# QF620 Stochastic Modelling in Finance

## Assignment 1/4

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### 1(a)

Since $X \sim N(\mu, \sigma^2)$, use MGF to find $\mathbb{E}[e^X]$,

$$\mathbb{E}[e^{\theta X}] = M_X(\theta) = exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right)$$

Using $\theta = 1$,

$$\mathbb{E}[e^X] = M_X(1) = exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

### 1(b)

Taking MGF and using $\theta = 2$,

$$\mathbb{E}[e^{2X}] = M_X(2) = exp\left(2\mu + 2\sigma^2\right)$$

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### 2(a)

Since $W_t \sim N(0, t)$, use MGF to find $\mathbb{E}[e^{W_t}]$,

$$\mathbb{E}[e^{\theta W_t}] = M_{W_t}(\theta)$$

$$\mathbb{E}[e^{W_t}] = M_{W_t}(1) = exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right) = exp\left(\frac{1}{2}t\right)$$

### 2(b)

$$\mathbb{E}[e^{\sigma W_t}] = M_{W_t}(\sigma) = exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right) = exp\left(\frac{1}{2}\sigma^2 t\right)$$

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### 3

Calculate risk neutral probabilities $p^*$ and $q^*$ using *Cox-Ross-Rubinstein* formulation and choose $d = \frac{1}{u}$

$$p^* = \frac{(1+r) - d}{u - d}; \quad q^* = 1 - p^*$$

$p^*$ = 36% and $q^*$ = 64%

Calculate the exercise prices at maturity ($t = 2$)

Calculate the expected price with exercise prices at $t = 2$ for the option using risk neutral probabilities for $t = 1$ then $t = 0$

$$V_E^n = \frac{1}{1+r}\mathbb{E}_n^*[V_{n+1}] = \frac{1}{1+r}\left(p^* \times V_{n+1}^u + q^* \times V_{n+1}^d\right)$$

## For Put

$$V_A^n = max\left\{\frac{1}{1+r}\mathbb{E}_n^*[V_{n+1}] = \frac{1}{1+r}\left(p^* \times V_{n+1}^u + q^* \times V_{n+1}^d\right), (K - S_n)^+\right\}$$

## For Call

$$V_A^n = max\left\{\frac{1}{1+r}\mathbb{E}_n^*[V_{n+1}] = \frac{1}{1+r}\left(p^* \times V_{n+1}^u + q^* \times V_{n+1}^d\right), (S_n - K)^+\right\}$$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | a) | European Put | | | | | | | c) | European Call | | | | |
| S0 | 5 | | t=0 | | t=1 | | t=2 | | | | t=0 | | t=1 | | t=2 |
| u | 2 | | | | | | | | | | | | | | |
| d | 0.5 | | | | | | $ 20.00 | | | | | | | | $ 20.00 |
| r | 4% | | | | | | $ - | | | | | | | | $ 10.00 |
| K | 10 | | | | $ 10.00 | | | | | | | | $ 10.00 | | |
| p* | 36% | | | | $ 3.08 | | | | | | | | $ 3.46 | | |
| q* | 64% | | $ 5.00 | | | | $ 5.00 | | | | $ 5.00 | | | | $ 5.00 |
| | | | $ 5.44 | | | | $ 5.00 | | | | $ 1.20 | | | | $ - |
| | | | | | $ 2.50 | | | | | | | | $ 2.50 | | |
| | | | | | $ 7.12 | | | | | | | | $ - | | |
| | | | | | | | $ 1.25 | | | | | | | | $ 1.25 |
| | | | | | | | $ 8.75 | | | | | | | | $ - |
| | | | | | | | | | | | | | | | |
| | | b) | American Put | | | | | | | d) | American Call | | | | |
| | | | t=0 | | t=1 | | t=2 | | | | t=0 | | t=1 | | t=2 |
| | | | | | | | | | | | | | | | |
| | | | | | | | $ 20.00 | | | | | | | | $ 20.00 |
| | | | | | | | $ - | | | | | | | | $ 10.00 |
| | | | | | $ 10.00 | | | | | | | | $ 10.00 | | |
| | | | | | $ 3.08 | | | | | | | | $ 3.46 | | |
| | | | $ 5.00 | | | | $ 5.00 | | | | $ 5.00 | | | | $ 5.00 |
| | | | $ 5.68 | | | | $ 5.00 | | | | $ 1.20 | | | | $ - |
| | | | | | $ 2.50 | | | | | | | | $ 2.50 | | |
| | | | | | $ 7.50 | | | | | | | | $ - | | |
| | | | | | | | $ 1.25 | | | | | | | | $ 1.25 |
| | | | | | | | $ 8.75 | | | | | | | | $ - |

(a) $5.44(b)$5.68 (c) $1.20(d)$1.20

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# 4(a)

Given that $W_1 > 0$ and $W_2 < 0$

$$W_1 \sim N(0,1); \quad W_2 \sim N(0,2); \quad W_2 - W_1 \sim N(0,1)$$

$$\begin{aligned} P(W_2 < 0 | W_1 > 0) &= P(W_2 < W_1 \cap [|W_2 - W_1| > |W_1 - W_0|] \ \ |W_1 > 0) \\ &= P(W_2 < W_1 | W_1 > 0) \times P(|W_2 - W_1| > |W_1 - W_0| \ \ |W_1 > 0) \\ &= \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Since $W_1 - W_0$ and $W_2 - W_1$ are independent

## 4(b)

$P(W_1 \times W_2 < 0)$ is when $W_1$ and $W_2$ have opposite signs and there is 50% probability that either is more or less than 0.

$$\begin{aligned} P(W_1 \times W_2 < 0) &= P(W_2 < 0|W_1 > 0) \times P(W_1 > 0) + P(W_2 > 0|W_1 < 0) \times P(W_1 < 0) \\ &= \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

## 4(c)

$$\begin{aligned} P(W_1 < 0 \cap W_2 < 0) &= P(W_2 < 0|W_1 < 0) \times P(W_1 < 0) \\ &= P(W_2 < W_1|W_1 < 0) \times P(W_1 < 0) \\ &+ P(W_2 > W_1 \cap |W_2 - W_1| < |W_1 - W_0|\ |W_1 < 0) \times P(W_1 < 0) \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{3}{8} \end{aligned}$$

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## 5(a)

Definitions for Brownian Motion:\ $W_0 = 0$\ $0 < s < t; \quad (W_t - W_s) \sim N(0, t-s)$\ \ $\mathbb{E}[W_t] = 0, W_t \sim N(0,t)$ \ then, $\mathbb{E}[(W_t - \mu)^2] = \mathbb{E}[W_t^2] = Var[W_t] = t$ and

$$V[(W_t - W_s)] = V[W_t] - V[W_s] = t - s$$

## 5(b)

By variance definition,

$$Var((W_t - W_s)^2) = \mathbb{E}[((W_t - W_s)^2)^2] - (\mathbb{E}[(W_t - W_s)^2])^2$$

\ First finding $\mathbb{E}[(W_t - W_s)^2]$

$$Var[(W_t - W_s)] = E[(W_t - W_s)^2] - E[(W_t - W_s)]^2 = E[(W_t - W_s)^2]$$

where $\mathbb{E}[W_t - W_s] = 0$. So,

$$E[(W_t - W_s)^2] = t - s \tag{1}$$

\ Next finding $\mathbb{E}[((W_t - W_s)^2)^2] = \mathbb{E}[(W_t - W_s)^4]$, Use MGF,

$$M_{W_t - W_s}(\theta) = exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right) = \exp\left(\frac{1}{2}\sigma^2\theta^2\right)$$

Compute derivatives up to 4th moment,

$$\frac{dM_{W_t - W_s}(\theta)}{d\theta} = \mathbb{E}[W_t - W_s] = \sigma\theta\exp\left(\frac{1}{2}\sigma^2\theta^2\right)$$

$$\frac{d^2 M_{W_t-W_s}(\theta)}{d\theta^2} = \mathbb{E}[(W_t - W_s)^2] = \sigma^4\theta^4 \exp\left(\frac{1}{2}\sigma^2\theta^2\right) + \sigma^2 \exp\left(\frac{1}{2}\sigma^2\theta^2\right)$$

$$\frac{d^3 M_{W_t-W_s}(\theta)}{d\theta^3} = \mathbb{E}[(W_t - W_s)^3] = \sigma^6\theta^3 \exp\left(\frac{1}{2}\sigma^2\theta^2\right) + 2\sigma^4\theta \exp\left(\frac{1}{2}\sigma^2\theta^2\right) + \sigma^4\theta \exp\left(\frac{1}{2}\sigma^2\theta^2\right.$$

$$\frac{d^4 M_{W_t-W_s}(\theta)}{d\theta^4} = \mathbb{E}[(W_t - W_s)^4] = \sigma^8\theta^4 \exp\left(\frac{1}{2}\sigma^2\theta^2\right) + 3\sigma^6\theta^2 \exp\left(\frac{1}{2}\sigma^2\theta^2\right) + 2\sigma^6\theta^2 \exp\left(\frac{1}{2}\sigma\right.$$
$$+ 2\sigma^4 \exp\left(\frac{1}{2}\sigma^2\theta^2\right) + \sigma^6\theta^2 \exp\left(\frac{1}{2}\sigma^2\theta^2\right) + \sigma^4 \exp\left(\frac{1}{2}\sigma^2\theta^2\right)$$

and taking $\frac{d^4 M_{W_t-W_s}(\theta)}{d\theta^4}$ when $\theta = 0, \sigma^2 = t - s$,

$$\mathbb{E}[(W_t - W_s)^4] = 3(t-s)^2 \tag{2}$$

From (1) and (2),

$$Var((W_t - W_s)^2) = \mathbb{E}[(W_t - W_s)^4] - (\mathbb{E}[(W_t - W_s)^2])^2 = 3(t-s)^2 - (t-s)^2 = 2(t-s)^2$$

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# 6

Finding $\mathbb{E}[|W_{t+\Delta t} - W_t|]$, given $W_t \sim N(0,t)$ and $(W_{t+\Delta t} - W_t) \sim N(0, \Delta t)$. \ Let $(W_{t+\Delta t} - W_t)$ be X, such that $X \sim N(0, \Delta t)$.

Function for p.d.f. is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ and $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$$\begin{aligned}
\mathbb{E}[|W_{t+\Delta t} - W_t|] &= \mathbb{E}[|X|] \\
&= 2\int_0^\infty X \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(X-\mu)^2}{2\sigma^2}} dx \\
&= 2\int_0^\infty X \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{X^2}{2\Delta t}} dx && (3) \\
&= -\sqrt{\frac{2\Delta t}{\pi}} \int_0^{-\infty} e^u du && (4) \\
&= -\sqrt{\frac{2\Delta t}{\pi}} [0 - 1] \\
&= \sqrt{\frac{2\Delta t}{\pi}}
\end{aligned}$$

(3) $\mu = 0$ and $\sigma^2 = \Delta t$ \ (4) substitute $u = -\frac{X^2}{2\Delta t}; \quad \frac{du}{dx} = -\frac{X}{\Delta t}; \quad xdx = -\Delta t du$