Geometry Processing

2 Discrete Differential Geometry

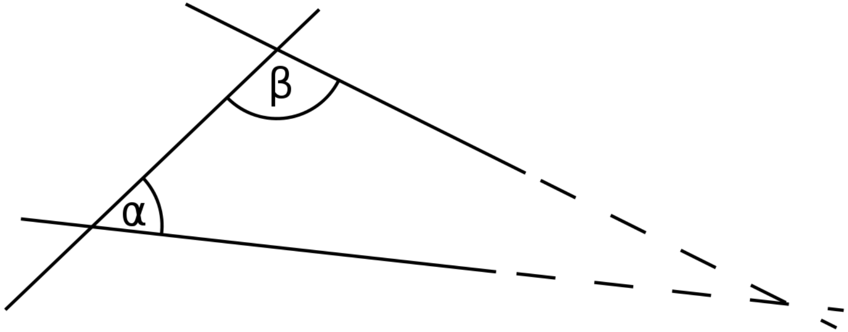
Ludwig-Maximilians-Universität München

Session 2: Discrete Differential Geometry

- Motivation
- Discrete Geometric Quantifies
  - Normals
  - Curvature
  - Laplace-Beltrami
- Summary
- Discussion
  - .OBJ Mesh Loader
  - Blender Python APIs
  - Blender BMesh Structure

Euclidean v.s. Non Euclidean: Parallel Postulate

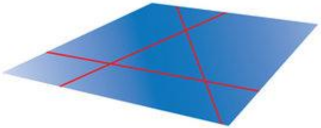
"In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point."



$$\alpha + \beta = \pi$$

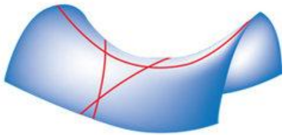
Euclid

Geometry principles works differently on curved spaces...



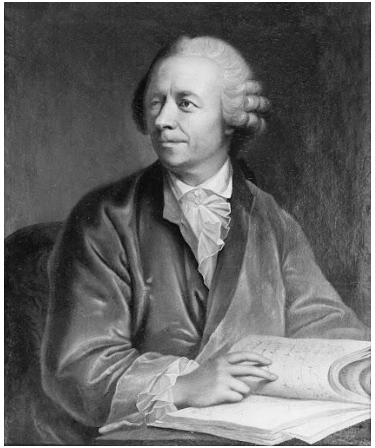
Euclidean

Elliptic



Hyperbolic

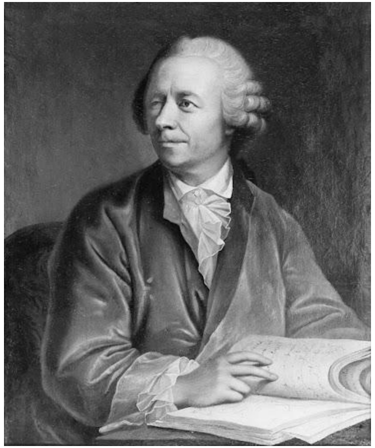
Differential Geometry



Leonhard
Euler

Carl
Gauss

Differential Geometry



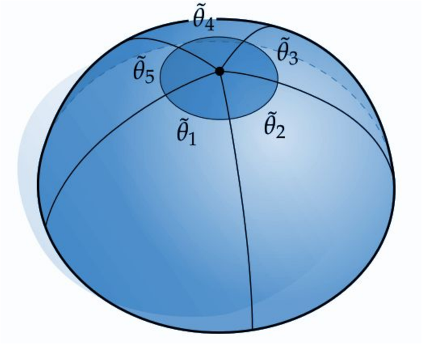
Leonhard
Euler

Carl
Gauss

Bernhard
Riemann

Example: Smooth Settings

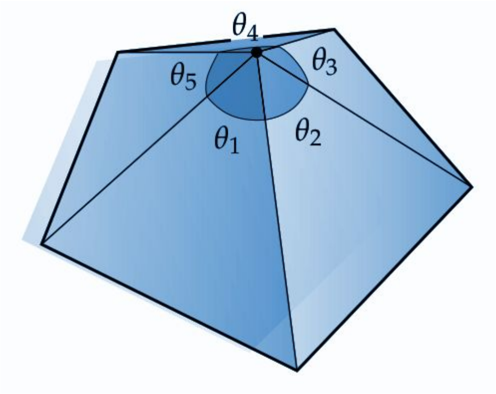
In smooth settings, the sum of the tip angle is always 2π



$$\sum \theta_i = 2\pi$$

Example: Discrete Settings

In discrete settings, the sum of the tip angle is not 2π but approximately 2π if we have infinite tessellated triangles

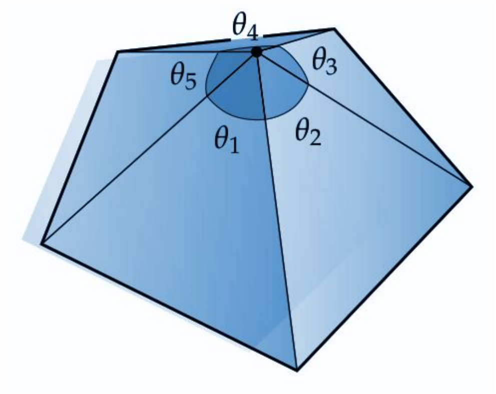


$$\sum \theta_i < 2\pi$$

(why?)

Example: Discrete Settings

In discrete settings, the sum of the tip angle is not 2π but approximately 2π if we have infinite tessellated triangles



$$\cancel{\sum \theta_i < 2\pi}$$

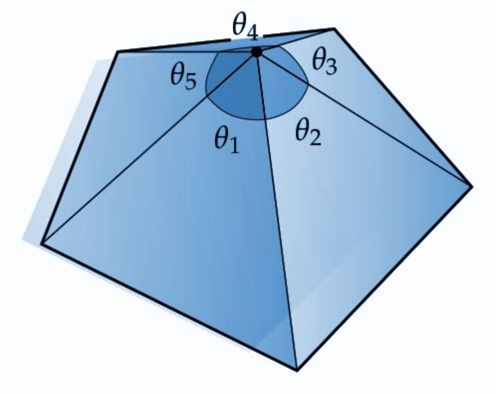
Redefine $\hat{\theta}_j = \theta_j \frac{2\pi}{\sum_i \theta_i}$

$$\Rightarrow \sum_j \hat{\theta}_j = \sum_j \theta_j \frac{2\pi}{\sum_i \theta_i} = \frac{2\pi}{\sum_i \theta_i} \sum_j \theta_j = 2\pi$$

By redefining the meaning of "angle", we preserved the geometric property that the sum of the tip angle is 2π

Example: Discrete Settings

In discrete settings, the sum of the tip angle is not 2π but approximately 2π if we have infinite tessellated triangles



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Caution (Ambiguity of Interpretation):

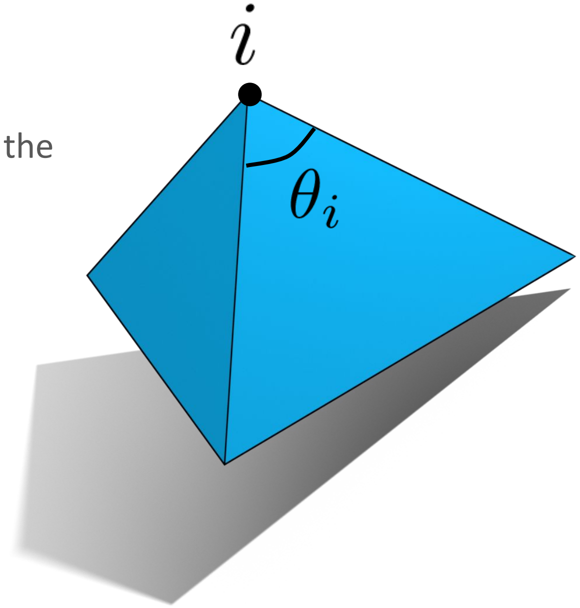
We are assuming the mesh is representing a smooth surface, but what if it is intended to represent a "hard" surface?

Angle Defect

Important idea: Represent how flatten (or how curved) around a vertex *i*

The ***angle defect*** at a vertex is the deviation of the sum of interior angles from the Euclidean angle sum of 2π:

$$\Omega_i = 2\pi - \sum \theta_i$$



Angle Defect

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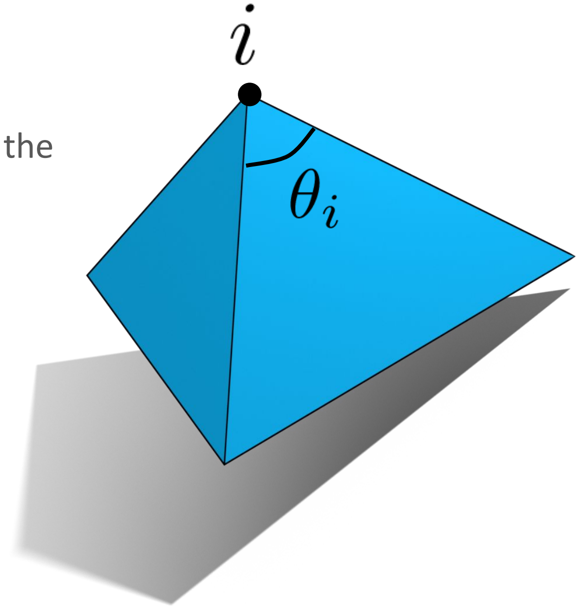

Discrete ***Gauss-Bonnet Theorem***:

For a simplicial surface, the total angle defect is

$$\sum_i \Omega_i = 2\pi\chi$$

(Euler Characteristic)

E.g. Given a convex polyhedra, the total angle defect is 4π (Try to verify this)

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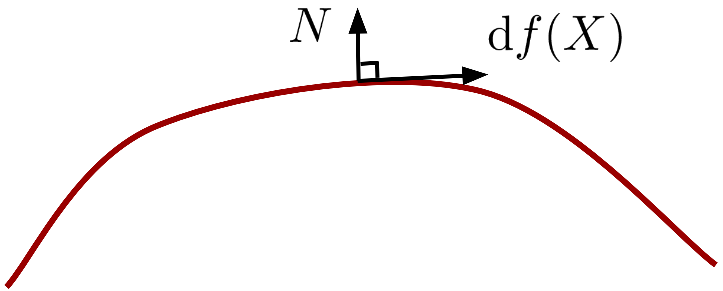
Normals

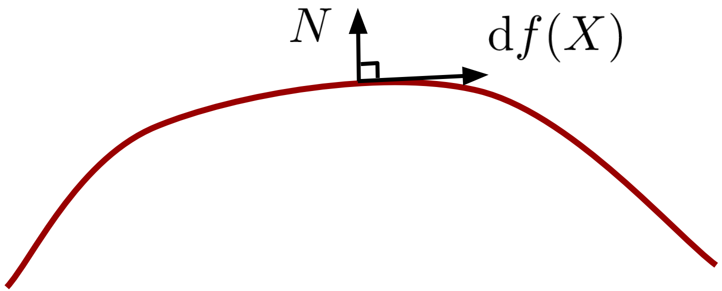
On a curve: *A vector is **normal** to a surface if it is orthogonal to all tangent vectors*

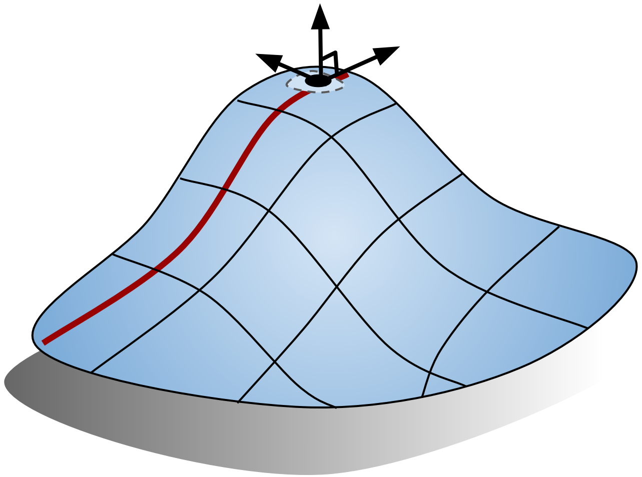


Normals

On a curve: *A vector is **normal** to a surface if it is orthogonal to all tangent vectors*

On a surface: *A normal is a unit vector along with the cross product of any given two tangent vectors*





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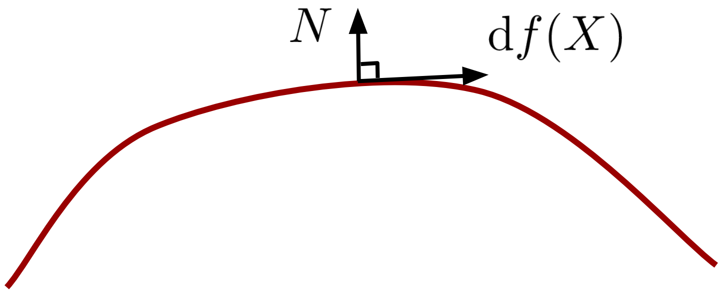
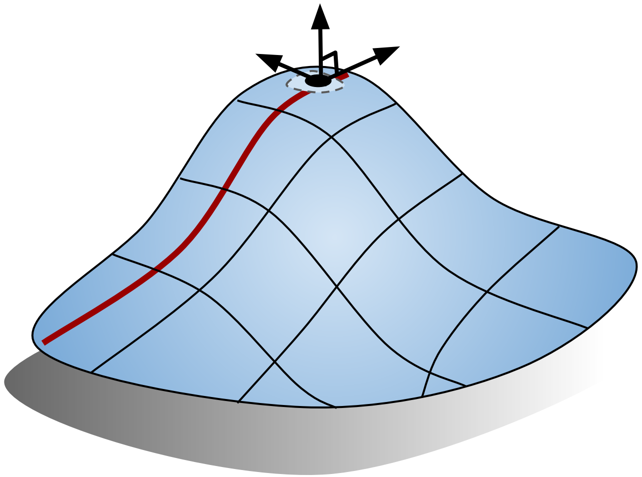
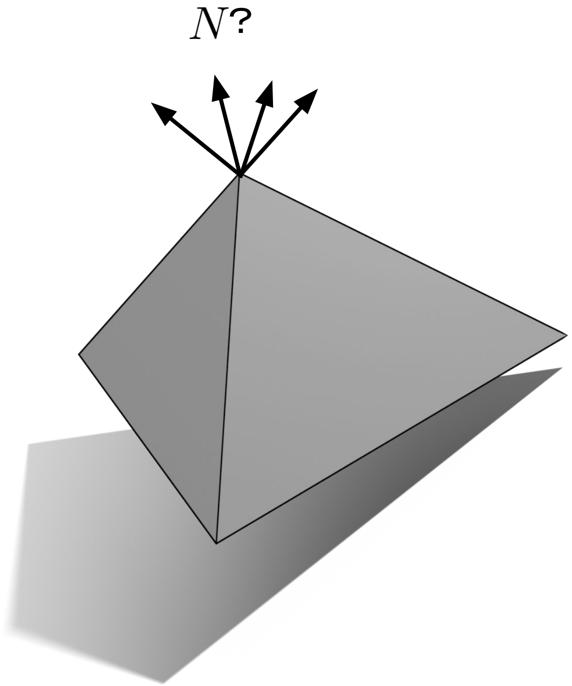


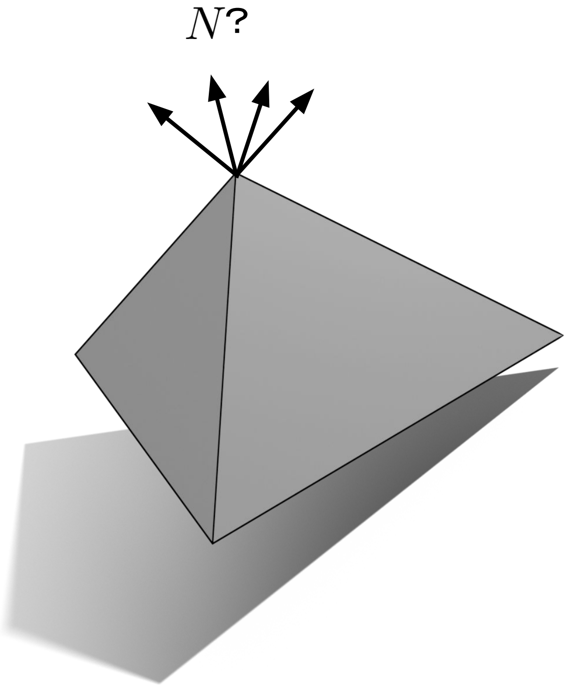


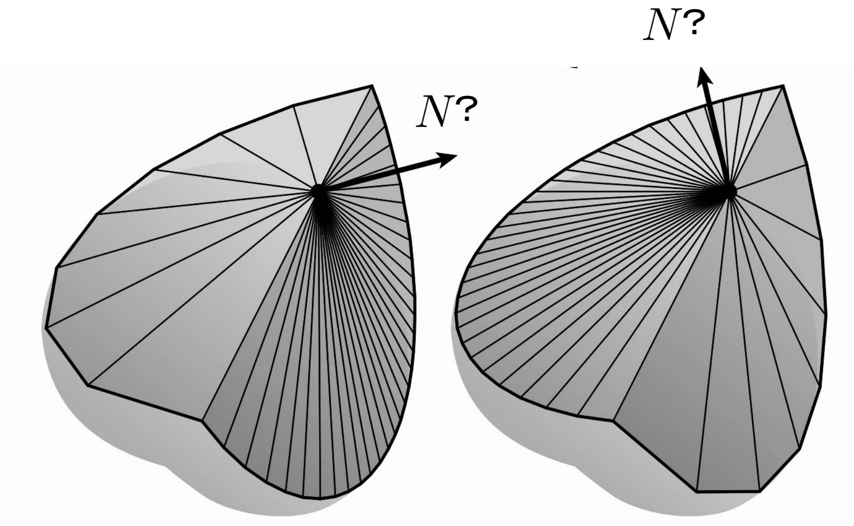
Q: How to discretize the definition on polygonal meshes?

Discrete Normals



Discrete Normals

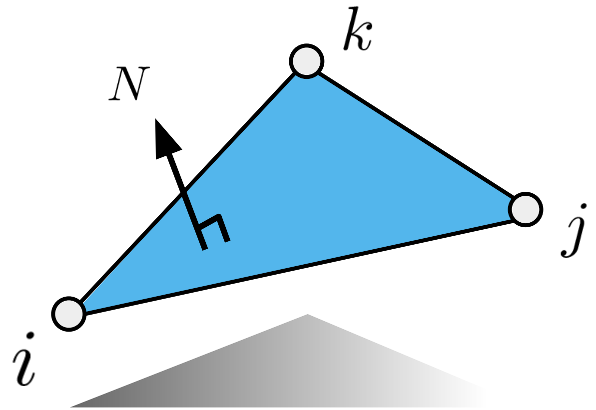




Face Normal

Face normals are well-defined:

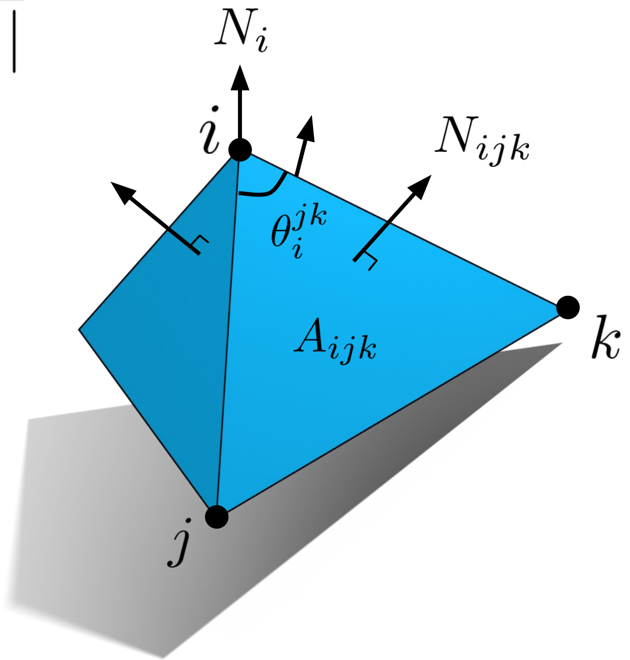
$$N = \frac{(f_j - f_i) \times (f_k - f_i)}{||(f_j - f_i) \times (f_k - f_i)||}$$



Vertex Normal

Basic idea: weighted average of the normal vectors of incident faces

$$N_i = \frac{\sum_i w_{ijk} N_{ijk}}{||\sum_i w_{ijk} N_{ijk}||}$$



Variances:

- Uniform (or Equally) Weighted $w_{ijk} = 1$

Vertex Normal

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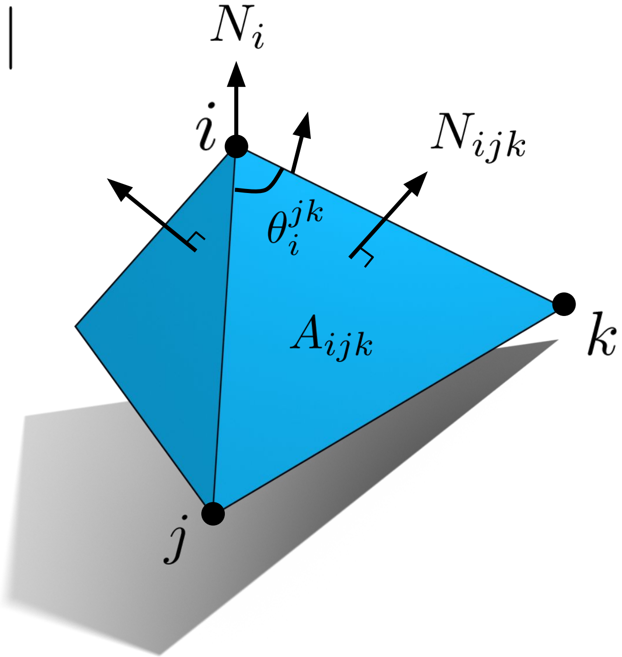

Variances:

- Uniform (or Equally) Weighted $w_{ijk} = 1$
- Area Weighted $w_{ijk} = A_{ijk}$

Vertex Normal

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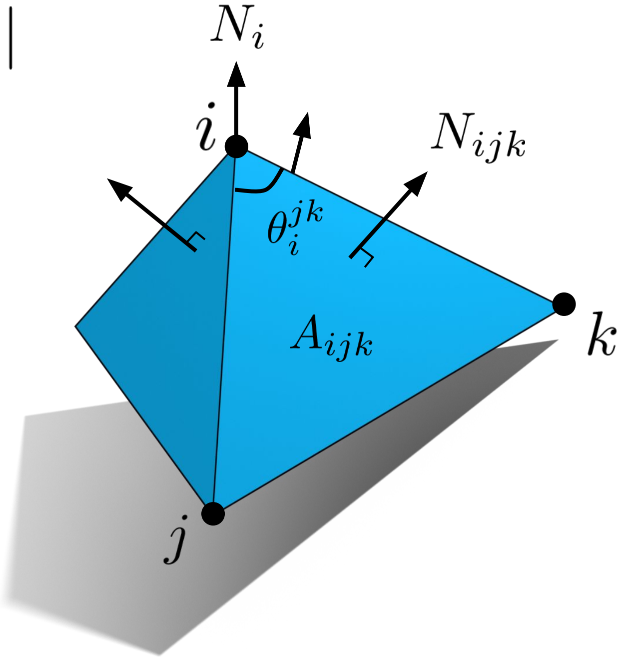


Variances:

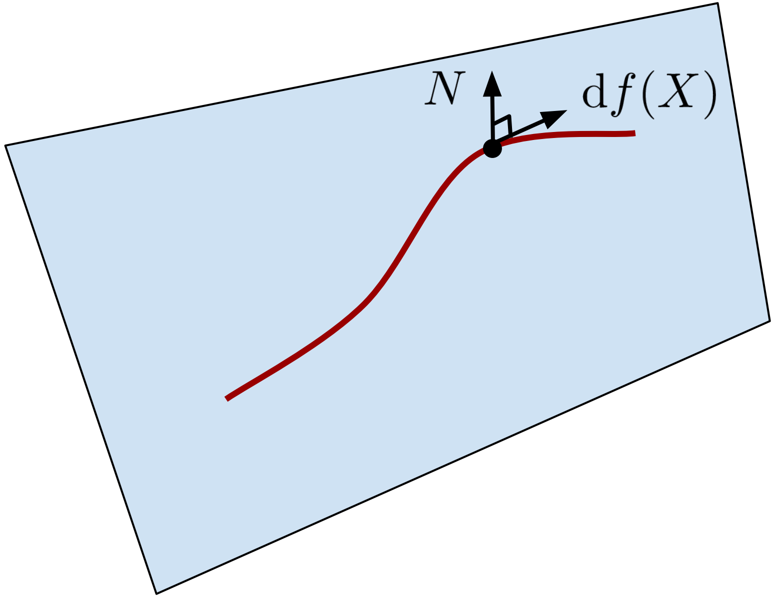
- Uniform (or Equally) Weighted $w_{ijk} = 1$
- **Area Weighted** $w_{ijk} = A_{ijk}$
- **Angle Weighted** $w_{ijk} = \theta_i^{jk}$
- …

Caution: face normal v.s. vertex normal v.s. normal interpolation

Curvature

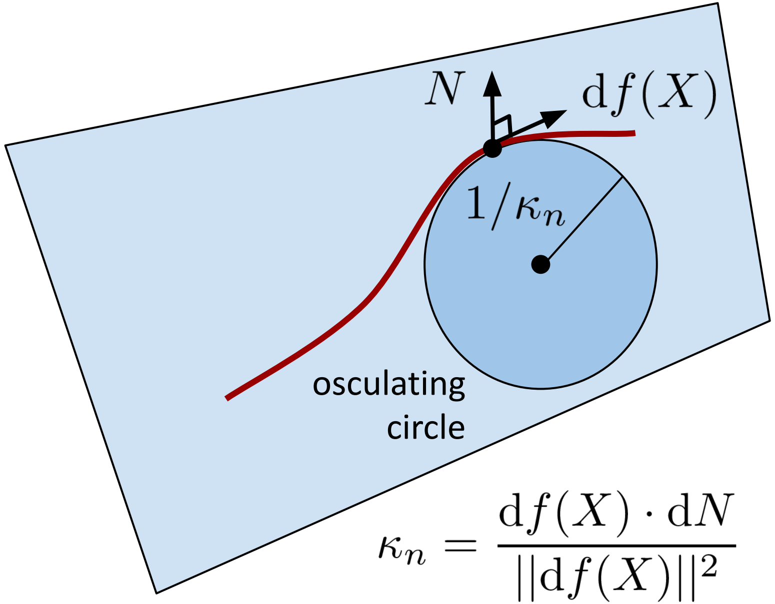
Intuitively, curvature describes "how much a curve bends", or

the rate of change in the tangent, or *second derivative*

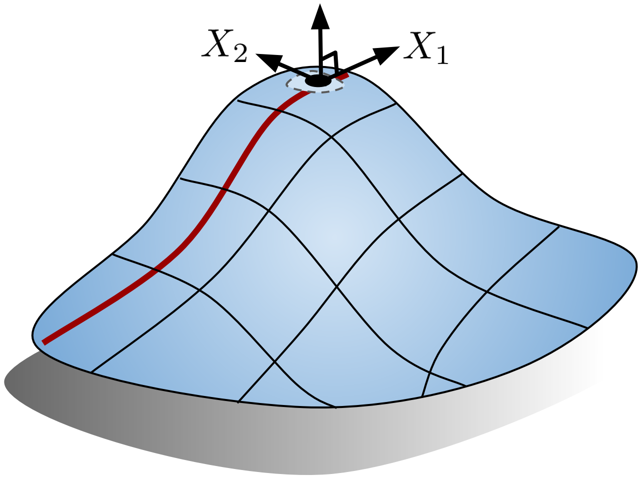


Normal Curvature κ_n

The rate at which normal is bending along a given tangent direction



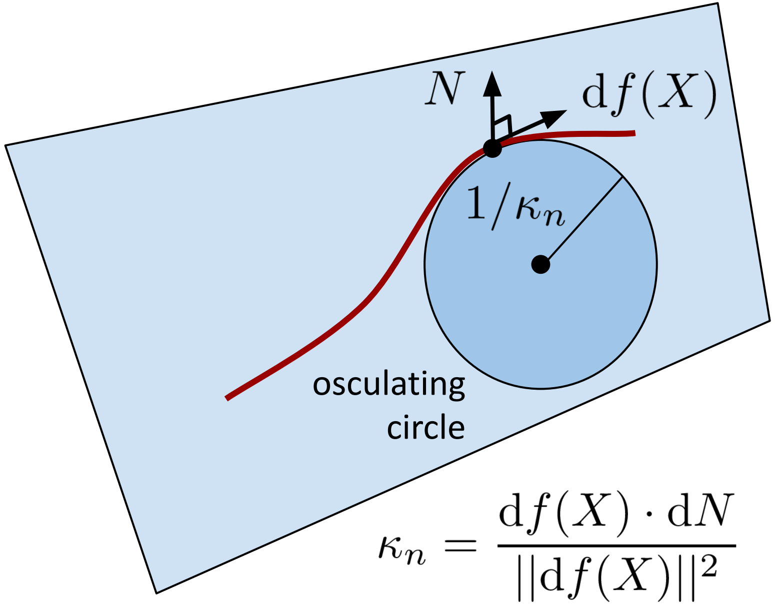
$$\kappa_n = \frac{\mathrm{d}f(X) \cdot \mathrm{d}N}{||\mathrm{d}f(X)||^2}$$



$$N = \frac{X_1 \times X_2}{||X_1 \times X_2||}$$

Normal Curvature κ_n

The rate at which normal is bending along a given tangent direction



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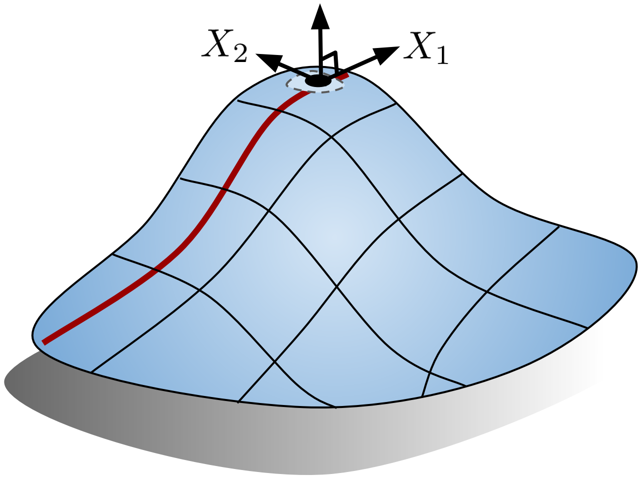
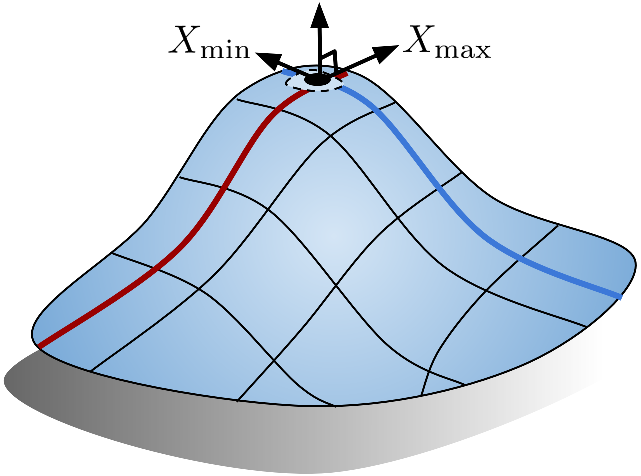


Q: ***which direction does the surface bend the most?***

Principal Curvature $\kappa_{\min}, \kappa_{\max}$

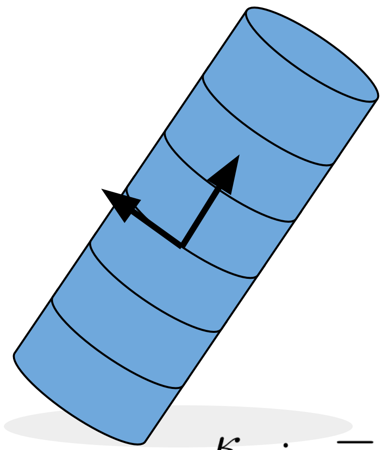
- **Principal directions:** Axes that describe the direction along which the normal changes the most/least
- **Principal curvatures:** along all directions, the two principal directions where normal curvature has minimum and maximum value respectively



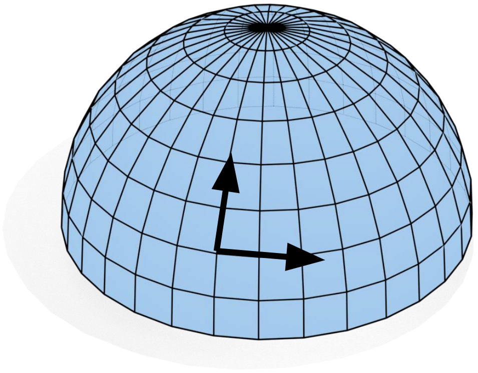
Some facts:

- (Euler's Theorem) principal directions are orthogonal
- $\mathrm{d}N = \kappa \mathrm{d}f(X)$

Principal Curvature: Examples



$$\kappa_{\min} = 0, \kappa_{\max} = 1/r$$



"Umbilic Points"

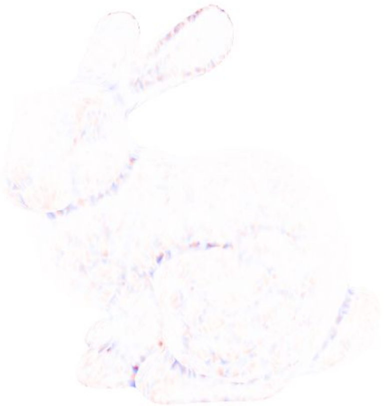
$$\kappa_{\min} = \kappa_{\max} = 1/r$$

Principal Curvature: Visualized

$\kappa_{\min}$

$\kappa_{\max}$

Gaussian and Mean Curvature



$$K = \kappa_1 \kappa_2$$

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

Q: Why Gaussian/Mean curvature is interesting to us?

Discussion Break: Smooth Curvature

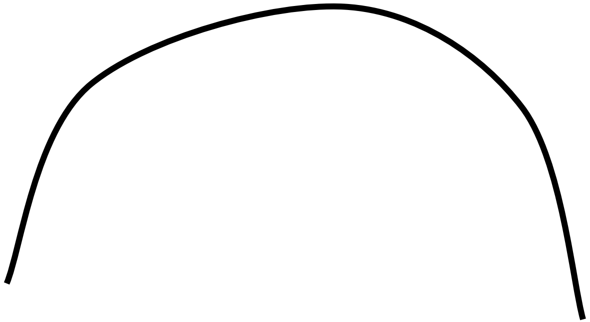
How do we actually compute curvatures in a discrete world?

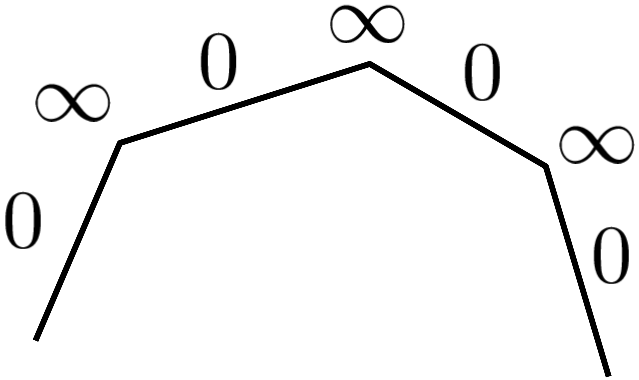
Discrete Curvature

Curvature is the change in normal direction as we travel along the curve

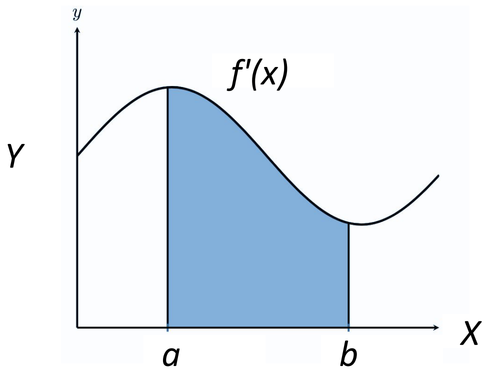
In discrete settings: No change along each edge ⇒ zero curvature?

$$\mathrm{d}N = \kappa \mathrm{d}f(X)$$



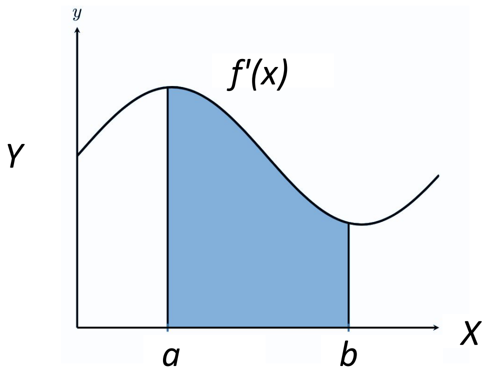


Revisit: Fundamental Theorem of Calculus



$$\int_a^b \mathrm{d}f = f(b) - f(a)$$

Revisit: Fundamental Theorem of Calculus and Stokes' Theorem



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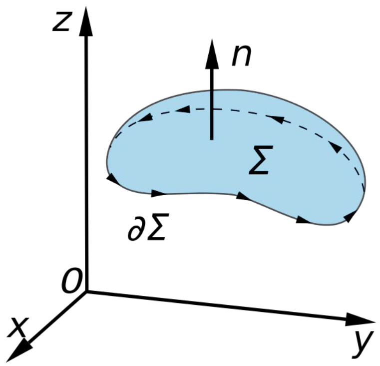


Insights from Stokes' Theorem:

The change we see on the outside is

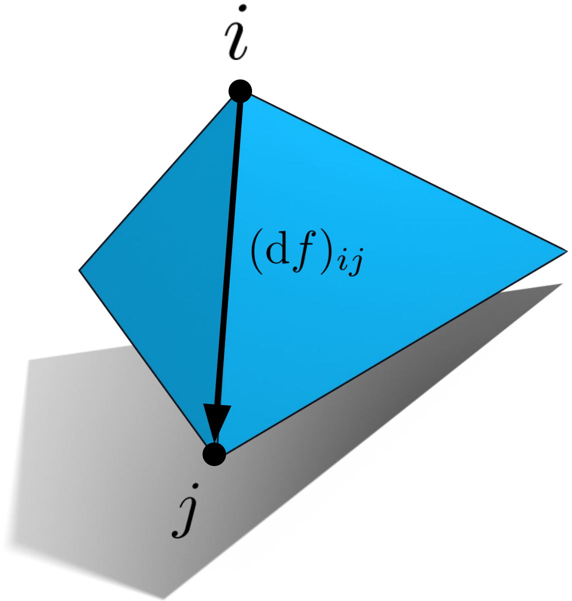
purely a function of the change within.

Example: Discrete *Differential*

Discrete differential *is just edge vectors in discrete settings*

$$(\mathrm{d}f)_{ij} = f_j - f_i$$

(Stokes' theorem)



Discrete *Principal Curvature*

$$K = \kappa_1 \kappa_2$$

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

$$\Rightarrow$$

$$\kappa_1 = ||H|| - \sqrt{H^2 - K}$$

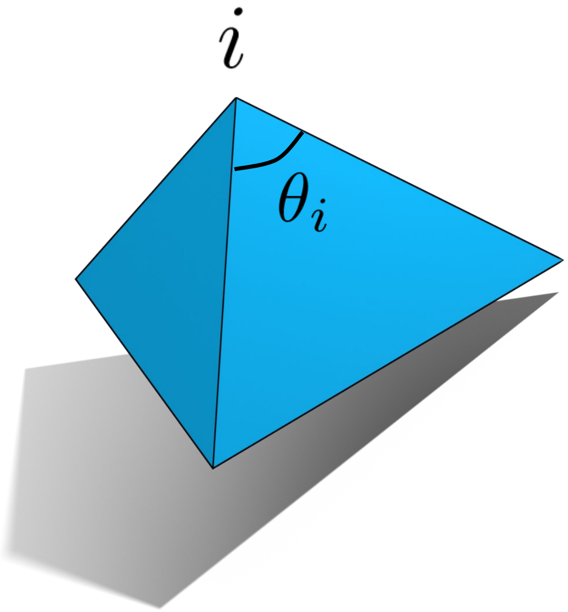
$$\kappa_2 = ||H|| + \sqrt{H^2 - K}$$

Then the question is: how to compute gaussian and mean curvature?

Discrete *Gaussian Curvature*

We already know how to compute Gaussian curvature: the angle defect is a good approximation

$$K = \Omega_i = 2\pi - \sum \theta_i$$



Discrete Mean Curvature

?

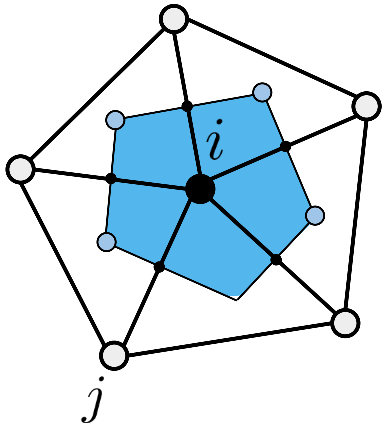
Laplacian

Basic idea: Laplacian is (scalar) deviation from local average

$$\Delta f = \nabla \cdot \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

(Discrete) Laplacian is the divergence of gradient

$$(\Delta f)_i = w_i \sum_{ij} w_{ij}(f_j - f_i)$$



Cotangent Formula

A more accurate discretization of the Laplace-Beltrami operator

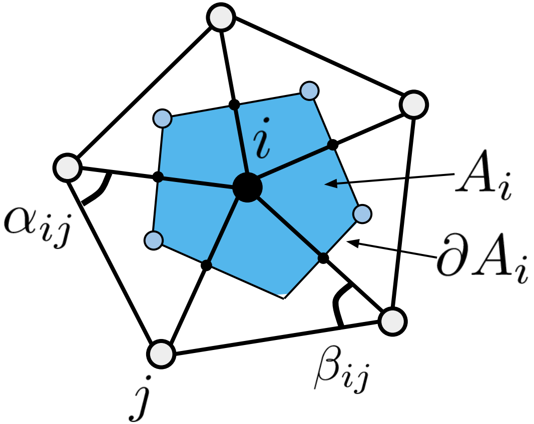
Basic idea: integrate the divergence of the gradient of a piecewise linear function over a *local averaging region* *(e.g. voronoi cell)*:

$$\int_{A_i} \Delta f \mathrm{d}A = \int_{\partial A_i} \nabla f \cdot N \mathrm{d}s$$

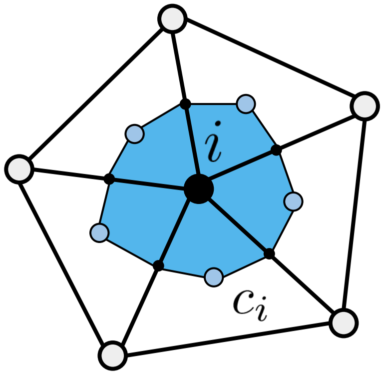
(Stokes' theorem)

One can prove:

$$\int_{A_i} \Delta f \mathrm{d}A = \frac{1}{2} \sum_{ij} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)$$

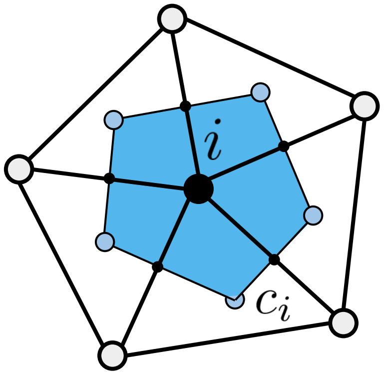


Local Averaging Region



Barycentric Cell

c_i = barycenter



Voronoi Cell

c_i = circumcenter

The *Laplace-Beltrami Operator*

The discrete version of the Laplace operator, of a function at a vertex *i* is given as

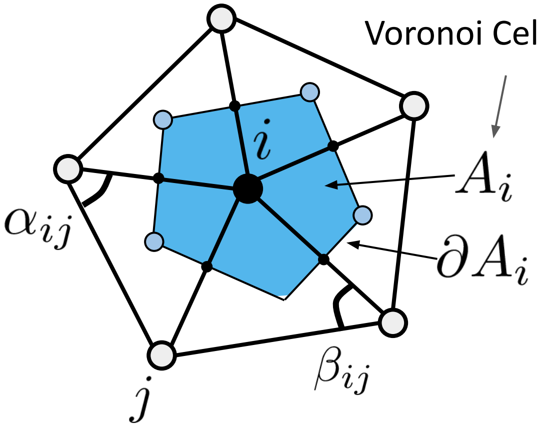
$$(\Delta f)_i = w_i \sum_{ij} w_{ij}(f_j - f_i)$$

This (cotan-version) is the most widely used discretization of the Laplace-Beltrami operator for geometry processing :

$$(\Delta f)_i = \frac{1}{2A_i} \sum_{ij} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)$$

The mean curvature is tightly related to the cotan Laplace-Beltrami:

$$H = \frac{1}{2} ||(\Delta f)_i||$$



Discrete Mean Curvature

The Laplace-Beltrami operator is tightly related to the mean curvature:

$$||\Delta f|| = 2||H||||N|| \Rightarrow ||H|| = \frac{1}{2}||(\Delta f)_i||$$

Implementation thinking: Is it necessary to keep the ½ factor?

Computing Discrete Curvatures

Mean curvature: via Laplace-Beltrami (length of Laplacian vector)

$$||H|| = \frac{1}{2}||(\Delta f)_i||$$

Gaussian curvature: via angle defect

$$K = \Omega_i$$

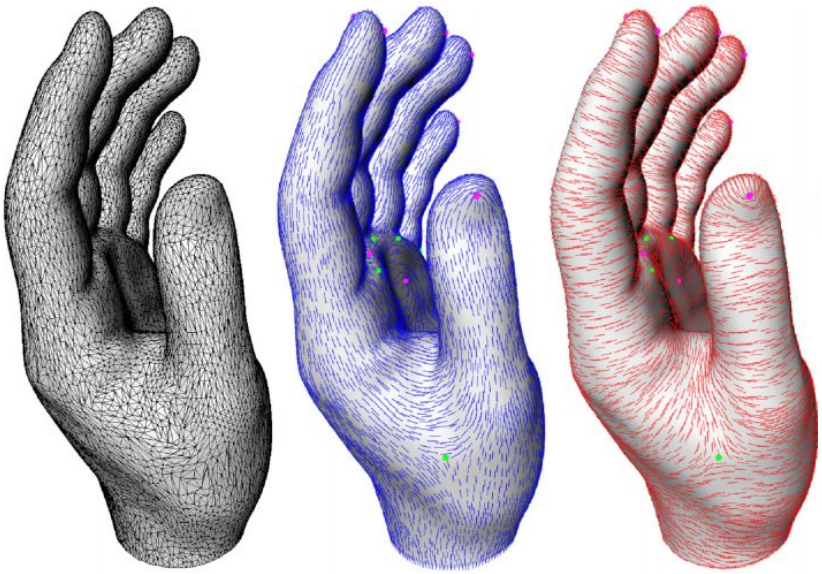
Principal curvature: via Gaussian and Mean

$$\kappa_1 = ||H|| - \sqrt{H^2 - K}$$
$$\kappa_2 = ||H|| + \sqrt{H^2 - K}$$

Application: Anisotropic Remeshing [Alliez et al. 2003]

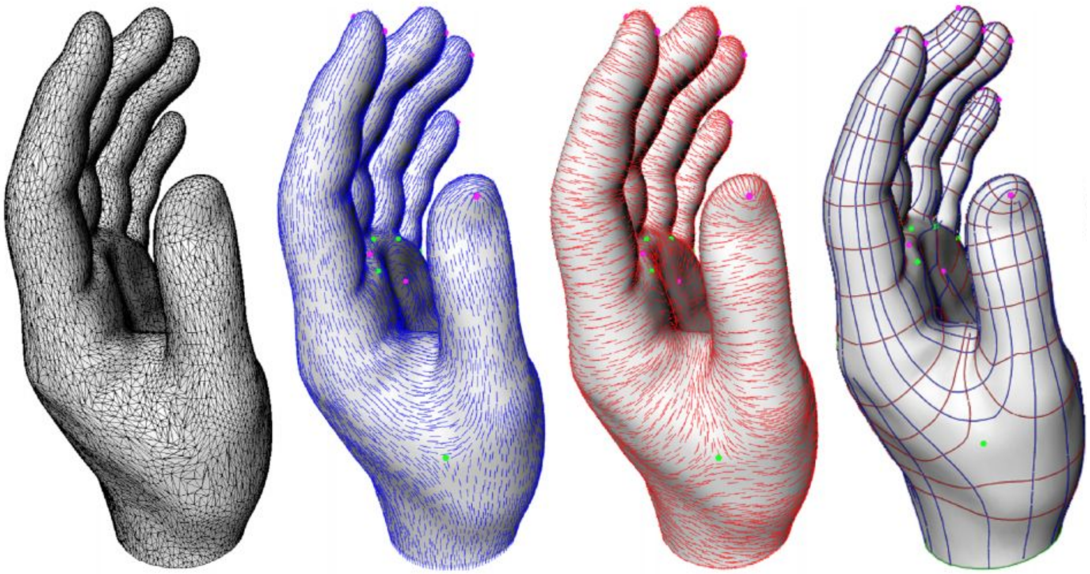
Input Mesh

Application: Anisotropic Remeshing [Alliez et al. 2003]



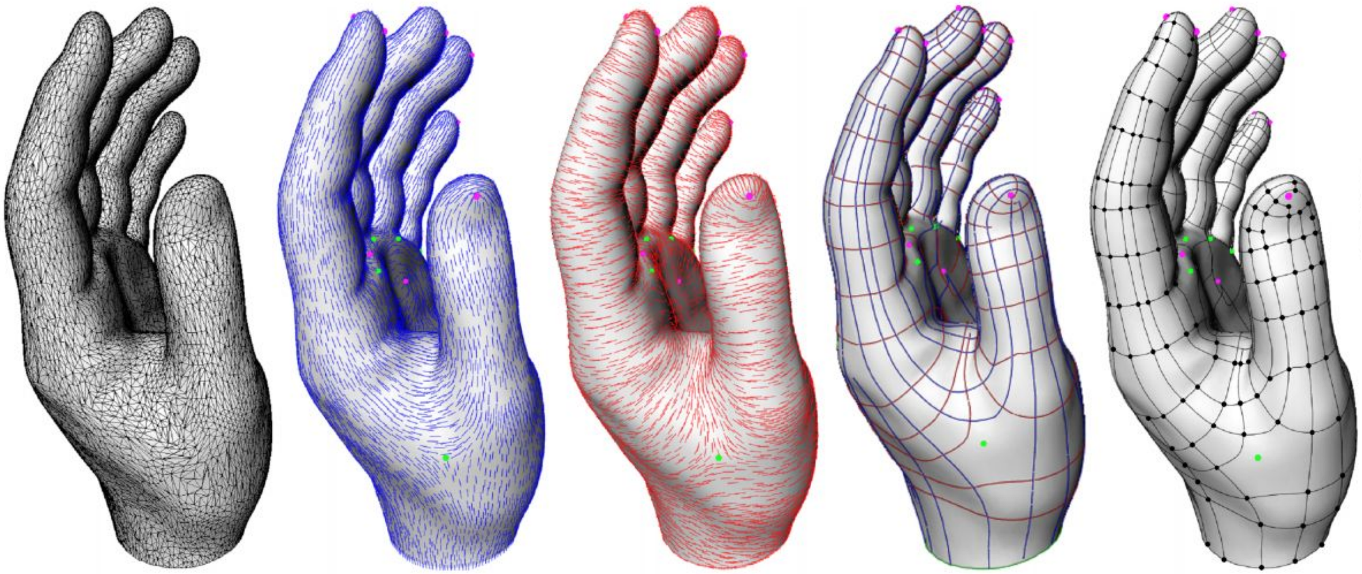
Input Mesh Kmin Kmax

Application: Anisotropic Remeshing [Alliez et al. 2003]



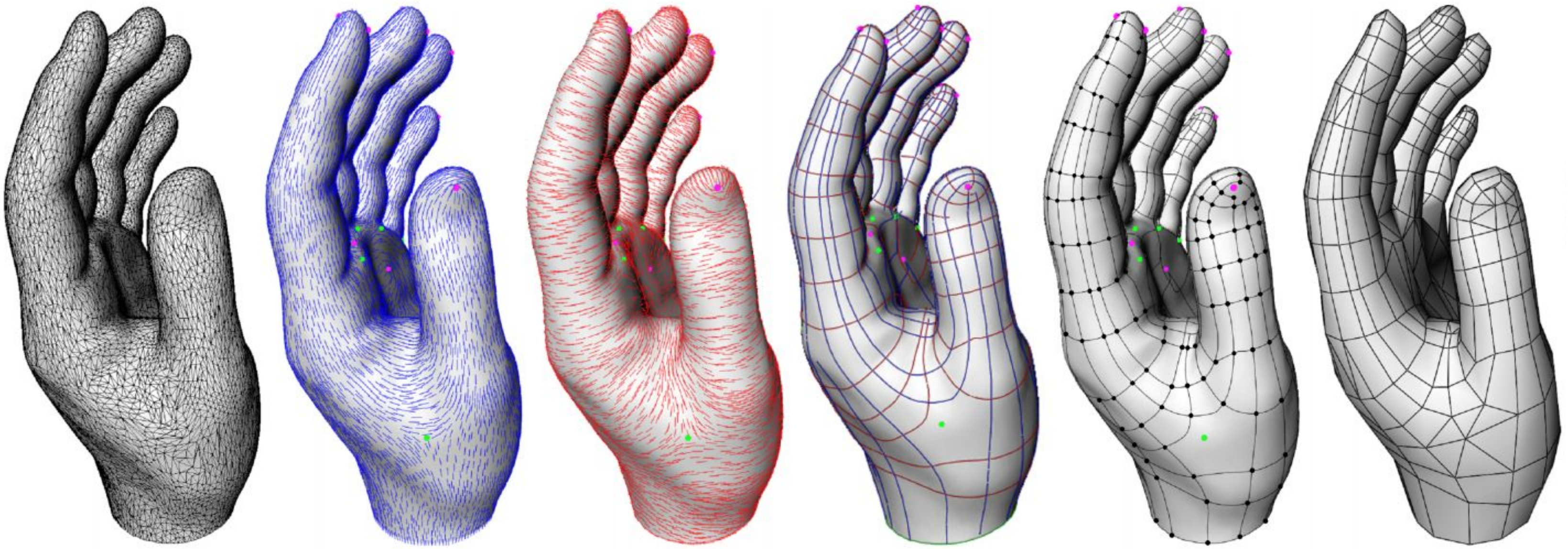
Input Mesh Kmin Kmax Sampling

Application: Anisotropic Remeshing [Alliez et al. 2003]



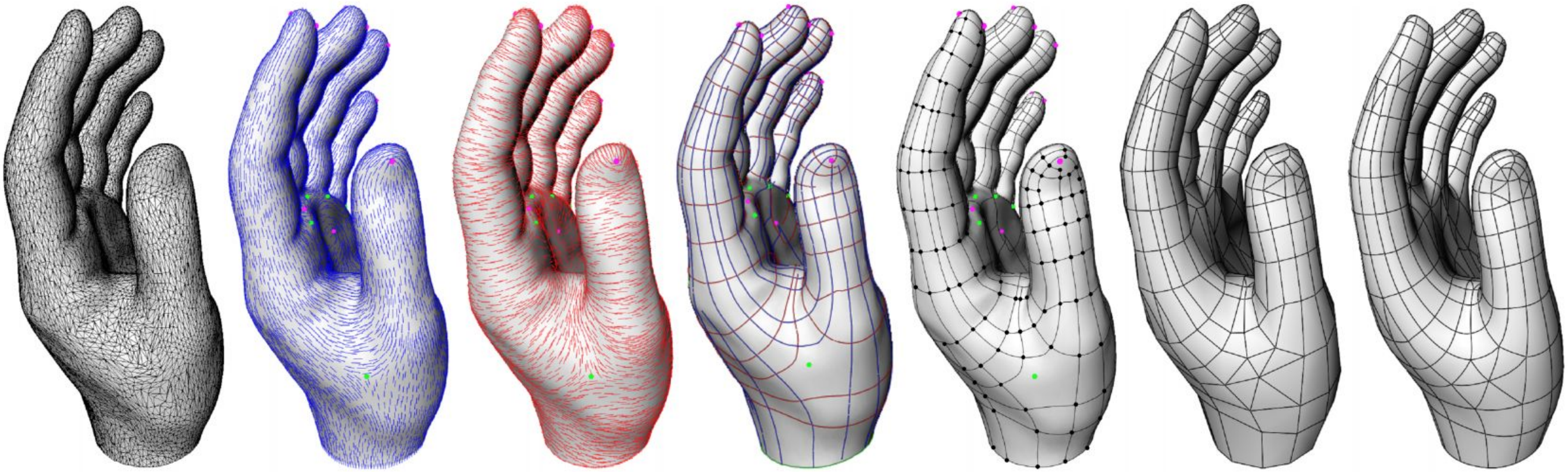
Input Mesh Kmin Kmax Sampling Meshing

Application: Anisotropic Remeshing [Alliez et al. 2003]



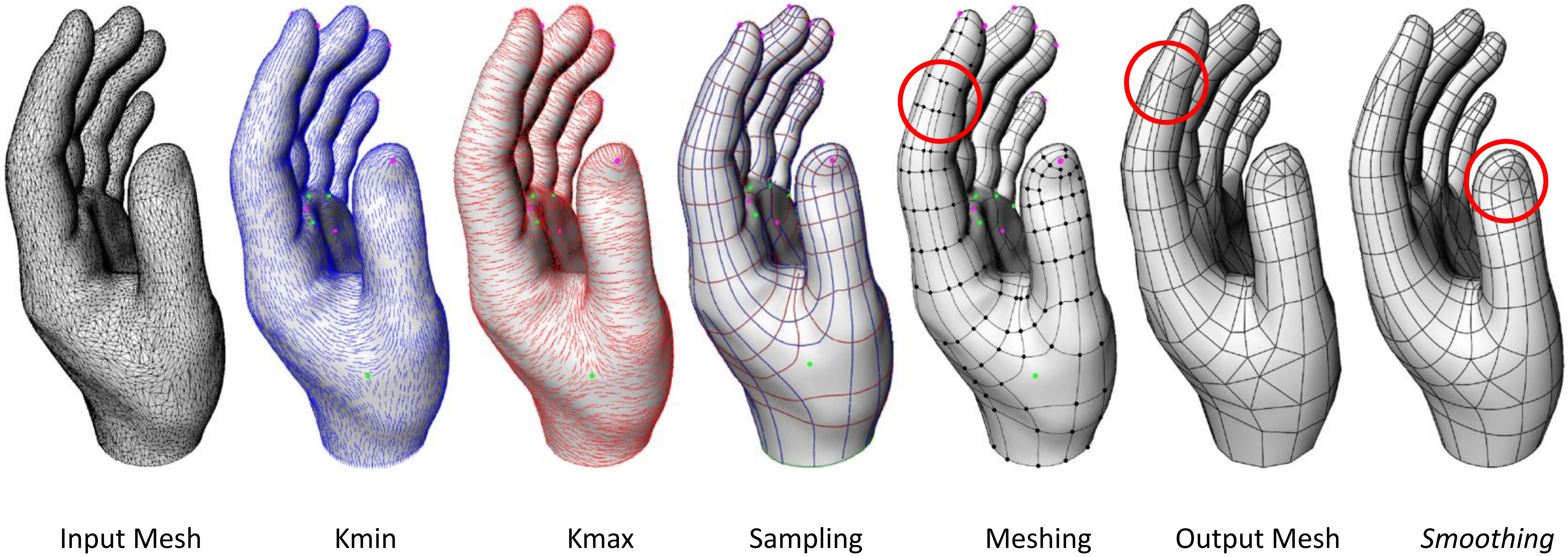
Input Mesh Kmin Kmax Sampling Meshing Output Mesh

Application: Anisotropic Remeshing [Alliez et al. 2003]



Input Mesh | Kmin | Kmax | Sampling | Meshing | Output Mesh | *Smoothing*

Application: Anisotropic Remeshing [Alliez et al. 2003]



* This is not an novel idea, we will see more advances later in our remeshing session. 😀

Session 2: Discrete Differential Geometry

- Motivation
- Discrete Geometric Quantifies
  - Normals
  - Curvature
  - Laplace-Beltrami
- Summary
- Discussion
  - .OBJ Mesh Loader
  - Blender Python APIs
  - Blender BMesh Structure

Summary

- Different discretized definition of a geometry quantity preserves different properties
- Curvature and the Laplacian are the core tools for geometry processing
- No free lunch (again): Compare discretized version to its smooth setting, not all properties can be preserved in discrete settings. Understand the landscape of possibilities of when we should apply a certain definition in a context

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Implementing A Mesh Loader

Questions

- What information must be loaded for a minimum implementation?
- How to define a data structure?
- How's the performance of loading?
- What was changed from constructing triangle soup to halfedge representation?

Blender + Python

https://docs.blender.org/api/current/

Key Concepts

Types: bpy.types

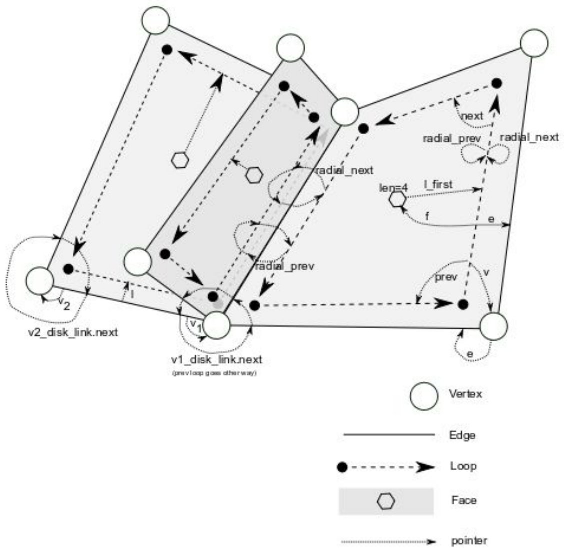
Data: bpy.data

Operator: bpy.ops

Context: bpy.context

BMesh

BMesh: A Non-Manifold Boundary Representation



https://wiki.blender.org/w/images/0/06/Dev-BMesh-Structures.png

Blender's Mesh Editing APIs

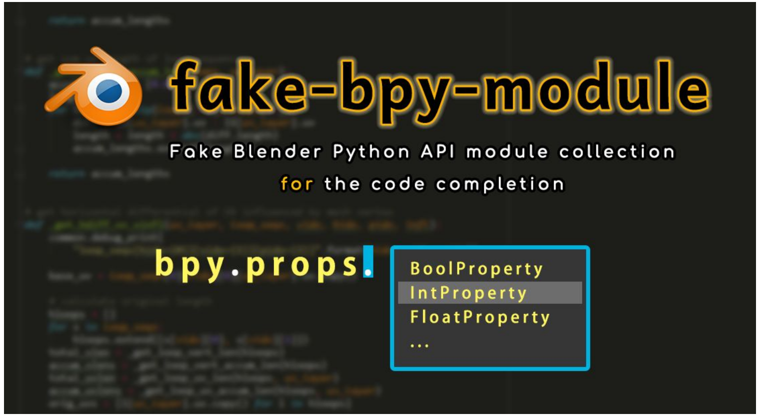
Low-level Operators

Mid-level Operators

Top-level Operators

Code Completion

```
pip install fake-bpy-module-2.93
```



https://github.com/nutti/fake-bpy-module

Further Readings (Mesh Structures)

Paul Bourke. Data Formats: 3D, Audio, Image. http://paulbourke.net/dataformats/

Weiler, K.J. : The Radial Edge Structure: A Topological Representation for Non-Manifold Geometric Modeling. in Geometric Modeling for CAD Applications, Springer Verlag, May 1986.

Further Readings (Discrete Differential Geometry)

Jin S, Lewis RR, West D. A comparison of algorithms for vertex normal computation. The visual computer. 2005 Feb 1;21(1-2):71-82.

Wardetzky, Max, et al. Discrete Laplace operators: no free lunch. Symposium on Geometry processing. 2007.

Keenan Crane. Discrete Differential Geometry: An Applied Introduction. 2020.