QF620 Assignment 3

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1.

(a)

$$\begin{aligned} dX_t &= \mu dt + \sigma dW_t \\ \int_0^t dX_s &= \int_0^t \mu \, ds + \int_0^t \sigma \, dW_s \end{aligned}$$

$$X_t = \mu t + \sigma W_t$$

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}[\mu t + \sigma W_t] \\ \mathbb{E}[X_t] &= \mu t \end{aligned}$$

$$\begin{aligned} \text{Var}[X_t] &= \text{Var}[\mu t + \sigma W_t] \\ &= \sigma^2 \text{Var}[W_t] \\ \text{Var}[X_t] &= \sigma^2 t \end{aligned}$$

$$X_t \sim N\left(\mu t, \, \sigma^2 t\right)$$

(b)

Let $f(X_t) = \log X_t$,

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW_t \\ X_t &= X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \end{aligned}$$

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}\left[X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}\right] \\ &= X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t} \mathbb{E}\left[e^{\sigma W_t}\right] \end{aligned}$$

$$\mathbb{E}[X_t] = X_0 e^{\mu t}$$

Since, by MGF $\mathbb{E}[e^{\sigma W_t}] = e^{\frac{1}{2}\sigma^2 t}$.

$$\begin{aligned} \text{Var}[X_t] &= \text{Var}\left[X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}\right] \\ &= X_0^2 e^{(2\mu - \sigma^2)t} \, \text{Var}\left[e^{\sigma W_t}\right] \\ &= X_0^2 e^{(2\mu - \sigma^2)t} \, \left[e^{\sigma^2 t}(e^{\sigma^2 t} - 1)\right] \end{aligned}$$

$$\text{Var}[X_t] = X_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$$

Where $\text{Var}(e^{\sigma W_t}) = \mathbb{E}[e^{2\sigma W_t}] - \mathbb{E}[e^{\sigma W_t}]^2 = e^{\sigma^2 t}(e^{\sigma^2 t} - 1)$ using MGF .

So,

$$X_t \sim N\left(X_0 e^{\mu t},\, X_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)\right)$$

(c)

Let $f(X_t, t) = e^{\kappa t} X_t$,

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

$$f_t(X_t, t) = \kappa e^{kt} X_t \quad f_x(X_t, t) = e^{\kappa t} \quad f_{xx}(X_t, t) = 0$$

$$\begin{aligned} df(X_t, t) &= \kappa e^{kt} X_t dt + e^{kt}[\kappa(\theta - X_t)dt + \sigma dW_t] \\ d(e^{\kappa t} X_t) &= e^{kt}\kappa\theta dt + e^{kt}\sigma dW_t \end{aligned}$$

$$\int_0^t d(e^{\kappa s} X_s) = \int_0^t e^{ks}\kappa\theta\, ds + \int_0^t e^{ks}\sigma\, dW_s$$

$$X_t = X_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{k(s-t)}\sigma\, dW_s$$

$$\mathbb{E}[X_t] = X_0 e^{-\kappa t} + \theta(1 - e^{-kt})$$

$$\mathrm{Var}[X_t] = \mathbb{E}\left[\left(\sigma \int_0^t e^{\kappa(s-t)} dW_s\right)^2\right] = \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(s-t)} ds\right]$$

$$\mathrm{Var}[X_t] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})$$

$$X_t \sim N\left(X_0 e^{-\kappa t} + \theta(1 - e^{-kt}),\, \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right)$$

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$$Z_t = \frac{X_t}{Y_t} = f(X_t, Y_t)$$

$$f_x = \frac{1}{Y_t} \quad f_{xx} = 0 \quad f_y = -\frac{X_t}{Y_t^2} \quad f_{yy} = \frac{2X_t}{Y_t^2} \quad f_{xy} = -\frac{1}{Y_t^2}$$

$$dZ_t = f_x dX_t + \frac{1}{2} f_{xx}(dX_t)^2 + f_y dY_t + \frac{1}{2} f_{yy}(dY_t)^2 + f_{xy} dX_t dY_t$$

$$dZ_t = \sigma^2 Z_t dt + \sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t - \sigma^2 Z_t \rho dt$$

Where $dW_t d\tilde{W}_t = \rho dt$ and ρ is correlation between the two Brownian motions.

(a)

When brownian motion are independent $\rho = 0$,

$$dZ_t = \sigma^2 Z_t dt + \sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t$$

(b)

When brownian motion are independent $\rho = 1$,

$$dZ_t = +\sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t$$

(c)

When brownian motion are independent $\rho = \rho$

$$dZ_t = \sigma^2 Z_t (1 - \rho) dt + \sigma Z_t dW_t - \sigma Z_t d\tilde{W}_t$$