ON τ-TILTING GRAPHS OF τ-TILTING FINITE ALGEBRAS

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Dedicated to our teacher Liangang Peng on the occasion of his 65th Birthday

Abstract. This note shows that the exchange graphs of support τ-tilting modules over τ-tilting finite algebras have the non-leaving-face property, namely, any geodesic connecting two vertices stays in the minimal face containing both.

1. Introduction

For a reasonably complete discussion of the history of abstract exchange graphs, see the introduction in [2]. Here, we just remark that the exchange graph of a cluster algebra encodes the combinatorics of cluster mutations and the exchange graphs stemming from representation theory such as exchange graphs of support τ-tilting modules over finite dimensional algebras play an important role in categorification of cluster algebras. In [11], along with Y. Zhou, we introduced the reachable-in-face property for abstract exchange graphs and showed that the exchange graph of support τ-tilting modules over a finite dimensional gentle algebra is connected and has the reachable-in-face property. The reachable-in-face property for exchange graphs of cluster algebras was conjectured by Fomin and Zelevinsky in 2003 and has been proved recently by Cao and Li [5, Theorem 10]. Another interesting phenomenon called the non-leaving-face property first appeared in [14]. It has transpired since that it naturally appears in many different contexts; these include generalized associahedra of finite types [7, 15] and exchange graphs arising from marked surfaces [3]. It is also known that there are examples related to the associahedron which do not satisfy the non-leaving-face property (cf. [7]). The non-leaving-face property implies the reachable-in-face property, but the converse is not true. In [10], we proved that for any cluster algebra, the exchange graph has the non-leaving-face property.

This note continues our study of (abstract) exchange graphs in [11, 10]. Here we study τ-tilting finite algebras, that is, finite dimensional algebras with finitely many isomorphism classes of indecomposable τ-rigid modules (the terminology will be recalled later). The main contribution of this note is Theorem 3.8, in which we prove that for τ-tilting finite algebras, the exchange graphs of support τ-tilting modules have the

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non-leaving-face property. As a consequence, we obtain a uniform proof for that W-permutahedron and W-associahedron have the non-leaving-face property, where W is a finite Weyl group.

2. Background on exchange graphs and τ-tilting theory

2.1. **Background on exchange graphs and non-leaving-face property.** As in [2], let V be a non-empty set with a reflexive and symmetric relation R. Two elements x and y of V are *compatible* if $(x, y) \in R$. Assume (V, R) satisfies the following conditions:

- All maximal subsets of pairwise compatible elements are finite and have the same cardinality, say n. We refer the subsets *clusters*;
- Any subset of $n-1$ pairwise compatible elements is contained in precisely two clusters.

The *exchange graph* $\mathcal{H}$ of (V, R) is defined to be the graph whose vertices are the clusters and where two vertices are joined by an edge if and only if their intersection has cardinality $n-1$. In fact, the exchange graph $\mathcal{H}$ has the structure of an abstract polytope. A *face* $\mathcal{F}$ of $\mathcal{H}$ is a full subgraph of $\mathcal{H}$ such that

- there is a subset U of pairwise compatible elements;
- the vertices of $\mathcal{F}$ are precisely the clusters containing U as a subset.

Clearly, $\mathcal{F}$ is uniquely determined by U and we denote it by $\mathcal{F}_U$. The inclusion of sets induces a partial order on faces. Namely, we say $\mathcal{F}_U \leq \mathcal{F}_V$ if $V \subseteq U$.

For two cluster v_1 and v_2, we write for $v_1 \text{ --- } v_2$ to indicate that they are linked by an edge. A path

$$v = v_1 \text{ --- } v_2 \text{ --- } \cdots \text{ --- } v_m = w$$

from v to w is a *geodesic* connecting vertices v and w, if the length of the path is minimal. We remark that the graph $\mathcal{H}$ may not be connected.

We now recall non-leaving-face property from [7].

Definition 2.1. If any geodesic connecting two vertices in $\mathcal{H}$ lies in the minimal face containing both, then we say that $\mathcal{H}$ has *non-leaving-face* property.

The following definition is useful to investigate the non-leaving-face property (cf. [14, 7, 15, 3]).

Definition 2.2. Let $\mathcal{H}$ be an exchange graph and $\mathcal{F}$ a face of $\mathcal{H}$. We say a map $P : \mathcal{H} \to \mathcal{F}$ is a *projection* if the following conditions are satisfied:

(P1) $P(v)$ is a vertex in $\mathcal{F}$ for each vertex $v \in \mathcal{H}$;
(P2) $P(v) = v$ whenever $v \in \mathcal{F}$;
(P3) P sends edges in $\mathcal{H}$ to edges or vertices in $\mathcal{F}$. Namely, if $v \text{ --- } w$ is an edge in $\mathcal{H}$, then either $P(v) = P(w)$ or $P(v) \text{ --- } P(w)$ is an edge of $\mathcal{F}$;
(P4) if $v \text{ --- } w$ is an edge in $\mathcal{H}$ such that $v \in \mathcal{F}$ but $w \notin \mathcal{F}$, then $P(v) = P(w)$.

The following result is obvious.

Lemma 2.3. *Let $\mathcal{H}$ be an exchange graph. If there exists a projection for each face $\mathcal{F}$, then $\mathcal{H}$ has non-leaving-face property.*

2.2. **Background on τ-tilting theory.** We recall the basic definitions concerning τ-tilting theory. Almost all of this material can be found in [1]. Let k be a field. Let A be a finite-dimensional k-algebra and $\mathbf{mod}\,A$ the category of finitely generated right A-modules. For a module M in $\mathbf{mod}\,A$, the symbol $|M|$ denotes the number of pairwise non-isomorphic indecomposable direct summands of M. We say that M is *basic* if the number of indecomposable direct summands of M equals $|M|$. For our purposes it is sufficient to consider basic modules unless otherwise stated.

Let τ be the Auslander-Reiten translation of $\mathbf{mod}\,A$. We say that M is *τ-rigid* if $\mathsf{Hom}_A(M, \tau M) = 0$. Let P be a projective A-module. We say that (M, P) is a *τ-rigid pair* if M is τ-rigid and $\mathsf{Hom}_A(P, M) = 0$. We call (M, P) a *support τ-tilting pair* (or simply, τ-tilting pair) if (M, P) is a τ-rigid pair and $|M|+|P| = |A|$. It is known that in this case P is uniquely determined by M and M is called a *support τ-tilting* A-module.

We say that a τ-rigid pair (M, P) is almost complete τ-tilting if $|M| + |P| = |A| - 1$. In this case, for an indecomposable module X, we say that $(X, 0)$ (respectively, $(0, X)$) is a complement of (U, P) if $(U \oplus X, P)$ (respectively, $(U, P \oplus X)$) is a τ-tilting pair, which we call a *completion* of (U, P). It is well-known now that any almost complete τ-tilting pair has precisely two complements [1]. Suppose that (X, R) and (X', R') are just the two complements of an almost complete τ-tilting pair (U, P), denote by $(M, Q) = (U, P) \oplus (X, R)$ and $(M', Q') = (U, P) \oplus (X', R')$, (M, Q) and (M', Q') are called mutations of each other, written as $\mu_{(X,R)}(M, Q) = (M', Q')$. Since Q (resp. Q') is uniquely determined by M (resp. M'), we also call M and M' are mutations of each other, simply written as $\mu_{(X,R)}(M) = M'$.

We now introduce the following relation R_A on the set of indecomposable τ-rigid pairs:

- two τ-rigid pairs (X_1, Q_1) and (X_2, Q_2) have the relation R_A if they are both direct summands of the same support τ-tilting pair (M, P).

According to [1], as in Section 2.1, we can associate with the relation R_A an exchange graph $\mathcal{H}_A$ whose vertices are basic support τ-tilting pairs and there is an edge between two non-isomorphic support τ-tilting pairs if and only if they are mutations of each other. Since each basic support τ-tilting pair is uniquely determined by the part of its support τ-tilting module, so the vertices of $\mathcal{H}_A$ can be think as the support τ-tilting A-modules and there is an edge between two non-isomorphic support τ-tilting modules if and only if they are mutations of each other.

3. Non-leaving-face property for τ-tilting finite algebras

3.1. τ-perpendicular categories and torsion classes. We retain all the notation of the preceding section. Recall that for any M in $\mathbf{mod}\, A$, we have a torsion class (that is, a subcategory closed under extensions, factor modules and isomorphisms)

$$^{\perp}M = \{X \in \mathbf{mod}A |\, \mathsf{Hom}(X, M) = 0\},$$

and a torsionfree class (that is, a subcategory closed under extensions, submodules and isomorphisms)

$$M^{\perp} = \{X \in \mathbf{mod}A |\, \mathsf{Hom}(M, X) = 0\}.$$

Let $\mathbf{Fac}\, M$(respectively, $\mathbf{Sub}\, M$) denote the subcategory of $\mathbf{mod}\, A$ consisting of factor modules (respectively, submodules) of direct sums of copies of M. Recall that, for torsion classes $\mathcal{T}$, being *functorially finite* is equivalent to the existence of M in $\mathbf{mod}\, A$ such that $\mathcal{T} = \mathbf{Fac}\, M$. We say that an A-module $M \in \mathcal{T}$ is Ext-projective in $\mathcal{T}$ if for all $N \in \mathcal{T}$ we have $\mathsf{Ext}^1_A(M, N) = 0$. We denote the full subcategory of $\mathbf{mod}\, A$ consisting of all modules which are direct sums of direct summands of M by $\mathsf{add}\, M$. It is well-known that for a functorially finite torsion class $\mathcal{T}$, there is a basic A-module $P(\mathcal{T}) \in \mathcal{T}$ such that $\mathbf{Fac}\, P(\mathcal{T}) = \mathcal{T}$ and $\mathsf{add}\, P(\mathcal{T})$ coincides with the class of Ext-projective A-modules in $\mathcal{T}$. We refer to $P(\mathcal{T})$ as the Ext-progenerator of $\mathcal{T}$. By [1], $P(\mathcal{T})$ is a support τ-tilting A-module. Now one can introduce a partial order in the set of support τ-tilting modules. Namely, if M and N are support τ-tilting A-modules, then

- $M \leq N$ if and only if $\mathbf{Fac}\, M \subseteq \mathbf{Fac}\, N$.

In particular, if M is a mutation of N, by [1], we either have $M \leq N$ or $N \leq M$.

Let (U, Q) be a basic τ-rigid pair of A-modules. We associate to (U, Q) a subcategory $\mathcal{W}_{(U,Q)}$ of $\mathbf{mod}\, A$ given by

$$\begin{aligned}\mathcal{W}_{(U,Q)} &= U^{\perp} \cap^{\perp} \tau U \cap Q^{\perp}\\ &= \{X \in \mathbf{mod}A |\, \mathsf{Hom}(U, X) = 0, \mathsf{Hom}(X, \tau U) = 0, \mathsf{Hom}(Q, X) = 0\}.\end{aligned}$$

We refer to $\mathcal{W}_{(U,Q)}$ as the *τ-perpendicular category* of (U, Q). It is easy to get that $\mathcal{W}_{(U,Q)}$ has the following basic property (cf. [13]).

Lemma 3.1. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathbf{mod}A$. If any two of X, Y and Z belong to $\mathcal{W}_{(U,Q)}$, then so does the third one.*

Note that $\mathcal{W}_{(U,Q)}$ is closed under extensions in $\mathbf{mod}\, A$, Lemma 3.1 implies that $\mathcal{W}_{(U,Q)}$ admits a natural structure of exact category such that admissible epimorphisms (respectively, admissible monomorphisms) in $\mathcal{W}_{(U,Q)}$ are exactly epimorphisms (respectively, monomorphisms) in $\mathbf{mod}\, A$ between modules in $\mathcal{W}_{(U,Q)}$. As in [13, Definition 3.9], a full subcategory $\mathcal{G}$ is called a torsion class in $\mathcal{W}_{(U,Q)}$ if for any admissible exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{W}_{(U,Q)}$, it satisfies the following conditions:

- $X, Z \in \mathcal{G}$ implies that $Y \in \mathcal{G}$.

- $Y \in \mathcal{G}$ implies that $Z \in \mathcal{G}$.

The following lemma is easy.

Lemma 3.2. *If $\mathcal{T}$ is a torsion class in $\mathbf{mod}\, A$, then $\mathcal{T} \cap \mathcal{W}_{(U,Q)}$ is a torsion class in $\mathcal{W}_{(U,Q)}$.*

We denote by $\mathbf{tors}\, A$ the set of torsion classes in $\mathbf{mod}\, A$ and by $\mathbf{tors}\, \mathcal{W}_{(U,Q)}$ the set of torsion classes in $\mathcal{W}_{(U,Q)}$. If $\mathcal{X}$ and $\mathcal{Y}$ are two subcategories of $\mathbf{mod}\, A$, then we denote by $\mathcal{X} * \mathcal{Y}$ the full subcategory of $\mathbf{mod} A$ given by all A-modules M such that there exist $X \in \mathcal{X}, Y \in \mathcal{Y}$ and a short exact sequence

$$0 \to X \to M \to Y \to 0.$$

Lemma 3.3. *If $\mathcal{G}$ is a torsion class in $\mathcal{W}_{(U,Q)}$, then $(\mathbf{Fac}\, U) * \mathcal{G}$ is a torsion class in $\mathbf{mod}\, A$ such that $\mathbf{Fac}\, U \subseteq (\mathbf{Fac}\, U) * \mathcal{G} \subseteq {}^{\perp}\tau U \cap Q^{\perp}$. Moreover, there is an order-preserving bijection*

$$\mathbf{red} : \{\mathcal{T} \in \mathbf{tors}\, A \mid \mathbf{Fac}\, U \subseteq \mathcal{T} \subseteq {}^{\perp}\tau U \cap Q^{\perp}\} \to \mathbf{tors}\, \mathcal{W}_{(U,Q)},$$

where $\mathbf{red}$ *is given by* $\mathbf{red}(\mathcal{T}) := \mathcal{T} \cap U^{\perp}$ *with inverse* $\mathbf{red}^{-1}(\mathcal{G}) := (\mathbf{Fac}\, U) * \mathcal{G}$.

Proof. Note that Q is a projective module, the same argument given for [13, Proposition 3.26, Theorem 3.12] also applies here. ☐

Remark 3.4. Let (U, Q) be a basic τ-rigid pair of A-modules and $\mathcal{T}$ a functorially finite torsion class in $\mathbf{mod}\, A$. Note that $P(\mathcal{T})$ is a support τ-tilting A-module. We refer to $(P(\mathcal{T}), Q')$ as the associated basic τ-tilting pair of $\mathcal{T}$. Then $\mathbf{Fac}\, U \subseteq \mathcal{T} \subseteq {}^{\perp}\tau U \cap Q^{\perp}$ if and only if (U, Q) is a direct summand of $(P(\mathcal{T}), Q')$.

3.2. **Main results.** In this section we consider τ-tilting finite algebras. First recall that a finite dimensional algebra A is τ-tilting finite if there are only finitely many isomorphism classes of basic τ-tilting A-modules ([8, Definition 1.1]). The most important thing for us is that according to [8, Theorem 3.8], any torsion class in $\mathbf{mod}\, A$ is functorially finite whence A is τ-tilting finite.

Let (U, Q) be a basic τ-rigid pair, let $\mathcal{F} = \mathcal{F}_{(U,Q)}$ be a face determined (U, Q), that is, a subgraph of $\mathcal{H}_A$ consisting of support τ-tilting A-pairs that contain (U, Q) as a direct summand. Define a map $P_{(U,Q)} : \mathcal{H}_A \to \mathcal{F}_{(U,Q)}$ given by

$$M \mapsto P((\mathbf{Fac}\, U) * ((\mathbf{Fac}\, M) \cap \mathcal{W}_{(U,Q)})) \tag{3.1}$$

where $P(\mathcal{T})$ is the Ext-progenerator of a torsion class $\mathcal{T}$.

Lemma 3.5. *If $M \in \mathcal{F}_{(U,Q)}$, then $P_{(U,Q)}(M) = M$.*

Proof. The condition $M \in \mathcal{F}_{(U,Q)}$ implies that M is a support τ-tilting A-module which contains U as a direct summand and $\mathsf{Hom}(Q, M) = 0$. An easy check shows that

$$\mathbf{Fac}\, U \subseteq \mathbf{Fac}\, M \subseteq {}^{\perp}\tau U \cap Q^{\perp}. \tag{3.2}$$

Then by Lemma 3.3,

$$\begin{aligned}P_{(U,Q)}(M) &= P((\mathbf{Fac}\,U) * (\mathbf{Fac}\,M \cap \mathcal{W}_{(U,Q)}))\\ &= P((\mathbf{Fac}\,U) * (\mathbf{Fac}\,M \cap U^{\perp}))\\ &= P((\mathbf{Fac}\,U) * (\mathbf{red}(\mathbf{Fac}\,M)))\\ &= P(\mathbf{red}^{-1}(\mathbf{red}(\mathbf{Fac}\,M)))\\ &= P(\mathbf{Fac}\,M) = M.\end{aligned}$$

□

Lemma 3.6. *If M — M' is an edge in $\mathcal{H}_A$, then either $P_{(U,Q)}(M) = P_{(U,Q)}(M')$ or $P_{(U,Q)}(M)$ — $P_{(U,Q)}(M')$ is an edge in $\mathcal{F}_{(U,Q)}$.*

Proof. We recall from [8, Example 3.5] that if M — M' is an edge in $\mathcal{H}_A$, then there are no torsion classes in $\mathbf{mod}\,A$ between $\mathbf{Fac}\,M'$ and $\mathbf{Fac}\,M$. Without loss of generality, we assume that $M' < M$. Recall that $\mathcal{W}_{(U,Q)} = U^{\perp} \cap^{\perp} \tau U \cap Q^{\perp}$ is the τ-perpendicular category of (U,Q). An easy check shows that there is no other torsion class $\mathcal{N} \in \mathbf{tors}\,\mathcal{W}_{(U,Q)}$ such that

$$(\mathbf{Fac}\,M') \cap \mathcal{W}_{(U,Q)} \subsetneq \mathcal{N} \subsetneq (\mathbf{Fac}\,M) \cap \mathcal{W}_{(U,Q)}. \tag{3.3}$$

As in [8, Theorem 3.1], if neither $P_{(U,Q)}(M) = P_{(U,Q)}(M')$ nor $P_{(U,Q)}(M)$ — $P_{(U,Q)}(M')$ is an edge in $\mathcal{F}_{(U,Q)}$, there exists some torsion class $\mathcal{T}$ such that

$$\mathbf{Fac}\,P_{(U,Q)}(M') \subsetneq \mathcal{T} \subsetneq \mathbf{Fac}\,P_{(U,Q)}(M).$$

So $\mathbf{Fac}\,U \subseteq \mathcal{T} \subseteq \tau U^{\perp} \cap Q^{\perp}$ and $\mathsf{red}\,(\mathcal{T}) = \mathcal{T} \cap U^{\perp} = \mathcal{T} \cap \mathcal{W}_{(U,Q)}$. By Lemma 3.3 the bijection $\mathbf{red}$ is order-preserving, we have

$$\mathbf{red}(\mathbf{Fac}\,P_{(U,Q)}(M')) \subsetneq \mathbf{red}(\mathcal{T}) \subsetneq \mathbf{red}(\mathbf{Fac}\,P_{(U,Q)}(M)). \tag{3.4}$$

Note that

$$\begin{aligned}\mathbf{red}(\mathbf{Fac}\,P_{(U,Q)}(M')) &= \mathbf{red}((\mathbf{Fac}\,U) * ((\mathbf{Fac}\,M') \cap \mathcal{W}_{(U,Q)}))\\ &= \mathbf{red}(\mathbf{red}^{-1}((\mathbf{Fac}\,M') \cap \mathcal{W}_{(U,Q)}))\\ &= (\mathbf{Fac}\,M') \cap \mathcal{W}_{(U,Q)},\end{aligned}$$

(3.4) implies that

$$(\mathbf{Fac}\,M') \cap \mathcal{W}_{(U,Q)} \subsetneq \mathcal{T} \cap \mathcal{W}_{(U,Q)} \subsetneq (\mathbf{Fac}\,M) \cap \mathcal{W}_{(U,Q)}, \tag{3.5}$$

which contradicts (3.3). □

Lemma 3.7. *If $M \in \mathcal{F}_{(U,Q)}$ and (X,R) is an indecomposable direct summand of (U,Q), then $P_{(U,Q)}(\mu_{(X,R)}(M)) = M$.*

Proof. We work with an edge M — $\mu_{(X,R)}(M)$ in $\mathcal{H}_A$. By Lemma 3.5, we just need to prove that $P_{(U,Q)}(\mu_{(X,R)}(M)) = P_{(U,Q)}(M)$.

We have either $\mu_{(X,R)}(M) < M$ or $M < \mu_{(X,R)}(M)$. Let's consider the former case first. Write $M = \bar{M} \oplus X$ and $\mu_{(X,R)}(M) = \bar{M} \oplus X'$. Clearly we have

$$(\mathbf{Fac}\, \bar{M}) \cap \mathcal{W}_{(U,Q)} \subseteq (\mathbf{Fac}\, M) \cap \mathcal{W}_{(U,Q)}.$$

Let Z be an object in $(\mathbf{Fac}\, M) \cap \mathcal{W}_{(U,Q)} = (\mathbf{Fac}\, M) \cap {}^{\perp}\tau U \cap U^{\perp} \cap Q^{\perp}$, then $Z \in \mathbf{Fac}\, M$ implies that there is a surjective map $(\bar{M} \oplus X)^r = M^r \twoheadrightarrow Z$. Note that X is a direct summand of U and $Z \in U^{\perp}$, we have $\mathsf{Hom}(X, Z) = 0$ and then a surjective map $\bar{M}^r \twoheadrightarrow Z$. That is, $Z \in (\mathbf{Fac}\, \bar{M}) \cap {}^{\perp}\tau U \cap U^{\perp} \cap Q^{\perp} = (\mathbf{Fac}\, \bar{M}) \cap \mathcal{W}_{(U,Q)}$. It follows that

$$(\mathbf{Fac}\, M) \cap \mathcal{W}_{(U,Q)} \subseteq (\mathbf{Fac}\, \bar{M}) \cap \mathcal{W}_{(U,Q)},$$

and so

$$(\mathbf{Fac}\, M) \cap \mathcal{W}_{(U,Q)} = (\mathbf{Fac}\, \bar{M}) \cap \mathcal{W}_{(U,Q)}. \tag{3.6}$$

Note that $\mu_{(X,R)}(M) = \bar{M} \oplus X' < M$, we have

$$(\mathbf{Fac}\, \bar{M}) \cap \mathcal{W}_{(U,Q)} \subseteq \mathbf{Fac}(\mu_{(X,R)}(M)) \cap \mathcal{W}_{(U,Q)} \subseteq (\mathbf{Fac}\, M) \cap \mathcal{W}_{(U,Q)}.$$

Then (3.6) implies that

$$(\mathbf{Fac}\, M) \cap \mathcal{W}_{(U,Q)} = \mathbf{Fac}(\mu_{(X,R)}(M)) \cap \mathcal{W}_{(U,Q)},$$

and so

$$P_{(U,Q)}(\mu_{(X,R)}(M)) = P_{(U,Q)}(M).$$

Now suppose that $M < \mu_{(X,R)}(M)$. Then

$$\mathbf{Fac}\, M \subseteq \mathbf{Fac}(\mu_{(X,R)}(M)).$$

Since $\mathbf{Fac}\, M \subseteq {}^{\perp}\tau U \cap Q^{\perp}$, we have

$$\mathbf{Fac}\, M \subseteq \mathbf{Fac}(\mu_{(X,R)}(M)) \cap {}^{\perp}\tau U \cap Q^{\perp} \subseteq \mathbf{Fac}(\mu_{(X,R)}(M)). \tag{3.7}$$

On the other hand, $\mu_{(X,R)}(M)$ is a mutation of M, there is no other torsion class between $\mathbf{Fac}\, M$ and $\mathbf{Fac}(\mu_{(X,R)}(M))$. Since the intersecion of two torsion classes is a torsion class, it follows from (3.7) that either

$$\mathbf{Fac}\, M = \mathbf{Fac}(\mu_{(X,R)}(M)) \cap {}^{\perp}\tau U \cap Q^{\perp}$$

or

$$\mathbf{Fac}(\mu_{(X,R)}(M)) \cap {}^{\perp}\tau U \cap Q^{\perp} = \mathbf{Fac}(\mu_{(X,R)}(M)).$$

But (U, Q) is not a direct summand of the basic τ-tilting pair associated with $\mathbf{Fac}\, \mu_{(X,R)}(M)$, then $\mathbf{Fac}(\mu_{(X,R)}(M)) \cap {}^{\perp}\tau U \cap Q^{\perp} \neq \mathbf{Fac}(\mu_{(X,R)}(M))$ (cf. Remark 3.4) and so

$$\mathbf{Fac}\, M = \mathbf{Fac}(\mu_{(X,R)}(M)) \cap {}^{\perp}\tau U \cap Q^{\perp}.$$

We have

$$\begin{aligned}(\mathbf{Fac}\,U) * (\mathbf{Fac}M \cap \mathcal{W}_{(U,Q)}) &= (\mathbf{Fac}\,U) * (\mathbf{Fac}(\mu_{(X,R)}(M)) \cap {}^{\perp}\tau U \cap Q^{\perp} \cap \mathcal{W}_{(U,Q)}) \\ &= (\mathbf{Fac}\,U) * (\mathbf{Fac}(\mu_{(X,R)}(M)) \cap \mathcal{W}_{(U,Q)}).\end{aligned}$$

That is, $P_{(U,Q)}(\mu_{(X,R)}(M)) = P_{(U,Q)}(M)$. $\square$

We are now ready to state the main result of this note.

Theorem 3.8. *Let A be a τ-tilting finite algebra. Then the exchange graph $\mathcal{H}_A$ of support τ-tilting A-modules has the non-leaving-face property.*

Proof. Let (U, Q) be a basic τ-rigid pair, $\mathcal{F}_{(U,Q)}$ be a face determined by (U, Q). By Lemma 3.5, Lemma 3.6 and Lemma 3.7, the map $P_{(U,Q)} : \mathcal{H}_A \to \mathcal{F}_{(U,Q)}$ given by (3.1) is a projection required in Lemma 2.3, finishing the proof. $\square$

Remark 3.9. As in [10], the image $P_{(U,Q)}(M)$ under the projection $P_{(U,Q)}$ can be understood as the Bongartz completion of (U, Q) with respect to M. Indeed, a notion of relative Bongartz completion for τ-tilting theory has been introduced in [6] under a mild condition. In the case of τ-tilting finite algebras, $P_{(Q,U)}(M)$ is precisely the left Bongartz completion of (U, Q) with respect to M.

Remark 3.10. The relative (co)-Bongartz completions have been well-studied for 2-Calabi-Yau triangulated categories with cluster-tilting objects in [4]. Combining results in [4, Section 4] with Lemma 2.3 implies that for any 2-Calabi-Yau tilted algebra, the τ-tilting graph has the non-leaving-face property. Results in [4, Section 4] can be generalized to any 2-Calabi-Yau category along with the work [16]. In particular, this implies that the exchange graph of support τ-tilting modules over the endomorphism algebra of a basic maximal rigid object in a 2-Calabi-Yau category has the non-leaving-face property.

3.3. **Application.** Let C be an $n \times n$ Cartan matrix of Dynkin type and W the associated Weyl group. Denote by $S \subset W$ the set of simple reflections. There are two classic polytopes, the *W-permutahedron* $\mathbf{Perm}(W)$ and the *W-associahedron* $\mathbf{Asso}(W)$, assoicatied to (W, S). The W-permutahedron $\mathbf{Perm}(W)$ has vertices in bijection with elements $w \in W$ and its faces are parametrized by the coset W/W_T for $T \subseteq S$, where W_T is the subgroup of W generated by elements in T. The W-associahedron $\mathbf{Asso}(W)$ has many different realizations, we refer to [15] for a definition and only mention that $\mathbf{Asso}(W)$ coincides with the exchange graph of a cluster algebra of finite type. As an application of Theorem 3.8, we obtain a uniform proof of the following result.

Corollary 3.11. [15, Theorem 3.2, Theorem 4.7] *W-permutahedra and W-associahedra have the non-leaving-face property.*

Proof. It is known that the exchange graph of a cluster algebra of finite type is isomorphic to the exchange graph of support τ-tilting modules of a τ-tilting finite algebra A. We conclude that W-associahedra have the non-leaving-face property by Theorem 3.8.

Now let $\mathbf{Perm}(W)$ be the W-permutahedron assoicated to (W, S) and C is the $n \times n$ Cartan matrix. Let D be a symmetrizer of C. Denote by $\Pi(C, D)$ the generalized preprojective algebra associated with (C, D) introduced by Geiß, Leclerc and Schröer [12], which is τ-tilting finite. It has been proved in [9] that there is a bijection between W and the set of basic support τ-tilting $\Pi(C, D)$-modules. For each $w \in W$, denote by I_w the corresponding support τ-tilting module. According to [9, Thoerem 5.6] and its proof, the map $w \mapsto I_{w^{-1}}$ induces an isomorphism of polytopes between $\mathbf{Perm}(W)$ and $\mathcal{H}_{\Pi(C,D)}$. It follows from Theorem 3.8 that the W-permutahedrom $\mathbf{Perm}(W)$ has the non-leaving-face property. □

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