On support τ-tilting graphs of finite-dimensional algebras

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based on joint work with S. Geng, P. Liu and Y. Zhou

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Contents

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- τ: Auslander-Reiten translation;
- $|M|$: number of non-isomorphic indecomposable direct summands of $M \in \operatorname{mod} A$.

Definition 1.1

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Remark 1.2

For a τ*-rigid pair* (M, P), $|M| + |P| \leq |A|$.

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A τ-rigid pair (M, P) is a τ-tilting pair if $|M| + |P| = |A|$.

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Remark 1.4

Each τ-rigid pair (M, P) can be completed into a τ-tilting pair, i.e., there is a τ-rigid pair (N, Q) such that $(M \oplus N, P \oplus Q)$ is a τ-tilting pair.

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Exam 1.5

$(A, 0)$ and $(0, A)$ are basic τ-tilting pairs.

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- (M, P) *is an* *almost complete τ-tilting pair* *if* $|M| + |P| = |A| - 1$;
- (M, P) *is a* *direct summand* *of a τ-rigid pair* (N, Q) *if* M *is a direct summand of* N *and* P *is a direct summand of* Q.

Theorem 1.7 (Adachi-Iyama-Reiten 2013)

Every basic almost complete τ-tilting pair is a direct summand of exactly two basic τ-tilting pairs.

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Remark 1.8

Let (M, P) be a basic almost complete τ-tilting pair, (N, Q) and (L, R) the two basic τ-tilting pairs which contains (M, P) as a direct summand. We say that (N, Q) and (L, R) are muations of each other.

Denote by sτ-tilt A the set of isomorphism classes of basic τ-tilting pairs of A.

Definition 1.9

The support τ-tilting graph $\mathcal{H}$(sτ-tilt A) *has a vertex set indexed by* sτ-tilt A*. For two basic support τ-tilting pairs (M, P) and (N, Q), there is an edge between (M, P) and (N, Q) iff (M, P) is a mutation of (N, Q).*

Exam 1.10

Let $A = k(1 \to 2)$. The AR quiver of mod A:

$$
\begin{tikzcd}[column sep=small]
 & P_2 \arrow[dr] & \\
P_1 \arrow[ur] \arrow[rr, dotted, no head] & & S_2
\end{tikzcd}
$$

sτ-tilt A:

$$(P_1 \oplus P_2, 0), (S_2 \oplus P_2, 0), (S_2, P_1), (P_1, P_2), (0, P_1 \oplus P_2).$$

$\mathcal{H}$(sτ-tilt A):

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\begin{tikzcd}[column sep=small]
 & & (P_1 \oplus P_2) \arrow[dll, no head] \arrow[drr, no head] & & \\
(P_1, P_2) \arrow[dr, no head] & & & & (S_2 \oplus P_2, 0) \arrow[dl, no head] \\
 & (0, P_1 \oplus P_2) \arrow[rr, no head] & & (S_2, P_1) &
\end{tikzcd}
$$

Remark 1.11

There is an abstract simplicial complex $\Delta(A)$ associated to A via τ-tilting theory. Namely, for $0 \leq d \leq |A| - 1$, the d-simplex Δ^d consists of sets $\{(M_1, P_1), \ldots, (M_{d+1}, P_{d+1})\}$ satisfying that

- $|M_i| + |P_i| = 1$;
- $(\bigoplus_{i=1}^{d+1} M_i, \bigoplus_{i=1}^{d+1} P_i)$ *is a basic τ-rigid pair.*

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The support τ-tilting graph $\mathcal{H}(\mathrm{s}\tau\text{-tilt } \mathrm{A})$ is the dual graph of $\Delta(A)$.

Remark 1.12

One can also endow $\mathcal{H}(\mathrm{s}\tau\text{-tilt } \mathrm{A})$ *with an orientation. In particular, there is a poset structure on* $\mathrm{s}\tau\text{-tilt } \mathrm{A}$*. Moreover,* $(A, 0)$ *is the uniquely maximal element and* $(0, A)$ *is the uniquely minimal element.*

Q1:Connected components

Question 2.1

To determine the number $c(A)$ of connected components of $\mathcal{H}(\mathsf{s}\tau\text{-tilt } A)$. In particular, when $c(A) = 1$?

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- 2-Calabi-Yau tilted algebras(=cluster-tilted algebras) arising from hereditary categories([BMRRT06, Hubery08, Fu-Geng 18]);

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- gentle algebras ([Fu-Geng-Liu-Zhou 21]);
- complete gentle algebras/complete special biserial algebras ([Asai 22]);
- skew-gentle algebras. More generally, endomorphism algebras of partial cluster-tilting objects in 2-Calabi-Yau categories associated to marked surfaces ([He-Zhou-Zhu]).

Q2:Reachable-in-face property

Definition 2.3

Let (M, P) *be a basic* τ*-rigid pair. The face* $\mathcal{F}_{(M,P)}$ *determined by* (M, P) *is the full subgraph of* $\mathcal{H}(\mathrm{s}\tau\text{-tilt } A)$ *consisting of basic* τ*-tilting pairs which admits* (M, P) *as a direct summand.*

Definition 2.4

The support τ*-tilting graph* $\mathcal{H}(\mathrm{s}\tau\text{-tilt } A)$ *or the algebra* A *has the* *reachable-in-face* *property, if for any* $T, T' \in \mathrm{s}\tau\text{-tilt } A$ *such that there is a path*

$$T \text{ —— } \bullet \text{ —— } \cdots \text{ —— } \bullet \text{ —— } T'$$

in $\mathcal{H}(\mathrm{s}\tau\text{-tilt } A)$*, then for any common direct summand* (L, Q) *of* T *and* T'*, there is a path*

$$T \text{ —— } \bullet \text{ —— } \cdots \text{ —— } \bullet \text{ —— } T'$$

in the face $\mathcal{F}_{(L,Q)}$*.*

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- gentle algebras ([Fu-Geng-Liu-Zhou 21]).

Reduction

Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category with a cluster-tilting object T.

For a rigid object M, denote by

$$^{\perp}M[1] := \{X \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(M, X) = 0\}.$$

Define the additve quotient category

$$\mathcal{C}_M := {}^{\perp}M[1] / \operatorname{add} M,$$

which is called the Iyama-Yoshino reduction of $\mathcal{C}$ with respect to M.

Reduction

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which is called the Iyama-Yoshino reduction of $\mathcal{C}$ with respect to M.

Theorem 3.1 (Iyama-Yoshino 08)

$\mathcal{C}_M$ is a 2-Calabi-Yau category with cluster-tilting objects.

Definition 3.2

The category $\mathcal{C}$ has the τ*-reachable property, if for any indecomposable rigid objects* $M, N \in \mathcal{C}$*, there is a sequence of indecomposable rigid objects*

$$M = X_0, X_1, \ldots, X_t = N$$

such that, for $0 \leq i < t$*,* $X_i \oplus X_{i+1}$ *are rigid.*
The category $\mathcal{C}$ has the totally τ*-reachable property, if for any rigid object* M *with* $|M| \leq |T| - 2$*,* $\mathcal{C}_M$ *has* τ*-reachable property.*

Denote by $A = \mathrm{End}_{\mathcal{C}}(T)$.

Theorem 3.3 (Fu-Geng-Liu-Zhou 21)

The category $\mathcal{C}$ has the totally τ-reachable property iff $c(A) = 1$.

Thank You!