QF620 Assignment 2

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1.

(a)

$$Z_t = \log(S_t) = f(S_t); \;\; f'(S_t) = \frac{1}{S_t}, \;\; f''(S_t) = -\frac{1}{S_t^2}$$
$$dZ_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

(b)

For S_t,

$$\int_0^t d(\log S_u) = \left(r - \frac{\sigma^2}{2}\right)\int_0^t du + \sigma \int_0^t dWu$$

$$\begin{aligned} S_t &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t} \\ {S_t}^2 &= {S_0}^2 e^{(2r-\sigma^2)t+2\sigma W_t} \end{aligned}$$

(c)

$$\begin{aligned} \mathbb{E}[S_t] &= \mathbb{E}\left[S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}\right] \\ &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t}\mathbb{E}\left[e^{\sigma W_t}\right] \\ &= S_0 e^{rt} \end{aligned}$$

Using MGF for $\mathbb{E}[e^{\sigma W_t}] = e^{\frac{\sigma^2 t}{2}}$.

$$\begin{aligned} \mathbb{E}[{S_t}^2] &= \mathbb{E}\left[{S_0}^2 e^{\left(2r-\frac{\sigma^2}{2}\right)t+2\sigma W_t}\right] \\ &= S_0 e^{(2r-\sigma^2)t}\mathbb{E}\left[e^{2\sigma W_t}\right] \\ &= S_0 e^{(2r+\sigma^2)t} \end{aligned}$$

Using MGF for $\mathbb{E}[e^{2\sigma W_t}] = e^{2\sigma^2 t}$.

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2.

Valuing Digital cash-or-nothing options

Let x^* be x where $S_T = K$.

$$\mathbb{E}[\mathbb{1}_{S_T>K}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathbb{1}_{S_T>K} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\,dx = \Phi\left(-x^*\right)$$

$V_{CashDigital}(0) = e^{-rT}\mathbb{E}[\mathbb{1}_{S_T>K}]$

BS - Digital Cash-or-nothing Options

$$x^{*BS} = \frac{\log\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$V_c^{BS} = e^{-rT}\Phi\left(-x^{*BS}\right)$$

3.

Valuing Digital Asset-or-nothing options

$$\mathbb{E}[S_T\mathbb{1}_{S_T>K}] = \frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_T\mathbb{1}_{S_T>K} e^{-\frac{x^2}{2}}\,dx$$

$V_{AssetDigital}(0) = e^{-rT}\mathbb{E}[S_T\mathbb{1}_{S_T>K}]$

BS - Digital Asset-or-nothing Options

$$\begin{aligned}
V_c^{BS} &= e^{-rT}\mathbb{E}[\mathbb{S}_{S_T>K}] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x-\frac{x^2}{2}}\,dx \\
&= \frac{S_0 e^{-\frac{\sigma^2 T}{2}}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{\sigma\sqrt{T}x-\frac{x^2}{2}}\,dx \\
&= S_0\Phi(-x^* + \sigma\sqrt{T}) \\
\\
&= S_0\Phi(d_1)
\end{aligned}$$

$$d_1 = \frac{\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$