QF620 Assignment 3

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1.

(a)

$$egin{aligned} dX_t &= \mu dt + \sigma dW_t \ \int_0^t dX_s &= \int_0^t \mu \, ds + \int_0^t \sigma \, dW_s \ X_t &= X_0 + \mu t + \sigma W_t \ \mathbb{E}[X_t] &= \mathbb{E}[X_0 + \mu t + \sigma W_t] \ \mathbb{E}[X_t] &= X_0 + \mu t \end{aligned} \ ext{Var}[X_t] &= Var[X_0 + \mu t + \sigma W_t] \ &= \sigma^2 \mathrm{Var}[W_t] \ \mathrm{Var}[X_t] &= \sigma^2 t \ X_t \sim N\left(X_0 + \mu t, \, \sigma^2 t
ight) \end{aligned}$$

(b)

Let $f(X_t) = \log X_t$,

$$egin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW_t \ X_t &= X_0 e^{\left(\mu - rac{\sigma^2}{2}
ight)t + \sigma W_t} \ \mathbb{E}[X_t] &= \mathbb{E}\left[X_0 e^{\left(\mu - rac{\sigma^2}{2}
ight)t + \sigma W_t}
ight] \ &= X_0 e^{\left(\mu - rac{\sigma^2}{2}
ight)t} \mathbb{E}\left[e^{\sigma W_t}
ight] \ \mathbb{E}[X_t] &= X_0 e^{\mu t} \end{aligned}$$

Since, by MGF $\mathbb{E}[e^{\sigma W_t}] = e^{\frac{1}{2}\sigma^2 t}.$

$$egin{aligned} ext{Var}[X_t] &= ext{Var}\left[X_0 e^{\left(\mu - rac{\sigma^2}{2}
ight)t + \sigma W_t}
ight] \ &= X_0^2 e^{\left(2\mu - \sigma^2
ight)t} \operatorname{Var}\left[e^{\sigma W_t}
ight] \ &= X_0^2 e^{\left(2\mu - \sigma^2
ight)t} \left[e^{\sigma^2 t}(e^{\sigma^2 t} - 1)
ight] \end{aligned}$$

$$ext{Var}[X_t] = X_0^2 e^{2\mu t} \left(e^{\sigma^2 t} - 1
ight)$$

Where $\mathrm{Var}(e^{\sigma W_t})=\mathbb{E}[e^{2\sigma W_t}]-\mathbb{E}[e^{\sigma W_t}]^2=e^{\sigma^2 t}(e^{\sigma^2 t}-1)$ using MGF .

$$X_t \sim N\left(X_0 e^{\mu t},\, X_0^2 e^{2\mu t} \left(e^{\sigma^2 t} - 1
ight)
ight)$$

(c)

Let $f(X_t, t) = e^{\kappa t} X_t$,

$$egin{aligned} dX_t &= \kappa(heta - X_t)dt + \sigma dW_t \ f_t(X_t,t) = & \kappa e^{kt} X_t \quad f_x(X_t,t) = e^{\kappa t} \quad f_{xx}(X_t,t) = 0 \ df(X_t,t) &= \kappa e^{kt} X_t dt + e^{kt} [\kappa(heta - X_t) dt + \sigma dW_t] \ d(e^{\kappa t} X_t) &= e^{kt} \kappa heta dt + e^{kt} \sigma dW_t \ \int_0^t d(e^{\kappa s} X_s) &= \int_0^t e^{ks} \kappa heta \, ds + \int_0^t e^{ks} \sigma \, dW_s \ X_t &= X_0 e^{-kt} + heta (1 - e^{-kt}) + \sigma \int_0^t e^{k(s-t)} \sigma \, dW_s \ \mathbb{E}[X_t] &= X_0 e^{-\kappa t} + heta (1 - e^{-kt}) \ \mathrm{Var}[X_t] &= \mathbb{E}\left[\left(\sigma \int_0^t e^{\kappa(s-t)} dW_s
ight)^2\right] &= \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(s-t)} ds
ight] \ \mathrm{Var}[X_t] &= \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa t}) \ X_t &\sim N\left(X_0 e^{-\kappa t} + heta (1 - e^{-kt}), \, \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa t})
ight) \end{aligned}$$

2

$$egin{aligned} Z_t &= rac{X_t}{Y_t} = f(X_t, Y_t) \ &f_x = rac{1}{Y_t} \quad f_{xx} = 0 \quad f_y = -rac{X_t}{Y_t^2} \quad f_{yy} = rac{2X_t}{Y_t^2} \quad f_{xy} = -rac{1}{Y_t^2} \ &dZ_t = f_x dX_t + rac{1}{2} f_{xx} (dX_t)^2 + f_y dY_t + rac{1}{2} f_{yy} (dY_t)^2 + f_{xy} dX_t dY_t \ &dZ_t = \sigma^2 Z_t dt + \sigma Z_t dW_t - \sigma Z_t d ilde{W}_t - \sigma^2 Z_t
ho dt \end{aligned}$$

Where $dW_t d\tilde{W}_t =
ho dt$ and ho is correlation between the two Brownian motions.

(a)

When brownian motion are independent $\rho = 0$,

$$dZ_t = \sigma^2 Z_t dt + \sigma Z_t dW_t - \sigma Z_t d ilde{W}_t$$

(b)

When brownian motion are independent $\rho = 1$,

$$dZ_t = \sigma Z_t dW_t - \sigma Z_t d ilde{W_t}$$

(c)

When brownian motion are independent ho=
ho

$$dZ_t = \sigma^2 Z_t (1-
ho) dt + \sigma Z_t dW_t - \sigma Z_t d ilde{W}_t$$