

# Bayesian Learning

## Basics and Advances in Modeling and Inference

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# Outline

- Overview
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  - Bayesian Inference
- Bayesian Inference
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    - Particle-Based Variational Inference
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  - Amortized Inference
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  - Markov Random Fields
- Topics
  - Online inference
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  - Bayesian Reinforcement Learning
  - Causality
- Future Work

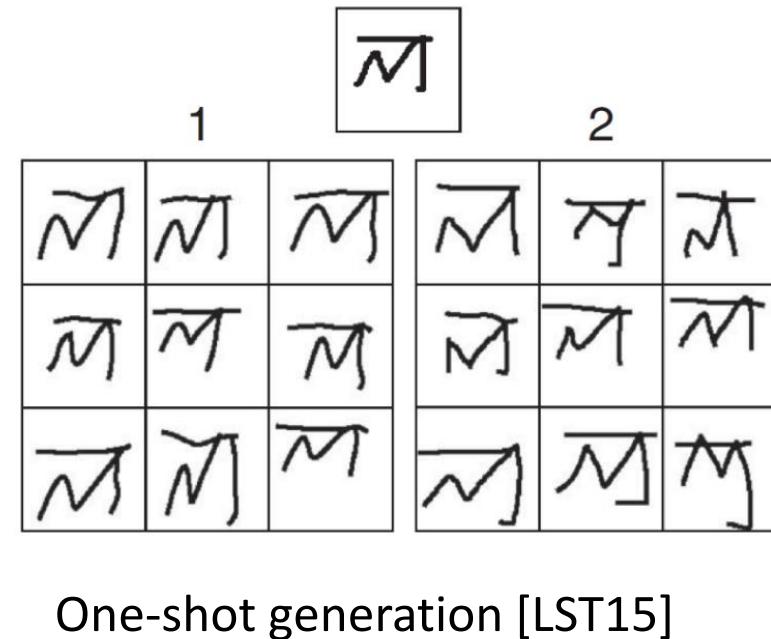
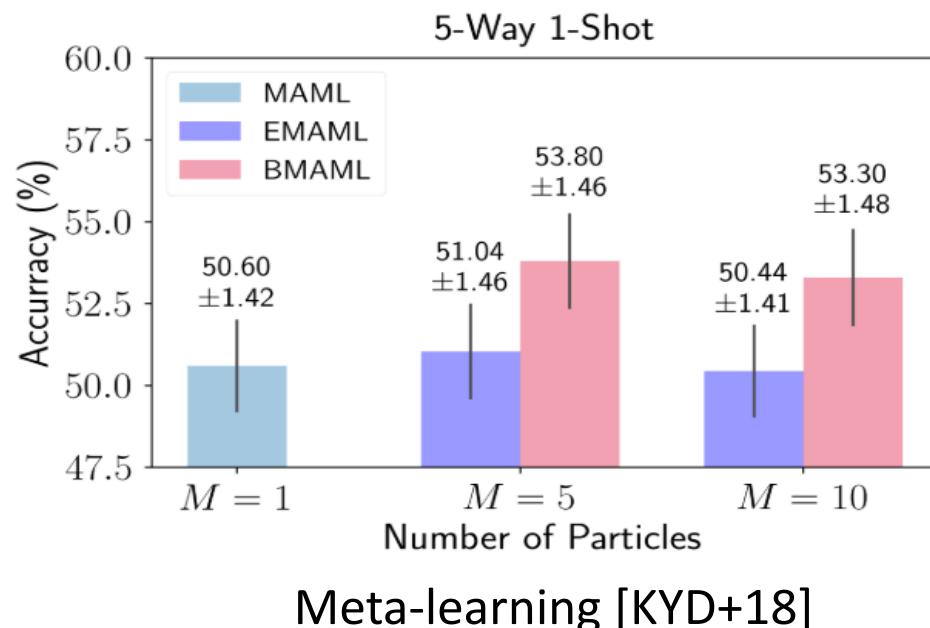
# Overview

## Bayesian Models

# Bayesian Model: Overview

The dependence between latent variable and data is probabilistic.

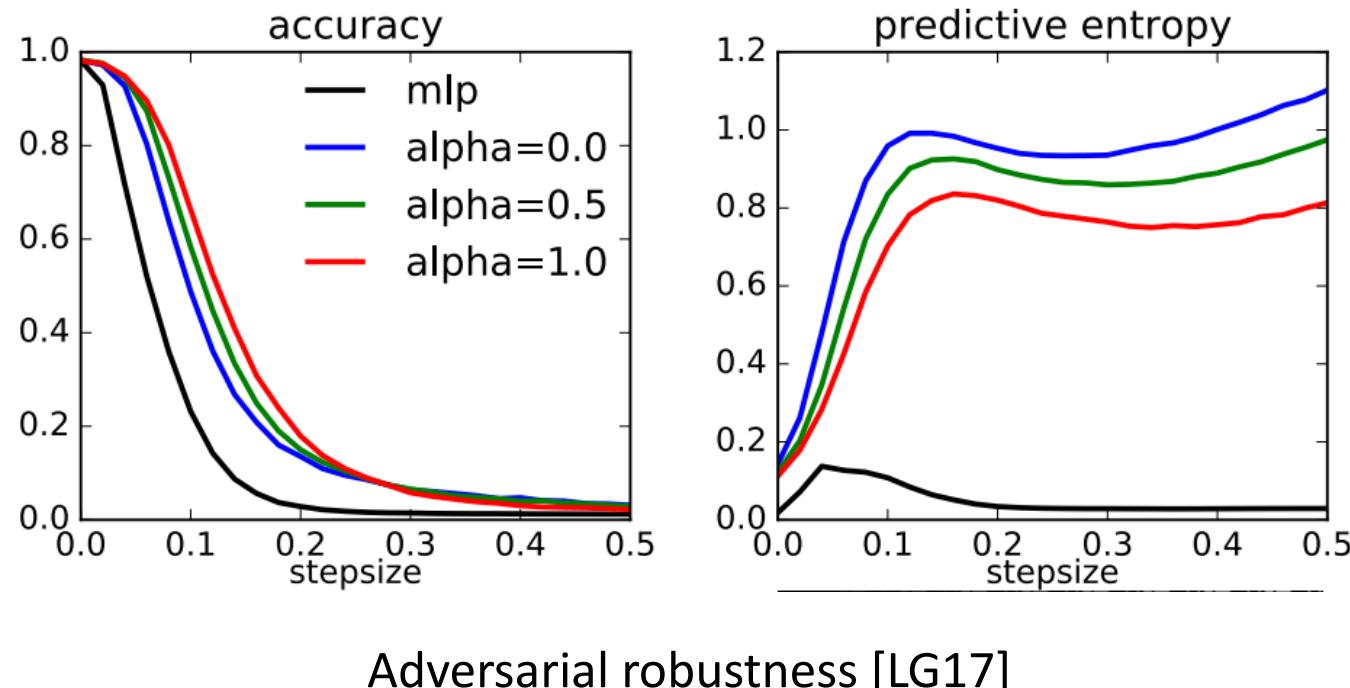
- Handles model complexity and uncertainty
- Robust to small data...



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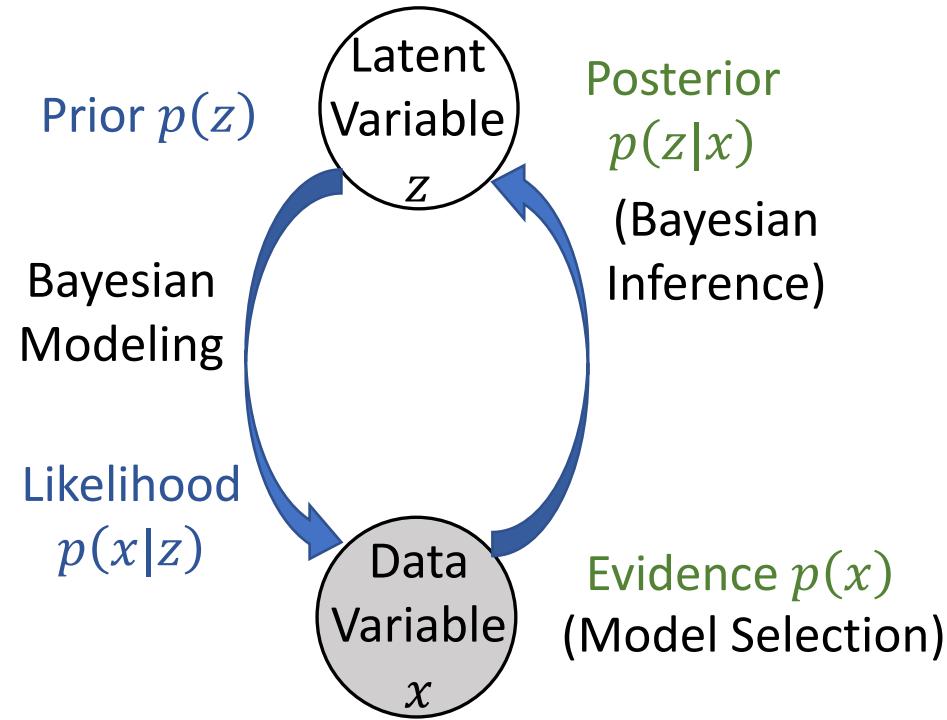
- Handles model complexity and uncertainty
- Robust to small data and adversarial attack



# Bayesian Model: Overview

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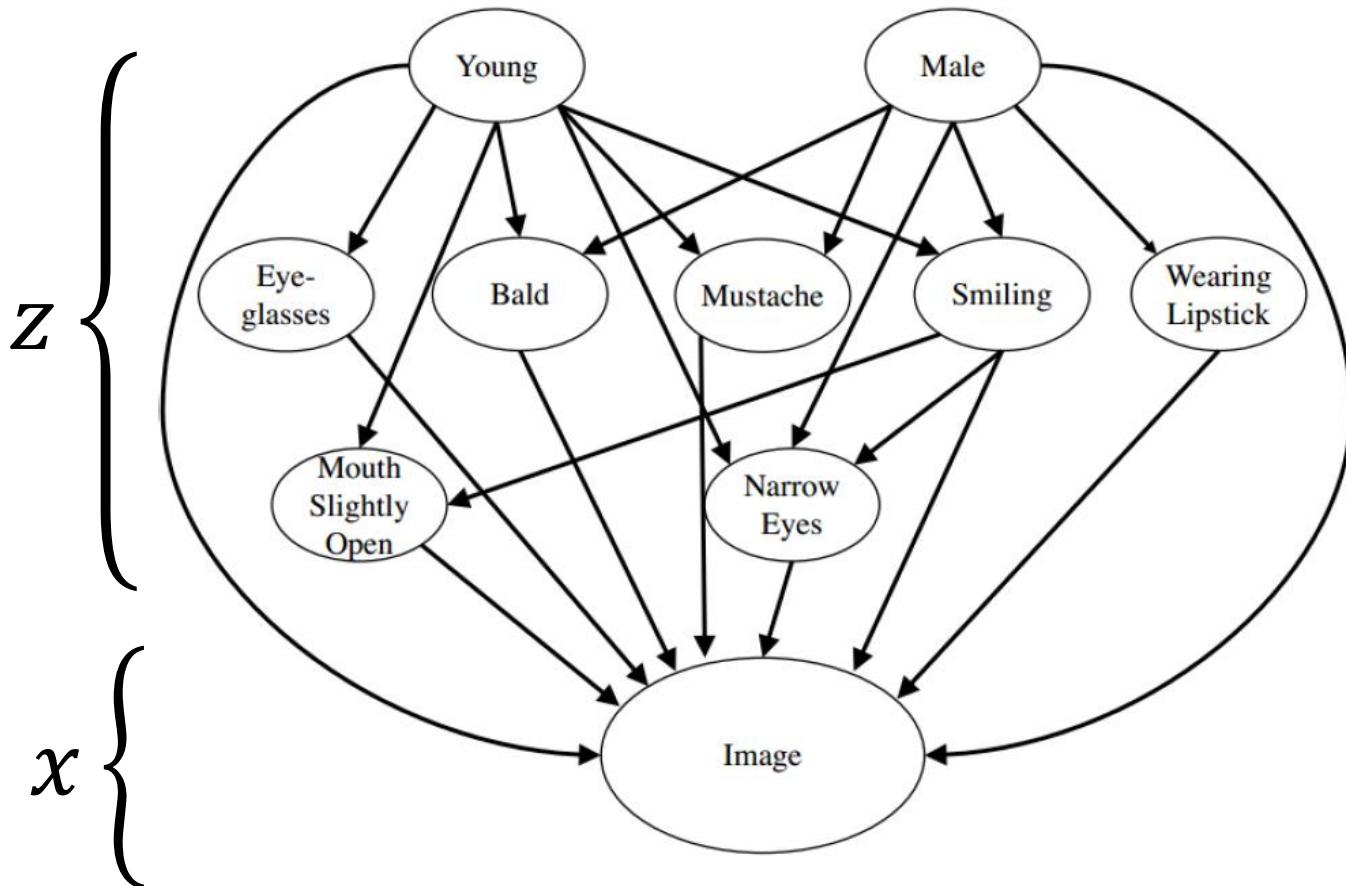
- Stable training process
- Principled and natural inference  $p(z|x)$  via Bayes' rule



# Bayesian Model: Overview

The dependence between latent variable and data is probabilistic.

- Natural to incorporate prior knowledge



[KSD+18]

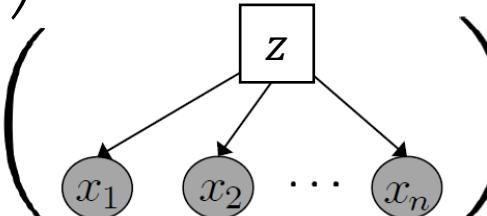
# Bayesian Model: Overview

The dependence between **latent variable** and data is probabilistic.

- Latent variable: Compact representation of the dependency.

- De Finetti's Theorem (1955)

- If  $(x_1, x_2, \dots)$  are *infinitely exchangeable*, then  $\exists$  r.v.  $z$  and  $p(\cdot | z)$  s.t.  $\forall n$ ,

$$p(x_1, \dots, x_n) = \int \left( \prod_{i=1}^n p(x_i | z) \right) p(z) dz .$$
$$p\left( \begin{matrix} x_1 & x_2 & \dots & x_n \end{matrix} \right) = \int_z p\left( \begin{matrix} z \\ x_1 & x_2 & \dots & x_n \end{matrix} \right)$$


*Infinite exchangeability:*

For all  $n$  and permutation  $\sigma$ ,  $p(x_1, \dots, x_n) = p(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ .

- For supervised modeling, relation between  $x$  and  $y$  can be modeled by  $z$  via e.g.,  $p_z(y|x)$  or  $y = f_z(x)$ .

# Bayesian Model: Overview

The dependence between **latent variable** and data is probabilistic.

- Latent variable: Abstract knowledge of the data.
  - Structured model:  $x$  (documents)  $\longleftrightarrow$   $z$  (topics)



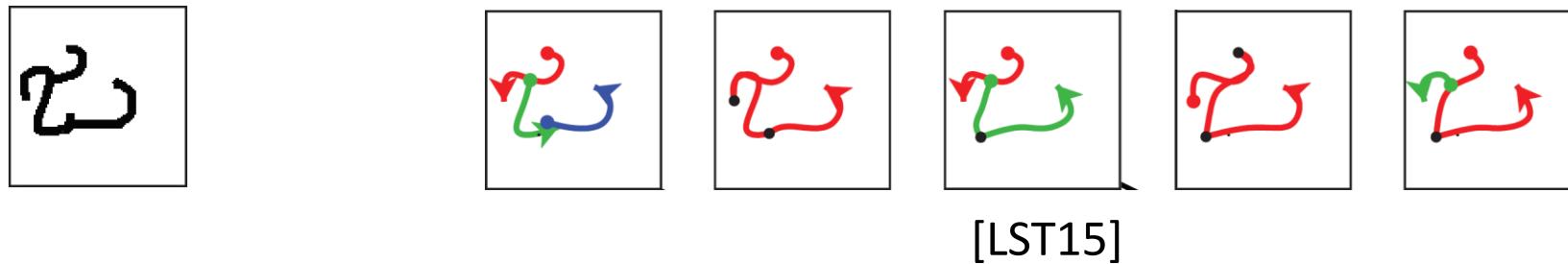
“ENGINES”	speed	product	introduced	designs	fuel
“ROYAL”	britain	queen	sir	earl	died
“ARMY”	commander	forces	war	general	military
“STUDY”	analysis	space	program	user	research
“PARTY”	act	office	judge	justice	legal
“DESIGN”	size	glass	device	memory	engine
“PUBLIC”	report	health	community	industry	conference
“CHURCH”	prayers	communion	religious	faith	historical

[PT13]

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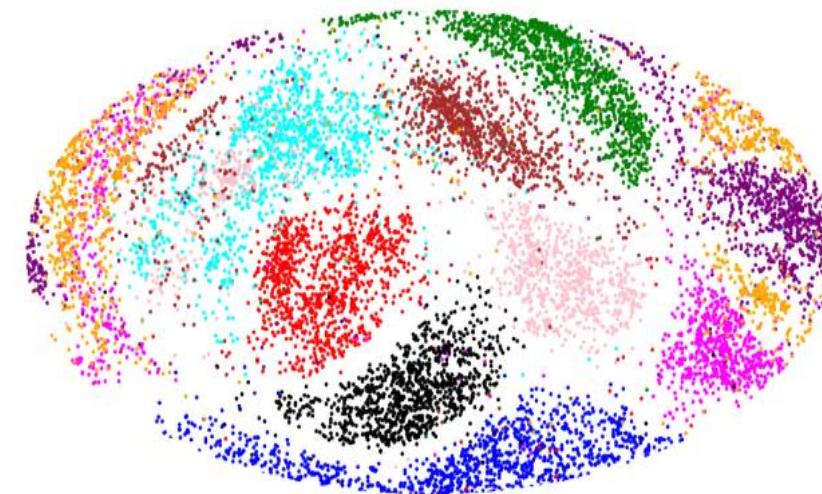
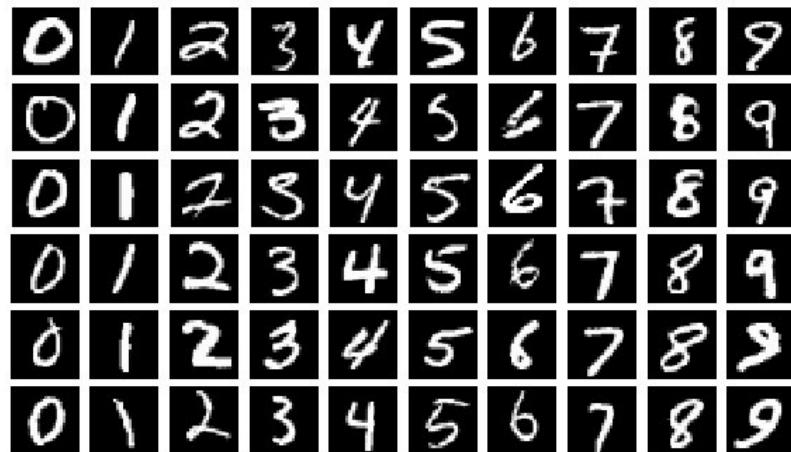
- Latent variable: Abstract knowledge of the data.
  - Structured model:  $x$  (character image)  $\longleftrightarrow$   $z$  (strokes)



# Bayesian Model: Overview

The dependence between **latent variable** and data is probabilistic.

- Latent variable: Abstract knowledge of the data.
  - Universal model:  $x$  (general data)  $\longleftrightarrow$   $z$  (general semantic feature)

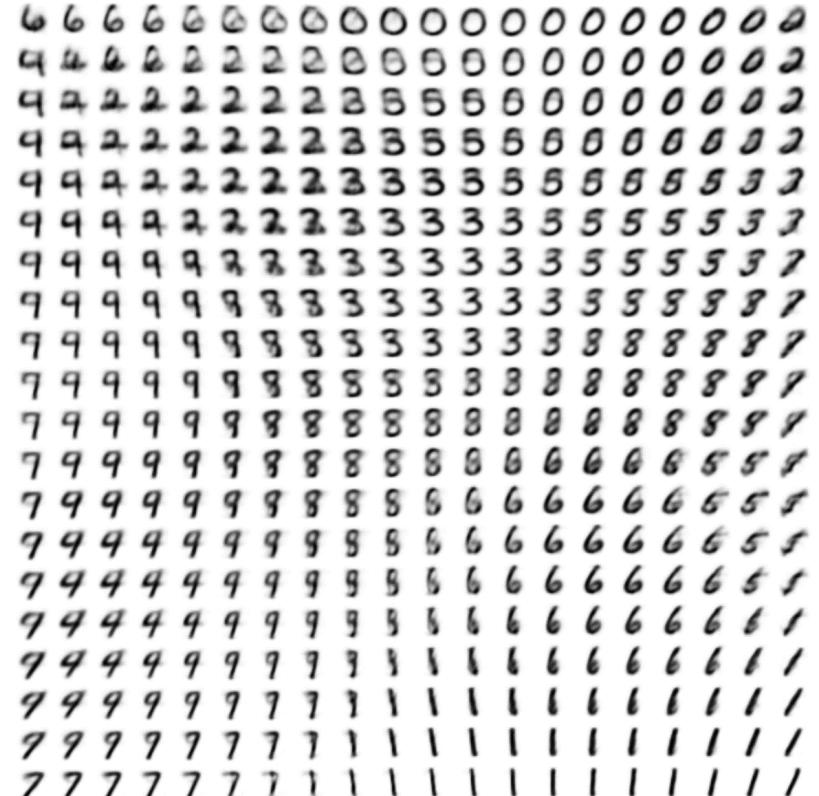
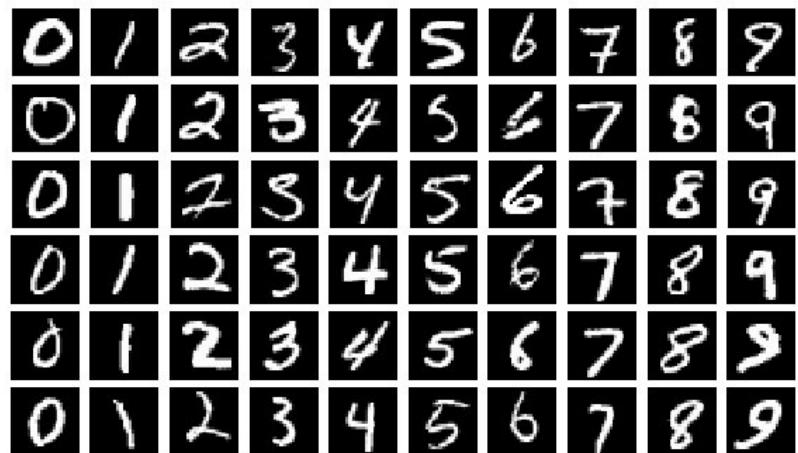


[DFD+18]

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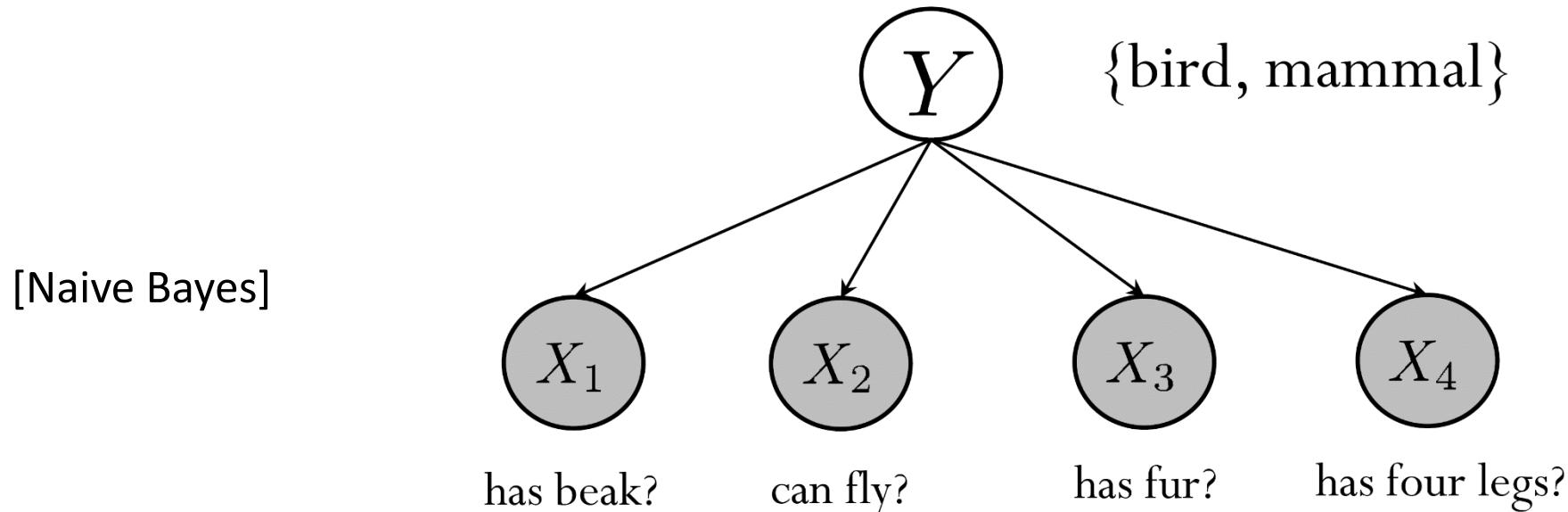


[KW14]

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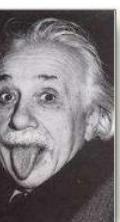
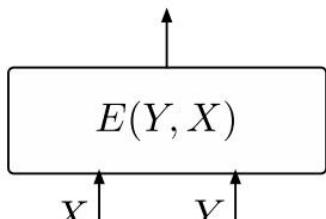
- Latent variable: Abstract knowledge of the data.
  - Supervised tasks:  $(x, y) \iff z = y$



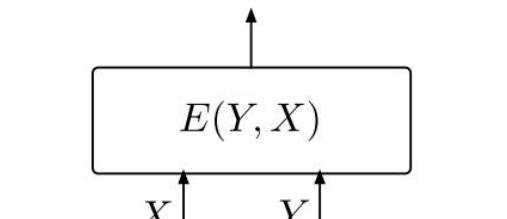
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The dependence between **latent variable** and data is probabilistic.

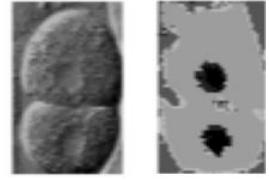
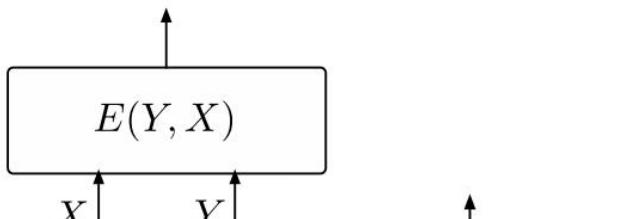
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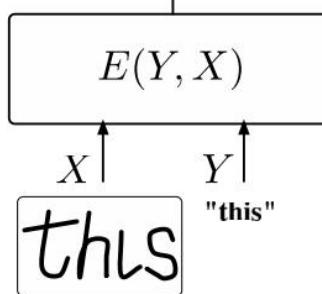
(a)



(b)

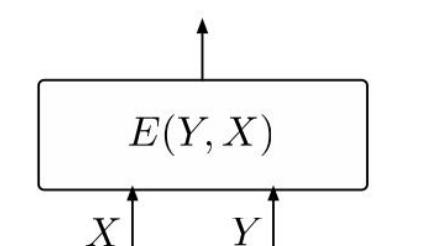


(c)



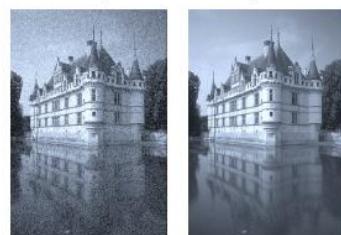
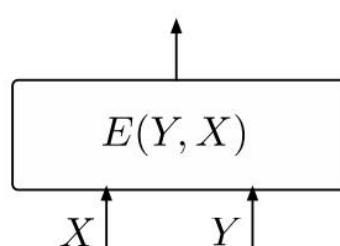
thus

(d)



"This is easy" (pronoun verb adj)

(e)



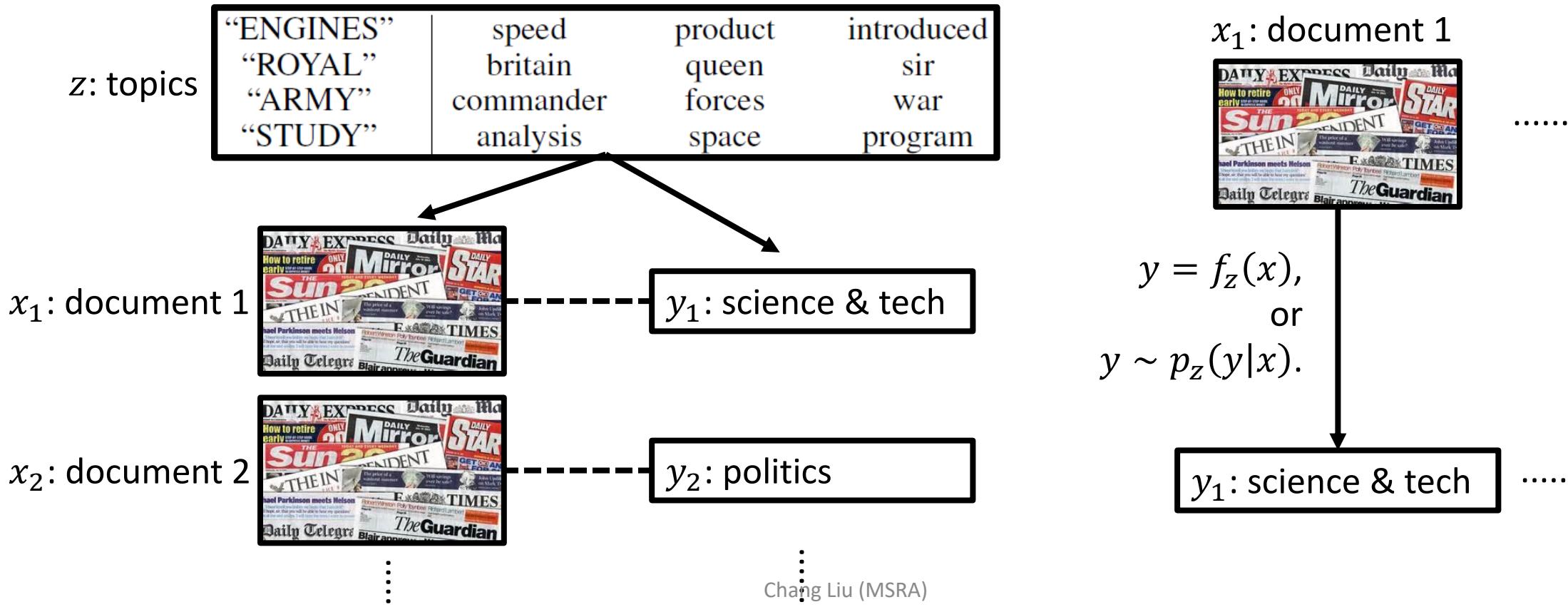
(f)

[LCH+06]

# Bayesian Model: Overview

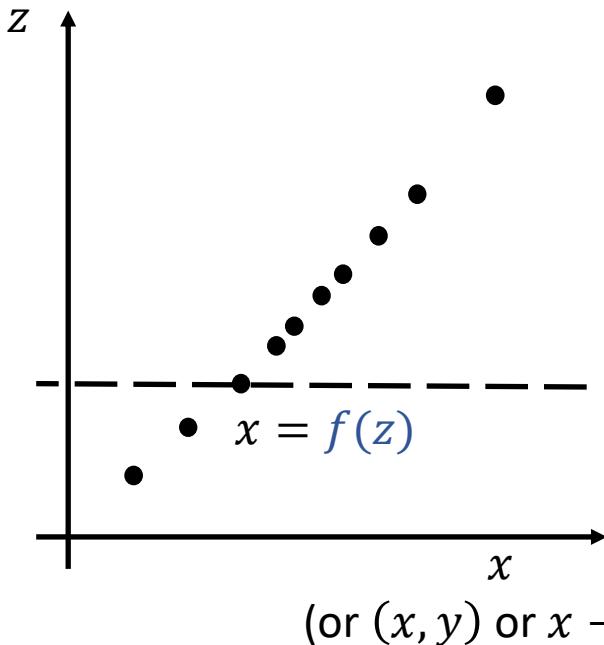
The dependence between **latent variable** and data is probabilistic.

- Latent variable: Abstract knowledge of the data.
  - Supervised tasks:  $(x, y) \longleftrightarrow z$

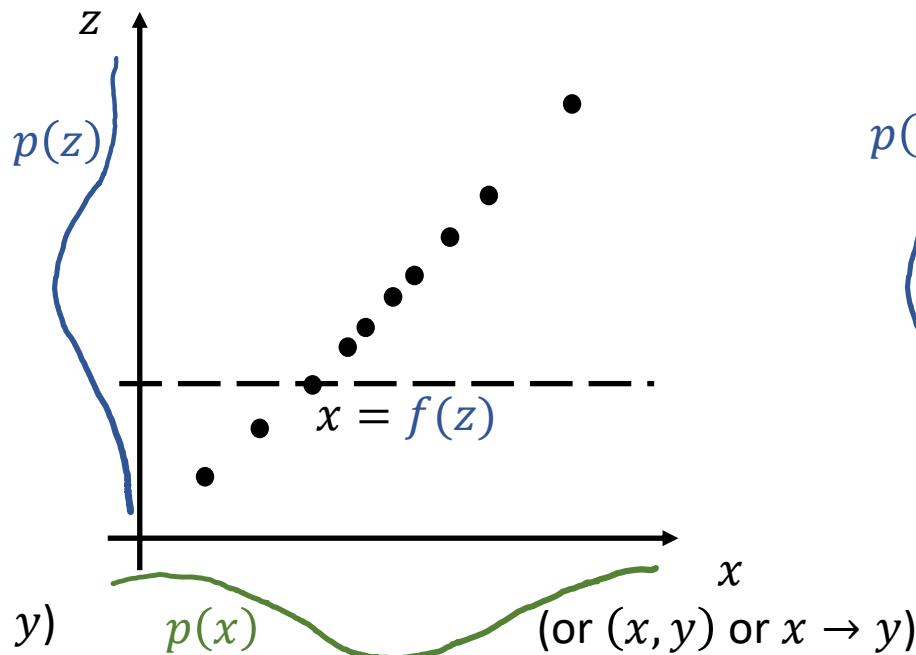


# Bayesian Model: Overview

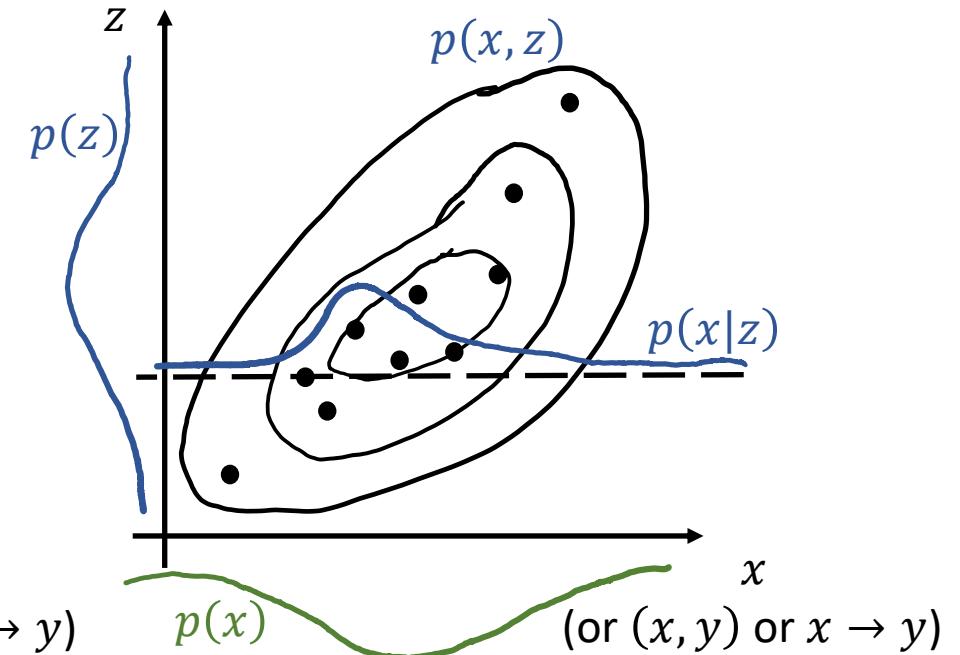
The dependence between latent variable and data is **probabilistic**.



Autoencoders  
(or Feedforward NNs)



Generative Adversarial Nets,  
Flow-Based Generative Models

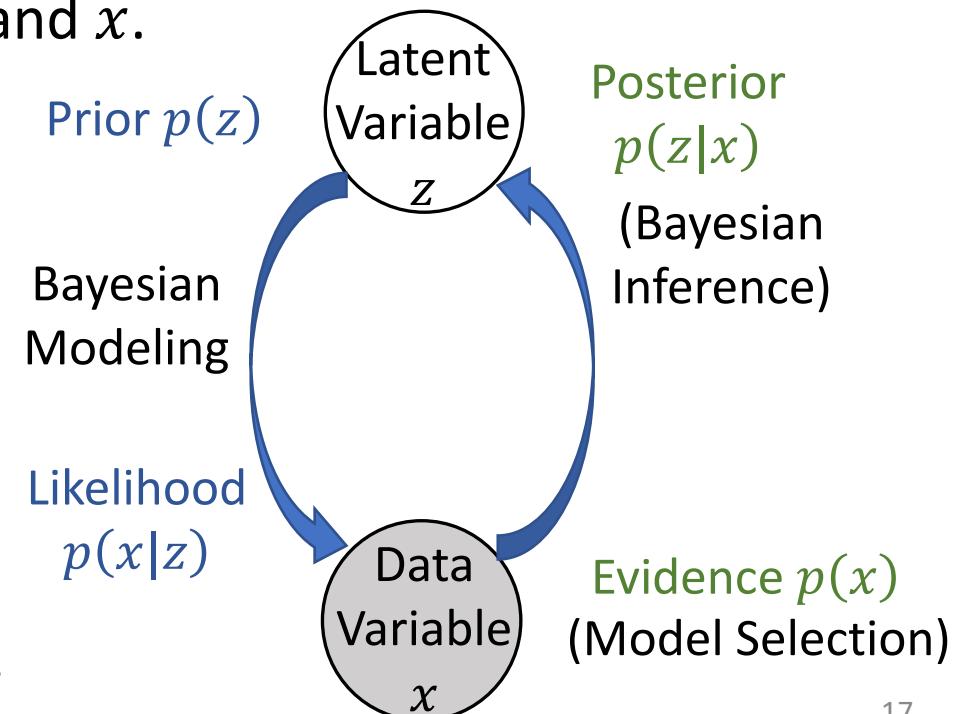


Bayesian Models

# Bayesian Model: Overview

The dependence between latent variable and data is **probabilistic**.

- Model structure: *Prior*  $p(z)$ , *Likelihood*  $p(x|z)$ .
- How it works:
  - $p(z)$  models the *initial belief* of  $z$ .
  - $p(x|z)$  models the dependence between  $z$  and  $x$ .
  - $p(z, x) = p(z)p(x|z)$  then models all the information about  $x$  and  $z$ . Particularly,
  - *Evidence*  $p(x) = \int p(z, x) dz$  gives the belief on  $x$ .
  - *Posterior*  $p(z|x) = \frac{p(z,x)}{p(x)} = \frac{p(z)p(x|z)}{\int p(z,x) dz}$  (by Bayes' rule) gives the *updated* information that observation  $x$  conveys to  $z$ .



# Bayesian Model: Overview

The dependence between latent variable and data is **probabilistic**.

- Model structure: *Prior  $p(z)$ , Likelihood  $p(x|z)$* .

- How it learns (*model selection*):

- Maximum likelihood estimation (equiv. to  $\min_{\theta} \text{KL}(\hat{p}, p_{\theta})$ ):

$$\begin{aligned}\theta^* &= \arg \max_{\theta} \mathbb{E}_{\hat{p}(x)} [\log p_{\theta}(x)] \\ &\approx \arg \max_{\theta} \frac{1}{N} \sum_{n=1}^N \log p_{\theta}(x^{(n)}).\end{aligned}$$

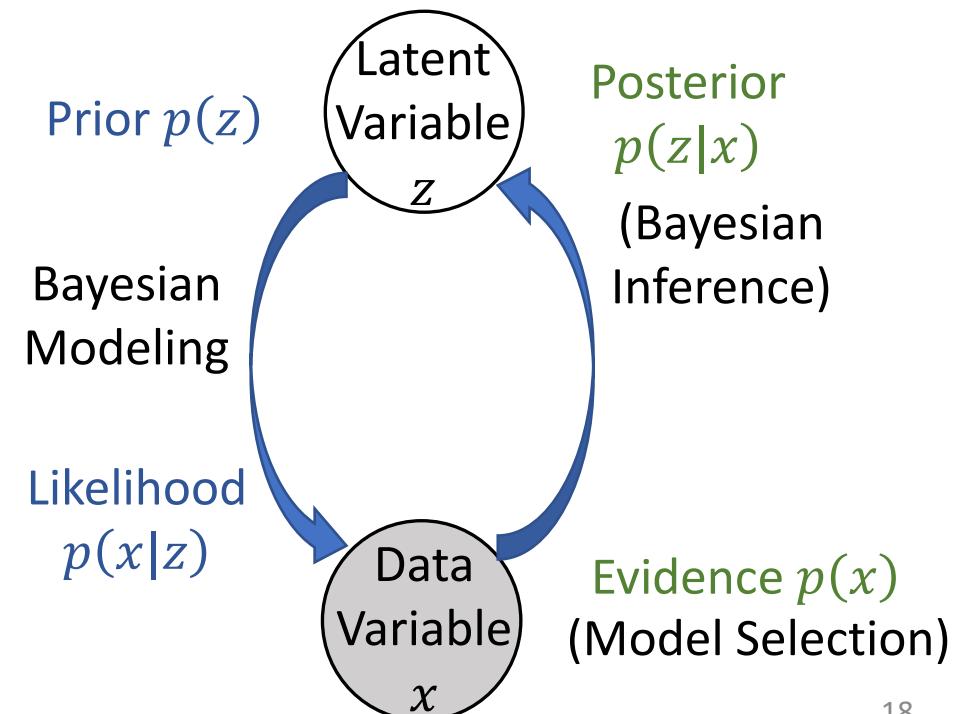
- How it serves:

- Feature/representation learning, prediction:

$p(z|x)$  (*Bayesian Inference*).

- Generation, prediction:

$$z_{\text{new}} \sim p(z|x), x_{\text{new}} \sim p(x|z_{\text{new}}).$$



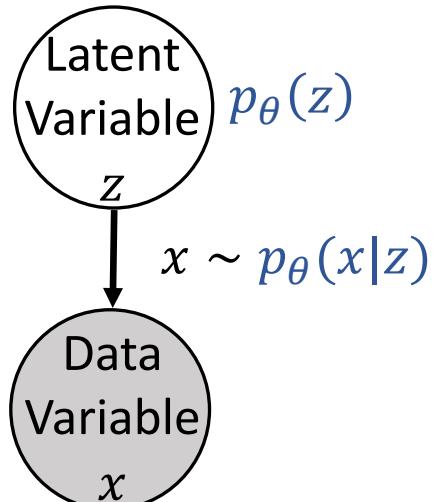
# Bayesian Model: Taxonomy

The dependence between latent variable and data is **probabilistic**.

Bayesian Network (BayesNet):

$p(x, z)$  is specified via  $p(z)$  and  $p(x|z)$ .

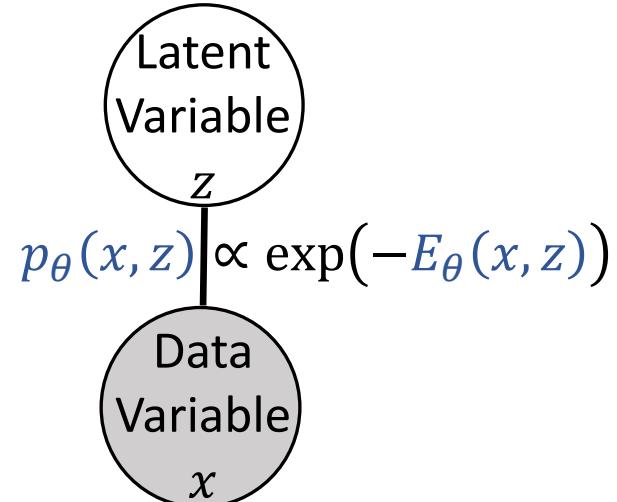
- Synonyms: Causal Networks, Directed Graphical Model
- Prior-likelihood interpretation
- Causality information encoded



Markov Random Field (MRF):

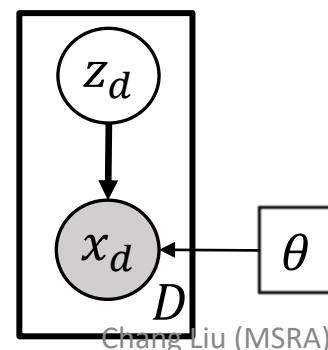
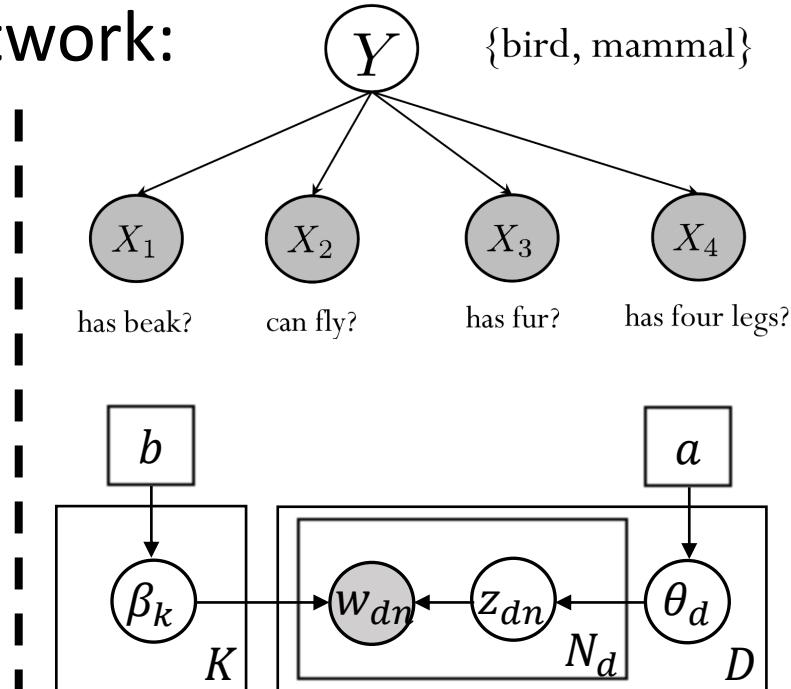
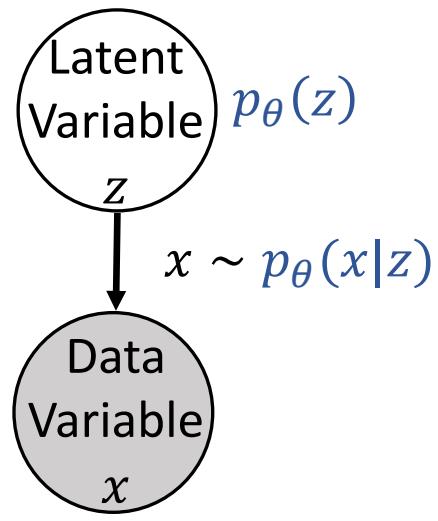
$p(x, z)$  is specified by an Energy function  $E_\theta(x, z)$ :  $p(x, z) \propto \exp(-E_\theta(x, z))$ .

- Synonyms: Energy-Based Model, Undirected Graphical Model
- Focus on modeling the relationship



# Bayesian Model: Taxonomy

- Bayesian Network:



Naive Bayes  $z = y$ :

$p(y), p(x_i|y)$ :  $\text{Bern}(\pi)$ .

Latent Dirichlet Allocation:

- Prior:  $p(\beta_k|b) = \text{Dir}(b)$ ,  
 $p(\theta_d|a) = \text{Dir}(a)$ ,  $p(z_{dn}|\theta_d) = \text{Mult}(\theta_d)$ .
- Likelihood:  $p(w_{dn}|z_{dn}, \beta) = \text{Mult}(\beta_{z_{dn}})$ .

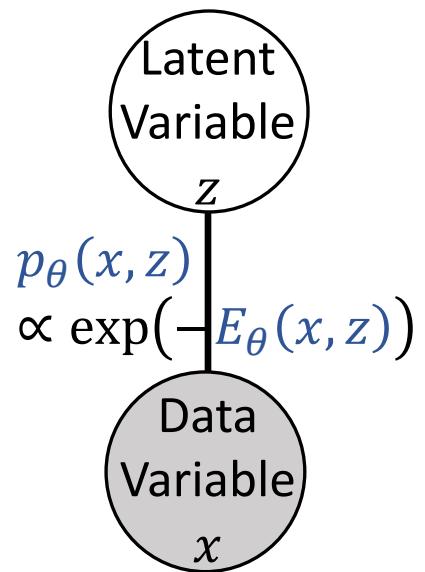
Variational Auto-Encoder

- Prior:  $p(z) = \mathcal{N}(0, I)$ .
- Likelihood:

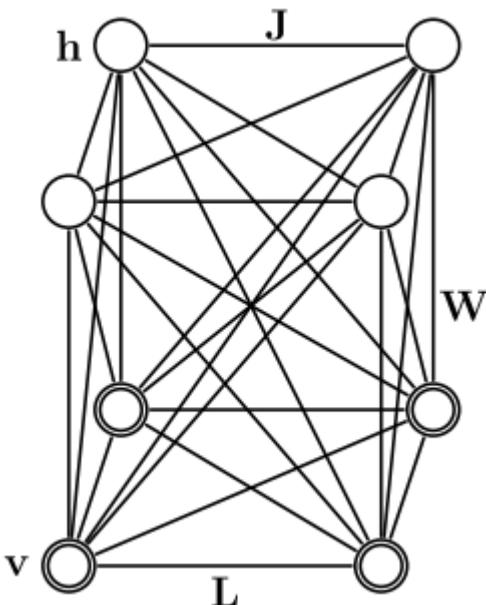
$$p(x_d|z_d) = \mathcal{N}\left(\text{NN}_{\theta}^{(1)}(z_d), \text{NN}_{\theta}^{(2)}(z_d)\right).$$

# Bayesian Model: Taxonomy

- Markov Random Field



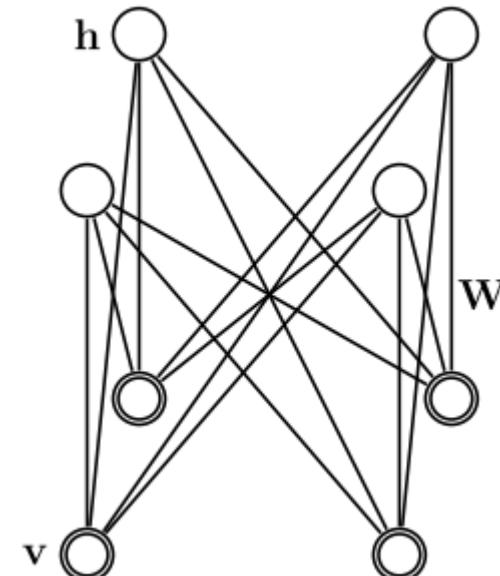
Boltzmann Machine [HS83]



$$E_\theta(v, h) = -\frac{1}{2}v^\top Lv - \frac{1}{2}h^\top Jh - v^\top Wh$$

Chang Liu (MSRA)

Restricted Boltzmann Machine  
(*Harmonium*) [Smo86]



$$E_\theta(v, h) = -v^\top Wh$$

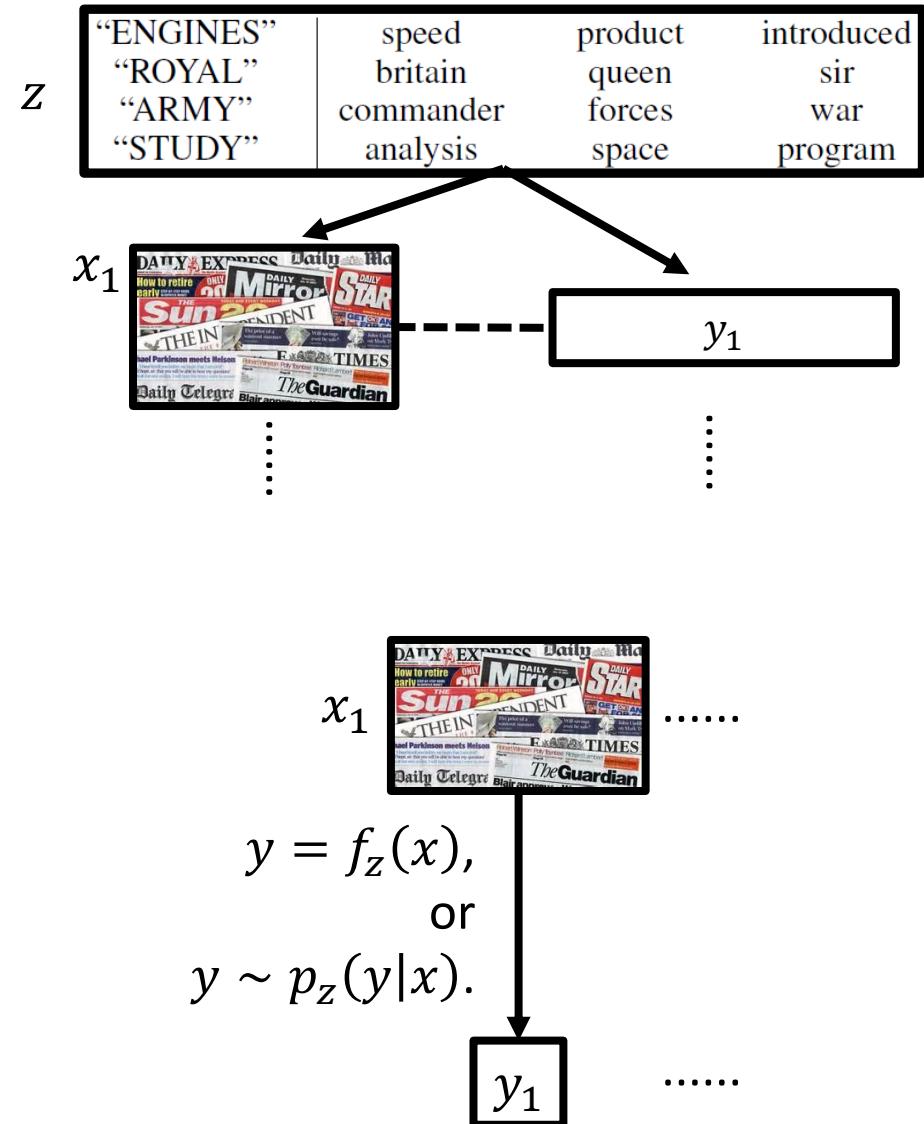
(may has a bias term  
 $b^{(v)^\top} v + b^{(h)^\top} h$ )

# Bayesian Model: Taxonomy

For supervised tasks,

- Generative model:
  - Model all the data variable.
  - $p(x, y)$  is available.

- Non-generative model:
  - Directly model the dependency of  $y$  on  $x$ .
  - Only  $p(y|x)$  is available.



# Bayesian Model: Taxonomy

For supervised tasks,

- Generative model:
  - Responsible knowledge of the data.

*“What I cannot create, I do not understand.”*

—Richard Feynman

- More data-efficient.  
For logistic regression (discriminative) and naive Bayes (generative) [NJ01],

$$\epsilon_{\text{Dis},N} \leq \epsilon_{\text{Dis},\infty} + O\left(\sqrt{\frac{d}{N}}\right), \epsilon_{\text{Gen},N} \leq \epsilon_{\text{Gen},\infty} + O\left(\sqrt{\frac{\log d}{N}}\right). \quad (N: \text{data size. } d: \text{dimension})$$

- Can leverage unlabeled data (semi-supv. learning).
- Non-generative model:

- Better results with big enough data.

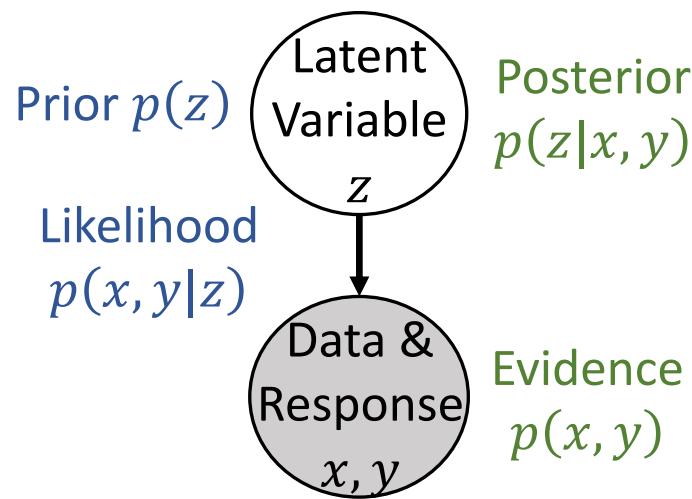
[NJ01] If  $x_1, \dots, x_N$  are independent given  $y$ , then  $\epsilon_{\text{Dis},\infty} \approx \epsilon_{\text{Gen},\infty}$ .

If not, then  $\epsilon_{\text{Dis},\infty} < \epsilon_{\text{Gen},\infty}$ .

# Bayesian Model: Taxonomy

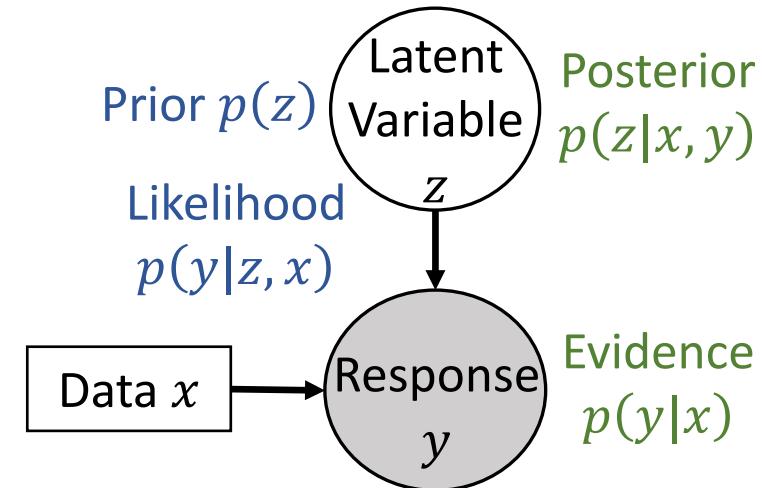
For supervised tasks,

Generative Bayesian Models



$$p(y^*|x^*, x, y) = \int p(y^*|z, x^*)p(z|x^*, x, y) dz.$$

Non-Generative Bayesian Models



$$p(y^*|x^*, x, y) = \int p(y^*|z, x^*)p(z|x, y) dz.$$

# Bayesian Model: Taxonomy

For supervised tasks,

Bayesian Models

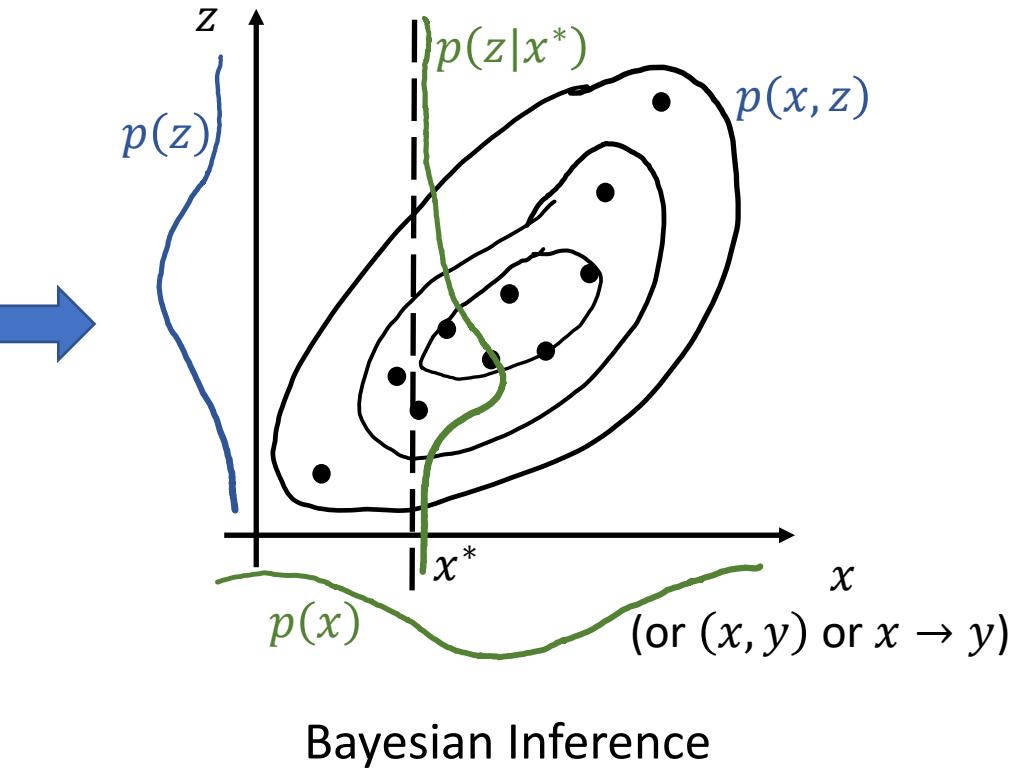
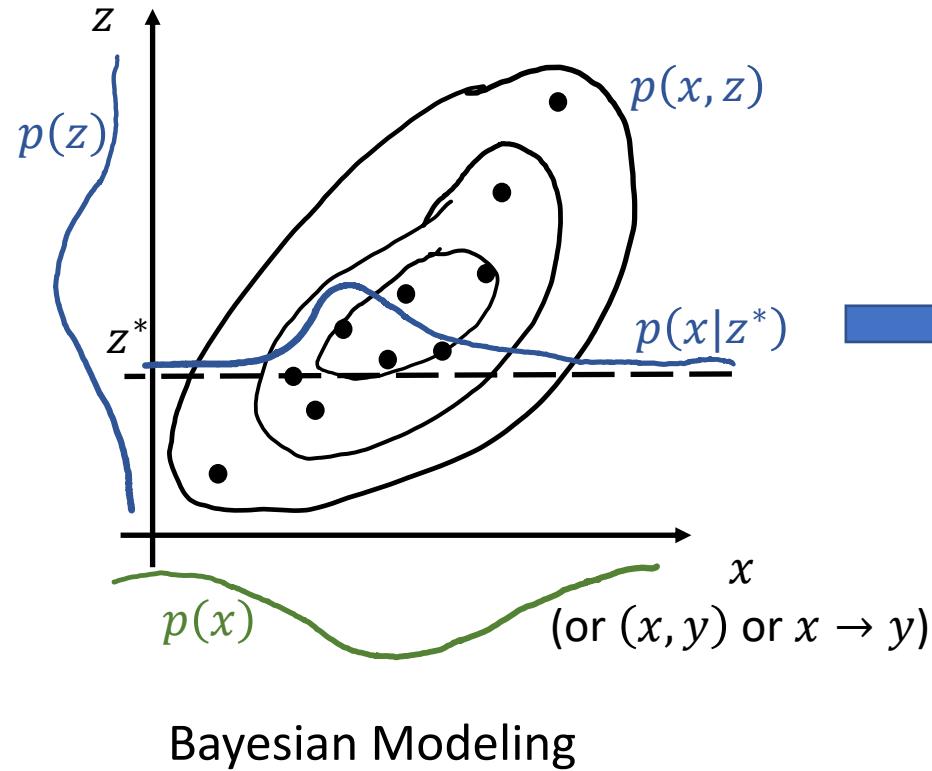
	Generative	Non-generative
Supervised	NB, sLDA, MedLDA, sVAE	BLR, BNN
Unsupervised	BayesNets (LDA, HMM, SBN, VAE), MRFs (BM, RBM, DBM, PoE)	(invalid task)

# Overview

## Bayesian Inference

# Bayesian Inference

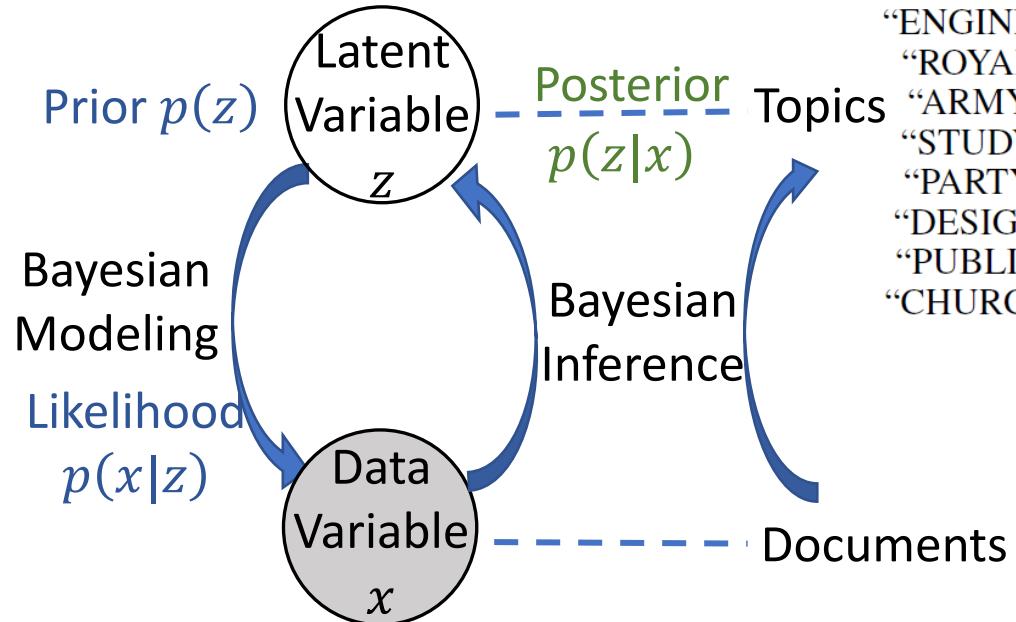
Estimate the posterior  $p(z|x)$ .



# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Motivation 1: extract knowledge/representation from data.



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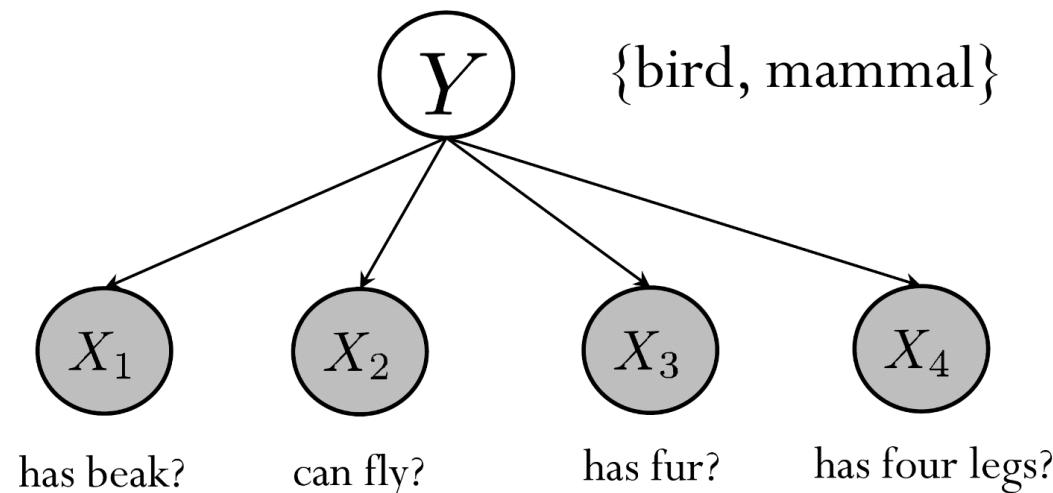


# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Motivation 1: extract knowledge/representation from data.

Naive Bayes:  $z = y$ .



$$p(y=0|x) = \frac{p(x|y=0)p(y=0)}{p(x|y=0)p(y=0) + p(x|y=1)p(y=1)}$$

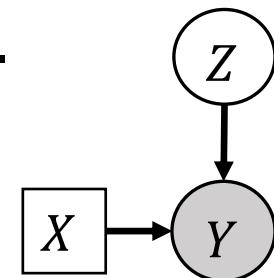
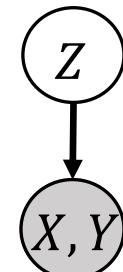
$f(x) = \arg \max_y p(y|x)$  achieves the lowest error  $\int p(y = (1 - f(x))|x) p(x) dx$ .

# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Motivation 2: prediction.

$$p(y^*|x^*, x, y) = \begin{cases} \int p(y^*|z, x^*)p(z|x^*, x, y) dz, \\ \int p(y^*|z, x^*)p(z|x, y) dz. \end{cases}$$



# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Motivation 3: facilitate model learning:  $\max_{\theta} \frac{1}{N} \sum_{n=1}^N \log p_{\theta}(x^{(n)})$ .
  - $p_{\theta}(x) = \int p_{\theta}(x|z)p_{\theta}(z) dz$  is *hard* to evaluate:
    - Closed-form integration is generally unavailable.
    - Numerical integration
      - Curse of dimensionality
      - Hard to optimize.
  - $\log p_{\theta}(x) = \log \mathbb{E}_{p(z)}[p_{\theta}(x|z)] \approx \log \frac{1}{N} \sum_n p_{\theta}(x|z^{(n)})$ ,  $\{z^{(n)}\} \sim p(z)$ .
    - Hard for  $p(z)$  to cover regions where  $p_{\theta}(x|z)$  is large.
    - $\log \frac{1}{N} \sum_n p_{\theta}(x|z^{(n)})$  is biased:
$$\mathbb{E} \left[ \log \frac{1}{N} \sum_n p_{\theta}(x|z^{(n)}) \right] \leq \log \mathbb{E} \left[ \frac{1}{N} \sum_n p_{\theta}(x|z^{(n)}) \right] = \log p_{\theta}(x).$$

# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Motivation 3: facilitate model learning:  $\max_{\theta} \frac{1}{N} \sum_{n=1}^N \log p_{\theta}(x^{(n)})$ .

An effective and practical learning approach:

- Introduce a *variational distribution*  $q(z)$ :

$$\begin{aligned}\log p_{\theta}(x) &= \mathcal{L}_{\theta}[q(z)] + \text{KL}(q(z), p_{\theta}(z|x)), \\ \mathcal{L}_{\theta}[q(z)] &\coloneqq \mathbb{E}_{q(z)}[\log p_{\theta}(z, x)] - \mathbb{E}_{q(z)}[\log q(z)].\end{aligned}$$

- $\mathcal{L}_{\theta}[q(z)] \leq \log p_{\theta}(x) \rightarrow$  Evidence Lower BOund (ELBO)!
- $\mathcal{L}_{\theta}[q(z)]$  is easier to compute ( $p_{\theta}(z, x)$  is available for BayesNets).
- (Variational) Expectation-Maximization Algorithm:

- Bayesian Inference
- 
- (a) E-step: Let  $\mathcal{L}_{\theta}[q(z)] \approx \log p_{\theta}(x), \forall \theta \Leftrightarrow \overbrace{\min_{q \in \mathcal{Q}} \text{KL}(q(z), p_{\theta}(z|x))}$ ;
- (b) M-step:  $\max_{\theta} \mathcal{L}_{\theta}[q(z)]$ .

# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- It is a hard problem
  - Closed form of  $p(z|x) \propto p(z)p(x|z)$  is generally intractable.
    - Numerical integration suffers from the curse of dimensionality.

# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- It is a hard problem
  - We care about *expectations* w.r.t  $p(z|x)$  (prediction, computing ELBO).
    - So that even if we know the closed form (e.g., by numerical integration), downstream tasks are still hard.
    - So that the Maximum *a Posteriori* (MAP) estimate

$$\arg \max_z \log p\left(z \middle| \{x^{(n)}\}_{n=1}^N\right) = \arg \max_z \log p(z) + \sum_{n=1}^N \log p(x^{(n)}|z)$$

does not help much for Bayesian tasks.

Modeling Method	Mathematical Problem
Parametric Method	Optimization
Bayesian Method	Bayesian Inference

# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Variational inference (VI)

Use a *tractable* variational distribution  $q(z)$  to approximate  $p(z|x)$ :

$$\min_{q \in Q} \text{KL}(q(z), p(z|x)).$$

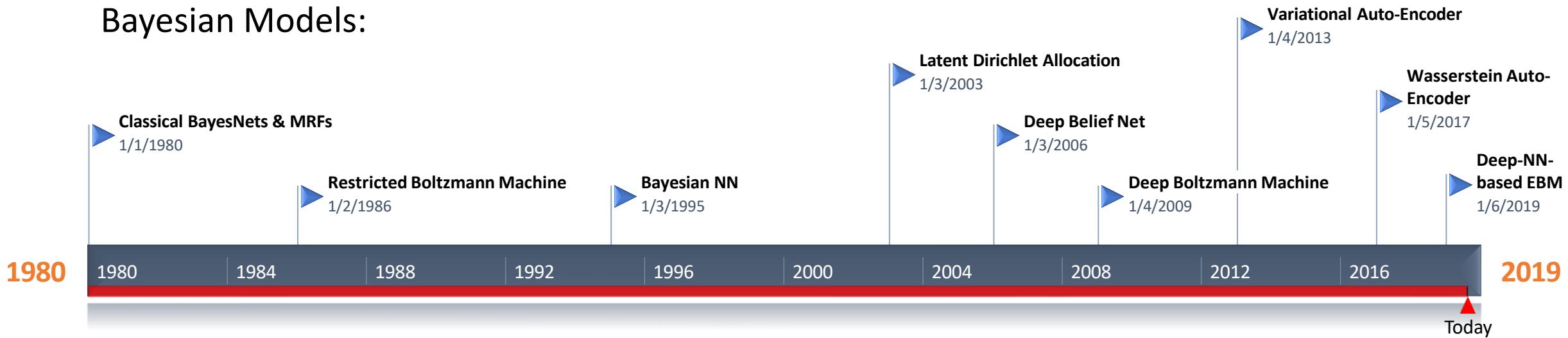
**Tractability:** known density function, or samples are easy to draw.

- Parametric VI: use a parameter  $\phi$  to represent  $q_\phi(z)$ .
- Particle-based VI: use a set of particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .
- Monte Carlo (MC)
  - Draw samples from  $p(z|x)$ .
  - Typically by simulating a *Markov chain* (i.e., MCMC) to release requirements on  $p(z|x)$ .

# Bayesian Learning

- Timelines

## Bayesian Models:



# Bayesian Learning

- Timelines

## Variational Inference:



## MCMC:



# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Parametric Variational Inference

# Bayesian Inference: Overview Recap.

Estimate the posterior  $p(z|x)$ .

- Variational inference (VI)

Use a *tractable* variational distribution  $q(z)$  to approximate  $p(z|x)$ :

$$\min_{q \in Q} \text{KL}(q(z), p(z|x)).$$

**Tractability:** known density function, or samples are easy to draw.

- Parametric VI: use a parameter  $\phi$  to represent  $q_\phi(z)$ .
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- Monte Carlo (MC)
  - Draw samples from  $p(z|x)$ .
  - Typically by simulating a *Markov chain* (i.e., MCMC) to release requirements on  $p(z|x)$ .

# Variational Inference

$$\min_{q \in \mathcal{Q}} \text{KL}(q(z), p(z|x)).$$

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

But  $\text{KL}(q_\phi(z), p_\theta(z|x))$  is hard to compute...

Recall  $\log p_\theta(x) = \mathcal{L}_\theta[q(z)] + \text{KL}(q(z), p_\theta(z|x)),$

so  $\min_\phi \text{KL}(q_\phi(z), p(z|x)) \Leftrightarrow \max_\phi \mathcal{L}_\theta[q_\phi(z)].$

The ELBO  $\mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)]$  is easier to compute.

# Variational Inference

$$\min_{q \in \mathcal{Q}} \text{KL}(q(z), p(z|x)).$$

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

- Information theory perspective of the ELBO: Bits-Back Coding [HV93].

- Average coding length for communicating  $x$  after communicating its code  $z$ :

$$\mathbb{E}_{q(z|x)}[-\log p(x|z)].$$

- Average coding length for communicating  $z$  under the bits-back coding:

$$\mathbb{E}_{q(z|x)}[-\log p(z)] - \mathbb{E}_{q(z|x)}[-\log q(z|x)].$$

The second term: the receiver knows the encoder  $q(z|x)$  that the sender uses.

- Average coding length for communicating  $x$  with the help of  $z$ :

$$\mathbb{E}_{q(z|x)}[-\log p(x|z) - \log p(z) + \log q(z|x)].$$

This coincides with the negative ELBO!

Maximize ELBO = Minimize averaged coding length under the bits-back scheme.

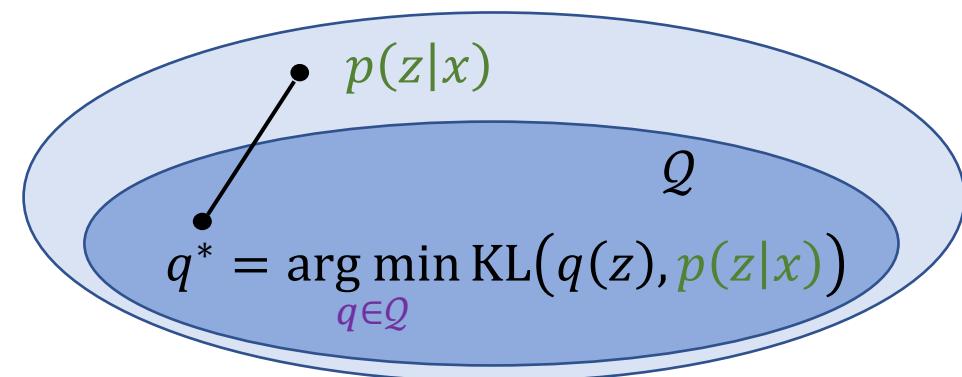
# Variational Inference

$$\min_{q \in \mathcal{Q}} \text{KL}(q(z), p(z|x)).$$

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

The Main Challenge:

- $\mathcal{Q}$  needs to enable practical computation of the ELBO,
- and needs to be as large/general/flexible as possible.



# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .  
$$\max_{\phi} \left( \mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)] \right).$$
- Explicit variational inference: specify the form of the density function  $q_\phi(z)$ .
  - Model-specific  $q_\phi(z)$ : ELBO( $\theta, \phi$ ) has closed form  
(e.g., [SJJ96] for SBN, [BNJ03] for LDA).

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} (\mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)]).$$

- Explicit variational inference: specify the form of the density function  $q_\phi(z)$ .

To be applicable to any model (model-agnostic  $q_\phi(z)$ ):

- [GHB12]: mixture of Gaussian  $q_\phi(z) = \frac{1}{N} \sum_{n=1}^N \mathcal{N}(z | \mu_n, \sigma_n^2 I)$ .

$$\text{Blue} = \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{\mathcal{N}(\mu_n, \sigma_n^2 I)}[f(z)]$$

$$\approx \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{\mathcal{N}(\mu_n, \sigma_n^2 I)}[\text{Taylor}_2(f, \mu_n)] = \frac{1}{N} \sum_{n=1}^N f(\mu_n) + \frac{\sigma_n^2}{2} \text{tr}(\nabla^2 f(\mu_n)),$$

$$\text{Red} \geq -\frac{1}{N} \sum_{n=1}^N \log \sum_{j=1}^N \mathcal{N}(\mu_n | \mu_j, (\sigma_n^2 + \sigma_j^2)I) + \log N.$$

- [RGB14]: mean-field  $q_\phi(z) = \prod_{d=1}^D q_{\phi_d}(z_d)$ .

- $\nabla_\theta \mathcal{L}_\theta[q_\phi] = \mathbb{E}_{q_\phi(z)}[\nabla_\theta \log p_\theta(z, x)]$ .

- $\nabla_\phi \mathcal{L}_\theta[q_\phi] = \mathbb{E}_{q_\phi(z)}[(\nabla_\phi \log q_\phi(z))(\log p_\theta(z, x) - \log q_\phi(z))]$

(similar to REINFORCE [Wil92]) (with variance reduction).

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} (\mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)]).$$

- Explicit variational inference: specify the form of the density function  $q_\phi(z)$ .

To be more flexible and model-agnostic:

- [RM15, KSJ+16]: define  $q_\phi(z)$  by a generative model:

$$z \sim q_\phi(z) \Leftrightarrow z = g_\phi(\epsilon), \epsilon \sim q(\epsilon),$$

where  $g_\phi$  is invertible (flow model).

Density function  $q_\phi(z)$  is known!

$$q_\phi(z) = q\left(\epsilon = g_\phi^{-1}(z)\right) \left| \frac{\partial g_\phi^{-1}}{\partial z} \right|. \quad (\text{rule of change of variables})$$

$$\mathcal{L}_\theta[q_\phi] = \mathbb{E}_{q(\epsilon)} \left[ \log p_\theta(z, x) \Big|_{z=g_\phi(\epsilon)} - \log q_\phi(z) \Big|_{z=g_\phi(\epsilon)} \right].$$

# Variational Inference

$$\min_{q \in \mathcal{Q}} \text{KL}(q(z), p(z|x)).$$

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .
  - Expectation Propagation [Min01]:

$p(z|{\{x^{(n)}\}}_{n=1}^N) \propto p(z) \prod_{n=1}^N p(x^{(n)}|z)$ , so let  $q(z) = p(z) \prod_{n=1}^N q_{\phi_n}(z)$ , and

$$\arg \min_{\phi_n} \text{KL}(\tilde{q}_n(z), q(z)) = \arg \min_{\phi_n} \mathbb{E}_{\tilde{q}_n} [\log q_{\phi_n}(z)],$$

where  $\tilde{q}_n(z) \propto \underbrace{\frac{q(z)}{q_{\phi_n}(z)}}_{\text{cavity distr.}} p(x^{(n)}|z) = p(z) p(x^{(n)}|z) \prod_{n' \neq n} q_{\phi_{n'}}(z)$ .

- Local objective. Also has a global objective reformulation.
- For  $q_{\phi_n}(z) \propto h(z) \exp(\phi_n^\top T_n(z))$ , EP  $\rightarrow$  Solve  $\mathbb{E}_{\tilde{q}_n}[T_n(z)] = \mathbb{E}_{q_{\phi_n}}[T_n(z)]$  for  $\phi_n$ .
- Stochastic EP [LHT15].

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} \left( \mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)] \right).$$

- Implicit variational inference: define  $q_\phi(z)$  by a generative model:

$$z \sim q_\phi(z) \Leftrightarrow z = g_\phi(\epsilon), \epsilon \sim q(\epsilon),$$

where  $g_\phi$  is a general function.

- More flexible than explicit VIs.
- Samples are easy to draw, but density function  $q_\phi(z)$  is unavailable.

$$\mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q(\epsilon)} \left[ \log p_\theta(x|z) \Big|_{z=g_\phi(\epsilon)} \right] - \mathbb{E}_{q(\epsilon)} \left[ \log \frac{q_\phi(z)}{p(z)} \Big|_{z=g_\phi(\epsilon)} \right].$$

Key Problem:

- Density Ratio Estimation  $r(z) := \frac{q_\phi(z)}{p(z)}$ .
- Gradient Estimation  $\nabla \log q(z)$ .

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} \left( \mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)] \right).$$

- Implicit variational inference

Density Ratio Estimation:

- [MNG17]:  $\log r = \arg \max_T \mathbb{E}_{q_\phi(z)}[\log \sigma(T(z))] + \mathbb{E}_{p(z)}[\log(1 - \sigma(T(z)))]$ .

Also used in [MSJ+15, Hus17, TRB17].

- [SSZ18a]:  $r = \arg \min_{\hat{r}} \frac{1}{2} \mathbb{E}_p[(\hat{r} - r)^2]$ , so

$$r \approx \arg \min_{\hat{r} \in \mathcal{H}} \frac{1}{2} \mathbb{E}_p[(\hat{r} - r)^2] + \frac{\lambda}{2} \|\hat{r}\|_{\mathcal{H}}^2 \approx \frac{1}{\lambda N_q} \mathbf{1}^\top K_q - \frac{1}{\lambda N_p N_q} \mathbf{1}^\top K_{qp} \left( \frac{1}{N_p} K_{pp} + \lambda I \right)^{-1} K_p,$$

where  $K_p(z)_j = K(z_j^{(p)}, z)$ ,  $(K_{qp})_{ij} = K(z_i^{(q)}, z_j^{(p)})$ ,  $\{z_i^{(q)}\}_{i=1}^{N_q} \sim q_\phi(z)$ ,  $\{z_j^{(p)}\}_{j=1}^{N_p} \sim p(z)$ .

- Other techniques: (Bickel et al., 2007), KLIEP (Sugiyama et al., 2008), LSIF (Kanamori et al., 2009), (Hido et al., 2011), (Sugiyama et al., 2012).

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} \left( \mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)] \right).$$

- Implicit variational inference

Gradient Estimation:

- [VLBM08]:

$$\begin{aligned} v^*(x, z) &= \operatorname{argmin}_v \mathbb{E}_{x, z \sim q(x, z), \eta \sim \mathcal{N}_{\sigma_n}} \|v(x, z + \eta) - z\|^2 \\ &\approx z + \sigma_n^2 \frac{\partial \log q(z|x)}{\partial z} \quad \text{as } \sigma_n \rightarrow 0 \end{aligned} \tag{7}$$

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} \left( \mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)] \right).$$

- Implicit variational inference

Gradient Estimation:

- [LT18]:

- Stein's identity: w/ bound. cond. for  $\mathbf{h}: \mathbb{R}^d \rightarrow \mathbb{R}^{d'}, \mathbb{E}_q[\mathbf{h}(x) \nabla_{\mathbf{x}} \log q(\mathbf{x})^\top + \nabla_{\mathbf{x}} \mathbf{h}(x)] = \mathbf{0}$

- MC approx.:  $-\frac{1}{K} \mathbf{H} \mathbf{G} + \text{err} = \overline{\nabla_{\mathbf{x}} \mathbf{h}}$ .

So define the approximator:  $\hat{\mathbf{G}}_V^{\text{Stein}} := \arg \min_{\hat{\mathbf{G}} \in \mathbb{R}^{K \times d}} \|\overline{\nabla_{\mathbf{x}} \mathbf{h}} + \frac{1}{K} \mathbf{H} \hat{\mathbf{G}}\|_F^2 + \frac{\eta}{K^2} \|\hat{\mathbf{G}}\|_F^2$

Solution:  $\hat{\mathbf{G}}_V^{\text{Stein}} = -(\mathbf{K} + \eta \mathbf{I})^{-1} \langle \nabla, \mathbf{K} \rangle$  where  $\mathbf{K} := \mathbf{H}^\top \mathbf{H}$ ,  $\langle \nabla, \mathbf{K} \rangle_{ij} := \sum_{k=1}^K \nabla_{x_j^k} \mathbf{K}_{ik}$ .

- Directly select a kernel (e.g., RBF kernel), instead of  $\mathbf{h}$ .
- Out-of-sample prediction:

$$\nabla_{\mathbf{y}} \log q(\mathbf{y})^\top \approx - \left( \mathbf{K}_{\mathbf{y}\mathbf{y}} + \eta - \mathbf{K}_{\mathbf{y}\mathbf{X}} (\mathbf{K} + \eta \mathbf{I})^{-1} \mathbf{K}_{\mathbf{X}\mathbf{y}} \right)^{-1} \left( \mathbf{K}_{\mathbf{y}\mathbf{X}} \hat{\mathbf{G}}_V^{\text{Stein}} - (\mathbf{K}_{\mathbf{y}\mathbf{X}} (\mathbf{K} + \eta \mathbf{I})^{-1} + \mathbf{1}^\top) \nabla_{\mathbf{y}} \mathcal{K}(\cdot, \mathbf{y}) \right).$$

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} \left( \mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)] \right).$$

- Implicit variational inference

Gradient Estimation:

- [SSZ18b]:

- The Nyström Method: approximation to the eigenfunctions  $\{\psi_i\}_{i=1}^\infty$  of kernel  $k$  in  $L_q^2$ :

$\hat{\psi}_i(x) = \frac{\sqrt{M}}{\lambda_i} \sum_{m=1}^M k(x, x^{(m)}) u_{im}$ , where  $\{x^{(m)}\}_{m=1}^M \sim q$ , and  $\{u_i\}_{i=1}^M, \{\lambda_i\}_{i=1}^M$  are the eigenvectors and eigenvalues of the Gram matrix  $(k(x^{(m)}, x^{(n)}))_{m,n}$ .

- $\partial_d \log q(x) = \sum_{j=1}^\infty \beta_{dj} \psi_j(x), \beta_{dj} = \mathbb{E}_q[\psi_j \partial_d \log q] = -\mathbb{E}_q[\partial_d \psi_j]$ . So:

$$\partial_d \log q(x) \approx \sum_{j=1}^K \hat{\beta}_{dj} \hat{\psi}_j(x), \hat{\beta}_{dj} = -\frac{1}{M} \sum_{n=1}^M \partial_d \hat{\psi}_j(x^{(n)}).$$

- Faster out-of-sample prediction than [LT18].

Assumed in  $L_q^2$

boundary  
cond. on  $k$

# Variational Inference

- Parametric variational inference: use a parameter  $\phi$  to represent  $q_\phi(z)$ .

$$\max_{\phi} \left( \mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)] \right).$$

- Implicit variational inference

Gradient Estimation:

- [SSZ18b]:

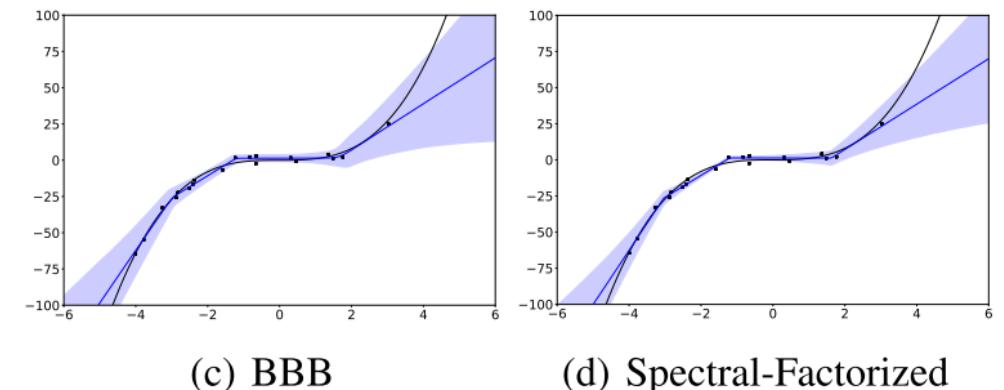
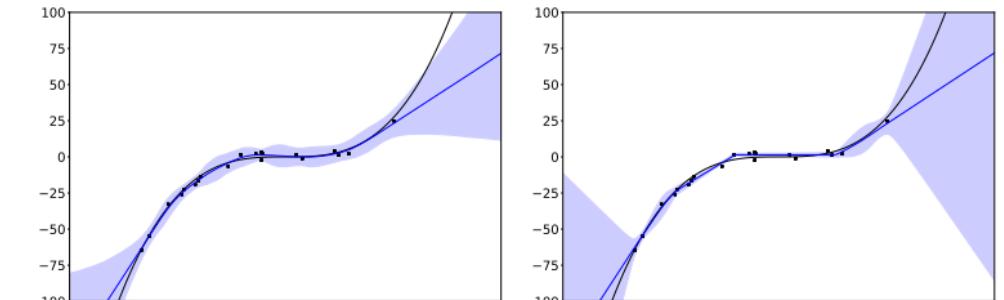
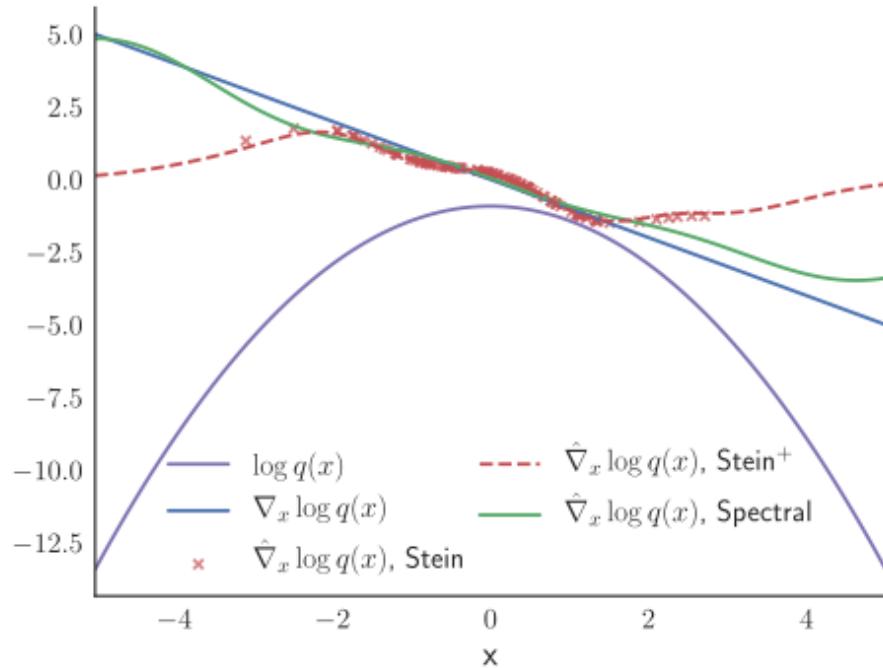


Figure 1. Gradient estimates of the log density of  $\mathcal{N}(0, 1)$ . Chang Liu (MSRA)

# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Particle-Based Variational Inference

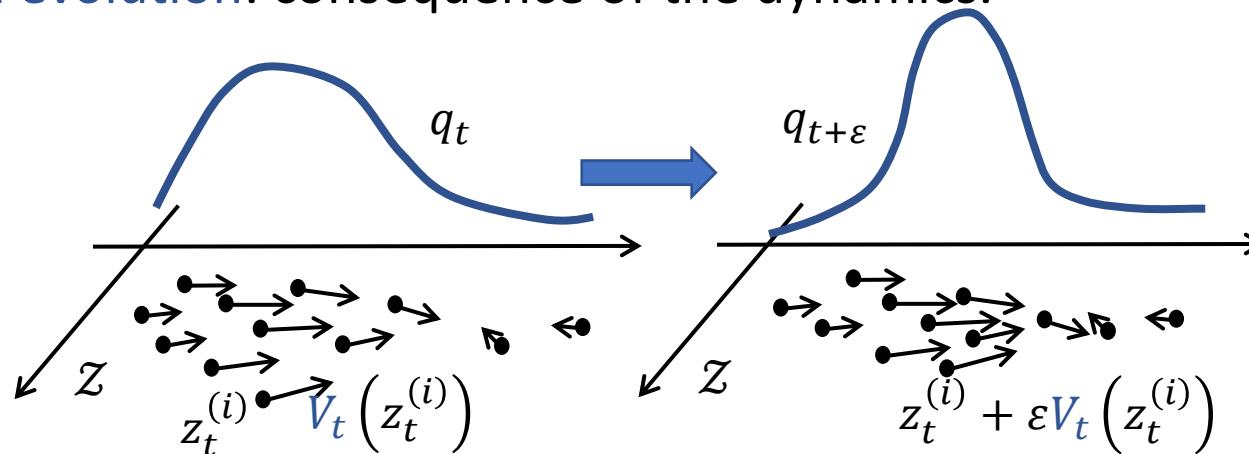
# Variational Inference

$$\min_{q \in \mathcal{Q}} \text{KL}(q, p).$$

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .  
Non-parametric  $q$ : more particles, more flexible.
  - Stein Variational Gradient Descent (SVGD) [LW16]:

Update the particles by a dynamics  $\frac{dz_t}{dt} = V_t(z_t)$  so that KL decreases.

- Distribution evolution: consequence of the dynamics.



$$\partial_t q_t = -\text{div}(q_t V_t) = -\nabla \cdot (q_t V_t) \text{ (continuity Eq.)}$$

# Variational Inference

$$\min_{q \in \mathcal{Q}} \text{KL}(q, p).$$

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .

- Stein Variational Gradient Descent (SVGD) [LW16]:

Update the particles by a dynamics  $\frac{dz_t}{dt} = V_t(z_t)$  so that **KL decreases**.

- **Decrease KL:**

$$V_t^* := \arg \max_{V_t} \left\{ -\frac{d}{dt} \text{KL}(q_t, p) = \mathbb{E}_{q_t} \underbrace{[V_t \cdot \nabla \log p + \nabla \cdot V_t]}_{\text{Stein Operator } \mathcal{A}_p[V_t]} \right\}.$$

For tractability,

$$\begin{aligned} V_t^{\text{SVGD}} &:= \max \cdot \arg \max_{V_t \in \mathcal{H}^D, \|V_t\|=1} \mathbb{E}_{q_t} [V_t \cdot \nabla \log p + \nabla \cdot V_t] \\ &= \mathbb{E}_{q(z')} [K(z', \cdot) \nabla_{z'} \log p(z') + \nabla_{z'} K(z', \cdot)]. \end{aligned}$$

Update rule:  $z^{(i)} += \varepsilon [\sum_j K_{ij} \nabla_{z^{(j)}} \log p(z^{(j)}) + \sum_j \nabla_{z^{(j)}} K_{ij}]$ .

$= \sum_j (z^{(i)} - z^{(j)}) K_{ij}$   
 for Gaussian Kernel:  
 Repulsive force!

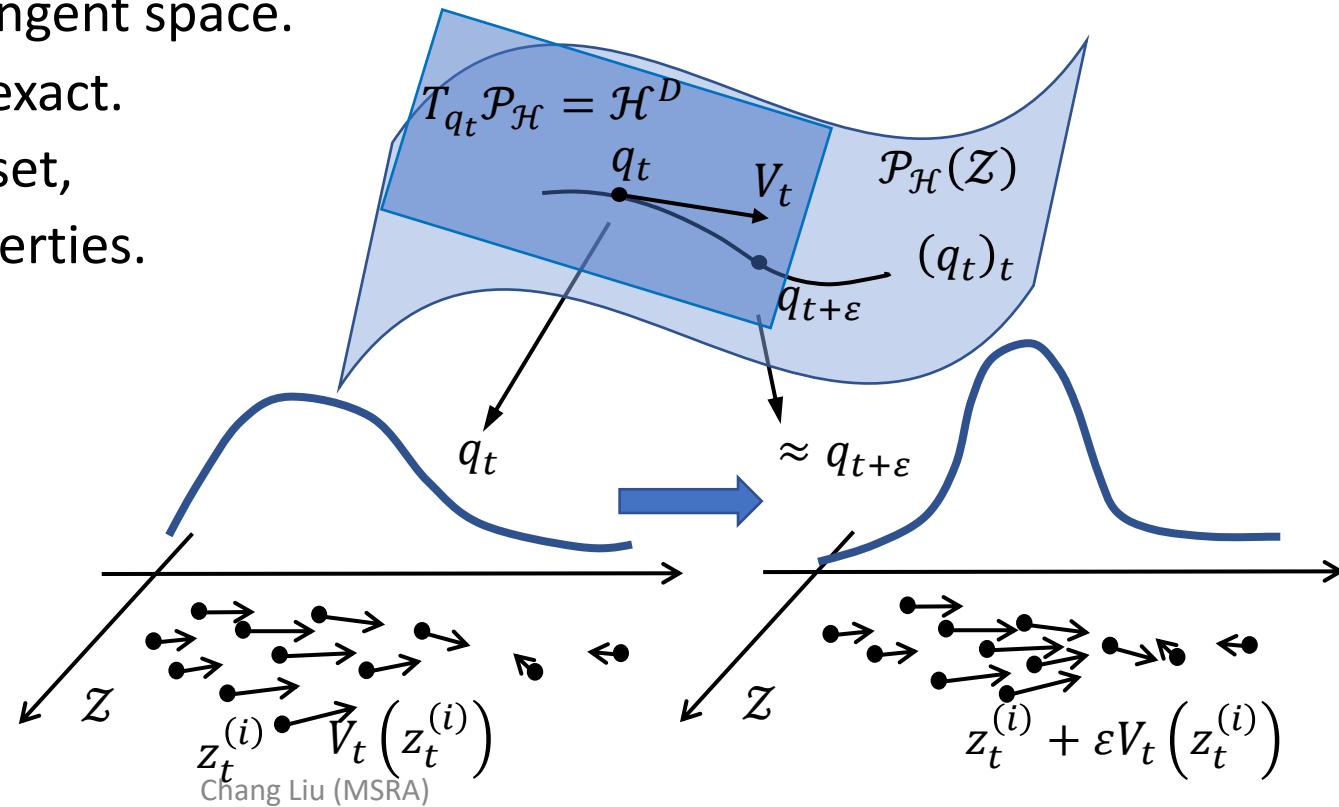
# Variational Inference

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .

- SVGD as Gradient Flow [Liu17]:

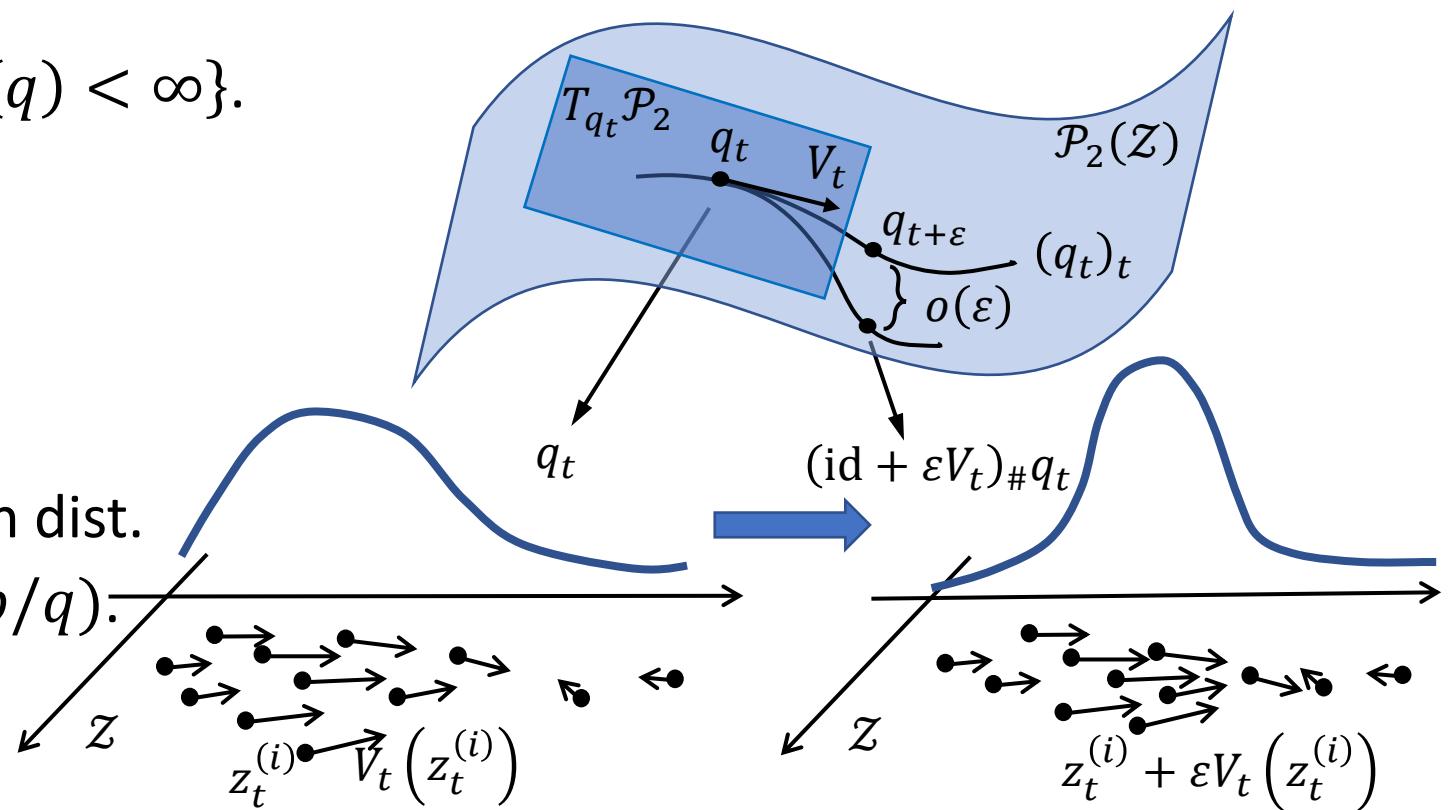
$V_t^{\text{SVGD}}$  is the gradient of  $\text{KL}(q_t, p)$  on  $\mathcal{P}_{\mathcal{H}}(\mathcal{Z})$  at  $q_t$ .

- $\mathcal{P}_{\mathcal{H}}(\mathcal{Z})$ : has  $\mathcal{H}^D$  as its tangent space.
- Explanation of  $V_t^{\text{SVGD}}$  is exact.
- $\mathcal{P}_{\mathcal{H}}(\mathcal{Z})$  is unknown as a set, and lacks appealing properties.



# Variational Inference

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .
  - Wasserstein Gradient Flow [BB00, Ott01, Vil08, AGS08]:
    - Wasserstein space:
    - $\mathcal{P}_2(\mathcal{Z}) := \{\text{distr. on } \mathcal{Z} \mid \text{Var}(q) < \infty\}$ .
    - Wasserstein tangent vector  
 $\Leftrightarrow$  vector field on  $\mathcal{Z}$ .
    - $T_q \mathcal{P}_2 = \overline{\{\nabla \varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{\mathcal{L}_q^2}$ ,  
 $\langle V, U \rangle_{T_q \mathcal{P}_2} = \mathbb{E}_q[V \cdot U]$ .
    - Consistent with Wasserstein dist.
    - $-\text{grad}_q \text{KL}(q, p) = \nabla \log(p/q)$ .
    - $\text{Exp}_q(V) = (\text{id} + V)_\# q$ .



# Variational Inference

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .
  - Wasserstein Gradient Flow (WGF)

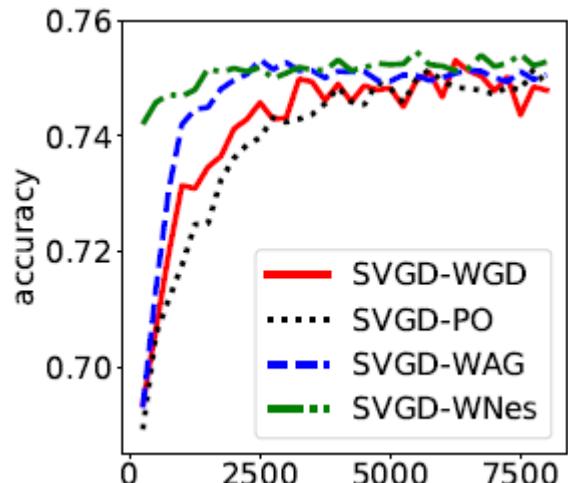
$$V := -\text{grad}_q \text{KL}(q, p) = \nabla \log(p/q), \\ z^{(i)} += \varepsilon V(z^{(i)}).$$

$$V(z^{(i)}) \approx$$

- Blob [CZW+18]:  $\nabla_{z^{(i)}} \log p(z^{(i)}) - \frac{\sum_j \nabla_{z^{(i)}} K_{ij}}{\sum_k K_{ik}} - \sum_j \frac{\nabla_{z^{(i)}} K_{ij}}{\sum_k K_{jk}}$ .
- GFSD [LZC+19]:  $\nabla_{z^{(i)}} \log p(z^{(i)}) - \frac{\sum_j \nabla_{z^{(i)}} K_{ij}}{\sum_k K_{ik}}$ .
- GFSF [LZC+19]:  $\nabla_{z^{(i)}} \log p(z^{(i)}) + \sum_{j,k} (K^{-1})_{ik} \nabla_{z^{(j)}} K_{kj}$ .
- [LZC+19]: particle-based VIs approximate WGF with a *compulsory smoothing* assumption, in either of the two *equivalent* forms of *smoothing the density* (Blob, GFSD) or *smoothing functions* (SVGD, GFSF).

# Variational Inference

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .
  - Acceleration on the Wasserstein space [LZC+19]:
    - Apply Riemannian Nesterov's methods to  $\mathcal{P}_2(\mathcal{Z})$ .



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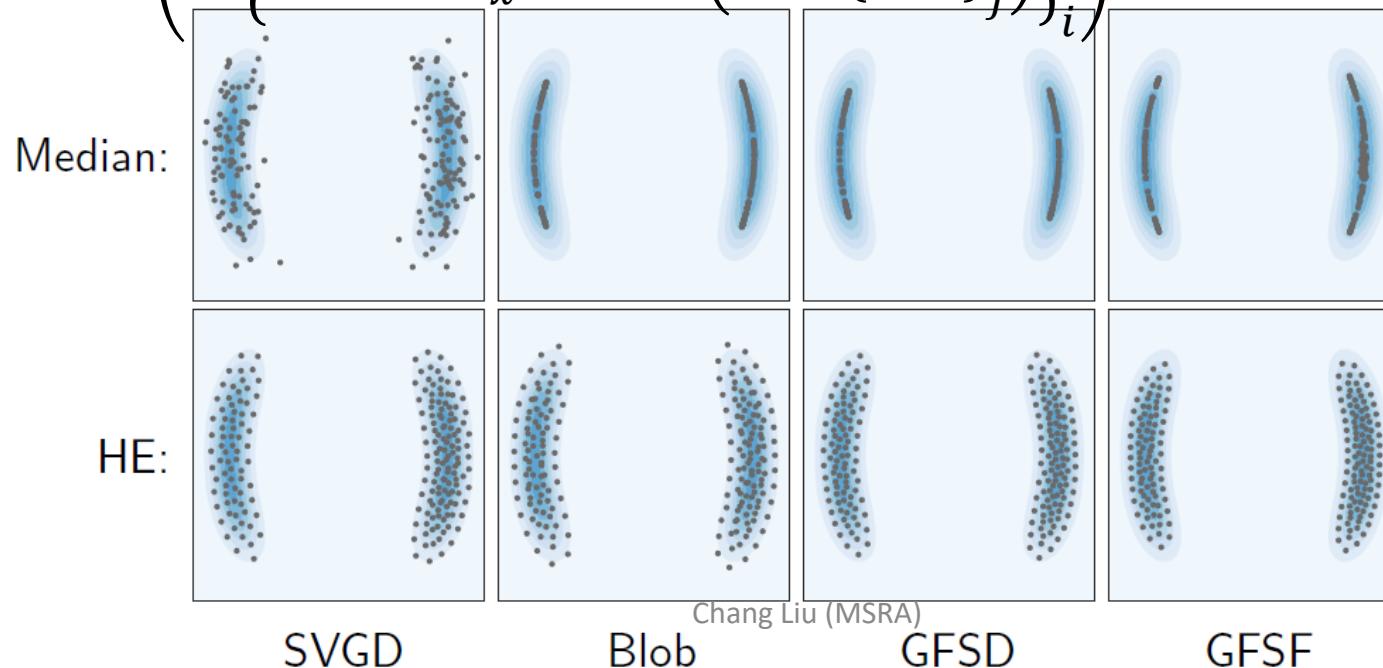
**Algorithm 1** The acceleration framework with Wasserstein Accelerated Gradient (WAG) and Wasserstein Nesterov's method (WNes)

---

```
1: WAG: select acceleration factor  $\alpha > 3$ ;  
    WNes: select or calculate  $c_1, c_2 \in \mathbb{R}^+$  (Appendix C.2);  
2: Initialize  $\{x_0^{(i)}\}_{i=1}^N$  distinctly; let  $y_0^{(i)} = x_0^{(i)}$ ;  
3: for  $k = 1, 2, \dots, k_{\max}$ , do  
4:   for  $i = 1, \dots, N$ , do  
5:     Find  $v(y_{k-1}^{(i)})$  by SVGD/Blob/GFSD/GFSF;  
6:      $x_k^{(i)} = y_{k-1}^{(i)} + \varepsilon v(y_{k-1}^{(i)})$ ;  
7:      $y_k^{(i)} = x_k^{(i)} +$   
          
$$\begin{cases} \text{WAG: } \frac{k-1}{k}(y_{k-1}^{(i)} - x_{k-1}^{(i)}) + \frac{k+\alpha-2}{k}\varepsilon v(y_{k-1}^{(i)}); \\ \text{WNes: } c_1(c_2 - 1)(x_k^{(i)} - x_{k-1}^{(i)}); \end{cases}$$
  
8:   end for  
9: end for  
10: Return  $\{x_{k_{\max}}^{(i)}\}_{i=1}^N$ .
```

# Variational Inference

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .
  - Kernel bandwidth selection:
    - Median [LW16]: median of pairwise distances of the particles.
    - HE [LZC+19]: the two approx. to  $q_{t+\varepsilon}(x)$ , i.e.,  $\tilde{q}\left(x; \{x^{(j)}\}_j\right) + \varepsilon \Delta_x \tilde{q}\left(x; \{x^{(j)}\}_j\right)$  (Heat Eq.) and  $\tilde{q}\left(x; \left\{x^{(i)} - \varepsilon \nabla_{x^{(i)}} \log \tilde{q}\left(x^{(i)}; \{x^{(j)}\}_j\right)\right\}_i\right)$  (particle evolution), should match.



# Variational Inference

- Particle-based variational inference: use particles  $\{z^{(i)}\}_{i=1}^N$  to represent  $q(z)$ .
  - Add particles dynamically: [CMG18, FCSS18].
  - Solve the Wasserstein gradient by optimal transport: [CZ17, CZW+18].
  - Manifold support space: [LZ18].
  - Asymptotic analysis: SVGD [Liu17] ( $N \rightarrow \infty, \varepsilon \rightarrow 0$ ).
  - Non-asymptotic analysis
    - w.r.t  $\varepsilon$ : e.g., [RT96] (as WGF).
    - w.r.t  $N$ : [CMG18, FCSS18, ZZC18].

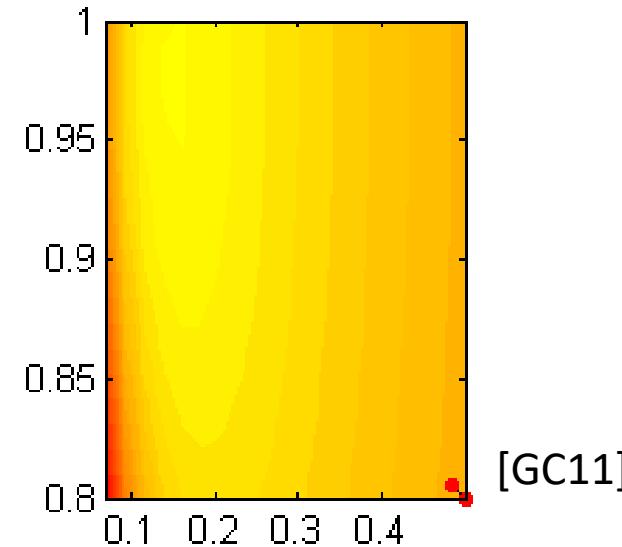
# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- MCMC

# MCMC

- Monte Carlo
  - Directly draw (i.i.d.) samples from  $p(z|x)$ .
  - Almost always impossible to directly do so.
- Markov Chain Monte Carlo (MCMC):
  - Simulate a Markov chain whose stationary distribution is  $p(z|x)$ .
  - Easier to implement: only requires unnormalized  $p(z|x)$  (e.g.,  $p(z, x)$ ).
  - Asymptotically accurate.
  - Drawback/Challenge: sample auto-correlation.
    - Less effective than i.i.d. samples.



# MCMC

- A fantastic MCMC animation site: <https://chi-feng.github.io/mcmc-demo/>

## The Markov-chain Monte Carlo Interactive Gallery

Click on an algorithm below to view interactive demo:

- Random Walk Metropolis Hastings
- Adaptive Metropolis Hastings [1]
- Hamiltonian Monte Carlo [2]
- No-U-Turn Sampler [2]
- Metropolis-adjusted Langevin Algorithm (MALA) [3]
- Hessian-Hamiltonian Monte Carlo (H2MC) [4]
- Stein Variational Gradient Descent (SVGD) [5]
- Nested Sampling with RadFriends (RadFriends-NS) [6]

View the source code on github: <https://github.com/chi-feng/mcmc-demo>.

# MCMC

## Classical MCMC

- Metropolis-Hastings framework [MRR+53, Has70]:

Draw  $z^* \sim q(z^*|z^{(k)})$  and take  $z^{(k+1)}$  as  $z^*$  with probability

$$\min\left\{1, \frac{q(z^{(k)}|z^*)p(z^*|x)}{q(z^*|z^{(k)})p(z^{(k)}|x)}\right\},$$

else take  $z^{(k+1)}$  as  $z^{(k)}$ .

- Importance sampling [Nea01, ADDJ03]:

$$\mathbb{E}_{p(z|x)}[f(z)] = \mathbb{E}_{q(z)} \left[ f(z) \frac{p(z|x)}{q(z)} \right].$$

Draw sample  $z^{(i)}$  from  $q(z)$  and assign it with importance/weight

$$\frac{p(z^{(i)}|x)}{q(z^{(i)})} \propto \frac{p(z^{(i)}, x)}{q(z^{(i)})}.$$

# MCMC

## Classical MCMC

- Gibbs sampling [GG87]:

Iteratively sample from conditional distributions, which are easier to draw:

$$z_1^{(1)} \sim p\left(z_1 \middle| z_2^{(0)}, z_3^{(0)}, \dots, z_d^{(0)}, x\right),$$

$$z_2^{(1)} \sim p\left(z_2 \middle| z_1^{(1)}, z_3^{(0)}, \dots, z_d^{(0)}, x\right),$$

$$z_3^{(1)} \sim p\left(z_3 \middle| z_1^{(1)}, z_2^{(1)}, \dots, z_d^{(0)}, x\right),$$

...

$$z_i^{(k+1)} \sim p\left(z_i \middle| z_1^{(k+1)}, \dots, z_{i-1}^{(k+1)}, z_{i+1}^{(k)}, \dots, z_d^{(k)}, x\right).$$

# MCMC

## Dynamics-based MCMC

- Simulates a jump-free continuous-time Markov process (dynamics):

$$dz = \underbrace{b(z) dt}_{\text{drift}} + \underbrace{\sqrt{2D(z)} dB_t(z)}_{\text{diffusion}},$$

$\Delta z = b(z)\varepsilon + \mathcal{N}(0, 2D(z)\varepsilon) + o(\varepsilon),$

with appropriate  $b(z)$  and  $D(z)$  so that  $p(z|x)$  is kept stationary/invariant.

- Informative transition using gradient  $\nabla_z \log p(z|x)$ .
- Some are compatible with *stochastic gradient* (SG): more efficient.

$$\nabla_z \log p(z|x) = \nabla_z \log p(z) + \sum_{d \in \mathcal{D}} \nabla_z \log p(x_d|z),$$

$$\tilde{\nabla}_z \log p(z|x) = \nabla_z \log p(z) + \frac{|\mathcal{D}|}{|\mathcal{S}|} \sum_{d \in \mathcal{S}} \nabla_z \log p(x_d|z), \quad \mathcal{S} \subset \mathcal{D}.$$

# MCMC

## Dynamics-based MCMC

- Langevin dynamics [Lan08]:

$$dz = \nabla \log p \, dt + \sqrt{2} \, dB_t(z).$$

Algorithm (also called Metropolis Adapted Langevin Algorithm) [RS02]:

$$z^{(k+1)} = z^{(k)} + \varepsilon \nabla \log p(z^{(k)}|x) + \mathcal{N}(0, 2\varepsilon),$$

followed by an MH step.

# MCMC

## Dynamics-based MCMC

- Hamiltonian Dynamics:  $\begin{cases} dz = \Sigma^{-1}r dt, \\ dr = \nabla \log p dt. \end{cases}$
- Algorithm: Hamiltonian Monte Carlo [DKPR87, Nea11, Bet17]

Draw  $r \sim \mathcal{N}(0, \Sigma)$  and simulate  $K$  steps:

$$\begin{cases} r^{(k+1/2)} = r^{(k)} + (\varepsilon/2) \nabla_z \log p(z^{(k)}|x), \\ z^{(k+1)} = z^{(k)} + \varepsilon \Sigma^{-1} r^{(k+1/2)}, \\ r^{(k+1)} = r^{(k+1/2)} + (\varepsilon/2) \nabla_z \log p(z^{(k+1)}|x), \end{cases}$$

and do an MH step, for one sample of  $z$ .

- Störmer-Verlet (leap-frog) integrator:
  - Makes MH ratio close to 1.
  - Higher-order simulation error.
- More distant exploration than LD (less auto-correlation).

# MCMC

Dynamics-based MCMC: using stochastic gradient (SG).

- Langevin dynamics is compatible with SG [WT11, CDC15, TTV16].
- Hamiltonian Monte Carlo is **incompatible** with SG [CFG14, Bet15]: the stationary distribution is changed.
- Stochastic Gradient Hamiltonian Monte Carlo [CFG14]:

$$\begin{cases} dz = \Sigma^{-1} r dt, \\ dr = \nabla \log p dt - C \Sigma^{-1} r dt + \sqrt{2C} dB_t(r). \end{cases}$$

- Asymptotically, stationary distribution is  $p$ .
- Non-asymptotically (with Euler integrator), gradient noise is of higher-order of Brownian-motion noise.

# MCMC

Dynamics-based MCMC: using stochastic gradient.

- Stochastic Gradient Nose-Hoover Thermostats [DFB+14] (scalar  $C$ ):

$$\left\{ \begin{array}{l} dz = \Sigma^{-1} r dt, \\ dr = \nabla \log p dt - \xi r dt + \sqrt{2C\Sigma} dB_t(r), \\ d\xi = \left( \frac{1}{D} r^\top \Sigma^{-1} r - 1 \right) dt. \end{array} \right.$$

- Thermostats  $\xi \in \mathbb{R}$ : adaptively balance the gradient noise and the Brownian-motion noise.

# MCMC

## Dynamics-based MCMC

- For manifold support space:
  - LD: [GC11]
  - HMC: [GC11, BSU12, BG13, LSSG15]
  - SGLD: [PT13]
  - SGHMC: [MCF15, LZS16]
  - SGNHT: [LZS16]
- Different kinetic energy (other than Gaussian):
  - Monomial Gamma [ZWC+16, ZCG+17].
- Fancy Dynamics:
  - Relativistic: [LPH+16]
  - Magnetic: [TRGT17]

# MCMC

## Dynamics-based MCMC

- Complete recipe for the dynamics [MCF15]:

For any skew-symmetric matrix  $Q$  and pos. semi-def. matrix  $D$ , the dynamics

$$\begin{aligned} dz &= b(z) dt + \sqrt{2D(z)} dB_t(z), \\ b_i &= \frac{1}{p} \sum_j \partial_j \left( p(D_{ij} + Q_{ij}) \right), \end{aligned}$$

keeps  $p$  stationary/invariant.

- The inverse also holds:
  - Any dynamics that keeps  $p$  stationary can be cast into this form.
  - If  $D$  is pos. def., then  $p$  is the unique stationary distribution.
- Integrators and their non-asymptotic analysis (with SG): [CDC15].

# MCMC and Particle-Based VI

- MCMC: dynamics that keeps  $p$  invariant.  
ParVI: dynamics that drives  $q_t$  to minimize  $\text{KL}(q_t, p)$ .
- LD: induces gradient flow of KL on  $\mathcal{P}_2$  [JKO98].  
ParVI: simulates gradient flow of KL on  $\mathcal{P}_2$  [CZW+18, LZC+19].
  - Convergence intuition
    - Exponential asymptotic convergence for convex KL (log-concave  $p$ ) [e.g., RT96, CB17].
  - Robust to SG.

# MCMC and Particle-Based VI

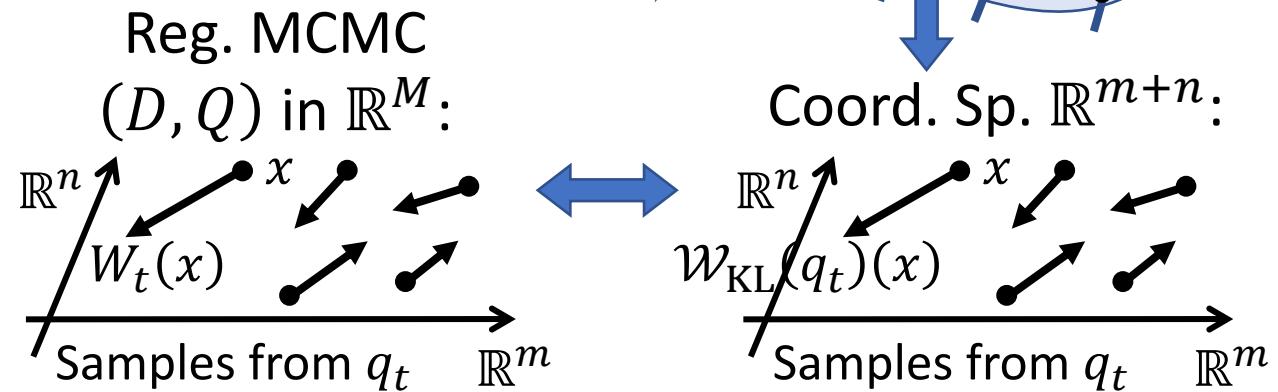
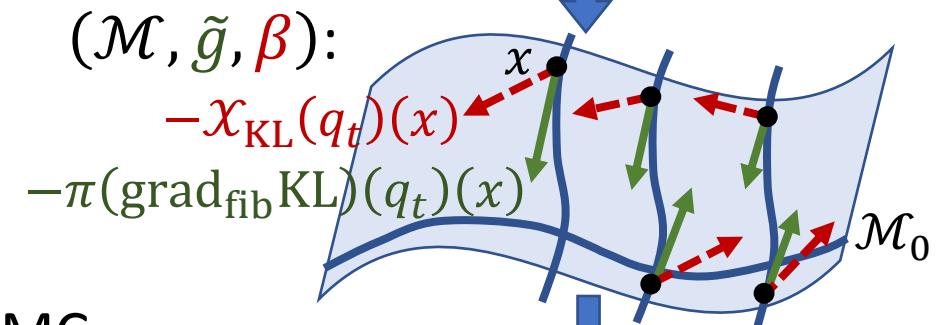
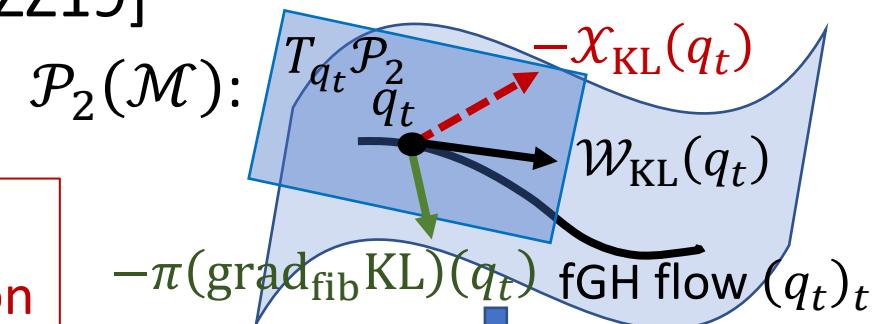
- General MCMC: fiber-Gradient Hamiltonian flow [LZZ19]

$$\mathcal{W}_{\text{KL}}(q_t) = -\pi(\text{grad}_{\text{fib}} \text{KL})(q_t) - \chi_{\text{KL}}(q_t)$$

Hamiltonian flow of KL on  $\mathcal{P}_2$ :  
keeps KL invariant; helps exploration

fiber-gradient flow of KL on  $\mathcal{P}_2$ :  
minimizes KL on each fiber  
(matches conditional distributions)

- Convergence intuitions for general MCMCs.
- New ParVIs using more efficient dynamics other than LD.



# Bayesian Inference: Comparison

	Parametric VI	Particle-Based VI	MCMC
Asymptotic Accuracy	No	Yes	Yes
Approximation Flexibility	Limited	Unlimited	Unlimited
Empirical Convergence Speed	High	High	Low
Particle Efficiency	(Do not apply)	High	Low
High-Dimensional Efficiency	High	Low	High

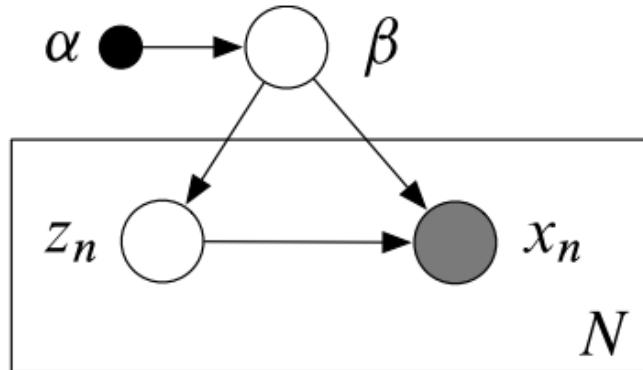
# Bayesian Inference

Estimate the posterior  $p(z|x)$ .

- Amortized Inference

# Amortized Inference

- Latent Variable: local and global.



$$p(x, \textcolor{blue}{z}, \textcolor{red}{\beta} | \alpha) = p(\beta | \alpha) \prod_{n=1}^N p(x_n, \textcolor{blue}{z}_n | \textcolor{red}{\beta}).$$

- For  $\beta$  to be global and  $z$  to be local,  $p(x_n, \textcolor{blue}{z}_n | x_{-n}, \textcolor{blue}{z}_{-n}, \textcolor{red}{\beta}, \alpha) = p(x_n, \textcolor{blue}{z}_n | \textcolor{red}{\beta}, \alpha)$ . [HBWP13]
- Typical inference task:  $p(\textcolor{red}{\beta}, \{\textcolor{blue}{z}_n\}_{n=1}^N | \{x_n\}_{n=1}^N)$ .
- More concerned task:
  - $p(\textcolor{red}{\beta} | \{x_n\}_{n=1}^N)$ : global representation of the whole data set.
  - $p(\textcolor{blue}{z}_n | x_n, \textcolor{red}{\beta})$ : fast  $x_n \rightarrow \textcolor{blue}{z}_n$  (feature map) desired (also global information).

# Amortized Inference

- Latent Variable: local and global.

$$p(x, z, \beta | \alpha) = p(\beta | \alpha) \prod_{n=1}^N p(x_n, z_n | \beta).$$

- Amortized variational inference: parametric, explicit

- [HBWP13]: stochastic variational inference.

- Assume  $p(\beta | x, z, \alpha) = h(\beta) \exp\{\eta_g(x, z, \alpha)^\top t(\beta) - a_g(\eta_g(x, z, \alpha))\}$

$$p(z_{nj} | x_n, z_{n,-j}, \beta) = h(z_{nj}) \exp\{\eta_\ell(x_n, z_{n,-j}, \beta)^\top t(z_{nj}) - a_\ell(\eta_\ell(x_n, z_{n,-j}, \beta))\}.$$

- Take variational distribution:

$$q(z, \beta) = q(\beta | \lambda) \prod_{n=1}^N \prod_{j=1}^J q(z_{nj} | \phi_{nj}). \quad q(\beta | \lambda) = h(\beta) \exp\{\lambda^\top t(\beta) - a_g(\lambda)\}$$
$$q(z_{nj} | \phi_{nj}) = h(z_{nj}) \exp\{\phi_{nj}^\top t(z_{nj}) - a_\ell(\phi_{nj})\}.$$

Then:  $\nabla_\lambda \mathcal{L} = \nabla_\lambda^2 a_g(\lambda) (\mathbb{E}_q[\eta_g(x, z, \alpha)] - \lambda)$ .  $\nabla_{\phi_{nj}} \mathcal{L} = \nabla_{\phi_{nj}}^2 a_\ell(\phi_{nj}) (\mathbb{E}_q[\eta_\ell(x_n, z_{n,-j}, \beta)] - \phi_{nj})$ .

Algorithm: • Sample a data point  $x_n$  uniformly from the data set.

- $\phi_{nj} = \mathbb{E}_{q_\lambda q_{\phi_{n,-j}}} [\eta_\ell(x_n, z_{n,-j}, \beta)], j = 1, \dots, J; \hat{\lambda} = \mathbb{E}_{q_{\phi_n}} [\eta_g(x_n^{(N)}, z_n^{(N)}, \alpha)]$ .
- $\lambda \leftarrow (1 - \rho_t)\lambda + \rho_t \hat{\lambda}$ .

# Amortized Inference

- Latent Variable: local and global.

$$p(x, z, \beta | \alpha) = p(\beta | \alpha) \prod_{n=1}^N p(x_n, z_n | \beta).$$

- Amortized variational inference: parametric, explicit

- [KW14, RMW14, RM15]: use a reparameterizable encoder for  $q_\phi(z_n | x_n)$  with available p.d.f.:

$$z_n = g_\phi(x_n, \epsilon), \epsilon \sim q(\epsilon).$$

- Objective for inferring  $z$ :

$$\begin{aligned} & \mathbb{E}_{\hat{p}(x)}[\text{ELBO}(x)] \\ & \approx \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{q_\phi(z_n | x_n)} [\log p_\theta(x_n, z_n, \beta)] - \mathbb{E}_{q_\phi(z_n | x_n)} [\log q_\phi(z_n | x_n)] \\ & = \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{q(\epsilon)} [\log p_\theta(x_n, g_\phi(x_n, \epsilon), \beta) - \log q_\phi(g_\phi(x_n, \epsilon) | x_n)]. \end{aligned}$$

Estimate the gradient by samples of  $q(\epsilon)$ .

# Amortized Inference

- Latent Variable: local and global.

$$p(x, z, \beta | \alpha) = p(\beta | \alpha) \prod_{n=1}^N p(x_n, z_n | \beta).$$

- Amortized variational inference: parametric, explicit

- [KW14, RMW14, RM15]: use a reparameterizable encoder for  $q_\phi(z_n | x_n)$  with available p.d.f.:

$$z_n = g_\phi(x_n, \epsilon), \epsilon \sim q(\epsilon).$$

- For inferring  $\beta$ ,

$$\begin{aligned} \mathcal{L}[q(\beta)] &= \mathbb{E}_{q(\beta)} [\log p(\{x_n\}_{n=1}^N, \beta) - \log q(\beta)] \\ &\geq \mathbb{E}_{q(\beta)} \left[ \sum_{n=1}^N \text{ELBO}(x_n) - \log q(\beta) \right]. \end{aligned}$$

# Amortized Inference

- Latent Variable: local and global.

$$p(x, z, \beta | \alpha) = p(\beta | \alpha) \prod_{n=1}^N p(x_n, z_n | \beta).$$

- Amortized variational inference: parametric, implicit

- [MNG17]:

$$\mathbb{E}_{\hat{p}(x)}[\text{ELBO}(x)] = \mathbb{E}_{\hat{p}(x)q_\phi(z|x)} \left[ \log p_\theta(x|z) - \log \frac{q_\phi(z|x)}{p_\theta(z)} \right].$$

- [Hus17, TRB17]:

$$\mathbb{E}_{\hat{p}(x)}[\text{ELBO}(x)] = -\mathbb{E}_{\hat{p}(x)q_\phi(z|x)} \left[ \log \frac{\hat{p}(x)q_\phi(z|x)}{p_\theta(z, x)} \right] + \mathbb{E}_{\hat{p}(x)}[\log \hat{p}(x)].$$

$q_\phi(z|x)$  is reparameterizable.

Estimate the **density ratio** by adversarial learning.

# Amortized Inference

- Latent Variable: local and global.

$$p(x, z, \beta | \alpha) = p(\beta | \alpha) \prod_{n=1}^N p(x_n, z_n | \beta).$$

- Amortized MCMC:

- [LTL17]: use MCMC update rule to guide a generator  $q_\phi(z|x)$ :

$$\phi_{t+1} = \arg \min_{\phi} \sum_{N=1}^N D \left( \underbrace{(\mathcal{K}_T)_\# q_{\phi_t}(z_n | x_n)}_{q_T}, q_{\phi_t}(z_n | x_n) \right),$$

where  $\mathcal{K}_T$  is the  $T$ -step transition of an MCMC method.

$q_T$  is easy to sample.

- $D = \text{KL}$ :  $\phi_{t+1} = \arg \min_{\phi} - \sum_{n=1}^N \mathbb{E}_{q_T} [\log q_{\phi}(z_n | x_n)].$
  - $D = \text{JS}$ : adversarial learning

$$\text{D}_{\text{JS}}[q_T || q] \geq \text{D}_{\text{adv}}[\{\mathbf{z}_T^k\} || \{\mathbf{z}_0^k\}] = \frac{1}{K} \sum_{k=1}^K \log \sigma(d_{\psi}(\mathbf{z}_T^k | \mathbf{x})) + \frac{1}{K} \sum_{k=1}^K \log(1 - \sigma(d_{\psi}(\mathbf{z}_0^k | \mathbf{x}))),$$

# Amortized Inference

- Latent Variable: local and global.

$$p(x, z, \beta | \alpha) = p(\beta | \alpha) \prod_{n=1}^N p(x_n, z_n | \beta).$$

- Amortized particle-based VI:

- [FWL17]: generator  $q_\phi(z|x)$ :  $z = g_\phi(\epsilon, x), \epsilon \sim q(\epsilon)$ .

$$\phi_{t+1} = \arg \min_{\phi} \sum_{n=1}^N \mathbb{E}_{q(\epsilon)} \left[ \|g_\phi(x_n, \epsilon) - K_1 g_{\phi_t}(x_n, \epsilon)\|_2^2 \right],$$

where  $K_1$  is the one-step update of SVGD.

$$\phi_{t+1} \approx \phi_t + \varepsilon \sum_{n=1}^N \sum_{i=1}^M \nabla_\phi g_{\phi_t}(x_n, \epsilon_i) V^{\text{SVGD}} \left( g_{\phi_t}(x_n, \epsilon_i) \right).$$

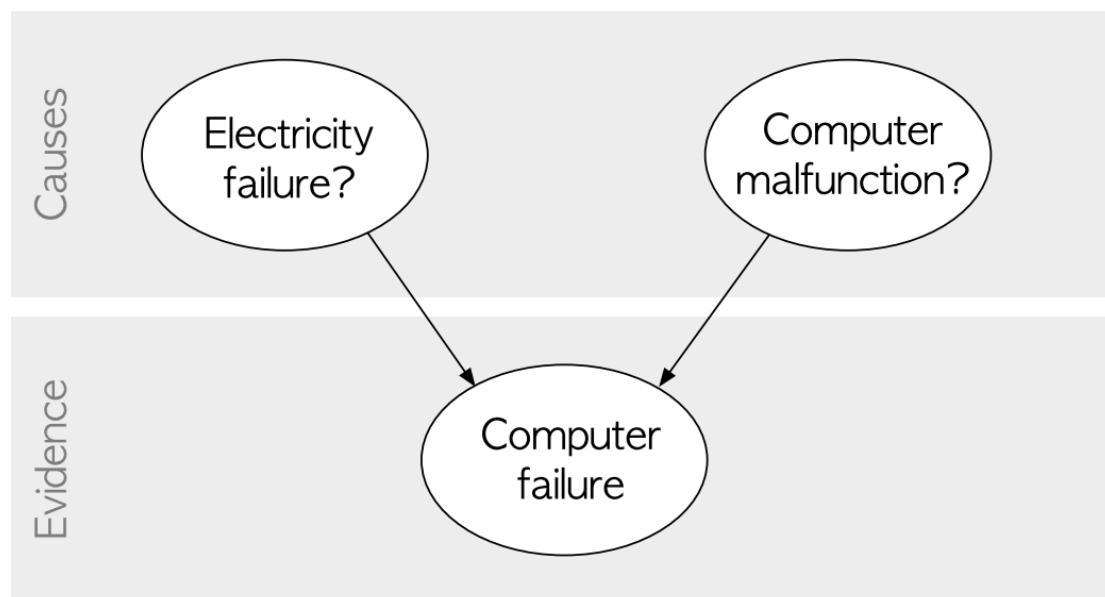
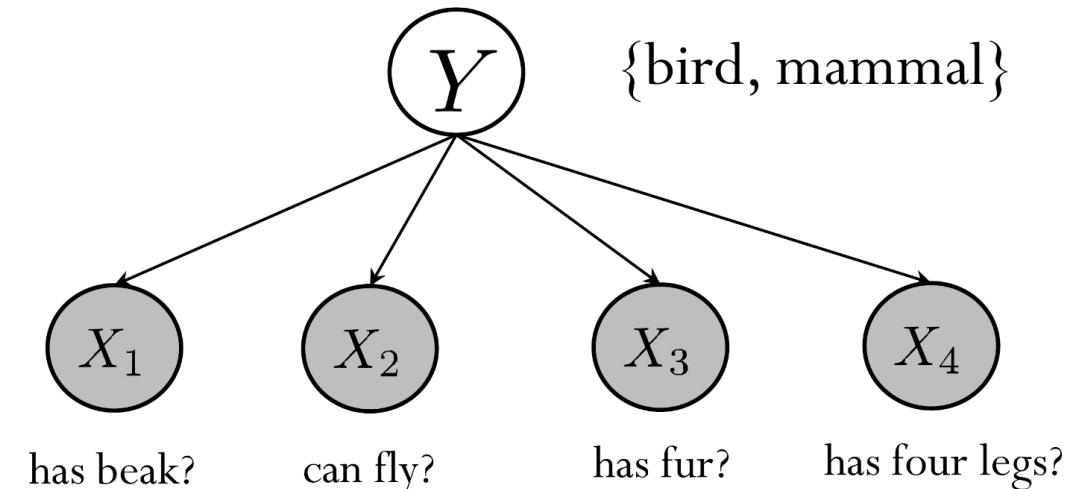
$$\text{For } \beta, \nabla_\beta \log p(\beta | \{x_n\}_{n=1}^N) = \sum_{n=1}^N \mathbb{E}_{p(z_n | \beta, x_n)} [\nabla_\beta \log p(x_n, z_n, \beta)].$$

# Bayesian Models

## Bayesian Networks

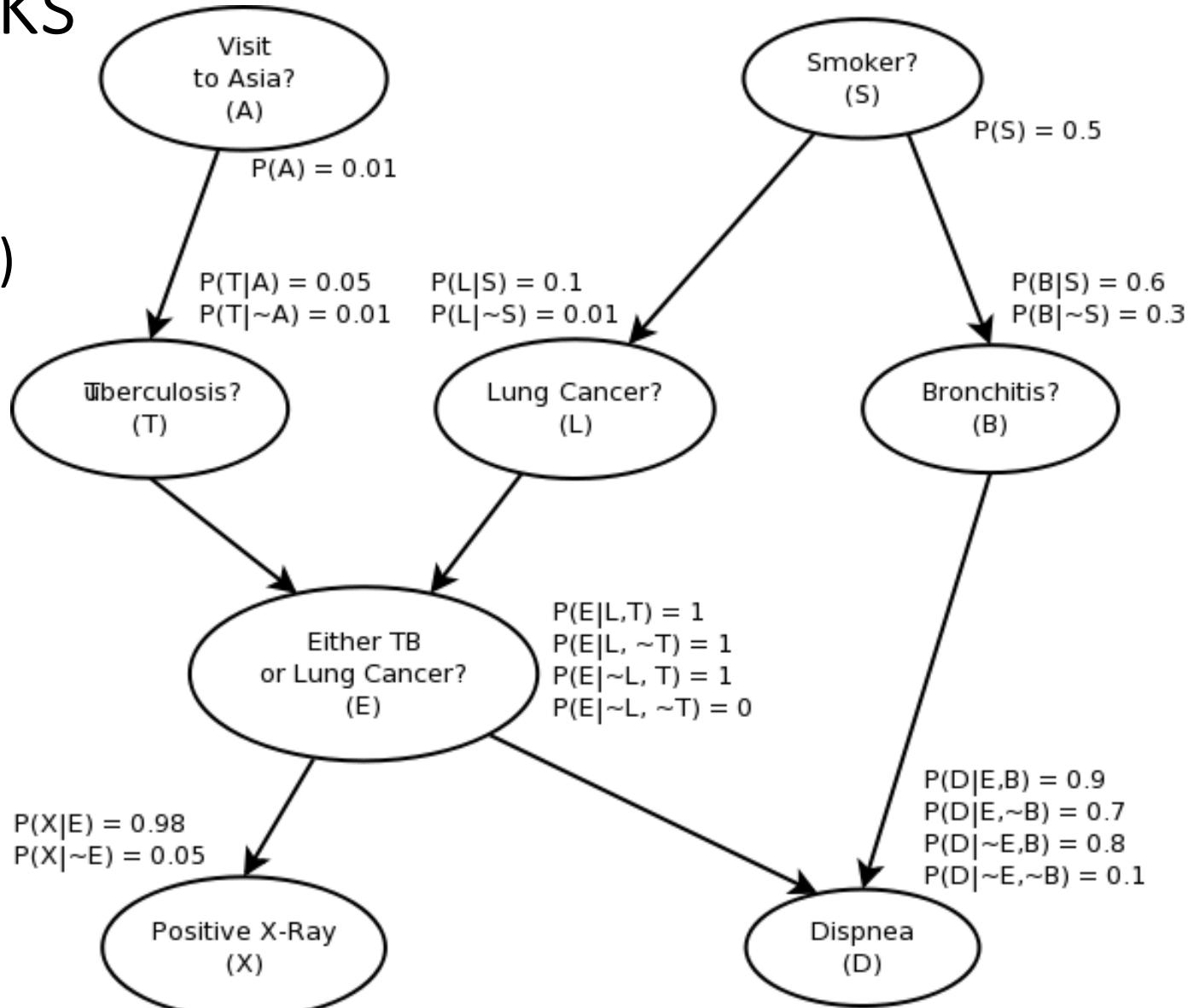
# Bayesian Networks

- Classical BayesNets  
(Causal Networks /  
Directed Graphical Models)



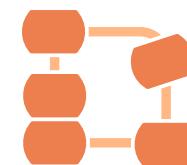
# Bayesian Networks

- Classical BayesNets  
(Causal Networks /  
Directed Graphical Models)



# Bayesian Networks

- Classical BayesNets
  - Interpretable and incorporates domain knowledge.
  - Variable Dependency <=====> Graph Structure
    - Markov blanket, independency map, ...
  - Approximate Bayes Computation (ABC) for large networks
    - Message passing, belief propagation, expectation propagation, ...
- General BayesNets:
  - Simple graph structure, but complex likelihood.
  - Reduce human knowledge requirement, increase model flexibility.
  - Sigmoid Belief Networks, Latent Dirichlet Allocation, Variational Auto-Encoders, ...
  - Programming Tools:
    - [ZhuSuan](#) [SCZ+17]: probabilistic programming library.
      - Easy Bayesian modeling, especially with deep neural nets.
      - Various convenient Bayesian inference tools.
      - Python interface.



ZHUSUAN

# Bayesian Models

Bayesian Networks

- Sigmoid Belief Networks

# Sigmoid Belief Networks

- Model structure [Nea92]:

$$P(S_i = s_i \mid S_j = s_j : j < i) = \sigma\left(s_i^* \sum_{j < i} s_j w_{ij}\right).$$

$$\begin{aligned} P(\tilde{S} = \tilde{s}) &= \prod_i P(S_i = s_i \mid S_j = s_j : j < i) \\ &= \prod_i \sigma\left(s_i^* \sum_{j < i} s_j w_{ij}\right). \end{aligned}$$

- Inference with MCMC: Gibbs sampling from  $P_w(h|v)$  ( $s = (h, v)$ ) [Nea92].

$$\begin{aligned} P(S_i = x \mid S_j = s_j : j \neq i) \\ \propto \sigma\left(x^* \sum_{j < i} s_j w_{ij}\right) \prod_{j > i} \sigma\left(s_j^*\left(x w_{ji} + \sum_{k < j, k \neq i} s_k w_{jk}\right)\right). \end{aligned}$$

- Learning [Nea92]:  $\nabla_w \log P_w(v) = \mathbb{E}_{P_w(h|v)}[\nabla_w \log P_w(h, v)]$ .

Gibbs sampling

# Sigmoid Belief Networks

- Sigmoid Belief Networks [Nea92]

$$P(S_i = s_i \mid S_j = s_j : j < i) = \sigma\left(s_i^* \sum_{j < i} s_j w_{ij}\right).$$

- Variational Inference: mean-field VI [SJJ96]

- Mean-field variational distribution:  $Q(H|V) = \prod_{i \in H} \mu_i^{S_i} (1 - \mu_i)^{1 - S_i}$

- ELBO:

$$\begin{aligned} \ln P(V) &\geq \sum_{ij} J_{ij} \mu_i \mu_j + \sum_i h_i \mu_i - \sum_i \left\langle \ln \left[ 1 + e^{\sum_j J_{ij} S_j + h_i} \right] \right\rangle \\ &\quad - \sum_i [\mu_i \ln \mu_i + (1 - \mu_i) \ln(1 - \mu_i)], \end{aligned}$$

where  $\langle \cdot \rangle$  indicates an expectation value over the mean field distribution.

With further relaxation, the gradient can be estimated.

# Sigmoid Belief Networks

- Sigmoid Belief Networks [Nea92]

$$P(S_i = s_i \mid S_j = s_j : j < i) = \sigma\left(s_i^* \sum_{j < i} s_j w_{ij}\right).$$

- Variational Inference: wake-sleep [HDFN95]

- Variational distribution: recognition model  $Q_\phi(h|x)$  (e.g., also an SBN).
  - Wake phase: update the original model.

$$\nabla_\theta \mathcal{L}(x) \approx \frac{1}{n} \sum_{i=1}^n \nabla_\theta \log P_\theta(x, h^{(i)})$$

where  $h^{(i)} \sim Q_\phi(h|x)$ .

- Sleep phase: update the recognition model.

$$\nabla_\phi \mathcal{L}(x) = E_{P_\theta(x,h)} [\nabla_\phi \log Q_\phi(h|x)].$$

- Intuition: learn the two models symmetrically; coordinate-descent.
    - “... does not optimize a well-defined objective function and is not guaranteed to converge.” -- [MG14]

Same as  
grad. of ELBO

Different from  
grad. of ELBO

# Sigmoid Belief Networks

- Sigmoid Belief Networks [Nea92]

$$P(S_i = s_i \mid S_j = s_j : j < i) = \sigma\left(s_i^* \sum_{j < i} s_j w_{ij}\right). \quad (11)$$

- Variational Inference: NVIL [MG14]

- Variational distribution: recognition model  $Q_\phi(h|x)$  (e.g., also an SBN).
  - Learning: same as the wake phase

$$\nabla_\theta \mathcal{L}(x) \approx \frac{1}{n} \sum_{i=1}^n \nabla_\theta \log P_\theta(x, h^{(i)})$$

- Inference: grad. of the ELBO:

$$\begin{aligned} \nabla_\phi \mathcal{L}(x) &= E_Q[(\log P_\theta(x, h) - \log Q_\phi(h|x)) \\ &\quad \times \nabla_\phi \log Q_\phi(h|x)], \end{aligned}$$

(since  $\nabla_\phi \mathbb{E}_{q_\phi}[f] = \mathbb{E}_{q_\phi}[f \nabla_\phi \log q_\phi]$ .)

- Variance reduction required.

Subtract the mean  $C_\psi(x)$ , divide by the variance, simplification using the structure.

# Bayesian Models

Bayesian Networks

- Topic Models

# Latent Dirichlet Allocation

Separate *global* latent feature of dataset and *local* representation of each datum.

- Model Structure [BNJ03]:

- Data variable:

Words/Documents  $w = \{w_{dn}\}_{n=1:N_d, d=1:D}$ .

- Latent variables:

- Global: topics  $\beta = \{\beta_k\}$ .

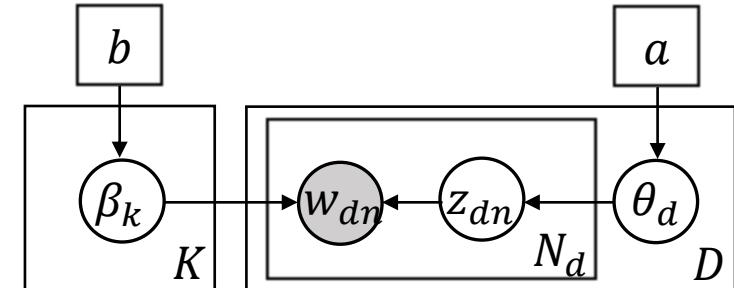
- Local: topic proportions  $\theta = \{\theta_d\}$ , topic assignments  $z = \{z_{dn}\}$ .

- Prior:  $p(\beta_k|b) = \text{Dir}(b)$ ,  $p(\theta_d|a) = \text{Dir}(a)$ ,  $p(z_{dn}|\theta_d) = \text{Mult}(\theta_d)$ .

- Likelihood:  $p(w_{dn}|z_{dn}, \beta) = \text{Mult}(\beta_{z_{dn}})$ .

- Joint distribution:  $p(\beta, \theta, z, w) =$

$$\left( \prod_{k=1}^K \text{Dir}(\beta_k|b) \right) \left( \prod_{d=1}^D \text{Dir}(\theta_d|a) \left( \prod_{n=1}^{N_d} \text{Mult}(z_{dn}|\theta_d) \text{Mult}(w_{dn}|\beta_{z_{dn}}) \right) \right).$$



# Latent Dirichlet Allocation

Variational inference [BNJ03]:

- Take variational distribution (mean-field approximation):

$$q_{\lambda, \gamma, \phi}(\beta, \theta, z) := \prod_{k=1}^K \text{Dir}(\beta_k | \lambda_k) \prod_{d=1}^D \text{Dir}(\theta_d | \gamma_d) \prod_{n=1}^{n_d} \text{Mult}(z_{dn} | \phi_{dn}).$$

- ELBO( $\lambda, \gamma, \phi; a, b$ ) is available in closed form.
- E-step: update  $\lambda, \gamma, \phi$  by maximizing ELBO;
- M-step: update  $a, b$  by maximizing ELBO.

# Latent Dirichlet Allocation

MCMC: Gibbs sampling [GS04]

$$p(\beta, \theta, z, w) = AB \left( \prod_{k,w} \beta_{kw}^{N_{kw} + b_w - 1} \right) \left( \prod_{d,k} \theta_{dk}^{N_{kd} + a_k - 1} \right),$$
$$N_{kw} = \sum_{d=1}^D \sum_{n=1}^{N_d} \mathbb{I}(w_{dn} = w, z_{dn} = k), N_{kd} = \sum_{n=1}^{N_d} \mathbb{I}(z_{dn} = k).$$

- $\beta$  and  $\theta$  can be collapsed:

$$p(z, w) = \iint p(\beta, \theta, z, w) d\beta d\theta$$
$$= AB \left( \prod_k \frac{\prod_w \Gamma(N_{kw} + b_w)}{\Gamma(N_k + W\bar{b})} \right) \left( \prod_d \frac{\prod_k \Gamma(N_{kd} + a_k)}{\Gamma(N_d + K\bar{a})} \right).$$

- Unacceptable cost to directly compute  $p(z|w) = p(z|w)/p(w)$ !

# Latent Dirichlet Allocation

MCMC: Gibbs sampling [GS04]

$$p(z, w) = AB \left( \prod_k \frac{\prod_w \Gamma(N_{kw} + b_w)}{\Gamma(N_k + W\bar{b})} \right) \left( \prod_d \frac{\prod_k \Gamma(N_{kd} + a_k)}{\Gamma(N_d + K\bar{a})} \right).$$

- Use Gibbs sampling: iteratively sample from

$$p(z_{11}^{(1)} \mid z_{12}^{(0)}, z_{13}^{(0)}, z_{14}^{(0)}, \dots, z_{DN_D}^{(0)}, w),$$

$$p(z_{12}^{(1)} \mid z_{11}^{(1)}, z_{13}^{(0)}, z_{14}^{(0)}, \dots, z_{DN_D}^{(0)}, w),$$

$$p(z_{13}^{(1)} \mid z_{11}^{(1)}, z_{12}^{(1)}, z_{14}^{(0)}, \dots, z_{DN_D}^{(0)}, w),$$

$$\dots, p(z_{dn}^{(l+1)} \mid z_{11}^{(l+1)}, \dots, z_{d(n-1)}^{(l+1)}, z_{d(n+1)}^{(l)}, \dots, w) =: p(z_{dn} \mid z^{-dn}, w).$$

$$p(z_{dn} = k \mid z^{-dn}, w) \propto \frac{N_{kw}^{-dn} + b_w}{N_k^{-dn} + W\bar{b}} (N_{kd}^{-dn} + a_k).$$

# Latent Dirichlet Allocation

MCMC: Gibbs sampling [GS04]

$$p(z, w) = AB \left( \prod_{k'} \frac{\prod_{w'} \Gamma(N_{k'w'} + b_{w'})}{\Gamma(N_{k'} + W\bar{b})} \right) \left( \prod_{d'} \frac{\prod_{k'} \Gamma(N_{k'd'} + a_{k'})}{\Gamma(N_{d'} + K\bar{a})} \right)$$

(Denote  $w_{dn}$  as  $w$ :

$$\begin{aligned} &= AB \prod_{k'} \frac{(\prod_{w' \neq w} \Gamma(N_{k'w'} + b_{w'})) \cdot \Gamma(N_{k'w}^{-dn} + \mathbb{I}(z_{dn} = k') + b_w)}{\Gamma(N_{k'}^{-dn} + \mathbb{I}(z_{dn} = k') + W\bar{b})} \cdot \left( \prod_{d' \neq d} \frac{\prod_{k'} \Gamma(N_{k'd'} + a_{k'})}{\Gamma(N_{d'} + K\bar{a})} \right) \cdot \frac{\prod_{k'} \Gamma(N_{k'd}^{-dn} + \mathbb{I}(z_{dn} = k') + a_{k'})}{\Gamma(N_d + K\bar{a})} \\ &= AB \prod_{k'} \frac{(\prod_{w' \neq w} \Gamma(N_{k'w'} + b_{w'})) \cdot \Gamma(N_{k'w}^{-dn} + b_w) \cdot (N_{k'w}^{-dn} + b_w)^{\mathbb{I}(z_{dn} = k')}}{\Gamma(N_{k'}^{-dn} + W\bar{b}) \cdot (N_{k'}^{-dn} + W\bar{b})^{\mathbb{I}(z_{dn} = k')}} \\ &\quad \cdot \left( \prod_{d' \neq d} \frac{\prod_{k'} \Gamma(N_{k'd'} + a_{k'})}{\Gamma(N_{d'} + K\bar{a})} \right) \cdot \frac{\prod_{k'} \Gamma(N_{k'd}^{-dn} + a_{k'}) \cdot (N_{k'd}^{-dn} + a_{k'})^{\mathbb{I}(z_{dn} = k')}}{\Gamma(N_d + K\bar{a})} \\ &= AB \left( \prod_{k'} \frac{(\prod_{w' \neq w} \Gamma(N_{k'w'} + b_{w'})) \cdot \Gamma(N_{k'w}^{-dn} + b_w)}{\Gamma(N_{k'}^{-dn} + W\bar{b})} \right) \cdot \prod_{k'} \left( \frac{N_{k'w}^{-dn} + b_w}{N_{k'}^{-dn} + W\bar{b}} \right)^{\mathbb{I}(z_{dn} = k')} \\ &\quad \cdot \left( \left( \prod_{d' \neq d} \frac{\prod_{k'} \Gamma(N_{k'd'} + a_{k'})}{\Gamma(N_{d'} + K\bar{a})} \right) \cdot \frac{\prod_{k'} \Gamma(N_{k'd}^{-dn} + a_{k'})}{\Gamma(N_d + K\bar{a})} \right) \cdot \prod_{k'} (N_{k'd}^{-dn} + a_{k'})^{\mathbb{I}(z_{dn} = k')} \\ &\Rightarrow p(z_{dn} = k | z^{-dn}, w) \propto \frac{N_{kw}^{-dn} + b_w}{N_k^{-dn} + W\bar{b}} (N_{kd}^{-dn} + a_k). \end{aligned}$$

# Latent Dirichlet Allocation

MCMC: Gibbs sampling [GS04]

- For  $\beta$ , use the MAP estimate:

$$\hat{\beta} = \arg \max_{\beta} \log p(\beta|w).$$

Estimate  $p(\beta|w) = \mathbb{E}_{p(z|w)}[p(\beta, z, w)]$  with one sample of  $z$  from  $p(z|w)$ :

$$\Rightarrow \hat{\beta}_k = \frac{N_{kw} + b_w - 1}{N_k + W\bar{b} - W} \approx \frac{N_{kw} + b_w}{N_k + W\bar{b}}.$$

- For  $\theta$ , use the MAP estimate:

$$\hat{\theta}_{dk} = \frac{N_{kd} + a_k - 1}{N_d + K\bar{a} - K} \approx \frac{N_{kd} + a_k}{N_d + K\bar{a}}.$$

# Latent Dirichlet Allocation

MCMC: Stochastic Gradient Riemannian Langevin Dynamics [PT13]

$$dx = G^{-1} \nabla \log p \ dt + \nabla \cdot G^{-1} \ dt + \mathcal{N}(0, 2G^{-1} \ dt).$$

- To draw from  $p(\beta|w)$ ,

$$\begin{aligned}\nabla_\beta \log p(\beta|w) &= \frac{1}{p(\beta|w)} \nabla_\beta \int p(\beta, z|w) \ dz = \int \frac{1}{p(\beta|w)} \nabla_\beta p(\beta, z|w) \ dz \\ &= \int \frac{p(\beta, z|w)}{p(\beta|w)} \frac{\nabla_\beta p(\beta, z|w)}{p(\beta, z|w)} \ dz = \mathbb{E}_{p(z|\beta, w)} [\nabla_\beta \log p(\beta, z, w)].\end{aligned}$$

- $p(\beta, z, w)$  is available in closed form.
- $p(z|\beta, w)$  can be drawn using Gibbs sampling.
- Each  $\beta_k$  is on a simplex: use reparameterization to convert to the Euclidean space (that's where  $G$  comes from), e.g.,  $\beta_{kw} = \frac{\pi_{kw}}{\sum_w \pi_{kw}}$ .

# Latent Dirichlet Allocation

MCMC: Stochastic Gradient Riemannian Langevin Dynamics [PT13]

$$dx = G^{-1} \nabla \log p \ dt + \nabla \cdot G^{-1} \ dt + \mathcal{N}(0, 2G^{-1} \ dt).$$

- Various parameterizations:

Parameterisation	Reduced-Mean	Reduced-Natural	Expanded-Mean	Expanded-Natural
$\theta$	$\theta_k = \pi_k$	$\theta_k = \log \frac{\pi_k}{1 - \sum_{k=1}^{K-1} \pi_k}$	$\pi_k = \frac{ \theta_k }{\sum_{k=1}^K  \theta_k }$	$\pi_k = \frac{e^{\theta_k}}{\sum_{k=1}^K e^{\theta_k}}$
$\nabla_\theta \log p(\theta   \mathbf{x})$	$\frac{n+\alpha}{\theta} - \mathbf{1} \frac{n_K + \alpha - 1}{\pi_K}$	$n + \alpha - (n. + K\alpha) \pi$	$\frac{n+\alpha-1}{\theta} - \frac{n.}{\theta.} - \mathbf{1}$	$n + \alpha - n. \pi - e^\theta$
$G(\theta)$	$n. \left( \text{diag}(\theta)^{-1} + \frac{1}{1 - \sum_k \theta_k} \mathbf{1} \mathbf{1}^T \right)$	$\frac{1}{n.} (\text{diag}(\pi) - \pi \pi^T)$	$\text{diag}(\theta)^{-1}$	$\text{diag}(e^\theta)$
$G^{-1}(\theta)$	$\frac{1}{n.} (\text{diag}(\theta) - \theta \theta^T)$	$n. \left( \text{diag}(\pi)^{-1} + \frac{1}{1 - \sum_k \pi_k} \mathbf{1} \mathbf{1}^T \right)$	$\text{diag}(\theta)$	$\text{diag}(e^{-\theta})$
$\sum_{k=1}^D \left( G^{-1} \frac{\partial G}{\partial \theta_k} G^{-1} \right)_{jk}$	$K\theta_j - 1$	$\frac{1}{\pi_j^2} - \frac{K-1}{(1 - \sum_k \pi_k)^2}$	$-1$	$e^{-\theta_j}$
$\sum_{k=1}^D \left( G^{-1}(\theta) \right)_{jk} \text{Tr} \left( G^{-1}(\theta) \frac{\partial G}{\partial \theta_k} \right)$	$K\theta_j - 1$	$\frac{1}{\pi_j^2} - \frac{K-1}{(1 - \sum_k \pi_k)^2}$	$-1$	$e^{-\theta_j}$

# Latent Dirichlet Allocation

$$\nabla_{\beta} \log p(\beta|w) = \mathbb{E}_{p(z|\beta, w)} [\nabla_{\beta} \log p(\beta, z, w)].$$

Gibbs Sampling

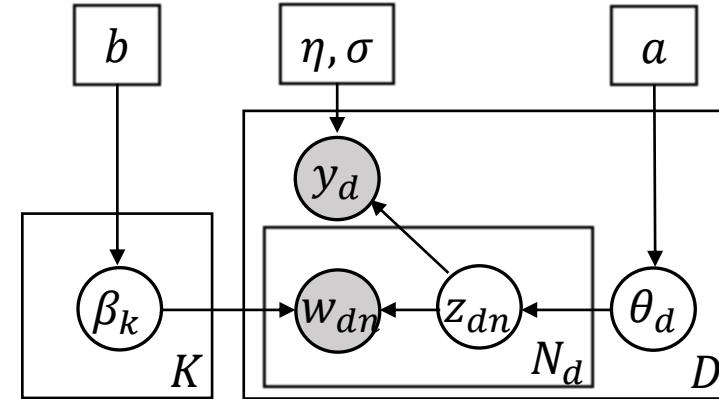
Closed-form known

- This makes dynamics-based MCMCs and particle-based VIs applicable.
  - Stochastic Gradient Nose-Hoover Thermostats [DFB+14], Stochastic Gradient Riemann Hamiltonian Monte Carlo [MCF15].
  - Accelerated and Hamiltonian particle-based VI [LZC+19, LZZ19]

# Supervised Latent Dirichlet Allocation

Model structure [MB08]:

- Generating process:
  - Draw topics:  $\beta_k \sim \text{Dir}(b), k = 1, \dots, K;$
  - For each document  $d,$ 
    - Draw topic proportion  $\theta_d \sim \text{Dir}(a);$
    - For each word  $n$  in document  $d,$ 
      - Draw topic assignment  $z_{dn} \sim \text{Mult}(\theta_d);$
      - Draw word  $w_{dn} \sim \text{Mult}(z_{dn}).$
    - Draw the response  $y_d \sim \mathcal{N}(\eta^\top \bar{z}_d, \sigma^2), \bar{z}_d := \frac{1}{N_d} \sum_{n=1}^{N_d} z_{dn}$  (one-hot).



$$p(\beta, \theta, z, w, y)$$

$$= \left( \prod_{k=1}^K \text{Dir}(\beta_k | b) \right) \left( \prod_{d=1}^D \text{Dir} \left( \prod_{n=1}^{N_d} \text{Mult}(z_{dn} | \theta_d) \text{Mult}(w_{dn} | \beta_{z_{dn}}) \right) \mathcal{N}(y_d | \eta^\top \bar{z}_d, \sigma^2) \right).$$

# Supervised Latent Dirichlet Allocation

Variational inference [MB08]: similar to LDA.

- Same variational distribution

$$q_{\lambda, \gamma, \phi}(\beta, \theta, z) := \prod_{k=1}^K \text{Dir}(\beta_k | \lambda_k) \prod_{d=1}^D \text{Dir}(\theta_d | \gamma_d) \prod_{n=1}^{n_d} \text{Mult}(z_{dn} | \phi_{dn}).$$

$\text{ELBO}(\lambda, \gamma, \phi; a, b, \eta, \sigma^2)$  is available in closed form.

- E-step: update  $\lambda, \gamma, \phi$  by maximizing ELBO.
- M-step: update  $a, b, \eta, \sigma^2$  by maximizing ELBO.
- Prediction: given a new document  $w_d$ ,

$$\hat{y}_d := \mathbb{E}_{p(y_d|w_d)}[y_d] = \eta^\top \mathbb{E}_{p(z_d|w_d)}[\bar{z}_d] \approx \eta^\top \mathbb{E}_{q(z_d|w_d)}[\bar{z}_d].$$

First do inference (find  $q(z_d|w_d)$  i.e.  $\phi_d$ ), then estimate  $\hat{y}_d$ .

# Supervised Latent Dirichlet Allocation

Variational inference with posterior regularization [ZAX12]

- Regularized Bayesian Posterior (RegBayes) [ZCX14]:

- Recall:  $p(z|x^{(n)}, y^{(n)}) = \arg \min_{q(z)} \{-\mathcal{L}[q] = \text{KL}(q(z), p(z)) - \sum_n \mathbb{E}_q[\log p(x^{(n)}, y^{(n)}|z)]\}.$

- **Regularize** posterior towards better prediction:

$$\min_{q(z)} \text{KL}(q(z), p(z)) - \sum_n \mathbb{E}_q[\log p(x^{(n)}, y^{(n)}|z)] + \lambda \ell(q(z); \{x^{(n)}, y^{(n)}\}).$$

- Maximum entropy discrimination LDA (MedLDA) [ZAX12]:

- $\ell(q; \{w^{(n)}, y^{(n)}\}) = \sum_n \ell_\varepsilon(y^{(n)} - \hat{y}^{(n)}(q, w^{(n)})) = \sum_n \ell_\varepsilon(y^{(n)} - \eta^\top \mathbb{E}_{q(z^{(n)}|w^{(n)})}[\bar{z}^{(n)}]),$

where  $\ell_\varepsilon(r) = \max\{0, |r| - \varepsilon\}$  is the hinge (max-margin) loss.

- Facilitates both prediction and topic representation.

# Bayesian Models

Bayesian Networks

- Variational Auto-Encoders

# Variational Auto-Encoder

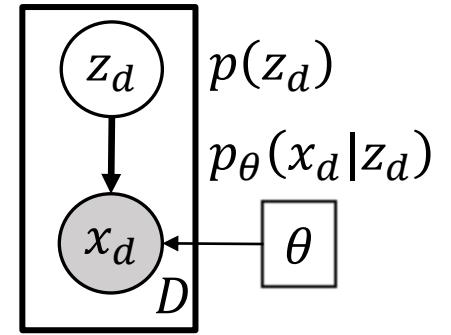
More *flexible* Bayesian model using *deep learning* tools.

- Model structure (decoder) [KW14]:

$$z_d \sim p(z_d) = \mathcal{N}(z_d | 0, I),$$

$$x_d \sim p_\theta(x_d | z_d) = \mathcal{N}(x_d | \mu_\theta(z_d), \Sigma_\theta(z_d)),$$

where  $\mu_\theta(z_d)$  and  $\Sigma_\theta(z_d)$  are modeled by neural networks.



# Variational Auto-Encoder

- Variational inference (encoder) [KW14]:

$$q_\phi(z|x) := \prod_{d=1}^D q_\phi(z_d|x_d) = \prod_{d=1}^D \mathcal{N}(z_d | \nu_\phi(x_d), \Gamma_\phi(x_d)),$$

where  $\nu_\phi(x_d), \Gamma_\phi(x_d)$  are also NNs.

- Amortized inference: approximate local posteriors  $\{p(z_d|x_d)\}_{d=1}^D$  globally by  $\phi$ .
- Objective:

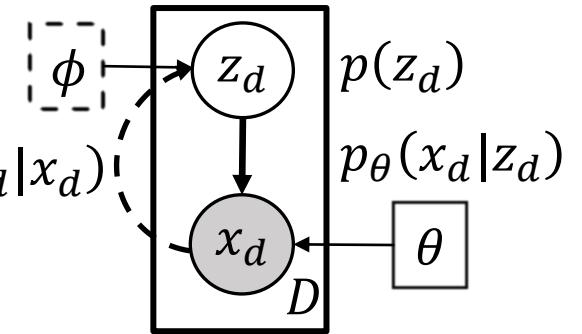
$$\mathbb{E}_{\hat{p}(x)}[\text{ELBO}(x)] \approx \frac{1}{D} \sum_{d=1}^D \mathbb{E}_{q_\phi(z_d|x_d)} [\log p_\theta(z_d)p_\theta(x_d|z_d) - \log q_\phi(z_d|x_d)].$$

- Gradient estimation with the reparameterization trick:

$$z_d \sim q_\phi(z_d|x_d) \Leftrightarrow z_d = g_\phi(x_d, \epsilon) := \nu_\phi(x_d) + \epsilon \sqrt{\Gamma_\phi(x_d)}, \epsilon \sim q(\epsilon) = \mathcal{N}(\epsilon|0, I).$$

Smaller variance than REINFORCE-like estimations.

$$\nabla_{\phi,\theta} \mathbb{E}_{\hat{p}(x)}[\text{ELBO}(x)] \approx \frac{1}{D} \sum_{d=1}^D \mathbb{E}_{q(\epsilon)} \left[ \nabla_{\phi,\theta} \left( \log p_\theta(z_d)p_\theta(x_d|z_d) - \log q_\phi(z_d|x_d) \Big|_{z_d=g_\phi(x_d, \epsilon)} \right) \right].$$



# Variational Auto-Encoder

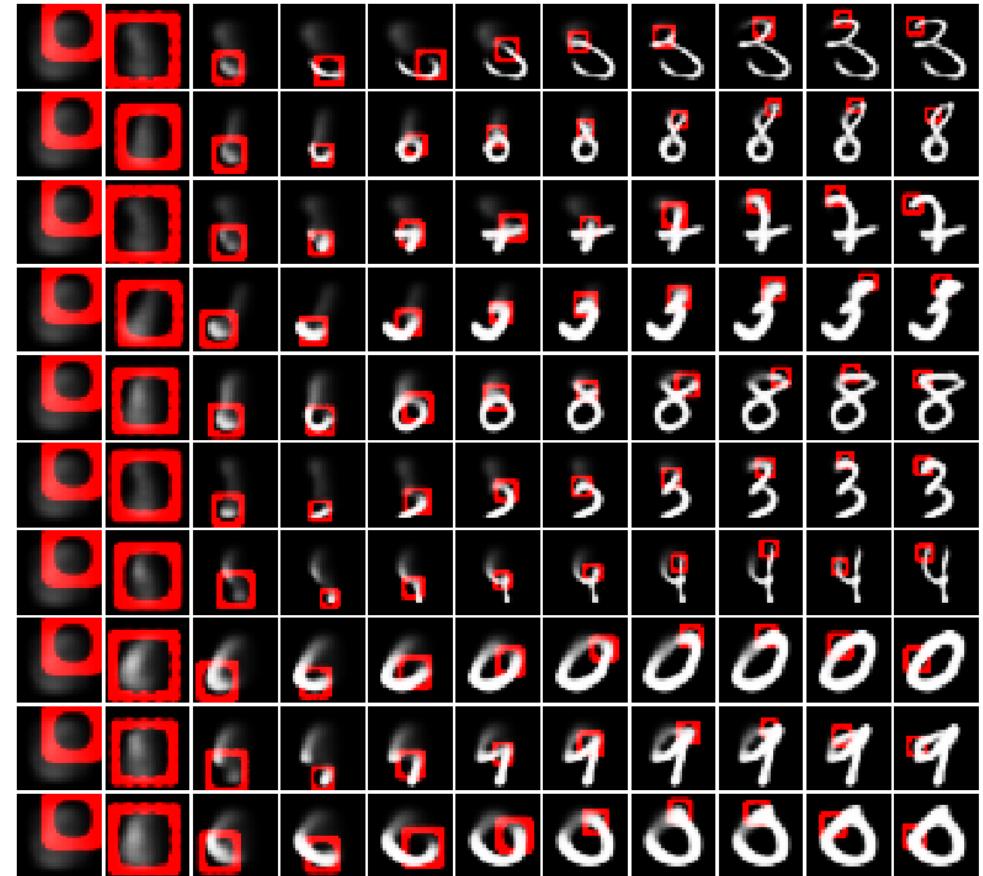
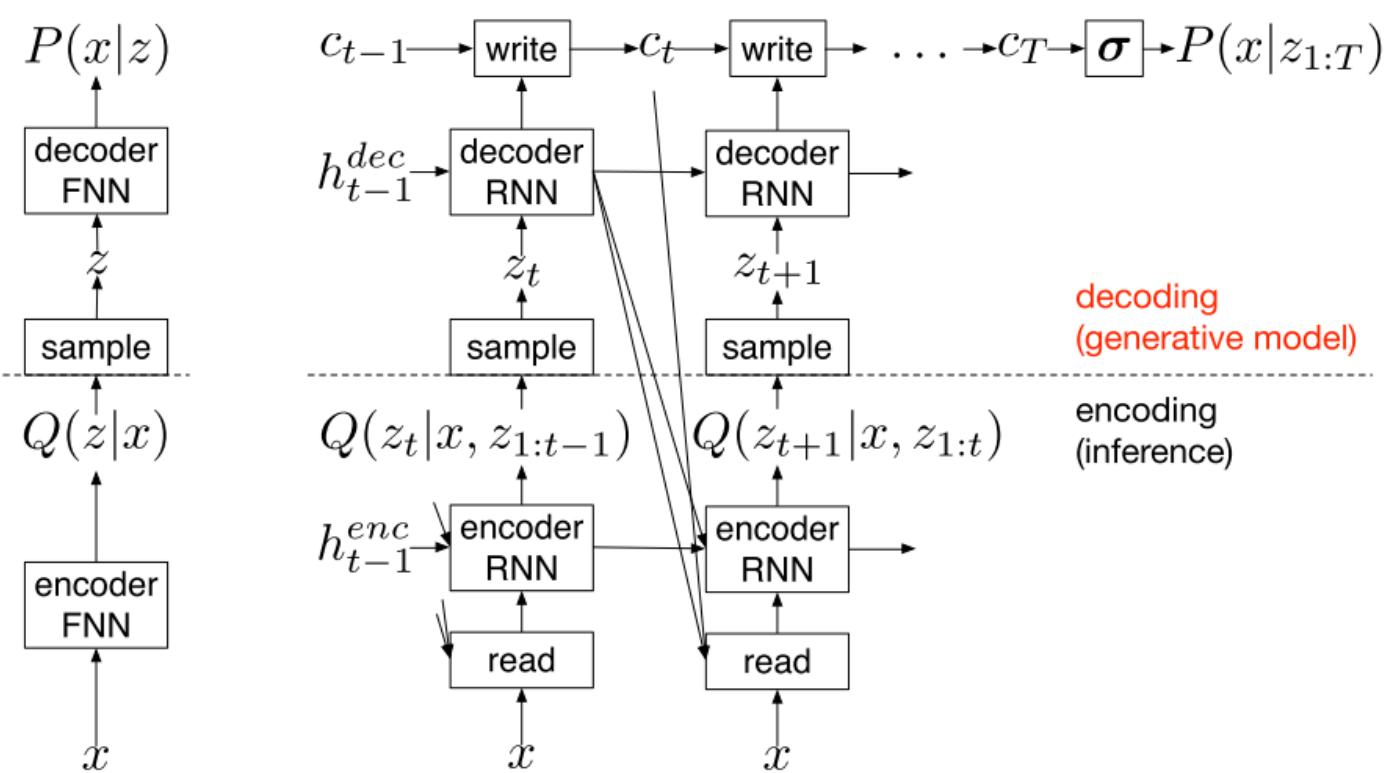
- Generation results [KW14]



6	6	6	6	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	4	4	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	3	5	5	5	5	5	5	5	5	5
9	9	2	2	2	2	2	2	2	2	3	5	5	5	5	5	5	5	5	5
9	9	4	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3
9	9	9	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3
9	9	9	9	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
9	9	9	9	9	8	3	3	3	3	3	3	3	3	3	3	3	3	3	3
7	9	9	9	9	9	8	3	3	3	3	3	3	3	3	3	3	3	3	3
7	9	9	9	9	9	8	8	3	3	3	3	3	3	3	3	3	3	3	3
7	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8
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7	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	9	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8
7	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8
7	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8
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7	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	8
7	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8
7	9	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8
7	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8
7	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8
7	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8
7	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8	8
7	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8
7	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8
7	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9

# Variational Auto-Encoder

- With spatial attention structure [GDG+15]



Time →

# Variational Auto-Encoder

- Inference with importance-weighted ELBO [BGS15]

- ELBO:  $\mathcal{L}_\theta[q_\phi(z)] = \mathbb{E}_{q_\phi(z)}[\log p_\theta(z, x)] - \mathbb{E}_{q_\phi(z)}[\log q_\phi(z)].$
- A tighter lower bound:

$$\mathcal{L}_\theta^{(k)}[q_\phi] := \mathbb{E}_{z^{(1)}, \dots, z^{(k)} \sim \text{i.i.d. } q_\phi} \left[ \log \frac{1}{k} \sum_{i=1}^k \frac{p_\theta(z^{(k)}, x)}{q_\phi(z^{(k)})} \right].$$

Ordering relation:

$$\mathcal{L}_\theta[q_\phi] = \mathcal{L}_\theta^{(1)}[q_\phi] \leq \mathcal{L}_\theta^{(2)}[q_\phi] \leq \dots \leq \mathcal{L}_\theta^{(\infty)}[q_\phi] = \log p_\theta(x).$$

If  $\frac{p(z, x)}{q(z|x)}$  is bounded.

3	2	6	5	8	8	1	9
5	3	2	8	1	3	1	2
7	2	8	5	2	5	7	3
8	1	0	1	4	7	1	0
6	2	2	3	7	3	0	
4	8	1	4	8	9	6	6
0	9	2	7	9	9	2	8
5	1	7	3	9	0	9	9
6	8	1	9	1	0	6	5
5	7	8	4	1	6	9	7

# Variational Auto-Encoder

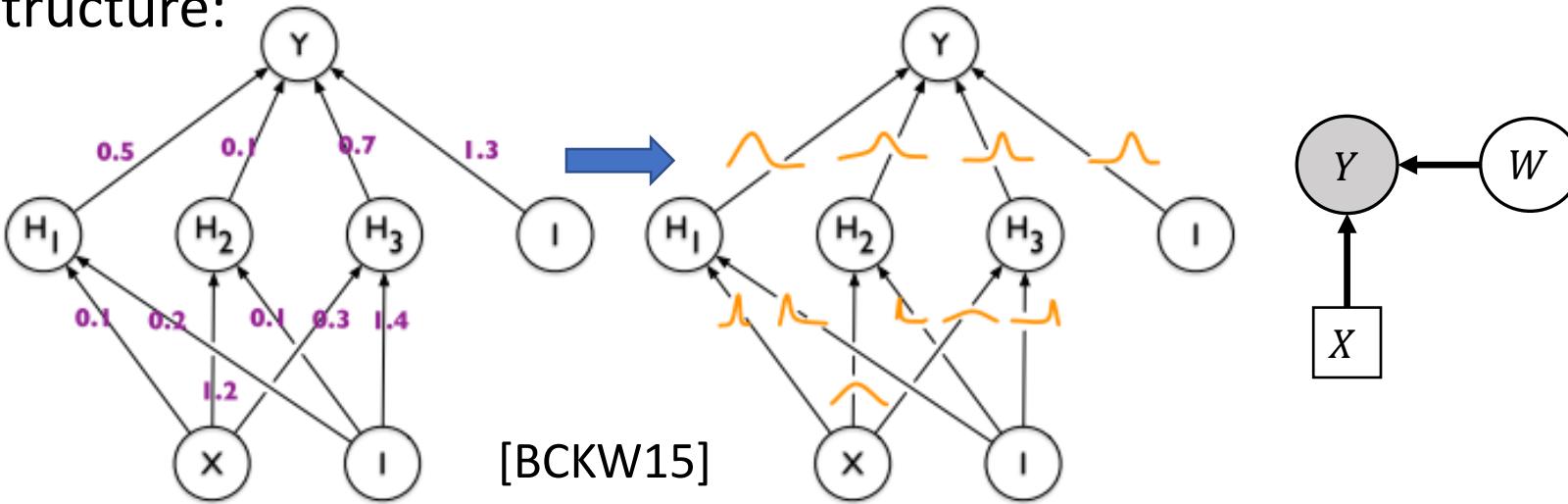
- Parametric Variational Inference: towards more flexible approximations.
  - Explicit VI:
    - Normalizing flows [RM15, KSJ+16]  
(concatenation of a sequence of invertible mappings).
  - Implicit VI:
    - Adversarial Auto-Encoder [MSJ+15], Adversarial Variational Bayes [MNG17],  
Wasserstein Auto-Encoder [TBGS17], [SSZ18a], [LT18], [SSZ18b].
- MCMC [LTL17] and Particle-Based VI [FWL17, PGH+17]:
  - Train the encoder as a sample generator.
  - Amortize the update on samples to  $\phi$ .

# Bayesian Models

## Bayesian Neural Networks

# Bayesian Neural Networks

Model Structure:



- Latent variable:  $W$ , the parameter (weights) of an NN.
- Prior:  $p(W)$  (e.g., diagonal Gaussian).
- Likelihood:  $p(Y|W, X) = \prod_{d=1}^D p(Y_d|W, X_d)$ ,  
 $p(Y_d|W, X_d)$  is specified by the NN.
- Equiv. to Gaussian process for infinite width [Nea95].
- Benefits: avoid overfitting (similar to dropout [SHK+14]); robust to adversarial attack.

# Bayesian Neural Networks

Variational Inference: parametric, explicit

Bayes by Backprop [BCKW15]

- Take  $q_\phi(W)$  s.t.  $w \sim q_\phi(W) \Leftrightarrow W = f_\phi(\epsilon), \epsilon \sim q(\epsilon)$ .  
The form of  $q_\phi(W)$  is required.
  - Example:  $W \sim \mathcal{N}(\phi_1, e^{2\phi_2}) \Leftrightarrow W = \phi_1 + e^{\phi_2}\epsilon, \epsilon \sim \mathcal{N}(0, I)$ .
  - $\text{ELBO}(\phi) = \mathbb{E}_{q_\phi(W)}[L(W, \phi)] = \mathbb{E}_{q(\epsilon)}[L(f_\phi(\epsilon), \phi)]$ ,  
where  $L(W, \phi) = \log p(W) + \sum_{d=1}^D \log p(Y_d | W, X_d) - \log q_\phi(W)$  is available in closed form.

$$\nabla_\phi \mathcal{L}(\phi) = \mathbb{E}_{q(\epsilon)}[\nabla_W L \cdot \nabla_\phi f_\phi(\epsilon) + \nabla_\phi L].$$

# Bayesian Neural Networks

Variational Inference: parametric, implicit

- [MNG17, SSZ18a, LT18, SSZ18b, ...]
- Using EP: [HA15]

Variational Inference: particle-based

- All required is  $\nabla_W \log p(W|Y, X) = \nabla_W \log p(W) + \sum_{d=1}^D \nabla_W \log p(Y_d|W, X_d)$ .
- [LW16, CZW+18, LZC+19, LZZ19, ...]

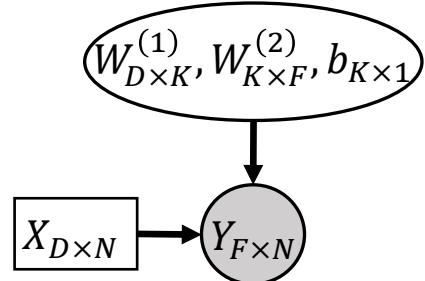
MCMC:

- For dynamics-based MCMC, all required is  $\nabla_W \log p(W|Y, X) = \nabla_W \log p(W) + \sum_{d=1}^D \nabla_W \log p(Y_d|W, X_d)$ .
- [Nea11, CFG14, DFB+14, ...]

# Bayesian Neural Networks

- Dropout as VI for BNN [GG15]

BNN model:



$$p(W^{(1)}) = \mathcal{N}(0, I), p(W^{(2)}) = \mathcal{N}(0, I), p(b) = \mathcal{N}(0, I),$$

$$p(Y|W^{(1)}, W^{(2)}, b, X) = \mathcal{N}\left(\frac{1}{\sqrt{K}}W^{(2)\top}\sigma(W^{(1)\top}X + b), \frac{1}{\tau}I\right).$$



Variational Distribution:

$$q(W^{(1)}, W^{(2)}, b) = q(b) \prod_{d=1}^D q(W_d^{(1)}) \prod_{k=1}^K q(W_k^{(2)}),$$

- $b = m + \delta \varepsilon^{(3)}, \varepsilon^{(3)} \sim \mathcal{N}(0, I_K),$
- $W_d^{(1)} = z_d^{(1)} \odot (M_d^{(1)} + \delta \varepsilon_d^{(1)}) + (1 - z_d^{(1)}) \odot \delta \varepsilon_d^{(1)},$   
 $\varepsilon_d^{(1)} \sim \mathcal{N}(0, I_K), z_d^{(1)} \sim [\text{Bern}(p^{(1)})]_{1 \times K},$
- $W_k^{(2)} = z_k^{(2)} \odot (M_k^{(2)} + \delta \varepsilon_k^{(2)}) + (1 - z_k^{(2)}) \odot \delta \varepsilon_k^{(2)},$   
 $\varepsilon_k^{(2)} \sim \mathcal{N}(0, I_F), z_k^{(2)} \sim [\text{Bern}(p^{(2)})]_{1 \times F}.$



$$\text{ELBO}(M^{(1)}, M^{(2)}, m) = \mathbb{E}_q[\log p(Y|W^{(1)}, W^{(2)}, b, X)] - \text{KL}(q, p(W^{(1)})p(W^{(2)})p(b))$$

$$= -\frac{\tau}{2} \sum_{n=1}^N \|Y_n - \hat{Y}_n\|_2^2 - \frac{p^{(1)}}{2} \|M^{(1)}\|_F^2 - \frac{p^{(2)}}{2} \|M^{(2)}\|_F^2 - \frac{1}{2} \|m\|_2^2 + C(\tau, \sigma),$$

$$\hat{Y}_n = \frac{1}{\sqrt{K}} \hat{W}^{(2)\top} \sigma(\hat{W}^{(1)\top} X_n + \hat{b}) \xrightarrow{\delta \rightarrow 0} \frac{1}{\sqrt{K}} (\hat{z}^{(2)} \odot M^{(2)})^\top \sigma((\hat{z}^{(1)} \odot M^{(1)})^\top X_n + m),$$

where  $\hat{W}^{(1)}, \hat{W}^{(2)}, \hat{b} \sim q, \hat{z} \sim \text{Bern}.$

$K \gg 1, \delta \rightarrow 0$

Obj. for training  
with dropout!

# Bayesian Models

## Markov Random Fields

# Markov Random Fields

Specify the joint distribution  $p_\theta(x, z)$  by an **energy function**  $E_\theta(x, z)$ .

$$p_\theta(x, z) = \frac{1}{Z_\theta} \exp(-E_\theta(x, z)), Z_\theta = \int \exp(-E_\theta(x', z')) \, dx' dz'.$$

- Only relationship and no causality:  $p(x, z)$  is either  $p(z)p(x|z)$  or  $p(x)p(z|x)$ .
  - Simpler modeling, yet still flexible.
  - Learning is harder.
- Synonyms: Energy-Based Model, Undirected Graphical Model.
- Etymology: Statistical Mechanics.

*The distribution over the microstate  $s$  of a canonical ensemble (system with fixed temperature  $T$ , volume and particle number) is called the **Boltzmann distribution**:*

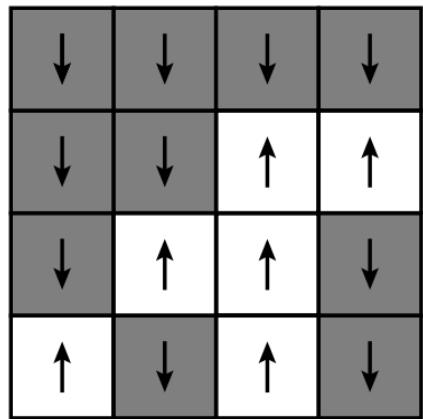
$$p(s) \propto \exp\left(-\frac{1}{k_B T} H(s)\right),$$

*where  $H(s)$  is the Hamiltonian of the system, which almost always represents the **energy** of the system. [Set06]*

# Markov Random Fields

- Gallery (alternative symbols:  $v \Leftrightarrow x, h \Leftrightarrow z, s \Leftrightarrow x \text{ or } z$ )

Ising model [Isi25]



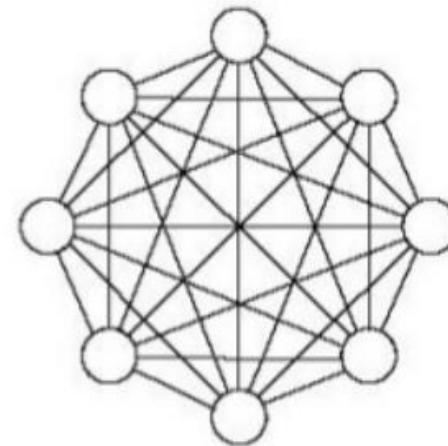
$$\mathcal{H} = - \sum_{\langle ij \rangle} J s_i s_j - H \sum_i s_i.$$

↑

[Set06]

Sum over nearest  
neighbors

Hopfield Network [Hop82]



$$E = -\frac{1}{2} \sum_{i,j} w_{ij} s_i s_j + \sum_i \theta_i s_i$$

(Slides by Kharagorgiev)

# Markov Random Fields

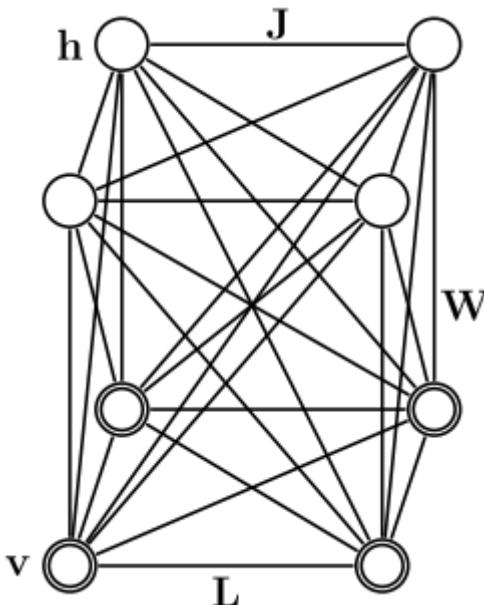
- Gallery (alternative symbols:  $v \Leftrightarrow x, h \Leftrightarrow z, s \Leftrightarrow x$  or  $z$ )

Boltzmann Machine [HS83]

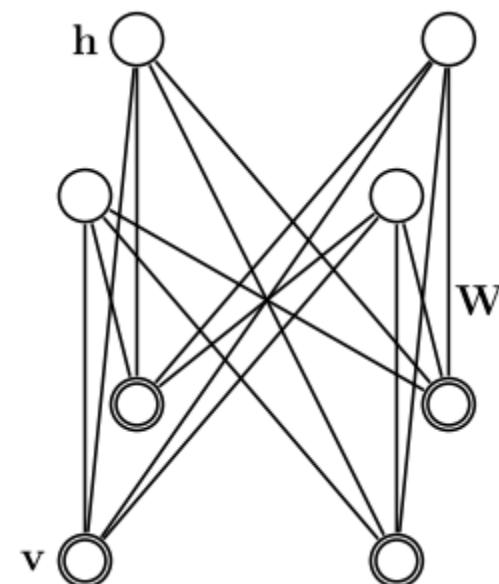
Restricted Boltzmann Machine

Products of Experts [Hin02]

(*Harmonium*) [Smo86]



$$E_{\theta}(v, h) = -\frac{1}{2}v^T Lv - \frac{1}{2}h^T Jh - v^T Wh$$



$$E_{\theta}(v, h) = -v^T Wh$$

(may has a bias term  
 $b^{(v)^T} v + b^{(h)^T} h$ )

Chang Liu (MSRA)

$$p(\mathbf{d} | \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_n) = \frac{\prod_m f_m(\mathbf{d} | \boldsymbol{\theta}_m)}{\sum_{\mathbf{c}} \prod_m f_m(\mathbf{c} | \boldsymbol{\theta}_m)},$$

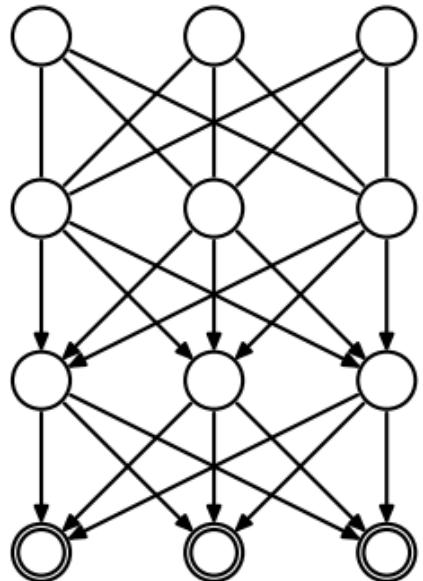
Each  $f_m(d|\theta_m)$  is an expert.

# Markov Random Fields

- Gallery (alternative symbols:  $v \Leftrightarrow x, h \Leftrightarrow z, s \Leftrightarrow x$  or  $z$ )

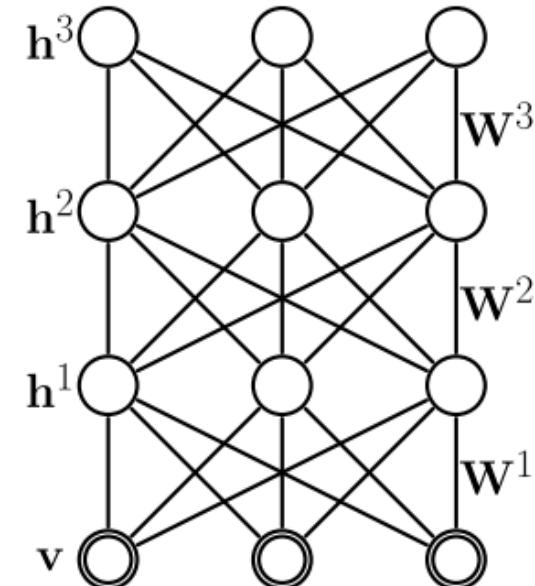
Deep Belief Network [HOT06]

(hybrid of directed and undirected)



$$\begin{aligned} & p(v, h^{(1)}, \dots, h^{(L)}) \\ = & p(v|h^{(1)})p(h^{(1)}|h^{(2)}) \\ & \dots p(h^{(L-2)}|h^{(L-1)})p(h^{(L-1)}, h^{(L)}) \end{aligned}$$

Deep Boltzmann Machine [SH09]



$$\begin{aligned} & E_\theta(v, h^{(1)}, \dots, h^{(L)}) \\ = & E_{W^{(1)}}(v, h^{(1)}) + \sum_{l=2}^L E_{W^{(l)}}(h^{(l-1)}, h^{(l)}) \end{aligned}$$

# Markov Random Fields

Specify the joint distribution  $p_\theta(x, z)$  by an energy function  $E_\theta(x, z)$ .

$$p_\theta(x, z) = \frac{1}{Z_\theta} \exp(-E_\theta(x, z)), Z_\theta = \int \exp(-E_\theta(x', z')) \, dx' dz'.$$

- Bayesian Inference: generally same as BayesNets.
  - Some MRFs can do this *exactly*, e.g., RBM:

$$p(\mathbf{x}, \mathbf{h}; \boldsymbol{\theta}) = \frac{\exp(\mathbf{x}^T \mathbf{W} \mathbf{h} + \mathbf{x}^T \mathbf{b}^{(1)} + \mathbf{h}^T \mathbf{b}^{(2)})}{\mathcal{Z}(\boldsymbol{\theta})}$$



$$\begin{aligned} p(x_k | \mathbf{h}; \boldsymbol{\theta}) &= \text{Bern}(\sigma(\mathbf{W}_{k:} \mathbf{h} + b_k^{(1)})) \\ p(h_k | \mathbf{x}; \boldsymbol{\theta}) &= \text{Bern}(\sigma(\mathbf{x}^T \mathbf{W}_{:k} + b_k^{(2)})) \end{aligned}$$

# Markov Random Fields

Specify the joint distribution  $p_\theta(x, z)$  by an energy function  $E_\theta(x, z)$ .

$$p_\theta(x, z) = \frac{1}{Z_\theta} \exp(-E_\theta(x, z)), Z_\theta = \int \exp(-E_\theta(x', z')) dx' dz'.$$

- Learning: MLE, i.e.,  $\max_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)]$ .

Harder than BayesNets! Even  $p_\theta(x, z)$  is unavailable.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x)p_\theta(z|x)}[\nabla_\theta E_\theta(x, z)] + \mathbb{E}_{p_\theta(x,z)}[\nabla_\theta E_\theta(x, z)].$$

(augmented) data distribution

model distribution

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x)p_\theta(z|x)}[\nabla_\theta E_\theta(x, z)] + \mathbb{E}_{p_\theta(x,z)}[\nabla_\theta E_\theta(x, z)].$$

Bayesian Inference

generation

- Generation: harder than BayesNets! Rely on MCMC or train a generator.

# Markov Random Fields

- Learning: harder than BayesNets! Even  $p_\theta(x, z)$  is unavailable.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)} [\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x) \underset{\text{↑}}{p_\theta(z|x)}} [\nabla_\theta E_\theta(x, z)] + \mathbb{E}_{\underset{\text{↑}}{p_\theta(x,z)}} [\nabla_\theta E_\theta(x, z)].$$

Bayesian Inference      Generation

- [HS83]: use Gibbs sampling for both inference and generation (for Boltzmann machine).

$$E(\mathbf{v}, \mathbf{h}; \theta) = -\frac{1}{2}\mathbf{v}^\top \mathbf{L}\mathbf{v} - \frac{1}{2}\mathbf{h}^\top \mathbf{J}\mathbf{h} - \mathbf{v}^\top \mathbf{W}\mathbf{h}, \quad \Delta \mathbf{W} = \alpha (\mathbb{E}_{P_{\text{data}}}[\mathbf{v}\mathbf{h}^\top] - \mathbb{E}_{P_{\text{model}}}[\mathbf{v}\mathbf{h}^\top]),$$

$$p(h_j = 1 | \mathbf{v}, \mathbf{h}_{-j}) = \sigma \left( \sum_{i=1}^D W_{ij} v_i + \sum_{m=1 \setminus j}^P J_{jm} h_m \right), \quad \Delta \mathbf{L} = \alpha (\mathbb{E}_{P_{\text{data}}}[\mathbf{v}\mathbf{v}^\top] - \mathbb{E}_{P_{\text{model}}}[\mathbf{v}\mathbf{v}^\top]),$$

$$p(v_i = 1 | \mathbf{h}, \mathbf{v}_{-i}) = \sigma \left( \sum_{j=1}^P W_{ij} h_j + \sum_{k=1 \setminus i}^D L_{ik} v_k \right), \quad \Delta \mathbf{J} = \alpha (\mathbb{E}_{P_{\text{data}}}[\mathbf{h}\mathbf{h}^\top] - \mathbb{E}_{P_{\text{model}}}[\mathbf{h}\mathbf{h}^\top]),$$

(Clipped from [SH09])

# Markov Random Fields

- Learning: harder than BayesNets! Even  $p_\theta(x, z)$  is unavailable.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x)p_\theta(z|x)}[\nabla_\theta E_\theta(x, z)] + \mathbb{E}_{p_\theta(x,z)}[\nabla_\theta E_\theta(x, z)].$$

↑    ↑  
 Bayesian Inference                                      Generation

- [Hin02]: estimation with  $k$ -step MCMC approximates the gradient of  $k$ -step *Contrastive Divergence* (CD- $k$ ).

(for Products of Experts (no lat. var.))

$$\begin{aligned} \text{CD}_k &\coloneqq \text{KL}(P^0 \parallel P_\theta^\infty) - \text{KL}(P_\theta^k \parallel P_\theta^\infty), \\ P^0(x) &= \hat{p}(x), P_\theta^k(x) \coloneqq P^0(x) P_\theta(x^{(k)}|x). \end{aligned}$$

← *k-step transition of MCMC from data to model.*

$$\begin{aligned} -\frac{\partial}{\partial \boldsymbol{\theta}_m}(P^0 \parallel P_\theta^\infty - P_\theta^1 \parallel P_\theta^\infty) &= \left\langle \frac{\partial \log f_{\theta_m}}{\partial \boldsymbol{\theta}_m} \right\rangle_{P^0} - \left\langle \frac{\partial \log f_{\theta_m}}{\partial \boldsymbol{\theta}_m} \right\rangle_{P_\theta^1} \\ &\quad + \frac{\partial P_\theta^1}{\partial \boldsymbol{\theta}_m} \frac{\partial(P_\theta^1 \parallel P_\theta^\infty)}{\partial P_\theta^1}. \end{aligned}$$

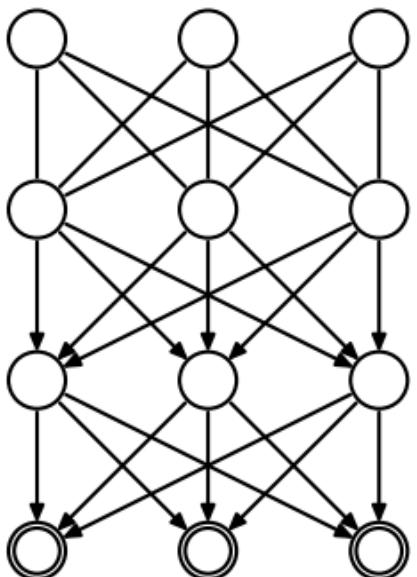
# Markov Random Fields

- Learning: harder than BayesNets! Even  $p_\theta(x, z)$  is unavailable.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)} [\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x) \underset{\uparrow}{p_\theta(z|x)}} [\nabla_\theta E_\theta(x, z)] + \mathbb{E}_{\underset{\uparrow}{p_\theta(x,z)}} [\nabla_\theta E_\theta(x, z)].$$

Bayesian Inference                                      Generation

- [HOT06]: training a stack of RBMs layer-wise corresponds to a DBN.



**Algorithm 1 (Recursive greedy learning procedure for the deep belief network):**

- 1: Fit the parameters  $W^{(1)}$  of the first-layer RBM to data.
- 2: Fix the parameter vector  $W^{(1)}$ , and use samples  $\mathbf{h}^{(1)}$  from  $Q(\mathbf{h}^{(1)}|\mathbf{v}) = P(\mathbf{h}^{(1)}|\mathbf{v}, W^{(1)})$  as the data for training the next layer of binary features with an RBM.
- 3: Fix the parameters  $W^{(2)}$  that define the second layer of features, and use the samples  $\mathbf{h}^{(2)}$  from  $Q(\mathbf{h}^{(2)}|\mathbf{h}^{(1)}) = P(\mathbf{h}^{(2)}|\mathbf{h}^{(1)}, W^{(2)})$  as the data for training the third layer of binary features.
- 4: Proceed recursively for the next layers.

[Sal15]

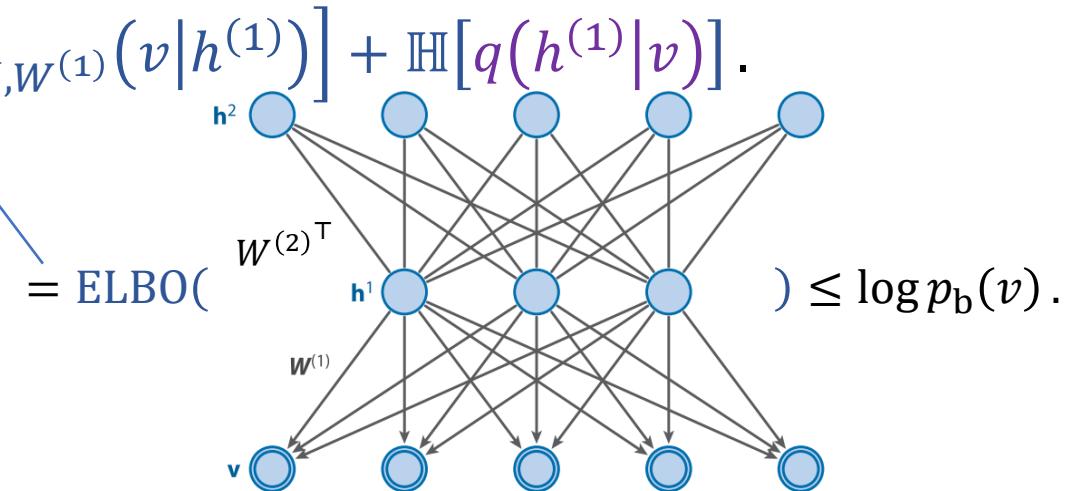
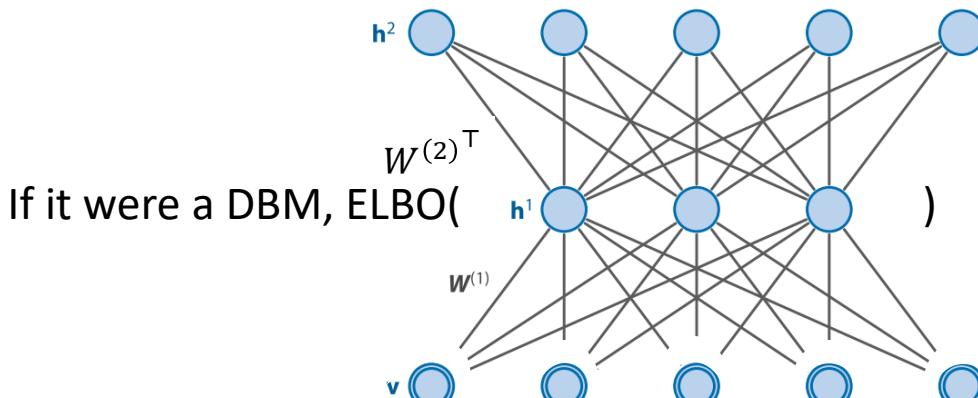
# Markov Random Fields

- Learning: harder than BayesNets! Even  $p_\theta(x, z)$  is unavailable.
  - [HOT06]: training a stack of RBMs layer-wise corresponds to a DBN.
  - Interpretation: after training the first RBM, training the second RBM is to:

$$\max_{W^{(2)}} \mathbb{E}_{p_{\text{RBM}, W^{(1)}}(h^{(1)}|v)} [\log p_{\text{RBM}, W^{(2)}}(h^{(1)})],$$

which is equivalent to

$$\max_{W^{(2)}} \mathbb{E}_{q(h^{(1)}|v)} [\log p_{\text{RBM}, W^{(2)}}(h^{(1)}) + \log p_{\text{SBN}, W^{(1)}}(v|h^{(1)})] + \mathbb{H}[q(h^{(1)}|v)].$$



$$= \mathbb{E}_{q(h^{(1)}|v)} [\log p_{\text{RBM}, W^{(2)}}(h^{(1)}) + \log p_{\text{SBN}, W^{(1)}}(v|h^{(1)})] + \mathbb{H}[q(h^{(1)}|v)] \\ + \mathbb{E}_{q(h^{(1)}|v)} [\log p_{\text{RBM}, W^{(1)}}(h^{(1)})] + \log Z_{W^{(2)}} - \log Z_{W^{(1)}, W^{(2)}}.$$

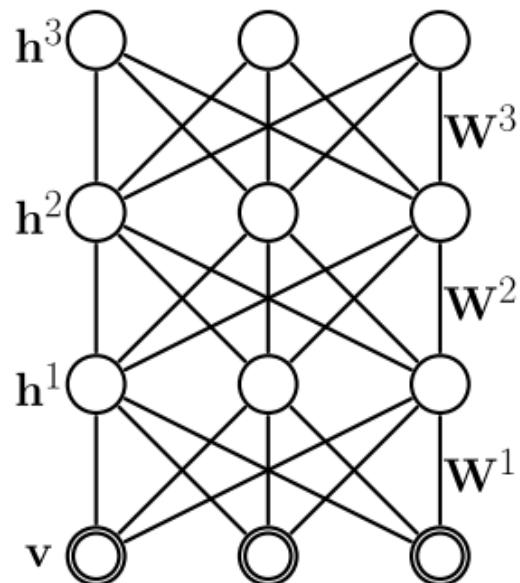
# Markov Random Fields

- Learning: harder than BayesNets! Even  $p_\theta(x, z)$  is unavailable.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x)p_\theta(z|x)}[\nabla_\theta E_\theta(x, z)] + \mathbb{E}_{p_\theta(x,z)}[\nabla_\theta E_\theta(x, z)].$$

↑  
Bayesian Inference                      ↑  
    Generation

- [SH09]: Deep Boltzmann Machine.



- Initialize the weights by running DBN in both directions.
- Fine-tune with standard training methods of a Boltzmann machine.

# Markov Random Fields

Deep-Nets-Based EBMs:

No latent variable;  $E_\theta(x)$  is modeled by a neural network.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x)}[\nabla_\theta E_\theta(x)] + \mathbb{E}_{p_\theta(x')}[\nabla_\theta E_\theta(x')].$$

- [KB16]: learn a generator

$$x \sim q_\phi(x) \Leftrightarrow z \sim q(z), x = g_\phi(z),$$

to mimic the generation from  $p_\theta(x)$ :

$$\arg \min_{\phi} \text{KL}(q_\phi, p_\theta) = \arg \min_{\phi} \mathbb{E}_{q(z)} \left[ E_\theta(g_\phi(z)) \right] - \underbrace{\mathbb{H}[q_\phi]}_{\substack{\text{approx. by batch} \\ \text{normalization Gaussian}}}$$



# Markov Random Fields

Deep-Nets-Based EBMs:

No latent variable;  $E_\theta(x)$  is modeled by a neural network.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x)}[\nabla_\theta E_\theta(x)] + \mathbb{E}_{p_\theta(x')}[\nabla_\theta E_\theta(x')].$$

- [ZML16]: interpret the discriminator  $D: \mathcal{X} \rightarrow \mathbb{R}^+$  as an energy function.

$$\begin{aligned} \min_D \mathbb{E}_{\hat{p}(x)p(z)} \mathcal{L}_D(x, z) &= D(x) + [m - D(G(z))]^+ \\ \min_G \mathbb{E}_{p(z)} \mathcal{L}_G(z) &= D(G(z)) \end{aligned}$$

- Theoretical guarantee: for Nash equilibrium state,  $p_{G^*}(x) = \hat{p}(x)$  a.e.
- Interpretation as EBM:
  - Obj. of  $D = \int_x \left( p_{data}(x)D(x) + p_{G^*}(x)[m - D(x)]^+ \right) dx$ . Similar gradient to  $\mathbb{E}_{\hat{p}}[\log p_\theta]$ .
  - Obj. of  $G = \mathbb{E}_{p_G}[D] = -\text{KL}(p_G, p_\theta) - \mathbb{H}[p_G]$ .

# Markov Random Fields

Deep-Nets-Based EBMs:

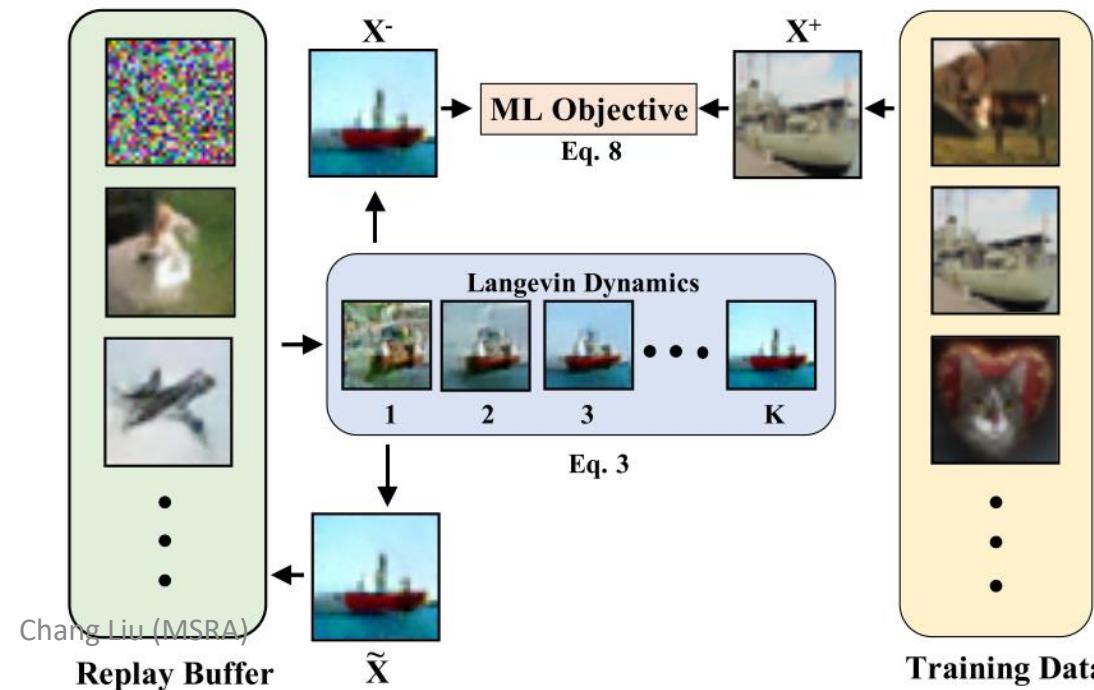
No latent variable;  $E_\theta(x)$  is modeled by a neural network.

$$\nabla_\theta \mathbb{E}_{\hat{p}(x)}[\log p_\theta(x)] = -\mathbb{E}_{\hat{p}(x)}[\nabla_\theta E_\theta(x)] + \mathbb{E}_{p_\theta(x')}[\nabla_\theta E_\theta(x')].$$

- [DM19]: estimate  $\mathbb{E}_{p_\theta}[\cdot]$  using samples drawn by the Langevin Dynamics

$$x^{(k+1)} = x^{(k)} - \varepsilon \nabla_x E_\theta(x^{(k)}) + \mathcal{N}(0, 2\varepsilon).$$

- Replay buffer for initializing the LD chain.
- $L_2$ -regularization on the energy function.



# Markov Random Fields

## Deep-Nets-Based EBMs:

- [DM19]



ImageNet32x32 Generation

Chang Liu (MSRA)

Model	Inception	FID
<b>CIFAR-10 Unconditional</b>		
PixelCNN (Van Oord et al., 2016)	4.60	65.93
PixelIQN (Ostrovski et al., 2018)	5.29	49.46
EBM (single)	6.02	40.58
DCGAN (Radford et al., 2016)	6.40	37.11
WGAN + GP (Gulrajani et al., 2017)	6.50	36.4
EBM (10 historical ensemble)	6.78	38.2
SNGAN (Miyato et al., 2018)	<b>8.22</b>	21.7
<b>CIFAR-10 Conditional</b>		
Improved GAN	8.09	-
EBM (single)	8.30	37.9
Spectral Normalization GAN	<b>8.59</b>	25.5
<b>ImageNet 32x32 Conditional</b>		
PixelCNN	8.33	33.27
PixelIQN	10.18	22.99
EBM (single)	<b>18.22</b>	<b>14.31</b>
<b>ImageNet 128x128 Conditional</b>		
ACGAN (Odena et al., 2017)	28.5	-
EBM* (single)	28.6	43.7
SNGAN	<b>36.8</b>	<b>27.62</b>

# Topics

## Online Inference

# Online Inference

Update est. of  $p(z|x_{1:n})$  to approx.  $p(z|x_{1:(n+1)})$  in  $O(1)$  computation.

- Online nature of Bayes' rule:  $p(z|x_{1:(n+1)}) \propto p(z|x_{1:n})p(x_{n+1}|z)$ .
- Variational inference
  - Streaming Variational Bayes [BBW+13]:  
Let  $q_{n+1}$  approx.  $q_n(z)p(x_{n+1}|z)$  (i.e., inference with prior  $q_n(z)$ ).
  - Stochastic-optimization-based method [HBB10]:
    - 1) solve global param.  $\hat{\lambda}$  with  $n$ -replicated  $x_{n+1}$ ;
    - 2)  $\lambda_{n+1} = (1 - \varepsilon_{n+1})\lambda_n + \varepsilon_{n+1}\hat{\lambda}$  with Robbins-Monro sequence  $\{\varepsilon_n\}_n$ .
  - Online Bayesian passive-aggressive [SZ14]: regularize posterior with **supervision**.
    - Online PA:  $z_{n+1} = \arg \min_z \frac{1}{2} \|z - z_n\|^2 + \lambda \ell(z; x_{n+1}, y_{n+1})$ .
    - Online BayesPA:  $q_{n+1} = \arg \min_q \text{KL}(q, q_n) - \mathbb{E}_q [\log p(x_{n+1}|z)] + \lambda \ell(q; x_{n+1}, y_{n+1})$ .
  - $q(z)$  needs to be explicit.

# Online Inference

Update est. of  $p(z|x_{1:n})$  to approx.  $p(z|x_{1:(n+1)})$  in  $O(1)$  computation.

- Online nature of Bayes' rule:  $p(z|x_{1:(n+1)}) \propto p(z|x_{1:n})p(x_{n+1}|z)$ .
- MCMC:

- Importance-sampling-based [Cho02]:

if  $\{z^{(i)}\}_i \sim p(z|x_{1:n})$ , then weighted particles

$$\left\{ z^{(i)} : \frac{p(z^{(i)}|x_{1:(n+1)})}{p(z^{(i)}|x_{1:n})} = p(x_{n+1}|z^{(i)}) \right\}_i \sim p(z|x_{1:(n+1)}).$$

- *Resample* the particles according to the weight;
- *Rejuvenate* by an MCMC to avoid particle degeneracy.

- Particle MCMC [ADH10]:

- Update particles by a transition proposal  $q(z_{n+1}|z_n, x_{n+1})$ ;
- Adjust the particles by an MH step.

# Topics

## Asymmetry of KL

# Asymmetry of KL

To approximate distribution  $p$  with distribution  $q$  (typically parameterized as  $q_\theta$ ),

- Forward KL / Inclusive KL:

$$\begin{aligned}\arg \min_q \text{KL}(p, q) &= \arg \min_q \mathbb{E}_p \left[ \log \left( \frac{p}{q} \right) \right] \\ &= \arg \min_q \mathbb{E}_p [\log q].\end{aligned}$$

- Used in MLE.
- $q(z) \neq 0$  wherever  $p(z) \neq 0$   
(otherwise huge penalty) [TOB16, Hus15].  
 $\text{supp}(q)$  includes  $\text{supp}(p)$ .
- Explains unrealistic or blurry generation  
of MLE-trained autoregressive models  
and ELBO-trained VAEs.

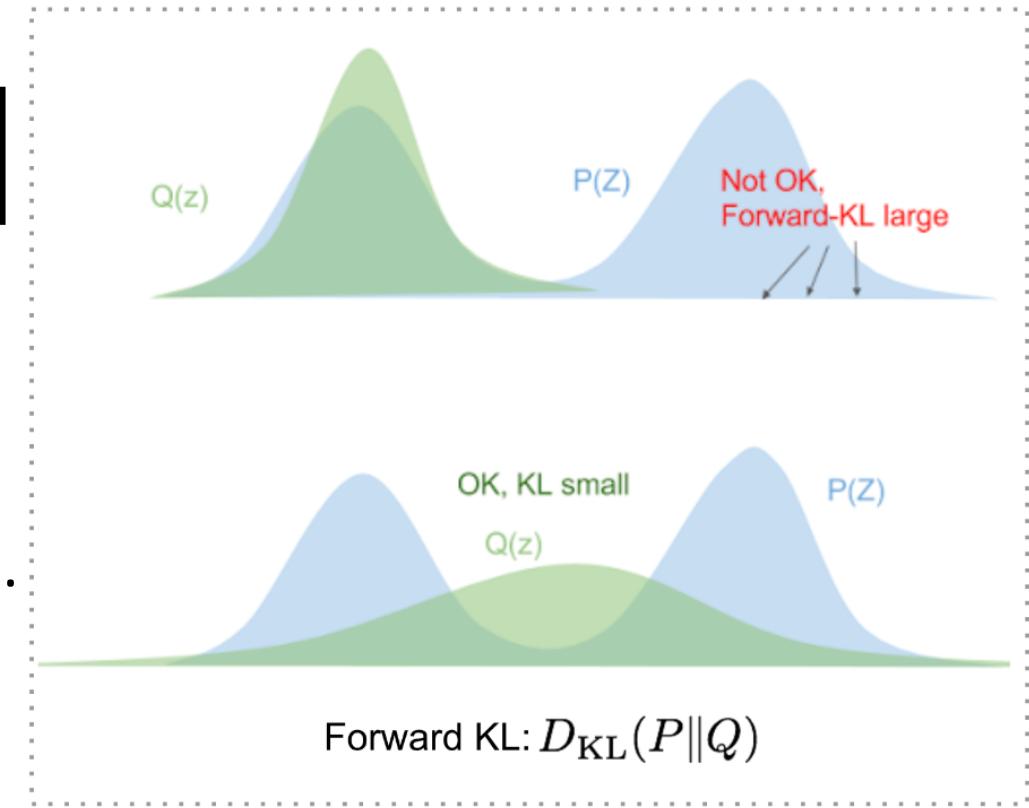


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Chang Liu (MSRA) [<https://lilianweng.github.io/lil-log/2018/08/12/from-autoencoder-to-beta-vae.html>]

# Asymmetry of KL

To approximate distribution  $p$  with distribution  $q$  (typically parameterized as  $q_\theta$ ),

- Forward KL / Inclusive KL:

$$\arg \min_q \text{KL}(p, q).$$

- Moment seeking behavior:

$$\text{Let } q_\phi(z) = \exp(\phi^\top T(z) - A(\phi)) h(z).$$

$$\arg \min_\phi \text{KL}(p, q_\phi)$$

$$= \arg \min_\phi \phi^\top \mathbb{E}_p[T(z)] - A(\phi).$$

$$\text{grad. obj.} = \mathbb{E}_p[T(z)] - \mathbb{E}_{q_\phi}[T(z)].$$

- Intractable gradient  $\nabla_\phi \text{KL}(p, q_\phi)$  with unnormalized  $p \propto \tilde{p}$ .

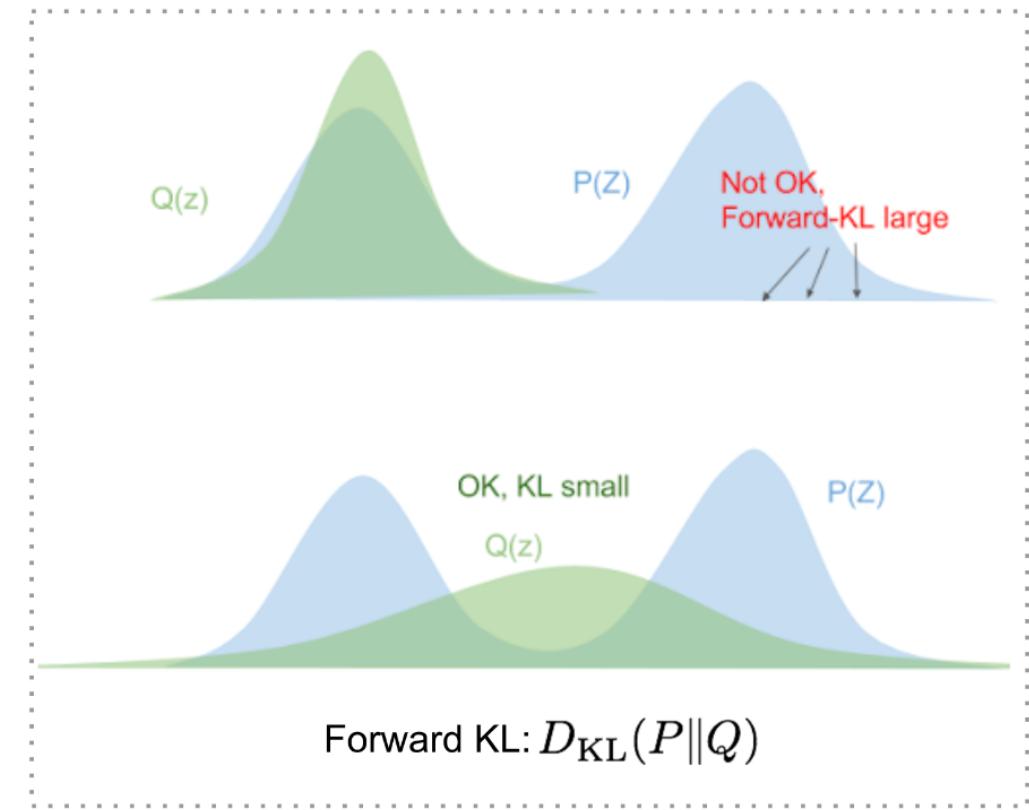


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- Reversed KL / Exclusive KL:

$$\arg \min_q \text{KL}(q, p) = \arg \min_q \mathbb{E}_q \left[ \log \left( \frac{q}{p} \right) \right].$$

- Tractable gradient  $\nabla_\phi \text{KL}(q_\phi, p)$  with unnormalized  $p \propto \tilde{p}$ :

$$\nabla_\phi \text{KL}(q_\phi, p) = \mathbb{E}_{q_\phi} \left[ (\nabla_\phi \log q_\phi) \log \frac{q_\phi}{\tilde{p}} \right].$$

(since  $\nabla_\phi \mathbb{E}_{q_\phi}[f] = \mathbb{E}_{q_\phi}[f \nabla_\phi \log q_\phi]$ ,  
and  $\nabla_\phi \mathbb{E}_{q_\phi}[C] = \nabla_\phi C = 0$ )

- Used in variational inference  
(except for Expectation Propagation).

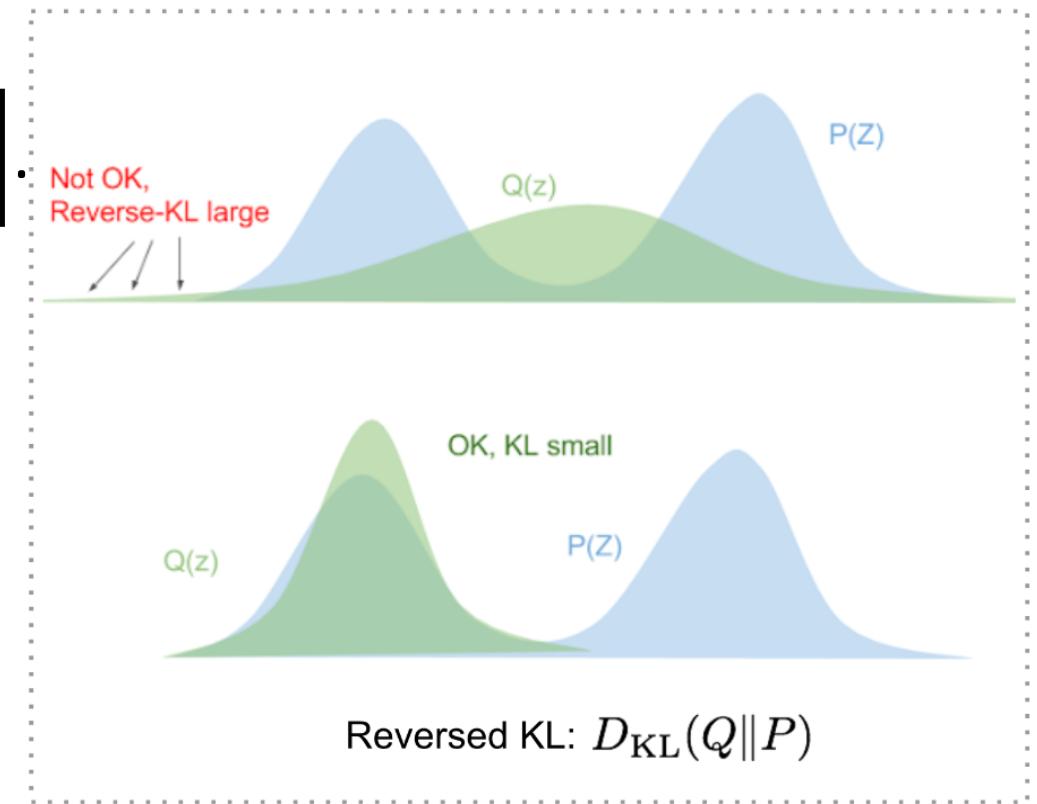


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(otherwise huge penalty) [TOB16, Hus15].  
 $\text{supp}(q)$  tends to exclude  $\text{supp}(p)$ .

- Mode seeking behavior.
- Jensen-Shannon divergence behaves more like Reversed KL [Hus15].
- Explains underestimated variance of variational inference and realistic but mode-collapsed generation of GANs.

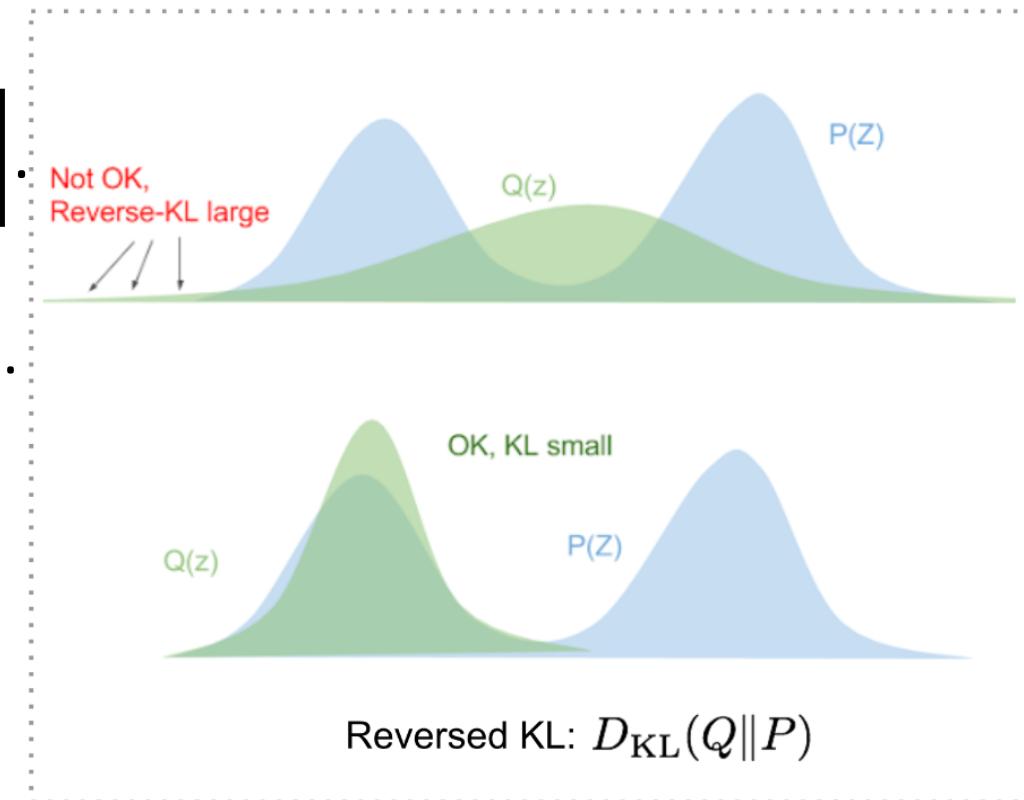


Figure: [<https://blog.evjang.com/2016/08/variational-bayes.html>],

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Chang Liu (MSRA)

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# Asymmetry of KL

- Adversarial Symmetric Variational Autoencoder [PWH+17]

- Original VAE objective:

$\hat{p}(x)$

$$\mathbb{E}_{q(x)}[\text{ELBO}(x)] = -\text{KL}\left(q_\phi(z, x), p_\theta(z, x)\right) + \mathbb{E}_{q(x)}[\log q(x)].$$

- For inference (i.e.,  $q_\phi$ ): mode-seeking/collapsing.
  - For learning (i.e.,  $p_\theta$ ): over-covering, unrealistic/blurry generation.

$$\mathbb{E}_{\hat{p}(x)}[\text{ELBO}(x)] \leq \mathbb{E}_{\hat{p}}[\log p_\theta(x)] = -\text{KL}(\hat{p}, p_\theta) + \text{const.}$$

- Symmetrized objective:

$$\begin{aligned}\mathcal{L}_x + \mathcal{L}_z &:= \mathbb{E}_{q(x)}[\log p_\theta(x)] - \text{KL}\left(q_\phi(z, x), p_\theta(z, x)\right) \\ &\quad + \mathbb{E}_{p_\theta(z)}[\log q_\phi(z)] - \text{KL}\left(p_\theta(z, x), q_\phi(z, x)\right).\end{aligned}$$

- Optimize by **adversarial density ratio estimation** [MNG17] and reparameterization:

$$\mathcal{L}_x = \mathbb{E}_{q_\phi(z, x)} \left[ \log \frac{p_\theta(x)p_\theta(z)}{q_\phi(z, x)} + \log p_\theta(x|z) \right].$$

Similar optimization for  $\mathcal{L}_z$ .

# Asymmetry of KL

- Dual Discriminator Generative Adversarial Nets [NLVP17]

- The JS divergence

$$\text{JS}(q_\phi, \hat{p}) := (G_\phi)_\#(p(z)) = \frac{1}{2} \text{KL}\left(q_\phi, \frac{q_\phi + \hat{p}}{2}\right) + \frac{1}{2} \text{KL}\left(\hat{p}, \frac{q_\phi + \hat{p}}{2}\right)$$

behaves similarly [Hus15] as the Reversed KL,  $\text{KL}(q_\phi, \hat{p})$ , thus mode-collapse.

- Symmetrized objective:

$$\mathcal{L}(\phi) = \alpha(\log \alpha - 1) + \beta(\log \beta - 1) + \alpha \text{KL}(\hat{p}, q_\phi) + \beta \text{KL}(q_\phi, \hat{p}).$$

- Optimize by adversarial training with two discriminators:

$$\begin{aligned} \min_{\phi} \max_{D_1, D_2} \mathcal{J}(\phi, D_1, D_2) &:= \alpha \mathbb{E}_{\hat{p}(x)} [\log D_1(x)] + \mathbb{E}_{p(z)} \left[ -D_1(G_\phi(z)) \right] \\ &\quad + \mathbb{E}_{\hat{p}(x)} [-D_2(x)] + \beta \mathbb{E}_{p(z)} \left[ \log D_2(G_\phi(z)) \right]. \\ \mathcal{J}(\phi, D_1^*, D_2^*) &= \mathcal{L}(\phi). \end{aligned}$$

# Asymmetry of KL

- Rényi's  $\alpha$ -divergence:  $R_\alpha(p, q) := \frac{1}{\alpha-1} \log \int p(z)^\alpha q(z)^{1-\alpha} dz$  ( $\alpha > 0, \alpha \neq 1$ ).

- $R_\alpha(p, q) = \frac{\alpha}{1-\alpha} R_{1-\alpha}(q, p)$ .
- $R_{-\infty}(p, q) = -R_{+\infty}(q, p)$ .
- $R_\alpha(p, q) \leq 0$  for  $\alpha < 0$ .
- $R_\alpha(p, q)$  is continuous and non-decreasing w.r.t  $\alpha$ .

$\alpha$	Definition	Notes
$\alpha \rightarrow 1$	$\int p(\boldsymbol{\theta}) \log \frac{p(\boldsymbol{\theta})}{q(\boldsymbol{\theta})} d\boldsymbol{\theta}$	<i>Kullback-Leibler (KL) divergence</i> , used in VI ( $\text{KL}[q  p]$ ) and EP ( $\text{KL}[p  q]$ )
$\alpha = 0.5$	$-2 \log(1 - \text{Hell}^2[p  q])$	function of the square <i>Hellinger distance</i>
$\alpha \rightarrow 0$	$-\log \int_{p(\boldsymbol{\theta}) > 0} q(\boldsymbol{\theta}) d\boldsymbol{\theta}$	zero when $\text{supp}(q) \subseteq \text{supp}(p)$ (not a divergence)
$\alpha = 2$	$-\log(1 - \chi^2[p  q])$	proportional to the $\chi^2$ -divergence
$\alpha \rightarrow +\infty$	$\log \max_{\boldsymbol{\theta} \in \Theta} \frac{p(\boldsymbol{\theta})}{q(\boldsymbol{\theta})}$	<i>worst-case regret</i> in <i>minimum description length principle</i> [24]

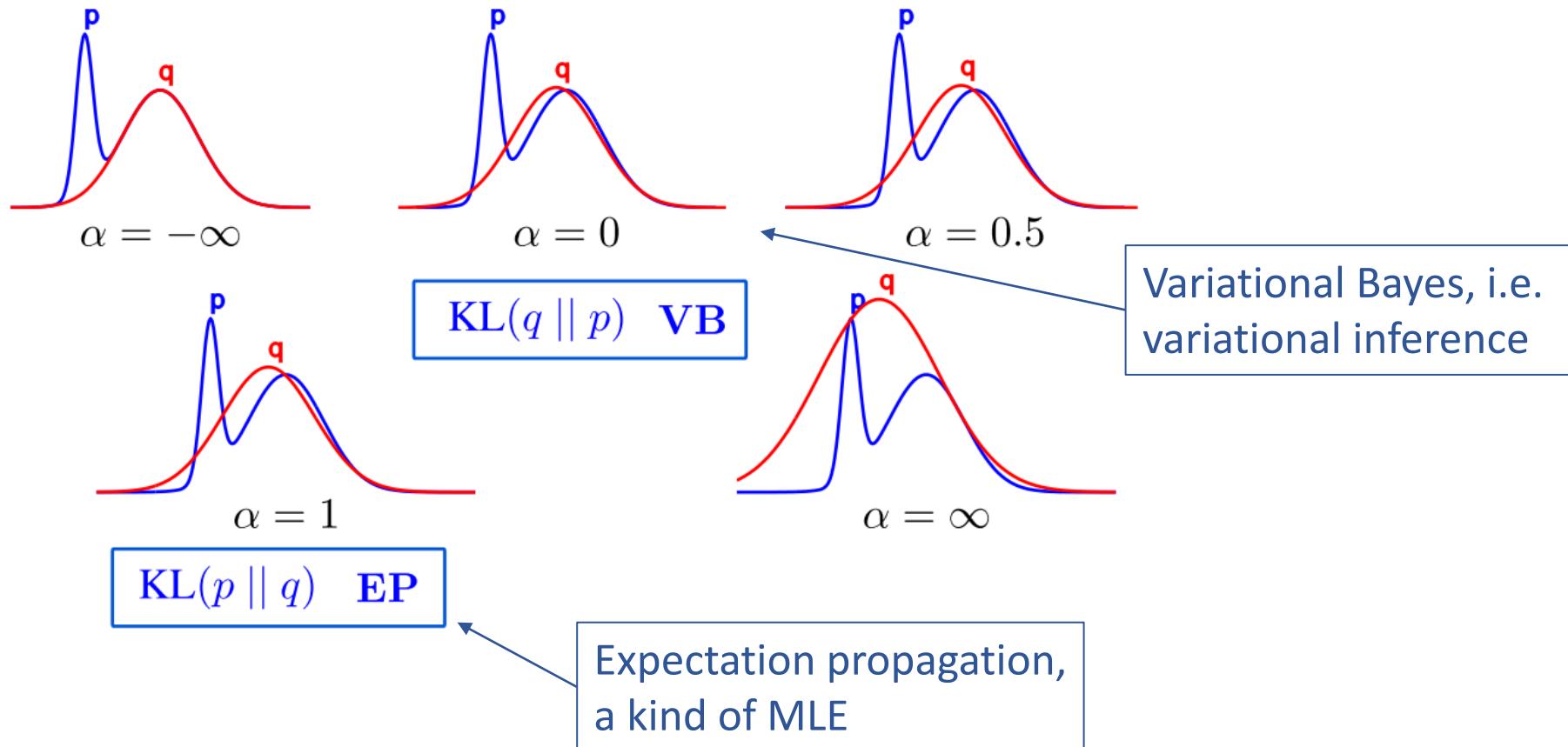
- $\log p(x) = \text{KL}(q(z), p(z|x)) + \underbrace{\mathbb{E}_{q(z)} \left[ \log \frac{p(x,z)}{q(z)} \right]}_{=: \mathcal{E}[q](x), \text{ELBO, free energy}}.$
- $\log p(x) = R_\alpha(q(z), p(z|x)) + \underbrace{\frac{1}{1-\alpha} \log \mathbb{E}_{q(z)} \exp \left[ (1-\alpha) \left( \log \frac{p(x,z)}{q(z)} \right) \right]}_{=: \mathcal{E}_\alpha[q](x), \alpha\text{-free energy, or variational Rényi bound}}$  [LT16].

# Asymmetry of KL

- Amari's  $\alpha$ -divergence [Ama85]:  $D_\alpha(p, q) := \frac{1}{\alpha(1-\alpha)} \left( 1 - \int p(z)^\alpha q(z)^{1-\alpha} dz \right)$ .
  - Exclusive KL divergence:  $D_0(p, q) = \text{KL}(q, p)$ .
  - Hellinger distance:  $D_{1/2}(p, q) = 4\text{Hel}^2(p, q) := 2 \int \left( \sqrt{p(z)} - \sqrt{q(z)} \right)^2 dz$ .
  - Inclusive KL divergence:  $D_1(p, q) = \text{KL}(p, q)$ .
  - $D_\alpha(p, q) = \frac{1}{\alpha(1-\alpha)} \left( 1 - e^{(\alpha-1)R_\alpha(p,q)} \right) = \frac{1}{\alpha(1-\alpha)} \left( 1 - e^{-\alpha R_{1-\alpha}(q,p)} \right)$ .
- Variational inference with  $\alpha$ -divergence:
  - For  $p(z) = \frac{f(z)}{Z}$ ,  $\partial_\phi D_\alpha(p, q_\phi) = -\frac{z^\alpha}{\alpha} \int \left( \frac{f(z)}{q_\phi(z)} \right)^\alpha \partial_\phi q_\phi(z) dz = -\frac{z^\alpha}{\alpha} \mathbb{E}_{q_\phi(z)} \left[ \exp \left( \alpha \log \frac{f(z)}{q_\phi(z)} \right) \partial_\phi \log q_\phi(z) \right]$ .
  - $\alpha$ -free energy [HLR+16]:  $\mathcal{L}_\alpha(q) := -\frac{1}{\alpha} \sum_n \log \mathbb{E}_q \left[ \left( p(y^{(n)}|z, x^{(n)}) p(z)^{\frac{1}{N}} / q(z)^{\frac{1}{N}} \right)^\alpha \right]$ .
  - Minimization in the EP style ( $q$  in exponential family): [HLR+16].
  - Minimization using variational dropout: [LG17].

# Asymmetry of KL

- Amari's  $\alpha$ -divergence [Ama85]:  $D_\alpha(p, q) := \frac{1}{\alpha(1-\alpha)} \left( 1 - \int p(z)^\alpha q(z)^{1-\alpha} dz \right)$ .
  - (From [HLR+16]) When optimizing a unimodal model  $q$ ,



# Asymmetry of KL

- Variational inference with alpha-divergence:

- $\alpha$ -free energy [HLR+16]:  $\mathcal{L}_\alpha(q) := -\frac{1}{\alpha} \sum_n \log \mathbb{E}_q \left[ \left( p(y^{(n)} | z, x^{(n)}) p(z)^{\frac{1}{N}} / q(z)^{\frac{1}{N}} \right)^\alpha \right]$ .

- Minimization using variational dropout [LG17]:

- With  $q(z) := \frac{1}{Z_q} \tilde{q}(z) (\tilde{q}(z)/p(z))^{\frac{\alpha}{N-\alpha}}$ :

Rényi divergence

$$\mathcal{L}_\alpha(q) = R_\beta(\tilde{q}(z), p(z)) - \frac{1}{\alpha} \sum_n \log \mathbb{E}_{\tilde{q}} \left[ p(y^{(n)} | z, x^{(n)})^\alpha \right], \beta := \frac{N}{N-\alpha}.$$

- For  $\frac{\alpha}{N} \rightarrow 0$ :  $q \rightarrow \tilde{q}$ ,  $R_\beta(\tilde{q}(z), p(z)) \rightarrow \text{KL}(\tilde{q}(z), p(z))$ , and  $\mathcal{L}_\alpha(q) \rightarrow \text{KL}(q(z), p(z)) - \frac{1}{\alpha} \sum_n \log \mathbb{E}_q \left[ p(y^{(n)} | z, x^{(n)})^\alpha \right] =: \tilde{\mathcal{L}}_\alpha(q)$ .

- Variational Dropout Inference:

$$z \sim q(z) \Leftrightarrow z_i = b_i \odot (M_i + \epsilon_i) + (1 - b_i) \odot \epsilon_i, b_i \sim \text{Bern}(p_i), \epsilon_i \sim \mathcal{N}(0, \delta^2).$$

$$\tilde{\mathcal{L}}_\alpha(q) \approx \sum_i \frac{p_i}{2} \|M_i\|_F^2 - \frac{1}{\alpha} \text{LogSumExp}_n \left( -\frac{\alpha\tau}{2} \|y^{(n)} - \hat{y}^{(n)}\|_2^2 \right) + \frac{ND}{2} \log \tau.$$

# Topics

## Bayesian Reinforcement Learning

(far from complete)

# Bayesian Reinforcement Learning

- Model-based Bayesian RL:  
update model with observations using Bayes' rule.
  - POMDP, Bayesian MDP, Bayesian Bandits, BEETLE...

# Bayesian Reinforcement Learning

- Model-free Bayesian RL

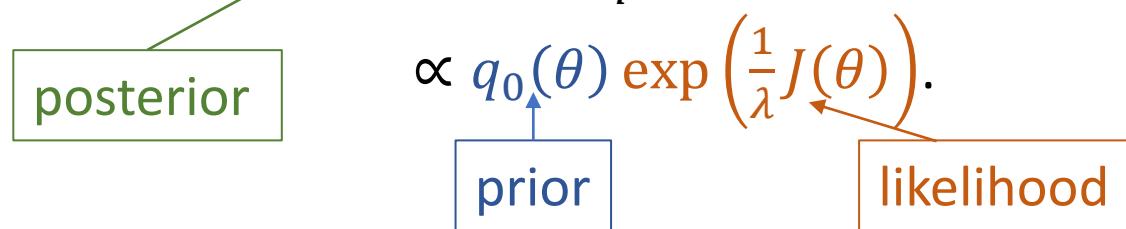
- Policy Gradient:

$$\theta^* = \arg \max_{\theta} \{J(\theta) := \mathbb{E}_{s_0, a_0, \dots} [\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t)]\}, \text{ where } a_t \sim \pi_{\theta}(a|s_t).$$

- Entropy Regularized Policy Gradient:

- Model the policy by parameter  $\theta$  and treat  $\theta \sim q(\theta)$  as a r.v. [LRLP17]:

$$q^*(\theta) = \arg \max_q \mathbb{E}_{q(\theta)}[J(\theta)] - \alpha \text{KL}(q, q_0).$$



- Bayesian “inference” by SVGD.
    - Helps exploration; increases data efficiency.

# Bayesian Reinforcement Learning

- Model-free Bayesian RL
  - Entropy Regularized Policy Gradient:
    - Directly regularize the policy [HTAL17]:

$$\pi^* = \arg \max_{\pi} \sum_t \mathbb{E}_{(s_t, a_t) \sim p(s_t)} [\pi(a_t | s_t)] [r(s_t, a_t) + \alpha \mathbb{H}[\pi(\cdot | s_t)]].$$

Target  
Q-value

- Soft Q-Iteration: using soft Bellman Eq.

$$J_Q(\theta) = \mathbb{E}_{s_t \sim q_{s_t}, a_t \sim q_{a_t}} \left[ \frac{1}{2} \left( \hat{Q}_{\text{soft}}^{\bar{\theta}}(s_t, a_t) - Q_{\text{soft}}^{\theta}(s_t, a_t) \right)^2 \right]$$

- Actor-critic with amortized SVGD: Bayesian inference interpretation.

$$\begin{aligned} \pi_{\text{MaxEnt}}^*(a_t | s_t) &= \exp \left( \frac{1}{\alpha} (Q_{\text{soft}}^*(s_t, a_t) - V_{\text{soft}}^*(s_t)) \right) \\ J_{\pi}(\phi; s_t) &= D_{\text{KL}} \left( \pi^{\phi}(\cdot | s_t) \parallel \exp \left( \frac{1}{\alpha} (Q_{\text{soft}}^{\theta}(s_t, \cdot) - V_{\text{soft}}^{\theta}) \right) \right). \end{aligned}$$

- Wasserstein gradient flow version [ZCLC18]:
  - Use a generator  $g_{\phi}(a|s)$  (amortization) that minimizes the KL for every  $s$ .

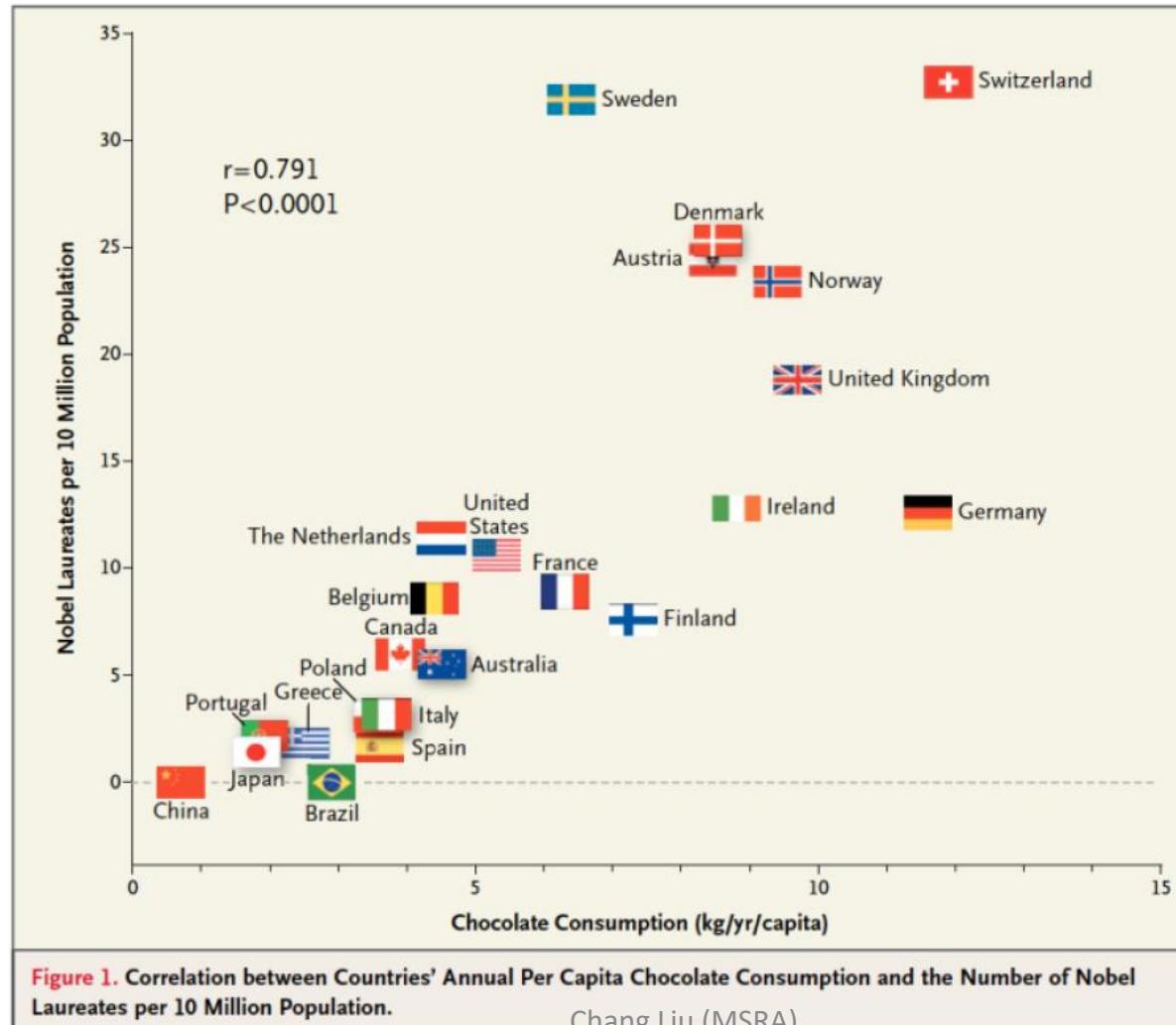
# Topics

## Causality

Clipped from the slides “Causality” (2018) of:  
Ruifei Cui (Radboud University Nijmegen)

# Causality

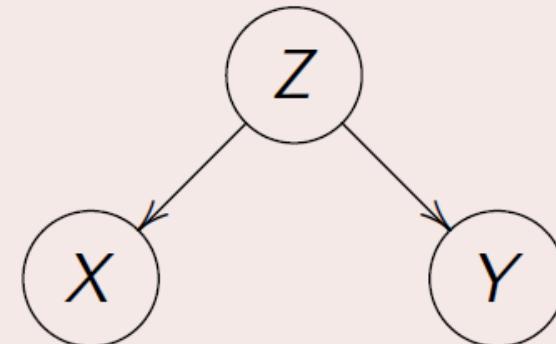
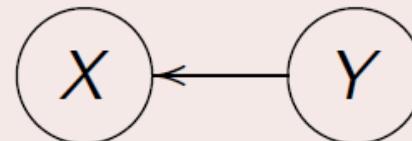
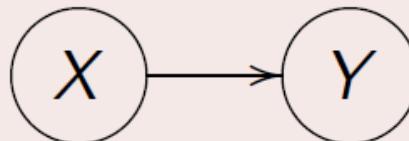
- Correlation unnecessarily means Causation!



# Causality

- Correlation unnecessarily means Causation!
  - Correlation is symmetric. Causality is not.
    - Markov Random Fields do not convey causality information.
  - Causality  $\Rightarrow$  Correlation

Three cases that incur correlation



# Causality

## Probabilistic Inference (traditional machine learning)

- Models the **distribution** of the data
- Focuses on predicting consequences of **observations**

## Causal Inference

- Models the **mechanism** that generates the data
- Also allows to predict results of **interventions**

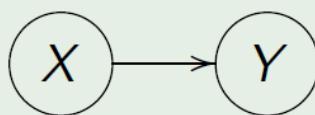
The fundamental difference is that Probabilistic Inference talks about **observing** while Causal Inference talks about **intervening**.

# Causality

## Definition (Interventional Distribution)

The resulting distribution of  $Y$  after setting  $X = x$  is called the **interventional distribution**, denoted by  $p(Y|\text{do}(X = x))$ . It is different from  $p(Y|X = x)$  in general.

### Example



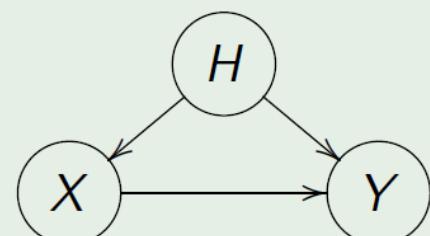
$$p(y|\text{do}(X = x))$$

=

$$p(y|X = x)$$

Yes!

### Example



$$p(y|\text{do}(X = x)) = \int p(h)p(y|x, h)dh$$

≠

$$p(y|X = x) = \int p(h|x)p(y|x, h)dh$$

No!

$X$  is **set** to the value  $x$ .  
( $x$  does not provide information on  $H$ ).

$X$  may be caused by  $H$ , so  $x$  provides information on  $H$ , which in turn also affects  $Y$ .

# Causality

## Theorem (Back-door Criterion (Pearl, 2000) )

For an ordered pair of variables  $(X, Y)$  and a set of variables  $\mathbf{H}$  in a DAG  $\mathcal{G}$  where  $X, Y \notin \mathbf{H}$ , if

- no vertex in  $\mathbf{H}$  is a descendant of  $X$ ;
- $\mathbf{H}$  blocks all the *back-door* paths between  $X$  and  $Y$  (i.e., the path that starts an arrowhead at  $X$ );

then the *interventional distribution* can be obtained by adjusting for  $\mathbf{H}$ :

$$p(y|\text{do}(X = x)) = \int p(\mathbf{h})p(y|x, \mathbf{h})d\mathbf{h}.$$

For the special case  $\mathbf{H} = \emptyset$ , this reduces to

$$p(y|\text{do}(X = x)) = p(y|X = x).$$

# Causality

## Definition (Causal Effect)

The causal effect of  $X$  on  $Y$  is defined as

- $X$  is binary:  $\mathbb{E}(Y|\text{do}(X = 1)) - \mathbb{E}(Y|\text{do}(X = 0))$
- $X$  is continuous:  $\frac{\partial}{\partial x} \mathbb{E}(Y|\text{do}(X = x))$

# Causality

## Simpson's Paradox

You are investigating the effectiveness of a drug against a deadly disease. You are given access to data collected by health insurance companies about their customers, based on which you have the following table.

	Recovery	No recovery	Total	Recovery rate
Drug	20	20	40	... %
No drug	16	24	40	... %
Total	36	44	80	

- The recovery rates for customers with drug and without drug are 50% and 40% respectively.
- If you were a doctor, would you recommend your patients to take the drug?

# Causality

## Simpson's Paradox

Upon closer inspection of the data, you notice something peculiar when you group patients according to gender.

Males	Recovery	No recovery	Total	Recovery rate
Drug	18	12	30	... %
No drug	7	3	10	... %
Total	25	15	40	

Females	Recovery	No recovery	Total	Recovery rate
Drug	2	8	10	... %
No drug	9	21	30	... %
Total	11	29	40	

- For males, with drug: 60%; no drug: 70%.
- For females, with drug: 20%; no drug: 30%.
- Would you recommend your patients to take the drug?

# Causality

- The probability of recovery ( $R = 1$ ) is higher for patients that took the drug ( $D = 1$ ):

$$p(R = 1|D = 1) > p(R = 1|D = 0)$$

- For both male and female patients, the relation is opposite:

$$p(R = 1|D = 1, G = \text{male}) < p(R = 1|D = 0, G = \text{male}),$$

$$p(R = 1|D = 1, G = \text{female}) < p(R = 1|D = 0, G = \text{female})$$

# Causality

- ① The research question of interest should be if

$$p(R = 1|\text{do}(D = 1)) > p(R = 1|\text{do}(D = 0)),$$

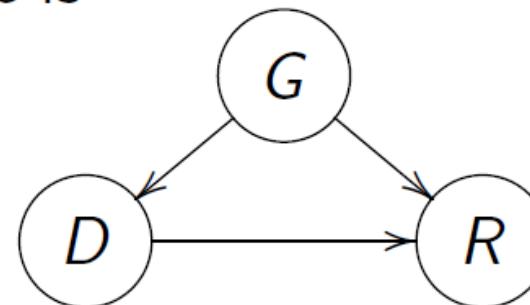
rather than if

$$p(R = 1|D = 1) > p(R = 1|D = 0).$$

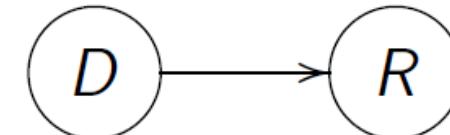
$D$  is SET to the value.  
Not caused by  $G$ .

$D$  gives information about  $G$ ,  
which also affects  $R$ .

- ② The true causal structure is



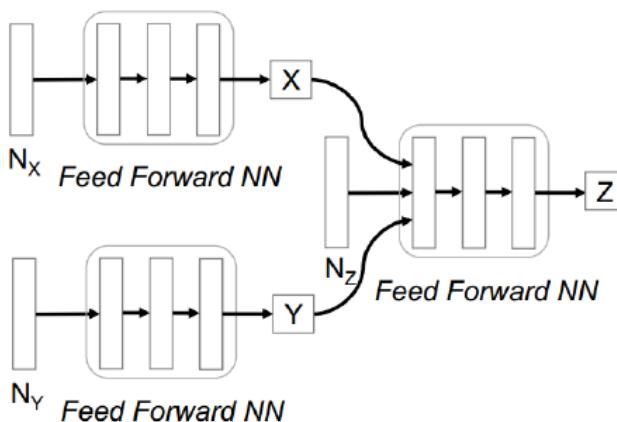
rather than



# Causality

## Causal Generative Adversarial Networks (CausalGAN)

- CausalGAN<sup>2</sup> aims to learn a **causal implicit generative model** (CiGM) with adversarial training: models that allow sampling from not only the true observational but also the **interventional distributions**.
- The key idea is to design the generator according to a causal graph. For example, given the graph  $X \rightarrow Z \leftarrow Y$ , we have the generator:

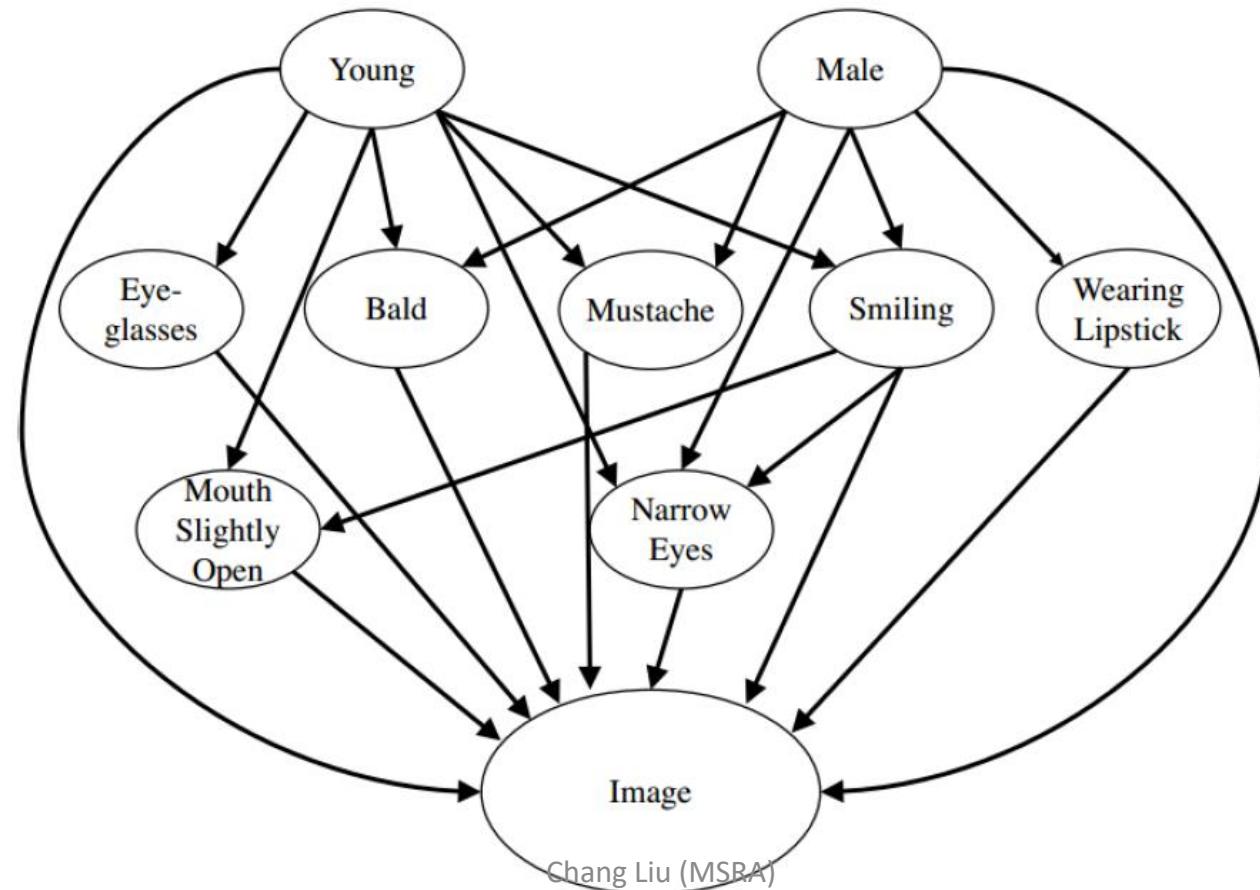


<sup>2</sup>Murat et al., "CausalGAN." ICLR, 2018.

# Causality

## An Example of CausalGAN

The generator is designed according to the following causal graph:



# Causality

## An Example of CausalGAN

Resulting images generated by the trained model:

Top row: Intervene Mustache=1; Bottom: Condition Mustache=1



Conditional GANs  
[MO14, OOS17] tend  
to behave like this.

- Top row contains both males and females because intervening on 'Mustache = 1' does not change the distribution of 'Male'.
- Bottom row only contains males because conditioning on 'Mustache = 1' tells the system 'Male = 1'.

# Topics not Covered

- Gaussian Process
- Bayesian Optimization
- Bayesian Nonparametrics
- Semi-Supervised Learning
- Classical ABCs: message passing, belief propagation
- Dynamic models (e.g., Hidden Markov Models)
- ...

# Future Work

# Future Work

- Decrease auto-correlation in MCMC explicitly.
- High-dimensional degeneracy issue of particle-base VI.
- Online variational inference with more flexible  $q$ .
- Online MCMC with more effective transition.

Thanks!

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