Optimal Recommender System Design

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This version: October 20, 2021

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Abstract

Intermediaries like Amazon and Google recommend products and services to consumers for which they receive compensation from the recommended sellers. Nevertheless, consumers will find these recommendations useful if they are informative about the quality of the match between the sellers' offerings and the consumer's needs. The intermediary would like the consumer to purchase the product from the seller who pays the most for a recommendation but is constrained because consumers will not follow the recommendation unless it is in their interest to do so. I frame the intermediary's problem as a mechanism design problem in which the mechanism designer cannot directly choose the outcome, but must encourage the consumer to choose the desired outcome. I show that in the optimal mechanism, the recommended seller has the largest non-negative virtual willingness to pay adjusted for the cost of persuasion. The optimal mechanism can be implemented via a handicap auction.

I use this model to examine the regulatory question of whether platforms should be allowed to use data reflecting sellers' private information, such as margins and bidding history. The use of data always benefits the intermediary, but can either benefit or harm the consumers and sellers. A special class of data is interpreted as the intermediary monopolizing a product market with private label products, and this is shown to benefit the consumer. I also examine a welfare-maximizing mechanism: relative to the revenue-maximizing mechanism, it reduces the intermediary's revenue but increases the consumer surplus and sellers' profits. An alternative interpretation of the model as a search engine is discussed.

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1 Introduction

A consumer often relies on an intermediary's recommendations in deciding which product to purchase. Sellers of each product pays the intermediary in exchange for recommending their products. The intermediary designs and commits to a recommendations rule to maximize its revenue while persuading the consumer to take the recommendation the intermediary gives. For example, Amazon recommends sponsored products based on the consumer's preference, but only if the sellers pay for it; Google and Facebook display advertisements that the target consumer would likely to like, but do so only if the advertiser wins the auction. The intermediary's problem of designing a pair of a recommendations rule and a payment rule to maximize its revenue is what I call by an *optimal recommender system design* problem.

An intermediary collects and uses additional information about sellers to further raise its revenue. Amazon often demands receipts from third-party sellers to prove their products' authenticity. The receipts often contain sensitive information such as from where at what prices the products are purchased, which would enable Amazon to sell the identical products without leaving any margin to third-party sellers[1]. Google is accused of using past bidding data to raise advertisers' winning bids[2]. Facebook is accused of using data collected from sellers to better promote its own services[3]. Regulators have initiated a series of antitrust investigations on intermediaries' use of additional information about sellers[4]. At the heart of the investigation, there is a question of whether the intermediary's collection and use of additional information benefits or harms the consumer, intermediary and sellers.

This paper studies a model of selling a recommender system and the welfare consequences of the intermediary using additional information about sellers. In the model, there are three types of players: a consumer, N sellers and an intermediary. The consumer may choose from one of the N products from each seller or his outside option. But the consumer *does not know his preference* over the products. Instead, the intermediary *knows the consumer's preference*

[1] A report by *U.S. House Judiciary Committee's Subcommittee on Antitrust, Commercial, and Administrative Law*, "Investigation of Competition in Digital Markets," https://judiciary.house.gov/uploadedfiles/competition_in_digital_markets.pdf?utm_campaign=4493-519

[2] A coverage by *The Wall Street Journal*, "Google's Secret 'Project Bernanke' Revealed in Texas Antitrust Case," https://www.wsj.com/articles/googles-secret-project-bernanke-revealed-in-texas-antitrust-case-11618097760

[3] A press release by *The European Commission*, "Antitrust: Commission opens investigation into possible anticompetitive conduct of Facebook," https://ec.europa.eu/commission/presscorner/detail/en/IP_21_2848

[4] The European Commission has launched an antitrust investigation against Amazon (https://ec.europa.eu/commission/presscorner/detail/en/ip_20_2077) and Facebook (https://ec.europa.eu/commission/presscorner/detail/en/IP_21_2848). Ten states led by Texas have sued Google for its anti-competitive practice in online advertisement market (https://www.wsj.com/articles/states-sue-google-over-digital-ad-practices-11608146817?mod=article$_inline$) with Project Bernanke being one of them.

better than the consumer himself and monetizes this knowledge by collecting payments from sellers in exchange for recommending their products. Sellers have private information about their *private willingness to pay*. The intermediary designs and commits to recommendations rules that persuades the consumer to take the recommended action, and transfer rules that elicits the private information of sellers in order to maximize its revenue.

I first characterize a revenue-maximizing recommender system. An optimal recommender system is characterized by recommending the product with the highest non-negative virtual willingness to pay adjusted for the cost of persuasion. The intermediary makes the revenue by inducing the consumer to purchase the product of the seller with high willingness to pay. To better incentivize the high type sellers to report their types truthfully, the intermediary has to commit to recommending the products with high probabilities if the sellers report as high types and commit to recommending other products or the outside option otherwise. Whether the intermediary can commit to such recommendations rule depends on whether the intermediary can persuade the consumer to take the recommended options. Whenever the intermediary recommends an option that is bad for the consumer, it lowers the consumer's incentive to obey the recommendation. To keep the consumer incentivized to obey the recommendation, the intermediary sometimes has to recommend options that are good for the consumer, even though they are not necessarily the most profitable for the intermediary. That is, the persuasion is costly. The intermediary charges the monetary value of the persuasion cost onto the sellers, and recommends the product with the highest virtual willingness to pay adjusted for the cost of persuasion (Theorem 1.a and Theorem 1.b). The optimal recommender system can be implemented through a version of handicap auction (Theorem 12).

I then analyze how allowing the intermediary to use *additional information* and learn partially about sellers' private information changes recommendations rule and welfare of the consumer, sellers and intermediary. Additional information changes the intermediary's preference over whether to recommend products over the outside option, and which product to recommend, and hence, the optimal recommender system accordingly (Theorem 3.a and Theorem 3.b). The intermediary's preference over products is summarized by *rates of substitution* (Proposition 1). When information rent is small, whether additional information increases rates of substitution completely determines whether the consumer is harmed (Theorem 4) and provides a sufficient condition for the seller to be harmed (Theorem 5 and Corollary 1) from the intermediary's use of additional information. Under a general environment, whether additional information increases rates of substitution still provides sufficient conditions under which additional information benefits or harms the consumer (Theorem 7) and sellers (Theorem 8 and Corollary 2) when the value of the outside option is low. In any case, additional information always increases the intermediary's revenue (Theorem 6).

The remainder of this paper proceeds as follows. Section 1.1 discusses related literature; Section 2 describes our model. Section 3 characterizes the optimal recommender system and its implementation. Section 4 characterizes how additional information changes the optimal recommender system and its welfare consequences. Section 5 discusses alternations and relaxation of the assumptions. Section 6 concludes. All proofs are collected in the Appendix.

1.1 Literature Review

Sales of Information

This paper contributes to the emerging literature of the sale of information by a monopolistic information intermediary. Starting with Admati and Pfleiderer (1986) and Admati and Pfleiderer (1990), many of the papers in this literature focuses on how to sell information where the information buyer receives the information for a better decision making. The monopolistic information intermediary sells experiments to a decision maker who has private information about the states of the world Bergemann, Bonatti, and Smolin (2018), statistics to a decision maker who has private information about what kinds of information it needs (Segura-Rodriguez (2021)) and consumer segments to a producer who then uses to better price-discriminate the consumer (Yang (2021)).

In contrast, the information buyer does not receive the information directly, but pays only to influence on information provision onto other parties. The closest to my paper is Yang (2019) that studies an intermediary who designs a recommendations rule, a transfer rule and a pricing rule over a single product and seller. He finds that three business models that govern the pricing rule are revenue equivalent if and only if the gains from trade are large enough. Our paper differs from his paper in that we study the problem of recommendations over *multiple* products and the outside option instead of a *single* product. That the intermediary recommends over multiple products has an economic importance as one source, and the only source for the consumer with a low outside option value, of consumer surplus is that the intermediary can better distinguish between ex-ante identical products. Inderst and Ottaviani (2012a) and Aridor and Gonçalves (2021) also analyze the problem of information buyer paying the intermediary to influence on the other parties, but information buyers in their setups do not have private information.

Persuasion

Between the intermediary and the consumer, the intermediary serves as a persuader. That the persuader mixes favorable and unfavorable states to induce the receiver to take favorable action is widely studied (Kamenica and Gentzkow (2011), Bergemann and Morris (2019)) in various contexts including financial advisor (Rayo and Segal (2010), Inderst and Ottaviani (2012b), Onuchic (2021)), platform (Yang (2019), Aridor and Gonçalves (2021)) and politics (Alonso and Câmara (2016)).

What has not been studied is that the intermediary's preference over the consumer's actions are endogenously given by the payment functions that it sets with the sellers. Furthermore, the persuader's preference is state-dependent where the state space itself is multi-dimensional and possibly infinite and multiple actions. Each of these has been a limiting factor in analyzing persuasion and communications problems. To my knowledge, there has been no paper with the persuader's endogenous preference. The persuader's preference is often simplified to be independent of the states (Lipnowski and Ravid (2020)), depend only on the posterior mean (Gentzkow and Kamenica (2016), Dworczak and Martini (2019)) or semi uppercontinuous in beliefs (Dworczak and Kolotilin (2019), Dizdar and Kováč (2020)). The state space or action space are often simplified to be finite (Kamenica and Gentzkow (2011)) or even binary (Rayo and Segal (2010), Alonso and Câmara (2016), Kolotilin (2018), Onuchic (2021), Aridor and Gonçalves (2021)). Without these assumptions, the three popular tools in Bayesian persuasion are not always tractable: concavification (Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Kolotilin (2018), Galperti and Perego (2018), Dworczak and Kolotilin (2019), Dworczak and Martini (2019), Dizdar and Kováč (2020)). In this paper, I demonstrate that a persuasion problem can still be tractably analyzed by focusing on a class of recommendations rule called *value-switching monotone* after lifting *all* of these restrictions as long as we impose a symmetry on the primitives and monotonicity in preference.

Auction and information provision

Between the intermediary and the sellers, the intermediary serves as an auctioneer selling the right to be recommended. Without the persuasion, the intermediary's problem is an optimal auction design problem (Myerson (1981)). That the intermediary persuades with its preference given by the payment functions implies that the object of sales, recommendations probabilities, is designed endogenously. Cramton (2013) points out a potential importance of the object design in conjunction with the auction design in the context of spectrum and combinatorial clock auction.

In designing the auction, the intermediary jointly designs an optimal provision of the private information it has (Milgrom and Weber (1982)). Interestingly, under the optimal recommender system, the intermediary *fully discloses* the information it has to the sellers, because the change in the sellers' willingness to pay after learning about the consumer's valuations can fully be extracted after appropriately adjusting for the persuasion cost. That the intermediary fully discloses its private information over any partially revealing information is first observed by Eső and Szentes (2007). Indeed, the optimal recommender system can be implemented by the handicap auction from Eső and Szentes (2007) adjusted for the cost of persuasion.

Lastly, there is a literature on selling online advertisements. Edelman, Ostrovsky, and Schwarz (2007) promotes *positions* as the unique feature of the online advertisements and shows that the generalized second price auction over the positions has a unique perfect Bayesian equilibrium under which advertisers attain the same position and payoff under Vickery-Clark-Groves mechanism (Vickrey (1961), Clarke (1971), Groves (1973)). Position auctions study a particular class of multigood auction in which per-click bidding is used over the positions, where the positions are often ranked by their profitability (number of clicks). Athey and Ellison (2011) studies a position auction with consumers' search decisions over the positions, and emphasize the positions' role as an informational intermediary that signals how valuable each position is to the consumers. I focus on the role of the online advertisements as an informational intermediary with fully flexible set of mechanisms. Positions and per-click bidding are interpreted as an indirect mechanism with additional structures on bidders' payoffs and message space.

2 Model

2.1 Setup

There is a consumer, N sellers and an intermediary. The consumer may choose from one of the N products from each seller or his outside option. The consumer has valuation $v_i \in \mathbb{R}_+$ for each product $i \in \mathcal{N}$ that the consumer does not know *a priori*. The consumer only knows that each v_i is independently and identically drawn from a distribution F that has a compact support $\mathcal{V}$. The value of the consumer's outside option is $v_0 \in \mathbb{R}_+$ and is common knowledge. The consumer's payoff is quasi-linear in the valuation, so that if he takes each action $i \in \mathcal{N} \cup \{0\}$ with probability r_i, his payoff is

$$\sum_{i \in \mathcal{N} \cup \{0\}} v_i r_i. \tag{1}$$

Each seller has private willingness to pay θ_i and value-dependent willingness to pay $w(v_i)$. The private willingness to pay θ_i is seller-specific or product-specific information that the seller privately knows such as its marginal cost[5], and is independently drawn from a common

[5] For example, if each product's price p_i is public knowledge and marginal cost c_i is each seller's private

distribution G. The distribution G is absolutely continuous on an interval $[\underline{\theta}, \bar{\theta}] = \Theta$ with $-\infty < \underline{\theta} < \bar{\theta} < \infty$.

The value-dependent willingness to pay $w(v_i)$ is a part of the seller's profit that is strictly increasing in the consumer's valuation v_i. This is a reduced form way to capture any interactions between the consumer and sellers including return rate, conversion rate and price-discrimination[6]. However, sellers do not know v_i but the distribution F from which v_i is drawn from[7]. If the seller pays t_i to the intermediary in exchange for recommending product i to the consumer $\boldsymbol{v} = (v_1, \cdots, v_N) \in \mathcal{V}^N = \boldsymbol{\mathcal{V}}$ with probability r_i, the seller's profit is

$$(\theta_i + w(v_i))r_i - t_i.$$

The intermediary knows the consumer's preference better than the consumer himself. The intermediary knows the consumer's valuation $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$. Before learning $\boldsymbol{v}$, the intermediary can design and commit to an information structure $(\sigma, \mathcal{S})$ where

$$\sigma : \boldsymbol{\mathcal{V}} \to \Delta\mathcal{S}$$

from which the consumer learns about $\boldsymbol{v}$. In exchange for providing information that sellers want the consumer to learn, the intermediary collects transfers according to $\boldsymbol{t} = (t_1, \cdots, t_N) : \boldsymbol{\mathcal{V}} \to \mathbb{R}^N$ from the sellers, and earn a revenue of

$$\sum_{i \in \mathcal{N}} t_i.$$

The intermediary also has technology to create a marketplace through which the sellers can interact with the consumer and only through it. That is, the sellers cannot interact with the consumer without the intermediary.

Assumptions

The following two regularity assumptions about the sellers' preference and information are maintained throughout the paper:

Assumption 1 (Positive Willingness to Pay). *$\theta + w(v) > 0$ for all $\theta \in \Theta$ and $v \in \mathcal{V}$.*

knowledge, then private willingness to pay is $\theta_i = p_i - c_i$.

[6] A consumer with a higher v_i may have a lower return rate, which leads to a higher profit and is captured in a strictly increasing $w(v)$. Similarly, a consumer with a higher v_i may have a higher conversion rate from clicking advertisements to purchasing the products and may be charged with a higher price, each of which contributes to a higher profit for the seller. Instead of modeling each of the specific instances, we summarize such interactions into $w(v)$ that strictly increases in v.

[7] Combined with the assumption that the intermediary knows v_i, $w(v_i)$ may be interpreted as a part of the seller's profit that intermediary knows better than the seller himself.

Assumption 2 (Myerson's Regularity). $\theta - \frac{1-G(\theta)}{g(\theta)}$ *strictly increases in* θ.

Assumption 1 states that sellers always prefer selling their products to the consumer than not selling theirs. Assumption 2 is a typical Myerson's regularity condition under which virtual private willingness to pay strictly increases.

The following two assumptions describes two different informational environments with varying sizes of information rent.

Assumption 3.a (Small Information Rent). $\inf_{\theta\in\Theta, v\in\mathcal{V}} \theta - \frac{1-G(\theta)}{g(\theta)} + w(v) > 0$.

Assumption 3.b (Large Information Rent). $\inf_{\theta\in\Theta, v\in\mathcal{V}} \theta - \frac{1-G(\theta)}{g(\theta)} + w(v) < 0 < \sup_{\theta\in\Theta, v\in\mathcal{V}} \theta - \frac{1-G(\theta)}{g(\theta)} + w(v)$.

Let us the inverse hazard rates $\frac{1-G(\theta)}{g(\theta)}$ by *state-dependent information rent* the intermediary pays to sellers in order to elicit their private information. Assumption 3.a considers an informational environment under which state-dependent information rent is small enough, so that the intermediary always prefer recommending products over the outside option $\inf_{\theta\in\Theta, v\in\mathcal{V}} \theta - \frac{1-G(\theta)}{g(\theta)} + w(v) > 0$. Assumption 3.b considers an informational environment under which state-dependent information rent is large enough to prevent some product recommendations $\inf_{\theta\in\Theta, v\in\mathcal{V}} \theta - \frac{1-G(\theta)}{g(\theta)} + w(v) < 0$, but not enough to prevent all product recommendations $\sup_{\theta\in\Theta, v\in\mathcal{V}} \theta - \frac{1-G(\theta)}{g(\theta)} + w(v) > 0$. If the state-dependent information rent is so large that $\sup_{\theta\in\Theta, v\in\mathcal{V}} \theta - \frac{1-G(\theta)}{g(\theta)} + w(v) < 0$, then the intermediary shuts down its business.

2.2 Recommender Systems

The intermediary runs a mechanism under which sellers can send messages. The intermediary then collects transfers from sellers and provides information to the consumer. The consumer updates his belief on valuations of the products, and chooses one of the products or the outside option.

Applying revelation principle arguments from mechanism design (Myerson (1981)) and information design (Bergemann and Morris (2019)), it can be shown that it is without loss of generality in restricting attention to incentive compatible, individually rational and obedient *recommender systems* under which *1)* the sellers report their types truthfully; *2)* the intermediary collects transfers from the sellers and recommends one of the products or the outside option to the consumer; *3)* the consumer obeys the recommendation.

Formally, a pair $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ is a recommender system where $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ is a *recommendations rule* that states the probability of recommending $i \in \{0\} \cup \mathcal{N}$ such that $\sum_{i\in\{0\}\cup\mathcal{N}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$, and $t_i(\boldsymbol{v}, \boldsymbol{\theta})$ is a *transfer* that the seller $i \in \mathcal{N}$ makes to the intermediary when the true state of the world is $(\boldsymbol{v}, \boldsymbol{\theta})$.

A recommender system is incentive compatible if for all $i \in \mathcal{N}$ and $\theta_i, \theta_i' \in \Theta$

$$\begin{aligned} IC_i : &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \Big[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - t_i(\boldsymbol{v}, \boldsymbol{\theta})\Big] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) \\ &\geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \Big[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}) - t_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i})\Big] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}), \end{aligned} \tag{2}$$

individually rational if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta$

$$IR_i : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \Big[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - t_i(\boldsymbol{v}, \boldsymbol{\theta})\Big] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) \geq 0, \tag{3}$$

and obedient if for all $i, j \in \mathcal{N} \cup \{0\}$,

$$OB_{ij} : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_i \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_j \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}). \tag{4}$$

That is, a recommender system is incentive compatible and individually rational if each seller finds it optimal to report truthfully and to participate while not knowing anything but for his own type θ_i, and are denoted by IC_i and IR_i. A recommender system is obedient if the consumer finds it optimal to take the recommneded option i over another option j while not knowing any of $(\boldsymbol{v}, \boldsymbol{\theta})$. This incentive constraint is called an *obedience constraint from i to j* and is denoted by OB_{ij}.

The intermediary designs and commit to a recommender system $(\boldsymbol{r}, \boldsymbol{t})$ that satisfies incentive compatibility, individual rationality and obedience constraints to maximize its expected revenue

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} t_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}).$$

A defining feature of my model is that the intermediary is solving two different kinds of problems at the same time. Between the intermediary and sellers, the problem is an auction problem under which the intermediary tries to raise the highest revenue from sellers by manipulating the probability of sales. Between the intermediary and consumer, the problem is a persuasion problem under which the intermediary persuades the consumer to take the option that the intermediary wants the consumer to take. In the persuasion problem, the intermediary's preference over recommendations is given by $\sum_{i \in \mathcal{N}} t_i(\boldsymbol{v}, \boldsymbol{\theta})$, an endogenously determined state-dependent preference with multi-dimensional, infinite state space.

I conclude this section with the timing of the game.

Timing of the Game

We summarize the description of the model with the timing of the game:

1. Intermediary offers and commits to a recommender system $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ where $\sum_{i \in \{0\} \cup \mathcal{N}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.
2. Each seller reports his type.
3. Intermediary observes the consumer's match values $\boldsymbol{v}$.
4. Intermediary recommends an action and collects transfers according to $(\boldsymbol{r}(\boldsymbol{v}, \boldsymbol{\theta}), \boldsymbol{t}(\boldsymbol{v}, \boldsymbol{\theta}))$.
5. Consumer gets a recommendation and takes an action.

3 Optimal Recommender System

In this section, the intermediary's optimal recommender system is characterized. The intermediary's problem is first recast as a Bayesian persuasion problem. An optimal recommendations rule to the Bayesian persuasion problem is characterized by a novel class of recommendations rule that I call *value-switching monotone*. A corresponding transfer function that supports the optimal recommendations as incentive compatible and individually rational is given. A version of handicap auction (Eső and Szentes (2007)) is an indirect mechanism to implement the optimal recommender system.

3.1 Intermediary as a Bayesian Persuader

Notice that the consumer is indifferent for any transfers $\boldsymbol{t}$ between the intermediary and the sellers as they do not directly impact the consumer's payoff (1). This means that a transfer $\boldsymbol{t}$ that supports a given recommendations rule $\boldsymbol{r}$ as incentive compatible and individually rational is characterized independently of the obedience constraints (4). The standard arguments from mechanism design (Myerson (1981)) may be applied to pin down sellers' interim transfer functions as a function of a recommendations rule, and obtain the following lemma.

Lemma 1. *Suppose that a recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^N$ maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \tag{5}$$

subject to obedience constraints (4) and monotonicity constraints, that is, for all $\theta_i > \theta_i'$

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \theta_i, \boldsymbol{\theta}_{-i}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d(\theta_i, \boldsymbol{\theta}_{-i})) \\ &\geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}} r_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d(\theta_i', \boldsymbol{\theta}_{-i})). \end{aligned} \tag{6}$$

Suppose also that

$$t_i(\boldsymbol{v},\boldsymbol{\theta}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i})d\tilde{\theta}_i. \tag{7}$$

Then, $(\boldsymbol{r},\boldsymbol{t})$ *is an optimal recommender system.*

Ignoring the monotonicity constraints (6), Lemma 1 recasts the intermediary's problem of designing a recommender system $(\boldsymbol{r},\boldsymbol{t})$ as a Bayesian persuasion problem with state-dependent preference over recommendations given by virtual willingness to pay $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$. Note that under small information rent environment as in Assumption 3.a, the intermediary always prefers recommending products over the outside option; but the intermediary sometimes prefers recommending the outside option under large information rent environment as in Assumption 3.b.

Another way to interpret Lemma 1 is that the intermediary is designing an optimal auction subject to obedience constraints. Without obedience constraints, the intermediary can force the consumer to take options whatever the intermediary wants the consumer to take. The intermediary then only cares about incentivizing sellers to report their types truthfully. The intermediary's problem reduces to an optimal auction design problem (Myerson (1981)). The optimal recommendations rule is to recommend a product with the highest non-negative virtual willingness to pay $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$, and the outside option if all sellers' virtual willingness to pays are negative. Note that under small information rent environment as in Assumption 3.a, the intermediary does not use the outside option to incentivize sellers to report truthfully; but the intermediary does use the outside option under large information rent environment as in Assumption 3.b.

With obedience constraints, however, the intermediary has to persuade. The above recommendations rule may fail to persuade consumer to take recommended option. Whenever it fails, the intermediary has to recommend options that are good for consumer, even though they are not the most profitable for the intermediary. The optimal way to incentivize the consumer to obey recommendations depends on which obedience constraints bind. A class of recommendations rule called *value-switching monotone* provides a systematic way to identify binding and non-binding constraints.

3.2 Value-Switching Monotone Recommendations Rule

The problem of maximizing (5) subject to obedience constraints (4) is a well-defined Bayesian persuasion problem where the intermediary's state-dependent preference over recommendations given by virtual willingness to pay $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$. However, that the intermediary has state-dependent preference over multi-dimensional and possibly infinite state space and multiple actions brings a difficulty in applying three popular tools in Bayesian persuasion literature: concavification (Aumann and Maschler (1995), Kamenica and Gentzkow (2011)), convex function characterization (Gentzkow and Kamenica (2016)) and duality (Kolotilin (2018), Dworczak and Kolotilin (2019))[8]. Instead, I take another approach: guess and verify using value-switching monotone recommendations rules.

Definition 1. A recommendations rule $\boldsymbol{r}$ is *value-switching monotone* if

1. $r_0(\boldsymbol{v},\boldsymbol{\theta})$ decreases in (v_i,θ_i) for all $i \in \mathcal{N}$.

2. $r_i(\boldsymbol{v},\boldsymbol{\theta})$ increases in (v_i,θ_i) for all $i \in \mathcal{N}$.

3. $r_i(\boldsymbol{v},\boldsymbol{\theta})$ decreases whenever v_j is switched with a larger v_i for all $i,j \in \mathcal{N}$, i.e. for all $i,j \in \mathcal{N}$, $(\boldsymbol{v}_{-ij},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}}_{-ij} \times \boldsymbol{\Theta}$ and $v > v'$,

$$r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \geq r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}).$$

Value-switching monotonicity is weaker than monotonicity which would replace the third condition with $r_i(\boldsymbol{v},\boldsymbol{\theta})$ decreasing in v_j for all $j \in \mathcal{N} \setminus \{i\}$. The optimal recommendations rule in Theorem 1.a is value-switching monotone, but not monotone. Value-switching monotonicity requires $r_i(\boldsymbol{v},\boldsymbol{\theta})$ to be increasing in θ_i but does not require any particular behavior with $\boldsymbol{\theta}_{-i}$.

The main challenge of departing away from concavification and duality methods lies in identifying binding obedience constraints. With $N+1$ options, there are $\frac{N(N+1)}{2}$ obedience constraints to check. This is a seemingly daunting task. The following two lemmas show that restricting attention to value-switching monotone recommendations rules is sufficient to identify which obedience constraints bind when.

Lemma 2. *Any value-switching monotone recommendations rule $\boldsymbol{r}$ satisfies obedience constraints between products, i.e. OB_{ij} for all $i,j \in \mathcal{N}$.*

Lemma 2 states that a value-switching monotone recommendations rule always satisfies obedience constraints between products. This immensely reduces the number of possibly

[8]Concavification has limited applicability when state space is large (Gentzkow and Kamenica (2016)). Convex function characterization necessarily assumes the sender's payoff to depend only on the expected value of the states (Gentzkow and Kamenica (2016)). Duality approach often assumes state space to be either an interval or discrete (Kolotilin (2018), Galperti and Perego (2018)), and the sender's payoff to be semi upper-continuous in beliefs (Dworczak and Kolotilin (2019), Dworczak and Martini (2019), Dizdar and Kováč (2020)), which are not necessarily the case in this environment.

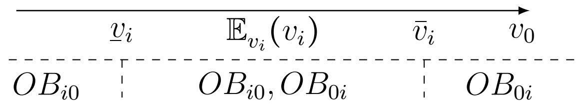

Figure 1: A graphical illustration of obedience constraints out of OB_{i0} and OB_{0i} that are satisfied in each region.

binding constraints to check from $\frac{N(N+1)}{2}$ to $2N$. For any value-switching monotone recommendations rule $\boldsymbol{r}$, it is necessary and sufficient to check two types of obedience constraints: obedience constraints from outside option to products,

$$OB_{0i} : \int_{\mathcal{V}} \int_{\boldsymbol{\Theta}} (v_0 - v_i) r_0(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0, \tag{8}$$

and those from products to outside option.

$$OB_{i0} : \int_{\mathcal{V}} \int_{\boldsymbol{\Theta}} (v_i - v_0) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{9}$$

I now examine for each given value-switching monotone recommendations rule, whether each of OB_{i0} and OB_{0i} is satisfied for what values of v_0.

Lemma 3. *Let $\boldsymbol{r}$ be a value-switching monotone recommendations rule. For any $i \in \mathcal{N}$, there are $0 \leq \underline{v}_i \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}_i \leq \infty$ such that*

1. *OB_{i0} is satisfied if and only if $v_0 \leq \bar{v}_i$,*
2. *OB_{0i} is satisfied if and only if $v_0 \geq \underline{v}_i$,*

where

$$\bar{v}_i = \begin{cases} \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))} & \text{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) > 0 \\ \infty & \text{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) = 0 \end{cases}$$

and

$$\underline{v}_i = \begin{cases} \mathbb{E}_{v_i}(v_i) + \frac{Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_0(\boldsymbol{v},\boldsymbol{\theta}))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta}))} & \text{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) > 0 \\ 0 & \text{if } \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) = 0 \end{cases}.$$

Note that $Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_0(\boldsymbol{v},\boldsymbol{\theta})) \leq 0 \leq Cov_{\boldsymbol{v},\boldsymbol{\theta}}(v_i, r_i(\boldsymbol{v},\boldsymbol{\theta}))$ since $r_i(\boldsymbol{v},\boldsymbol{\theta})$ increases in v_i and $r_0(\boldsymbol{v},\boldsymbol{\theta})$ decreases in v_i, so that $\underline{v}_i \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}_i$. The intuition behind Lemma 3 is simple. As the value of the outside option v_0 increases, it becomes harder to persuade the consumer to purchase a product, and easier to take the outside option. Consequently, for a given $\boldsymbol{r}$, OB_{i0} is satisfied if and only if v_0 is small enough; OB_{0i} if and only if v_0 is large enough. See Figure 1.

The thresholds $\bar{v}_i$ and $\underline{v}_i$ are given by conditional expected value of the product i conditioning on recommending the product i and the outside option, respectively, whenever the conditional expectations are well-defined, that is, $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1) > 0$ and $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta}) = 1) > 0$. The conditions $v_0 \leq \bar{v}_i$ and $v_0 \geq \underline{v}_i$ are, respectively, equivalent to OB_{i0} and OB_{0i}. If $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1) = 0$, the intermediary never recommends the product i almost surely, so that the intermediary does not worry about incentivizing the consumer to purchase product i over the outside option. In other words, OB_{i0} is always satisfied for any v_0, and hence, $\bar{v}_i = \infty$. Similarly, if $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta}) = 1) = 0$, the intermediary never recommends the outside option almost surely, so that OB_{0i} is always satisfied for all v_0, and hence, $\underline{v}_i = 0$.

Last, but not the least, it always is that $\underline{v}_i \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}_i$. In particular, if $v_0 = \mathbb{E}_{v_i}(v_i)$, the consumer is ex-ante indifferent between all options, so that the consumer is obedient to an *unconstrained optimal recommendations rule* $\boldsymbol{\rho}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$

$$\rho_i^*(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \mathcal{M} \text{ and } \theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) \geq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{10}$$

where $\mathcal{M} = \{i \in \mathcal{M} \mid \arg\max_{j\in\mathcal{N}}\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\}\}$. An unconstrained optimal recommendations rule is obtained by maximizing (5) without taking obedience constraints (4) and monotonicity constraints (6) into account. Under $\boldsymbol{\rho}^*$, the intermediary only cares about incentivizing sellers to report their private information truthfully, and hence, recommends an option with the highest non-negative virtual willingness to pay $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$ as in Myerson (1981). Note that $\boldsymbol{\rho}^*$ satisfies monotonicity constraints (6), is value-switching monotone and $\rho_i^*(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $i \in \mathcal{N} \cup \{0\}$ almost surely.

Let $\bar{v}^* = \bar{v}_i$ and $\underline{v}^* = \underline{v}_i$. Since $\boldsymbol{\rho}^*$ is symmetric[9], the thresholds $\bar{v}_i$ and $\underline{v}_i$ are identical across all products $i \in \mathcal{N}$, so that $\bar{v}^*$ and $\underline{v}^*$ are well-defined. Under a small information rent environment as in Assumption 3.a,the intermediary always prefers recommending one of products over the outside option, so that $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(\rho_0^*(\boldsymbol{v},\boldsymbol{\theta}) = 1) = 0$ and $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(\rho_i^*(\boldsymbol{v},\boldsymbol{\theta}) = 1) > 0$ for any $i \in \mathcal{N}$, so that

$$\begin{cases} \bar{v}^* = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid \rho_i^*(\boldsymbol{v},\boldsymbol{\theta}) = 1) \\ \underline{v}^* = 0 \end{cases}. \tag{11}$$

Under a large information rent environment as in Assumption 3.b, the intermediary some-

[9] A recommendations rule $\boldsymbol{r}$ is symmetric if for any $i \in \mathcal{N}$, any bijective function $\iota : \mathcal{N} \to \mathcal{N}$ and any $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = r_{\iota(i)}(\boldsymbol{v}^\iota,\boldsymbol{\theta}^\iota)$$

where $(\boldsymbol{v}^\iota,\boldsymbol{\theta}^\iota)$ is such that $v^\iota_{\iota(i)} = v_i$ and $\theta^\iota_{\iota(i)} = \theta_i$ for all $i \in \mathcal{N}$.

times prefers recommending products over the outside option, and sometimes otherwise, so that $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(\rho_0^*(\boldsymbol{v},\boldsymbol{\theta}) = 1) > 0$ and $\Pr_{\boldsymbol{v},\boldsymbol{\theta}}(\rho_i^*(\boldsymbol{v},\boldsymbol{\theta}) = 1) > 0$ for any $i \in \mathcal{N}$, so that

$$\begin{cases} \bar{v}^* = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid \rho_i^*(\boldsymbol{v},\boldsymbol{\theta}) = 1) \\ \underline{v}^* = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid \rho_0^*(\boldsymbol{v},\boldsymbol{\theta}) = 1) \end{cases}. \tag{12}$$

3.3 Optimal Recommender System

The intermediary's problem, ignoring monotonicity constraints (6), is to maximize the expected sum of virtual willingness to pay (5) subject to at most one kind of obedience constraints (8) and (9). Since both the objective function (5) and constraints (4) are linear in recommendations rules $\boldsymbol{r}$, an optimal recommendations rule may be obtained by applying the method of Lagrangian. That is, taking a point-wise maximization with appropriately chosen Lagrangian multipliers. An optimal recommender system when $v_0 \in (\underline{v}, \bar{v})$ is characterized below.

Theorem 1.a. *Let $v_0 \in (\underline{v}, \bar{v})$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for each $i \in \mathcal{N}$,*

$$r_i^*(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{ \underbrace{\theta_j - \dfrac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)}_{\text{virtual willingness to pay}} - \underbrace{\ell_j^*(\boldsymbol{v})}_{\substack{\text{cost of} \\ \text{persuasion}}}, 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where $\mathcal{M} = \arg\max_{j\in\mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v}) \right\}$, and

$$\ell_i^*(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^*, \bar{v}^*] \\ \lambda_1^*(v_0) \cdot (v_0 - v_i) & \text{if } v_0 > \bar{v}^* \\ \lambda_2^*(v_0) \cdot \sum_{k\in\mathcal{N}} (v_0 - v_k) & \text{if } v_0 < \underline{v}^* \end{cases} \tag{14}$$

where $\lambda_1^*(v_0)$ and $\lambda_2^*(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively. Let $\boldsymbol{t}$ be as in (7). Then, $\boldsymbol{r}^*$ is value-switching monotone and $r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $i \in \mathcal{N} \cup \{0\}$ almost surely, and $(\boldsymbol{r}^*, \boldsymbol{t})$ is an optimal recommender system.

The optimal recommendations rule $\boldsymbol{r}^*$ is characterized by recommending a product with the highest non-negative virtual willingness to pay adjusted for the cost of persuasion, $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) - \ell_i^*(\boldsymbol{v})$, and the outside option if the adjusted virtual willingness to pay is negative for all sellers. When $v_0 \in [\underline{v}^*, \bar{v}^*]$, the cost of persuasion is zero $\ell_i^*(\boldsymbol{v})$. In other words, the optimal recommendations rule (13) is the same as the unconstrained optimal recommendations rule (10), i.e. $\boldsymbol{r}^* = \boldsymbol{\rho}^*$. When $v_0 > \bar{v}^*$, $\boldsymbol{\rho}^*$ violates OB_{i0}. In order to keep the consumer incentivized to purchase a product i when recommended, the intermediary recommends product i more often when its valuation v_i is high and less often otherwise than it would have under $\boldsymbol{\rho}^*$. This means that the intermediary recommends products even though it does not necessarily has the highest virtual willingness to pay $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$. The optimal way to resolve the trade-off between recommending the product with the highest virtual willingness to pay and keeping the consumer incentivized to purchase obey recommendations is given by adjusting the virtual willingness to pay by $\ell_i^*(\boldsymbol{v}) = \lambda_1(v_0) \cdot (v_0 - v_i)$ which is a monetary cost of marginally violating OB_{i0} and recommending a product i. Since the intermediary wants to minimize the loss of revenue from the trade-off, the obedience constraint OB_{i0} binds. When $v_0 < \underline{v}^*$, $\boldsymbol{r}^*$ violates OB_{0i}. The intermediary recommends the outside option 0 more often when all of products have low valuation v_i and less often otherwise than it would have under $\boldsymbol{\rho}^*$, which is captured in the cost of persuasion $\ell_i^*(\boldsymbol{v}) = \lambda_2(v_0) \cdot \sum_{\in \mathcal{N}} (v_0 - v_k)$ and OB_{0i} binds.

When information rent is small as in Assumption 3.a, $\underline{v}^* = 0$ by (11), so that the cost of persuasion (14) reduces to

$$\ell_i^*(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^*, \bar{v}^*] \\ \lambda_1(v_0) \cdot (v_0 - v_i) & \text{if } v_0 > \bar{v}^* \end{cases}.$$

The cost of persuasion related to OB_{0i}, $\ell_i(\boldsymbol{v}) = \lambda_2(v_0) \cdot \sum_{k \in \mathcal{N}} (v_0 - v_k)$ disappears because, under an small information environment, the intermediary always prefers recommending products over the outside option unless recommending the outside option is required for persuasion, and hence, OB_{0i} are irrelevant constraints. When information rent is large as in Assumption 3.b, $\underline{v} \leq \underline{v}^* \leq \mathbb{E}_{v_i}(v_i) \leq \bar{v}^* \leq \bar{v}$ by (12), so that the cost of persuasion is as in (14).

For outside option values that are always above or below the value of the products, the optimal recommender system is characterized in the following theorem.

Theorem 1.b. *1. Let $v_0 > \bar{v}$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that*

$$r_0^*(\boldsymbol{v}, \boldsymbol{\theta}) = 1$$

for all $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$. Let $\boldsymbol{t}$ be as in (7). Then, $\boldsymbol{r}^*$ is value-switching and $r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $i \in \mathcal{N} \cup \{0\}$ almost surely, and $(\boldsymbol{r}^*, \boldsymbol{t})$ is an optimal recommender system.

2. Let $v_0 < \underline{v}$. Let $\boldsymbol{r}^* : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for *each* $i \in \mathcal{N}$,

$$r_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^*|} & \textit{if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) \right\} \\ 0 & \textit{otherwise} \end{cases} \tag{15}$$

where $\mathcal{M}^* = \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) \right\}$ *and*

$$\ell_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = 0 \textit{ for all } i \in \mathcal{N} \textit{ and } (\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \tag{16}$$

Let $\boldsymbol{t}$ *be as in* (7). *Then,* $\boldsymbol{r}^*$ *is value-switching monotone and* $r_i(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ *for some* $i \in \mathcal{N} \cup \{0\}$ *almost surely, and* $(\boldsymbol{r}, \boldsymbol{t})$ *is an optimal recommender system.*

When $v_0 > \bar{v}$, the consumer always prefers the outside option over the products. For such a consumer, the only obedient recommendations rule is to always recommend the outside option. When $v_0 < \underline{v}$, the consumer always prefers products over the outside option, but does not know which product the consumer prefers the most. The intermediary is restricted to recommend products only, but not the outside option. Consequently, the intermediary always recommends the product with the highest virtual willingness to pay, even though it may be negative.

When $v_0 = \bar{v}$, the intermediary is restricted to recommending outside option except for when $v_i = \bar{v}$ for some $i \in \mathcal{N}$. Conditioning on such $\boldsymbol{v}$, the intermediary recommends a product with the highest non-negative virtual willingness to pay among the products that have the valuation of $\bar{v}$. When $v_0 = \underline{v}$, the intermediary is restricted to recommending products except for when $v_i = \underline{v}$ for all $i \in \mathcal{N}$. Conditioning on such $\boldsymbol{v}$, the intermediary recommends a product with the highest non-negative virtual willingness to pay.

Rate of Substitution. The optimal recommendations rules (13) and (15) may equivalently characterized by *rates of substitution*. A rate of substitution $RS_{ij}(\boldsymbol{v}, \boldsymbol{\theta})$ between products i and j at a given point $(\boldsymbol{v}, \boldsymbol{\theta})$ is

$$RS_{ij}(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{\frac{\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)}\right)}{\theta_i - \theta_j}}{\frac{\left(w(v_i) + \ell_i(\boldsymbol{v})\right) - \left(w(v_j) + \ell_j(\boldsymbol{v})\right)}{v_i - v_j}}.$$

A rate of substitution $RS_{ij}(\boldsymbol{v}, \boldsymbol{\theta})$ is an amount of a change in valuation v needed for the intermediary to be indifferent between recommending i and j to compensate for their difference in their private willingness to pay θ. Consider two products i and j with (v_i, θ_i), (v_j, θ_j) and $\theta_i > \theta_j$. The intermediary would be indifferent between recommending i and j if and only if i has lower v_i as much as $-RS_{ij}(\boldsymbol{v},\boldsymbol{\theta})(\theta_i-\theta_j)$, that is,

$$v_i - v_j = -RS_{ij}(\boldsymbol{v},\boldsymbol{\theta})(\theta_i-\theta_j).$$

If $v_i - v_j$ is larger than $-RS_{ij}(\boldsymbol{v},\boldsymbol{\theta})(\theta_i-\theta_j)$, then the intermediary prefers recommending i; if smaller, then j. At a given state $(\boldsymbol{v},\boldsymbol{\theta})$, the intermediary prefers recommending product $i \in \mathcal{N}$ over all other products if and only if

$$v_i - v_j \geq -RS_{ij}(\boldsymbol{v},\boldsymbol{\theta})(\theta_i-\theta_j) \ \forall j \in \mathcal{N}.$$

Using this property, the intermediary's optimal recommendations rule $\boldsymbol{r}^*$ can equivalently be characterized by the following two-step procedure.

Proposition 1. *Let $v_0 \in (\underline{v},\bar{v})$ or $v_0 < \underline{v}$, and $\boldsymbol{\ell}^*$ be as in (14) or (16), respectively. Let $\boldsymbol{r}^{RS}$ be a recommendations rule that is constructed by*

1. *If $\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j(\boldsymbol{v}) < 0 \ \forall j \in \mathcal{N}$, then $r_0(\boldsymbol{v},\boldsymbol{\theta}) = 1$.*
2. *Otherwise,*

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^{RS}|} & \textit{if } v_i - v_j \geq -RS_{ij}(\boldsymbol{v},\boldsymbol{\theta})(\theta_i-\theta_j) \ \forall j \in \mathcal{N} \\ 0 & \textit{otherwise} \end{cases} \tag{17}$$

where $\mathcal{M}^{RS} = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j(\boldsymbol{v})\right\}$

Then,

$$\boldsymbol{r}^{RS} = \boldsymbol{r}^*.$$

The recommendations rule in Proposition 1 is constructed by two-steps: First, if the adjusted marginal revenue is negative for all $j \in \mathcal{N}$, then recommend the outside option. Second, if there is at least one product whose marginal revenue is non-negative, then recommend the product that the intermediary prefers the most according to the rates of substitution as in (17). In case of uniform distribution with $w(v) = v$ and $v_0 = \mathbb{E}_{v_i}(v_i)$, the rate of substitution is

$$RS_{ij}(\boldsymbol{v},\boldsymbol{\theta}) = 2$$

for all $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.

Note that the larger the rates of substitutions are, the more likely the intermediary recommends a product with a higher θ than v.

3.4 Implementation

We conclude this section by showing that the optimal recommender system can be implemented using a handicap auction (Eső and Szentes (2007)) after adjusting for the cost of persuasion $\boldsymbol{\ell}(\boldsymbol{v})$. The handicap auction consists of two rounds. In the first round, the sellers purchase a price premium from the menu designed and provided by the intermediary. In the second round, the intermediary first *fully discloses* information about the consumer's valuations $\boldsymbol{v}$ as well as the associated cost of persuasion $\boldsymbol{\ell}(\boldsymbol{v})$, and opens an auction. In the auction, the seller with the highest bid wins, but pays the premium and the cost of persuasion on top of the second highest bid. The winner's product gets recommended to the consumer.

Definition 2 (Modified Handicap Auction)**.** In the first round, the seller i with θ_i chooses a price premium τ at a fee $c(\tau)$ where c is a fee-schedule published by the intermediary. Other sellers $-i$ and the consumer do not know which premium the seller i has chosen.

In the second round, the consumer's valuation $\boldsymbol{v}$ is publicly announced to the sellers. Then, a modified second price auction with reservation price 0, the premium $\tau(\theta_i)$ and the persuasion cost $\ell_i(\boldsymbol{v})$ follows. The seller i with his bid b_i wins the auction if and only if $b_i > \max_{j \neq i} b_j$. The winner pays the second highest price $\max_{j \neq i} b_j$ in addition to the premium $\tau(\theta_i)$ and the persuasion cost $\ell_i(\boldsymbol{v})$ to the intermediary. The winner's product is recommended to the consumer.

Lemma 4. *Suppose each seller i with θ_i chooses a price premium $\tau(\theta_i)$ in the first round. In the second round of the handicap auction, it is a weakly dominant strategy for seller i with price premium τ^{θ_i} and persuasion cost $\ell_i(\boldsymbol{v})$ to bid $b_i = \theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v})$.*

We assume that the sellers play according to their weakly dominant strategy in the second round. Then, the handicap auction may be represented by $(\tau, c, \ell_i)_{i \in \mathcal{N}}$. The following theorem states that the optimal recommendations rule can be implemented with a handicap auction with the identical cost of persuasion function under the optimal recommendations rule.

Theorem 2. *Let $v_0 \in (\underline{v}, \bar{v})$. Suppose that $\frac{1-G(\theta_i)}{g_i(\theta_i)}$ weakly decreases in θ_i. The intermediary can implement the optimal recommendations rule with the same revenue via a handicap auction $(c, \tau, \ell_i)_{i \in \mathcal{N}}$ where*

$$\tau(\theta_i) = \frac{1 - G(\theta_i)}{g(\theta_i)}, \tag{18}$$

$c(\theta_i)$ is defined by

$$c(\theta_i) = E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v}) - b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right] - \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i},\boldsymbol{v}}\left[\mathbf{1}_{\{\tilde{\theta} + w(v_i) - \tau(\tilde{\theta}) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right] d\tilde{\theta} \tag{19}$$

and the cost of persuasion $(\ell_i)_{i\in\mathcal{N}}$ from (14).

In the first round, a menu of premium and fee is offered to the sellers. A seller with high willingness to pay, expecting to win with a high probability, chooses a pair of low premium and high fee with which the sellers pay less when they wins in the second round. The premium the seller pays amounts to the information rent $\frac{1-G(\theta)}{g(\theta)}$.

In the second round, the information about the consumer's valuations $\boldsymbol{v}$ is announced, which is followed by a second price auction with the premium and the cost of persuasion. Note that the cost of persuasion $\ell_i(\boldsymbol{v})$ increases in v_i. Whenever the intermediary recommends a product with a low valuation, it becomes harder to persuade the consumer to purchase the product, and forces the intermediary to forego another chance to persuade the consumer to purchase the product that is good for the intermediary but bad for the consumer. The intermediary charges the monetary value of the cost of persuasion $\ell_i(\boldsymbol{v})$ to the seller in order to compensate itself for the forgone opportunity.

Optimal Information Provision

The intermediary has private information about the consumer's valuations and strategically releases the information to maximize its revenue while interacting with both sides of the two-sided market. The way it releases its information, however, is starkly different between the two sides of the markets. To the consumer, it mixes the high valuation state and low valuation state to induce the consumer to purchase the product whose seller has a high willingness to pay. In particular, the intermediary never fully discloses neither the consumer's valuation nor the sellers' private willingness to pay to the consumer, but mix the states in making recommendations. In contrast, the intermediary fully discloses the consumer's valuation to the sellers before starting the auction. This is because the additional willingness to pay the seller obtains by learning about the consumer's surplus can then be fully extracted by the intermediary.

4 Additional Information

This section analyzes how the intermediary's use of additional information about sellers' private willingness to pay affects on the consumer surplus, intermediary's revenue and sellers' profits. The intermediary's problem with additional information is analyzed in a similar way as in the baseline model without additional information. I recast the use of additional information as a change in the intermediary's preference, analyze how the change in preference changes the optimal recommendations rules, and provide sufficient conditions under which additional information benefits or harms each player of the market.

The intermediary observes *additional signals* $\boldsymbol{z} = (z_1, \cdots, z_N)$ about sellers' private information $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_N)$. Each $z_i \in \mathcal{Z} \subset \mathbb{R}$ is independently drawn from a common distribution $H(\cdot \mid \theta_i)$ conditioning on each θ_i, and is common knowledge between a seller i and the intermediary, but not known to others[10]. Let $\mathcal{H} = \{H(\cdot \mid \theta)\}_{\theta \in \Theta}$ be *additional information*, a collection of distribution functions conditioning on each $\theta \in \Theta$. Let $\mathcal{Z}(\theta)$ be a support of $H(\cdot \mid \theta)$, and $\Theta(z)$ be a collection of states at which z is generated with a positive probability. Below, I present two examples of additional information.

Example 1 (Perfectly revealing additional information)**.** Additional information $\mathcal{H}$ is *perfectly revealing* if $\mathcal{Z} = [\underline{\theta}, \bar{\theta}]$ and

$$H(z \mid \theta) = \begin{cases} 1 & \text{if } z \geq \theta \\ 0 & \text{if } z < \theta \end{cases}$$

with $\mathcal{Z}(\theta) = \{\theta\}$ and $\Theta(z) = \{z\}$. □

Example 2 (Lower censorship additional information)**.** Additional information $\mathcal{H}$ is *lower censorship* if it reveals θ if $\theta \geq \theta^*$, but does not reveal otherwise. Formally, $\mathcal{Z} = \{z_0\} \cup [\theta^*, \bar{\theta}]$ where $z_0 < \theta^*$, and

$$H(z \mid \theta) = \begin{cases} 1 & \text{if } \theta \geq \theta^* \text{ and } z \geq \theta, \text{ or } \theta < \theta^* \text{ and } z \geq z_0 \\ 0 & \text{otherwise} \end{cases}$$

with $\mathcal{Z}(\theta) = \{\theta\}$ when $\theta \geq \theta^*$ and $\mathcal{Z}(\theta) = \{z_0\}$ when $\theta < \theta^*$, and $\Theta(z) = \{z\}$ when $z \geq \theta^*$ and $\Theta(z) = [\underline{\theta}, \theta^*]$ and $z = z_0$ □

The intermediay's recommender system is $(\mathbf{r}, \mathbf{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ such that $\sum_{i \in \mathcal{N} \cup \{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. A recommender system is incentive

[10]More generally, it may be assumed that each i observes a signal ζ_i about additional signals about others $\boldsymbol{z}_{-i}$ without affecting any of the results. The signal ζ_i may be uninformative about $\boldsymbol{z}_{-i}$ as in here, may be completely revealing or may be related with $\boldsymbol{z}_{-i}$ in any arbitrary way.

compatible if for all $i \in \mathcal{N}$ and $\theta_i, \theta_i' \in \Theta$

$$
\begin{aligned}
IC_i : &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} \Big[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \Big] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) \boldsymbol{H}(d\boldsymbol{z}_{-i}) \\
&\geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}} \Big[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}, \boldsymbol{z}) - t_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}, \boldsymbol{z}) \Big] \boldsymbol{F}(d\boldsymbol{v}_{-i}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) \boldsymbol{H}(d\boldsymbol{z}_{-i}),
\end{aligned}
\tag{20}
$$

where $\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) = \frac{\boldsymbol{H}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i}) \boldsymbol{G}(d\boldsymbol{\theta}_{-i})}{\int_{\boldsymbol{\Theta}_{-i}} \boldsymbol{H}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i}) \boldsymbol{G}(d\boldsymbol{\theta}_{-i})}$, individually rational if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta$

$$
IR_i : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} \Big[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \Big] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) \boldsymbol{H}(d\boldsymbol{z}_{-i}) \geq 0,
\tag{21}
$$

and obedient if for all $i, j \in \mathcal{N} \cup \{0\}$,

$$
OB_{ij} : \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} v_i \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \geq \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}} v_j \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})
\tag{22}
$$

where $\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) = \frac{\boldsymbol{H}(d\boldsymbol{z} \mid \boldsymbol{\theta}) \boldsymbol{G}(d\boldsymbol{\theta})}{\int_{\boldsymbol{\Theta}} \boldsymbol{H}(d\boldsymbol{z} \mid \boldsymbol{\theta}) \boldsymbol{G}(d\boldsymbol{\theta})}$.

Note that this formulation nests the intermediary's problem without additional information in Section 2 as having an access to additional signals $\boldsymbol{z}$ but being restricted to ignore them, i.e. being able to use recommender system $(\boldsymbol{r}, \boldsymbol{t})$ that satisfy *invariance constraints*

$$
\boldsymbol{r}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \boldsymbol{r}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}') \text{ for all } \boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\mathcal{Z}}.
\tag{23}
$$

and

$$
\boldsymbol{t}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \boldsymbol{t}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}') \text{ for all } \boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\mathcal{Z}}.
$$

for all $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\mathcal{Z}}$, on top of the other constraints (20), (21) and (22).

With additional information, the intermediary maximizes its expected revenue

$$
\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})
$$

using the recommender system $(\boldsymbol{r}, \boldsymbol{t})$ that satisfies incentive compatibility (20), individual rationality (21) and obedience constraints (22). Applying the usual argument, the intermediary's problem reduces to maximizing

$$
\int_{\mathcal{V} \times \boldsymbol{\Theta} \times \mathcal{S}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})
\tag{24}
$$

satisfying obedience constraints (22) and monotonicity constraints (25), i.e. for all $i \in \mathcal{N}$ and $\theta_i > \theta_i'$

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}} r_i(\boldsymbol{v},\theta_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d(\theta_i,\boldsymbol{\theta}_{-i})\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \\ &\geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}} r_i(\boldsymbol{v},\theta_i',\boldsymbol{\theta}_{-i},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d(\theta_i',\boldsymbol{\theta}_{-i})\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \end{aligned} \tag{25}$$

Using value-switching recommendations rule, the analysis from Section 3 may be applied to characterize a symmetric optimal recommendations rule. Let $\boldsymbol{\rho}^A$ be an unconstrained optimal recommendations rule with additional information that is obtained by maximizing (24) without any constraints. Let $0 \leq \bar{v}^I \leq \mathbb{E}_{v_i}(v_i) \leq \underline{v}^I \leq \infty$ be such that

$$\begin{cases} \bar{v}^A & = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid \rho_i^A(\boldsymbol{v},\boldsymbol{\theta}) = 1) \\ \underline{v}^A & = 0 \end{cases}$$

if the information rent is small $\inf_{v\in\mathcal{V},\theta\in\Theta,z\in\mathcal{Z}} \theta - \frac{1-G(\theta|z)}{g(\theta|z)} + w(v) > 0$, and

$$\begin{cases} \bar{v}^A & = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid \rho_i^A(\boldsymbol{v},\boldsymbol{\theta}) = 1) \\ \underline{v}^A & = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid \rho_0^A(\boldsymbol{v},\boldsymbol{\theta}) = 1) \end{cases}$$

if the information rent is large $\inf_{v\in\mathcal{V},\theta\in\Theta,z\in\mathcal{Z}} \theta - \frac{1-G(\theta|z)}{g(\theta|z)} + w(v) < 0 < \sup_{v\in\mathcal{V},\theta\in\Theta,z\in\mathcal{Z}} \theta - \frac{1-G(\theta|z)}{g(\theta|z)} + w(v)$.

A symmetric optimal recommender system when $v_0 \in (\underline{v}, \bar{v})$ is given in the following theorem.

Theorem 3.a. *Let $v_0 \in (\underline{v}, \bar{v})$. Let $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ be a recommendations rule such that for each $i \in \mathcal{N}$,*

$$r_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = \begin{cases} \frac{1}{|\mathcal{M}^A|} & \textit{if } i \in \arg\max_{j\in\mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1-G(\theta_j \mid z_j)}{g(\theta_j \mid z_j)} + w(v_j)}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^A(\boldsymbol{v})}_{\substack{\textit{cost of} \\ \textit{persuasion}}}, 0 \right\} \\ 0 & \textit{otherwise} \end{cases} \tag{26}$$

where $\mathcal{M}^A = \arg\max_{j\in\mathcal{N}}\left\{\theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j) - \ell_j^A(\boldsymbol{v})\right\}$, *and*

$$\ell_i^A(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^A, \bar{v}^A] \\ \lambda_1^A(v_0)\cdot(v_0 - v_i) & \text{if } v_0 > \bar{v}^A \\ \lambda_2^A(v_0)\cdot\sum_{k\in\mathcal{N}}(v_0 - v_k) & \text{if } v_0 < \underline{v}^A \end{cases} \tag{27}$$

where $\lambda_1^A(v_0)$ *and* $\lambda_2^A(v_0)$ *are Lagrangian multipliers for* OB_{i0} *and* OB_{0i} *that may vary depending on* v_0*, respectively. Let* $\boldsymbol{t}$ *be given by*

$$t_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})d\tilde{\theta}_i. \tag{28}$$

for each i*. Then,* $\boldsymbol{r}^A$ *is value-switching monotone and is* $r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1$ *for some* $i \in \mathcal{N}\cup\{0\}$ *almost surely, and* $(\boldsymbol{r}^A,\boldsymbol{t})$ *is an optimal recommender system.*

Symmetric optimal recommender systems when $v_0 > \bar{v}$ and $v_0 < \underline{v}$ are given in the following theorem.

Theorem 3.b. *1. Let* $v_0 > \bar{v}$*. Let* $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta} \to [0,1]^{N+1}\times\mathbb{R}^N$ *be a recommendations rule such that*

$$r_0^A(\boldsymbol{v},\boldsymbol{\theta}) = 1$$

for all $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}$*. Let* $\boldsymbol{t}$ *be as in* (7)*. Then,* $(\boldsymbol{r}^A,\boldsymbol{t})$ *is an optimal recommender system.*

2. Let $v_0 < \underline{v}$*. Let* $\boldsymbol{r}^A : \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta} \to [0,1]^{N+1}\times\mathbb{R}^N$ *be a recommendations rule such that for each* $i \in \mathcal{N}$*,*

$$r_i^A(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^A|} & \text{if } i \in \arg\max_{j\in\mathcal{N}}\left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\right\} \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

where $\mathcal{M}^A = \arg\max_{j\in\mathcal{N}}\left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\right\}$ *and*

$$\ell_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 0 \text{ for all } i \in \mathcal{N} \text{ and } (\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta} \tag{30}$$

Let $\boldsymbol{t}$ *be as in* (28)*. Then,* $\boldsymbol{r}^A$ *is value-switching monotone and* $r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1$ *for some* $i \in \mathcal{N}\cup\{0\}$ *almost surely, and* $(\boldsymbol{r}^A,\boldsymbol{t})$ *is an optimal recommender system.*

It remains to analyze how an optimal recommendations rule with additional information $\boldsymbol{r}^A$ is different from that without additional information $\boldsymbol{r}^*$, and how does the difference impact on consumer surplus, intermediary's revenue and sellers' profits. I begin the analysis with recasting the use of additional information as a change the intermediary's preference and defining a class of additional information that I call by *regular*.

4.1 Additional Information as a Change in Intermediary's Preference

Additional information changes the intermediary's state-dependent preference over recommendations, but leaves the constraints (22) and (25) unchanged. To see what this means, notice that the intermediary's problem without additional information is to maximize

$$\int_{\mathcal{V}\times\boldsymbol{\Theta}\times\mathcal{S}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \tag{31}$$

using recommendations rules subject to obedience constraints (22), monotonicity constraints (25) and invariance constraints (23). In this problem, the invariance constraints (23) turns out to be redundant: The integrands of both the objective function (31) and constraints (22) and (25) do not depend on $\boldsymbol{z}$, so that an optimal recommendations rule without being constrained to the invariance (23) still satisfies the constraint. Ignoring the invariance constraints, the only difference between the intermediary's problem without and with additional information is the integrand, $\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)$ and $\theta_i - \frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} + w(v_i)$, which are the intermediary's preferences over recommendations.

Additional information changes the intermediary's preference in two different ways: 1) which product to recommend out of the products; 2) whether to recommend products over the outside option. The following class of additional information captures the change in intermediary's preference in a tractable manner.

Definition 3. Additional information $\mathcal{H}$

1. *has positive rates of substitution* if for all $z, z' \in \mathcal{Z}$, $\theta \in \Theta(z)$ and $\theta' \in \Theta(z')$,

$$1 - \frac{\frac{1-G(\theta|z)}{g(\theta|z)} - \frac{1-G(\theta'|z')}{g(\theta'|z')}}{\theta - \theta'} > 0.$$

2. *increases (decreases) rates of substitution* if for all $z,' \in \mathcal{Z}$, $\theta \in \Theta(z)$ and $\theta' \in \Theta(z')$,

$$1 - \frac{\frac{1-G(\theta|z)}{g(\theta|z)} - \frac{1-G(\theta'|z')}{g(\theta'|z')}}{\theta - \theta'} \geq (\leq) 1 - \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}.$$

3. *reduces state-dependent information rent* if for all $z \in \mathcal{Z}$ and $\theta \in \Theta(z)$

$$\frac{1 - G(\theta \mid z)}{g(\theta \mid z)} \leq \frac{1 - G(\theta)}{g(\theta)}$$

4. is *regular* if it satisfies all of the three.

The first condition states that the intermediary strictly prefers recommending a product with a higher private willingness to pay everything else equal. The positive rates of substitution is a generalized notion of Myerson's regularity that requires $\theta - \frac{1-G(\theta)}{g(\theta)}$ to be strictly increasing, which is equivalent to $1 - \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'} > 0$ for all $\theta, \theta' \in \Theta$. The second condition states that the rates of substitution are increased or decreased uniformly across all private willingness to pay and additional signals. With increased (decreased) rates of substitution, the intermediary prefers recommending a product with a higher θ over a lower θ' more (less) so than without the additional information. The third condition states that state-dependent information rent is decreased for all private willingness to pay and additional information. Finally, I say additional information is *regular* if the additional information satisfies all of the three conditions.

Example 1, cont. Let $\mathcal{H}$ be a perfectly revealing additional information. Let G be any distribution with monotone $\frac{1-G(\theta)}{g(\theta)}$. For each $z \in \mathcal{Z}$, it perfectly reveals that $\theta = z$. Since sellers no longer have private information, the intermediary pays no information rent, so that the state-dependent information rent conditioning on each $z \in \mathcal{Z}$ and $\theta \in \Theta(z) = \{z\}$ is 0, i.e.

$$\frac{1 - G(\theta \mid z)}{g(\theta \mid z)} = 0 \leq \frac{1 - G(\theta)}{g(\theta)},$$

and therefore, reduces state-dependent information rent. Since state-dependent information rent after observing an additional signal is always 0, the rate of substitution is

$$1 - \frac{\frac{1-G(\theta|z)}{g(\theta|z)} - \frac{1-G(\theta'|z')}{g(\theta'|z')}}{\theta - \theta'} = 1 > 0.$$

for all $z, z' \in \mathcal{Z}$, $\theta \in \Theta(z)$ and $\theta' \in \Theta(z')$. Therefore, the perfectly revealing additional information has positive rate of substitution, and increases (decreases) rate of substitution if the inverse hazard rate before observing the additional information were increasing (decreasing), $\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'} \geq (\leq)\ 0$ for any $\theta, \theta' \in \Theta$. With the monotonicity of $\frac{1-G(\theta)}{g(\theta)}$, the perfectly revealing additional information is regular.

□

Example 2, cont. Let $\mathcal{H}$ be a lower censorship additional information with θ^*. Let G be a distribution that has a decreasing inverse hazard rate $\frac{1-G(\theta)}{g(\theta)}$ on $[\underline{\theta}, \bar{\theta}]$ and has a density function such that for some neighborhood $B(\underline{\theta})$ of $\underline{\theta}$, $\inf_{\theta \in B(\underline{\theta})} g(\theta) > 0$ and $\sup_{\theta \in B(\underline{\theta})} g'(\theta) < \infty$. This nests a rich class of distributions including uniform distribution, linear virtual valuation distribution, (truncated) normal distribution, (truncated) exponential distribution and unimodal distribution with appropriate restrictions.

Let $\theta^* \in (\underline{\theta}, \bar{\theta})$. For any $z \in \mathcal{Z}$ and $\theta \in \Theta(z)$, the state-dependent information rent is

$$\frac{1 - G(\theta \mid z)}{g(\theta \mid z)} = \begin{cases} 0 & \text{if } \theta \geq \theta^* \\ \frac{G(\theta^*) - G(\theta)}{g(\theta)} & \text{otherwise} \end{cases},$$

from which it follows that lower censorship additional information reduces state-dependent information rent. Furthermore, in Appendix B.3, it is shown that for sufficiently small $\theta^* \in (\underline{\theta}, \bar{\theta})$, for all $z, z' \in \mathcal{Z}$, $\theta \in \Theta(z)$ and $\theta \in \Theta(z')$,

$$0 < 1 - \frac{\frac{1-G(\theta|z)}{g(\theta|z)} - \frac{1-G(\theta'|z')}{g(\theta'|z')}}{\theta - \theta'} \leq 1 - \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'},$$

meaning that the additional information has positive rates of substitution and decreases them. Therefore, the lower censorship additional information with sufficiently small $\theta^* \in (\underline{\theta}, \bar{\theta})$ is regular. □

4.2 Small Information Rent Environment

In analyzing how additional information impacts on consumer surplus, sellers' profits and the intermediary's revenue, I first focus on regular additional information in a small information rent environment as in Assumption 3.a. In this environment, I show that whether additional information increases rates of substitution completely determines whether additional information harms the consumer, and play an important role in characterizing whether additional information necessarily harms the sellers.

Consumer Surplus

The consumer surplus under a recommendations rule $\boldsymbol{r}$ at v_0 is

$$CS(v_0; \boldsymbol{r}) = \int_{\mathcal{V}} \int_{\Theta} \left[\sum_{i \in \{0\} \cup \mathcal{N}} (v_i - u^*) r_i(\boldsymbol{v}, \boldsymbol{\theta}) \right] d\boldsymbol{G} d\boldsymbol{F}$$

where $u^* = \max(v_0, \mathbb{E}_{v_i}(v_i))$ is the consumer's optimal payoff without the recommendations.

Regular additional information either increases or decreases rates of substitution. With increased (decreased) rates of substitution, intermediary is more (less) likely to recommend products with higher θ instead of higher v. Consumer is better off when product i is recommended because v_i is high, not because θ_i is high. Consequently, additional information decreases (increases) consumer surplus if it increases (decrease) rates of substitution.

Theorem 4. *Consider a small information rent environment. Let $\mathcal{H}$ be any regular additional information. Additional information decreases (increases) consumer surplus if it increases (decrease) rates of substitution.*

With increased (decreased) rates of substitution, the intermediary is more (less) likely to recommend products with higher private willingness to pay θ instead of those with higher valuation v. The consumer is better off when a product is recommended because it has high valuation v, not because it has high private willingness to pay θ. Consequently, additional information decreases (increases) consumer surplus if it increases (decreases) rates of substitution.

Example 1, cont. Perfectly revealing additional information *decreases* (increases) rates of substitution if and only if $\frac{1-G(\theta)}{g(\theta)}$ *decreases* (increases) in θ. By Theorem 4, perfectly revealing additional information *increases* (decreases) consumer surplus for all v_0 if $\frac{1-G(\theta)}{g(\theta)}$ *decreases* (increases) in θ. □

Recall that decreasing $\frac{1-G(\theta)}{g(\theta)}$ is a common assumption employed in the literature of mechanism design as it includes a rich class of distributions including uniform distribution, normal distribution, exponential distribution, and more generally, any log-concave distributions. Perfectly revealing additional information is the most informative information the intermediary can acquire about sellers. Example 1, cont implies that for many commonly used distributions having decreasing inverse hazard rates, additional information always increases the consumer surplus for *all* v_0.

Example 2, cont. Lower censorship additional information always *decreases* rates of substitution. By Theorem 4, lower censorship additional information always *increases* consumer surplus for all v_0. □

Sellers' Profits

A seller i's ex-ante expected profit without additional information is

$$\Pi_i^* = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \frac{1-G(\theta_i)}{g(\theta_i)} r_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \tag{32}$$

and with additional information is

$$\Pi_i^A = \int_{\mathcal{V}\times\Theta\times\mathcal{Z}} \frac{1-G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} r_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \tag{33}$$

The intermediary's use of additional information does not necessarily harm sellers' profits. Additional information may decrease information rent when it prevented the intermediary from recommending products, and thereby, increase sellers' profits by increasing the chance of recommending products over the outside option. To illustrate this intuition, consider the following example.

Example 3. Consider an environment where $\mathcal{N} = \{1,2\}$, θ_i is independently drawn from a uniform distribution over $[0,1]$ and v_i is independently drawn from a distribution F that has a support $\mathcal{V} = \{\underline{v},\bar{v}\}$ with $w(\underline{v}) = 2$ and $w(\bar{v}) = 3$. Let $v_0 = \bar{v}$ so that the consumer is obedient to recommendations only if $v_i = \bar{v}$ whenever the intermediary recommends $i \in \mathcal{N}$. The virtual willingness to pay is $\theta - \frac{1-G(\theta)}{g(\theta)} + w(v) = 2\theta - 1 + w(v) > 0$ for any $v \in \mathcal{V}$, so that it is a small information rent environment. The intermediary always prefers recommending one of the products, but the consumer is obedient to recommendations only if the intermediary recommends the product only when the product's valuation is $\bar{v}$.

Suppose that the intermediary does not have additional information. If both products' valuations are $\underline{v}$, then the intermediary recommends the outside option. If one of the product has $\bar{v}$ but the other has $\underline{v}$, then the intermediary recommends the product with $\bar{v}$ if and only if $\theta > \frac{1}{2}$. If both products have $\bar{v}$, then the intermediary i over j and the outside option if

$$\theta_i > \max\left(\theta_j, \frac{1}{2}\right), \tag{34}$$

and the outside option over products if

$$\max(\theta_i, \theta_j) \leq \frac{1}{2}. \tag{35}$$

Suppose that the intermediary has a *partitional* additional information that informs whether each θ_i is above or below $\frac{1}{2}$, i.e. $\mathcal{Z} = \{z_L, z_H\} \subset \mathbb{R}^1$ with $z_L < z_H$ such that

$$H(z \mid \theta) = \begin{cases} 1 & \theta \leq \frac{1}{2} \text{ and } z \geq z_L \text{ and } \theta > \frac{1}{2} \text{ and } z \geq z_H \\ 0 & \text{otherwise} \end{cases}.$$

Conditioning on z_H, $\theta > \frac{1}{2}$ and the seller's virtual willingness to pay is the same as before by

$2\theta - 1 > 0$; conditioning on z_L, $\theta \leq \frac{1}{2}$ and the seller's virtual willingness to pay is

$$\theta - \frac{\frac{1}{2} - \theta}{1} = 2\theta - \frac{1}{2}.$$

If the intermediary observes $\underline{v}$ for at least one of the products, the intermediary's recommendations rule remain the same as before. Suppose that the both products have $\bar{v}$. If the intermediary observes z_H for both sellers, then the virtual willingness to pay remain the same as before and hence so is the intermediary's recommendations rule as in (35) and (35). If the intermediary observes z_H for one seller but z_L for the other, then the virtual willingness to pay increases for z_L as it reduces the state-dependent information rent, and hence, the intermediary recommends the product with z_L more often than before. If the intermediary observes z_L for both sellers, then the virtual willingness to pay increases, so that the intermediary starts to recommending products over the outside option.

Note that when the additional signal is z_H for both sellers, the additional information have reduced the state-dependent information rent when it prevented the intermediary from recommending products, and thereby, increased sellers' profits by increasing the instances of recommending products over the outside option. In this example, this effect is large enough that sellers gain higher profits with additional information. □

In a small information rent environment with regular additional information, I provide a systematic analysis on the impact of additional information on sellers' profits, and a sufficient condition under which additional information harms sellers. Note that two objects change from (32) to (33): The recommendations rule from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$ and state-dependent information rent from $\frac{1-G(\theta_i)}{g(\theta_i)}$ to $\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)}$. It is helpful to separate the change in total profit by each of the changes. To this end, define a fictitious expected profit function obtained by fixing state-dependent information rent at $\frac{1-G(\theta_i)}{g(\theta_i)}$ but changing the recommendations rule changes from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$

$$\Pi_i^I = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \frac{1 - G(\theta_i)}{g(\theta_i)} r_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}). \tag{36}$$

The total change in the seller's profit $\Pi_i^A - \Pi_i^*$ can be decomposed into two terms,

$$\underbrace{\Pi_i^A - \Pi_i^*}_{\text{total change}} = \underbrace{(\Pi_i^A - \Pi_i^I)}_{\text{information rent effect}} - \underbrace{(\Pi_i^I - \Pi_i^*)}_{\substack{\text{recommendations} \\ \text{rule effect}}}$$

where *recommendations rule effect* $\Pi_i^I - \Pi_i^*$ captures the change in profit caused by a change in recommendations rule from $\boldsymbol{r}^*$ to $\boldsymbol{r}^A$, and *information rent effect* $\Pi_i^A - \Pi_i^I$ captures the change in profit caused by a change in state-dependent information rent the intermediary pays to sellers with and without additional information. I say the recommendations rule effect increases (decreases) all sellers' profits if $\Pi_i^I - \Pi_i^* \geq (\leq)\ 0$ The following theorem characterizes how each effect changes the profit.

Theorem 5. *Consider a small information rent environment. Let $\mathcal{H}$ be any regular additional information and $v_0 \leq \mathbb{E}_{v_i}(v_i)$.*

1. *Recommendations rule effect increases (decreases) all sellers' profits if one of the following conditions is satisfied:*
   - *(a) Additional information increases (decreases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ increases (decreases) in θ.*
   - *(b) Additional information decreases (increases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.*
2. *Information rent effect always decreases all sellers' profits.*

With increased (decreased) rates of substitution, the intermediary is more (less) likely to recommend products with higher private willingness to pay θ instead of those with higher valuation v. If $\frac{1-G(\theta)}{g(\theta)}$ is increasing in θ, then the seller is better off when a product is recommended because it has a high private willingness to pay θ, not because it has high valuation v. Consequently, additional information increases (decreases) sellers' profits if it increases (decreases) rates of substitution. On the other hand, if $\frac{1-G(\theta)}{g(\theta)}$ decreases, then the seller is better off when a product is recommended when it has a low private willingness to pay, and hence, additional information increases (decreases) sellers' profits if it decreases (increases) rates of substitution.

This immediately provides sufficient conditions under which additional information decreases sellers' profits.

Corollary 1. *Consider a small information rent environment. Let $\mathcal{H}$ be any regular additional information and $v_0 \leq \mathbb{E}_{v_i}(v_i)$. Additional information decreases sellers' profits if one of the following conditions is satisfied:*

1. *Additional information increases rates of substitution and $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ.*
2. *Additional information decreases rates of substitution and $\frac{1-G(\theta)}{g(\theta)}$ increases in θ.*

Example 1, cont. Perfectly revealing additional information always decreases sellers' profits to 0. □

Example 2, cont. By Corollary 1, lower censorship additional information decreases sellers' profits if $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. □

4.3 General Environment

This section examines the impact of additional information under a general informational environment. An informational environment is general if information rent may be small as in Assumption 3.a or large as in Assumption 3.b. Additional information always increases the intermediary's revenue, because the intermediary can always choose to ignore additional information. Additional information may increase, decrease or not change consumer surplus. For a consumer with low value of outside option, whether additional information increases or decreases consumer surplus solely depends on rates of substitution. Additional information may increase or decrease sellers' profits: reduced information rent decreases sellers' profits conditioning on recommending products, it allows the intermediary to recommend products more often which then increases sellers' profits. A sufficient condition under which the additional information decreases the sellers' profits is provided.

Intermediary's Revenue

The intermediary can always choose to ignore additional information, which would reduce the problem with additional information to the problem without the additional information. By revealed preference, the intermediary's revenue is always higher with additional information than without the additional information.

Theorem 6. *Any additional information increases the intermediary's revenue.*

Consumer Surplus

When information rent is large as in Assumption 3.b, the intermediary does not necessarily prefer recommending products over the outside option even in the absence of obedience constraints. That is, when incentivizing sellers to report their private willingness to pay, the intermediary may prefer recommending the outside option over products.

Theorem 7. *Let G be any distribution and $\mathcal{H}$ be regular additional information.*

1. *Let $v_0 < \underline{v}$. Additional information increases (decreases) the consumer surplus if it decreases (increases) sensitivity.*

2. *Let $v_0 > \bar{v}$. Additional information does not change the consumer surplus.*

When $v_0 < \underline{v}$, the lowest possible value for products is better than that of the outside option. The only obedient recommendations is to always recommend products possibly except for when all products has the lowest value $\underline{v}$ as well. Since the intermediary is forced to recommend products over outside option both with and without additional information, the intermediary's recommendation solely depends on rates of substitution as in the small information rent environment. Consequently, whether additional information increases or decreases the consumer surplus only depends on whether the additional information decreases or increases rates of substitution.

When $v_0 > \bar{v}$, the highest possible value for products is lower than that of the outside option. The only obedient recommendations is to always recommend outside options possibly except for when all products has the highest value $\bar{v}$ as well. Since the intermediary is forced to recommend outside option which is what the consumer would have taken without recommendations, the consumer surplus is 0 both with and without additional information. Consequently, additional information does not change the consumer surplus.

Example 1, cont. By Theorem 4, perfectly revealing additional information *increases* (decreases) consumer surplus for all v_0 if $\frac{1-G(\theta)}{g(\theta)}$ *decreases* (increases) in θ. □

Example 2, cont. Let G and $\mathcal{H}$ as in Example 2 with any $\theta^* \in (\underline{\theta}, \bar{\theta})$. For sufficiently small θ^*, the additional information is revenue-informative and regular. For a higher θ^*, the additional information is regular but not revenue-informative. Since the additional information decreases rate of substitution, by Theorem 7, the additional information increases consumer surplus for $v_0 \le \underline{v}$, but does not change consumer surplus for $v_0 \ge \bar{v}$. □

Sellers' Profits

When $v_0 < \underline{v}$, the intermediary is restricted to recommend products over the outside option. The same analysis as in Theorem 5 and Corollary 1 can be applied.

Theorem 8. *Let $\mathcal{H}$ be any regular additional information and $v_0 < \underline{v}$.*

1. *Recommendations rule effect increases (decreases) all sellers' profits if one of the following conditions is satisfied:*

   (a) *Additional information increases (decreases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ increases (decreases) in θ.*

   (b) *Additional information decreases (increases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.*

2. *Information rent effect always decreases all sellers' profits.*

Corollary 2. *Let $\mathcal{H}$ be any regular additional information and $v_0 < \underline{v}$. Additional information decreases sellers' profits if one of the following conditions is satisfied:*

1. *Additional information increases rates of substitution and $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ.*

2. *Additional information decreases rates of substitution and $\frac{1-G(\theta)}{g(\theta)}$ increases in θ.*

5 Discussion

5.1 Constrained Welfare Maximization

Consider the following welfare function which is a weighted sum of the consumer welfare and firms' profits,

$$\alpha \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) + (1-\alpha)\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} (\theta_i + w(v_i)) r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}), \tag{37}$$

where $\alpha \in (0,1)$. A recommender system $(\boldsymbol{r}^\alpha, \boldsymbol{t}^\alpha)$ is *constrained α-welfare maximizing* if it maximizes (37) subject to incentive compatibility (2), individual rationality (3) and obedience constraints (4). How does the α-welfare maximizing recommender system $(\boldsymbol{r}^\alpha, \boldsymbol{t}^\alpha)$ differ from the revenue-maximizing optimal recommender system $(\boldsymbol{r}^*, \boldsymbol{t}^*)$ from Theorem 1.a, and what are the welfare implications?

Note first that transfer $\boldsymbol{t}^\alpha$ is irrelevant to the welfare as long as it supports a given recommendations rule incentive compatible and individual. In particular, $\boldsymbol{t}^\alpha$ may be assumed to be (7) without loss. Rewriting (37)

$$\alpha \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} v_0 r(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) + \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i + w(v_i) + \frac{\alpha}{1-\alpha} v_i\right) r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}). \tag{38}$$

At a given $(\boldsymbol{v},\boldsymbol{\theta})$, the rate of substitution between $i \in \mathcal{N}$ and $j \in \mathcal{N}$ under (38) is

$$\frac{\frac{\theta_i-\theta_j}{\theta_i-\theta_j}}{\frac{w(v_i-w(v_j))}{v_i-v_j} + \frac{\alpha}{1-\alpha}} = \frac{1}{\frac{w(v_i-w(v_j))}{v_i-v_j} + \frac{\alpha}{1-\alpha}} \tag{39}$$

whereas the rate of substitution under the baseline case (5) is

$$\frac{\frac{\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)}\right) - \left(\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)}\right)}{\theta_i-\theta_j}}{\frac{w(v_i)-w(v_j)}{v_i-v_j}} = \frac{1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i-\theta_j}}{\frac{w(v_i)-w(v_j)}{v_i-v_j}}. \tag{40}$$

Comparing (39) and (40), it follows that rates of substitutions are higher under the welfare maximization (38) than under the revenue maximization (5) if the state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ is decreasing in θ. Applying the same logic as in the additional information case, I derive the following results.

Theorem 9. *Consider a small information environment. Let $\alpha \in (0,1)$ and G be such that $\frac{1-G(\theta)}{g(\theta)}$ decreases in θ. Relative to the revenue maximizing recommender system $(\boldsymbol{r}^*, \boldsymbol{t}^*)$,* *under the α-welfare maximizing recommender system $(\boldsymbol{r}^\alpha, \boldsymbol{t}^\alpha)$,*

1. *Consumer surplus is higher for all v_0.*
2. *Sellers' profits are higher for all $v_0 \leq \mathbb{E}_{v_i}(v_i)$.*
3. *Intermediary's revenue is lower for all $v_0 \leq \mathbb{E}_{v_i}(v_i)$.*

Theorem 9 states that under the α-welfare maximizing recommendations rule, it recommends a product with higher valuation over that with higher private willingness to pay more often, which would increase the consumer surplus and sellers' profits increase, but decreases the intermediary's revenue. Under a general distribution, the similar results are obtained for $v_0 < \underline{v}$.

5.2 Search Engine Interpretation

Another way to interpret a recommender system with some modifications is as a search engine. Consider an environment where there are $N_p + N_o$ search items. For each of the first N_p search items, there is an advertiser (seller) $i \in \mathcal{N}_p = \{1, \ldots, N_p\}$ who is willing to pay for a paid search for his item (recommendation). For the later N_o items, owners of the item $i \in \mathcal{N}_o = \{N_p + 1, \ldots, N_p + N_o\}$ are not willing to pay. Let each type of items be called *paid search* and *organic search* items. As in the baseline case, each seller has private willingness to pay θ_i and value-dependent willingness to pay $w_p(v_i)$ that are additively separable. If an advertiser i pays t_i for his item to be searched with probability r_i, the advertiser's profit is

$$(\theta_i + w_p(v_i))r_i - t_i$$

where $w_p(.)$ is any strictly increasing function. The private willingness to pay θ_i is independently drawn from a common distribution G that is absolutely continuous and has support $[\underline{\theta}, \bar{\theta}]$ with $-\infty < \underline{\theta} < \bar{\theta} < \infty$.

The user (the consumer) does not know the valuation of each search item, but only knows that v_i is independently drawn from a common distribution F_p if $i \in \mathcal{N}_p$, and from F_o if $i \in \mathcal{N}_o$. For simplicity, assume that $F_p(v) = \tilde{F}(v - \mu_p)$ and $F_o(v) = \tilde{F}(v - \mu_o)$ where $\mu_p, \mu_o \in \mathbb{R}^1$ are mean shifters of F_p and F_o, respectively. Let $\boldsymbol{v}_p = (v_1, \ldots, v_{N_p}) \in \boldsymbol{\mathcal{V}}_o = \mathcal{V}^{N_p}$, $\boldsymbol{v}_o = (v_{N_p+1}, \ldots, v_{N_p+N_o}) \in \boldsymbol{\mathcal{V}}_o = \mathcal{V}^{N_o}$, $\boldsymbol{v} = (\boldsymbol{v}_p, \boldsymbol{v}_o) \in \boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o$, $\boldsymbol{F} = \boldsymbol{F}_p \times \boldsymbol{F}_o$ and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_{N_p}) \in \boldsymbol{\Theta} = \Theta^{N_p}$.

The intermediary knows about the value of the options better than the user himself. The intermediary privately observes $\boldsymbol{v}$ and designs how to provide information to the user while raising revenue from advertisers, that I call by a search engine (a recommender system).

Since a search engine always returns some output, I assume that there is no outside option. By revelation principle from mechanism design and information design, it is without loss to assume that the intermediary's search engine is $(\boldsymbol{r},\boldsymbol{t}) : \boldsymbol{\mathcal{V}}_p \times \boldsymbol{\mathcal{V}}_o \times \boldsymbol{\Theta} \to [0,1]^{N_p+N_o} \times \mathbb{R}^{N_p}$ with $\sum_{i\in\mathcal{N}_p\cup\mathcal{N}_o} r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) = 1$. The intermediary collects revenue from advertisers, but also has reputation concerns that incentivizes the intermediary to search the right item for the user. Formally, the intermediary's objective function is a weighted sum of revenue and value of the recommendations

$$\int_{\boldsymbol{\mathcal{V}}_p\times\boldsymbol{\mathcal{V}}_o\times\boldsymbol{\Theta}} \left(\sum_{i\in\mathcal{N}_p} t_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) + \alpha \sum_{i\in\mathcal{N}_p\cup\mathcal{N}_o} w_o(v_i) r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) \right) \boldsymbol{F}_p(d\boldsymbol{v}_p)\boldsymbol{F}_o(d\boldsymbol{v}_o)\boldsymbol{G}(d\boldsymbol{\theta})$$

where $\alpha \in (0,1)$ is a relative weight between the revenue and value of the recommendations, and $w_o(.)$ is any strictly increasing function. After applying the usual arguments, the intermediary's problem is reduced to a Bayesian persuasion problem of maximizing

$$\int_{\boldsymbol{\mathcal{V}}_p\times\boldsymbol{\mathcal{V}}_o\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}_p\cup\mathcal{N}_o} \psi_i^E(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) \boldsymbol{F}_p(d\boldsymbol{v}_p)\boldsymbol{F}_o(d\boldsymbol{v}_o)\boldsymbol{G}(d\boldsymbol{\theta})$$

where

$$\psi_i^E(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) = \begin{cases} \theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w_p(v_i) + w_o(v_i) & \text{if } i \in \mathcal{N}_p \\ w_o(v_i) & \text{if } i \in \mathcal{N}_o \end{cases}$$

subject to obedience constraints, i.e. OB_{ij} for any $i,j \in \mathcal{N}_p \cup \mathcal{N}_o$

$$\int_{\boldsymbol{\mathcal{V}}_p\times\boldsymbol{\mathcal{V}}_o\times\boldsymbol{\Theta}} (v_i - v_j)\, r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta}) \boldsymbol{F}_p(d\boldsymbol{v}_p)\boldsymbol{F}_o(d\boldsymbol{v}_o)\boldsymbol{G}(d\boldsymbol{\theta}) \geq 0$$

and monotonicity constraints, that is, for all $\theta_i > \theta_i'$

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}}_p\times\boldsymbol{\mathcal{V}}_o\times\boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \theta_i, \boldsymbol{\theta}_{-i}) \boldsymbol{F}_p(d\boldsymbol{v}_p)\boldsymbol{F}_o(d\boldsymbol{v}_o)\boldsymbol{G}(d(\theta_i, \boldsymbol{\theta}_{-i})) \\ &\geq \int_{\boldsymbol{\mathcal{V}}_p\times\boldsymbol{\mathcal{V}}_o\times\boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \theta_i', \boldsymbol{\theta}_{-i}) \boldsymbol{F}_p(d\boldsymbol{v}_p)\boldsymbol{F}_o(d\boldsymbol{v}_o)\boldsymbol{G}(d(\theta_i', \boldsymbol{\theta}_{-i})). \end{aligned} \tag{41}$$

Define a value-switching monotone recommendations rule as in the following.

Definition 4. A recommendations rule $\boldsymbol{r}$ is value-switching monotone if

1. $r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta})$ increases in (v_i, θ_i) for all $i \in \mathcal{N}_p \cup \mathcal{N}_o$.

2. $r_i(\boldsymbol{v}_p, \boldsymbol{v}_o, \boldsymbol{\theta})$ decreases in v_j whenever $i \in \mathcal{N}_p$ and $j \in \mathcal{N}_o$, or, $i \in \mathcal{N}_o$ and $j \in \mathcal{N}_p$.

3. $r_i(\boldsymbol{v}, \boldsymbol{\theta})$ decreases whenever v_j is switched with a larger v_i for all $i,j \in \mathcal{N}_p$ or $i,j \in \mathcal{N}_o$,

i.e. $(\boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}}_{-ij} \times \boldsymbol{\Theta}$ and $v > v'$,

$$r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \geq r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}).$$

We can apply the same logic as before. Any value-switching monotone recommendations rule always satisfy OB_{ij} for $i, j \in \mathcal{N}_p$ and $i, j \in \mathcal{N}_o$; satisfy all of the obedience constraints if μ_p and μ_o are similar to each other; violates OB_{op} if μ_p is significantly larger than μ_o; violates OB_{po} if μ_p is significantly smaller than μ_o, where $p \in \mathcal{N}_p$ and $o \in \mathcal{N}_o$. An optimal search engine is characterized by two numbers $\underline{\mu}^E \leq 0 \leq \bar{\mu}^E$

$$r_i^E(\boldsymbol{v}, \boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^E|} & \text{if } i \in \arg\max_{j \in \mathcal{N}} \left\{\psi_j^E(\boldsymbol{v}, \boldsymbol{\theta}) - \ell_j^E(\boldsymbol{v}), 0\right\} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{M}^E = \arg\max_{j \in \mathcal{N}} \left\{\psi_j^E(\boldsymbol{v}, \boldsymbol{\theta}) + w_j - \ell_j^E(\boldsymbol{v})\right\}$, for $i \in \mathcal{N}_p$

$$\ell_i^E(\boldsymbol{v}) = \begin{cases} 0 & \text{if } \mu_p - \mu_o \geq \underline{\mu}^E \\ \lambda_2^E(v_0) \cdot \sum_{o \in \mathcal{N}_o}(v_o - v_i) & \text{if } \mu_p - \mu_o < \underline{\mu}^E \end{cases}$$

and for $i \in \mathcal{N}_o$

$$\ell_i^E(\boldsymbol{v}) = \begin{cases} \lambda_1^E(v_0) \cdot \sum_{p \in \mathcal{N}_p}(v_p - v_i) & \text{if } \mu_p - \mu_o > \bar{\mu}^E \\ 0 & \text{if } \mu_p - \mu_o \leq \bar{\mu}^E \end{cases}$$

where $\lambda_1^E(v_0)$ and $\lambda_2^E(v_0)$ are Lagrangian multipliers for OB_{po} and OB_{op} that may vary depending on v_0, respectively. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}) = (\theta_i + w_p(v_i))r_i(\boldsymbol{v}, \boldsymbol{\theta}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}) d\tilde{\theta}_i$$

for each i.

5.3 Relaxing v_0 as Common Knowledge

The value of the outside option v_0 has been assumed to be a constant that is commonly known to all players. One way to relax this assumption is to assume that v_0 is common knowledge between the intermediary and consumer, but not to the sellers. Sellers instead believe that v_0 is drawn from a distribution F_0. All results continue to hold identically under this assumption. Below I explore two different ways to break the symmetric knowledge of

v_0 between the inter mediary and the consumer: One under which v_0 is private knowledge of the intermediary; the other that of the consumer.

5.3.1 Value of Outside Option as Private Knowledge of Intermediary

Suppose that only the intermediary observes v_0 that is drawn from a distribution F_0. For simplicity[11], assume that $F_0(v_0) = F(v_0 - \mu_0)$ where $\mu_0 \in \mathbb{R}^1$ is a mean shifter of F_0. If $\mu_0 = 0$, then the distribution for the outside option F_0 is identical to those of other products F; if $\mu_0 > 0$, then F_0 first-order stochastically dominates F; if $\mu_0 < 0$, then F_0 is first-order stochastically dominated by F. All of the results in Section 3 and Section 4 continues to hold after replacing v_0 with μ_0.

5.3.2 Value of Outside Option as Private Knowledge of Consumer

Suppose that only the consumer observes v_0 that is drawn from a distribution F_0 that has a full support on the real line. Suppose that the intermediary offers the same recommender system to the consumer of all types, and the consumer of each type decides whether to follow the recommendation. This is equivalent to the intermediary designing the set $\mathcal{V}_0^{NR} \subset \mathbb{R}^1$ of the consumer types who will obey recommendations on top of designing a recommender system itself. For any given recommender system, a consumer with high v_0 disobeys when recommended with a product; a consumer with low v_0 disobeys when recommended with the outside option. Consequently, the intermediary's problem reduces to designing a recommender system and picking up two thresholds $\underline{v}^{NR} \leq \bar{v}^{NR}$ such that the consumer obeys if and only if $v_0 \in [\underline{v}^{NR}, \bar{v}^{NR}]$.

The intermediary faces another layer of trade-offs, setting target population $\mathcal{V}_0^{NR} = [\underline{v}^{NR}, \bar{v}^{NR}]$, on top of the trade-off between raising revenue and keeping the consumer incentivized to obey for each consumer $v_0 \in \mathcal{V}_0^{NR}$. Which population to target depends on whether the intermediary chooses to recommend the outside option with positive probability or not.

If the intermediary does not recommend the outside option almost surely, then any consumer with bad enough outside option is obedient to the recommender system. In particular, an optimal recommender system $(\boldsymbol{r}^{NR}, \boldsymbol{t}^{NR})$ is characterized by $\bar{v}^{NR} \geq \mathbb{E}_{v_i}(v_i)$ such that the consumer is obedient to $\boldsymbol{r}^{NR}$ if and only if $v_0 \in \mathcal{V}_0^{NR} = (-\infty, \bar{v}^{NR}]$, and

$$r_i^{NR}(\boldsymbol{v}, \boldsymbol{\theta}) = \frac{1}{|\mathcal{M}^{NR}|} \text{ if } i \in \arg\max_{j \in \mathcal{N}} \left\{ \theta_j - \frac{1 - G(\theta_j)}{g(\theta_j)} + w_j \right\} \tag{42}$$

[11] All results here can be generalized to any family distributions $F_0(v_0; \mu_0)$ where $\mu_0 \in \mathbb{R}^1$ is an index such that $F_0(v_0; \mu_0)$ first-order stochastically dominates $F_0(v_0; \mu_0')$ whenever $\mu_0 > \mu_0'$.

where $\mathcal{M}^{NR} = \arg\max_{j\in\mathcal{N}}\left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j\right\}$. Note that there is no cost of persuasion because none of the obedience constraints bind. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i^{NR}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{w})d\tilde{\theta}_i. \tag{43}$$

If the intermediary recommends the outside option with positive probability, then the optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ is characterized by $\underline{v} \le \underline{v}^{NR} = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1) \le \mathbb{E}_{v_i}(v_i) \le \bar{v}^{NR} = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1) \le \bar{v}^{NR} \le \bar{v}$such that the consumer is obedient to $\boldsymbol{r}^{NR}$ if and only if $v_0 \in \mathcal{V}_0^{NR} = [\underline{v}^{NR},\bar{v}^{NR}]$. Furthermore, OB_{0i} binds at $\underline{v}^{NR}$ and OB_{i0} binds at $\bar{v}^{NR}$, so that

$$r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^{NR}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}}\left\{\underbrace{\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^{NR}(\boldsymbol{v})}_{\substack{\text{cost of}\\ \text{persuasion}}}, 0\right\} \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

where $\mathcal{M}^{NR} = \arg\max_{j\in\mathcal{N}}\left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j - \ell_j^{NR}(\boldsymbol{v})\right\}$, and

$$\ell_i^{NR}(\boldsymbol{v}) = \lambda_1(\bar{v}^{NR})(\bar{v}^{NR} - v_i) - \lambda_2(\underline{v}^{NR})\sum_{k\in\mathcal{N}}(\bar{v}^{NR} - v_k)$$

where $\lambda_1^{NR}(v_0)$ and $\lambda_2^{NR}(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively, and $\underline{v}^{NR}$ and $\bar{v}^{NR}$ are the thresholds constructed from the uncontrained optimal recommendations rule. An optimal transfer rule is given as in (43).

Theorem 10. *Suppose that the consumer privately observes v_0 that is drawn from a distribution F_0. An optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ takes one of the following two structures:*

1. *The intermediary always recommends one of products. An optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ is as in (42) and (43). The consumer is obedient if and only if $v_0 \le \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1)$.*

2. *The intermediary sometimes recommends the outside option. An optimal recommender system $(\boldsymbol{r}^{NR},\boldsymbol{t}^{NR})$ is as in (44) and (43). The consumer is obedient if and only if $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_0^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1) \le v_0 \le \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(v_i \mid r_i^{NR}(\boldsymbol{v},\boldsymbol{\theta}) = 1)$.*

5.4 Relaxing Intermediary's Private Knowledge of $w(v)$

The value-dependent willingness to pay $w(v)$ has been assumed to be a deterministic function of v_i. Combined with the assumption that only the intermediary knows $\boldsymbol{v}$, this entails that the intermediary privately knows sellers' value-dependent willingness to pay $w(v)$ that sellers themselves do not know.

This assumption may be relaxed in two different ways. The first is to assume that each seller learns the value of $w(v_i)$ even though he does not know v_i. With the strict monotonicity of $w(v_i)$, learning the value of $w(v_i)$ is equivalent to learning v_i and hence having v_i as common knowledge between the intermediary and seller i. This does not change the optimal recommendations rule: the intermediary discloses v_i to and extract the entire value-dependent willingness to pay $w(v_i)$ from each seller i in the baseline model under which the seller does not know v_i. Therefore, the optimal recommendations rule letting seller to learn $w(v_i)$ does not change the optimal recommender system.

Another way to relax the assumption is to assume that each seller privately observes a value-dependent willingness to pay in the following way similar to Eső and Szentes (2007): Suppose that the intermediary can disclose[12] v_i to a seller i. Upon disclosure, the seller i privately observes w_i independently drawn from a common distribution $W(. \mid v_i)$ conditioning on v_i, where $W(. \mid v)$ first-order dominates $W(. \mid v')$ whenever $v > v'$. Without disclosure, the seller does not learn any information about v_i and w_i.

Following Eső and Szentes (2007), it can be shown that the intermediary completely discloses its private information $\boldsymbol{v}$ under an optimal recommender system and obtains the same expected revenue as if the intermediary could observe $\boldsymbol{w}$ using the modified handicap auction obtained by replacing $w(v_i)$ with w_i from Theorem 12. A sketch of the proof is presented here.

Let $v_0 \in (\underline{v}, \bar{v})$ and G be such that $\frac{1-G(\theta)}{g(\theta)}$ decreases.. Suppose that the intermediary has disclosed v_i to each seller i. A recommender system is now extended to $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{W}} \to [0,1]^{N+1} \times \mathbb{R}^N$ with $\sum_{i \in \mathcal{N}\{0\}} r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) = 1$ for all $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{w}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{W}}$ where $\mathcal{W}$ is a support

[12] An alternate setup under which the intermediary can provide any arbitrary information to sellers, instead of being restricted to disclosing or not, leads to the same conclusion. The revenue from (46) and (47) still is an upper bound of the intermediary's revenue which can be attained by first fully disclosing $\boldsymbol{v}$ to sellers and then running the modified handicap auction from Theorem

of $W(.\mid .)$ and $\boldsymbol{\mathcal{W}} = \mathcal{W}^N$. An optimal recommendations rule is given by

$$r_i^W(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}^W|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{ \underbrace{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j}_{\textit{virtual willingness to pay}} - \underbrace{\ell_j^W(\boldsymbol{v})}_{\substack{\text{cost of}\\ \text{persuasion}}}, 0 \right\} \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

where $\mathcal{M}^W = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w_j - \ell_j^W(\boldsymbol{v})\right\}$, and

$$\ell_i^W(\boldsymbol{v}) = \begin{cases} 0 & \text{if } v_0 \in [\underline{v}^W, \bar{v}^W] \\ \lambda_1^W(v_0)\cdot(v_0 - v_i) & \text{if } v_0 > \bar{v}^W \\ \lambda_2^W(v_0)\cdot \sum_{k\in\mathcal{N}}(v_0 - v_k) & \text{if } v_0 < \underline{v}^W \end{cases} \tag{46}$$

where $\lambda_1^W(v_0)$ and $\lambda_2^W(v_0)$ are Lagrangian multipliers for OB_{i0} and OB_{0i} that may vary depending on v_0, respectively, and $\underline{v}^W$ and $\bar{v}^W$ are the thresholds constructed from the uncontrained optimal recommendations rule. An optimal transfer rule is given by $\boldsymbol{t}$ such that

$$t_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) = (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{w}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v},\tilde{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{w})d\tilde{\theta}_i \tag{47}$$

for each i.

The revenue from (46) and (47) clearly is an upper bound of the intermediary's revenue. This revenue can be attained by first fully disclosing $\boldsymbol{v}$ to sellers and then running a modified handicap auction as the following.

Theorem 11. *Let $v_0 \in (\underline{v}, \bar{v})$. Suppose that $\frac{1-G(\theta_i)}{g_i(\theta_i)}$ weakly decreases in θ_i. The intermediary can implement the optimal recommendations rule with the same revenue via a handicap auction $(c,\tau,\ell_i)_{i\in\mathcal{N}}$ where*

$$\tau(\theta_i) = \frac{1-G(\theta_i)}{g(\theta_i)}, \tag{48}$$

$c(\theta_i)$ is defined by

$$\begin{aligned} c(\theta_i) =& E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] \\ &- \int_{\underline{\theta}}^{\theta_i} E_{\theta_{-i},\boldsymbol{v}}\left[\mathbf{1}_{\{\tilde{\theta} + w(v_i) - \tau(\tilde{\theta}) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right]d\tilde{\theta} \end{aligned} \tag{49}$$

and the cost of persuasion $(\ell_i^W)_{i\in\mathcal{N}}$ from (46).

6 Conclusion

In this paper, I study a model of an intermediary that uses its private information about the consumer to persuade the consumer to take recommended options while raising revenue from sellers. The optimal recommender system is characterized by recommending a product with the highest non-negative virtual willingness to pay adjusted for the cost of persuasion, where the cost of persuasion is the shadow price of each binding constraints, and can be implemented by a version of handicap auction. Allowing the intermediary to use additional information about sellers' private information always increases the intermediary's revenue, but its impact on consumer surplus and sellers' profits is ambiguous. I provide sufficient conditions under which additional information increases or decreases consumer surplus and sellers' profits using rates of substitution.

There are several directions for future works. First, to focus solely on informational aspects of the recommender system and impact of additional information, the prices of products have been assumed to be given fixed. Endogenizing prices introduce various economic and technical questions. Should the prices be set by the intermediary or by sellers and when? How does the recommender system affects on sellers' price competition and consumer surplus? How is persuasion strategy characterized when the consumer's payoff is endogenously determined by prices? Answering these questions will help us to have a better understanding of the recommender system and regulations over it. Second, allowing the consumer to have ex-ante asymmetric preference over products and private information about his preference would be an interesting direction to explore. The consumer has multi-dimensional private information about his preference. The key challenge lies in tractably characterizing recommendations rules that incentivizes the consumer to report its type truthfully. Last but not the least, analysing competing intermediaries is another important direction both for theory and practice. Competing mechanism designers and persuaders are generally known to be a hard problem. Characterizing how competing intermediaries, with consumers and sellers endogenously choosing the intermediaries to interact with, designs their services would be another interesting issue both for economic theorists and regulators.

A Proofs for Baseline Case

A.1 Lemma 1

For a seller $i \in \mathcal{N}$ with θ_i reports truthfully as θ_i, let

$$\Pi(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} [(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}) - t_i(\boldsymbol{v}, \boldsymbol{\theta})] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit, and

$$Q(\theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected probability of recommending i's product.

Let us first prove the following lemma that characterizes incentive compatible and individually recommender system.

Lemma 5. *A recommender system $(\boldsymbol{r}, \boldsymbol{t}) : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^{N+1} \times \mathbb{R}^N$ is incentive compatible, individually rational and obedient if and only if for each $i \in \mathcal{N}$, for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and $z_i \in \mathcal{Z}$,*

$$Q_i(\theta_i) \text{ is increasing in } \theta_i, \tag{50}$$

$$\Pi_i(\theta_i) = \Pi_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i) d\tilde{\theta}_i, \tag{51}$$

$$\Pi_i(\underline{\theta}) \geq 0 \tag{52}$$

and for all $i, j \in \mathcal{N} \cup \{0\}$,

$$\begin{aligned} &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \theta_i, \boldsymbol{\theta}_{-i}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) \\ \geq &\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} r_i(\boldsymbol{v}, \theta_i', \boldsymbol{\theta}_{-i}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}). \end{aligned} \tag{53}$$

Proof. **Necessity:** Let $\theta_i > \hat{\theta}_i$. Let

$$\pi_i(\hat{\theta}_i; \theta_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i}} \left[(\theta_i + w(v_i)) r_i(\boldsymbol{v}, \hat{\theta}_i, \boldsymbol{\theta}_{-i}) - t_i(\boldsymbol{v}, \hat{\theta}_i, \boldsymbol{\theta}_{-i})\right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit of the seller with θ_i when he reports as $\hat{\theta}_i$. Note that

$$\pi_i(\hat{\theta}_i; \theta_i) = \Pi_i(\hat{\theta}_i) + (\theta_i - \hat{\theta}_i) Q_i(\hat{\theta}_i)$$

Similarly,

$$\pi_i(\theta_i;\hat{\theta}_i) = \Pi_i(\theta_i) + (\theta_i - \hat{\theta}_i)Q_i(\theta_i).$$

Incentive compatibility $\Pi_i(\theta_i) \geq \pi_i(\hat{\theta}_i;\theta_i)$ and $\Pi_i(\hat{\theta}_i) \geq \pi_i(\theta_i;\hat{\theta}_i)$, which in turn implies

$$(\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i) \leq \Pi_i(\theta_i) - \Pi_i(\hat{\theta}_i) \leq (\theta_i - \hat{\theta}_i)Q_i(\theta_i).$$

By the above inequality, Q_i is weakly increasing and hence is integrable, which then implies (51).

Individual rationality is equivalent to $\Pi_i(\theta_i) \geq 0$ for all $i \neq 0$ and $\theta_i \in [\underline{\theta}, \bar{\theta}]$, from which $\Pi_i(\underline{\theta}) \geq 0$ for all i follows.

Sufficiency: Let $\theta_i \neq \hat{\theta}_i$. From (51) and the monotonicity of Q_i,

$$\begin{aligned}
\Pi_i(\theta_i) &= \Pi_i(\hat{\theta}) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i \\
&\geq \Pi_i(\hat{\theta}) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\hat{\theta}_i)d\tilde{\theta}_i \\
&= \Pi_i(\hat{\theta}) + (\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i) \\
&= \pi_i(\hat{\theta}_i;\theta_i) \qquad (54)
\end{aligned}$$

Since $\theta_i \neq \hat{\theta}_i$ are arbitrary, (54) implies incentive compatibility.

Since $Q_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$, (51) implies that $\Pi_i(\theta_i)$ increases in θ_i, and hence, individual rationality is satisfied if $\Pi_i(\underline{\theta}) \geq 0$. □

By Lemma 9, for any $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and $z_i \in \mathcal{Z}$,

$$\Pi_i(\theta_i) = \Pi_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i.$$

The expected transfer of the seller i with θ_i is

$$T_i(\theta_i) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}} (\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i)d\tilde{\theta}_i - \Pi_i(\underline{\theta}). \qquad (55)$$

By the usual argument of the change of variables, for each $z_i \in \mathcal{Z}$, we have

$$\begin{aligned}
&\int_{\Theta} T_i(\theta_i)G(d\theta_i) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \left(\theta_i + w(v_i) - \frac{1-G(\theta_i)}{g(\theta_i)}\right) r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) - \Pi_i(\underline{\theta}),
\end{aligned}$$

so that the intermediary's expected revenue is

$$
\begin{aligned}
&\sum_{i\in\mathcal{N}}\int_{\Theta} T_i(\theta_i)G(d\theta_i)\\
&=\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i+w(v_i)-\frac{1-G(\theta_i)}{g(\theta_i)}\right)r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})-\sum_{i\in\mathcal{N}}\Pi_i(\underline{\theta}). \qquad (56)
\end{aligned}
$$

The intermediary's problem is to maximize (56) using a recommender system $(\boldsymbol{r},\boldsymbol{t})$ subject to monotonicity constraints (50), payoff equivalence constraints (51), non-negativity constraints (52) and obedience constraints (53). Note that for any given recommendations rule $\boldsymbol{r}$ satisfying (50) and (53), any transfer $\boldsymbol{t}$ such that $\pi_i(\underline{\theta})=0$ for all $i\in\mathcal{N}$ and whose interim transfer satisfies (55) maximizes (5) while satisfying (51) and (52). Transfer (7) is one of such.

It remains to find an optimal recommendations rule $\boldsymbol{r}$. Since $\Pi_i(\underline{\theta})=0$ for all $i\in\mathcal{N}$ independent of $\boldsymbol{r}$, it immediately follows that a recommendations rule $\boldsymbol{r}$ that maximizes (5) subject to (50) and (53), together with the corresponding transfer (7), maximizes the intermediary's expected revenue subject to (50), (51), (52) and (53).

A.2 Lemma 2

The obedience constraint from a product $i\in\mathcal{N}$ to another $j\in\mathcal{N}$ is

$$
\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}}\int_{\boldsymbol{\Theta}}(v_i-v_j)r_i(v_i,v_j,\boldsymbol{v}_{-ij},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\
&=\int_{v>v'}\int_{\boldsymbol{\mathcal{V}}_{-ij}}\int_{\boldsymbol{\Theta}}(v-v')\Big(r_i(v_i=v,v_j=v',\boldsymbol{v}_{-ij},\boldsymbol{\theta})-r_i(v_i=v',v_j=v,\boldsymbol{v}_{-ij},\boldsymbol{\theta})\Big)\boldsymbol{F}_{ij}(d(v,v'))\boldsymbol{F}_{-ij}(d\boldsymbol{v}_{-ij})\boldsymbol{G}(d\boldsymbol{\theta})\\
&\geq 0
\end{aligned}
$$

where the first equality follows from the assumption that $\boldsymbol{F}$ is symmetric and hence $\boldsymbol{F}_{ij}(d(v,v'))=\boldsymbol{F}_{ij}(d(v',v))$, and the last inequality follows from $v>v'$ and $r_i(v_i=v,v_j=v',\boldsymbol{v}_{-ij},\boldsymbol{\theta})-r_i(v_i=v',v_j=v,\boldsymbol{v}_{-ij},\boldsymbol{\theta})\geq 0$ by the value-switching monotonicity of $\boldsymbol{r}$.

A.3 Lemma 3

1. Let $i\in\mathcal{N}$. OB_{i0} is
$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i-v_0)r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\geq 0. \qquad (57)$$

Define $R_i(v_i) = \int_{\boldsymbol{\mathcal{V}}_{-i}\times\boldsymbol{\Theta}} r_i(v_i,\boldsymbol{v}_{-i},\boldsymbol{\theta})\boldsymbol{F}_{-i}(d\boldsymbol{v}_{-i})\boldsymbol{G}(d\boldsymbol{\theta})$ which is increasing in v_i. Then, (57) is equivalent to

$$Cov_{v_i}(v_i, R_i(v_i)) + (\mathbb{E}_{v_i}(v_i) - v_0)\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) \geq 0. \tag{58}$$

The first term $Cov_{v_i}(v_i, R_i(v_i))$ is non-negative since $R_i(v_i)$ is increasing in v_i. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) = 0$, then (58) always holds, so that $\bar{v}_i = \infty$. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta})) > 0$, then (58) holds if and only if $v_0 \leq \bar{v}_i$ where $\bar{v}_i = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{v_i}(v_i,R_i(v_i))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))} \geq \mathbb{E}_{v_i}(v_i)$. In any case, there is $\bar{v}_i \geq \mathbb{E}_{v_i}(v_i)$ such that OB_{i0} holds if and only if $v_0 \leq \bar{v}_i$.

2. Let $i \in \mathcal{N}$. OB_{0i} is
$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} (v_0 - v_i) r_0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \geq 0. \tag{59}$$
Define $R_0(v_i) = \int_{\boldsymbol{\mathcal{V}}_{-i}\times\boldsymbol{\Theta}} r_0(v_i,\boldsymbol{v}_{-i},\boldsymbol{\theta})\boldsymbol{F}_{-i}(d\boldsymbol{v}_{-i})\boldsymbol{G}(d\boldsymbol{\theta})$ which is decreasing in v_i. Then, (59) is equivalent to

$$-Cov_{v_i}(v_i, R_0(v_i)) - (\mathbb{E}_{v_i}(v_i) - v_0)\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) \geq 0. \tag{60}$$

The first term $Cov_{v_i}(v_i, R_0(v_i))$ is non-positive since $R_0(v_i)$ is decreasing in v_i. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) = 0$, then (60) always holds, so that $\underline{v}_i = 0$. If $\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_0(\boldsymbol{v},\boldsymbol{\theta})) > 0$, then (60) holds if and only if $v_0 \geq \underline{v}_i$ where $\underline{v}_i = \mathbb{E}_{v_i}(v_i) + \frac{Cov_{v_i}(v_i,R_i(v_i))}{\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}}(r_i(\boldsymbol{v},\boldsymbol{\theta}))} \leq \mathbb{E}_{v_i}(v_i)$. In any case, there is $\underline{v}_i \leq \mathbb{E}_{v_i}(v_i)$ such that OB_{i0} holds if and only if $v_0 \geq \bar{v}_i$.

A.4 Theorem 1.a

I first show that a symmetric recommender system attains the optimal revenue. Recall that recommendations rule $\boldsymbol{r}$ is symmetric if for any $i \in \mathcal{N}$, any bijective function $\iota : \mathcal{N} \to \mathcal{N}$ and any $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = r_{\iota(i)}(\boldsymbol{v}^{\iota},\boldsymbol{\theta}^{\iota})$$

where $(\boldsymbol{v}^{\iota},\boldsymbol{\theta}^{\iota})$ is such that $v^{\iota}_{\iota(i)} = v_i$ and $\theta^{\iota}_{\iota(i)} = \theta_i$ for all $i \in \mathcal{N}$.

Lemma 6. *For each obedient recommendations rule $\boldsymbol{r}$, there is a symmetric recommendations rule $\boldsymbol{r}^0$ that is obedient and attains the same revenue as $\boldsymbol{r}$.*

Proof. Let us first construct a symmetric recommendations rule $\boldsymbol{r}^0$ from any given obedient recommendations rule $\boldsymbol{r}$. Let $\boldsymbol{r}$ be a recommendations rule. Let

$$\mathcal{I}^{\mathcal{N}} = \{\iota^{\dagger} \mid \iota : \mathcal{N} \to \mathcal{N} \text{ is a bijective function}\}$$

be a set of all permutation functions on $\mathcal{N}$. For each $\iota^\dagger \in \mathcal{I}^{\mathcal{N}}$, let $\boldsymbol{r}^{\iota^\dagger}$ be a recommendations rule obtained by permutating $\boldsymbol{r}$ according to $\iota^\dagger$, i.e. for each $i \in \mathcal{N}$,

$$r_i^{\iota^\dagger}(\boldsymbol{v},\boldsymbol{\theta}) = r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger}).$$

Then, $\boldsymbol{r}^\dagger$ satisfies obedience constraints as well. Define another recommendatiosn rule $\boldsymbol{r}^0$ such that for each $i \in \mathcal{N}$ and $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i^0(\boldsymbol{v},\boldsymbol{\theta}) = \frac{1}{N!}\sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r^{\iota^\dagger}(\boldsymbol{v},\boldsymbol{\theta}) = \frac{1}{N!}\sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger}). \tag{61}$$

To prove that $\boldsymbol{r}^0$ is symmetric, it is sufficient to show that for any bijection $\iota \in \mathcal{I}^{\mathcal{N}}$ and $(\boldsymbol{v},\boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$

$$r_i^0(\boldsymbol{v},\boldsymbol{\theta}) = r_{\iota(i)}^0(\boldsymbol{v}^{\iota},\boldsymbol{\theta}^{\iota}). \tag{62}$$

To show this, note that

$$r_{\iota(i)}^0(\boldsymbol{v}^{\iota},\boldsymbol{\theta}^{\iota}) = \frac{1}{N!}\sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger\circ\iota(i)}(\boldsymbol{v}^{\iota^\dagger\circ\iota},\boldsymbol{\theta}^{\iota^\dagger\circ\iota}) \tag{63}$$

where $\iota^\dagger \circ \iota$ is a composition of two permuntation functions. Note that there is a bijective mapping between $\mathcal{I}^{\mathcal{N}}$ and $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$. To show this, since both $\mathcal{I}^{\mathcal{N}}$ and $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ are finite sets, it is sufficient to show that $\mathcal{I}^{\mathcal{N}} = \{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$. To show the equality, first note that $\iota^\dagger \circ \iota$ is a composition of two bijective mappings and hence is a bijection, i.e. $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\} \subset \mathcal{I}^{\mathcal{N}}$. To show the other inclusion, let $\tilde{\iota} \in \mathcal{I}^{\mathcal{N}}$. Since both $\tilde{\iota}$ and ι are bijection over the same finite space, I can define $\iota^\dagger = \tilde{\iota} \cdot \iota^{-1}$ which is a composition of two bijections and hence a well-defined bijection over $\mathcal{N}$. By construction, for each $j \in \mathcal{N}$, $\iota^\dagger \circ \iota(j) = \iota^\dagger(\iota(j)) = \tilde{\iota}(j)$, and hence $\tilde{\iota} = \iota^\dagger \circ \iota$ for some $\iota^\dagger \in \mathcal{I}^{\mathcal{N}}$. In other words, $\tilde{\iota} \in \{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ and hence $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\} \supset \mathcal{I}^{\mathcal{N}}$, which gives the desired equality. That there is a bijective mapping between $\{\iota^\dagger \circ \iota \mid \iota^\dagger \in \mathcal{I}^{\mathcal{N}}\}$ and $\mathcal{I}^{\mathcal{N}}$ implies that the right-hand side of (63) is

$$\frac{1}{N!}\sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger\circ\iota(i)}(\boldsymbol{v}^{\iota^\dagger\circ\iota},\boldsymbol{\theta}^{\iota^\dagger\circ\iota}) = \frac{1}{N!}\sum_{\iota^\dagger \in \mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger}) = r_i^0(\boldsymbol{v},\boldsymbol{\theta}),$$

and therefore, (62) holds.

It remains to show that $\boldsymbol{r}^0$ is obedient and attains the equal revenue. These results follow from the linearity of the revenue and obedience constraints. For each $i, j \in \mathcal{N}$, the obedience constraint from i to j for $\boldsymbol{r}^0$ is

$$\begin{aligned}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i-v_j)r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i-v_j)\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v^{\iota^\dagger}_{\iota^\dagger(i)}-v^{\iota^\dagger}_{\iota^\dagger(j)})r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\ &\geq 0 \end{aligned}\tag{64}$$

where the third equality follows from the definition that $v^{\iota^\dagger}_{\iota^\dagger(i)}=v_i$ and $\theta^{\iota^\dagger}_{\iota^\dagger(i)}=\theta_i$, and the last inequality follows from the fact that $\boldsymbol{r}^{\iota^\dagger}$ is obtained by permutating $\boldsymbol{r}$ which is obedient, and hence, so is $\boldsymbol{r}^{\iota^\dagger}$.

For $i\in\mathcal{N}$, the obedience constraint from i to 0 is

$$\begin{aligned}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i-v_0)r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_i-v_0)\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v^{\iota^\dagger}_{\iota^\dagger(i)}-v_0)r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\ &\geq 0 \end{aligned}\tag{65}$$

where the last inequality follows from OB_{i0} for each $\boldsymbol{r}^{\iota^\dagger}$, so that OB_{i0} is satisfied for $\boldsymbol{r}^0$. The obedience constraint from 0 to i is

$$\begin{aligned}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0-v_i)r_0^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0-v_i)\left(1-\sum_{j\in\mathcal{N}}r_j^0(\boldsymbol{v},\boldsymbol{\theta})\right)\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0-v_i)\left(1-\sum_{j\in\mathcal{N}}\frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}r_{\iota^\dagger(j)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\right)\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0-v^{\iota^\dagger}_{\iota^\dagger(i)})\left(1-\sum_{j\in\mathcal{N}}r_{\iota^\dagger(j)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\right)\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\\ &= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}(v_0-v^{\iota^\dagger}_{\iota^\dagger(i)})r_0(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger})\\ &\geq 0 \end{aligned}\tag{66}$$

where the last inequality follows from OB_{0i} for each $\boldsymbol{r}^{\iota^\dagger}$, so that OB_{0i} is satisfied for $\boldsymbol{r}^0$. By (64), (65) and (66), $\boldsymbol{r}^0$ is obedient.

It remains to verify that $\boldsymbol{r}$ and $\boldsymbol{r}^0$ attain the same revenue. Note that every $\boldsymbol{r}^{\iota^\dagger}$ has the same revenue as $\boldsymbol{r}$ because $\boldsymbol{r}^{\iota}$ is obtained by permutating $\boldsymbol{r}$ according to $\iota^\dagger$. Consequently, their average must be the same as the revenue obtained by $\boldsymbol{r}$ as shown below.

$$\begin{aligned}
&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta^{\iota^\dagger}_{\iota^\dagger(i)} - \frac{1-G(\theta^{\iota^\dagger}_{\iota^\dagger(i)})}{g(\theta^{\iota^\dagger}_{\iota^\dagger(i)})} + w(v^{\iota^\dagger}_{\iota^\dagger(i)})\right) r_{\iota^\dagger(i)}(\boldsymbol{v}^{\iota^\dagger},\boldsymbol{\theta}^{\iota^\dagger})\boldsymbol{F}(d\boldsymbol{v}^{\iota^\dagger})\boldsymbol{G}(d\boldsymbol{\theta}^{\iota^\dagger}) \\
&= \frac{1}{N!}\sum_{\iota^\dagger\in\mathcal{I}^{\mathcal{N}}} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \\
&= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i^0(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}).
\end{aligned}$$

□

By Lemma 6, there always exists a symmetric recommendations rule that maximizes the revenue. From here on, I focus on symmetric recommendations rules. The following lemma states that the following symmetric recommendations rule is value-switching monotone and recommends one of the options with certainty almost surely.

Lemma 7. *For each $i\in\mathcal{N}$, define*

$$\psi_i(\boldsymbol{v},\boldsymbol{\theta}) = \xi_i(v_i,\theta_i) + \xi_0(\boldsymbol{v})$$

where ξ_i strictly increases in (θ_i, v_i), and ξ_0 is common across the products and is symmetric and increases in $\boldsymbol{v}$ that could possibly be 0. Ignoring ties, let, for $i\in\mathcal{N}$,

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1 \text{ if } i = \arg\max_{j\in\mathcal{N}}\left(\psi_j(\boldsymbol{v},\boldsymbol{\theta}),0\right)$$

and $r_0(\boldsymbol{v},\boldsymbol{\theta}) = 1-\sum_{i\in\mathcal{N}} r_i(\boldsymbol{v},\boldsymbol{\theta})$. Then, $\boldsymbol{r}$ is value-switching monotone almost surely and $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j\in\mathcal{N}\cup\{0\}$ almost surely.

Proof. Since ξ_i strictly increases and ξ_0 increases in (v_i,θ_i), almost surely, for each $(\boldsymbol{v},\boldsymbol{\theta})\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}$, there is $i\in\mathcal{N}$ $\psi_i(\boldsymbol{v},\boldsymbol{\theta}) > \max_{j\in\mathcal{N}\setminus\{i\}}(\psi_j(\boldsymbol{v},\boldsymbol{\theta}),0)$ or $0 > \max_{j\in\mathcal{N}}\psi_j(\boldsymbol{v},\boldsymbol{\theta})$, so that $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j\in\{0\}\cup\mathcal{N}$ almost surely.

Let $\boldsymbol{\mathcal{W}} = \{(\boldsymbol{v},\boldsymbol{\theta})\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta} \mid r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1 \text{ for some } i\in\mathcal{N}\cup\{0\}\}$. Since $(\boldsymbol{v},\boldsymbol{\theta})\in\boldsymbol{\mathcal{W}}$ almost surely, to show almost sure value-switching monotonicity, it is sufficient to show establish that for any $i\in\mathcal{N}$, $(v_i',\boldsymbol{v}_{-i},\theta_i',\boldsymbol{\theta}_{-i})$, $(v_i,\boldsymbol{v}_{-i},\theta_i,\boldsymbol{\theta}_{-i})\in\boldsymbol{\mathcal{W}}$ such that $(v_i,\theta_i)\geq(v_i',\theta_i')$,

1. $r_0(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 0$ implies $r_0(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 0$,

2. $r_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 1$ implies $r_i(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 1$,

and for any $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$, $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ where $v > v'$,

3. $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$ implies $r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$ for any $v > v'$.

To show the first item, let $(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), (v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\mathcal{W}}$ be such that $(v_i', \theta_i') \leq (v_i, \theta_i)$ and $r_0(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) = 0$. By definition, $\max_{j \in \mathcal{N}} \psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) > 0$. Increasing from (v_i', θ_i') to (v_i, θ_i) increases $\psi_j(\boldsymbol{v}, \boldsymbol{\theta})$ for all $j \in \mathcal{N}$, so that $\max_{j \in \mathcal{N}} \psi_j(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \geq \max_{j \in \mathcal{N}} \psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}) > 0$, and hence, $r_0(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 0$.

To show the second item, let $(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), (v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) \in \boldsymbol{\mathcal{W}}$ be such that $(v_i', \theta_i') \leq (v_i, \theta_i)$ and $r_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}) = 1$. By definition, $\psi_i(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}) > \max_{j \in \mathcal{N} \setminus \{i\}}(\psi_j(v_i', \boldsymbol{v}_{-i}, \theta_i', \boldsymbol{\theta}_{-i}), 0)$. Increasing from (v_i', θ_i') to (v_i, θ_i) strictly increases ξ_i and increases ξ_0, but not ξ_j. Therefore, $\psi_i = \xi_i + \xi_0$ increases more than $\psi_j = \xi_j + \xi_0$ for all $j \in \mathcal{N} \setminus \{i\}$. Consequently, $\psi_i(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}) > \max_{j \in \mathcal{N} \setminus \{i\}}(\psi_j(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}), 0)$, and hence, $r_i(v_i, \boldsymbol{v}_{-i}, \theta_i, \boldsymbol{\theta}_{-i}) = 1$.

To show the last item, let $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$, $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{W}}$ where $v > v'$ and $r_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$. By definition, $\psi_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) < \max_{k \in \mathcal{N} \setminus \{i\}}(\psi_k(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}), 0)$. Changing from $(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$ to $(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta})$ only strictly decreases $\xi_i(v_i, \theta_i)$ and strictly increases $\xi_j(v_j, \theta_j)$ while leaving others (note that $\xi_0(\boldsymbol{v})$ remains the same by the symmetry of ξ_0). Therefore, only ψ_i strictly decreases and ψ_j strictly increases, while ψ_k remains the same for all $k \in \mathcal{N} \setminus \{i, j\}$. Consequently, $\psi_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) < \psi_i(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) < \max_{k \in \mathcal{N} \setminus \{i\}}(\psi_k(v_i = v, v_j = v', \boldsymbol{v}_{-ij}, \boldsymbol{\theta}), 0) \leq \max_{k \in \mathcal{N} \setminus \{i\}}(\psi_k(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}), 0)$, and hence, $r_i(v_i = v', v_j = v, \boldsymbol{v}_{-ij}, \boldsymbol{\theta}) = 0$. $\square$

By Lemma 6, there is a symmetric optimal recommender system. The rest of the proof focuses on constructing a symmetric optimal recommender system.

Let $v_0 \in [\underline{v}^*, \bar{v}^*]$. The unconstrained optimal recommendations rule obtained ignoring the obedience constraints $\boldsymbol{\rho}^*$ as in (10) is obedient and hence optimal. In other words, $\boldsymbol{r}^* = \boldsymbol{\rho}^*$ and $\ell_i^*(\boldsymbol{v}, \boldsymbol{\theta}) = 0$ for all $i \in \mathcal{N}$ and $(\boldsymbol{v}, \boldsymbol{\theta}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}$.

Let $v_0 > \bar{v}^*$. Then, $\boldsymbol{\rho}^*$ violates OB_{i0}. Also, since $v_0 > \bar{v}^* \geq \underline{v}^*$, any value-switching monotone recommendations rule such that $r_j(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely satisfies OB_{0i}. At an optimal symmetric value-switching monotone recommendations rule such that $r_j(\boldsymbol{v}, \boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely, OB_{i0} are binding; otherwise, none of the constraints bind which would imply that $\boldsymbol{\rho}^*$ is the optimal recommendations rule which is known to violate OB_{i0}. Taking the Lagrangian, the optimal recommendations rule is characterized by

$$r_i^*(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \lambda(v_0 - v_j), 0\right\} \\ 0 & \text{otherwise} \end{cases} \tag{67}$$

where $\mathcal{M} = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v})\right\}$, λ is a Lagrangian multiplier of OB_{i0} that makes OB_{i0} binding.

Let $v_0 < \underline{v}^*$. Then, $\boldsymbol{\rho}^*$ violates OB_{0i}. Also, since $v_0 < \underline{v}^* \leq \underline{v}^*$, any value-switching monotone recommendations rule such that $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely satisfies OB_{0i}. At an optimal symmetric value-switching monotone recommendations rule such that $r_j(\boldsymbol{v},\boldsymbol{\theta}) = 1$ for some $j \in \mathcal{N} \cup \{0\}$ almost surely, OB_{0i} are binding; otherwise, none of the constraints bind which would imply that $\boldsymbol{\rho}^*$ is the optimal recommendations rule which is known to violate OB_{0i}. Taking the Lagrangian, the optimal recommendations rule is characterized by

$$r_i^*(\boldsymbol{v},\boldsymbol{\theta}) = \begin{cases} \frac{1}{|\mathcal{M}|} & \text{if } i \in \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \lambda \sum_{k\in\mathcal{N}}(v_0 - v_k), 0\right\} \\ 0 & \text{otherwise} \end{cases} \tag{68}$$

where $\mathcal{M} = \arg\max_{j\in\mathcal{N}} \left\{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j) - \ell_j^*(\boldsymbol{v})\right\}$, λ is a Lagrangian multiplier of OB_{i0} that makes OB_{i0} binding.

A.5 Lemma 4

Consider a seller i with θ_i. Let

$$b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}) = \max_{j\neq i}(b_j^*(\boldsymbol{v},\theta_j), 0)$$

be the value that a seller i must bid above for the seller i to win the second stage auction given that others bid according to

$$b_j^*(\boldsymbol{v},\theta_j) = \theta_j + w(v_j) - \tau_i(\theta_i) - \ell_i(\boldsymbol{v}).$$

There are three cases to consider depending on the relative size of $\theta_i+w_i-\tau_i-l_i$ and $b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})$.

If $\theta_i+w(v_i)-\tau(\theta_i)-\ell_i(\boldsymbol{v}) > b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})$, then the seller i strictly prefers winning to drawing and losing, which is attained by bidding $b_i \in (b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}), \infty)$.

If $\theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v}) = b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})$, then the seller i strictly prefers winning and drawing to losing, which is attained by bidding $b_i \in [b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}), \infty)$.

If $\theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v}) < b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})$, then the seller i strictly prefers losing to winning and drawing which is attained by bidding $b_i \in [0, b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))$.

In every case, $b_i = \theta_i + w(v_i) - \tau(\theta_i) - \ell_i(\boldsymbol{v})$ is one of the best responses, and therefore, is weakly dominant strategy.

A.6 Theorem 12

This section characterizes incentive compatible and individually rational first stage mechanisms assuming that the sellers follow the weakly dominant strategy in the second round, and then characterizes an optimal mechanism. Arguments mostly follow those from Eső and Szentes (2007), but because $\mathcal{V}$ could either be discrete or continuum, unlike being always continuum in Eső and Szentes (2007), I generalize their arguments slightly in characterizing incentive compatible and individually rational mechanisms. For the sake of completeness, I present the entire proof, instead of the parts that have been generalized.

Let

$$\pi^H_i(\theta_i, \hat{\theta}_i) = E_{\boldsymbol{v},\theta_{-i}}\left[(\theta_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right] - c(\hat{\theta}_i),$$

be the seller i with θ_i reported truthfully in the first stage but reports as $\hat{\theta}_i$ in the second stage auction,

$$\Pi^H_i(\theta_i) = \pi^H_i(\theta_i, \theta_i)$$

be the interim payoff when i reports truthfully, and

$$Q^H_i(\theta_i, \hat{\theta}_i) = E_{\boldsymbol{v},\theta_{-i}}\left[\mathbf{1}_{\{\theta_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b^{**}_{-i}(\boldsymbol{v}, \boldsymbol{\theta}_{-i})\}}\right]$$

be the interim probability of i getting recommended when reporting as $\hat{\theta}_i$ in the second stage, where the superscript H is to denote that the payoffs are derived under the modified handicap auction.

Lemma 8. *A modified handicap auction is incentive compatible iff for all $i \neq 0$ and $\theta_i \in \Theta$,*

$$\Pi^H_i(\theta_i) = \Pi^H_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} Q^H_i(\tilde{\theta}_i, \tilde{\theta}_i) d\tilde{\theta}_i \tag{69}$$

and for all $\theta'_i, \theta''_i \in [\underline{\theta}, \bar{\theta}]$ such that $\theta'_i < \theta_i < \theta''_i$,

$$Q_i(\theta_i, \theta'_i) \leq Q_i(\theta_i, \theta_i) \leq Q_i(\theta_i, \theta''_i). \tag{70}$$

Proof. **Necessity ($\rightarrow$):**

Note that an incentive compatibility is equivalent to

$$\text{for all } i \text{ and } \hat{v}_i < v_i,\ \pi_i^H(v_i, v_i) \geq \pi_i^H(v_i, \hat{v}_i) \text{ and } \pi_i^H(\hat{v}_i, \hat{v}_i) \geq \pi_i^H(\hat{v}_i, v_i).$$

Without loss of generality, we assume $\hat{v}_i < v_i$. Define for $x, y \in [\underline{\theta}, \bar{}]$,

$$\Delta(x,y) = E_{\theta_{-i},\boldsymbol{v}}\left[(x + w(v_i) - \tau_i(y) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}) - w(v_i) + \tau_i(y) + \ell_i(\boldsymbol{v}) \in (\min(x,y), \max(x,y))\}}\right]$$

We can rewrite $\pi_i^H(v_i, \hat{v}_i)$ as

$$\begin{aligned}
\pi_i^H(v_i, \hat{v}_i) = {} & E_{\theta_{-i},\boldsymbol{v}}\left[(\hat{\theta}_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\hat{\theta}_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] - c(\hat{\theta}_i) \\
& + E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i - \hat{\theta}_i)\mathbf{1}_{\{\hat{\theta}_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}\}}\right] \\
& + E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v}) \geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}) \geq \hat{\theta}_i + w(v_i) - \tau_i(\hat{\theta}_i) - \ell_i(\boldsymbol{v})\}}\right. \\
= {} & \Pi_i^H(\hat{\theta}_i) + Q_i^H(\hat{\theta}_i, \hat{\theta}_i)(\theta_i - \hat{\theta}_i) + \Delta_i(\theta_i, \hat{\theta}_i).
\end{aligned}$$

Similarly,

$$\pi_i^H(\hat{v}_i, v_i) = \Pi_i^H(\hat{\theta}_i) - Q_i^H(\hat{\theta}_i, \hat{\theta}_i)(\theta_i - \hat{\theta}_i) - \Delta_i(\theta_i, \hat{\theta}_i).$$

Incentive compatibility is equivalent to, for all i and $\hat{\theta}_i < \theta_i$,

$$Q_i^H(\hat{\theta}_i, \hat{\theta}_i) + \frac{\Delta_i(\theta_i, \hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq \frac{\Pi_i^H(\theta_i) - \Pi_i^H(\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq Q_i^H(\theta_i, \theta_i) + \frac{\Delta_i(\hat{\theta}_i, \theta_i)}{\theta_i - \hat{\theta}_i}$$

Note that $\Delta_i(x,y) \geq 0$ if and only if $x \geq y$, so that $\Delta_i(\hat{\theta}_i, \theta_i) \leq 0 \leq \Delta_i(\theta_i, \hat{\theta}_i)$. Therefore, incentive compatibility implies $Q_i^H(\hat{\theta}_i, \hat{\theta}_i) \leq \frac{\Pi_i^H(\theta_i) - \Pi_i^H(\hat{\theta}_i)}{\theta_i - \hat{\theta}_i} \leq Q_i^H(\theta_i, \theta_i)$. Since $Q_i^H(\theta_i, \theta_i)$ increases in θ_i, it is integrable, which implies (69).

It remains to prove that (70) holds. Suppose $\hat{\theta}_i < \theta_i$. If $\tau(\hat{\theta}_i) \geq \tau(\theta_i)$, then $Q_i^H(\hat{\theta}_i, \hat{\theta}_i) \leq Q_i^H(\theta_i, \hat{\theta}_i) \leq Q_i^H(\theta_i, \theta_i)$. Suppose $\tau(\hat{\theta}_i) < \tau(\theta_i)$. Let

$$\epsilon_i(x,y) = \mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}) \in (\min(x - \tau(x), \tau(y)), \max(x - \tau(x), \tau(y)))\}}(x + w(v_i) - \tau(y) - b_{-i}^{**}{}_{-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})))$$

Rewrite

$$\begin{aligned}\pi_i^H(\theta_i,\hat\theta_i) =&\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i+w(v_i)-\tau(\theta_i)\geq b^{**}_{-i\,-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}(\theta_i+w(v_i)-\tau(\theta_i)-b^{**}_{-i\,-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})))-c(\theta_i)\\ &-\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i+w(v_i)-\tau(\theta_i)\geq b^{**}_{-i\,-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}(\tau(\theta_i)-\tau(\hat\theta_i)))+c(\theta_i)-c(\hat\theta_i)\\ &+\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i+w(v_i)-\tau(\theta_i)\geq b^{**}_{-i\,-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})>\theta_i+w(v_i)-\tau(\theta_i)\}}(\theta_i+w(v_i)-\tau(\hat\theta_i)-b^{**}_{-i\,-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})))\\ =&\Pi_i^H(\theta_i)+Q_i^H(\theta_i,\theta_i)(\tau_i(\theta_i)-\tau_i(\hat\theta_i))+c(\theta_i)-c(\hat\theta_i)+\epsilon_i(\theta_i,\hat\theta_i).\end{aligned}$$

Similarly,

$$\pi_i^H(\theta_i,\hat\theta_i)=\Pi_i^H(\hat\theta_i)+Q_i^H(\hat\theta_i,\hat\theta_i)(\tau_i(\theta_i)-\tau_i(\hat\theta_i))+c(\hat\theta_i)-c(\theta_i)-\epsilon_i(\hat\theta_i,\theta_i).$$

By incentive compatibility, $\pi_i^H(\theta_i,\hat\theta_i)-\pi_i^H(\theta_i,\theta_i)\leq 0\leq \pi_i^H(\hat\theta_i,\hat\theta_i)-\pi_i^H(\hat\theta_i,\theta_i)$, which is equivalent to

$$\pi_i^H(\theta_i,\theta_i)(\tau(\theta_i)-\tau^{\hat\theta_i})+\epsilon_i(\theta_i,\hat\theta_i)\leq c(\hat\theta_i)-c(\theta_i)\leq \pi_i^H(\hat\theta_i,\hat\theta_i)(\tau(\theta_i)-\tau^{\hat\theta_i})+\epsilon_i(\hat\theta_i,\theta_i). \quad (71)$$

Since $\hat\theta_i<\theta_i$ and $\tau(\hat\theta_i)<\tau(\theta_i)$, it follows that $\epsilon_i(\hat\theta_i,\theta_i)\leq 0\leq \epsilon_i(\theta_i,\hat\theta_i)$. Then, $\epsilon_i(\theta_i,\hat\theta_i)=\epsilon_i(\hat\theta_i,\theta_i)=0$; otherwise, (71) implies $Q_i^H(\theta_i,\theta_i)<Q_i^H(\hat\theta_i,\hat\theta_i)$, contradicting (A.6). Since $\epsilon_i(\theta_i,\hat\theta_i)=0$,

$$\mathbb{E}_{\boldsymbol{v},\boldsymbol{\theta}_{-i}}(\mathbf{1}_{\{\theta_i+w(v_i)-\tau(\hat\theta_i)\geq b^{**}_{-i\,-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})>\theta_i+w(v_i)-\tau(\theta_i)\}})=0,$$

which is equivalent to $\Pi_i^H(\theta_i,\hat\theta_i)-\Pi_i^H(\theta_i,\theta_i)$. Therefore, $\hat\theta_i<\theta_i$ implies $\Pi_i^H(\theta_i,\hat\theta_i)\leq\Pi_i^H(\theta_i,\theta_i)$ whether $\tau(\theta_i)\leq\tau(\hat\theta_i)$ or not, so that the first inequality of (70) holds. The other inequality may be shown similarly, which establishes (70).

Sufficiency (←): Suppose that the seller i with θ_i has reported itself as $\hat\theta_i$ when purchasing the premium. In the second stage, after learning $\boldsymbol{v}$, the payoff of reporting as θ_i' is

$$U_i(\theta_i,\theta_i'\mid\hat\theta_i,\boldsymbol{v})=E_{\theta_{-i}}\left[(\theta_i+w(v_i)-\tau_i(\hat\theta_i)-\ell_i(\boldsymbol{v})-b^{**}_{-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i'+w(v_i)-\tau_i(\hat\theta_i)-\ell_i(\boldsymbol{v})\geq b^{**}_{-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right]$$

Since the second price auction in the second stage is incentive compatible, for any $\theta_i\in(\underline{\theta},\bar\theta)$

$$\frac{\partial U_i^H}{\partial\theta_i}(\theta_i\mid\hat\theta_i,\boldsymbol{v})=Q_i^H(\theta_i\mid\hat\theta_i,\boldsymbol{v})$$

where $U_i^H(\theta_i\mid\hat\theta_i,\boldsymbol{v})=U_i(\theta_i,\theta_i\mid\hat\theta_i,\boldsymbol{v})$ and $Q_i^H(\theta_i\mid\hat\theta_i,\boldsymbol{v})=E_{\theta_{-i}}\left[\mathbf{1}_{\{\theta_i'+w(v_i)-\tau_i(\hat\theta_i)-\ell_i(\boldsymbol{v})\geq b^{**}_{-i}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right]$.

For any $\theta_i, \theta_i' \in (\underline{\theta}, \bar{\theta})$,

$$U_i^H(\theta_i \mid \hat{\theta}_i, \boldsymbol{v}) = U_i^H(\theta_i' \mid \hat{\theta}_i, \boldsymbol{v}) + \int_{\theta_i'}^{\theta_i} Q_i^H(\tilde{\theta} \mid \hat{\theta}_i, \boldsymbol{v}) d\tilde{\theta}.$$

The seller's interim payoff function in the first stage may be expressed as

$$\begin{aligned}
\pi_i^H(\theta_i, \hat{\theta}_i) &= E_{\boldsymbol{v}}\left[U_i^H(\theta_i \mid \hat{\theta}_i, \boldsymbol{v})\right] - c(\hat{\theta}_i) \\
&= E_{\boldsymbol{v}}\left[U_i^H(\hat{\theta}_i \mid \hat{\theta}_i, \boldsymbol{v})\right] + E_{\boldsymbol{v}}\left[\int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i \mid \hat{\theta}_i, \boldsymbol{v}) d\tilde{\theta}_i\right] - c(\hat{\theta}_i) \\
&= \Pi_i^H(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}, \hat{\theta}_i) d\tilde{\theta}.
\end{aligned}$$

Similarly,

$$\pi_i^H(\hat{\theta}_i, \theta_i) = \Pi_i^H(\theta_i) + \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}, \theta_i) d\tilde{\theta}.$$

Note that the incentive compatibility is equivalent to

$$\text{for all } i \text{ and } \hat{\theta}_i < \theta_i,\ \pi_i^H(\theta_i, \theta_i) \geq \pi_i^H(\theta_i, \hat{\theta}_i) \text{ and } \pi_i^H(\hat{\theta}_i, \hat{\theta}_i) \geq \pi_i^H(\hat{\theta}_i, \theta_i).$$

Note that

$$\begin{aligned}
&\pi_i^H(\theta_i, \theta_i) \geq \pi_i^H(\theta_i, \hat{\theta}_i) \\
&\text{iff } \Pi_i^H(\theta_i) \geq \Pi_i^H(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i, \hat{\theta}_i) d\tilde{\theta}_i \\
&\text{iff } \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) d\tilde{\theta}_i \geq \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i, \hat{\theta}_i) d\tilde{\theta}_i \qquad \text{(By (69))}
\end{aligned}$$

where the last inequality holds because $Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) \geq Q_i^H(\tilde{\theta}_i, \hat{\theta}_i)$ for all $\tilde{\theta}_i \geq \hat{\theta}_i$ by (70). Similarly,

$$\begin{aligned}
&\pi_i^H(\hat{\theta}_i, \hat{\theta}_i) \geq \pi_i^H(\hat{\theta}_i, \theta_i) \\
&\text{iff } \Pi_i^H(\hat{\theta}_i) \geq \Pi_i^H(\theta_i) + \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}_i, \theta_i) d\tilde{\theta}_i \\
&\text{iff } \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) d\tilde{\theta}_i \geq \int_{\theta_i}^{\hat{\theta}_i} Q_i^H(\tilde{\theta}_i, \theta_i) d\tilde{\theta}_i \qquad \text{(By (69))} \\
&\text{iff } \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) d\tilde{\theta}_i \leq \int_{\hat{\theta}_i}^{\theta_i} Q_i^H(\tilde{\theta}_i, \theta_i) d\tilde{\theta}_i
\end{aligned}$$

where the last inequality holds because $Q_i^H(\tilde{\theta}_i, \tilde{\theta}_i) \leq Q_i^H(\tilde{\theta}_i, \theta_i)$ for all $\tilde{\theta}_i \leq \theta_i$ by (70).

$\square$

Now I am ready to prove the theorem. If every seller of every type reports its willingness to pay truthfully under the handicap auction, then the intermediary recommends product i if and only if

$$r_i(\boldsymbol{v},\boldsymbol{\theta}) = 1 \text{ iff } \theta_i + w(v_i) - \frac{1-G_i(\theta_i)}{g_i(\theta_i)} - \ell_i(\boldsymbol{v}) > \max_{j\neq i,0}\left(\theta_j + w(v_j) - \frac{1-G_i(\theta_i)}{g_i(\theta_i)} - \ell_j(\boldsymbol{v}), 0\right)$$

which is the same recommendations rule as in the revenue maximizing recommender system.

It remains to verify that the handicap auction (18) and (19) is incentive compatible and individually rational. Since τ_i weakly decreases in θ_i, the monotonicity condition (70) holds. The interim payoff of the seller i with θ_i receives is

$$\Pi_i^H(\theta_i) = E_{\theta_{-i},\boldsymbol{v}}\left[(\theta_i + w(v_i) - \tau_i(\theta_i) - \ell_i(\boldsymbol{v}) - b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i}))\mathbf{1}_{\{\theta_i+w(v_i)-\tau_i(\theta_i)-\ell_i(\boldsymbol{v})\geq b_{-i}^{**}(\boldsymbol{v},\boldsymbol{\theta}_{-i})\}}\right] - c(\theta_i)$$

which implies that $\Pi_i^H(\underline{\theta}) = 0$. The handicap auction gives the same revenue as the revenue maximizing recommender system because the recommendation rules are identical and the lowest type's payoff is the same by $\Pi_i^*(\underline{\theta}) = 0 = \Pi_i^H(\underline{\theta})$, and therefore, $\Pi_i^*(\theta_i) = \Pi_i^H(\theta_i)$, which in turn implies that the expected payment from each recommender system must be identical between the two. The individual rationality trivially follows from the observation that the lowest payoff each seller gets $\Pi_i(\underline{\theta})$ is 0.

B Proofs for Additional Information

B.1 Optimal Recommender System with Additional Information

This section characterizes incentive compatible, individually rational and obedient recommender system, and recast the intermediary's problem to a Bayesian persuasion problem as in the baseline model.

For a seller $i \in \mathcal{N}$ with (θ_i, z_i) reporting truthfully as θ_i, let

$$\Pi(\theta_i, z_i) = \int_{\mathcal{V}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}_{-i}} \left[(\theta_i + w(v_i))r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) - t_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\right]\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit, and

$$Q(\theta_i, z_i) = \int_{\mathcal{V}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}_{-i}} r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i} \mid \boldsymbol{\theta}_{-i})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected probability of recommending i's product.

Lemma 9. *A recommender system* $(\boldsymbol{r},\boldsymbol{t}):\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}\to[0,1]^{N+1}\times\mathbb{R}^N$ *is incentive compatible, individually rational and obedient if and only if for each* $i\in\mathcal{N}$*, for all* $\theta_i\in[\underline{\theta},\bar{\theta}]$ *and* $z_i\in\mathcal{Z}$*,*

$$Q_i(\theta_i,z_i) \text{ is increasing in } \theta_i, \tag{72}$$

$$\Pi_i(\theta_i,z_i)=\Pi_i(\underline{\theta},z_i)+\int_{\underline{\theta}}^{\theta_i}Q_i(\tilde{\theta}_i,z_i)d\tilde{\theta}_i, \tag{73}$$

$$\Pi_i(\underline{\theta})\geq 0 \tag{74}$$

and for all $i,j\in\mathcal{N}\cup\{0\}$*,*

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}}r_i(\boldsymbol{v},\theta_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})\\ &\geq\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}}r_i(\boldsymbol{v},\theta_i',\boldsymbol{\theta}_{-i},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).\end{aligned} \tag{75}$$

Proof. **Necessity:** Let $\theta_i>\hat{\theta}_i$ and $z_i\in\mathcal{Z}$. Let

$$\pi_i(\hat{\theta}_i;\theta_i,z_i)=\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}_{-i}\times\boldsymbol{\mathcal{Z}}_{-i}}\left[(\theta_i+w(v_i))r_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})-t_i(\boldsymbol{v},\hat{\theta}_i,\boldsymbol{\theta}_{-i},\boldsymbol{z})\right]\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{H}_{-i}(d\boldsymbol{z}_{-i}\mid\boldsymbol{\theta}_{-i})\boldsymbol{G}_{-i}(\boldsymbol{\theta}_{-i})$$

be the expected profit of the seller with (θ_i,z_i) when he reports as $\hat{\theta}_i$. Note that

$$\pi_i(\hat{\theta}_i;\theta_i,z_i)=\Pi_i(\hat{\theta}_i,z_i)+(\theta_i-\hat{\theta}_i)Q_i(\hat{\theta}_i,z_i)$$

Similarly,

$$\pi_i(\theta_i;\hat{\theta}_i,z_i)=\Pi_i(\theta_i,z_i)+(\theta_i-\hat{\theta}_i)Q_i(\theta_i,z_i).$$

Incentive compatibility $\Pi_i(\theta_i,z_i)\geq\pi_i(\hat{\theta}_i;\theta_i,z_i)$ and $\Pi_i(\hat{\theta}_i,z_i)\geq\pi_i(\theta_i;\hat{\theta}_i,z_i)$, which in turn implies

$$(\theta_i-\hat{\theta}_i)Q_i(\hat{\theta}_i)\leq\Pi_i(\theta_i)-\Pi_i(\hat{\theta}_i)\leq(\theta_i-\hat{\theta}_i)Q_i(\theta_i,z_i).$$

By the above inequality, Q_i is weakly increasing and hence is integrable, which then implies (73).

Individual rationality is equivalent to $\Pi_i(\theta_i)\geq 0$ for all $i\neq 0$ and $\theta_i\in[\underline{\theta},\bar{\theta}]$, from which $\Pi_i(\underline{\theta})\geq 0$ for all i follows.

Sufficiency: Let $\theta_i \neq \hat{\theta}_i$. From (73) and the monotonicity of Q_i,

$$\begin{aligned}\Pi_i(\theta_i, z_i) &= \Pi_i(\hat{\theta}, z_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i \\ &\geq \Pi_i(\hat{\theta}, z_i) + \int_{\hat{\theta}_i}^{\theta_i} Q_i(\hat{\theta}_i, z_i) d\tilde{\theta}_i \\ &= \Pi_i(\hat{\theta}, z_i) + (\theta_i - \hat{\theta}_i) Q_i(\hat{\theta}_i, z_i) \\ &= \pi_i(\hat{\theta}_i; \theta_i, z_i) \end{aligned} \tag{76}$$

Since $\theta_i \neq \hat{\theta}_i$ are arbitrary, (76) implies incentive compatibility.

Since $Q_i(\theta_i) \geq 0$ for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$, (73) implies that $\Pi_i(\theta_i)$ increases in θ_i, and hence, individual rationality is satisfied if $\Pi_i(\underline{\theta}) \geq 0$. □

Lemma 10. *Suppose that a recommendations rule $\boldsymbol{r} : \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}} \to [0,1]^N$ maximizes*

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i) \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \tag{77}$$

subject to OB and monotonicity. Suppose also that

$$t_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) - \int_{\underline{\theta}}^{\theta_i} r_i(\boldsymbol{v}, \tilde{\theta}_i, \boldsymbol{\theta}_{-i}, \boldsymbol{z}) d\tilde{\theta}_i. \tag{78}$$

Then, $(\boldsymbol{r}, \boldsymbol{t})$ is an optimal recommender system.

Proof. By Lemma 9, for any $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and $z_i \in \mathcal{Z}$,

$$\Pi_i(\theta_i, z_i) = \Pi_i(\underline{\theta}, z_i) + \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i.$$

The expected transfer of the seller i with (θ_i, z_i) is

$$T_i(\theta_i, z_i) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}_{-i} \times \boldsymbol{\mathcal{Z}}_{-i}} (\theta_i + w(v_i)) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i} \mid \boldsymbol{z}_{-i}) \boldsymbol{H}(d\boldsymbol{z}_{-i}) - \int_{\underline{\theta}}^{\theta_i} Q_i(\tilde{\theta}_i, z_i) d\tilde{\theta}_i - \Pi_i(\underline{\theta}, z_i). \tag{79}$$

By the usual argument of the change of variables, for each $z_i \in \mathcal{Z}$, we have

$$\begin{aligned}&\int_{\Theta \times \mathcal{Z}} T_i(\theta_i, z_i) G(d\theta_i \mid z_i) H(dz_i) \\ &= \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \left(\theta_i + w(v_i) - \frac{1 - G(\theta_i)}{g(\theta_i)} \right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - \int_{\mathcal{Z}} \Pi_i(\underline{\theta}, z_i) H(dz_i),\end{aligned}$$

so that the intermediary's expected revenue is

$$\sum_{i\in\mathcal{N}}\int_{\Theta\times\mathcal{Z}} T_i(\theta_i,z_i)G(d\theta_i\mid z_i)H(dz_i)$$
$$=\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}\left(\theta_i+w(v_i)-\frac{1-G(\theta_i)}{g(\theta_i)}\right)r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})-\int_{\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}\Pi_i(\underline{\theta},z_i)\boldsymbol{H}(d\boldsymbol{z}). \tag{80}$$

The intermediary's problem is to maximize (80) using a recommender system $(\boldsymbol{r},\boldsymbol{t})$ subject to monotonicity constraints (72), payoff equivalence constraints (73), non-negativity constraints (74) and obedience constraints (75). Note that for any given recommendations rule $\boldsymbol{r}$ satisfying (72) and (75), any transfer $\boldsymbol{t}$ such that $\pi_i(\underline{\theta},z_i)=0$ for all $i\in\mathcal{N}$ and $z_i\in\mathcal{Z}$ and whose interim transfer satisfies (79) maximizes (80) while satisfying (73) and (74). Transfer (78) is one of such.

It remains to find an optimal recommendations rule $\boldsymbol{r}$. Since $\Pi_i(\underline{\theta},z_i)=0$ for all $i\in\mathcal{N}$ and $z_i\in\mathcal{Z}$ independent of $\boldsymbol{r}$, it immediately follows that a recommendations rule $\boldsymbol{r}$ that maximizes (77) subject to (72) and (75), together with the corresponding transfer (78), maximizes the intermediary's expected revenue subject to (72), (73), (74) and (75). □

B.2 Additional Information as a Change in the Intermediary's Preference

This section presents four equivalent ways to express the intermediary's problem without additional information, each interpreted as: 1. the intermediary's problem without additional information; 2. the intermediary's problem with additional information but with invariance constraints; 3. additional information as a change in preference; 4. additional information as relaxation of invariance constraints.

Lemma 11. *The followings are solution equivalent (after adjusting for invariance constraints related notations):*

1. *A recommendations rule without additional infromation* $\boldsymbol{r}:\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\to[0,1]^{N+1}\times\mathbb{R}^N$ *that maximizes*

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i-\frac{1-G(\theta_i)}{g(\theta_i)}+w(v_i)\right)r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \tag{81}$$

subject to monotonicity constraints without additional information, for all $i\in\mathcal{N}$ *and*

$\theta_i > \theta_i'$

$$\begin{aligned} &\int_{\mathcal{V}\times\mathbf{\Theta}_{-i}} r_i(\boldsymbol{v},\theta_i,\boldsymbol{\theta}_{-i})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) \\ \geq &\int_{\mathcal{V}\times\mathbf{\Theta}_{-i}} r_i(\boldsymbol{v},\theta_i',\boldsymbol{\theta}_{-i})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}_{-i}(d\boldsymbol{\theta}_{-i}) \end{aligned} \tag{82}$$

and obedience constraints without additional information, for all $i, j \in \mathcal{N} \cup \{0\}$,

$$\begin{aligned} &\int_{\mathcal{V}\times\mathbf{\Theta}} v_i r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}) \\ \geq &\int_{\mathcal{V}\times\mathbf{\Theta}} v_j r_i(\boldsymbol{v},\boldsymbol{\theta})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}). \end{aligned} \tag{83}$$

2. *A recommendations rule $\boldsymbol{r} : \mathcal{V} \times \mathbf{\Theta} \times \mathcal{Z} \to [0,1]^{N+1} \times \mathbb{R}^N$ that maximizes*

$$\int_{\mathcal{V}\times\mathbf{\Theta}\times\mathcal{Z}} \sum_{i\in\mathcal{N}} \left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right) r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \tag{84}$$

subject to monotonicity constraints (72), obedience constraints (75) and invariance constraints

$$r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}') \text{ for all } \boldsymbol{z},\boldsymbol{z}' \in \mathcal{Z}. \tag{85}$$

3. *A recommendations rule $\boldsymbol{r} : \mathcal{V} \times \mathbf{\Theta} \times \mathcal{Z} \to [0,1]^{N+1} \times \mathbb{R}^N$ that maximizes (84) subject to monotonicity constraints (72) and obedience constraints (75).*

4. *A recommendations rule $\boldsymbol{r} : \mathcal{V} \times \mathbf{\Theta} \times \mathcal{Z} \to [0,1]^{N+1} \times \mathbb{R}^N$ that maximizes (77) subject to monotonicity constraint (72), obedience constraints (75) and invariance constraints (85).*

The first is the intermediary's problem without additional information after substituting the expected transfer with virtual willingness to pay using the standard arguments. The second is a reformulation of the first under the setup with additional information. The third is stating that invariance constraints (85) are redundant in the second, because the integrands both in the objective function and the constraints do not depend on $\boldsymbol{z}$.

Note that the set of constraints are identical under the third and the intermediary's problem with additional information. The only difference between the two problems is the objective functions. In other words, additional information changes the intermediary's objective function from (84) to (77) subject to the *same* constraints, i.e. 'additional information as a change in the intermediary's preference,' the idea used for the consumer surplus analysis.

This means that the baseline problem without additional information can be understood as maximizing the same objective function but with added invariance constraints in relative to . That is, additional information is a *deletion* of invariance constraints withe the *same* objective function, i.e. 'additional information as a deletion of invariance constraints,' the idea used for the intermediary's revenue analysis.

Proof. **1** $\iff$ **2**: Once restricting attention to the recommendations rule satisfying the invariance constraints, the first problem and the second problem are identical, and hence, their solutions must be solution-equivalent.

2 $\iff$ **3**: The solution to the third problem is

$$r_i^{P3}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1 \text{ if } i = \arg\max_{j\in\mathcal{N}}\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i) + \ell_i(\boldsymbol{v}), 0\right)$$

where $\ell_i(\boldsymbol{v})$ is a cost of persuasion. Note that $\boldsymbol{r}^{P3}$ does not vary depending on $\boldsymbol{z}$, and hence, satisfies the invariance constraints. This is because neither the objective function (84) nor the obedience constraints (75) have integrands that depend on $\boldsymbol{z}$ whereas the monotonicity constraints (72) are automatically satisfied. Therefore, the solution to the third problem $\boldsymbol{r}^{P3}$ solves the second problem $\boldsymbol{r}^{P2}$.

2 $\iff$ **4**: Note that

$$\int_{\mathcal{Z}}\frac{1-G(\theta_i\mid z_i)}{g(\theta_i\mid z_i)}G(d\theta_i\mid z_i)H(dz_i) = 1 - \int_{\mathcal{Z}}G(\theta_i\mid z_i)H(dz_i) = 1-G(\theta_i) = \frac{1-G(\theta_i)}{g(\theta_i)}G(d\theta_i). \tag{86}$$

By (86), restricting attention to the recommendations rule satisfying the invariance constraints (85), the objective function in the fourth problem (77) becomes

$$\begin{aligned}&\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}}\left(\theta_i - \frac{1-G(\theta_i\mid z_i)}{g(\theta_i\mid z_i)} + w(v_i)\right)r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z})\\ &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}}\sum_{i\in\mathcal{N}}\left(\theta_i - \frac{1-G(\theta_i)}{g(\theta_i)} + w(v_i)\right)r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta})\end{aligned}$$

so that the fourth problem $P4$ becomes the same as the second problem $P2$. □

B.3 Proof for Lower Censorship Additional Information

Let $\mathcal{H}$ be lower censorship additional information. Let G be a twice continuously differentiable distribution that has a decreasing $\frac{1-G(\theta)}{g(\theta)}$ on Θ, and $g(\theta) > 0$ and $0 \le g'(\theta) < \infty$ on a neighborhood of $\underline{\theta}$.

For $z \in [\theta^*, \bar{\theta}]$, the signal fully reveals the state, $\Theta(z) = \{\theta\}$, and hence, the inverse hazard rate is 0. For $z = z_0$, the signal informs that $\theta \in \Theta(z) = [\underline{\theta}, \theta^*)$, and the inverse hazard rate is

$$\frac{1 - G(\theta \mid z_0)}{g(\theta \mid z_0)} = \frac{\int_\theta^{\bar{\theta}} g(\tilde{\theta}) 1_{\tilde{\theta} \in [\underline{\theta}, \theta^*)} d\tilde{\theta}}{g(\theta) 1_{\theta \in [\underline{\theta}, \theta^*)}} = \frac{G(\theta^*) - G(\theta)}{g(\theta)}.$$

Let $z(\theta)$ be a unique additional signal $z \in \mathcal{Z}$ that can be generated with a positive probability for each $\theta \in \Theta$. Define a conditional inverse hazard rate α

$$\alpha(\theta) = \frac{1 - G(\theta \mid z(\theta))}{g(\theta \mid z(\theta))} = \begin{cases} \frac{G(\theta^*) - G(\theta)}{g(\theta)} & \text{if } \theta < \theta^* \\ 0 & \text{if } \theta \geq \theta^*. \end{cases}$$

Now I show that for any $\epsilon > 0$, there is a small enough θ^* such that for all $z \in \mathcal{Z}$ and $\theta \in \Theta(z)$

$$\frac{1 - G(\theta \mid z)}{g(\theta \mid z)} < \epsilon, \tag{87}$$

and for any θ^*, for all $z, z' \in \mathcal{Z}$, $\theta \in \Theta(z)$ and $\theta' \in \Theta(z')$,

$$0 < 1 - \frac{\frac{1 - G(\theta|z)}{g(\theta|z)} - \frac{1 - G(\theta'|z')}{g(\theta'|z')}}{\theta - \theta'} \leq 1 - \frac{\frac{1 - G(\theta)}{g(\theta)} - \frac{1 - G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{88}$$

To show (87), let $\epsilon > 0$ be such that $\epsilon < \inf_{\theta \in \Theta, v \in \mathcal{V}} \theta + w(v)$. Let $\beta(\theta^*) = \sup_{\theta \in [\underline{\theta}, \theta^*]} \frac{1}{g(\theta)}$. Then, $\beta(\theta^*)$ is continuously increasing in θ^* and $\beta(\theta) \leq \beta(\theta^*) < \infty$ for sufficiently small θ^* and $\theta \leq \theta^*$. Therefore, for any θ^* and $\theta \leq \theta^*$

$$\frac{1 - G(\theta \mid z_0)}{g(\theta \mid z_0)} = \frac{G(\theta^*) - G(\theta)}{g(\theta)} \leq G(\theta^*) \beta(\theta^*).$$

Since both $G(\theta^*)$ and $\beta(\theta^*)$ increase in θ^* with $G(\theta^*) \to 0$ and $\beta(\theta^*) < \infty$ as $\theta^* \to \underline{\theta}$, for sufficiently small θ^*, $G(\theta^*)\beta(\theta^*) < \epsilon$. For $\theta \geq \theta^*$, $z(\theta) = \theta$ perfectly reveals θ, so that $\frac{1 - G(\theta|z(\theta))}{G(\theta|z(\theta))} = 0$. Therefore, for any given ϵ, for sufficiently small θ^*, (87) holds.

To show (88), let $\theta^* \in (\underline{\theta}, \bar{\theta})$. For $\theta, \theta' \geq \theta^*$, their inverse hazard rates are 0, so that

$$0 < 1 = 1 - \frac{\frac{1 - G(\theta|z(\theta))}{g(\theta|z(\theta))} - \frac{1 - G(\theta'|z(\theta'))}{g(\theta'|z(\theta'))}}{\theta - \theta'} \leq 1 - \frac{\frac{1 - G(\theta)}{g(\theta)} - \frac{1 - G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{89}$$

For $\theta, \theta' < \theta^*$, their additional signal is z_0, and

$$\frac{\frac{1-G(\theta|z_0)}{g(\theta|z_0)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = \frac{\frac{G(\theta^*)-G(\theta)}{g(\theta)} - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'}$$
$$= -\frac{\frac{1-G(\theta^*)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta - \theta'} + \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}.$$

Note that $\frac{1-G(\theta^*)}{g(\theta)}$ decreases in θ on $[\underline{\theta}, \theta^*]$ for sufficiently small θ^* because $g(\theta)$ increases in θ, implying $-\frac{\frac{1-G(\theta^*)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta-\theta'} \geq 0$ and hence,

$$1 - \frac{\frac{1-G(\theta|z_0)}{g(\theta|z_0)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} \leq 1 - \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{90}$$

Furthermore, since g is continuously differentiable, by Mean Value Theorem, there is θ'' between θ and θ' such that

$$1 - \frac{\frac{1-G(\theta|z_0)}{g(\theta|z_0)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = 1 - \frac{\frac{G(\theta^*)-G(\theta)}{g(\theta)} - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'}$$
$$= 1 - \frac{-g^2(\theta'') - (G(\theta^*) - G(\theta''))g'(\theta'')}{g^2(\theta'')}$$
$$= 2 + (G(\theta^*) - G(\theta''))\frac{g'(\theta'')}{g^2(\theta'')}. \tag{91}$$

Define $\gamma(\theta'') = \sup_{\tilde{\theta} \in [\underline{\theta}, \theta'']} \left| \frac{g'(\tilde{\theta})}{g^2(\tilde{\theta})} \right|$. Then, $\gamma(\theta'')$ is continuously increasing in θ'' and $\gamma(\theta'')$ and $\gamma(\theta''') \leq \gamma(\theta'') < \infty$ for all $\theta''' \leq \theta''$ for sufficiently small θ''. Since $\theta'' \leq \theta^*$, for any $\epsilon > 0$, for sufficiently small θ^*,

$$\left| (G(\theta^*) - G(\theta''))\frac{g'(\theta'')}{g^2(\theta'')} \right| \leq G(\theta^*) \left| \frac{g'(\theta^*)}{g^2(\theta^*)} \right| < \epsilon.$$

which implies that (91) is positive.

Lastly, consider $\theta \geq \theta^* \geq \theta'$. Then, $z(\theta) = \theta$ and $z(\theta') = z_0$, so that

$$\frac{\frac{1-G(\theta|\theta)}{g(\theta|\theta)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = \frac{0 - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'}$$
$$= -\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta - \theta'} + \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}.$$

For sufficiently small θ^*, g increases on $[\underline{\theta}, \theta^*]$, so that $g(\theta') \leq g(\theta^*)$, and

$$
\begin{aligned}
-\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta')}}{\theta - \theta'} &\geq -\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta^*)}}{\theta - \theta'} \\
&= -\frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta^*)}{g(\theta^*)}}{\theta - \theta^*} \frac{\theta - \theta^*}{\theta - \theta'} \\
&\geq 0
\end{aligned}
$$

where the last inequality is implied by $\frac{1-G(\theta)}{g(\theta)}$ decreasing in θ. Therefore,

$$
1 - \frac{\frac{1-G(\theta|\theta)}{g(\theta|\theta)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} \leq 1 - \frac{\frac{1-G(\theta)}{g(\theta)} - \frac{1-G(\theta')}{g(\theta')}}{\theta - \theta'}. \tag{92}
$$

Furthermore, for any θ^*,

$$
1 - \frac{\frac{1-G(\theta|\theta)}{g(\theta|\theta)} - \frac{1-G(\theta'|z_0)}{g(\theta'|z_0)}}{\theta - \theta'} = 1 - \frac{0 - \frac{G(\theta^*)-G(\theta')}{g(\theta')}}{\theta - \theta'} \geq 1 > 0. \tag{93}
$$

By (89), (90), (91), (92) and (93), it follows that (88).

Therefore, by (87) and (88), for sufficiently small $\epsilon > 0$ and θ^*, the lower censorship additional information with small $\theta^* \in (\underline{\theta}, \bar{\theta})$ is ϵ-informative and hence revenue-informative, is regular and decreases rate of substitution.

B.4 Proof for Theorem 4

The following lemma provides a sufficient condition under which the consumer surplus under one recommendations rule $\tilde{\boldsymbol{r}}$ is higher or lower than that under the other $\boldsymbol{r}^\dagger$. The lemma states that if $\boldsymbol{r}^\dagger$ almost surely recommends an option that is at least (at most) as good as options recommended by $\tilde{\boldsymbol{r}}$, then the consumer surplus under $\boldsymbol{r}^\dagger$ is higher (lower) than that under $\tilde{\boldsymbol{r}}$.

Define the consumer surplus under $\boldsymbol{r}$ at v_0 as

$$
CS(v_0; \boldsymbol{r}) = \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}} \left[v_0 r_0(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) + \sum_{i \neq 0} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \right] \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0)
$$

where

$$
u^*(v_0) = \max(v_0, \mathbb{E}(v_i))
$$

is the consumer's optimal payoff without recommendations.

Lemma 12. *If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^\dagger(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } v_j \geq (\leq)\ v_i \text{ for } i,j \in \mathcal{N} \cup \{0\},$$

almost surely, then $CS(v_0;\tilde{\boldsymbol{r}}) \leq (\geq)\ CS(v_0;\boldsymbol{r}^\dagger)$.

Proof. Suppose

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^\dagger(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } v_j \geq\ v_i \text{ for } i,j \in \mathcal{N} \cup \{0\},$$

almost surely. Let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \boldsymbol{\mathcal{Z}}$. Then,

$$\sum_{i\in\mathcal{N}\cup\{0\}} v_i \tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \leq \max_{i:\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} v_i \leq \min_{j:r_j^\dagger(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} v_j \leq \sum_{j\in\mathcal{N}\cup\{0\}} v_j r_j^\dagger(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})$$

so that

$$\begin{aligned} CS(v_0;\tilde{\boldsymbol{r}}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i \tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\ &\leq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} v_i r_i^\dagger(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\ &= CS(v_0;\boldsymbol{r}^\dagger). \end{aligned}$$

The other inequality may be shown similarly. ☐

Let $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ be optimal unconstrained recommendations rules without and with additional information. That is, $\boldsymbol{\rho}^*$ maximizes (84) subject to monotonicity constraints (72) and $\boldsymbol{\rho}^A$ maximizes (77) subject to monotonicity constraints (72).

Lemma 13. *Let* v_0 *at which both* $\boldsymbol{\rho}^*$ *and* $\boldsymbol{\rho}^A$ *are obedient. If additional information is revenue informative, regular and increases (decreases) rate of substitution, then*

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \geq (\leq)\ \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}). \tag{94}$$

Proof. Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Since the information rent is small as in Assumption 3.a, it follows that $\rho_0^* = 0$ and $\rho_0^A = 0$ almost surely. When one of the products is recommended, the rules are almost surely given by

$$\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i)\ \forall j \in \mathcal{N} \setminus \{i\}$$

$$\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i)\ \forall j \in \mathcal{N} \setminus \{i\}$$

To use Lemma 12, it is sufficient to show:

> if additional information increases (decreases) rates of substitution, $\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ for $i \in \mathcal{N}$ implies $\rho_j^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ only if $j \in \mathcal{N}$ and $v_j \leq (\geq)\ v_i$ almost surely.

To show this, let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}(\boldsymbol{z}) \times \boldsymbol{\mathcal{Z}}$ be at which $\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ for $i \in \mathcal{N}$. Almost surely, i is strictly preferred to $k \in \mathcal{N} \setminus \{i\}$. Since the additional information is regular, rates of substitution are positive, so that

$$\theta_i - \theta_k > (w(v_k) - w(v_i)) \Bigg/ \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right).$$

Suppose that additional information increases rates of substitution. Then,

$$1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j} \geq 1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}.$$

If $v_k > v_i$,

$$\theta_i - \theta_k > (w(v_k) - w(v_i)) \Bigg/ \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right) \geq (w(v_k) - w(v_i)) \Bigg/ \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k \in \mathcal{N} \setminus \{i\}$, $\rho_k^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 0$ if $v_k > v_i$. In other words, $\rho_j^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ only if $v_j \leq v_i$. By Lemma 12, it follows that $CS(v_0;\boldsymbol{\rho}^*) \geq CS(v_0;\boldsymbol{\rho}^A)$. Since the consumer's optimal payoff without recommendations is identical under both problems without and with additional information, this is equivalent to

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \boldsymbol{F}(d\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}).$$

To show the other case, suppose that additional information decreases rates of substitution. Then,

$$1-\frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)}-\frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i-\theta_j}\leq 1-\frac{\frac{1-G(\theta_i)}{g(\theta_i)}-\frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i-\theta_j}.$$

If $v_k < v_i$,

$$\theta_i-\theta_k > (w(v_k)-w(v_i))\Bigg/\left(1-\frac{\frac{1-G(\theta_i)}{g(\theta_i)}-\frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i-\theta_j}\right)\geq (w(v_k)-w(v_i))\Bigg/\left(1-\frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)}-\frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i-\theta_j}\right)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k\in\mathcal{N}\smallsetminus\{i\}$, $\rho_k^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 0$ if $v_k < v_i$. In other words, $\rho_j^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0$ only if $v_j \geq v_i$. By Lemma 12, it follows that $CS(v_0;\boldsymbol{\rho}^*) \geq CS(v_0;\boldsymbol{\rho}^A)$. Since the consumer's optimal payoff without recommendations is identical under both fictitious problem and problem with additional information, this is equivalent to

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}} v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \geq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}} v_i\rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).$$

□

Let $\bar{v}^* = \mathbb{E}(v_i \mid \rho_i^O(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1)$. Note that $\bar{v}* \geq \mathbb{E}_{v_i}(v_i)$. For $v_0 \leq \bar{v}^*$, $\boldsymbol{\rho}^O$ satisfies obedience constraints and hence $\boldsymbol{\rho}^*$ is optimal, that is, $\boldsymbol{r}^* = \boldsymbol{\rho}^*$ and $CS^O(v_0) > 0$. For $v_0 > \bar{v}^*$, $\boldsymbol{\rho}^*$ no longer satisfies obedience constraints. The obedience constraints from products to the outside option bind under $\boldsymbol{r}^*$ under which $CS(v_0;\boldsymbol{r}^*) = 0$. In particular, at $v_0 = \bar{v}^*$, $\boldsymbol{r}^* = \boldsymbol{\rho}^*$ and $CS^*(v_0) = 0$, implying that

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}} v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) = \bar{v}^*.$$

The consumer surplus without additional information is

$$CS^*(v_0) = \begin{cases}\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}} v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - \mathbb{E}_{v_i}(v_i) & \text{if } v_0 < \mathbb{E}_{v_i}(v_i)\\ \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}}\sum_{i\in\mathcal{N}} v_i\rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - v_0 & \text{if } v_0 \in [\mathbb{E}_{v_i}(v_i), \bar{v}_0^*]\\ 0 & \text{if } v_0 > \bar{v}_0^*\end{cases}$$

Similar analysis can be applied for $\boldsymbol{r}^A$. Let $\bar{v}^A = \mathbb{E}(v_i \mid \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) = 1)$. Note that $\bar{v}^A \geq \mathbb{E}_{v_i}(v_i)$. For $\bar{v}^A$, $\boldsymbol{\rho}^A$ satisfies obedience constraints and hence $\boldsymbol{\rho}^A$ is optimal, that is, $\boldsymbol{r}^A = \boldsymbol{\rho}^A$ and $CS^O(v_0) > 0$. For $v_0 > \bar{v}^A$, $\boldsymbol{\rho}^A$ no longer satisfies obedience constraints. The obedience constraints from products to the outside option bind under $\boldsymbol{r}^A$ under which

$CS(v_0; \boldsymbol{r}^A) = 0$. In particular, at $v_0 = \bar{v}^A$, $\boldsymbol{r}^A = \boldsymbol{\rho}^A$ and $CS^A(v_0) = 0$, implying that

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) = \bar{v}^A.$$

The consumer surplus under the problem with (revenue-informative) additional information problem is

$$CS^A(v_0) = \begin{cases} \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - \mathbb{E}_{v_i}(v_i) & \text{if } v_0 < v_p \\ \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - v_0 & \text{if } v_0 \in [v_p, \bar{v}^A] \\ 0 & \text{if } v_0 > \bar{v}^A \end{cases}$$

For $v_0 > \max(\bar{v}^*, \bar{v}^A)$, $CS^*(v_0) = CS^A(v_0) = 0$. For $v_0 \le \min(\bar{v}^*, \bar{v}^A)$, since the best value without recommendations $u^*(v_0)$ is identical without and with additional information,

$$CS^*(v_0) \ge (\le)\ CS^A(v_0)$$

if and only if

$$\int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^*(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) \ge (\le) \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}} v_i \rho_i^A(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}). \tag{95}$$

For $v_0 \in (\min(\bar{v}^*, \bar{v}^A), \max(\bar{v}^*, \bar{v}^A)]$, if $\bar{v}^* \le \bar{v}^A$, then $CS^*(v_0) = 0 \le CS^A(v_0)$; if $\bar{v}^* \ge \bar{v}^A$, then $CS^*(v_0) \ge 0 = CS^A(v_0)$. Therefore, $CS^*(v_0) \ge (\le)\ CS^A(v_0)$ if and only if $\bar{v}_0^O \ge (\le)\ \bar{v}_0^A$ which is equivalent to (95). Therefore, for any $v_0 \in \mathbb{R}^1_+$, $CS^O(v_0) \ge (\le)\ CS^A(v_0)$ if and only if (95). By Lemma 13, if additional information increases (decreases) rate of substitution, then (95) holds, and hence, $CS^O(v_0) \ge (\le)\ CS^A(v_0)$.

B.5 Proof for Theorem 5

Define a seller i's profit under recommendations rule $\boldsymbol{r}$ at v_0 by

$$\Pi_i(v_0; \boldsymbol{r}) = \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \frac{1 - G(\theta_i)}{g(\theta_i)} r_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}).$$

and sum of all sellers' expected profits by

$$\Pi(v_0; \boldsymbol{r}) = \sum_{i\in\mathcal{N}} \Pi_i(v_0; \boldsymbol{r}).$$

Lemma 14. *Let $\tilde{\boldsymbol{r}}$ and $\boldsymbol{r}^\dagger$ be recommendations rule that never recommends the outside option* *almost surely.*

1. *Suppose* $\frac{1-G(\theta)}{g(\theta)}$ *increases in* θ. *If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ only if } \theta_j \geq (\leq)\ \theta_i \textit{ for } i,j \in \mathcal{N}\cup\{0\},$$

almost surely, then $\Pi(v_0;\tilde{\boldsymbol{r}}) \leq (\geq)\ \Pi(v_0;\boldsymbol{r}^{\dagger})$.

2. *Suppose* $\frac{1-G(\theta)}{g(\theta)}$ *decreases in* θ. *If*

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \textit{ only if } \theta_j \leq (\geq)\ \theta_i \textit{ for } i,j \in \mathcal{N}\cup\{0\},$$

almost surely, then $\Pi(v_0;\tilde{\boldsymbol{r}}) \geq (\leq)\ \Pi(v_0;\boldsymbol{r}^{\dagger})$.

Proof. Suppose $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. Suppose

$$\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ implies } r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) > 0 \text{ only if } \theta_j \geq\ \theta_i \text{ for } i,j \in \mathcal{N}\cup\{0\},$$

almost surely. Let $(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) \in \boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}$. Then,

$$\begin{aligned}
\sum_{i\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)}\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z}) &\leq \max_{i:\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} \frac{1-G(\theta_i)}{g(\theta_i)} \\
&\leq \min_{j:r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})>0} \frac{1-G(\theta_j)}{g(\theta_j)} \\
&\leq \sum_{j\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_j)}{g(\theta_j)} r_j^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})
\end{aligned}$$

so that

$$\begin{aligned}
\Pi(v_0;\tilde{\boldsymbol{r}}) &= \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)}\tilde{r}_i(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\
&\leq \int_{\boldsymbol{\mathcal{V}}\times\boldsymbol{\Theta}\times\boldsymbol{\mathcal{Z}}} \sum_{i\in\mathcal{N}\cup\{0\}} \frac{1-G(\theta_i)}{g(\theta_i)} r_i^{\dagger}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{z})\boldsymbol{F}(d\boldsymbol{v})\boldsymbol{G}(d\boldsymbol{\theta}\mid\boldsymbol{z})\boldsymbol{H}(d\boldsymbol{z}) - u^*(v_0) \\
&= \Pi(v_0;\boldsymbol{r}^{\dagger}).
\end{aligned}$$

The other inequalities may be shown similarly. $\square$

Let $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ be optimal unconstrained recommendations rules without and with additional information. That is, $\boldsymbol{\rho}^*$ maximizes (84) subject to monotonicity constraints (72) and

$\boldsymbol{\rho}^A$ maximizes (77) subject to monotonicity constraints (72). Note that under small information rent environment, both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ does not recommend the outside option almost surely.

Lemma 15. *Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Suppose the additional information is regular. Then,*

$$\Pi(v_0; \boldsymbol{\rho}^*) \leq (\geq)\ \Pi(v_0; \boldsymbol{\rho}^A). \tag{96}$$

if one of the following conditions is satisfied:

1. *Additional information increases (decreases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ increases (decreases) in θ.*

2. *Additional information decreases (increases) rates of substitution and state-dependent information rent $\frac{1-G(\theta)}{g(\theta)}$ decreases (increases) in θ.*

Proof. Let v_0 at which both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient. Since the information rent is small as in Assumption 3.a, it follows that $\rho_0^* = 0$ and $\rho_0^A = 0$ almost surely. When one of the products is recommended, the rules are almost surely given by

$$\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i)\ \forall j \in \mathcal{N} \setminus \{i\}$$

$$\rho_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 1 \text{ if } \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_j) > w(v_j) - w(v_i)\ \forall j \in \mathcal{N} \setminus \{i\}$$

Suppose $\frac{1-G(\theta)}{g(\theta)}$ increases in θ. To use Lemma 14, it is sufficient to show:

> if additional information increases (decreases) rates of substitution, $\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ for $i \in \mathcal{N}$ implies $\rho_j^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ only if $j \in \mathcal{N}$ and $\theta_j \geq (\leq)\ \theta_i$ almost surely.

To show this, let $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \in \boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta}(\boldsymbol{z}) \times \boldsymbol{\mathcal{Z}}$ be at which $\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ for $i \in \mathcal{N}$. Almost surely, i is strictly preferred to $k \in \mathcal{N} \setminus \{i\}$. Since the additional information is regular, rates of substitution are positive, so that

$$\left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_k) > w(v_k) - w(v_i)$$

Suppose that additional information decreases rates of substitution. Then,

$$1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j} \geq 1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}.$$

If $\theta_k < \theta_i$,

$$\left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_k) \geq \left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j}\right)(\theta_i - \theta_k) > w(v_k) - w(v_i)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k \in \mathcal{N} \setminus \{i\}$, $\rho_k^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 0$ if $v_k > v_i$. In other words, $\rho_j^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ only if $\theta_j \geq \theta_i$. By Lemma 14, it follows that

$$\Pi(v_0; \boldsymbol{\rho}^*) \leq \Pi(v_0; \boldsymbol{\rho}^A).$$

To show the other case given an increasing $\frac{1-G(\theta)}{g(\theta)}$, suppose that additional information decreases rates of substitution. Then,

$$1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)}}{\theta_i - \theta_j} \leq 1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_j)}{g(\theta_j)}}{\theta_i - \theta_j}.$$

If $\theta_k > \theta_i$,

$$\left(1 - \frac{\frac{1-G(\theta_i|z_i)}{g(\theta_i|z_i)} - \frac{1-G(\theta_k|z_k)}{g(\theta_k|z_k)}}{\theta_i - \theta_k}\right)(\theta_i - \theta_k) \geq \left(1 - \frac{\frac{1-G(\theta_i)}{g(\theta_i)} - \frac{1-G(\theta_k)}{g(\theta_k)}}{\theta_i - \theta_k}\right)(\theta_i - \theta_k) > w(v_k) - w(v_i)$$

so that i is strictly preferred to k under $\boldsymbol{\rho}^A$ as well. Therefore, for any $k \in \mathcal{N} \setminus \{i\}$, $\rho_k^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = 0$ if $\theta_k > \theta_i$. In other words, $\rho_j^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) > 0$ only if $\theta_k \leq \theta_i$. By Lemma 12, it follows that

$$\Pi(v_0; \boldsymbol{\rho}^*) \geq \Pi(v_0; \boldsymbol{\rho}^A)$$

The other inequalities related to decreasing $\frac{1-G(\theta)}{g(\theta)}$ may be shown similarly. □

Let $v_0 \leq \mathbb{E}_{v_i}(v_i)$. Under a small information rent environment, both $\boldsymbol{\rho}^*$ and $\boldsymbol{\rho}^A$ are obedient, so that Lemma 15 applies straightforwardly to induce the desired results.

B.6 Proof for Theorem 6

Note that for each $z \in \mathcal{Z}$ and $\theta \in \Theta(z)$,

$$\begin{aligned}\int_{\mathcal{Z}} \left(\theta - \frac{1 - G(\theta \mid z)}{g(\theta \mid z)}\right) G(d\theta \mid z) H(dz) &= \theta g(\theta) - \int_{\mathcal{Z}} (1 - G(\theta \mid z)) H(dz) \\ &= \theta g(\theta) - \left(1 - \int_{\mathcal{Z}} \Pr(\tilde{\theta} \le \theta, z) dz\right) \\ &= \theta g(\theta) - (1 - G(\theta)) \\ &= \left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right) G(d\theta)\end{aligned}$$

and

$$\int_{\mathcal{Z}} v G(d\theta \mid z) H(dz) = v G(d\theta)$$

This means that the intermediary's problem without additional information, which is to maximize

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i)}{g(\theta)} + w(v_i)\right) r_i^\dagger(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta})$$

subject to obedience constraints

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_i r_i^\dagger(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \ge \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_j r_i^\dagger(\boldsymbol{v}, \boldsymbol{\theta}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta}) \text{ for all } i, j \in \mathcal{N} \cup \{0\}$$

is identical to maximizing the intermediary's problem with additional information

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} \sum_{i \in \mathcal{N}} \left(\theta_i - \frac{1 - G(\theta_i \mid z_i)}{g(\theta_i \mid z_i)} + w(v_i)\right) r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z})$$

subject to obedience constraints

$$\int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_i r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \ge \int_{\boldsymbol{\mathcal{V}} \times \boldsymbol{\Theta} \times \mathcal{Z}} v_j r_i(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) \boldsymbol{F}(\boldsymbol{v}) \boldsymbol{G}(d\boldsymbol{\theta} \mid \boldsymbol{z}) \boldsymbol{H}(d\boldsymbol{z}) \text{ for all } i, j \in \mathcal{N} \cup \{$$

and invariance constraints

$$\boldsymbol{r}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \boldsymbol{r}^\dagger(\boldsymbol{v}, \boldsymbol{\theta}) \text{ for some } \boldsymbol{r}^\dagger \text{ for all } \boldsymbol{z} \in \boldsymbol{\mathcal{Z}}.$$

The intermediary's problem with additional information is the same but without the invariance constraints. Since both have the same objective function but there is another set of constraints in the problem without the additional information, by revealed preference, the intermediary's revenue is higher.

B.7 Proof for Theorem 7

Suppose $v_0 < \underline{v}$. Since the consumer always prefers products over the outside option always, any obedient recommendations rule must always recommend one of the products. An optimal recommendations rule with this constraint but without additional information is given by for each $i \in \mathcal{N}$

$$\rho_i^*(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \frac{1}{|\mathcal{M}^*|} \text{ if } i \in \mathcal{M}^* \tag{97}$$

where $\mathcal{M}^* = \arg\max_{j \in \mathcal{N}} \{\theta_j - \frac{1-G(\theta_j)}{g(\theta_j)} + w(v_j)\}$, and that with additional information is given by

$$\rho_i^A(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{z}) = \frac{1}{|\mathcal{M}^A|} \text{ if } i \in \mathcal{M}^A \tag{98}$$

where $\mathcal{M}^A = \arg\max_{j \in \mathcal{N}} \{\theta_j - \frac{1-G(\theta_j|z_j)}{g(\theta_j|z_j)} + w(v_j)\}$. Both recommendations rules are completely determined by rates of substitutions, so that whether additional information increases the consumer surplus or not is completely determined by whether additional information decreases or increases the rates of substitution. Applying similar arguments as in Theorem 4 gives the desired result.

Suppose $v_0 > \bar{v}$. Since the consumer always prefers the outside option over products, any obedient recommendations rule must always recommend the outside option, without and with additional information, under which the consumer surplus is always 0. Therefore, the additional information does not change the consumer surplus.

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