State Prices

Wang Wei Mun

Lee Kong Chian School of Business
Singapore Management University

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Economic Environment

- Financial market consists of n risky assets, with initial price of P_i and (random) final payoff of $\tilde{X}_i$ for one share of asset i
- Financial market has $k \geq 2$ "states of nature", where each state corresponds to unique set of outcomes for final payoffs
- Let X_{si} be final payoff for one share of asset i in state s, and let $\mathbf{X}$ be $k \times n$ matrix that shows all possible outcomes:

$$\mathbf{X} = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kn} \end{bmatrix}$$

- Notice that each column of $\mathbf{X}$ represents different asset, while each row of $\mathbf{X}$ represents different state of nature

Complete Market

- Financial market is complete if $n \geq k$ and $\mathbf{X}$ has k linearly independent columns and rows $\implies$ $\mathbf{X}$ has rank k
- If $n > k$, then can form k portfolios with linearly independent payoffs: assume that $n = k \implies \mathbf{X}$ is invertible
- Let $\mathbf{Y} = [Y_1, \ldots, Y_k]'$ be any $k \times 1$ vector of desired final payoffs in each state of nature
- Let $\mathbf{N} = [N_1, \ldots, N_k]'$ be $k \times 1$ vector of required shares in each asset, in order to create portfolio with payoffs of $\mathbf{Y}$:

$$\mathbf{Y} = \mathbf{XN} \implies \mathbf{N} = \mathbf{X}^{-1}\mathbf{Y}$$

- Hence can create (unique) portfolio to deliver any set of desired payoffs, or replicate payoffs for any existing investment

State Prices

- Let $\mathbf{P} = [P_1, \ldots, P_k]'$ be $k \times 1$ vector of initial prices for one share of each asset: to avoid arbitrage, portfolio with final payoffs of $\mathbf{Y}$ must have initial price of $P_Y = \mathbf{P}'\mathbf{N} = \mathbf{P}'\mathbf{X}^{-1}\mathbf{Y}$
- Let $\mathbf{e}_s$ be $k \times 1$ vector of final payoffs for elementary security (or primitive security or Arrow–Debreu security) that delivers final payoff of one in state s, and zero in every other state
- Let p_s be initial price of elementary security for state s:

$$p_s = \mathbf{P}'\mathbf{X}^{-1}\mathbf{e}_s \quad \forall \quad s = 1, \ldots, k \implies$$
$$\begin{bmatrix} p_1 & \cdots & p_k \end{bmatrix} = \mathbf{P}'\mathbf{X}^{-1} \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_k \end{bmatrix} = \mathbf{P}'\mathbf{X}^{-1}$$

- Then p_s is known as state price for state s, which represents initial value of receiving final payoff of one in state s

Pricing Kernel

- There exists unique set of state prices in complete market
- Investors who are non-satiated will always be willing to pay for more consumption, so state prices must be strictly positive
- Let $\pi_s > 0$ be probability for state s, where $\sum_{s=1}^{k} \pi_s = 1$
- Initial price of portfolio with payoffs of $\mathbf{Y}$ can be expressed in terms of state prices, which is related to pricing kernel:

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \sum_{s=1}^{k} \pi_s \left(\frac{p_s}{\pi_s} \right) Y_s = E\left[\tilde{M} \tilde{Y} \right]$$

- Hence there exists unique pricing kernel in complete market, which must have value of $M_s = p_s / \pi_s > 0$ in state s

Risk-Neutral Probabilities

- Initial price of riskless asset with payoff of one in every state:

$$P_f = \sum_{s=1}^{k} p_s = \sum_{s=1}^{k} \pi_s M_s = E\left[\tilde{M}\right] = \frac{1}{R_f}$$

- Let $\widehat{\pi}_s = R_f p_s > 0$ for $s = 1, \ldots, k$, and interpret as set of (risk-adjusted) state probabilities since $\sum_{s=1}^{k} \widehat{\pi}_s = 1$
- Initial price of portfolio that delivers payoffs of $\mathbf{Y}$:

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \frac{1}{R_f} \sum_{s=1}^{k} \widehat{\pi}_s Y_s = \frac{1}{R_f} \widehat{E}\left[\tilde{Y}\right]$$

Risk-Neutral Probabilities

- Here $\widehat{E}[\cdot]$ is expectation under probability distribution of $\widehat{\pi}$
- All portfolios will have same expected return under probability distribution of $\widehat{\pi}$, which is equal to risk-free rate:

$$R_Y = \frac{1}{P_Y}\widehat{E}\left[\tilde{Y}\right] = R_f$$

- Interpret $\widehat{\pi}$ as risk-neutral probability distribution, for which pricing kernel is non-random: $\widehat{M}_s = R_f^{-1}$ for all s
- Then π represents physical probability distribution, for which (random) pricing kernel is given by investor's IMRS
- Hence risk-neutral pricing formula (using risk-neutral probability distribution) is equivalent to using state prices

Risk-Neutral Probabilities

- Risk-neutral probability distribution puts more (or less) weight on states where pricing kernel is above (or below) average:

$$\widehat{\pi}_s = R_f p_s = R_f M_s \pi_s = \left(\frac{M_s}{E\left[\tilde{M}\right]} \right) \pi_s$$

- Hence "bad" states (where consumption is low and marginal utility is high) are more likely to occur and "good" states are less likely to occur, under risk-neutral probability distribution
- Then $\widehat{\pi}$ is risk-adjusted probability distribution that eliminates risk premium and induces risk-neutral behaviour (where expected return is equal to risk-free rate for all assets)

Binomial Model

- Consider "binomial" model with two states of nature
- Risky stock has initial price of S, which will subsequently rise to uS or drop to dS, where $u > d$
- Riskless bond has initial price of $P_f = R_f^{-1}$, where $u > R_f > d$
- Vector of initial prices and matrix of final payoffs:

$$\mathbf{P} = \begin{bmatrix} S \\ P_f \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} uS & 1 \\ dS & 1 \end{bmatrix}$$

- Vector of state prices:

$$\begin{bmatrix} p_u & p_d \end{bmatrix} = \mathbf{P}'\mathbf{X}^{-1} = \begin{bmatrix} \dfrac{1 - dP_f}{u - d} & \dfrac{uP_f - 1}{u - d} \end{bmatrix}$$

Binomial Model

- Vector of risk-neutral probabilities:

$$\left[\begin{array}{cc} \hat{\pi}_u & \hat{\pi}_d \end{array}\right] = R_f \left[\begin{array}{cc} p_u & p_d \end{array}\right] = \left[\begin{array}{cc} \dfrac{R_f - d}{u - d} & \dfrac{u - R_f}{u - d} \end{array}\right]$$

- Initial price of portfolio that delivers final payoff of Y_u or Y_d:

$$P_Y = p_u Y_u + p_d Y_d = \frac{1}{R_f} \left(\hat{\pi}_u Y_u + \hat{\pi}_d Y_d \right)$$

- Binomial model is often used to price options: initial price will not be accurate with just one time period, but converges to true value as model is extended to more time periods

Example: Binomial Model

- Stock has initial price of 6 and final payoff of 10 or 5
- Riskless bond has risk-free rate of 1.05
- Vector of initial prices and matrix of final payoffs:

$$\mathbf{P} = \begin{bmatrix} 6 \\ \frac{1}{1.05} \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} 10 & 1 \\ 5 & 1 \end{bmatrix}$$

- Vector of state prices:

$$\begin{bmatrix} p_u & p_d \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & \frac{1}{1.05} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} 0.2476 & 0.7048 \end{bmatrix}$$

Example: Binomial Model

- Vector of risk-neutral probabilities:

$$\begin{bmatrix} \hat{\pi}_u & \hat{\pi}_d \end{bmatrix} = 1.05 \times \begin{bmatrix} 0.2476 & 0.7048 \end{bmatrix} = \begin{bmatrix} 0.26 & 0.74 \end{bmatrix}$$

- Call option gives option (but not obligation) to buy one share of stock (at end of time period) for "strike price" of 6
- Call option will have terminal value of 4 when stock has final payoff of 10, or 0 when stock has final payoff of 5
- Hence initial price of call option:

$$0.2476 \times 4 = \frac{0.26 \times 4}{1.05} = 0.99$$