Efficient Frontier

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Outline

1. Minimum-Variance Frontier

2. Frontier with Riskless Asset

3. Constant Absolute Risk Aversion

Asset Allocation vs Asset Pricing

- Asset allocation (or portfolio choice) is theory of how investor allocates wealth amongst financial assets
- For simplicity, assume investor is "price taker" $\implies$ scale of investment is small enough to not affect prices
- Also for simplicity, assume perfect "frictionless" financial market with no taxes or transaction costs, etc.
- Asset allocation goes hand-in-hand with asset pricing: if we know how all investors choose to allocate their wealth, then we can find equilibrium prices to balance supply and demand
- Harry Markowitz pioneered modern concept of asset allocation with theory of mean-variance-efficient frontier in 1952

Investment Environment

- Financial market consists of $n \geq 2$ risky tradable assets with normal returns (and no riskless asset)
- Let $\mathbf{R} = (R_1, \ldots, R_n)'$ be $n \times 1$ vector of expected returns
- Let $\mathbf{V}$ be $n \times n$ covariance matrix of returns, which consists of variances on diagonal and covariances on off-diagonal
- $\mathbf{V}$ must be symmetric: $\mathbf{V}' = \mathbf{V}$, and positive definite: $\mathbf{z}'\mathbf{V}\mathbf{z} > 0$ for any non-zero $n \times 1$ vector $\mathbf{z}$
- Assume no redundant assets, so returns must be linearly independent and covariance matrix must be invertible:

$$\exists \ \mathbf{V}^{-1} \text{ such that } \mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$$

- Then $\mathbf{V}^{-1}$ is also symmetric and positive definite

Portfolio Weights

- Let $\mathbf{w} = (w_1, \ldots, w_n)'$ be $n \times 1$ vector of portfolio weights, which represents proportion of investor's wealth allocated to each tradable financial asset

- No restriction on individual portfolio weights: positive weight indicates normal investment (or "long position") while negative weight indicates short-selling (or "short position")

- Only restriction is that portfolio weights must sum to one: $\mathbf{w}'\mathbf{e} = 1$, where $\mathbf{e} = (1, \ldots, 1)'$ is $n \times 1$ unit vector

- Investor can allocate more than available wealth into any individual financial asset, but cannot allocate more than available wealth in aggregate

- Then $\mathbf{w}'\mathbf{R}$ is expected return for investor's portfolio, and $\mathbf{w}'\mathbf{V}\mathbf{w} > 0$ is variance of return for investor's portfolio

Asset Allocation Problem – Part 1

- Investor's ultimate goal is to create "optimal" portfolio that maximises expected utility (of wealth)

- Requires knowledge of investor's utility function, which is difficult to observe in reality

- Since asset returns have normal distribution, investor's expected utility depends only on mean return and variance of return for investor's portfolio

- Risk-averse investor will always prefer higher mean return and lower variance of return

- Hence simpler problem is to identify all "efficient" or "frontier" portfolios that minimise risk for given mean return, which includes specific optimal portfolio for any risk-averse investor

Asset Allocation Problem – Part 2

- Portfolio weights for portfolio with mean return of R_p must satisfy two conditions:
  - Portfolio weights must sum to one: $\mathbf{w}'\mathbf{e} = 1$
  - Portfolio must have mean return of R_p: $\mathbf{w}'\mathbf{R} = R_p$
- Out of all portfolios that satisfy these two conditions, frontier portfolio has lowest variance of return: $\mathbf{w}'\mathbf{V}\mathbf{w}$
- In mathematical terms, finding portfolio weights for frontier portfolio is constrained minimisation problem with quadratic objective function and equality constraints
- Fortunately, this minimisation problem is guaranteed to have unique solution that is global minimum

Asset Allocation Problem – Part 3

- Solve constrained minimisation problem by adding Lagrange multipliers to objective function (or "Langragian") to ensure that equality constraints are satisfied:

$$\min_{\{\mathbf{w},\lambda,\gamma\}} \mathcal{L} = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left(R_p - \mathbf{w}'\mathbf{R}\right) + \gamma\left(1 - \mathbf{w}'\mathbf{e}\right)$$

- Here λ and γ are Lagrange multipliers for equality constraints on mean return and portfolio weights, respectively
- From economic perspective, Lagrange multiplier represents marginal cost ("shadow price") of relaxing constraint
- Alternative "dual problem" is to maximise mean return for specified variance of return, but more difficult to solve

Frontier Portfolios – Part 1

- Set partial derivative of Lagrangian to zero to get first-order optimality condition for portfolio weights of frontier portfolio:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \mathbf{V}\mathbf{w}^* - \lambda \mathbf{R} - \gamma \mathbf{e} = 0$$

- Use other first-order optimality conditions (for Lagrange multipliers) to confirm that equality constraints are satisfied:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = R_p - \mathbf{w}'\mathbf{R} = 0 \implies \mathbf{w}'\mathbf{R} = R_p$$
$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - \mathbf{w}'\mathbf{e} = 0 \implies \mathbf{w}'\mathbf{e} = 1$$

Frontier Portfolios – Part 2

- Pre-multiply first-order optimality condition for frontier portfolio weights by $\mathbf{V}^{-1}$ and rearrange:

$$\mathbf{V}\mathbf{w}^* - \lambda\mathbf{R} - \gamma\mathbf{e} = 0 \implies \mathbf{w}^* = \lambda\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{V}^{-1}\mathbf{e}$$

- Pre-multiply by $\mathbf{R}'$ and apply constraint for mean return:

$$\mathbf{R}'\mathbf{w}^* = \lambda\mathbf{R}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{R}'\mathbf{V}^{-1}\mathbf{e} = R_p$$

- Pre-multiply by $\mathbf{e}'$ and apply constraint for portfolio weights:

$$\mathbf{e}'\mathbf{w}^* = \lambda\mathbf{e}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{e}'\mathbf{V}^{-1}\mathbf{e} = 1$$

Frontier Portfolios – Part 3

- Solve simultaneous equations to find Lagrange multipliers:

$$\lambda = \frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2}; \qquad \gamma = \frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2}$$

- Here α is scalar, while ζ and δ are strictly positive scalars:

$$\alpha = \mathbf{R}'\mathbf{V}^{-1}\mathbf{e}; \qquad \zeta = \mathbf{R}'\mathbf{V}^{-1}\mathbf{R}; \qquad \delta = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

- Confirm that denominator is strictly positive:

$$(\alpha\mathbf{R} - \zeta\mathbf{e})'\,\mathbf{V}^{-1}\,(\alpha\mathbf{R} - \zeta\mathbf{e}) = \zeta\left(\zeta\delta - \alpha^2\right) > 0$$

Frontier Portfolios – Part 4

- Substitute for λ and γ to find portfolio weights for frontier portfolio with mean return of R_p:

$$\mathbf{w}^* = \left(\frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2}\right)\mathbf{V}^{-1}\mathbf{R} + \left(\frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2}\right)\mathbf{V}^{-1}\mathbf{e}$$

- Rearrange to get linear relationship: $\mathbf{w}^* = \mathbf{a} + \mathbf{b}R_p$, where:

$$\mathbf{a} = \frac{\zeta\mathbf{V}^{-1}\mathbf{e} - \alpha\mathbf{V}^{-1}\mathbf{R}}{\zeta\delta - \alpha^2}; \qquad \mathbf{b} = \frac{\delta\mathbf{V}^{-1}\mathbf{R} - \alpha\mathbf{V}^{-1}\mathbf{e}}{\zeta\delta - \alpha^2}$$
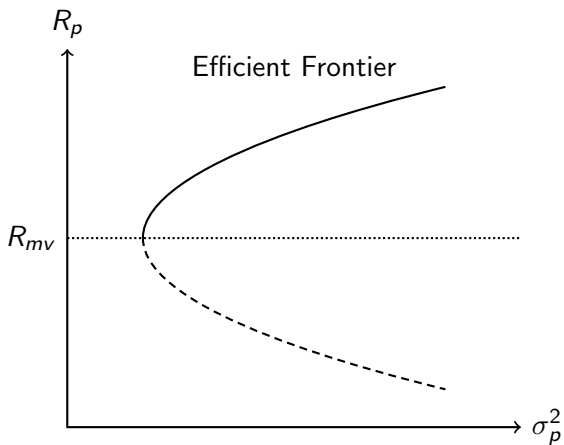
- Minimum-variance frontier consists of portfolios with lowest amount of risk, for different values of R_p

Efficient Frontier – Part 1

- Variance of return for frontier portfolio:

$$
\sigma_p^2 = (\mathbf{w}^*)' \, \mathbf{V} \mathbf{w}^* = (\mathbf{a} + \mathbf{b}R_p)' \, \mathbf{V} \, (\mathbf{a} + \mathbf{b}R_p)
$$
$$
= \frac{\delta R_p^2 - 2\alpha R_p + \zeta}{\zeta\delta - \alpha^2}
$$
$$
= \frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2} \, (R_p - R_{mv})^2
$$

- $R_{mv} = \frac{\alpha}{\delta}$ is mean return for global minimum-variance portfolio
- Minimum-variance frontier is parabola when plotted with variance of return on y-axis and expected return on x-axis
- Standard practice to flip axes, as shown on next slide

R_p

Efficient Frontier

R_{mv}

σ_p^2

Efficient Frontier – Part 2

- Top half of minimum-variance frontier (where $R_p \geq R_{mv}$) is known as efficient frontier, consisting of portfolios with highest mean return for given amount of risk

- If use standard deviation of return (instead of variance of return) on x-axis, then minimum-variance frontier is hyperbola with center at $(0, R_{mv})$ and asymptotes:

$$R_p = R_{mv} \pm \left(\zeta - \frac{\alpha^2}{\delta} \right)^{\frac{1}{2}} \sigma_p$$

- To maximise expected utility, investor will choose optimal portfolio where indifference curve is tangent to frontier

Efficient Frontier

Expected Return

Std Dev of Return
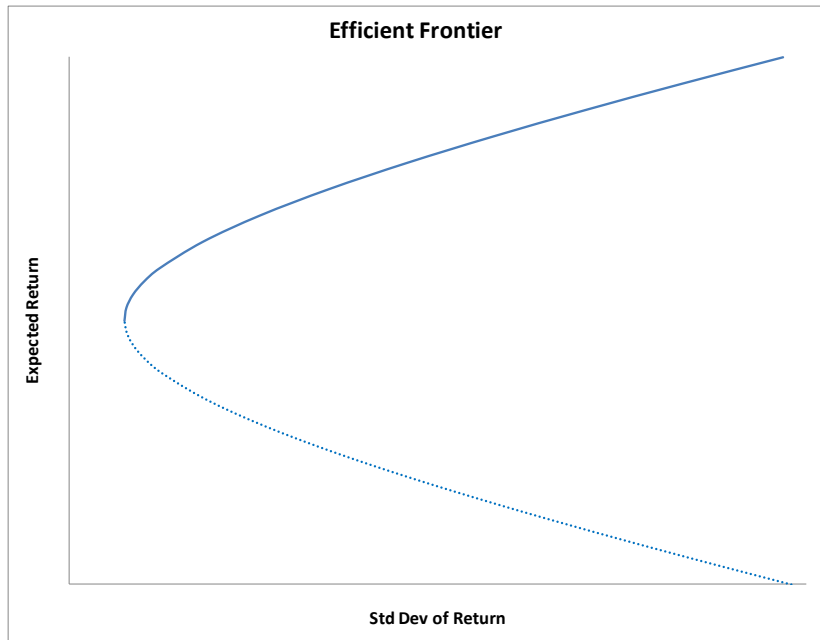
Portfolio Separation – Part 1

- Affine combination is linear combination with (positive or negative) coefficients that sum to one, so affine combination of two portfolios is also portfolio:

$$\left[\kappa \mathbf{w}_1 + (1 - \kappa)\,\mathbf{w}_2\right]' \mathbf{e} = \kappa \mathbf{w}_1' \mathbf{e} + (1 - \kappa)\,\mathbf{w}_2' \mathbf{e} = 1$$

- If p_1 and p_2 are both frontier portfolios, then affine combination of p_1 and p_2 is also frontier portfolio:

$$\begin{aligned}
\kappa \mathbf{w}_1^* + (1 - \kappa)\,\mathbf{w}_2^* &= \kappa\left(\mathbf{a} + \mathbf{b}R_{p_1}\right) + (1 - \kappa)\left(\mathbf{a} + \mathbf{b}R_{p_2}\right) \\
&= \mathbf{a} + \mathbf{b}\left[\kappa R_{p_1} + (1 - \kappa)\,R_{p_2}\right] \\
&= \mathbf{a} + \mathbf{b}R_{p_3}
\end{aligned}$$

Portfolio Separation – Part 2

- Here p_3 is frontier portfolio with mean return of
  $R_{p_3} = \kappa R_{p_1} + (1 - \kappa) R_{p_2}$
- No restriction on value of κ, so investor can generate entire minimum-variance frontier with different affine combinations of any two frontier portfolios
- Two-fund (or mutual fund) separation theorem: investor can construct optimal portfolio with appropriate affine combination of any two frontier portfolios
- In theory, more convenient since investor can generate efficient frontier and construct optimal portfolio without knowing **R** and **V** for individual risky assets

Orthogonal Frontier Portfolios – Part 1

- Covariance of return between two frontier portfolios:

$$
\begin{aligned}
\left(\mathbf{w}_1^*\right)' \mathbf{V} \mathbf{w}_2^* &= \left(\mathbf{a} + \mathbf{b} R_{p_1}\right)' \mathbf{V} \left(\mathbf{a} + \mathbf{b} R_{p_2}\right) \\
&= \frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2} \left(R_{p_1} - R_{mv}\right)\left(R_{p_2} - R_{mv}\right)
\end{aligned}
$$

- For given p_1, set covariance to zero to find mean return for p_2, which is frontier portfolio that is "orthogonal" to p_1:

$$
R_{p_2} = R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2 \left(R_{p_1} - R_{mv}\right)}
$$

- If p_1 is efficient, then p_2 must be "inefficient" (and vice versa)

Orthogonal Frontier Portfolios – Part 2

- Can measure slope at any point of minimum-variance frontier:

$$\frac{\partial R_p}{\partial \sigma_p} = \frac{\zeta\delta - \alpha^2}{\delta\left(R_p - R_{mv}\right)}\sigma_p$$

- Evaluate at (σ_{p_1}, R_{p_1}) to get slope of minimum-variance frontier at point corresponding to specific frontier portfolio p_1

- Hence equation for line (with unknown y-intercept of R_0) that is tangent to frontier at point corresponding to p_1:

$$R_p = R_0 + \left[\frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}\right]\sigma_p$$

Orthogonal Frontier Portfolios – Part 3

- Evaluate at (σ_{p_1}, R_{p_1}) and solve for y-intercept:

$$
\begin{aligned}
R_0 &= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}^2 \\
&= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\left[\frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2}\left(R_{p_1} - R_{mv}\right)^2\right] \\
&= R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2\left(R_{p_1} - R_{mv}\right)} \\
&= R_{p_2}
\end{aligned}
$$

- Hence y-intercept for tangent line at p_1 also shows mean return for frontier portfolio that is orthogonal to p_1

Outline

1. Minimum-Variance Frontier

2. Frontier with Riskless Asset

3. Constant Absolute Risk Aversion

Investment Environment with Riskless Asset

- Financial market consists of $n \geq 2$ risky assets (with normal returns) and riskless asset with risk-free rate of R_f
- Let **w** be vector of portfolio weights for risky assets, so that $1 - \mathbf{w}'\mathbf{e}$ is proportion of wealth invested in riskless asset
- If $\mathbf{w}'\mathbf{e} < 1$, then investor is lending money (to other investors, through bank) at risk-free rate
- If $\mathbf{w}'\mathbf{e} > 1$, then investor is borrowing money (from other investors, through bank) at risk-free rate
- Expected return for investor's portfolio:

$$\mathbf{w}'\mathbf{R} + \left(1 - \mathbf{w}'\mathbf{e}\right) R_f = R_f + \mathbf{w}' \left(\mathbf{R} - R_f \mathbf{e}\right)$$

- Here $\mathbf{R} - R_f \mathbf{e}$ represents $n \times 1$ vector of risk premiums

Asset Allocation with Riskless Asset – Part 1

- Lagrangian for asset allocation problem:

$$\min_{\{\mathbf{w}, \lambda\}} \mathcal{L} = \frac{1}{2} \mathbf{w}' \mathbf{V} \mathbf{w} + \lambda \left[R_p - R_f - \mathbf{w}' \left(\mathbf{R} - R_f \mathbf{e} \right) \right]$$

- Use optimality condition to find frontier portfolio weights:

$$\mathbf{V} \mathbf{w}^* - \lambda \left(\mathbf{R} - R_f \mathbf{e} \right) = 0 \implies \mathbf{w}^* = \lambda \mathbf{V}^{-1} \left(\mathbf{R} - R_f \mathbf{e} \right)$$

- Use expected return to solve for Lagrange multiplier:

$$R_p = R_f + \lambda \left(\mathbf{R} - R_f \mathbf{e} \right)' \mathbf{V}^{-1} \left(\mathbf{R} - R_f \mathbf{e} \right)$$
$$= R_f + \lambda \left(\zeta - 2\alpha R_f + \delta R_f^2 \right) \implies \lambda = \frac{R_p - R_f}{\zeta - 2\alpha R_f + \delta R_f^2}$$

Asset Allocation with Riskless Asset – Part 2

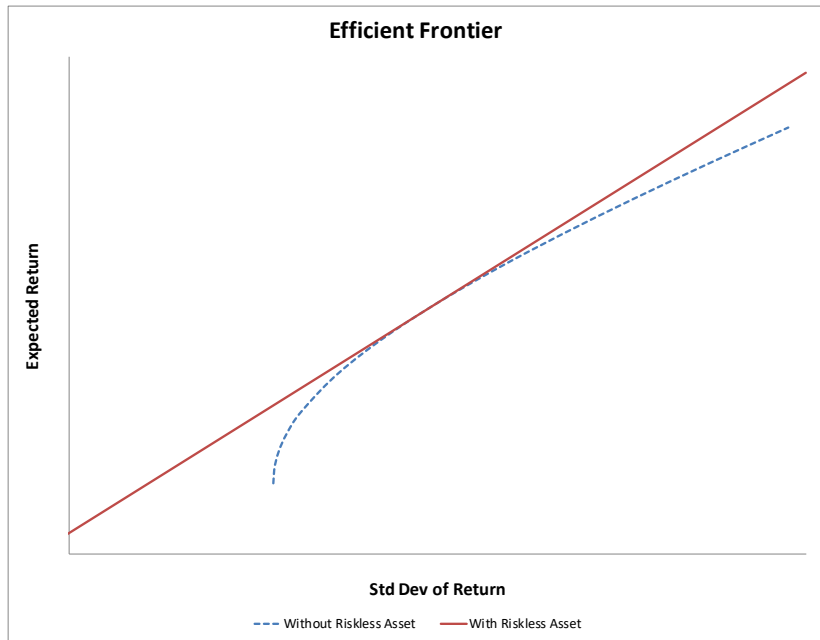- Variance of return for portfolio on minimum-variance frontier:

$$\sigma_p^2 = (\mathbf{w}^*)' \mathbf{V} \mathbf{w}^* = \lambda^2 (\mathbf{R} - R_f \mathbf{e})' \mathbf{V}^{-1} (\mathbf{R} - R_f \mathbf{e})$$
$$= \frac{(R_p - R_f)^2}{\zeta - 2\alpha R_f + \delta R_f^2}$$

- Minimum-variance frontier in (σ_p, R_p)-space:

$$R_p = R_f \pm \left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}} \sigma_p$$

- Hence efficient frontier consists of straight line with
  y-intercept of R_f and positive slope of $\left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}}$

Efficient Frontier

Expected Return

Std Dev of Return

- - - Without Riskless Asset —— With Riskless Asset

Portfolio Separation with Riskless Asset – Part 1

- If $R_f < R_{mv} = \frac{\alpha}{\delta}$, then efficient frontier (with riskless asset) is tangent to top half of risky-asset-only frontier

- To verify this result, consider adding riskless asset to existing frontier generated by n risky assets

- Capital allocation line (CAL) is line in (σ_p, R_p)-space joining riskless asset to any risky portfolio

- CAL shows mean return and std dev of return for different affine combinations of risky portfolio and riskless asset

- Slope of CAL shows Sharpe ratio for all affine combinations

- "Tangency" portfolio is unique risky portfolio where CAL is tangent to existing frontier generated by n risky assets

Portfolio Separation with Riskless Asset – Part 2

- Tangency portfolio must have highest Sharpe ratio out of all possible risky portfolios
- Set y-intercept to R_f, and use previous result for equation of tangent line to get mean return for tangency portfolio:

$$R_{tg} = R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2 (R_f - R_{mv})} = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha}$$

- Hence risk premium for tangency portfolio:

$$R_{tg} - R_f = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha} - R_f = -\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta (R_f - R_{mv})}$$

Portfolio Separation with Riskless Asset – Part 3

- Variance of return for tangency portfolio:

$$
\sigma_{tg}^2 = \frac{1}{\delta} + \frac{\delta \left(R_{tg} - R_{mv}\right)^2}{\zeta\delta - \alpha^2} = \frac{1}{\delta} + \frac{\zeta\delta - \alpha^2}{\delta^3 \left(R_f - R_{mv}\right)^2}
$$
$$
= \frac{1}{\delta}\left[1 + \frac{\zeta\delta - \alpha^2}{\left(\delta R_f - \alpha\right)^2}\right] = \frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta^2 \left(R_f - R_{mv}\right)^2}
$$

- Choose positive square root to get std dev of return:

$$
\sigma_{tg} = -\frac{\left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}}{\delta \left(R_f - R_{mv}\right)}
$$

Portfolio Separation with Riskless Asset – Part 4

- Hence slope of CAL and Sharpe ratio for tangency portfolio:

$$
\frac{R_{tg} - R_f}{\sigma_{tg}} = \left[-\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta \left(R_f - R_{mv} \right)} \right] \left[-\frac{\delta \left(R_f - R_{mv} \right)}{\left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}}} \right]
$$
$$
= \left(\zeta - 2\alpha R_f + \delta R_f^2 \right)^{\frac{1}{2}}
$$

- Confirms that CAL for tangency portfolio is efficient frontier (with riskless asset), since risk-averse investors prefer to hold risky portfolio with highest possible "reward-to-risk" ratio
- Opposite result for $R_f > R_{mv}$: tangency portfolio lies on bottom half of existing frontier and has lowest Sharpe ratio

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3. Constant Absolute Risk Aversion

CARA Utility: Economic Environment

- Financial market consists of $n \geq 2$ risky assets (with normal returns) and riskless asset with risk-free rate of R_f

- Let $\tilde{\mathbf{R}}$ be $n \times 1$ vector of asset returns, so portfolio return:

$$\tilde{R}_p = R_f + \mathbf{w}' \left(\tilde{\mathbf{R}} - R_f \mathbf{e} \right)$$

- Investor has constant absolute risk aversion, so let $b_r = bW_0$ be coefficient of relative risk aversion (at initial wealth)

- Hence investor's utility of (random) final wealth:

$$U\left(\tilde{W} \right) = -e^{-b\tilde{W}} = -e^{-b_r \frac{\tilde{W}}{W_0}} = -e^{-b_r \tilde{R}_p}$$

CARA Utility: Asset Allocation

- Asset returns have joint normal distribution, so utility of final wealth has lognormal distribution:

$$E\left[U\left(\tilde{W}\right)\right] = E\left[-e^{-b_r \tilde{R}_p}\right]$$
$$= -e^{-b_r\left[R_f + \mathbf{w}'(\mathbf{R} - R_f \mathbf{e})\right] + \frac{1}{2} b_r^2 \mathbf{w}' \mathbf{V} \mathbf{w}}$$

- Exponential function is monotonically increasing, so minimise (negative) exponent to maximise expected utility:

$$\max_{\mathbf{w}} \left\{ \mathbf{w}'\left(\mathbf{R} - R_f \mathbf{e}\right) - \frac{1}{2} b_r \mathbf{w}' \mathbf{V} \mathbf{w} \right\}$$

CARA Utility: Optimal Portfolio

- Use optimality condition to find optimal portfolio weights:

$$\mathbf{R} - R_f \mathbf{e} - b_r \mathbf{V} \mathbf{w}^* = 0 \implies \mathbf{w}^* = \frac{1}{b_r} \mathbf{V}^{-1} \left(\mathbf{R} - R_f \mathbf{e} \right)$$

- Pre-multiply by $W_0 \mathbf{e}'$ to find absolute (dollar) amount of wealth invested in risky assets:

$$W_0 \mathbf{e}' \mathbf{w}^* = \frac{1}{b} \left(\alpha - \delta R_f \right)$$

- Notice that W_0 doesn't show up on RHS $\implies$ all investors with same coefficient of absolute risk aversion will invest same (dollar) amount in risky assets, regardless of initial wealth