# Stochastic Discount Factor

Wang Wei Mun

Lee Kong Chian School of Business
Singapore Management University

October 9, 2023

# Consumption Choice Model

- Investor receives utility from regular consumption of goods and services, which is financed by investor's existing wealth
- Consider static (or "one-period") model in which investor only consumes at start and end of single time period
- Investor starts with initial wealth of $W_0$ and immediately consumes $C_0$, which leaves remaining wealth of $(W_0 - C_0)$
- Investor can invest remaining wealth in $n \geq 2$ risky assets, where $i$'th asset has initial price of $P_i$ and (random) final value of $\tilde{X}_i \implies$ (random) return of $\tilde{R}_i = \tilde{X}_i / P_i$
- One of the risky assets may in fact be riskless bond with (non-random) risk-free rate of $R_f$

# Portfolio Choice & Budget Constraint

- Let $w_i$ be proportion of investor's remaining wealth invested into $i$'th asset, subject to constraint: $\sum_{i=1}^{n} w_i = 1$
- Investor's final wealth depends on realised portfolio return:

$$\tilde{W}_1 = (W_0 - C_0) \sum_{i=1}^{n} w_i \tilde{R}_i$$

- No further opportunity for consumption after end of time period, so investor optimally chooses to consume final wealth:

$$\tilde{C}_1 = (W_0 - C_0) \sum_{i=1}^{n} w_i \tilde{R}_i$$

# Investor's Utility

- In general, investor's overall utility of consumption will depend on both initial and final consumption: $V\left(C_0, \tilde{C}_1\right)$
- For simplicity, assume that investor has time-separable utility of consumption $\implies$ investor's utility of consumption at given point of time is not affected by past or future consumption:

$$V\left(C_0, \tilde{C}_1\right) = U(C_0) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

- Here $\delta \in (0, 1)$ is subjective discount factor that reflects investor's rate of time preference (or impatience)
- Assume that $U(\cdot)$ is strictly increasing and concave $\implies$ investor will be non-satiated and risk averse

# Consumption and Portfolio Choice Problem

- At start of time period, investor chooses initial consumption of $C_0$ and portfolio weights of $w_i$ (for investment portfolio) so as to maximise overall utility, subject to relevant constraints:

$$\max_{C_0,\{w_i\}} \mathcal{L} = \left\{ U(C_0) + \delta E\left[ U\left( \tilde{C}_1 \right) \right] + \lambda \left( 1 - \sum_{i=1}^{n} w_i \right) \right\}$$

- First-order optimality condition for initial consumption, after applying chain rule since $\tilde{C}_1$ is function of $C_0$:

$$\frac{\partial \mathcal{L}}{\partial C_0} = 0 \implies U'(C_0^*) = \delta E\left[ U'\left( \tilde{C}_1^* \right) \sum_{i=1}^{n} w_i^* \tilde{R}_i \right]$$

# Optimal Asset Allocation – Part 1

- First-order optimality conditions for portfolio weights, after applying chain rule since $\tilde{C}_1$ is function of $w_i$:

$$\frac{\partial \mathcal{L}}{\partial w_i} = 0 \quad \Longrightarrow$$
$$\delta E\left[U'\left(\tilde{C}_1^*\right)\tilde{R}_i\right] = \frac{\lambda}{W_0 - C_0^*} \quad \forall \quad i = 1, \ldots, n$$

- All assets must have same expected marginal-utility-weighted return, based on marginal utility of optimal final consumption:

$$E\left[U'\left(\tilde{C}_1^*\right)\tilde{R}_i\right] = E\left[U'\left(\tilde{C}_1^*\right)\tilde{R}_j\right] \quad \forall \quad i, j$$

# Optimal Asset Allocation – Part 2

- Additional dollar invested in $i$'th asset produces return of $\tilde{R}_i$, which provides additional utility of $U'\left(\tilde{C}_1\right)\tilde{R}_i$ when consumed
- If $i$'th asset provides higher expected marginal-utility-weighted return, then investor will shift investment into $i$'th asset
- This leads to higher correlation between $\tilde{R}_i$ and $\tilde{C}_1 \implies \tilde{C}_1$ is more likely to be high when $\tilde{R}_i$ is high, and vice versa
- Then $U'\left(\tilde{C}_1\right)$ is more likely to be low when $\tilde{R}_i$ is high, and vice versa, since utility function is concave $\implies U''(\cdot) < 0$
- Expected marginal-utility-weighted return for $i$'th asset will fall, so investor will shift investment across risky assets until all assets have same expected marginal-utility-weighted return

# Intertemporal Allocation

- Use equality of expected marginal-utility-weighted returns to simplify optimality condition for initial consumption:

$$U'(C_0^*) = \sum_{i=1}^{n} w_i^* \left(\delta E\left[U'\left(\tilde{C}_1^*\right)\tilde{R}_i\right]\right) = \delta E\left[U'\left(\tilde{C}_1^*\right)\tilde{R}_i\right]$$

- LHS represents marginal utility from one unit of initial consumption, while RHS represents discounted expected marginal utility from $\tilde{R}_i$ units of final consumption
- Hence investor will shift between initial consumption and investment in final consumption to equalise marginal benefit
- Applies to all assets, as well as any combination of assets

# Asset Pricing Formula

- Rearrange to get asset pricing formula:

$$E\left[\delta \frac{U'\left(\tilde{C}_1^*\right)}{U'(C_0^*)} \tilde{R}_i\right] = 1 \quad \Longrightarrow \quad P_i = E\left[\delta \frac{U'\left(\tilde{C}_1^*\right)}{U'(C_0^*)} \tilde{X}_i\right]$$

- Here $\tilde{M} = \delta U'\left(\tilde{C}_1^*\right) / U'(C_0^*) > 0$ represents investor's intertemporal marginal rate of substitution (IMRS)
- Hence investor's IMRS acts as pricing kernel (or stochastic discount factor) that relates initial price to final value:

$$E\left[\tilde{M}\tilde{R}_i\right] = 1 \quad \Longrightarrow \quad P_i = E\left[\tilde{M}\tilde{X}_i\right]$$

# Consumption CAPM – Part 1

- Assume that riskless bond exists:

$$E\left[\tilde{M}R_f\right] = 1 \implies E\left[\tilde{M}\right] = R_f^{-1} > 0$$

- Expand expectation of product in asset pricing formula:

$$E\left[\tilde{M}\tilde{R}_i\right] = E\left[\tilde{M}\right] E\left[\tilde{R}_i\right] + \mathrm{Cov}\left[\tilde{M}, \tilde{R}_i\right] = 1$$

- Rearrange to get pricing formula for Consumption CAPM:

$$E\left[\tilde{R}_i\right] - R_f = -\frac{\mathrm{Cov}\left[\tilde{M}, \tilde{R}_i\right]}{E\left[\tilde{M}\right]} = -\frac{\mathrm{Cov}\left[U'\left(\tilde{C}_1^*\right), \tilde{R}_i\right]}{E\left[U'\left(\tilde{C}_1^*\right)\right]}$$

# Consumption CAPM – Part 2

- Suppose that $\tilde{R}_i$ has negative correlation with $U'\left(\tilde{C}_1^*\right)$
- Implies that asset return tends to be high when marginal utility of final consumption is low, and vice versa
- Hence investor is likely to receive more consumption when consumption is less valuable, and vice versa
- Asset has undesirable payoff characteristics, so investor will demand large risk premium for holding this "risky" asset
- Conversely, if $\tilde{R}_i$ has positive correlation with $U'\left(\tilde{C}_1^*\right)$, then investing in asset provides insurance against low consumption, so investor is willing to accept negative risk premium

# Volatility Bound – Part 1

- Let $\mu_M = E\left[\tilde{M}\right] = R_f^{-1}$ and $\text{Cov}\left[\tilde{M}, \tilde{R}_i\right] = \rho\sigma_M\sigma_i$, where $\rho$ is correlation coefficient between $\tilde{M}$ and $\tilde{R}_i$
- Apply to pricing formula for Consumption CAPM:

$$E\left[\tilde{R}_i\right] - R_f = -\frac{\rho\sigma_M\sigma_i}{\mu_M} \implies \frac{E\left[\tilde{R}_i\right] - R_f}{\sigma_i} = -\rho\frac{\sigma_M}{\mu_M}$$

- Use $\rho \in [-1, 1]$ to get Hansen–Jagannathan (H–J) bound:

$$\left|\frac{E\left[\tilde{R}_i\right] - R_f}{\sigma_i}\right| \leq \frac{\sigma_M}{\mu_M}$$

# Volatility Bound – Part 2

- LHS of H–J bound is Sharpe ratio of any risky asset (or portfolio), while RHS of H–J bound is "volatility ratio" of pricing kernel ($\approx$ standard deviation of pricing kernel)
- Hence volatility ratio of pricing kernel cannot be less than highest Sharpe ratio out of all possible portfolios
- Annual risk premium of around 7% and annual standard deviation of around 17% for U.S. stock market returns $\Longrightarrow$ Sharpe ratio of around 0.4, so pricing kernel is very volatile
- Pricing kernel has lower limit of zero but no upper limit $\Longrightarrow$ probability distribution must be heavily skewed to right side

# Power Utility – Part 1

- Consider investor with power utility of consumption:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} \implies$$

$$\tilde{M} = \delta \left( \frac{\tilde{C}_1^*}{C_0^*} \right)^{-\gamma} = \delta \exp\left[ -\gamma \ln \left( \frac{\tilde{C}_1^*}{C_0^*} \right) \right]$$

- Suppose that optimal consumption growth has lognormal distribution with mean of $\mu_c$ and variance of $\sigma_c^2$:

$$\ln \left( \frac{\tilde{C}_1^*}{C_0^*} \right) = \mu_c + \sigma_c \tilde{z}, \qquad \tilde{z} \sim N(0,1)$$

# Power Utility – Part 2

- Apply result for variance of pricing kernel to volatility ratio:

$$\begin{aligned}
\mathrm{Var}\left[\tilde{M}\right] &= E\left[\tilde{M}^2\right] - E\left[\tilde{M}\right]^2 \\
\implies \sigma_M^2 &= \mu_{M^2} - \mu_M^2 \\
\implies \frac{\sigma_M}{\mu_M} &= \sqrt{\frac{\mu_{M^2}}{\mu_M^2} - 1}
\end{aligned}$$

- Apply results for lognormal random variable:

$$\begin{aligned}
\mu_M &= \delta E\left[e^{-\gamma(\mu_c + \sigma_c \tilde{z})}\right] = \delta e^{-\gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2} = \eta \\
\mu_{M^2} &= \delta^2 E\left[e^{-2\gamma(\mu_c + \sigma_c \tilde{z})}\right] = \delta^2 e^{-2\gamma\mu_c + 2\gamma^2\sigma_c^2} = \eta^2 e^{\gamma^2\sigma_c^2}
\end{aligned}$$

# Power Utility – Part 3

- Substitute for $\mu_M$ and $\mu_{M^2}$ in equation for volatility ratio of pricing kernel, and apply $e^x \approx 1 + x$ for small values of $x$:

$$\frac{\sigma_M}{\mu_M} = \sqrt{\frac{\mu_{M^2}}{\mu_M^2} - 1} = \sqrt{e^{\gamma^2 \sigma_c^2} - 1} \approx \gamma \sigma_c$$

- Now substitute into result for H–J bound:

$$\frac{\sigma_M}{\mu_M} \approx \gamma \sigma_c \geq \left| \frac{E\left[\tilde{R}_i\right] - R_f}{\sigma_i} \right|$$

# Equity Premium Puzzle

- Sharpe ratio of around 0.4 for U.S. stock market
- $\sigma_c \approx 2\%$ based on real annual per capita consumption for post-war U.S. economy (i.e., after World War II)
- Hence investor must have $\gamma \gtrsim 20$, which is widely considered to be unreasonably high degree of (relative) risk aversion
- Equity premium puzzle: investor with time-separable power utility of consumption and lognormal consumption growth must have unreasonably high degree of risk aversion
- Risk aversion magnifies volatility of consumption growth, so must have high degree of risk aversion since consumption growth is very stable but pricing kernel is very volatile

# Skewness Bound

- Investor's optimal consumption growth has lognormal distribution $\implies$ small amount of negative (left) skewness
- So for investor with power utility of consumption, distribution for pricing kernel will have positive (right) skewness that increases with investor's (relative) risk aversion
- Empirical evidence suggests that probability distribution for pricing kernel should have large amount of positive skewness
- Hence investor must also have high degree of (relative) risk aversion to satisfy "skewness bound" for pricing kernel
- But what if empirical data on post-war consumption understates volatility and skewness of consumption growth?

# Rare Disasters – Part 1

- Now suppose that optimal consumption growth also contains random variable that represents effect of rare disasters:

$$\ln\left(\frac{\tilde{C}_1^*}{C_0^*}\right) = \mu_c + \sigma_c \tilde{z} + \tilde{\nu},$$

$$\tilde{\nu} = \begin{cases} \ln\phi & \text{with probability of } \pi \\ 0 & \text{with probability of } 1-\pi \end{cases}$$

- Here $\pi \in [0,1]$ is probability that rare disaster occurs
- Then $1-\phi$ is fraction of optimal consumption that is lost in event of disaster, where $\phi \in (0,1)$
- For simplicity, assume that $\tilde{\nu}$ is independent of $\tilde{z}$

# Rare Disasters – Part 2

- Disasters are events that result in great economic disruption, such as Great Depression, World War I, and World War II
- Other examples are outbreak of infectious and deadly viral pandemic, or asteroid striking Earth in densely populated area
- Historical data usually covers time periods without disasters, which makes consumption growth appear less volatile
- Moreover, excluding disasters severely understates amount of negative (left) skewness in consumption growth
- Robert Barro estimated that $\pi = 1.7\%$ and $\phi = 0.65$, based on survey of major disasters of 20th century

# Rare Disasters – Part 3

- Apply results for lognormal random variable:

$$\mu_M = \eta E\left[e^{-\gamma\tilde{\nu}}\right] = \eta\left\{1 + \pi\left(\phi^{-\gamma} - 1\right)\right\}$$
$$\mu_{M^2} = \eta^2 e^{\gamma^2\sigma_c^2} E\left[e^{-2\gamma\tilde{\nu}}\right] = \eta^2 e^{\gamma^2\sigma_c^2}\left\{1 + \pi\left(\phi^{-2\gamma} - 1\right)\right\}$$

- If $\gamma$ is reasonably small, then $\gamma^2\sigma_c^2 \approx 0$, and can apply $1 + x \approx e^x$ to remaining terms:

$$\mu_M \approx \eta e^{\pi\left(\phi^{-\gamma}-1\right)}$$
$$\mu_{M^2} \approx \eta^2 e^{\pi\left(\phi^{-2\gamma}-1\right)}$$
$$\Longrightarrow \frac{\sigma_M}{\mu_M} = \sqrt{\frac{\mu_{M^2}}{\mu_M^2} - 1} \approx \sqrt{e^{\pi\left(\phi^{-\gamma}-1\right)^2} - 1} \approx \sqrt{\pi}\left(\phi^{-\gamma} - 1\right)$$

# Rare Disasters – Part 4

- H–J bound is satisfied for $\gamma = 3.3$, based on Sharpe ratio of around 0.4 for U.S. stock market:

$$\gamma = 3.3 \implies \frac{\sigma_M}{\mu_M} \approx \sqrt{0.017}\left(0.65^{-3.3} - 1\right) = 0.41$$

- No equity premium puzzle since $\gamma = 3.3$ represents reasonable degree of (relative) risk aversion
- Intuition is that possibility of rare disasters greatly increases volatility and negative (left) skewness of consumption growth
- So only need small amount of magnification (via $\gamma$) to match volatility and negative (left) skewness of pricing kernel