Cheat sheet material

Statistics

Basic Stats

Expectation

$$\mathbb{E}[X] = p(outcome1) + (1-p)outcome2$$
$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

For i.i.d. with n different random X (i.e. X1+X2+...+Xn). Let Sn be sum of random outcomes,

$$\mathbb{E}[S_n] = n\mathbb{E}[X]$$
$$\mathbb{E}[S_n - S_m] = \mathbb{E}[S_n] - \mathbb{E}[S_m] \qquad \forall\, m < n$$

Variance

$$Var(X) = p(outcome1)^2 + (1-p)(outcome2)^2$$
$$Var(aX) = a^2 Var(X)$$
$$Var(S_n) = nVar(X) = Var(\sqrt{n}X)$$

Moment Generating Function - Normal Distribution

Use MGF to get exp of a geometric brownian motion

$$\mathbb{E}\left[e^{\theta X}\right] = M_x(\theta) = \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right)$$

Example Geometric Brownian motion has an expectation that is **not 0**

$$W_t \sim N(0,t)$$
$$\mathbb{E}\left[e^{\sigma W_t}\right] = \exp\left(\frac{1}{2}\sigma^2 t\right)$$

Use MGF to get expectation of X by differentiating and solving

$$M_x(\theta) = \mathbb{E}\left[e^{\theta X}\right] = 1 + \theta\mathbb{E}\left[X\right] + \frac{\theta^2}{2!}\mathbb{E}\left[X^2\right] + \frac{\theta^3}{3!}\mathbb{E}\left[X^3\right] + \ldots + \frac{\theta^n}{n!}\mathbb{E}\left[X^n\right] + \ldots$$
$$M_x'(0) = \mathbb{E}\left[X\right]$$
$$M_x^{(2)}(0) = \mathbb{E}\left[X^2\right]$$
$$M_x^{(3)}(0) = \mathbb{E}\left[X^3\right]$$
$$M_x^{(4)}(0) = \mathbb{E}\left[X^4\right]$$

Martingales

To prove martingale, show that Expectation of a future time is the same as the value in present time. This shows that it there is no tendency i.e. drift = 0

$$\mathbb{E}[X_m|X_n] = \mathbb{E}[X_n]$$
$$\mathbb{E}[S_n|m] = \mathbb{E}[S_m]$$

or when $dX_t = Adt + BdW_t$ where $A = 0$.

Covariance

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$
$$\mathrm{Cov}(X,Y) = \rho \cdot \sigma_X \cdot \sigma_Y$$
$$\mathrm{Var}(X+Y) = \sigma_x^2 + \sigma_y^2 + 2\,\sigma_x\sigma_y\mathrm{Cov}(X,Y)$$

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] + \mathrm{Cov}(X,Y)$$

Probability Density Function and moments

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$
$$\mathbb{E}[X^k] = \int_{-\infty}^{\infty} x^k\, f(x)\, dx$$

PDF

$$f(x) = \phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

CDF

$$F(x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

Normal Distribution

$$X \sim N(0,1)$$

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-\frac{x^2}{2}}\, dx$$

$$E[X] = 0, \quad E[X^2] = 1, \quad E[X^3] = 0, \quad E[X^4] = 3.$$
$$\text{Next, note that } (W_t - W_s)^4 \sim N(0,(t-s))^4 = (t-s)^2$$
$$N(0,1)^4 = (t-s)^2 X^4.$$
$$E[(W_t - W_s)^4] = E[(t-s)^2 X^4] = 3(t-s)^2$$

Risk Neutral pricing

For European option

$$\begin{aligned} C_0 &= \frac{1}{(1+r)^n}\mathbb{E}[C_n] \\ &= \frac{1}{(1+r)^n}\left[p_{up}C_n(\text{call px tick up}) + p_{down}C_n(\text{call px tick down})\right] \end{aligned}$$

also applies for cases when multiple branches, example:

Multinomial Call option pricing

4 outcomes in one time step
115Δ - 15, 105Δ - 5, 95Δ, 85Δ

- Use the extreme up/down ticks. 115Δ - 15 = 85Δ, Δ = 0.5
- C(K=100) = $p_4^* \times (115-100) + p_3^* \times (105-100) = 5$ => $p_4^* = \frac{1-p_3^*}{3}$

Binomial trees

$$p^* = \frac{(1+r)-d}{u-d}$$

0 < d < 1 + r < u holds because of no arbitrage.

For american options

$$V_n^A = \max\left[\frac{1}{1+r}[p^* \times V_{n+1}^u + q^* \times V_{n+1}^d], (K-S_n)^+\right]$$

Radon-Nikodym Derivative

$$\mathbb{E}^Q[S_2] = \mathbb{E}^P\left[S_2 \cdot \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \quad ; \quad \mathbb{E}^P[S_2] = \mathbb{E}^Q\left[S_2 \cdot \frac{d\mathbb{P}}{d\mathbb{Q}}\right]$$

conditional brownian motion

$$P(W_2 < 0 | W_1 > 0) = P\left[(W_2 < W_1 \cap |W_2 - W_1| > |W_1 - W_0|) \ |W_1 > 0\right] = \frac{1}{4}$$

Taylors Expansion

for a function of x

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

where df = f(x) - f(a), dx = x-a

$$\begin{aligned} f(x,t) = {} & f(x_0,t_0) \\ & + \frac{\partial f}{\partial x}(x_0,t_0)(x-x_0) + \frac{\partial f}{\partial t}(x_0,t_0)(t-t_0) \\ & + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,t_0)(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial t}(x_0,t_0)(x-x_0)(t-t_0) + \frac{\partial^2 f}{\partial t^2}(x_0,t_0)(t-t_0)^2\right) + \cdots \end{aligned}$$

for a function of x and t, f(x,t)

$$\Delta f = f'(a,b)\Delta x + f'(a,b)\Delta x + \frac{f''(a,b)}{2!}(\Delta x)^2 + \frac{f^{(3)}(a)}{3!}(\Delta x)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(\Delta x)^n + \cdots$$

Properties of Stochastic Integrals

1. $\mathbb{E}[I_T] = 0$
2. $E\left[I_T^2\right] = E\left[\left(\int_0^T f(u,W_u)\,dW_u\right)^2\right] = E\left[\int_0^T f(u,W_u)^2\,du\right]$
3. If f is deterministic $I_T \sim N\left(0, \int_0^T f(u)^2\,du\right)$
4. $E\left[\int_0^T f(u)\,dW_u \times \int_0^T g(s)\,dW_s\right] = E\left[\int_0^T f(u)g(u)\,du\right]$
5. Itô's Isometry theorem states that $E\left[\left(\int_0^T X_t\,dW_t\right)^2\right] = E\left[\int_0^T X_t^2\,dt\right]$
6. Ito integral: $\int_0^T W_t\,dW_t = \frac{W_T^2}{2} - \frac{T}{2}$ when $X_t = f(W_t) = W_t^2$
7. Ito integral can be used twice, to get solution

$$\int_0^T W_t dt = \int_0^T (T-t)^2 dW_t$$

$$\mathbb{V}\mathrm{ar}\left[\int_0^T W_t dt\right] = \mathbb{E}\left[\left(\int_0^T W_t dt\right)^2\right] = \mathbb{E}\left[\left(\int_0^T (T-t) dWt\right)^2\right] = \mathbb{E}\left[\int_0^T (T-t)^2 dt\right] = \frac{T^3}{3}$$

$$\begin{aligned} Var(\sigma_x {W_T}^x + \sigma_y {W_T}^x) &= \sigma_x^2 T + \sigma_y^2 T + 2\sigma_x\sigma_y \mathrm{Cov}({W_T}^x, {W_T}^y) \\ &= \sigma_x^2 T + \sigma_y^2 T + 2\sigma_x\sigma_y \mathrm{T}\ \mathrm{Cov}(\tilde{X}, \tilde{Y}) \end{aligned}$$

$$E[|W_{t+\Delta t} - W_t|] = \sqrt{\frac{\Delta t}{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}}\,dx = 2\sqrt{\frac{\Delta t}{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}}\,dx = 2\sqrt{\frac{\Delta t}{2\pi}} \int_0^{\infty} e^{-u}\,du = \sqrt{\frac{2\Delta t}{\pi}}$$

x* and option pricing for Bachelier vanilla call

$$S_0 + \sigma\sqrt{T}x - K > 0 \implies x > \frac{K - S_0}{\sigma\sqrt{T}} = x^*$$

$$\begin{aligned} V_0^{Bach} &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 - K) e^{-\frac{x^2}{2}} dx + \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \sigma\sqrt{T} x e^{-\frac{x^2}{2}} dx \\ &= e^{-rT}\left[(S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)\right] \end{aligned}$$

x* and option pricing for BS

$$BS_{call} : x > \frac{\log(\frac{K}{S_0}) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = x^*$$

$$\begin{aligned} V_0^{call} &= \frac{S_0 e^{-\frac{\sigma^2 T}{2}}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} e^{\frac{\sigma^2 T}{2}} dx - \frac{K e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= S_0 \Phi\left(\frac{\log\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - K e^{-rT} \Phi\left(\frac{\log\frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \end{aligned}$$

Stochastic Differential Equations

SDE Product Rule

$$df(X_t, Y_t) = \frac{\partial f}{\partial X} dX_t + \frac{\partial f}{\partial Y} dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial Y^2}(dY_t)^2 + \frac{\partial^2 f}{\partial X \partial Y} dX_t dY_t$$

Bachelier

Arithmetic Brownian Process

$$dS_t = \sigma dW_t$$

$$f(X_t) = \log(S_t)$$
$$S_T = S_0 + \sigma W_t$$

Black-Scholes

Geometric Brownian Process

$$dS_T = rS_T dt + \sigma S_T dW_t$$

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right]$$

Vasicek Model

mean reverting stochastic process

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$$

$$f(X_t) = e^{\kappa t} r_t$$
$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u$$

Displaced-Diffusion

combination of BS and Bach adjusted by β

$$dF_t = \sigma\left[\beta F_t + (1-\beta)F_0\right] dW_t^*$$

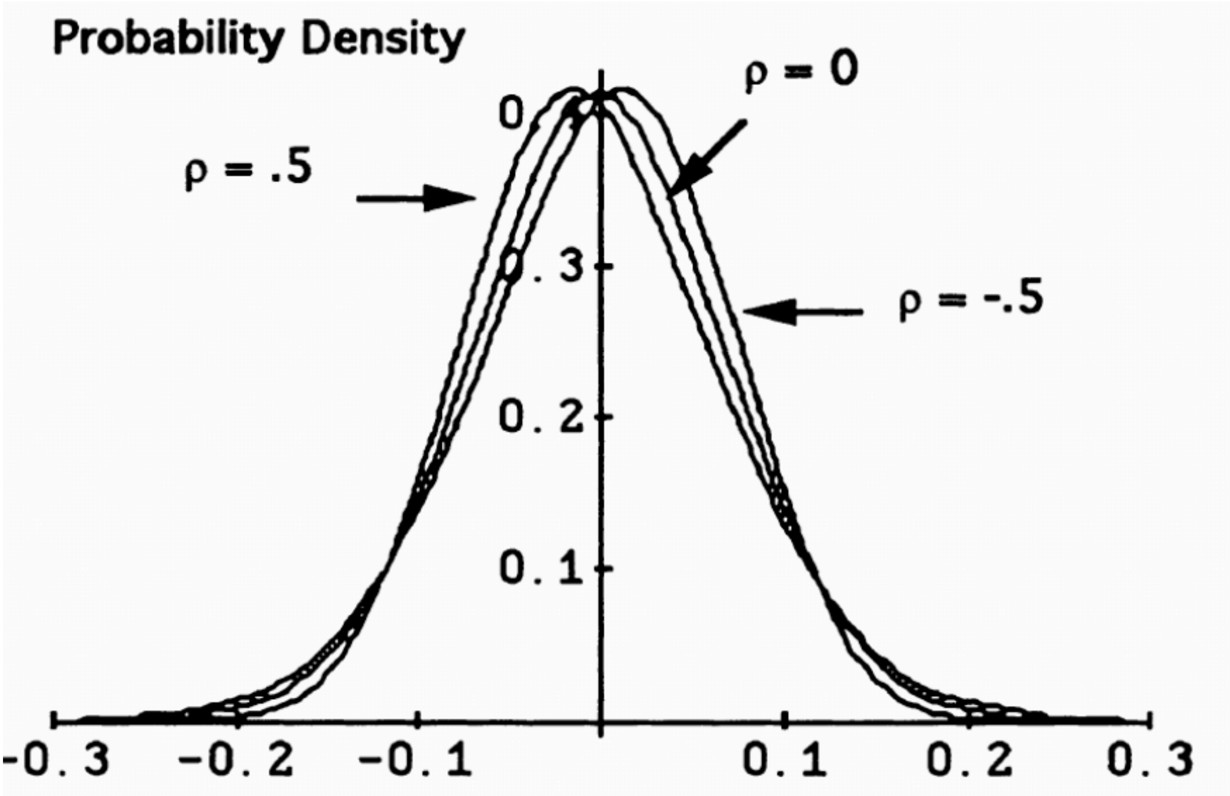
$$f(X_t) = \log(aF_t + b)$$
$$F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T^*} - \frac{1-\beta}{\beta} F_0$$

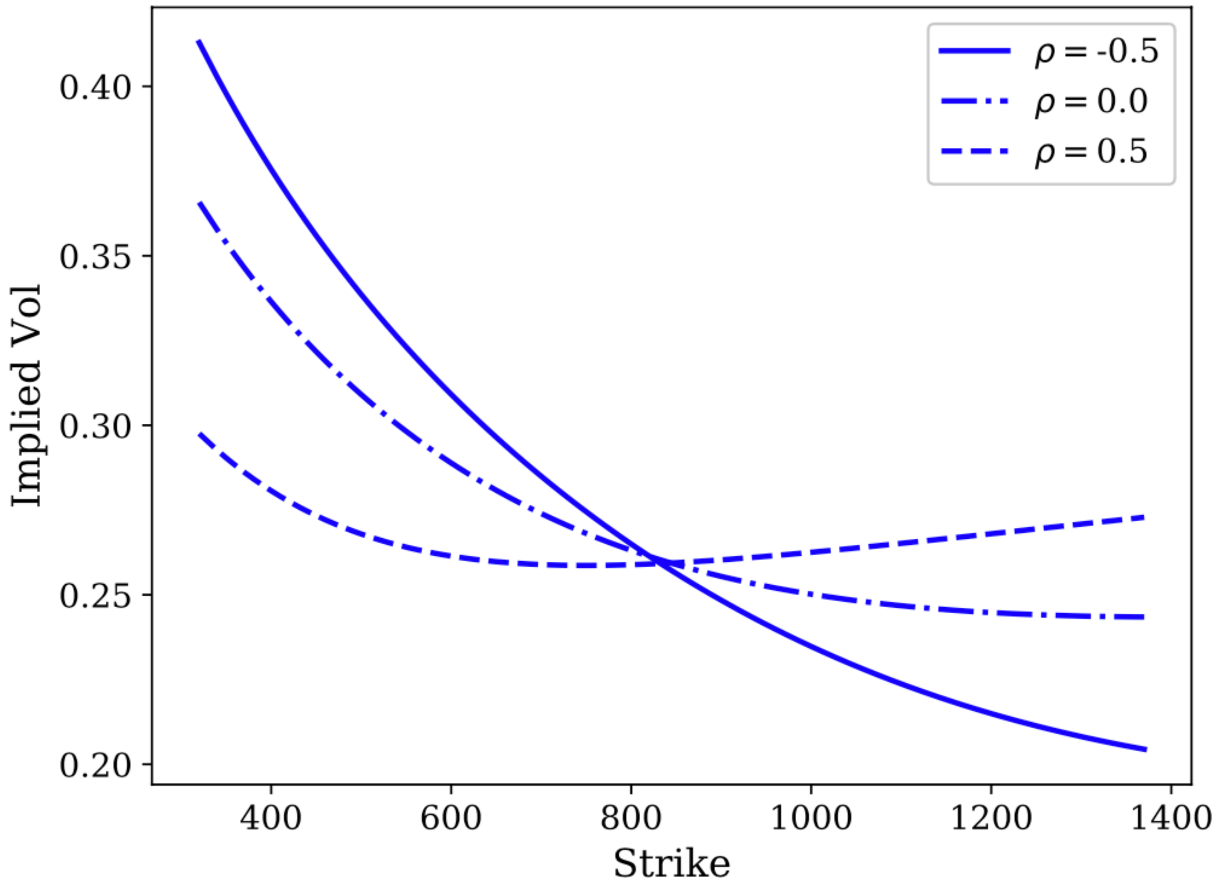
SABR parameters description: nu, rho, alpha

ρ Implication on Distribution

- The correlation parameter ρ is proportional to the skewness of stock returns.
- Intuitively, a negative correlation results in high volatility when the stock price drops, and this spreads the left tail of the probability density. The right tail is associated with low volatility and is not spread out.

- A negative correlation creates a fat left tail and a thin right tail in the stock return distribution.
  Implication on Pricing
  - This increases the prices of out-of-the-money puts and decreases the prices of out-of-the-money calls relative to the Black-Scholes model price.
  - Intuitively, out-of-the-money put options benefit substantially from a fat left tail.
  - A positive correlation will have completely opposite effects—it creates a fat right tail and a thin left tail.

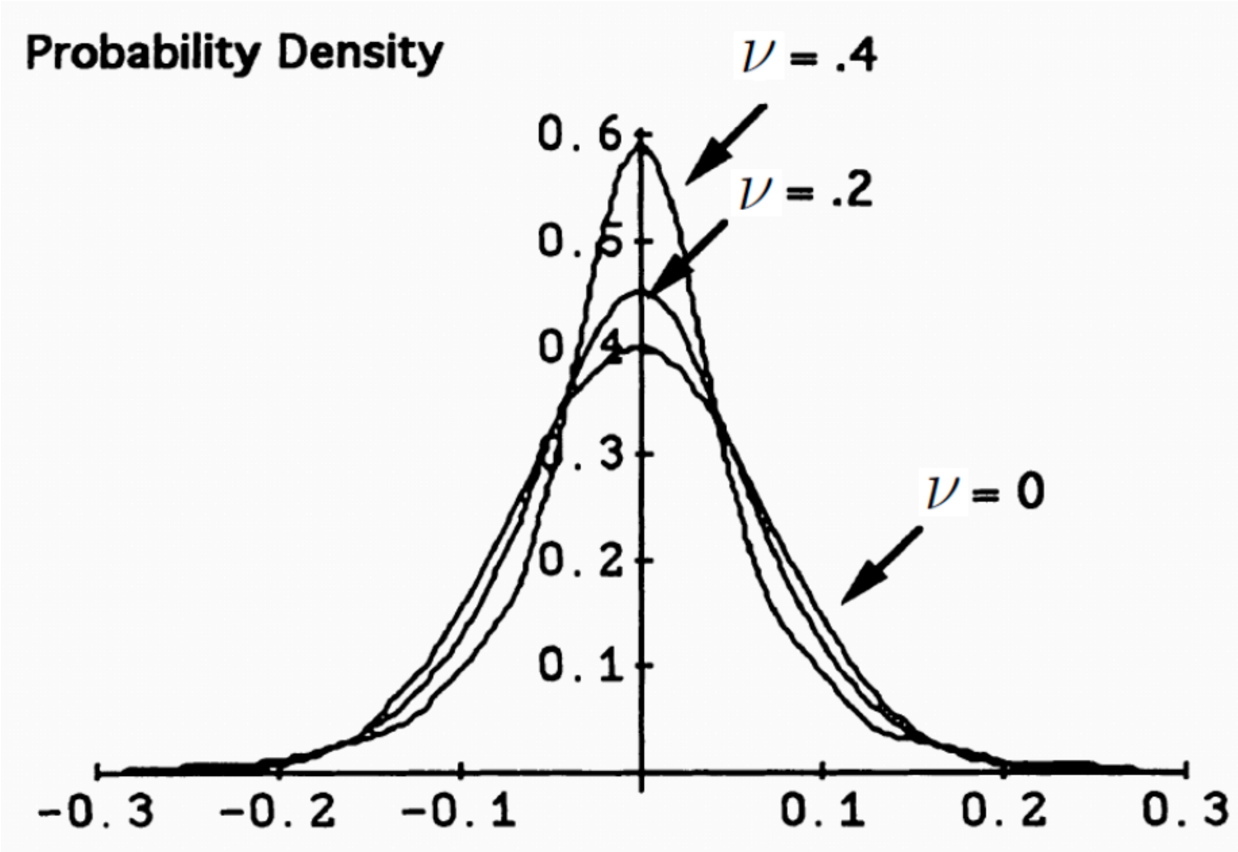


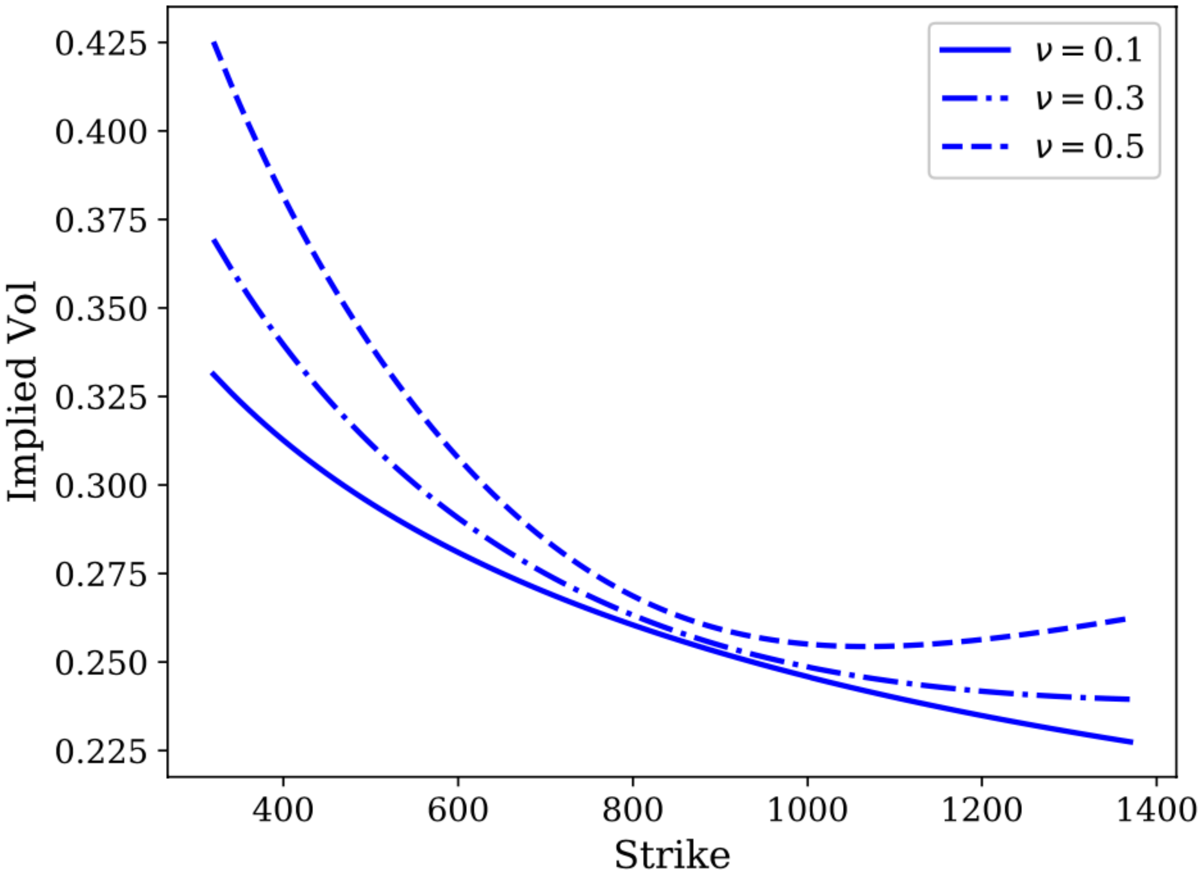


v Implication on Distribution

- When the volatility of volatility parameter is 0, we are back to a normal log-return distribution (if β = 0).
- Otherwise, it increases the kurtosis of stock returns, creating two fat tails in both ends of the distribution.

- This has the effect of raising out-of-the-money puts and out-of-the-money call prices. Implication on Pricing
- If volatility is uncorrelated with stock return, then increasing the volatility of volatility only increases the kurtosis of spot return.
- In this case, random volatility is associated with increases in the prices of far-from-the-money options relative to near-the-money options.
- In contrast, the correlation of volatility with the spot return produces skewness.





Girsanov

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t, \\
d\left(\frac{S_t}{B_t}\right) &= -\frac{S_t}{B_t^2} dB_t + \frac{1}{B_t} dS_t \\
&= \mu \frac{S_t}{B_t} dt + \sigma \frac{S_t}{B_t} dW_t \\
&= \sigma \frac{S_t}{B_t}\left(dW_t + \frac{\mu}{\sigma} dt\right).
\end{aligned}
$$

Under the Radon-Nikodym derivative

$$
\frac{dQ^*}{dP} = \exp\left(-\frac{1}{2}\int_0^t k^2 du - \int_0^t k dW_u\right), \quad k = \frac{\mu}{\sigma},
$$

we have

$$
W_t^* = W_t + \frac{\mu}{\sigma} t \implies dW_t^* = dW_t + \frac{\mu}{\sigma} dt.
$$

Static Replication

Integration by parts

$$
\int u dv = uv - \int v du
$$

Leibnitz rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f\left(x, v(x)\right)v'(x) - f\left(x, u(x)\right)u'(x) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

Static Replication of European Payoff

$$V_0 = e^{-rT}\mathbb{E}[h(S_T)] = \int_0^F h(K)\frac{\partial^2 P(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}dK$$

$$\begin{aligned}
\int_F^\infty h(K)\frac{d^2C(K)}{dK^2}dK &= \left[h(K)\frac{dC(K)}{dK}\right]_F^\infty - \int_F^\infty h'(K)\frac{dC(K)}{dK}dK \\
&= -h(F)\frac{dC(F)}{dK} - \int_F^\infty h'(K)\frac{dC(K)}{dK}dK \\
&= -h(F)\frac{dC(F)}{dK} - \left[h'(K)C(K)\right]_F^\infty + \int_F^\infty h''(K)C(K)dK \\
&= -h(F)\frac{dC(F)}{dK} + h'(F)C(F) + \int_F^\infty h''(K)C(K)dK
\end{aligned}$$

Carr-Madan

$$V_0 = e^{-rT}h(F) + \int_0^F h''(K)P(K)\,dK + \int_F^\infty h''(K)C(K)\,dK$$

Variance Swap

$$\mathbb{E}\left[\int_0^T \sigma_t^2\,dt\right] = 2e^{rT}\int_0^F \frac{P(K)}{K^2}\,dK + 2e^{rT}\int_F^\infty \frac{C(K)}{K^2}\,dK$$