Multi-Period Asset Pricing

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Multi-Period Setting

- Consider investor with lifetime of T time periods and time-separable utility of consumption:

$$V(C_0, \ldots, C_T) = E\left[\sum_{t=0}^{T} \delta^t U(C_t)\right]$$

- Here $\delta \in (0, 1)$ is subjective discount factor that reflects investor's rate of time preference, while $U(\cdot)$ is strictly increasing and concave utility function
- Investor is endowed with initial wealth of W_0 and trades in n risky assets with (random) return of $R_{i,t+1}$ over time interval from t to $t + 1$, for $i = 1, \ldots, n$ and $t = 0, \ldots, T - 1$

Intertemporal Allocation

- Investor can rebalance portfolio at start of each time interval, allocating proportion $w_{i,t}$ of remaining wealth of $(W_t - C_t)$ to asset i at time t, subject to $\sum_{i=1}^{n} w_{i,t} = 1$:

$$W_{t+1} = (W_t - C_t) \sum_{i=1}^{n} w_{i,t} R_{i,t+1}$$

- Optimality condition to maximise expected lifetime utility:

$$U'(C_t^*) = \delta E_t \left[U'\left(C_{t+1}^*\right) R_{i,t+1} \right]$$

- Here $E_t[\cdot]$ is expectation conditional on information at time t: investor uses all available information to make optimal choices

Pricing Kernel

- Investor will shift between consumption and investment at time t to equalise marginal benefit for all risky assets
- Divide through by $U'(C_t^*)$ to get asset pricing formula:

$$E_t\left[\delta \frac{U'\left(C_{t+1}^*\right)}{U'(C_t^*)} R_{i,t+1}\right] = E_t[M_{t+1} R_{i,t+1}] = 1$$

- Here $M_{t+1} = \delta U'\left(C_{t+1}^*\right) / U'(C_t^*)$ is (one-period) pricing kernel over time interval from time t to time $t+1$
- By extension, $M_{t,t+2} = M_{t+1} M_{t+2} = \delta^2 U'\left(C_{t+2}^*\right) / U'(C_t^*)$ is pricing kernel over time interval from time t to time $t+2$
- Hence $M_{t,t+i} = M_{t+1} \cdots M_{t+i} = \delta^i U'\left(C_{t+i}^*\right) / U'(C_t^*)$ is pricing kernel over time interval from time t to time $t+i$

Dividend Discount Model

- Pricing formula for "long-lived" asset that pays (random) dividend of D_{t+i} at time $t+i$, for all $i = 1, 2, \ldots$:

$$P_t = E_t\left[\sum_{i=1}^{T} M_{t,t+i}D_{t+i} + M_{t,t+T}P_{t+T}\right]$$

- Here P_t is initial price at time t, while P_{t+T} is terminal (liquidation) value at end of T-period investment horizon
- Assume that $E_t[M_{t,t+T}P_{t+T}] \to 0$ as $T \to \infty$, and extend to infinite investment horizon for investor with infinite lifetime:

$$P_t = E_t\left[\sum_{i=1}^{\infty} M_{t,t+i}D_{t+i}\right]$$

Endowment Economy

- Consider "endowment economy" where aggregate economic output grows randomly over time, and investor who invests in market portfolio receives share of aggregate economic output
- Investor immediately consumes any dividend that is received $\Longrightarrow$ aggregate consumption must be equal to aggregate dividend (from market portfolio): $\overline{C}_t = \overline{D}_t$ for all t
- If financial market is "complete" and frictionless, then there will be unique pricing kernel that prices all assets
- Hence there exists representative investor who consumes aggregate consumption and invests in market portfolio
- Prices and returns of underlying risky assets will shift until representative investor's choices become optimal choices

Power Utility

- Suppose that representative investor has power utility, where γ is (constant) coefficient of relative risk aversion:

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \implies M_{t,t+i} = \delta^i \left(\frac{C_{t+i}}{C_t}\right)^{-\gamma}$$

- Use C_t as aggregate consumption (at time t) for convenience
- Aggregate consumption is always equal to aggregate dividend, so price-dividend ratio for market portfolio:

$$\frac{P_t}{D_t} = E_t\left[\sum_{i=1}^{\infty} \delta^i \left(\frac{D_{t+i}}{D_t}\right)^{1-\gamma}\right]$$

Lognormal Growth: Economic Environment

- Suppose that aggregate consumption evolves as lognormal random walk with drift:

$$\ln C_{t+1} = \ln C_t + \mu + \sigma\epsilon_{t+1}$$

- Here $\epsilon_t \sim N(0,1)$ is independent and identically distributed (i.i.d.) random variable that captures economic fluctuations
- Hence continuously compounded aggregate consumption growth rate has normal distribution over every time interval
- Then μ represents long-run trend in continuous growth rate, while σ represents volatility of economic fluctuations
- Let $\rho = -\ln\delta$ be investor's rate of time preference

Lognormal Growth: Market Portfolio

- Dividend claim that delivers (single) aggregate dividend at time $t+1$ has constant price-dividend ratio:

$$\frac{P_{1,t}}{D_t} = E_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right] = E_t\left[\delta e^{(1-\gamma)(\mu+\sigma\epsilon_{t+1})}\right]$$
$$= e^{-\rho+(1-\gamma)\mu+\frac{1}{2}(1-\gamma)^2\sigma^2} = \theta$$

- Hence price-dividend ratio for dividend claim that delivers (single) aggregate dividend at time $t+i$:

$$\frac{P_{i,t}}{D_t} = E_t\left[\delta^i\left(\frac{C_{t+i}}{C_t}\right)^{1-\gamma}\right] = E_t\left[\prod_{j=0}^{i-1}\delta\left(\frac{C_{t+j+1}}{C_{t+j}}\right)^{1-\gamma}\right]$$

Lognormal Growth: Market Portfolio

- Consumption growth is i.i.d., so all dividend claims have constant price-dividend ratio:

$$\frac{P_{i,t}}{D_t} = \prod_{j=0}^{i-1} E_t\left[\delta\left(\frac{C_{t+j+1}}{C_{t+j}}\right)^{1-\gamma}\right] = \prod_{j=0}^{i-1} \frac{P_{1,t+j}}{D_{t+j}} = \theta^i$$

- Hence market portfolio will also have finite constant price-dividend ratio when $\theta < 1$:

$$\frac{P_t}{D_t} = \sum_{i=1}^{\infty} \frac{P_{i,t}}{D_t} = \sum_{i=1}^{\infty} \theta^i = \frac{\theta}{1-\theta}$$

Lognormal Growth: Market Portfolio

- Market portfolio also has constant expected return:

$$
\begin{aligned}
E_t[R_{t+1}] &= E_t\left[\frac{D_{t+1}+P_{t+1}}{P_t}\right] = \frac{D_t}{P_t}E_t\left[\frac{D_{t+1}}{D_t}\left(1+\frac{P_{t+1}}{D_{t+1}}\right)\right] \\
&= \frac{1-\theta}{\theta}E_t\left[\frac{D_{t+1}}{D_t}\left(1+\frac{\theta}{1-\theta}\right)\right] = \frac{1}{\theta}E_t\left[\frac{C_{t+1}}{C_t}\right] \\
&= e^{\rho+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2+\gamma\sigma^2}
\end{aligned}
$$

- First dividend claim has same mean return as market portfolio:

$$
E_t\left[\frac{D_{t+1}}{P_{1,t}}\right] = \frac{D_t}{P_{1,t}}E_t\left[\frac{D_{t+1}}{D_t}\right] = \frac{1}{\theta}E_t\left[\frac{C_{t+1}}{C_t}\right] = E_t[R_{t+1}]
$$

Lognormal Growth: Equity Premium

- Suppose there exists riskless asset that always delivers one unit of output in next time period:

$$P_{f,t} = E_t[M_{t+1}] \implies R_{f,t} = \frac{1}{P_{f,t}} = e^{\rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2}$$

- Hence continuously compounded equity premium:

$$\ln E_t[R_{t+1}] - \ln R_{f,t} = \gamma\sigma^2$$

- Equity premium puzzle: consumption growth is very smooth, with $\sigma \approx 2\%$ per year $\implies$ equity premium of only 4% per year even with $\gamma = 100$

Rare Disasters: Economic Environment

- Now suppose that aggregate consumption also contains i.i.d. random variable that represents effect of rare disaster:

$$\ln C_{t+1} = \ln C_t + \mu + \sigma \epsilon_{t+1} + \nu_{t+1},$$

$$\nu_t = \begin{cases} \ln \phi & \text{with probability of } \pi \\ 0 & \text{with probability of } 1 - \pi \end{cases}$$

- As before, use $\pi = 1.7\%$ and $\phi = 0.65$
- Price-dividend ratio for first dividend claim:

$$\frac{P_{1,t}}{D_t} = \theta E_t \left[e^{(1-\gamma)\nu_{t+1}} \right] = \theta \left\{ 1 + \pi \left(\phi^{1-\gamma} - 1 \right) \right\}$$

Rare Disasters: Equity Premium

- Market portfolio has same mean return as first dividend claim:

$$E_t[R_{t+1}] = \frac{D_t}{P_{1,t}} E_t\left[\frac{C_{t+1}}{C_t}\right] = \frac{e^{\mu+\frac{1}{2}\sigma^2}\left\{1+\pi\left(\phi-1\right)\right\}}{\theta\left\{1+\pi\left(\phi^{1-\gamma}-1\right)\right\}}$$

- Can use $\ln(1+x) \approx x$ as long as γ is reasonably small:

$$\ln E_t[R_{t+1}] \approx \rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 + \gamma\sigma^2 - \pi\phi\left(\phi^{-\gamma}-1\right)$$

- Risk-free rate for riskless bond:

$$R_{f,t} = E_t[M_{t+1}]^{-1} = \left\{1+\pi\left(\phi^{-\gamma}-1\right)\right\}^{-1} e^{\rho+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2}$$

Rare Disasters: Equity Premium

- Can also use $\ln(1+x) \approx x$ as long as γ is reasonably small:

$$\ln R_{f,t} \approx \rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 - \pi\left(\phi^{-\gamma} - 1\right)$$

- Hence continuously compounded equity premium:

$$\ln E_t[R_{t+1}] - \ln R_f \approx \gamma\sigma^2 + \pi\left(1-\phi\right)\left(\phi^{-\gamma} - 1\right)$$

- Equity premium of around 7.5% per year for $\gamma = 6$, which represents reasonable degree of (relative) risk aversion:

$$6 \times 0.02^2 + 0.017 \times 0.35 \times \left(0.65^{-6} - 1\right) = 7.5\%$$

Rare Disasters: Problematic Results

- Unfortunately, expected market return is negative for reasonable choice of subjective discount factor:

$$-\ln 0.99 + 0.1152 - 0.017 \times 0.65 \times \left(0.65^{-6} - 1\right) = -1.0\%$$

- Moreover, investors are so concerned about risk of disasters that they will accept large negative risk-free rate:

$$-\ln 0.99 + 0.1128 - 0.017 \times \left(0.65^{-6} - 1\right) = -8.6\%$$

- Rare disasters is not acceptable solution to equity premium puzzle in multi-period endowment economy with complete market and investors with constant relative risk aversion