{v}^{NR}$, so that

$$r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^{NR}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}}\left\{\underbrace{\theta_i - \dfrac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)}_{virtual\ willingness\ to\ pay} - \underbrace{\ell_j^{NR}(\boldsymbol{v})}_{\substack{\text{cost of}\\ \text{persuasion}}}, 0\right\} \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

where $\mathcal{M}^{NR} = \arg\max_{j\in\mathcal{N}}\left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j - \ell_j^{NR}(\boldsymbol{v})\right\}$, and

$$\ell_i^{NR}(\boldsymbol{v}) = \lambda_1(\bar{v}^{NR})(\bar{v}^{NR} - v_i) - \lambda_2(\underline{v}^{NR})\sum_{k\in\mathcal{N}}(\bar{v}^{NR} - v_k)$$

where $\lambda_1^{NR}(v_0)$ and $\lambda_2^{NR}(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively, and $\underline{v}^{NR}$ and $\bar{v}^{NR}$ are the thresholds constructed from the uncontrained optimal recommendations rule. An optimal transfer rule is given as in (41).

Theorem 12. *Suppose that the consumer privately observes v_0 that is drawn from a distribution F_0. An optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ takes one of the following two structures:*

1. *The intermediary always recommends one of products. An optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ is as in (40) and (41). The consumer is obedient if and only if $v_0 \leq \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1)$.*

2. *The intermediary sometimes recommends the outside option. An optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ is as in (42) and (41). The consumer is obedient if and only if $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1) \leq v_0 \leq \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1)$.*

7.4 Relaxing Intermediary's Private Knowledge of $w(v)$

The value-dependent willingness to pay $w(v)$ has been assumed to be a deterministic function of v_i. Combined with the assumption that only the intermediary knows $\boldsymbol{v}$, this entails that the intermediary privately knows sellers' value-dependent willingness to pay $w(v)$ that sellers themselves do not know.

This assumption may be relaxed in two different ways. The first is to assume that each seller learns the value of $w(v_i)$ even though he does not know v_i. With the strict monotonicity of $w(v_i)$, learning the value of $w(v_i)$ is equivalent to learning v_i and hence having v_i as common knowledge between the intermediary and seller i. This does not change the optimal recommendations rule: the intermediary discloses v_i to and extract the entire value-dependent willingness to pay $w(v_i)$ from each seller i in the baseline model under which the seller does not know v_i. Therefore, the optimal recommendations rule letting seller to learn $w(v_i)$ does not change the optimal recommender system.

Another way to relax the assumption is to assume that each seller privately observes a value-dependent willingness to pay in the following way similar to Eső and Szentes (2007): Suppose that the intermediary can disclose[18] v_i to a seller i. Upon disclosure, the seller i privately observes w_i independently drawn from a common distribution $W(. \mid v_i)$ conditioning on v_i, where $W(. \mid v)$ first-order dominates $W(. \mid v')$ whenever $v > v'$. Without disclosure, the seller does not learn any information about v_i and w_i.

Following Eső and Szentes (2007), it can be shown that the intermediary completely discloses its private information $\boldsymbol{v}$ under an optimal recommender system and obtains the same expected revenue as if the intermediary could observe $\boldsymbol{w}$ using the modified handicap auction obtained by replacing $w(v_i)$ with w_i from Theorem 9. A sketch of the proof is presented here.

Let $v_0 \in (\underline{v}, \bar{v})$ and G be such that $\frac{1-G(\theta)}{g(\theta)}$ decreases.. Suppose that the intermediary has disclosed v_i to each seller i. A recommender system is now extended to $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{W}} \to [0,1]^{N+1} \times \mathbb{R}^N$ with $\sum_{i \in \mathcal{N}\{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{W}}$ where $\mathcal{W}$ is a support

[18] An alternate setup under which the intermediary can provide any arbitrary information to sellers, instead of being restricted to disclosing or not, leads to the same conclusion. The revenue from (44) and (45) still is an upper bound of the intermediary's revenue which can be attained by first fully disclosing $\boldsymbol{v}$ to sellers and then running the modified handicap auction from Theorem 9

of $W(.\,|\,.)$ and $\boldsymbol{\mathcal{W}} = \mathcal{W}^N$. An optimal recommendations rule is given by

$$r_i^W(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^W|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^W(\boldsymbol{v})}_{\substack{\text{cost of}\\ \text{persuasion}}}, 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

where $\mathcal{M}^W = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j - \ell_j^W(\boldsymbol{v})\right\}$, and

$$\ell_i^W(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^W, \bar{v}^W] \\ \lambda_1^W(v_0)\cdot(v_0 - v_i) & \text{if } v_0 > \bar{v}^W \\ \lambda_2^W(v_0)\cdot\sum_{k\in\mathcal{N}}(v_0 - v_k) & \text{if } v_0 < \underline{v}^W \end{cases} \tag{44}$$

where $\lambda_1^W(v_0)$ and $\lambda_2^W(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively, and $\underline{v}^W$ and $\bar{v}^W$ are the thresholds constructed from the uncontrained optimal recommendations rule. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{w})d\tilde{\theta}_i \tag{45}$$

for each i.

The revenue from (44) and (45) clearly is an upper bound of the intermediary's revenue. This revenue can be attained by first fully disclosing $\boldsymbol{v}$ to sellers and then running a modified handicap auction as the following.

Theorem 13. *Let $v_0 \in (\underline{v}, \bar{v})$. Suppose that $\frac{1-G(\theta_i)}{g_i(\theta_i)}$ weakly decreases in θ_i. The intermediary can implement the optimal recommendations rule with the same revenue via a handicap auction $(c, p, \ell_i)_{i\in\mathcal{N}}$ where*

$$p(\theta_i) = \frac{1-G(\theta_i)}{g(\theta_i)}, \tag{46}$$

$c(\theta_i)$ is defined by

$$\begin{aligned} c(\theta_i) =& E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - p(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - p(\theta_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] \\ &- \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i},\boldsymbol{v}}\left[\mathbf{1}_{\{\tilde{\theta} + w(v_i) - p(\tilde{\theta}) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right]d\tilde{\theta} \end{aligned} \tag{47}$$

and the cost of persuasion $(\ell_i^W)_{i\in\mathcal{N}}$ from (44).

8 Conclusion

In this paper, I study a monopolistic intermediary designing a recommender system. I frame the intermediary's problem as a revenue-maximizing mechanism design problem of allocating one unit of sales to one of multiple sellers, but with a constraint of having to rely on the consumer to choose the outcome. I reformulate the intermediary's revenue-maximizing mechanism design problem as a Bayesian persuasion problem. Using value-switching monotone recommendations rules, I find that the intermediary recommends the product of the seller with the highest virtual willingness to pay adjusted by the cost of persuasion.

I use this model to explore policy-relevant questions. First, I characterize the types of seller data that benefit or harm consumers and sellers. Second, I find that the optimal direct mechanism protects consumer privacy, but consumer data is leaked to sellers under other implementations. Lastly, I show that the welfare-maximizing recommender system increases consumer surplus, but reduces the joint profit of the intermediary and sellers.

There are several directions for future works. To start, endogenizing prices leads to a number of economic and technical questions. Should the prices be set by the intermediary or by sellers, and at what timing? How does the recommender system affects price competition among sellers? How should the pricing and recommendations rule jointly be determined? Furthermore, allowing consumers to have ex-ante asymmetric, privately known preferences over products would also be an interesting direction. In this extension, the consumer has multi-dimensional private information about his preference. The key challenge lies in tractably characterizing the recommendations rules that make consumers report their type truthfully. Finally, considering competition among intermediaries is another important direction for both theory and practice. Although analyzing competition among mechanism designers and persuaders is generally difficult, characterizing the optimal recommender system in this environment would provide valuable insights into the strategic interactions of intermediaries with growing capabilities and influence.

A Proofs for Section 3 and Section 4

A.1 Lemma 2

For a seller $i \in \mathcal{N}$ with θ_i reports truthfully as θ_i, let

$$\Pi(\theta_i) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} [(\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta}) - t_i(\boldsymbol{v},\boldsymbol{\theta})]\,\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit, and

$$Q_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected probability of recommending i's product.

Let us first prove the following lemma that characterizes incentive compatible and individually recommender system.

Lemma 10. *A recommender system $(\boldsymbol{r},\boldsymbol{t}) : \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}} \to [0,1]^{N+1}\times\mathbb{R}^N$ is incentive compatible, individually rational and obedient if and only if for each $i \in \mathcal{N}$, for all $\theta_i \in [\underline{\theta},\bar{\theta}]$ and $z_i \in \mathcal{Z}$,*

$$Q_i(\theta_i) \text{ is increasing in } \theta_i, \tag{48}$$

$$\Pi_i(\theta_i) = \Pi_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i, \tag{49}$$

$$\Pi_i(\underline{\theta}) \geq 0 \tag{50}$$

and for all $i, j \in \mathcal{N}\cup\{0\}$,

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} v_i r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &\geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} v_i r_j(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}).\end{aligned} \tag{51}$$

Proof. **Necessity:** Let $\theta_i > \hat{\theta}_i$. Let

$$\pi_i(\hat{\theta}_i;\theta_i) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} \left[(\theta_i + w(v_i))r_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i}) - t_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i})\right]\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit of the seller with θ_i when he reports as $\hat{\theta}_i$. Note that

$$\pi_i(\hat{\theta}_i;\theta_i) = \Pi_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i)$$

Similarly,

$$\pi_i(\theta_i;\hat{\theta}_i) = \Pi_i(\theta_i) + (\theta_i - \hat{\theta}_i)Q_i(\theta_i).$$

Incentive compatibility $\Pi_i(\theta_i) \geq \pi_i(\hat{\theta}_i;\theta_i)$ and $\Pi_i(\hat{\theta}_i) \geq \pi_i(\theta_i;\hat{\theta}_i)$, which in turn implies

$$(\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i) \leq \Pi_i(\theta_i) - \Pi_i(\hat{\theta}_i) \leq (\theta_i - \hat{\theta}_i)Q_i(\theta_i).$$

By the above inequality, Q_i is weakly increasing and hence is integrable, which then implies (49).

Individual rationality is equivalent to $\Pi_i(\theta_i) \geq 0$ for all $i \neq 0$ and $\theta_i \in [\underline{\theta}, \bar{\theta}]$, from which $\Pi_i(\underline{\theta}) \geq 0$ for all i follows.

Sufficiency: Let $\theta_i \neq \hat{\theta}_i$. From (49) and the monotonicity of Q_i,

$$\begin{aligned}
\Pi_i(\theta_i) &= \Pi_i(\hat{\theta}) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i \\
&\geq \Pi_i(\hat{\theta}) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\hat{\theta}_i)d\tilde{\theta}_i \\
&= \Pi_i(\hat{\theta}) + (\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i) \\
&= \pi_i(\hat{\theta}_i;\theta_i)
\end{aligned} \tag{52}$$

Since $\theta_i \neq \hat{\theta}_i$ are arbitrary, (52) implies incentive compatibility.

Since $Q_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$, (49) implies that $\Pi_i(\theta_i)$ increases in θ_i, and hence, individual rationality is satisfied if $\Pi_i(\underline{\theta}) \geq 0$. □

By Lemma 13, for any $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and $z_i \in \mathcal{Z}$,

$$\Pi_i(\theta_i) = \Pi_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i.$$

The expected transfer of the seller i with θ_i is

$$T_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i - \Pi_i(\underline{\theta}). \tag{53}$$

By the usual argument of the change of variables, for each $z_i \in \mathcal{Z}$, we have

$$\begin{aligned}
&\int_{\Theta} T_i(\theta_i)G(d\theta_i) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \left(\theta_i + w(v_i) - \frac{1-G(\theta_i)}{g(\theta_i)}\right) r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) - \Pi_i(\underline{\theta}),
\end{aligned}$$

so that the intermediary's expected revenue is

$$
\begin{aligned}
&\sum_{i\in\mathcal{N}} \int_{\Theta} T_i(\theta_i) G(d\theta_i) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i + w(v_i) - \frac{1-G(\theta_i)}{g(\theta_i)}\right) r_i(\boldsymbol{v},\boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \Pi_i(\underline{\theta}).
\end{aligned} \tag{54}
$$

The intermediary's problem is to maximize (54) using a recommender system $(\boldsymbol{r},\boldsymbol{t})$ subject to monotonicity constraints (48), payoff equivalence constraints (49), non-negativity constraints (50) and obedience constraints (51). Note that for any given recommendations rule $\boldsymbol{r}$ satisfying (48) and (51), any transfer $\boldsymbol{t}$ such that $\pi_i(\underline{\theta}) = 0$ for all $i \in \mathcal{N}$ and whose interim transfer satisfies (53) maximizes (7) while satisfying (49) and (50). Transfer (8) is one of such.

It remains to find an optimal recommendations rule $\boldsymbol{r}$. Since $\Pi_i(\underline{\theta}) = 0$ for all $i \in \mathcal{N}$ independent of $\boldsymbol{r}$, it immediately follows that a recommendations rule $\boldsymbol{r}$ that maximizes (7) subject to (48) and (51), together with the corresponding transfer (8), maximizes the intermediary's expected revenue subject to (48), (49), (50) and (51).

A.2 Lemma 3

The obedience constraint from a product $i \in \mathcal{N}$ to another $j \in \mathcal{N}$ is

$$
\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}} \int_{\boldsymbol{\Theta}} (v_i - v_j) r_i(v_i, v_j, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \int_{v>v'} \int_{\boldsymbol{\mathcal{V}}_{-ij}} \int_{\boldsymbol{\Theta}} (v - v') \Big(r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) - r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \Big) \boldsymbol{F}_{ij}(d(v,v')) \boldsymbol{F}_{-ij}(d\boldsymbol{v}_{-ij}) \boldsymbol{G}(d\boldsymbol{\theta}) \\
&\geq 0
\end{aligned}
$$

where the first equality follows from the assumption that $\boldsymbol{F}$ is symmetric and hence $\boldsymbol{F}_{ij}(d(v,v')) = \boldsymbol{F}_{ij}(d(v',v))$, and the last inequality follows from $v > v'$ and $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) - r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \geq 0$ by the value-switching monotonicity of $\boldsymbol{r}$.

A.3 Lemma 4

1. Let $i \in \mathcal{N}$. OB_{i0} is

$$
\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_i - v_0) r_i(\boldsymbol{v},\boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{55}
$$

Define $R_i(v_i) = \int_{\boldsymbol{\mathcal{V}}_{-i}\times\boldsymbol{\Theta}} r_i(v_i, \boldsymbol{v}_{-i}, \boldsymbol{\theta})\boldsymbol{F}_{-i}(d\boldsymbol{v}_{-i})\boldsymbol{G}(d\boldsymbol{\theta})$ which is increasing in v_i. Then, (55) is equivalent to

$$Cov_{v_i}(v_i, R_i(v_i)) + (\mathbb{E}_{v_i}(v_i) - v_0)\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v}, \boldsymbol{\theta})) \geq 0. \tag{56}$$

The first term $Cov_{v_i}(v_i, R_i(v_i))$ is non-negative since $R_i(v_i)$ is increasing in v_i. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v}, \boldsymbol{\theta})) = 0$, then (56) always holds, so that $\bar{v}_i = \infty$. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v}, \boldsymbol{\theta})) > 0$, then (56) holds if and only if $v_0 \leq \bar{v}_i$ where $\bar{v}_i = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{v_i}(v_i, R_i(v_i))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))} \geq \mathbb{E}_{v_i}(v_i)$. In any case, there is $\bar{v}_i \geq \mathbb{E}_{v_i}(v_i)$ such that OB_{i0} holds if and only if $v_0 \leq \bar{v}_i$.

2. Let $i \in \mathcal{N}$. OB_{0i} is

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) r_0(\boldsymbol{v}, \boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{57}$$

Define $R_0(v_i) = \int_{\boldsymbol{\mathcal{V}}_{-i}\times\boldsymbol{\Theta}} r_0(v_i, \boldsymbol{v}_{-i}, \boldsymbol{\theta})\boldsymbol{F}_{-i}(d\boldsymbol{v}_{-i})\boldsymbol{G}(d\boldsymbol{\theta})$ which is decreasing in v_i. Then, (57) is equivalent to

$$-Cov_{v_i}(v_i, R_0(v_i)) - (\mathbb{E}_{v_i}(v_i) - v_0)\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v}, \boldsymbol{\theta})) \geq 0. \tag{58}$$

The first term $Cov_{v_i}(v_i, R_0(v_i))$ is non-positive since $R_0(v_i)$ is decreasing in v_i. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v}, \boldsymbol{\theta})) = 0$, then (58) always holds, so that $\underline{v}_i = 0$. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v}, \boldsymbol{\theta})) > 0$, then (58) holds if and only if $v_0 \geq \underline{v}_i$ where $\underline{v}_i = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{v_i}(v_i, R_i(v_i))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))} \leq \mathbb{E}_{v_i}(v_i)$. In any case, there is $\underline{v}_i \leq \mathbb{E}_{v_i}(v_i)$ such that OB_{i0} holds if and only if $v_0 \geq \bar{v}_i$.

A.4 Theorem 1.a

I first show that a symmetric recommender system attains the optimal revenue. Recall that recommendations rule $\boldsymbol{r}$ is symmetric if for any $i \in \mathcal{N}$, any bijective function $\iota : \mathcal{N} \to \mathcal{N}$ and any $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i(\boldsymbol{v}, \boldsymbol{\theta}) = r_{\iota(i)}(\boldsymbol{v}^\iota, \boldsymbol{\theta}^\iota)$$

where $(\boldsymbol{v}^\iota, \boldsymbol{\theta}^\iota)$ is such that $v^\iota_{\iota(i)} = v_i$ and $\theta^\iota_{\iota(i)} = \theta_i$ for all $i \in \mathcal{N}$.

Lemma 11. *For each obedient recommendations rule $\boldsymbol{r}$, there is a symmetric recommendations rule $\boldsymbol{r}^0$ that is obedient and attains the same revenue as $\boldsymbol{r}$.*

Proof. Let us first construct a symmetric recommendations rule $\boldsymbol{r}^0$ from any given obedient recommendations rule $\boldsymbol{r}$. Let $\boldsymbol{r}$ be a recommendations rule. Let

$$\mathcal{I}^{\mathcal{N}} = \{\iota^\dagger \mid \iota : \mathcal{N} \to \mathcal{N} \text{ is a bijective function}\}$$

be a set of all permutation functions on $\mathcal{N}$. For each $\iota^\dagger \in \mathcal{I}^{\mathcal{N}}$, let $\boldsymbol{r}^{\iota^\dagger}$ be a recommendations rule obtained by permutating $\boldsymbol{r}$ according to $\iota^\dagger$, i.e. for each $i \in \mathcal{N}$,

$$r_i^{\iota^\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) = r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger}, \boldsymbol{\theta}^{\iota^\dagger}).$$

Then, $\boldsymbol{r}^\dagger$ satisfies obedience constraints as well. Define another recommendatiosn rule $\boldsymbol{r}^0$ such that for each $i \in \mathcal{N}$ and $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i^0(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r^{\iota^\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger}, \boldsymbol{\theta}^{\iota^\dagger}). \tag{59}$$

To prove that $\boldsymbol{r}^0$ is symmetric, it is sufficient to show that for any bijection $\iota \in \mathcal{I}^{\mathcal{N}}$ and $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i^0(\boldsymbol{v}, \boldsymbol{\theta}) = r_{\iota(i)}^0(\boldsymbol{v}^{\iota}, \boldsymbol{\theta}^{\iota}). \tag{60}$$

To show this, note that

$$r_{\iota(i)}^0(\boldsymbol{v}^{\iota}, \boldsymbol{\theta}^{\iota}) = \frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger \circ \iota(i)}(\boldsymbol{v}^{\iota^\dagger \circ \iota}, \boldsymbol{\theta}^{\iota^\dagger \circ \iota}) \tag{61}$$

where $\iota^\dagger \circ \iota$ is a composition of two permuntation functions. Note that there is a bijective mapping between $\mathcal{I}^{\mathcal{N}}$ and $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$. To show this, since both $\mathcal{I}^{\mathcal{N}}$ and $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ are finite sets, it is sufficient to show that $\mathcal{I}^{\mathcal{N}} = \{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$. To show the equality, first note that $\iota^\dagger \circ \iota$ is a composition of two bijective mappings and hence is a bijection, i.e. $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\} \subset \mathcal{I}^{\mathcal{N}}$. To show the other inclusion, let $\tilde{\iota} \in \mathcal{I}^{\mathcal{N}}$. Since both $\tilde{\iota}$ and ι are bijection over the same finite space, I can define $\iota^\dagger = \tilde{\iota} \cdot \iota^{-1}$ which is a composition of two bijections and hence a well-defined bijection over $\mathcal{N}$. By construction, for each $j \in \mathcal{N}$, $\iota^\dagger \circ \iota(j) = \iota^\dagger(\iota(j)) = \tilde{\iota}(j)$, and hence $\tilde{\iota} = \iota^\dagger \circ \iota$ for some $\iota^\dagger \in \mathcal{I}^{\mathcal{N}}$. In other words, $\tilde{\iota} \in \{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ and hence $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\} \supset \mathcal{I}^{\mathcal{N}}$, which gives the desired equality. That there is a bijective mapping between $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ and $\mathcal{I}^{\mathcal{N}}$ implies that the right-hand side of (61) is

$$\frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger \circ \iota(i)}(\boldsymbol{v}^{\iota^\dagger \circ \iota}, \boldsymbol{\theta}^{\iota^\dagger \circ \iota}) = \frac{1}{N!} \sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger}, \boldsymbol{\theta}^{\iota^\dagger}) = r_i^0(\boldsymbol{v}, \boldsymbol{\theta}),$$

and therefore, (60) holds.

It remains to show that $\boldsymbol{r}^0$ is obedient and attains the equal revenue. These results follow from the linearity of the revenue and obedience constraints. For each $i, j \in \mathcal{N}$, the obedience constraint from i to j for $\boldsymbol{r}^0$ is

$$\begin{aligned}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i - v_j)r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i - v_j)\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v^{\iota^\dagger}_{\iota^\dagger(i)} - v^{\iota^\dagger}_{\iota^\dagger(j)})r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\ &\geq 0 \end{aligned}\tag{62}$$

where the third equality follows from the definition that $v^{\iota^\dagger}_{\iota^\dagger(i)} = v_i$ and $\theta^{\iota^\dagger}_{\iota^\dagger(i)} = \theta_i$, and the last inequality follows from the fact that $\boldsymbol{r}^{\iota^\dagger}$ is obtained by permutating $\boldsymbol{r}$ which is obedient, and hence, so is $\boldsymbol{r}^{\iota^\dagger}$.

For $i \in \mathcal{N}$, the obedience constraint from i to 0 is

$$\begin{aligned}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i - v_0)r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i - v_0)\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v^{\iota^\dagger}_{\iota^\dagger(i)} - v_0)r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\ &\geq 0 \end{aligned}\tag{63}$$

where the last inequality follows from OB_{i0} for each $\boldsymbol{r}^{\iota^\dagger}$, so that OB_{i0} is satisfied for $\boldsymbol{r}^0$. The obedience constraint from 0 to i is

$$\begin{aligned}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0 - v_i)r_0^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0 - v_i)\left(1 - \sum_{j\in\mathcal{N}} r_j^0(\boldsymbol{v},\boldsymbol{\theta})\right)\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0 - v_i)\left(1 - \sum_{j\in\mathcal{N}}\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(j)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\right)\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0 - v^{\iota^\dagger}_{\iota^\dagger(i)})\left(1 - \sum_{j\in\mathcal{N}} r_{\iota^\dagger(j)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\right)\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0 - v^{\iota^\dagger}_{\iota^\dagger(i)})r_0(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\ &\geq 0 \end{aligned}\tag{64}$$

where the last inequality follows from OB_{0i} for each $\boldsymbol{r}^{\iota^\dagger}$, so that OB_{0i} is satisfied for $\boldsymbol{r}^0$. By (62), (63) and (64), $\boldsymbol{r}^0$ is obedient.

It remains to verify that $\boldsymbol{r}$ and $\boldsymbol{r}^0$ attain the same revenue. Note that every $\boldsymbol{r}^{\iota^\dagger}$ has the same revenue as $\boldsymbol{r}$ because $\boldsymbol{r}^{\iota}$ is obtained by permutating $\boldsymbol{r}$ according to $\iota^\dagger$. Consequently, their average must be the same as the revenue obtained by $\boldsymbol{r}$ as shown below.

$$
\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i-\frac{1-G(\theta_i)}{g(\theta_i)}+w(v_i)\right)r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&=\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i-\frac{1-G(\theta_i)}{g(\theta_i)}+w(v_i)\right)\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&=\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta^{\iota^\dagger}_{\iota^\dagger(i)}-\frac{1-G(\theta^{\iota^\dagger}_{\iota^\dagger(i)})}{g(\theta^{\iota^\dagger}_{\iota^\dagger(i)})}+w(v^{\iota^\dagger}_{\iota^\dagger(i)})\right)r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\
&=\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i-\frac{1-G(\theta_i)}{g(\theta_i)}+w(v_i)\right)r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&=\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i-\frac{1-G(\theta_i)}{g(\theta_i)}+w(v_i)\right)r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}).
\end{aligned}
$$

□

By Lemma 11, there always exists a symmetric recommendations rule that maximizes the revenue. From here on, I focus on symmetric recommendations rules. The following lemma states that the following symmetric recommendations rule is value-switching monotone and recommends one of the options with certainty almost surely.

Lemma 12. *For each $i\in\mathcal{N}$, define*

$$\psi_i(\boldsymbol{v},\boldsymbol{\theta})=\xi_i(v_i,\theta_i)+\xi_0(\boldsymbol{v})$$

where ξ_i strictly increases in (θ_i,v_i), and ξ_0 is common across the products and is symmetric and increases in $\boldsymbol{v}$ that could possibly be 0. Ignoring ties, let, for $i\in\mathcal{N}$,

$$r_i(\boldsymbol{v},\boldsymbol{\theta})=1 \text{ if } i=\operatorname*{arg\,max}_{j\in\mathcal{N}}\big(\psi_j(\boldsymbol{v},\boldsymbol{\theta}),0\big)$$

and $r_0(\boldsymbol{v},\boldsymbol{\theta})=1-\sum_{i\in\mathcal{N}}r_i(\boldsymbol{v},\boldsymbol{\theta})$. Then, $\boldsymbol{r}$ is value-switching monotone almost surely and $r_j(\boldsymbol{v},\boldsymbol{\theta})=1$ for some $j\in\mathcal{N}\cup\{0\}$ almost surely.

Proof. Since ξ_i strictly increases and ξ_0 increases in (v_i,θ_i), almost surely, for each $(\boldsymbol{v},\boldsymbol{\theta})\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}$, there is $i\in\mathcal{N}$ $\psi_i(\boldsymbol{v},\boldsymbol{\theta})>\max_{j\in\mathcal{N}\setminus\{i\}}(\psi_j(\boldsymbol{v},\boldsymbol{\theta}),0)$ or $0>\max_{j\in\mathcal{N}}\psi_j(\boldsymbol{v},\boldsymbol{\theta})$, so that $r_j(\boldsymbol{v},\boldsymbol{\theta})=1$ for some $j\in\{0\}\cup\mathcal{N}$ almost surely.

Let $\boldsymbol{\mathcal{W}}=\{(\boldsymbol{v},\boldsymbol{\theta})\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\mid r_i(\boldsymbol{v},\boldsymbol{\theta})=1 \text{ for some } i\in\mathcal{N}\cup\{0\}\}$. Since $(\boldsymbol{v},\boldsymbol{\theta})\in\boldsymbol{\mathcal{W}}$ almost surely, to show almost sure value-switching monotonicity, it is sufficient to show establish that for any $i\in\mathcal{N}$, $(v_i',\boldsymbol{v}_{-i},\theta_i',\boldsymbol{\theta}_{-i})$, $(v_i,\boldsymbol{v}_{-i},\theta_i,\boldsymbol{\theta}_{-i})\in\boldsymbol{\mathcal{W}}$ such that $(v_i,\theta_i)\geq(v_i',\theta_i')$,

1. $r_0(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 0$ implies $r_0(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 0$,

2. $r_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 1$ implies $r_i(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 1$,

and for any $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$, $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ where $v > v'$,

3. $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$ implies $r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$ for any $v > v'$.

To show the first item, let $(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), (v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\mathcal{W}}$ be such that $(v_i', \theta_i') \leq (v_i, \theta_i)$ and $r_0(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 0$. By definition, $\max_{j \in \mathcal{N}} \psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) > 0$. Increasing from (v_i', θ_i') to (v_i, θ_i) increases $\psi_j(\boldsymbol{v}, \boldsymbol{\theta})$ for all $j \in \mathcal{N}$, so that $\max_{j \in \mathcal{N}} \psi_j(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \geq \max_{j \in \mathcal{N}} \psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) > 0$, and hence, $r_0(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 0$.

To show the second item, let $(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), (v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\mathcal{W}}$ be such that $(v_i', \theta_i') \leq (v_i, \theta_i)$ and $r_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}) = 1$. By definition, $\psi_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}) > \max_{j \in \mathcal{N} \setminus \{i\}}(\psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), 0)$. Increasing from (v_i', θ_i') to (v_i, θ_i) strictly increases ξ_i and increases ξ_0, but not ξ_j. Therefore, $\psi_i = \xi_i + \xi_0$ increases more than $\psi_j = \xi_j + \xi_0$ for all $j \in \mathcal{N} \setminus \{i\}$. Consequently, $\psi_i(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}) > \max_{j \in \mathcal{N} \setminus \{i\}}(\psi_j(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}), 0)$, and hence, $r_i(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 1$.

To show the last item, let $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$, $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ where $v > v'$ and $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$. By definition, $\psi_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) < \max_{k \in \mathcal{N} \setminus \{i\}}(\psi_k(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}), 0)$. Changing from $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$ to $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$ only strictly decreases $\xi_i(v_i, \theta_i)$ and strictly increases $\xi_j(v_j, \theta_j)$ while leaving others (note that $\xi_0(\boldsymbol{v})$ remains the same by the symmetry of ξ_0). Therefore, only ψ_i strictly decreases and ψ_j strictly increases, while ψ_k remains the same for all $k \in \mathcal{N} \setminus \{i, j\}$. Consequently, $\psi_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) < \psi_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) < \max_{k \in \mathcal{N} \setminus \{i\}}(\psi_k(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}), 0) \leq \max_{k \in \mathcal{N} \setminus \{i\}}(\psi_k(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}), 0)$, and hence, $r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$. □

By Lemma 11, there is a symmetric optimal recommender system. The rest of the proof focuses on constructing a symmetric optimal recommender system.

Let $v_0 \in [\underline{v}^*, \bar{v}^*]$. The unconstrained optimal recommendations rule obtained ignoring the obedience constraints $\boldsymbol{\rho}^*$ as in (11) is obedient and hence optimal. In other words, $\boldsymbol{r}^* = \boldsymbol{\rho}^*$ and $\ell_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = 0$ for all $i \in \mathcal{N}$ and $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.

Let $v_0 > \bar{v}^*$. Then, $\boldsymbol{\rho}^*$ violates OB_{i0}. Also, since $v_0 > \bar{v}^* \geq \underline{v}^*$, any value-switching monotone recommendations rule such that $r_j(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely satisfies OB_{0i}. At an optimal symmetric value-switching monotone recommendations rule such that $r_j(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely, OB_{i0} are binding; otherwise, none of the constraints bind which would imply that $\boldsymbol{\rho}^*$ is the optimal recommendations rule which is known to violate OB_{i0}. Taking the Lagrangian, the optimal recommendations rule is characterized by

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \lambda(v_0 - v_j), 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

where $\mathcal{M} = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v}) \right\}$, λ is a Lagrangian multiplier of OB_{i0} that makes OB_{i0} binding.

Let $v_0 < \underline{v}^*$. Then, $\boldsymbol{\rho}^*$ violates OB_{0i}. Also, since $v_0 < \underline{v}^* \leq \underline{v}^*$, any value-switching monotone recommendations rule such that $r_j(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely satisfies OB_{0i}. At an optimal symmetric value-switching monotone recommendations rule such that $r_j(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely, OB_{0i} are binding; otherwise, none of the constraints bind which would imply that $\boldsymbol{\rho}^*$ is the optimal recommendations rule which is known to violate OB_{0i}. Taking the Lagrangian, the optimal recommendations rule is characterized by

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \lambda \sum_{k \in \mathcal{N}} (v_0 - v_k), 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{66}$$

where $\mathcal{M} = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v}) \right\}$, λ is a Lagrangian multiplier of OB_{i0} that makes OB_{i0} binding.

B Proofs for Section 5

B.1 Optimal Recommender System with Additional Information

This section characterizes incentive compatible, individually rational and obedient recommender system, and recast the intermediary's problem to a Bayesian persuasion problem as in the baseline model.

For a seller $i \in \mathcal{N}$ with (θ_i, z_i) reporting truthfully as θ_i, let

$$\Pi(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} \left[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit, and

$$Q(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected probability of recommending i's product.

Lemma 13. *A recommender system* $(\boldsymbol{r},\boldsymbol{t}):\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}\to[0,1]^{N+1}\times\mathbb{R}^N$ *is incentive compatible, individually rational and obedient if and only if for each* $i\in\mathcal{N}$, *for all* $\theta_i\in[\underline{\theta},\bar{\theta}]$ *and* $z_i\in\mathcal{Z}$,

$$Q_i(\theta_i,z_i) \text{ is increasing in } \theta_i, \tag{67}$$

$$\Pi_i(\theta_i,z_i)=\Pi_i(\underline{\theta},z_i)+\int_{\underline{\theta}}^{\theta_i}Q_i(\tilde{\theta}_i,z_i)d\tilde{\theta}_i, \tag{68}$$

$$\Pi_i(\underline{\theta})\geq 0 \tag{69}$$

and for all $i,j\in\mathcal{N}\cup\{0\}$,

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}}r_i(\boldsymbol{v},\theta_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})\\ \geq&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}}r_i(\boldsymbol{v},\theta_i',\boldsymbol{\theta}_{-i},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).\end{aligned} \tag{70}$$

Proof. **Necessity:** Let $\theta_i>\hat{\theta}_i$ and $z_i\in\mathcal{Z}$. Let

$$\pi_i(\hat{\theta}_i;\theta_i,z_i)=\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}_{-i}}\left[(\theta_i+w(v_i))r_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})-t_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})\right]\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i}\mid\boldsymbol{\theta}_{-i})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit of the seller with (θ_i,z_i) when he reports as $\hat{\theta}_i$. Note that

$$\pi_i(\hat{\theta}_i;\theta_i,z_i)=\Pi_i(\hat{\theta}_i,z_i)+(\theta_i-\hat{\theta}_i)Q_i(\hat{\theta}_i,z_i)$$

Similarly,

$$\pi_i(\theta_i;\hat{\theta}_i,z_i)=\Pi_i(\theta_i,z_i)+(\theta_i-\hat{\theta}_i)Q_i(\theta_i,z_i).$$

Incentive compatibility $\Pi_i(\theta_i,z_i)\geq\pi_i(\hat{\theta}_i;\theta_i,z_i)$ and $\Pi_i(\hat{\theta}_i,z_i)\geq\pi_i(\theta_i;\hat{\theta}_i,z_i)$, which in turn implies

$$(\theta_i-\hat{\theta}_i)Q_i(\hat{\theta}_i)\leq\Pi_i(\theta_i)-\Pi_i(\hat{\theta}_i)\leq(\theta_i-\hat{\theta}_i)Q_i(\theta_i,z_i).$$

By the above inequality, Q_i is weakly increasing and hence is integrable, which then implies (68).

Individual rationality is equivalent to $\Pi_i(\theta_i)\geq 0$ for all $i\neq 0$ and $\theta_i\in[\underline{\theta},\bar{\theta}]$, from which $\Pi_i(\underline{\theta})\geq 0$ for all i follows.

Sufficiency: Let $\theta_i \neq \hat{\theta}_i$. From (68) and the monotonicity of Q_i,

$$\begin{aligned}
\Pi_i(\theta_i, z_i) &= \Pi_i(\hat{\theta}, z_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i \\
&\geq \Pi_i(\hat{\theta}, z_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\hat{\theta}_i, z_i) d\tilde{\theta}_i \\
&= \Pi_i(\hat{\theta}, z_i) + (\theta_i - \hat{\theta}_i) Q_i(\hat{\theta}_i, z_i) \\
&= \pi_i(\hat{\theta}_i; \theta_i, z_i) \qquad (71)
\end{aligned}$$

Since $\theta_i \neq \hat{\theta}_i$ are arbitrary, (71) implies incentive compatibility.

Since $Q_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$, (68) implies that $\Pi_i(\theta_i)$ increases in θ_i, and hence, individual rationality is satisfied if $\Pi_i(\underline{\theta}) \geq 0$. □

Lemma 14. *Suppose that a recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^N$ maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \qquad (72)$$

subject to OB and monotonicity. Suppose also that

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}, \boldsymbol{z}) d\tilde{\theta}_i. \qquad (73)$$

Then, $(\boldsymbol{r}, \boldsymbol{t})$ is an optimal recommender system.

Proof. By Lemma 13, for any $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and $z_i \in \mathcal{Z}$,

$$\Pi_i(\theta_i, z_i) = \Pi_i(\underline{\theta}, z_i) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i.$$

The expected transfer of the seller i with (θ_i, z_i) is

$$T_i(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) \boldsymbol{H}(d\boldsymbol{z}_{-i}) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i - \Pi_i(\underline{\theta}, z_i). \qquad (74)$$

By the usual argument of the change of variables, for each $z_i \in \mathcal{Z}$, we have

$$\begin{aligned}
&\int_{\Theta \times \mathcal{Z}} T_i(\theta_i, z_i) G(d\theta_i \mid z_i) H(dz_i) \\
&= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \left(\theta_i + w(v_i) - \frac{1 - G(\theta_i)}{g(\theta_i)} \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - \int_{\mathcal{Z}} \Pi_i(\underline{\theta}, z_i) H(dz_i),
\end{aligned}$$

so that the intermediary's expected revenue is

$$
\begin{aligned}
&\sum_{i \in \mathcal{N}} \int_{\Theta \times \mathcal{Z}} T_i(\theta_i, z_i) G(d\theta_i \mid z_i) H(dz_i) \\
&= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i + w(v_i) - \frac{1 - G(\theta_i)}{g(\theta_i)} \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - \int_{\boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \Pi_i(\underline{\theta}, z_i) \boldsymbol{H}(d\boldsymbol{z}).
\end{aligned}
\tag{75}
$$

The intermediary's problem is to maximize (75) using a recommender system $(\boldsymbol{r}, \boldsymbol{t})$ subject to monotonicity constraints (67), payoff equivalence constraints (68), non-negativity constraints (69) and obedience constraints (70). Note that for any given recommendations rule $\boldsymbol{r}$ satisfying (67) and (70), any transfer $\boldsymbol{t}$ such that $\pi_i(\underline{\theta}, z_i) = 0$ for all $i \in \mathcal{N}$ and $z_i \in \mathcal{Z}$ and whose interim transfer satisfies (74) maximizes (75) while satisfying (68) and (69). Transfer (73) is one of such.

It remains to find an optimal recommendations rule $\boldsymbol{r}$. Since $\Pi_i(\underline{\theta}, z_i) = 0$ for all $i \in \mathcal{N}$ and $z_i \in \mathcal{Z}$ independent of $\boldsymbol{r}$, it immediately follows that a recommendations rule $\boldsymbol{r}$ that maximizes (72) subject to (67) and (70), together with the corresponding transfer (73), maximizes the intermediary's expected revenue subject to (67), (68), (69) and (70). $\square$

B.2 Additional Information as a Change in the Intermediary's Preference

This section presents four equivalent ways to express the intermediary's problem without additional information, each interpreted as: 1. the intermediary's problem without additional information; 2. the intermediary's problem with additional information but with invariance constraints; 3. additional information as a change in preference; 4. additional information as relaxation of invariance constraints.

Lemma 15. *The followings are solution equivalent (after adjusting for invariance constraints related notations):*

1. *A recommendations rule without additional infromation $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ that maximizes*

$$
\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \tag{76}
$$

subject to monotonicity constraints without additional information, for all $i \in \mathcal{N}$ and

$\theta_i > \theta_i'$

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v},\theta_i,\boldsymbol{\theta}_{-i})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i})\\ &\geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v},\theta_i',\boldsymbol{\theta}_{-i})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i})\end{aligned} \tag{77}$$

and obedience constraints without additional information, for all $i, j \in \mathcal{N} \cup \{0\}$,

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} v_i r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &\geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} v_j r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}).\end{aligned} \tag{78}$$

2. *A recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ that maximizes*

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \tag{79}$$

subject to monotonicity constraints (67), obedience constraints (70) and invariance constraints

$$r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}') \text{ for all } \boldsymbol{z},\boldsymbol{z}' \in \boldsymbol{\mathcal{Z}}. \tag{80}$$

3. *A recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ that maximizes (79) subject to monotonicity constraints (67) and obedience constraints (70).*

4. *A recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ that maximizes (72) subject to monotonicity constraint (67), obedience constraints (70) and invariance constraints (80).*

The first is the intermediary's problem without additional information after substituting the expected transfer with virtual willingness to pay using the standard arguments. The second is a reformulation of the first under the setup with additional information. The third is stating that invariance constraints (80) are redundant in the second, because the integrands both in the objective function and the constraints do not depend on $\boldsymbol{z}$.

Note that the set of constraints are identical under the third and the intermediary's problem with additional information. The only difference between the two problems is the objective functions. In other words, additional information changes the intermediary's objective function from (79) to (72) subject to the *same* constraints, i.e. 'additional information as a change in the intermediary's preference,' the idea used for the consumer surplus analysis.

This means that the baseline problem without additional information can be understood as maximizing the same objective function but with added invariance constraints in relative to . That is, additional information is a *deletion* of invariance constraints withe the *same* objective function, i.e. 'additional information as a deletion of invariance constraints,' the idea used for the intermediary's revenue analysis.

Proof. **1** $\iff$ **2**: Once restricting attention to the recommendations rule satisfying the invariance constraints, the first problem and the second problem are identical, and hence, their solutions must be solution-equivalent.

2 $\iff$ **3**: The solution to the third problem is

$$r_i^{P3}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1 \text{ if } i = \arg\max_{j\in\mathcal{N}}\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) + \ell_i(\boldsymbol{v}), 0\right)$$

where $\ell_i(\boldsymbol{v})$ is a cost of persuasion. Note that $\boldsymbol{r}^{P3}$ does not vary depending on $\boldsymbol{z}$, and hence, satisfies the invariance constraints. This is because neither the objective function (79) nor the obedience constraints (70) have integrands that depend on $\boldsymbol{z}$ whereas the monotonicity constraints (67) are automatically satisfied. Therefore, the solution to the third problem $\boldsymbol{r}^{P3}$ solves the second problem $\boldsymbol{r}^{P2}$.

2 $\iff$ **4**: Note that

$$\int_{\mathcal{Z}} \frac{1-G(\theta_i\mid z_i)}{g(\theta_i\mid z_i)} G(d\theta_i\mid z_i)H(dz_i) = 1 - \int_{\mathcal{Z}} G(\theta_i\mid z_i)H(dz_i) = 1-G(\theta_i) = \frac{1-G(\theta_i)}{g(\theta_i)}G(d\theta_i). \tag{81}$$

By (81), restricting attention to the recommendations rule satisfying the invariance constraints (80), the objective function in the fourth problem (72) becomes

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}\left(\theta_i - \frac{1-G(\theta_i\mid z_i)}{g(\theta_i\mid z_i)} + w(v_i)\right) r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})\\ &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\end{aligned}$$

so that the fourth problem $P4$ becomes the same as the second problem $P2$. □

B.3 Proof for Lower Censorship Additional Information

Let $\mathcal{H}$ be lower censorship additional information. Let G be a twice continuously differentiable distribution that has a decreasing $\frac{1-G(\theta)}{g(\theta)}$ on Θ, and $g(\theta) > 0$ and $0 \le g'(\theta) < \infty$ on a neighborhood of $\underline{\theta}$.

For $z \in [\theta^*, \bar{\theta}]$, the signal fully reveals the state, $\Theta(z) = \{\theta\}$, and hence, the inverse hazard rate is 0. For $z = z_0$, the signal informs that $\theta \in \Theta(z) = [\underline{\theta}, \theta^*)$, and the inverse hazard rate is

$$\frac{1 - G(\theta \mid z_0)}{g(\theta \mid z_0)} = \frac{\int_{\theta}^{\bar{\theta}} g(\tilde{\theta}) 1_{\tilde{\theta} \in [\underline{\theta}, \theta^*)} d\tilde{\theta}}{g(\theta) 1_{\theta \in [\underline{\theta}, \theta^*)}} = \frac{G(\theta^*) - G(\theta)}{g(\theta)}.$$

Let $z(\theta)$ be a unique additional signal $z \in \mathcal{Z}$ that can be generated with a positive probability for each $\theta \in \Theta$. Define a conditional inverse hazard rate α as

$$\alpha(\theta) = \frac{1 - G(\theta \mid z(\theta))}{g(\theta \mid z(\theta))} = \begin{cases} \frac{G(\theta^*) - G(\theta)}{g(\theta)} & \text{if } \theta < \theta^* \\ 0 & \text{if } \theta \geq \theta^*. \end{cases}$$

Now I show that for any $\epsilon > 0$, there is a small enough θ^* such that for all $z \in \mathcal{Z}$ and $\theta \in \Theta(z)$

$$\frac{1 - G(\theta \mid z)}{g(\theta \mid z)} < \epsilon, \tag{82}$$

and for any θ^*, for all $z, z' \in \mathcal{Z}$, $\theta \in \Theta(z)$ and $\theta' \in \Theta(z')$,

$$0 < 1 - \frac{\frac{1 - G(\theta | z)}{g(\theta | z)} - \frac{1 - G(\theta' | z')}{g(\theta' | z')}}{\theta - \theta'} \leq 1 - \frac{\frac{1 - G(\theta)}{g(\theta)} - \frac{1 - G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{83}$$

To show (82), let $\epsilon > 0$ be such that $\epsilon < \inf_{\theta \in \Theta, v \in \mathcal{V}} \theta + w(v)$. Let $\beta(\theta^*) = \sup_{\theta \in [\underline{\theta}, \theta^*]} \frac{1}{g(\theta)}$. Then, $\beta(\theta^*)$ is continuously increasing in θ^* and $\beta(\theta) \leq \beta(\theta^*) < \infty$ for sufficiently small θ^* and $\theta \leq \theta^*$. Therefore, for any θ^* and $\theta \leq \theta^*$

$$\frac{1 - G(\theta \mid z_0)}{g(\theta \mid z_0)} = \frac{G(\theta^*) - G(\theta)}{g(\theta)} \leq G(\theta^*) \beta(\theta^*).$$

Since both $G(\theta^*)$ and $\beta(\theta^*)$ increase in θ^* with $G(\theta^*) \to 0$ and $\beta(\theta^*) < \infty$ as $\theta^* \to \underline{\theta}$, for sufficiently small θ^*, $G(\theta^*)\beta(\theta^*) < \epsilon$. For $\theta \geq \theta^*$, $z(\theta) = \theta$ perfectly reveals θ, so that $\frac{1 - G(\theta | z(\theta))}{G(\theta | z(\theta))} = 0$. Therefore, for any given ϵ, for sufficiently small θ^*, (82) holds.

To show (83), let $\theta^* \in (\underline{\theta}, \bar{\theta})$. For $\theta, \theta' \geq \theta^*$, their inverse hazard rates are 0, so that

$$0 < 1 = 1 - \frac{\frac{1 - G(\theta | z(\theta))}{g(\theta | z(\theta))} - \frac{1 - G(\theta' | z(\theta'))}{g(\theta' | z(\theta'))}}{\theta - \theta'} \leq 1 - \frac{\frac{1 - G(\theta)}{g(\theta)} - \frac{1 - G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{84}$$

For $\theta, \theta' < \theta^*$, their additional signal is z_0, and

$$\frac{\frac{1-G(\theta|z_0)}{g(\theta|z_0)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = \frac{\frac{G(\theta^*)-G(\theta)}{g(\theta)} - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'}$$
$$= -\frac{\frac{1-G(\theta^*)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta - \theta'} + \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}.$$

Note that $\frac{1-G(\theta^*)}{g(\theta)}$ decreases in θ on $[\underline{\theta}, \theta^*]$ for sufficiently small θ^* because $g(\theta)$ increases in θ, implying $-\frac{\frac{1-G(\theta^*)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta-\theta'} \geq 0$ and hence,

$$1 - \frac{\frac{1-G(\theta|z_0)}{g(\theta|z_0)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} \leq 1 - \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{85}$$

Furthermore, since g is continuously differentiable, by Mean Value Theorem, there is θ'' between θ and θ' such that

$$1 - \frac{\frac{1-G(\theta|z_0)}{g(\theta|z_0)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = 1 - \frac{\frac{G(\theta^*)-G(\theta)}{g(\theta)} - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'}$$
$$= 1 - \frac{-g^2(\theta'') - (G(\theta^*) - G(\theta''))g'(\theta'')}{g^2(\theta'')}$$
$$= 2 + (G(\theta^*) - G(\theta''))\frac{g'(\theta'')}{g^2(\theta'')}. \tag{86}$$

Define $\gamma(\theta'') = \sup_{\tilde{\theta} \in [\underline{\theta}, \theta'']} \left| \frac{g'(\tilde{\theta})}{g^2(\tilde{\theta})} \right|$. Then, $\gamma(\theta'')$ is continuously increasing in θ'' and $\gamma(\theta'')$ and $\gamma(\theta''') \leq \gamma(\theta'') < \infty$ for all $\theta''' \leq \theta''$ for sufficiently small θ''. Since $\theta'' \leq \theta^*$, for any $\epsilon > 0$, for sufficiently small θ^*,

$$\left| (G(\theta^*) - G(\theta''))\frac{g'(\theta'')}{g^2(\theta'')} \right| \leq G(\theta^*) \left| \frac{g'(\theta^*)}{g^2(\theta^*)} \right| < \epsilon.$$

which implies that (86) is positive.

Lastly, consider $\theta \geq \theta^* \geq \theta'$. Then, $z(\theta) = \theta$ and $z(\theta') = z_0$, so that

$$\frac{\frac{1-G(\theta|\theta)}{g(\theta|\theta)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = \frac{0 - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'}$$
$$= -\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta - \theta'} + \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}.$$

For sufficiently small θ^*, g increases on $[\underline{\theta}, \theta^*]$, so that $g(\theta') \leq g(\theta^*)$, and

$$
\begin{aligned}
-\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta - \theta'} &\geq -\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta^*)}}{\theta - \theta'} \\
&= -\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta^*)}}{\theta - \theta^*} \frac{\theta - \theta^*}{\theta - \theta'} \\
&\geq 0
\end{aligned}
$$

where the last inequality is implied by $\frac{1-G(\theta)}{g(\theta)}$ decreasing in θ. Therefore,

$$
1 - \frac{\frac{1-G(\theta|\theta)}{g(\theta|\theta)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} \leq 1 - \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{87}
$$

Furthermore, for any θ^*,

$$
1 - \frac{\frac{1-G(\theta|\theta)}{g(\theta|\theta)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = 1 - \frac{0 - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'} \geq 1 > 0. \tag{88}
$$

By (84), (85), (86), (87) and (88), it follows that (83).

Therefore, by (82) and (83), for sufficiently small $\epsilon > 0$ and θ^*, the lower censorship additional information with small $\theta^* \in (\underline{\theta}, \bar{\theta})$ is ϵ-informative and hence revenue-informative, is regular and decreases rate of substitution.

B.4 Proof for Theorem 3

The following lemma provides a sufficient condition under which the consumer surplus under one recommendations rule $\tilde{\boldsymbol{r}}$ is higher or lower than that under the other $\boldsymbol{r}^\dagger$. The lemma states that if $\boldsymbol{r}^\dagger$ almost surely recommends an option that is at least (at most) as good as options recommended by $\tilde{\boldsymbol{r}}$, then the consumer surplus under $\boldsymbol{r}^\dagger$ is higher (lower) than that under $\tilde{\boldsymbol{r}}$.

Define the consumer surplus under $\boldsymbol{r}$ at v_0 as

$$
CS(v_0; \boldsymbol{r}) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \left[v_0 r_0(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) + \sum_{i \neq 0} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0)
$$

where

$$
u^*(v_0) = \max(v_0, \mathbb{E}(v_i))
$$

is the consumer's optimal payoff without recommendations.

Lemma 16. *If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0 \textit{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0 \textit{ only if } v_j \geq (\leq)\ v_i \textit{ for } i,j\in\mathcal{N}\cup\{0\},$$

almost surely, then $CS(v_0;\tilde{\boldsymbol{r}}) \leq (\geq)\ CS(v_0;\boldsymbol{r}^{\dagger})$.

Proof. Suppose

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0 \text{ only if } v_j \geq\ v_i \text{ for } i,j\in\mathcal{N}\cup\{0\},$$

almost surely. Let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}$. Then,

$$\sum_{i\in\mathcal{N}\cup\{0\}} v_i\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \leq \max_{i:\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} v_i \leq \min_{j:r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} v_j \leq \sum_{j\in\mathcal{N}\cup\{0\}} v_j r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})$$

so that

$$\begin{aligned}
CS(v_0;\tilde{\boldsymbol{r}}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0)\\
&\leq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i r_i^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0)\\
&= CS(v_0;\boldsymbol{r}^{\dagger}).
\end{aligned}$$

The other inequality may be shown similarly. □

Let $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ be optimal unconstrained recommendations rules without and with additional information. That is, $\boldsymbol{\rho}^*$ maximizes (79) subject to monotonicity constraints (67) and $\boldsymbol{\rho}^A$ maximizes (72) subject to monotonicity constraints (67).

Lemma 17. *Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. If additional information is revenue informative, regular and increases (decreases) rate of substitution, then*

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \geq (\leq) \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \tag{89}$$

Proof. Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Since the information rent is small as in Assumption **??**, it follows that $\rho_0^* = 0$ and $\rho_0^A = 0$ almost surely. When one of the products is recommended, the rules are almost surely given by

$$\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i) \ \forall j \in \mathcal{N} \smallsetminus \{i\}$$

$$\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i) \ \forall j \in \mathcal{N} \smallsetminus \{i\}$$

To use Lemma 16, it is sufficient to show:

> if additional information increases (decreases) rates of substitution, $\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ for $i \in \mathcal{N}$ implies $\rho_j^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ only if $j \in \mathcal{N}$ and $v_j \le (\ge)\ v_i$ almost surely.

To show this, let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}(\boldsymbol{z}) \times \boldsymbol{\mathcal{Z}}$ be at which $\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ for $i \in \mathcal{N}$. Almost surely, i is strictly preferred to $k \in \mathcal{N} \smallsetminus \{i\}$. Since the additional information is regular, rates of substitution are positive, so that

$$\theta_i - \theta_k > (w(v_k) - w(v_i)) \Big/ \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right).$$

Suppose that additional information increases rates of substitution. Then,

$$1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j} \ge 1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}.$$

If $v_k > v_i$,

$$\theta_i - \theta_k > (w(v_k) - w(v_i)) \Big/ \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right) \ge (w(v_k) - w(v_i)) \Big/ \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k \in \mathcal{N} \smallsetminus \{i\}$, $\rho_k^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 0$ if $v_k > v_i$. In other words, $\rho_j^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ only if $v_j \le v_i$. By Lemma 16, it follows that $CS(v_0; \boldsymbol{\rho}^*) \ge CS(v_0; \boldsymbol{\rho}^A)$. Since the consumer's optimal payoff without recommendations is identical under both problems without and with additional information, this is equivalent to

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \ge \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}).$$

To show the other case, suppose that additional information decreases rates of substitution. Then,

$$1-\frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)}-\frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i-\theta_j}\leq 1-\frac{\frac{1-G(\theta_i)}{g(\theta_i)}-\frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i-\theta_j}.$$

If $v_k < v_i$,

$$\theta_i-\theta_k>(w(v_k)-w(v_i))\Bigg/\left(1-\frac{\frac{1-G(\theta_i)}{g(\theta_i)}-\frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i-\theta_j}\right)\geq(w(v_k)-w(v_i))\Bigg/\left(1-\frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)}-\frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i-\theta_j}\right)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k\in\mathcal{N}\setminus\{i\}$, $\rho_k^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})=0$ if $v_k<v_i$. In other words, $\rho_j^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0$ only if $v_j\geq v_i$. By Lemma 16, it follows that $CS(v_0;\boldsymbol{\rho}^*)\geq\ CS(v_0;\boldsymbol{\rho}^A)$. Since the consumer's optimal payoff without recommendations is identical under both fictitious problem and problem with additional information, this is equivalent to

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})\geq\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).$$

□

Let $\bar{v}^*=\mathbb{E}(v_i\mid\rho_i^O(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})=1)$. Note that $\bar{v}*\geq\mathbb{E}_{v_i}(v_i)$. For $v_0\leq\bar{v}^*$, $\boldsymbol{\rho}^O$ satisfies obedience constraints and hence $\boldsymbol{\rho}^*$ is optimal, that is, $\boldsymbol{r}^*=\boldsymbol{\rho}^*$ and $CS^O(v_0)>0$. For $v_0>\bar{v}^*$, $\boldsymbol{\rho}^*$ no longer satisfies obedience constraints. The obedience constraints from products to the outside option bind under $\boldsymbol{r}^*$ under which $CS(v_0;\boldsymbol{r}^*)=0$. In particular, at $v_0=\bar{v}^*$, $\boldsymbol{r}^*=\boldsymbol{\rho}^*$ and $CS^*(v_0)=0$, implying that

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})=\bar{v}^*.$$

The consumer surplus without additional information is

$$CS^*(v_0)=\begin{cases}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})-\mathbb{E}_{v_i}(v_i) & \text{if } v_0<\mathbb{E}_{v_i}(v_i)\\ \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})-v_0 & \text{if } v_0\in[\mathbb{E}_{v_i}(v_i),\bar{v}_0^*]\\ 0 & \text{if } v_0>\bar{v}_0^*\end{cases}$$

Similar analysis can be applied for $\boldsymbol{r}^A$. Let $\bar{v}^A=\mathbb{E}(v_i\mid\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})=1)$. Note that $\bar{v}^A\geq\mathbb{E}_{v_i}(v_i)$. For $\bar{v}^A$, $\boldsymbol{\rho}^A$ satisfies obedience constraints and hence $\boldsymbol{\rho}^A$ is optimal, that is, $\boldsymbol{r}^A=\boldsymbol{\rho}^A$ and $CS^O(v_0)>0$. For $v_0>\bar{v}^A$, $\boldsymbol{\rho}^A$ no longer satisfies obedience constraints. The obedience constraints from products to the outside option bind under $\boldsymbol{r}^A$ under which

$CS(v_0;\boldsymbol{r}^A) = 0$. In particular, at $v_0 = \bar{v}^A$, $\boldsymbol{r}^A = \boldsymbol{\rho}^A$ and $CS^A(v_0) = 0$, implying that

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) = \bar{v}^A.$$

The consumer surplus under the problem with (revenue-informative) additional information problem is

$$CS^A(v_0) = \begin{cases} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - \mathbb{E}_{v_i}(v_i) & \text{if } v_0 < v_p \\ \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - v_0 & \text{if } v_0 \in [v_p, \bar{v}^A] \\ 0 & \text{if } v_0 > \bar{v}^A \end{cases}$$

For $v_0 > \max(\bar{v}^*, \bar{v}^A)$, $CS^*(v_0) = CS^A(v_0) = 0$. For $v_0 \le \min(\bar{v}^*, \bar{v}^A)$, since the best value without recommendations $u^*(v_0)$ is identical without and with additional information,

$$CS^*(v_0) \ge (\le)\ CS^A(v_0)$$

if and only if

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \ge (\le) \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \tag{90}$$

For $v_0 \in (\min(\bar{v}^*, \bar{v}^A), \max(\bar{v}^*, \bar{v}^A)]$, if $\bar{v}^* \le \bar{v}^A$, then $CS^*(v_0) = 0 \le CS^A(v_0)$; if $\bar{v}^* \ge \bar{v}^A$, then $CS^*(v_0) \ge 0 = CS^A(v_0)$. Therefore, $CS^*(v_0) \ge (\le)\ CS^A(v_0)$ if and only if $\bar{v}_0^O \ge (\le)\ \bar{v}_0^A$ which is equivalent to (90). Therefore, for any $v_0 \in \mathbb{R}_+^1$, $CS^O(v_0) \ge (\le)\ CS^A(v_0)$ if and only if (90). By Lemma 17, if additional information increases (decreases) rate of substitution, then (90) holds, and hence, $CS^O(v_0) \ge (\le)\ CS^A(v_0)$.

B.5 Proof for Theorem 4

Define a seller i's profit under recommendations rule $\boldsymbol{r}$ at v_0 by

$$\Pi_i(v_0;\boldsymbol{r}) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \frac{1-G(\theta_i)}{g(\theta_i)} r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).$$

and sum of all sellers' expected profits by

$$\Pi(v_0;\boldsymbol{r}) = \sum_{i\in\mathcal{N}} \Pi_i(v_0;\boldsymbol{r}).$$

Lemma 18. *Let $\tilde{\boldsymbol{r}}$ and $\boldsymbol{r}^\dagger$ be recommendations rule that never recommends the outside option* *almost surely.*

1. *Suppose* $\frac{1-G(\theta)}{g(\theta)}$ *increases in* θ. *If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ only if } \theta_j \geq (\leq)\ \theta_i \textit{ for } i,j \in \mathcal{N} \cup \{0\},$$

   almost surely, then $\Pi(v_0;\tilde{\boldsymbol{r}}) \leq (\geq)\ \Pi(v_0;\boldsymbol{r}^{\dagger})$.

2. *Suppose* $\frac{1-G(\theta)}{g(\theta)}$ *decreases in* θ. *If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ only if } \theta_j \leq (\geq)\ \theta_i \textit{ for } i,j \in \mathcal{N} \cup \{0\},$$

   almost surely, then $\Pi(v_0;\tilde{\boldsymbol{r}}) \geq (\leq)\ \Pi(v_0;\boldsymbol{r}^{\dagger})$.

Proof. Suppose $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. Suppose

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } \theta_j \geq\ \theta_i \text{ for } i,j \in \mathcal{N} \cup \{0\},$$

almost surely. Let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. Then,

$$\begin{aligned}
\sum_{i \in \mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)}\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) &\leq \max_{i:\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} \frac{1-G(\theta_i)}{g(\theta_i)} \\
&\leq \min_{j:r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} \frac{1-G(\theta_j)}{g(\theta_j)} \\
&\leq \sum_{j \in \mathcal{N}\cup\{0\}} \frac{1-G(\theta_j)}{g(\theta_j)} r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})
\end{aligned}$$

so that

$$\begin{aligned}
\Pi(v_0;\tilde{\boldsymbol{r}}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)}\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\
&\leq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)} r_i^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\
&= \Pi(v_0;\boldsymbol{r}^{\dagger}).
\end{aligned}$$

The other inequalities may be shown similarly. □

Let $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ be optimal unconstrained recommendations rules without and with additional information. That is, $\boldsymbol{\rho}^*$ maximizes (79) subject to monotonicity constraints (67) and

$\boldsymbol{\rho}^A$ maximizes (72) subject to monotonicity constraints (67). Note that under small information rent environment, both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ does not recommend the outside option almost surely.

Lemma 19. *Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Suppose the additional information is regular. Then,*

$$\Pi(v_0; \boldsymbol{\rho}^*) \le (\ge)\ \Pi(v_0; \boldsymbol{\rho}^A). \tag{91}$$

if one of the following conditions is satisfied:

1. *Additional information increases (decreases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ increases (decreases) in θ.*

2. *Additional information decreases (increases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.*

Proof. Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Since the information rent is small as in Assumption **??**, it follows that $\rho_0^* = 0$ and $\rho_0^A = 0$ almost surely. When one of the products is recommended, the rules are almost surely given by

$$\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i)\ \forall j \in \mathcal{N} \setminus \{i\}$$

$$\rho_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i)\ \forall j \in \mathcal{N} \setminus \{i\}$$

Suppose $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. To use Lemma 18, it is sufficient to show:

> if additional information increases (decreases) rates of substitution, $\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ for $i \in \mathcal{N}$ implies $\rho_j^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ only if $j \in \mathcal{N}$ and $\theta_j \ge (\le)\ \theta_i$ almost surely.

To show this, let $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}(\boldsymbol{z}) \times \boldsymbol{\mathcal{Z}}$ be at which $\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ for $i \in \mathcal{N}$. Almost surely, i is strictly preferred to $k \in \mathcal{N} \setminus \{i\}$. Since the additional information is regular, rates of substitution are positive, so that

$$\left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_k) > w(v_k) - w(v_i)$$

Suppose that additional information decreases rates of substitution. Then,

$$1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j} \ge 1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}.$$

If $\theta_k < \theta_i$,

$$\left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_k) \geq \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_k) > w(v_k) - w(v_i)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k \in \mathcal{N} \setminus \{i\}$, $\rho_k^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 0$ if $v_k > v_i$. In other words, $\rho_j^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ only if $\theta_j \geq \theta_i$. By Lemma 18, it follows that

$$\Pi(v_0; \boldsymbol{\rho}^*) \leq \Pi(v_0; \boldsymbol{\rho}^A).$$

To show the other case given an increasing $\frac{1-G(\theta)}{g(\theta)}$, suppose that additional information decreases rates of substitution. Then,

$$1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j} \leq 1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}.$$

If $\theta_k > \theta_i$,

$$\left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_k|z_k)}{g(\theta_k|z_k)}}{\theta_i - \theta_k}\right)(\theta_i - \theta_k) \geq \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_k)}{g(\theta_k)}}{\theta_i - \theta_k}\right)(\theta_i - \theta_k) > w(v_k) - w(v_i)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k \in \mathcal{N} \setminus \{i\}$, $\rho_k^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 0$ if $\theta_k > \theta_i$. In other words, $\rho_j^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ only if $\theta_k \leq \theta_i$. By Lemma 16, it follows that

$$\Pi(v_0; \boldsymbol{\rho}^*) \geq \Pi(v_0; \boldsymbol{\rho}^A)$$

The other inequalities related to decreasing $\frac{1-G(\theta)}{g(\theta)}$ may be shown similarly. $\square$

Let $v_0 \leq \mathbb{E}_{v_i}(v_i)$. Under a small information rent environment, both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient, so that Lemma 19 applies straightforwardly to induce the desired results.

B.6 Proof for Theorem 5

Note that for each $z \in \mathcal{Z}$ and $\theta \in \Theta(z)$,

$$\begin{aligned}\int_{\mathcal{Z}} \left(\theta - \frac{1 - G(\theta \mid z)}{g(\theta \mid z)}\right) G(d\theta \mid z) H(dz) &= \theta g(\theta) - \int_{\mathcal{Z}} (1 - G(\theta \mid z)) H(dz) \\ &= \theta g(\theta) - \left(1 - \int_{\mathcal{Z}} \Pr(\tilde{\theta} \le \theta, z) dz\right) \\ &= \theta g(\theta) - (1 - G(\theta)) \\ &= \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) G(d\theta)\end{aligned}$$

and

$$\int_{\mathcal{Z}} v G(d\theta \mid z) H(dz) = v G(d\theta)$$

This means that the intermediary's problem without additional information, which is to maximize

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta)} + w(v_i)\right) r_i^{\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta})$$

subject to obedience constraints

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_i r_i^{\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \ge \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_j r_i^{\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \text{ for all } i, j \in \mathcal{N} \cup \{0\}$$

is identical to maximizing the intermediary's problem with additional information

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i)\right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})$$

subject to obedience constraints

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \ge \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_j r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \text{ for all } i, j \in \mathcal{N} \cup \{0\}$$

and invariance constraints

$$\boldsymbol{r}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \boldsymbol{r}^{\dagger}(\boldsymbol{v}, \boldsymbol{\theta}) \text{ for some } \boldsymbol{r}^{\dagger} \text{ for all } \boldsymbol{z} \in \boldsymbol{\mathcal{Z}}.$$

The intermediary's problem with additional information is the same but without the invariance constraints. Since both have the same objective function but there is another set of constraints in the problem without the additional information, by revealed preference, the intermediary's revenue is higher.

B.7 Proof for Theorem 6

Suppose $v_0 < \underline{v}$. Since the consumer always prefers products over the outside option always, any obedient recommendations rule must always recommend one of the products. An optimal recommendations rule with this constraint but without additional information is given by for each $i \in \mathcal{N}$

$$\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \frac{1}{|\mathcal{M}^*|} \text{ if } i \in \mathcal{M}^* \tag{92}$$

where $\mathcal{M}^* = \arg\max_{j \in \mathcal{N}} \{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\}$, and that with additional information is given by

$$\rho_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \frac{1}{|\mathcal{M}^A|} \text{ if } i \in \mathcal{M}^A \tag{93}$$

where $\mathcal{M}^A = \arg\max_{j \in \mathcal{N}} \{\theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j)\}$. Both recommendations rules are completely determined by rates of substitutions, so that whether additional information increases the consumer surplus or not is completely determined by whether additional information decreases or increases the rates of substitution. Applying similar arguments as in Theorem 3 gives the desired result.

Suppose $v_0 > \bar{v}$. Since the consumer always prefers the outside option over products, any obedient recommendations rule must always recommend the outside option, without and with additional information, under which the consumer surplus is always 0. Therefore, the additional information does not change the consumer surplus.

C Proofs for Section 6

C.1 Lemma 7

Let $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. Suppose each seller $j \in \mathcal{N} \setminus \{i\}$ bids b_j and the seller i is charged with a price premium p_i and a cost of persuastion $\ell_i(\boldsymbol{v})$. If the seller i bids b_i and wins, then her ex-post payoff is

$$\theta_i + w(v_i) - \Big(\max_{j \in \mathcal{N} \setminus \{i\}} (b_j, 0) + p_i + \ell_i(\boldsymbol{v}) \Big).$$

Now I show that $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$ is a weakly dominant strategy.

If $\theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v}) > \max_{j \in \mathcal{N} \setminus \{i\}}(b_j, 0)$, then the seller i's ex-post payoff after winning is positive, so that she prefers winning over losing and drawing. Bidding $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$ results in winning.

If $\theta_i + w(v_i) - p_i(\theta_i, \boldsymbol{v}) - \ell_i(\boldsymbol{v}) < \max_{j \in \mathcal{N} \setminus \{i\}}(b_j, 0)$, then the seller i's ex-post payoff after winning is negative, so that she prefers losing over winning and drawing. Bidding $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$ results in losing.

If $\theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v}) = \max_{j \in \mathcal{N} \setminus \{i\}}(b_j, 0)$, then the seller i's ex-post payoff after winning is zero, so that she is indifferent among winning, losing and drawing. Bidding $b_i = \theta_i + w(v_i) - p_i$ results in any of those.

Therefore, it is weakly dominant to bid $b_i = \theta_i + w(v_i) - p_i - \ell_i(\boldsymbol{v})$.

C.2 Lemma 8

For reference, let

$$\Pi_i^H(\theta_i) = \Pi_i^H(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) d\tilde{\theta}_i \text{ for all } i \neq 0 \text{ and } \theta_i \in \Theta \tag{94}$$

and

$$Q_i(\theta_i, \theta_i') \leq Q_i(\theta_i, \theta_i) \leq Q_i(\theta_i, \theta_i'') \text{ for all } \theta_i', \theta_i'' \in [\underline{\theta}, \bar{\theta}] \text{ such that } \theta_i' < \theta_i < \theta_i''. \tag{95}$$

Necessity ($\rightarrow$):

Note that an incentive compatibility is equivalent to

$$\text{for all } i \text{ and } \hat{\theta}_i < \theta_i,\ \pi_i^H(\theta_i, \theta_i) \geq \pi_i^H(\theta_i, \hat{\theta}_i) \text{ and } \pi_i^H(\hat{\theta}_i, \hat{\theta}_i) \geq \pi_i^H(\hat{\theta}_i, \theta_i).$$

Without loss of generality, we assume $\hat{\theta}_i < \theta_i$. Define for $x, y \in [\underline{\theta}, \bar{\theta}]$,

$$\Delta(x, y) = E_{\theta_{-i}, \boldsymbol{v}}\left[(x + w(v_i) - p_i(y) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) - w(v_i) - p_i(y) + \ell_i(\boldsymbol{v}) \in (\min(x,y), \max(x,y))\}}\right]$$

We can rewrite $\pi_i^H(\theta_i, \hat{\theta}_i)$ as

$$\begin{aligned}
\pi_i^H(\theta_i, \hat{\theta}_i) = {} & E_{\theta_{-i}, \boldsymbol{v}}\left[(\hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right] - c(\hat{\theta}_i) \\
& + E_{\theta_{-i}, \boldsymbol{v}}\left[(\theta_i - \hat{\theta}_i)\mathbf{1}_{\{\hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}\}}\right] \\
& + E_{\theta_{-i}, \boldsymbol{v}}\left[(\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) \geq \hat{\theta}_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v})\}}\right] \\
= {} & \Pi_i^H(\hat{\theta}_i) + Q_i^H(\hat{\theta}_i, \hat{\theta}_i)(\theta_i - \hat{\theta}_i) + \Delta_i(\theta_i, \hat{\theta}_i).
\end{aligned}$$

Similarly,

$$\pi_i^H(\hat{\theta}_i, \theta_i) = \Pi_i^H(\hat{\theta}_i) - Q_i^H(\hat{\theta}_i, \hat{\theta}_i)(\theta_i - \hat{\theta}_i) - \Delta_i(\theta_i, \hat{\theta}_i).$$

Incentive compatibility is equivalent to, for all i and $\hat{\theta}_i < \theta_i$,

$$Q_i^H(\hat{\theta}_i, \hat{\theta}_i) + \frac{\Delta_i(\theta_i, \hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq \frac{\Pi_i^H(\theta_i) - \Pi_i^H(\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq Q_i^H(\theta_i, \theta_i) + \frac{\Delta_i(\hat{\theta}_i, \theta_i)}{\theta_i - \hat{\theta}_i}$$

Note that $\Delta_i(x, y) \geq 0$ if and only if $x \geq y$, so that $\Delta_i(\hat{\theta}_i, \theta_i) \leq 0 \leq \Delta_i(\theta_i, \hat{\theta}_i)$. Therefore, incentive compatibility implies $Q_i^H(\hat{\theta}_i, \hat{\theta}_i) \leq \frac{\Pi_i^H(\theta_i) - \Pi_i^H(\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq Q_i^H(\theta_i, \theta_i)$. Since $Q_i^H(\theta_i, \theta_i)$ increases in θ_i, it is integrable, which implies (94).

It remains to prove that (95) holds. Suppose $\hat{\theta}_i < \theta_i$. If $p(\hat{\theta}_i) \geq p(\theta_i)$, then $Q_i^H(\hat{\theta}_i, \hat{\theta}_i) \leq Q_i^H(\theta_i, \hat{\theta}_i) \leq Q_i^H(\theta_i, \theta_i)$. Suppose $p(\hat{\theta}_i) < p(\theta_i)$. Let

$$\begin{aligned}\epsilon_i(x, y) = \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}\Big[&\mathbf{1}_{\{b_{-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) \in (\min(x - p_i(x) - \ell_i(\boldsymbol{v}), p(y) + \ell_i(\boldsymbol{v})), \max(x - p(x) - \ell_i(\boldsymbol{v}), p(y) + \ell_i(\boldsymbol{v})))\}} \\ &\cdot (x + w(v_i) - p_i(y) - \ell_i(\boldsymbol{v}) - b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\Big].\end{aligned}$$

Rewrite

$$\begin{aligned}\pi_i^H(\theta_i, \hat{\theta}_i) =& \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})} \geq b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\})(\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))] - c(\theta_i) \\ &- \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})} \geq b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\})(p(\theta_i) - p(\hat{\theta}_i))] + c(\theta_i) - c(\hat{\theta}_i) \\ &+ \mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}\Big[\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}) \geq b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) > \theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})\}} \\ &\qquad \cdot (\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i\,-i}^{**}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\Big] \\ =& \Pi_i^H(\theta_i) + Q_i^H(\theta_i, \theta_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + c(\theta_i) - c(\hat{\theta}_i) + \epsilon_i(\theta_i, \hat{\theta}_i).\end{aligned}$$

Similarly,

$$\pi_i^H(\theta_i, \hat{\theta}_i) = \Pi_i^H(\hat{\theta}_i) + Q_i^H(\hat{\theta}_i, \hat{\theta}_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + c(\hat{\theta}_i) - c(\theta_i) - \epsilon_i(\hat{\theta}_i, \theta_i).$$

By incentive compatibility, $\pi_i^H(\theta_i, \hat{\theta}_i) - \pi_i^H(\theta_i, \theta_i) \leq 0 \leq \pi_i^H(\hat{\theta}_i, \hat{\theta}_i) - \pi_i^H(\hat{\theta}_i, \theta_i)$, which is equivalent to

$$\pi_i^H(\theta_i, \theta_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + \epsilon_i(\theta_i, \hat{\theta}_i) \leq c(\hat{\theta}_i) - c(\theta_i) \leq \pi_i^H(\hat{\theta}_i, \hat{\theta}_i)(p_i(\theta_i) - p_i(\hat{\theta}_i)) + \epsilon_i(\hat{\theta}_i, \theta_i). \tag{96}$$

Since $\hat{\theta}_i < \theta_i$ and $p(\hat{\theta}_i) < p(\theta_i)$, it follows that $\epsilon_i(\hat{\theta}_i, \theta_i) \leq 0 \leq \epsilon_i(\theta_i, \hat{\theta}_i)$. Then, $\epsilon_i(\theta_i, \hat{\theta}_i) =$

$\epsilon_i(\hat{\theta}_i, \theta_i) = 0$; otherwise, (96) implies $Q_i^H(\theta_i, \theta_i) < Q_i^H(\hat{\theta}_i, \hat{\theta}_i)$, contradicting (C.2). Since $\epsilon_i(\theta_i, \hat{\theta}_i) = 0$,

$$\mathbb{E}_{\boldsymbol{v}, \boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i\,-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}) > \theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v})\}}) = 0,$$

which is equivalent to $\Pi_i^H(\theta_i, \hat{\theta}_i) = \Pi_i^H(\theta_i, \theta_i)$. Therefore, $\hat{\theta}_i < \theta_i$ implies $\Pi_i^H(\theta_i, \hat{\theta}_i) \leq \Pi_i^H(\theta_i, \theta_i)$ whether $p(\theta_i) \leq p(\hat{\theta}_i)$ or not, so that the first inequality of (95) holds. The other inequality may be shown similarly, which establishes (95).

Sufficiency (←): Suppose that the seller i with θ_i has reported itself as $\hat{\theta}_i$ when purchasing the premium. In the second stage, after learning $\boldsymbol{v}$, the payoff of reporting as θ_i' is

$$U_i(\theta_i, \theta_i'; \hat{\theta}_i, \boldsymbol{v}) = E_{\theta_{-i}}\left[(\theta_i + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i' + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right]$$

Since the second price auction in the second stage is incentive compatible, for any $\theta_i \in (\underline{\theta}, \bar{\theta})$

$$\frac{\partial U_i^H}{\partial \theta_i}(\theta_i; \hat{\theta}_i, \boldsymbol{v}) = Q_i^H(\theta_i \mid \hat{\theta}_i, \boldsymbol{v})$$

where $U_i^H(\theta_i; \hat{\theta}_i, \boldsymbol{v}) = U_i(\theta_i, \theta_i; \hat{\theta}_i, \boldsymbol{v})$ and $Q_i^H(\theta_i; \hat{\theta}_i, \boldsymbol{v}) = E_{\theta_{-i}}\left[\mathbf{1}_{\{\theta_i' + w(v_i) - p_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right]$. For any $\theta_i, \theta_i' \in (\underline{\theta}, \bar{\theta})$,

$$U_i^H(\theta_i; \hat{\theta}_i, \boldsymbol{v}) = U_i^H(\theta_i'; \hat{\theta}_i, \boldsymbol{v}) + \int_{\theta_i'}^{\theta_i} Q_i^H(\tilde{\theta}; \hat{\theta}_i, \boldsymbol{v}) d\tilde{\theta}.$$

The seller's interim payoff function in the first stage may be expressed as

$$\begin{aligned}
\pi_i^H(\theta_i, \hat{\theta}_i) &= E_{\boldsymbol{v}}\left[U_i^H(\theta_i; \hat{\theta}_i, \boldsymbol{v})\right] - c(\hat{\theta}_i) \\
&= E_{\boldsymbol{v}}\left[U_i^H(\hat{\theta}_i; \hat{\theta}_i, \boldsymbol{v})\right] + E_{\boldsymbol{v}}\left[\int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i; \hat{\theta}_i, \boldsymbol{v}) d\tilde{\theta}_i\right] - c(\hat{\theta}_i) \\
&= \Pi_i^H(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}, \hat{\theta}_i) d\tilde{\theta}.
\end{aligned}$$

Similarly,

$$\pi_i^H(\hat{\theta}_i, \theta_i) = \Pi_i^H(\theta_i) + \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}, \theta_i) d\tilde{\theta}.$$

Note that the incentive compatibility is equivalent to

$$\text{for all } i \text{ and } \hat{\theta}_i < \theta_i,\ \pi_i^H(\theta_i,\theta_i) \geq \pi_i^H(\theta_i,\hat{\theta}_i) \text{ and } \pi_i^H(\hat{\theta}_i,\hat{\theta}_i) \geq \pi_i^H(\hat{\theta}_i,\theta_i).$$

Note that

$$\begin{aligned}
&\pi_i^H(\theta_i,\theta_i) \geq \pi_i^H(\theta_i,\hat{\theta}_i)\\
&\text{ iff } \Pi_i^H(\theta_i) \geq \Pi_i^H(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\hat{\theta}_i)d\tilde{\theta}_i\\
&\text{ iff } \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)d\tilde{\theta}_i \geq \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\hat{\theta}_i)d\tilde{\theta}_i && \text{(By (94))}
\end{aligned}$$

where the last inequality holds because $Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i) \geq Q_i^H(\tilde{\theta}_i,\hat{\theta}_i)$ for all $\tilde{\theta}_i \geq \hat{\theta}_i$ by (95). Similarly,

$$\begin{aligned}
&\pi_i^H(\hat{\theta}_i,\hat{\theta}_i) \geq \pi_i^H(\hat{\theta}_i,\theta_i)\\
&\text{ iff } \Pi_i^H(\hat{\theta}_i) \geq \Pi_i^H(\theta_i) + \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}_i,\theta_i)d\tilde{\theta}_i\\
&\text{ iff } \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)d\tilde{\theta}_i \geq \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}_i,\theta_i)d\tilde{\theta}_i && \text{(By (94))}\\
&\text{ iff } \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i)d\tilde{\theta}_i \leq \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i,\theta_i)d\tilde{\theta}_i
\end{aligned}$$

where the last inequality holds because $Q_i^H(\tilde{\theta}_i,\tilde{\theta}_i) \leq Q_i^H(\tilde{\theta}_i,\theta_i)$ for all $\tilde{\theta}_i \leq \theta_i$ by (95).

C.3 Theorem 9

Given the handicap auction $(p_i^*, c_i^*, \ell_i^*)_{i\in\mathcal{N}}$, if every seller of every type reports its willingness to pay truthfully under the handicap auction, then the intermediary recommends product i if and only if

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1 \text{ iff } \theta_i + w(v_i) - \frac{1-G_i(\theta_i)}{g_i(\theta_i)} - \ell_i^*(\boldsymbol{v}) > \max_{j\neq i,0}\left(\theta_j + w(v_j) - \frac{1-G_i(\theta_i)}{g_i(\theta_i)} - \ell_j^*(\boldsymbol{v}), 0\right)$$

which is the same recommendations rule as in the revenue maximizing recommender system.

It remains to verify that the handicap auction (33) and (34) is incentive compatible and individually rational. Since p_i weakly decreases in θ_i, the monotonicity condition (95) holds. The interim payoff of the seller i with θ_i receives is

$$\Pi_i^H(\theta_i) = E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - p_i(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i+w(v_i)-p_i(\theta_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] - c(\theta_i)$$

which implies that $\Pi_i^H(\underline{\theta}) = 0$. The handicap auction gives the same revenue as the revenue maximizing recommender system because the recommendation rules are identical and the lowest type's payoff is the same by $\Pi_i^*(\underline{\theta}) = 0 = \Pi_i^H(\underline{\theta})$, and therefore, $\Pi_i^*(\theta_i) = \Pi_i^H(\theta_i)$, which in turn implies that the expected payment from each recommender system must be identical between the two. The individual rationality trivially follows from the observation that the lowest payoff each seller gets $\Pi_i(\underline{\theta})$ is 0.

References

Admati, A. R. and P. Pfleiderer (1986): "A Monopolistic Market for Information," *Journal of Economic Theory*, 39, 400–438.

——— (1990): "Direct and Indirect Sale of Information," *Econometrica*, 58, 901–928.

Alonso, R. and O. Câmara (2016): "Persuading Voters," *American Economic Review*, 106, 3590–3605.

Aridor, G. and D. Gonçalves (2021): "Recommenders' Originals: The Welfare Effects of the Dual Role of Platforms as Producers and Recommender Systems," *Working Paper*, 59.

Athey, S. and G. Ellison (2011): "Position Auctions with Consumer Search," *The Quarterly Journal of Economics*, 126, 1213–1270.

Aumann, R. J. and M. Maschler (1995): "Repeated Games with Incomplete Information," *MIT Press*.

Bergemann, D., A. Bonatti, and A. Smolin (2018): "The Design and Price of Information," *American Economic Review*, 108, 1–48.

Bergemann, D. and S. Morris (2019): "Information Design: A Unified Perspective," *Journal of Economic Literature*, 57, 44–95.

Clarke, E. H. (1971): "Multipart Pricing of Public Goods," *Public Choice*, 11, 17–33.

de Cornière, A. and G. Taylor (2019): "A Model of Biased Intermediation," *The RAND Journal of Economics*, 50, 854–882.

Dizdar, D. and E. Kováč (2020): "A Simple Proof of Strong Duality in the Linear Persuasion Problem," *Games and Economic Behavior*, 122, 407–412.

Dworczak, P. (2020): "Mechanism Design With Aftermarkets: Cutoff Mechanisms," *Econometrica*, 88, 2629–2661.

Dworczak, P. and A. Kolotilin (2019): "The Persuasion Duality," *SSRN Electronic Journal*.

Dworczak, P. and G. Martini (2019): "The Simple Economics of Optimal Persuasion," *Journal of Political Economy*, 56.

Edelman, B., M. Ostrovsky, and M. Schwarz (2007): "Internet Advertising and the Generalized Second-Price Auction: Selling Billions of Dollars Worth of Keywords," *American Economic Review*, 97, 18.

Eső, P. and B. Szentes (2007): "Optimal Information Disclosure in Auctions and the Handicap Auction," *The Review of Economic Studies*, 74, 705–731.

Fang, H. and S. J. Kim (2021): "Data Neutrality and Market Competition," *Working Paper*.

Galperti, S. and J. Perego (2018): "A Dual Perspective on Information Design," *Working Paper*.

Gentzkow, M. and E. Kamenica (2016): "A Rothschild-Stiglitz Approach to Bayesian Persuasion," *American Economic Review*, 106, 597–601.

Groves, T. (1973): "Incentives in Teams," *Econometrica*, 41, 617–631.

Hagiu, A., T.-H. Teh, and J. Wright (2020): "Should Platforms Be Allowed to Sell on Their Own Marketplaces?" *Working Paper*.

Hagiu, A. and J. Wright (2015): "Marketplace or Reseller? — Management Science," *Management Science*.

——— (2019): "Controlling vs. Enabling," *Management Science*, 65, 577–595.

Howard, B. (2010): "The Value of Behavioral Targeting," *Network Advertising Initiative*, 61.

Hu, Y. J., J. Shin, and Z. Tang (2016): "Incentive Problems in Performance-Based Online Advertising Pricing: Cost per Click vs. Cost per Action," *Management Science*, 62, 2022–2038.

Inderst, R. and M. Ottaviani (2012): "Competition through Commissions and Kickbacks," *American Economic Review*, 102, 780–809.

Kamenica, E. and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

Kang, Z. Y. and E. Muir (2021): "Regulating Vertically Structured Platforms," *Working Paper*.

Kolotilin, A. (2018): "Optimal Information Disclosure: A Linear Programming Approach," *Theoretical Economics*, 13, 607–635.

Korolova, A. (2010): "Privacy Violations Using Microtargeted Ads: A Case Study," in *2010 IEEE International Conference on Data Mining Workshops*, 474–482.

Madsen, E. and N. Vellodi (2021): "Insider Imitation," *Working Paper*.

Mitchell, M. (2021): "Free Ad(Vice): Internet Influencers and Disclosure Regulation," *The RAND Journal of Economics*, 52, 3–21.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

——— (1982): "Optimal Coordination Mechanisms in Generalized Principal–Agent Problems - ScienceDirect," *Journal of Mathematical Economics*, 10.

——— (1983): "Bayesian Equilibrium and Incentive-Compatibility: An Introduction," *Working Paper*.

Rayo, L. and I. Segal (2010): "Optimal Information Disclosure," *Journal of Political Economy*, 118, 949–987.

Segura-Rodriguez, C. (2021): "Selling Data," *Working Paper*.

Varian, H. R. (2007): "Position Auctions," *International Journal of Industrial Organization*, 25, 1163–1178.

Venkatadri, G., A. Andreou, Y. Liu, A. Mislove, K. P. Gummadi, P. Loiseau, and O. Goga (2018): "Privacy Risks with Facebook's PII-Based Targeting: Auditing a Data Broker's Advertising Interface," in *2018 IEEE Symposium on Security and Privacy (SP)*, 89–107.

Vickrey, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *The Journal of Finance*, 16, 8–37.

Yang, K. H. (2019): "Equivalence in Business Models for Informational Intermediaries," *Working Paper*.

——— (2021): "Selling Consumer Data for Profit: Optimal Market-Segmentation Design and Its Consequences," *Working Paper*.