Efficient Frontier Revisited

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Outline

Sampling Error – Part 1

- Previously assumed that investors know exact mean return for all tradable assets, but not true in reality
- Assume that excess returns for n tradable risky assets has joint normal distribution with covariance matrix of $\mathbf{\Sigma}$:

$$\tilde{\mathbf{R}}_{\mathbf{e}} \sim N(\tilde{\boldsymbol{\mu}}, \mathbf{\Sigma})$$

- Here $\tilde{\boldsymbol{\mu}}$ is $n \times 1$ vector of (unobservable) population risk premiums, which has independent normal distribution with covariance matrix of $\mathbf{\Sigma}_{\mu}$:

$$\tilde{\boldsymbol{\mu}} \sim N(\boldsymbol{\pi}, \mathbf{\Sigma}_{\mu})$$

Sampling Error – Part 2

- Here $\boldsymbol{\pi}$ is $n \times 1$ vector of (observable) sample risk premiums, which provides noisy estimate of population risk premiums
- For simplicity, assume that $\boldsymbol{\Sigma}_{\mu} = \tau \boldsymbol{\Sigma}$, where τ is constant
- In practice, often set $\tau = 1/m$, where m is number of data points used to estimate $\boldsymbol{\Sigma}$
- Reflects standard error of sample mean, when used as estimate of population mean
- Hence joint normal distribution for excess returns, expressed in terms of (observable) sample risk premiums:

$$\tilde{\mathbf{R}}_{\mathbf{e}} \sim N\Big(\boldsymbol{\pi}, (1+\tau)\,\boldsymbol{\Sigma}\Big)$$

Black–Litterman Model

- Standard model of efficient frontier uses historical data to estimate sample risk premiums
- But returns are volatile and data is limited, so estimate of sample mean tends to have large standard error
- Frontier portfolio weights tend to be sensitive: small change in inputs will produce large shift in location of efficient frontier
- Tangency portfolio weights often imply extreme positions
- Moreover, changes in inputs often produce unintuitive effects: higher risk premium for one asset may produce big swing in tangency portfolio weight for all other assets
- Fischer Black and Robert Litterman developed model (at Goldman Sachs) to improve on standard efficient frontier

Implied Risk Premiums

- Assume that (representative) investor with constant absolute risk aversion optimally chooses to invest in market portfolio
- Use observed weights of risky assets in market portfolio to determine implied (equilibrium) risk premiums:

$$\boldsymbol{\pi} = \lambda \boldsymbol{\Sigma} \mathbf{w}_m$$

- Here λ is coefficient of relative risk aversion (based on initial wealth) for investor with constant absolute risk aversion
- Calibrate λ using market risk premium or Sharpe ratio:

$$\lambda = \frac{\mathbf{w}'_m \boldsymbol{\pi}}{\mathbf{w}'_m \boldsymbol{\Sigma} \mathbf{w}_m} = \frac{R_m - R_f}{\sigma_m^2} = \frac{S_m}{\sigma_m}$$

Investor Views

- Black–Litterman model can also incorporate investor's "views" on (absolute or relative) risk premiums of risky assets
- Suppose that investor has $k \geq 1$ views on expected returns
- Let $\mathbf{P}$ be $k \times n$ matrix of asset weights corresponding to investor's views, and let $\mathbf{Q}$ be $k \times 1$ vector of expected returns corresponding to investor's views
- Also let $\boldsymbol{\Omega}$ be $k \times k$ covariance matrix based on confidence of investor's views
- For simplicity, assume that $\boldsymbol{\Omega}$ is diagonal matrix, so if investor is equally confident in all views, then $\boldsymbol{\Omega}$ will be identity matrix
- Caters to high-net-worth investors, who often have strong opinions on (absolute or relative) performance of risky assets

Example: Investor Views

- Suppose that investible universe consists of three risky assets (or asset classes, or mutual funds)
- Investor expects annual risk premium of 5% for first asset
- Investor also expects second asset to outperform third asset by 100 basis points (i.e., one percentage point) per year
- Then $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}$
- First row of $\mathbf{P}$ represents absolute view: weights sum to one
- Second row of $\mathbf{P}$ represents relative view: weights sum to zero
- Views can apply to portfolios, such as: equal-weighted combination of first two assets will earn risk premium of 5%
- Must be careful to avoid redundant or inconsistent views

Updated Return Distribution

- Bayes' theorem is used to update probability of given hypothesis H when new evidence E is observed:

$$\Pr(H|E) = \frac{\Pr(E|H)}{\Pr(E)}\Pr(H)$$

- Black–Litterman model uses Bayes' theorem to incorporate investor's views into distribution of excess returns
- Conditional on investor's views, excess returns have normal distribution of $N(\hat{\boldsymbol{\pi}}, \mathbf{M})$, where:

$$\hat{\boldsymbol{\pi}} = \boldsymbol{\pi} + \tau\boldsymbol{\Sigma}\mathbf{P}' \left(\tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} (\mathbf{Q} - \mathbf{P}\boldsymbol{\pi}),$$
$$\mathbf{M} = \boldsymbol{\Sigma} + \left(\frac{1}{\tau}\boldsymbol{\Sigma}^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}$$

Outline

Skewed Returns – Part 1

- Efficient frontier assumes normal returns $\implies$ ignores higher moments of return distribution, such as skewness and kurtosis
- Risk-averse investor faces trade-off between minimising variance of return vs maximising skewness of return
- But optimal trade-off depends on investor's preferences $\implies$ no easy way to incorporate skewness of return into objective function, such that it can apply to all risk-averse investors
- Alternative approach is to identify frontier portfolios that minimise downside risk, for given target mean return
- So if frontier portfolios minimise below-target semi-variance, then tangency portfolio will maximise Sortino ratio

Skewed Returns – Part 2

- Unfortunately, below-target semi-variance also implies specific trade-off between minimising variance vs maximising skewness
- Trade-off between skewness and variance can be adjusted by using lower partial moment in place of semi-variance:

$$\mathrm{LPM}\left(\tilde{R}_i; \tilde{R}_t, \kappa\right) = E\left[\min\left\{\tilde{R}_i - \tilde{R}_t,\ \ 0\right\}^{\kappa}\right]$$

- Here κ relates to investor's risk tolerance: $\kappa = 1$ is appropriate for risk-neutral investor, while $\kappa > 1$ is appropriate for (increasingly) risk-averse investor
- Larger values of κ also seem to correspond to stronger emphasis on maximising skewness of return

Three-Moment CAPM – Part 1

- Alan Kraus and Robert Litzenberger extended CAPM to account for exposure to skewness risk:

$$E\left(\tilde{R}_i\right) - R_f = \beta_i \pi_1 + \gamma_i \pi_2$$

- Here β_i is usual covariance coefficient that measures amount of exposure to variance risk, while γ_i is coskewness coefficient that measures amount of exposure to skewness risk:

$$\gamma_i = \frac{\sigma_{imm}}{\sigma_m^3} = \frac{E\left[\left(\tilde{R}_i - \mu_i\right)\left(\tilde{R}_m - \mu_m\right)^2\right]}{E\left(\tilde{R}_m - \mu_m\right)^3}$$

Three-Moment CAPM – Part 2

- Allocating more weight to i'th asset in market portfolio will increase skewness of market return when $\gamma_i > 0$, or reduce skewness of market return when $\gamma_i < 0$
- Using empirical data from U.S. financial markets, Kraus and Litzenberger found that $\pi_1 > 0 \implies$ investors demand economic compensation for exposure to variance risk
- Kraus and Litzenberger also found that $\pi_2 < 0 \implies$ investors demand economic compensation for exposure to negative skewness, but will pay for exposure to positive skewness
- Three-moment CAPM is widely used in insurance industry, which requires proper pricing of exposure to skewness risk