

# On support $\tau$ -tilting graphs of finite-dimensional algebras

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based on joint work with S. Geng, P. Liu and Y. Zhou

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- $\tau$ : Auslander-Reiten translation;
- $|M|$ : number of non-isomorphic indecomposable direct summands of  $M \in \text{mod } A$ .



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- A pair  $(M, P)$  with  $M \in \text{mod } A$  and  $P \in \text{proj } A$  is a  $\tau$ -rigid pair if  $M$  is  $\tau$ -rigid and  $\text{Hom}_A(P, M) = 0$ ;

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## Remark 1.2

For a  $\tau$ -rigid pair  $(M, P)$ ,  $|M| + |P| \leq |A|$ .

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### Remark 1.4

Each  $\tau$ -rigid pair  $(M, P)$  can be completed into a  $\tau$ -tilting pair, i.e., there is a  $\tau$ -rigid pair  $(N, Q)$  such that  $(M \oplus N, P \oplus Q)$  is a  $\tau$ -tilting pair.

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### Exam 1.5

$(A, 0)$  and  $(0, A)$  are basic  $\tau$ -tilting pairs.

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- $(M, P)$  is a *direct summand* of a  $\tau$ -rigid pair  $(N, Q)$  if  $M$  is a direct summand of  $N$  and  $P$  is a direct summand of  $Q$ .

## Theorem 1.7 (Adachi-Iyama-Reiten 2013)

*Every basic almost complete  $\tau$ -tilting pair is a direct summand of exactly **two** basic  $\tau$ -tilting pairs.*

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## Remark 1.8

*Let  $(M, P)$  be a basic almost complete  $\tau$ -tilting pair,  $(N, Q)$  and  $(L, R)$  the two basic  $\tau$ -tilting pairs which contains  $(M, P)$  as a direct summand. We say that  $(N, Q)$  and  $(L, R)$  are mutations of each other.*

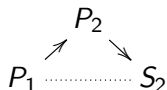
Denote by  $\text{s}\tau\text{-tilt } A$  the set of isomorphism classes of basic  $\tau$ -tilting pairs of  $A$ .

### Definition 1.9

The *support  $\tau$ -tilting graph*  $\mathcal{H}(\text{s}\tau\text{-tilt } A)$  has a vertex set indexed by  $\text{s}\tau\text{-tilt } A$ . For two basic support  $\tau$ -tilting pairs  $(M, P)$  and  $(N, Q)$ , there is an edge between  $(M, P)$  and  $(N, Q)$  iff  $(M, P)$  is a mutation of  $(N, Q)$ .

## Exam 1.10

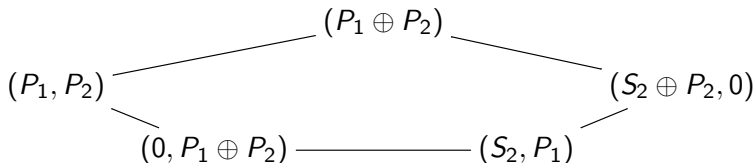
Let  $A = k(1 \rightarrow 2)$ . The AR quiver of  $\text{mod } A$ :



$s_T$ -tilt  $A$ :

$$(P_1 \oplus P_2, 0), (S_2 \oplus P_2, 0), (S_2, P_1), (P_1, P_2), (0, P_1 \oplus P_2).$$

$\mathcal{H}(s_T\text{-tilt } A)$ :



## Remark 1.11

*There is an abstract simplicial complex  $\Delta(A)$  associated to  $A$  via  $\tau$ -tilting theory. Namely, for  $0 \leq d \leq |A| - 1$ , the  $d$ -simplex  $\Delta^d$  consists of sets  $\{(M_1, P_1), \dots, (M_{d+1}, P_{d+1})\}$  satisfying that*

- $|M_i| + |P_i| = 1$ ;
- $(\bigoplus_{i=1}^{d+1} M_i, \bigoplus_{i=1}^{d+1} P_i)$  is a basic  $\tau$ -rigid pair.



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The support  $\tau$ -tilting graph  $\mathcal{H}(\text{s}\tau\text{-tilt } A)$  is the dual graph of  $\Delta(A)$ .

## Remark 1.12

*One can also endow  $\mathcal{H}(s\tau\text{-tilt } A)$  with an orientation. In particular, there is a poset structure on  $s\tau\text{-tilt } A$ . Moreover,  $(A, 0)$  is the uniquely maximal element and  $(0, A)$  is the uniquely minimal element.*

# Q1: Connected components

## Question 2.1

*To determine the number  $c(A)$  of connected components of  $\mathcal{H}(s\tau\text{-tilt } A)$ .  
In particular, when  $c(A) = 1$ ?*

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- gentle algebras ([Fu-Geng-Liu-Zhou 21]);
- complete gentle algebras/complete special biserial algebras ([Asai 22]);
- skew-gentle algebras. More generally, endomorphism algebras of partial cluster-tilting objects in 2-Calabi-Yau categories associated to marked surfaces ([He-Zhou-Zhu]).

## Q2: Reachable-in-face property

### Definition 2.3

Let  $(M, P)$  be a basic  $\tau$ -rigid pair. The *face*  $\mathcal{F}_{(M,P)}$  determined by  $(M, P)$  is the full subgraph of  $\mathcal{H}(\text{s}\tau\text{-tilt } A)$  consisting of basic  $\tau$ -tilting pairs which admits  $(M, P)$  as a direct summand.

## Definition 2.4

The support  $\tau$ -tilting graph  $\mathcal{H}(\text{s}\tau\text{-tilt } A)$  or the algebra  $A$  has the *reachable-in-face* property, if for any  $T, T' \in \text{s}\tau\text{-tilt } A$  such that there is a path

$$T \text{ --- } \bullet \text{ --- } \dots \text{ --- } \bullet \text{ --- } T'$$

in  $\mathcal{H}(\text{s}\tau\text{-tilt } A)$ , then for any common direct summand  $(L, Q)$  of  $T$  and  $T'$ , there is a path

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in the face  $\mathcal{F}_{(L,Q)}$ .

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# Reduction

Let  $\mathcal{C}$  be a 2-Calabi-Yau triangulated category with a cluster-tilting object  $T$ .

For a rigid object  $M$ , denote by

$${}^{\perp}M[1] := \{X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^1(M, X) = 0\}.$$

Define the additive quotient category

$$\mathcal{C}_M := {}^{\perp}M[1] / \text{add } M,$$

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**Theorem 3.1 (Iyama-Yoshino 08)**

*$\mathcal{C}_M$  is a 2-Calabi-Yau category with cluster-tilting objects.*

## Definition 3.2

The category  $\mathcal{C}$  has the  $\tau$ -reachable property, if for any indecomposable rigid objects  $M, N \in \mathcal{C}$ , there is a sequence of indecomposable rigid objects

$$M = X_0, X_1, \dots, X_t = N$$

such that, for  $0 \leq i < t$ ,  $X_i \oplus X_{i+1}$  are rigid.

The category  $\mathcal{C}$  has the totally  $\tau$ -reachable property, if for any rigid object  $M$  with  $|M| \leq |T| - 2$ ,  $\mathcal{C}_M$  has  $\tau$ -reachable property.



Denote by  $A = \text{End}_{\mathcal{C}}(T)$ .

### Theorem 3.3 (Fu-Geng-Liu-Zhou 21)

*The category  $\mathcal{C}$  has the totally  $\tau$ -reachable property iff  $c(A) = 1$ .*

Thank You!