VE527

Computer-Aided Design of Integrated Circuits

Two-level Logic Synthesis

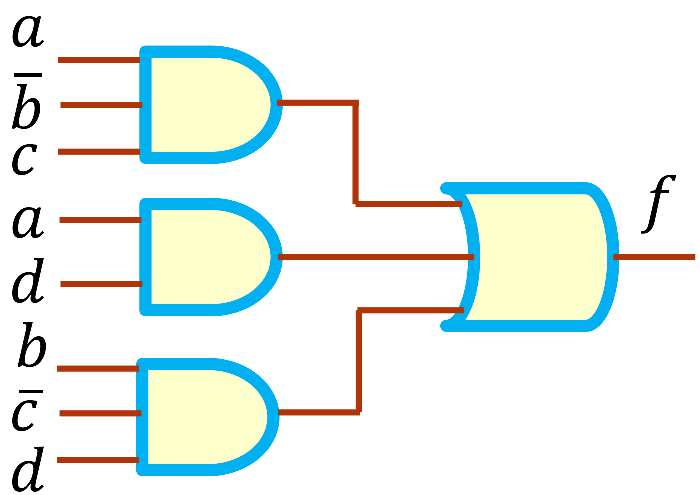
Outline

- Introduction
- Quine-McCluskey Algorithm
  - Phase I
  - Phase II
- Heuristic Method
  - Overview
  - Expand step

Two-Level Logic Design

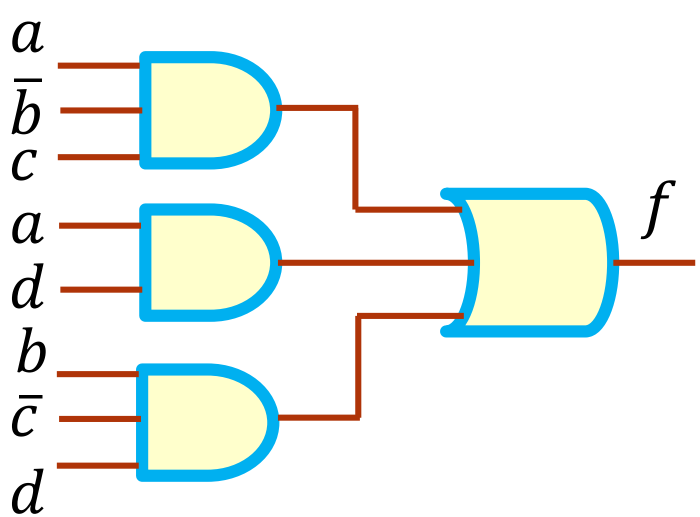
- A logic implementation based on **sum-of-products (SOP)** expression.
  - First level: **<u>many</u>** AND gates.
  - Second level: **<u>one</u>** "big" OR gate.

$$f = a\bar{b}c + ad + b\bar{c}d$$



Two-Level Minimization

- Trying to find **minimal** SOP logic.
  - Want: **fewest input wires**.
  - These input wires called **literals**: a variable in true or complemented form. Thus, we want to minimize **number of literals**.



$$f = a\bar{b}c + ad + b\bar{c}d$$

3 product terms, 8 literals

Number of literals are good metric for the complexity of SOP function.

Two-Level Minimization

- None of the following methods you know from **fundamental digital design** is practical.
  - **Boolean algebra:** Hard with many variables. Can’t tell when have a good solution.
  - **Karnaugh maps**: Can only deal with up to 6 variables.

- We need a systematic computational method.

Outline

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- Quine-McCluskey Algorithm
  - Phase I
  - Phase II
- Heuristic Method
  - Overview
  - Expand step

Quine-McCluskey Algorithm

- Solve the 2-level minimization problem
  - Find an SOP with the **minimal number of literals**.
  - Developed by Walliam Quine and Edward McCluskey in 1956.

- Basic idea: Eliminate as many literals as possible by systematically applying $xy + x\bar{y} = x$.
  - A "**computational**" version of Karnaugh map.

Implicants and Prime Implicants

- **Implicant** of a Boolean function F: A <u>**product term**</u> P that "**implies**" F.
  - I.e., whenever P is 1, F also takes value 1.
  - Example: abc and ab are implicants of $abc + ab\bar{c} + \bar{a}c$.
- **Prime implicant**: an <u>**implicant**</u> that cannot be covered by a "**larger**" implicant.
  - "**Larger**" means have <u>**fewer**</u> literals.
  - Example: ab is a prime implicant of $abc + ab\bar{c} + \bar{a}c$, while abc is not.
  - Corresponds to a largest circle we can draw in Karnaugh maps.

<u>Claim</u>: an SOP with the **minimal number of literals** only contains **prime implicants**.

Two Major Steps

1. Find all **prime implicants** of the function.
   - Just like finding all "largest" circles in Karnaugh map.

2. Use those prime implicants in a **prime implicant chart** to find a **minimal** set of prime implicants that covers the function.
   - Just like selecting a minimal number of "largest" circles in Karnaugh map to cover all 1's.

Phase I: Find All Prime Implicants

- Group the minterms according to the number of 1s in the minterm.
  - This way we only have to compare minterms from **adjacent** groups.
- Example: $f(a,b,c,d) = \sum m(0,1,2,5,6,7,8,9,10,14)$

group 0	0	0000
group 1	1	0001
	2	0010
	8	1000
group 2	5	0101
	6	0110
	9	1001
	10	1010
group 3	7	0111
	14	1110

Phase I: Find All Prime Implicants

- Combining group 0 and group 1.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	
	8	1000	
group 2	5	0101	
	6	0110	
	9	1001	
	10	1010	
group 3	7	0111	
	14	1110	

Column II

0,1	000-

Phase I: Find All Prime Implicants

- Combining group 0 and group 1.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	
group 2	5	0101	
	6	0110	
	9	1001	
	10	1010	
group 3	7	0111	
	14	1110	

Column II

0,1	000-
0,2	00-0

Phase I: Find All Prime Implicants

- Combining group 0 and group 1.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

Group	Minterm	Binary	
group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	
	6	0110	
	9	1001	
	10	1010	
group 3	7	0111	
	14	1110	

Column II

Minterms	Product
0,1	000-
0,2	00-0
0,8	-000

Phase I: Find All Prime Implicants

- Question: Does it make sense to combine group 0 with group 2 or 3?
  - No. There are at least two bits that are different.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	
	6	0110	
	9	1001	
	10	1010	
group 3	7	0111	
	14	1110	

Column II

0,1	000-
0,2	00-0
0,8	-000

So, the next step is to combine group 1 with group 2.

Phase I: Find All Prime Implicants

- Combining group 1 and group 2.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

Group	Minterm	Binary	
group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	
	9	1001	
	10	1010	
group 3	7	0111	
	14	1110	

Column II

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01

Phase I: Find All Prime Implicants

- Combining group 1 and group 2.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	
	9	1001	
	10	1010	
group 3	7	0111	
	14	1110	

Column II

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01

No combining

Phase I: Find All Prime Implicants

- Combining group 1 and group 2.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	
	9	1001	✓
	10	1010	
group 3	7	0111	
	14	1110	

Column II

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01
1,9	-001

Phase I: Find All Prime Implicants

- Combining group 1 and group 2.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	
	9	1001	✓
	10	1010	
group 3	7	0111	
	14	1110	

Column II

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01
1,9	-001

No combining

Phase I: Find All Prime Implicants

- Combining group 1 and group 2.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	
	9	1001	✓
	10	1010	
group 3	7	0111	
	14	1110	

No combining

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Column II

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01
1,9	-001

Phase I: Find All Prime Implicants

- Combining group 1 and group 2.
  - If two products can be combined, check mark them, and record the resulting products.

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	✓
	9	1001	✓
	10	1010	✓
group 3	7	0111	
	14	1110	

Column II

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01
1,9	-001
2,6	0-10
2,10	-010
8,9	100-
8,10	10-0

Phase I: Find All Prime Implicants

- Combining group 2 and group 3.
  - If two products can be combined, check mark them, and record the resulting products.

Column I				Column II	
group 0	0	0000	✓	0,1	000-
group 1	1	0001	✓	0,2	00-0
	2	0010	✓	0,8	-000
	8	1000	✓	1,5	0-01
group 2	5	0101	✓	1,9	-001
	6	0110	✓	2,6	0-10
	9	1001	✓	2,10	-010
	10	1010	✓	8,9	100-
group 3	7	0111	✓	8,10	10-0
	14	1110		5,7	01-1

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Phase I: Find All Prime Implicants

- Combining group 2 and group 3.

Column I

Group	Minterm	Binary	
group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	✓
	9	1001	✓
	10	1010	✓
group 3	7	0111	✓
	14	1110	✓

Column II

0,1	000-
0,2	00-0
0,8	-000
1,5	0-01
1,9	-001
2,6	0-10
2,10	-010
8,9	100-
8,10	10-0
5,7	01-1
6,7	011-
6,14	-110
10,14	1-10

Phase I: Find All Prime Implicants

- Divide column II into groups based on numbers of ones.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-
	0,2	00-0
	0,8	-000
group 1	1,5	0-01
	1,9	-001
	2,6	0-10
	2,10	-010
	8,9	100-
	8,10	10-0
group 2	5,7	01-1
	6,7	011-
	6,14	-110
	10,14	1-10

Phase I: Find All Prime Implicants

- Combining group 0 and group 1 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-
	0,2	00-0
	0,8	-000
group 1	1,5	0-01
	1,9	-001
	2,6	0-10
	2,10	-010
	8,9	100-
	8,10	10-0
group 2	5,7	01-1
	6,7	011-
	6,14	-110
	10,14	1-10

Column III

No combining

Phase I: Find All Prime Implicants

- Combining group 0 and group 1 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-
	0,2	00-0
	0,8	-000
group 1	1,5	0-01
	1,9	-001
	2,6	0-10
	2,10	-010
	8,9	100-
	8,10	10-0
group 2	5,7	01-1
	6,7	011-
	6,14	-110
	10,14	1-10

Column III

No combining

Phase I: Find All Prime Implicants

- Combining group 0 and group 1 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-
	0,2	00-0
	0,8	-000
group 1	1,5	0-01
	1,9	-001
	2,6	0-10
	2,10	-010
	8,9	100-
	8,10	10-0
group 2	5,7	01-1
	6,7	011-
	6,14	-110
	10,14	1-10

Column III

No combining

Phase I: Find All Prime Implicants

- Combining group 0 and group 1 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-
	0,2	00-0
	0,8	-000
group 1	1,5	0-01
	1,9	-001
	2,6	0-10
	2,10	-010
	8,9	100-
	8,10	10-0
group 2	5,7	01-1
	6,7	011-
	6,14	-110
	10,14	1-10

Column III

No combining

Phase I: Find All Prime Implicants

- Combining group 0 and group 1 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-	✓
	0,2	00-0	
	0,8	-000	
group 1	1,5	0-01	
	1,9	-001	
	2,6	0-10	
	2,10	-010	
	8,9	100-	✓
	8,10	10-0	
group 2	5,7	01-1	
	6,7	011-	
	6,14	-110	
	10,14	1-10	

Column III

0,1,8,9	-00-

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Phase I: Find All Prime Implicants

- Combining group 0 and group 1 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-	✓
	0,2	00-0	✓
	0,8	-000	✓
group 1	1,5	0-01	
	1,9	-001	✓
	2,6	0-10	
	2,10	-010	✓
	8,9	100-	✓
	8,10	10-0	✓
group 2	5,7	01-1	
	6,7	011-	
	6,14	-110	
	10,14	1-10	

Column III

0,1,8,9	-00-
0,2,8,10	-0-0
0,8,1,9	-00-
0,8,2,10	-0-0

Phase I: Find All Prime Implicants

- Combining group 1 and group 2 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-	✓
	0,2	00-0	✓
	0,8	-000	✓
group 1	1,5	0-01	
	1,9	-001	✓
	2,6	0-10	
	2,10	-010	✓
	8,9	100-	✓
	8,10	10-0	✓
group 2	5,7	01-1	
	6,7	011-	
	6,14	-110	
	10,14	1-10	

Column III

0,1,8,9	-00-
0,2,8,10	-0-0
0,8,1,9	-00-
0,8,2,10	-0-0

No combining

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Phase I: Find All Prime Implicants

- Combining group 1 and group 2 of Column II.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

group 0	0,1	000-	✓
	0,2	00-0	✓
	0,8	-000	✓
group 1	1,5	0-01	
	1,9	-001	✓
	2,6	0-10	✓
	2,10	-010	✓
	8,9	100-	✓
	8,10	10-0	✓
group 2	5,7	01-1	
	6,7	011-	
	6,14	-110	✓
	10,14	1-10	✓

Column III

0,1,8,9	-00-
0,2,8,10	-0-0
0,8,1,9	-00-
0,8,2,10	-0-0
2,6,10,14	--10
2,10,6,14	--10

Phase I: Find All Prime Implicants

- Divide column III into groups based on numbers of ones.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

0,1	000-	✓
0,2	00-0	✓
0,8	-000	✓
1,5	0-01	
1,9	-001	✓
2,6	0-10	✓
2,10	-010	✓
8,9	100-	✓
8,10	10-0	✓
5,7	01-1	
6,7	011-	
6,14	-110	✓
10,14	1-10	✓

Column III

group 0	0,1,8,9	-00-
	0,2,8,10	-0-0
	0,8,1,9	-00-
	0,8,2,10	-0-0
group 1	2,6,10,14	--10
	2,10,6,14	--10

No more combinations in column III are possible, so we stop here.

Phase I: Find All Prime Implicants

- Eliminate **repeated** combinations.

Column I

0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II

0,1	000-	✓
0,2	00-0	✓
0,8	-000	✓
1,5	0-01	
1,9	-001	✓
2,6	0-10	✓
2,10	-010	✓
8,9	100-	✓
8,10	10-0	✓
5,7	01-1	
6,7	011-	
6,14	-110	✓
10,14	1-10	✓

Column III

0,1,8,9	-00-
0,2,8,10	-0-0
~~0,8,1,9~~	~~-00-~~
~~0,8,2,10~~	~~-0-0~~
2,6,10,14	--10
~~2,10,6,14~~	~~--10~~

Phase I: Find All Prime Implicants

- **<u>Claim</u>**: those products that have NOT been check-marked are prime implicants, because they cannot be combined any more.

Column I		
0	0000	✓
1	0001	✓
2	0010	✓
8	1000	✓
5	0101	✓
6	0110	✓
9	1001	✓
10	1010	✓
7	0111	✓
14	1110	✓

Column II		
0,1	000-	✓
0,2	00-0	✓
0,8	-000	✓
1,5	0-01	⇐
1,9	-001	✓
2,6	0-10	✓
2,10	-010	✓
8,9	100-	✓
8,10	10-0	✓
5,7	01-1	⇐
6,7	011-	⇐
6,14	-110	✓
10,14	1-10	✓

Column III		
0,1,8,9	-00-	⇐
0,2,8,10	-0-0	⇐
2,6,10,14	--10	⇐

Phase I Done!

Outline

- Introduction
- Quine-McCluskey Algorithm
  - Phase I
  - Phase II
- Heuristic Method
  - Overview
  - Expand step

Phase II:
Select A Minimum Set of Prime Implicants

- Build the **prime implicant chart**.
  - Rows: All prime implicants we obtained in Phase I.
  - Columns: All minterms in the original function.

Minterms

Prime Implicants	0	1	2	5	6	7	8	9	10	14
(0,1,8,9)	X	X					X	X		
(0,2,8,10)	X		X				X		X	
(2,6,10,14)			X		X				X	X
(1,5)		X		X						
(5,7)				X		X				
(6,7)					X	X				

Next step: find all **essential** prime implicants.

Essential Prime Implicants

- **Essential prime implicants** are prime implicants that cover a minterm of the function that no other prime implicants are able to cover.
- How can we find essential prime implicants?
  - Look for minterm that is covered by only one prime implicant.
  - The covering prime implicant is essential.

		0	1	2	5	6	7	8	9	10	14
ess.	(0,1,8,9)	X	X					X	X		
	(0,2,8,10)	X		X				X		X	
ess.	(2,6,10,14)			X		X				X	X
	(1,5)		X		X						
	(5,7)				X		X				
	(6,7)					X	X				

Essential Prime Implicants

- Why **essential prime implicants**?
  - Claim: an SOP with the **minimal** number of literals must contain each essential prime implicant.

Phase II:
Select A Minimum Set of Prime Implicants

- Once a product term is included in the solution, all the minterms covered by that term are covered.
- Therefore we may now mark the covered minterms and only focus on prime implicants that cover some **<u>not-yet-covered</u>** minterms.

		0	1	2	5	6	7	8	9	10	14
ess.	(0,1,8,9)	X	X					X	X		
	(0,2,8,10)	X		X				X		X	
ess.	(2,6,10,14)			X		X				X	X
	(1,5)		X		X						
	(5,7)				X		X				
	(6,7)					X	X				

Minterms 5 and 7 haven't been covered. We only need to focus on prime implicants (1,5), (5,7), and (6,7).

Phase II:
Select A Minimum Set of Prime Implicants

		0	1	2	5	6	7	8	9	10	14
ess.	(0,1,8,9)	X	X					X	X		
	(0,2,8,10)	X		X				X		X	
ess.	(2,6,10,14)			X		X				X	X
	(1,5)		X		X						
	(5,7)				X		X				
	(6,7)					X	X				

- Prime implicants (1,5), (5,7), and (6,7) are **non-essential** prime implicants.
  - Now we must choose **enough** non-essential prime implicants to cover the remaining minterms.
  - What strategy should we use?

Phase II:
Select A Minimum Set of Prime Implicants

		0	1	2	5	6	7	8	9	10	14
ess.	(0,1,8,9)	X	X					X	X		
	(0,2,8,10)	X		X				X		X	
ess.	(2,6,10,14)			X		X				X	X
	(1,5)		X		X						
✓	(5,7)				X		X				
	(6,7)					X	X				

Done: the minimal SOP is $f = \bar{b}\bar{c} + c\bar{d} + \bar{a}bd$

- Heuristics: choose the prime implicant that covers the most minterms.
  - Then mark covered minterms.
  - Repeat the above procedure if necessary.

Does this strategy always wok?

Cyclic Prime Implicant Chart

- Example: $f(a,b,c) = \sum m(0,1,2,5,6,7)$

	abc	
0	000	✓
1	001	✓
2	010	✓
5	101	✓
6	110	✓
7	111	✓

	abc
0,1	00-
0,2	0-0
1,5	-01
2,6	-10
5,7	1-1
6,7	11-

Minterms

Prime Implicants	0	1	2	5	6	7
(0,1)	X	X				
(0,2)	X		X			
(1,5)		X		X		
(2,6)			X		X	
(5,7)				X		X
(6,7)					X	X

- Are there any essential prime implicants? **No!**
- Further, all prime implicants cover the same number of minterms.
- Then, how shall we proceed? **By trial and error.**

Cyclic Prime Implicant Chart

	abc	
0	000	✓
1	001	✓
2	010	✓
5	101	✓
6	110	✓
7	111	✓

	abc
0,1	00-
0,2	0-0
1,5	-01
2,6	-10
5,7	1-1
6,7	11-

	0	1	2	5	6	7
(0,1)	X	X				
(0,2)	X		X			
(1,5)		X		X		
(2,6)			X		X	
(5,7)				X		X
(6,7)					X	X

- Pick product (0,1).
- Now products (2,6), (5,7), and (6,7) cover the same most number of minterms.
  - Trial and error again: Pick product (2,6).
- Next: pick product (5,7)

<u>Done</u>: the minimal SOP is $f = \bar{a}\bar{b} + b\bar{c} + ac$

Cyclic Prime Implicant Chart

	abc	
0	000	✓
1	001	✓
2	010	✓
5	101	✓
6	110	✓
7	111	✓

	abc
0,1	00-
0,2	0-0
1,5	-01
2,6	-10
5,7	1-1
6,7	11-

	0	1	2	5	6	7
(0,1)	X	X				
(0,2)	X		X			
(1,5)		X		X		
(2,6)			X		X	
(5,7)				X		X
(6,7)					X	X

- Let's try another set of prime implicants: Pick product (0,2).
- Now products (1,5), (5,7), and (6,7) cover the same most number of minterms.
  - Trial and error again: Pick product (1,5).
- Next: pick product (6,7)

<u>Done</u>: the minimal SOP is $f = \bar{a}\bar{c} + \bar{b}c + ab$

Cyclic Prime Implicant Chart

$$f = \bar{a}\bar{b} + b\bar{c} + ac$$
$$f = \bar{a}\bar{c} + \bar{b}c + ab$$

- Which minimal form is better?
  - Both are best.

- Often we are interested in examining all minimal forms for a given function.
  - Thus we need an algorithm to do so.

Petrick's Method

- Given: a prime implicant chart.
- Goal: determine all sum-of-products solutions that contains the **minimal number of product terms**.

- Step 1: Label all the rows in the chart with a Boolean variable

		0	1	2	5	6	7
P1	(0,1)	X	X				
P2	(0,2)	X		X			
P3	(1,5)		X		X		
P4	(2,6)			X		X	
P5	(5,7)				X		X
P6	(6,7)					X	X

Pi = 1 means the i-th prime implicant is selected

Petrick's Method

- Step 2: We want to build a function on variables P1, P2, etc. to encode **all** prime implicant covers: for any Pi combination that lets function be 1, it corresponds to a valid cover

		0	1	2	5	6	7
P1	(0,1)	X	X				
P2	(0,2)	X		X			
P3	(1,5)		X		X		
P4	(2,6)			X		X	
P5	(5,7)				X		X
P6	(6,7)					X	X

- Note: The first column has an **X** in rows P1 and P2.
  - Therefore we must include one of these rows in order to cover minterm 0. Thus the following term must be in P:

$$P1+P2$$

Petrick's Method

- Step 2: We want to build a function on variables P1, P2, etc. to encode **all** prime implicant covers: for any Pi combination that lets function be 1, it corresponds to a valid cover

		0	1	2	5	6	7
P1	(0,1)	X	X				
P2	(0,2)	X		X			
P3	(1,5)		X		X		
P4	(2,6)			X		X	
P5	(5,7)				X		X
P6	(6,7)					X	X

- Note: The second column has an **X** in rows P1 and P3.
  - Therefore we must include one of these rows in order to cover minterm 1. Thus the following term must be in P:

$$P1+P3$$

• • •

Petrick's Method

		0	1	2	5	6	7
P1	(0,1)	X	X				
P2	(0,2)	X		X			
P3	(1,5)		X		X		
P4	(2,6)			X		X	
P5	(5,7)				X		X
P6	(6,7)					X	X

P=1 ↔ all minterms are covered

- The final function P is AND of all the sums:

$$P = (P1+P2)(P1+P3)(P2+P4)(P3+P5)(P4+P6)(P5+P6)$$

$$= [(P1+P2)(P1+P3)][(P2+P4)(P4+P6)][(P3+P5)(P5+P6)]$$

$$= (P1+P2\ P3)(P4+P2\ P6)(P5+P3\ P6)$$

$$= P1\ P4\ P5 + P1\ P3\ P4\ P6 + P1\ P2\ P5\ P6 + \cancel{P1\ P2\ P3\ P6}$$

$$+ P2\ P3\ P4\ P5 \cancel{+ P2\ P3\ P4\ P6} \cancel{+ P2\ P3\ P5\ P6} + P2\ P3\ P6$$

Petrick's Method

		0	1	2	5	6	7
P1	(0,1)	X	X				
P2	(0,2)	X		X			
P3	(1,5)		X		X		
P4	(2,6)			X		X	
P5	(5,7)				X		X
P6	(6,7)					X	X

P = P1 P4 P5 + P1 P3 P4 P6 + P1 P2 P5 P6
+ P2 P3 P4 P5 + P2 P3 P6

- What does the above equation mean?
  - It says that to cover all the minterms we must include the terms in lines P1, P4, and P5, **or** we must include lines P1, P3, P4, and P6, **or** we must include lines P1, P2, P5, and P6, …

Petrick's Method

		0	1	2	5	6	7
P1	(0,1)	X	X				
P2	(0,2)	X		X			
P3	(1,5)		X		X		
P4	(2,6)			X		X	
P5	(5,7)				X		X
P6	(6,7)					X	X

$$P = P1\ P4\ P5 + P1\ P3\ P4\ P6 + P1\ P2\ P5\ P6 + P2\ P3\ P4\ P5 + P2\ P3\ P6$$

- What are the choices with the minimal number of products?
  - P1, P4, P5 $\Longrightarrow f = \bar{a}\bar{b} + b\bar{c} + ac$
  - **Or** P2, P3, P6 $\Longrightarrow f = \bar{a}\bar{c} + \bar{b}c + ab$

Quine-McCluskey Algorithm
Summary

- Two Major Steps:
  1. Find all **prime implicants** of the function.
  2. Use those prime implicants in a **prime implicant chart** to find a **minimal** set of prime implicants that covers the function.
     - **Essential prime implicants**
     - Petrick's method

- Quine-McCluskey algorithm gives **exact** minimal solution.
- … but it has **exponential** complexity on number of inputs.
  - Not practical for large problem.
  - How can we do better?

Heuristic Methods

Outline

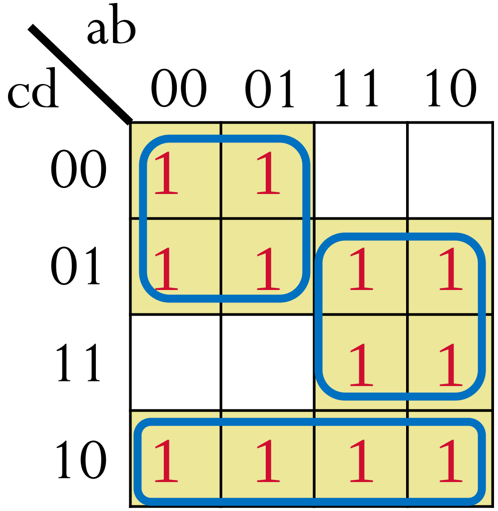
- Introduction
- Quine-McCluskey Algorithm
  - Phase I
  - Phase II
- Heuristic Method
  - Overview
  - Expand step

Better Strategy

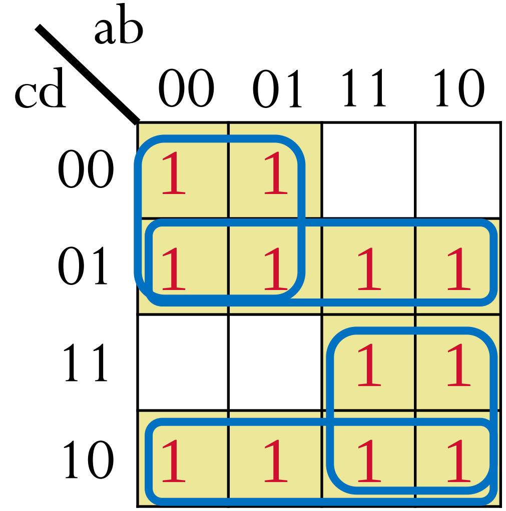
- **Big idea #1**: **Don't** try for the **best**, **perfect** answer. Just get a **good** answer.

- **Big idea #2**: **Iterative improvement**. From one answer, **reshape** the solution to discover a (possibly better) answer. Continue until no more improvement.

Example: Best vs "Good Enough"



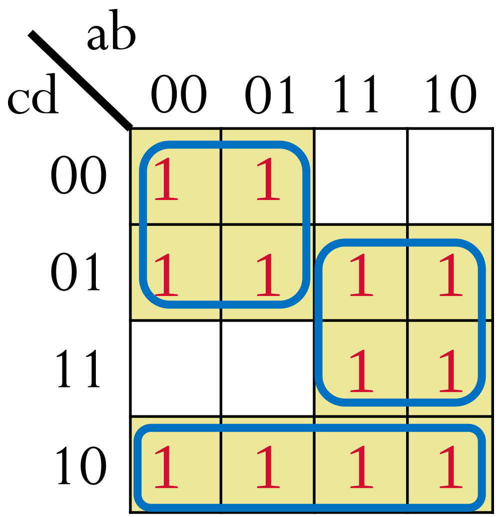
Best



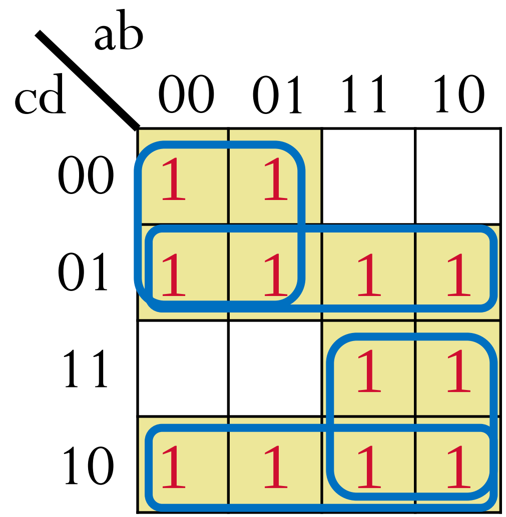
Good

- Comparing the two solutions:
  - Both are made of product terms ("**cubes**") that are "**as big as possible**", i.e., **prime implicants**.
  - We <u>**insist on**</u> this, because best solution is composed of **cover of prime implicants**.

Example: Best vs "Good Enough"



Best



Good

- <u>Note</u>: Neither solution can be improved by **removing** a prime
  - Both solutions are "**irredundant**".
  - We also **<u>insist on</u>** this.

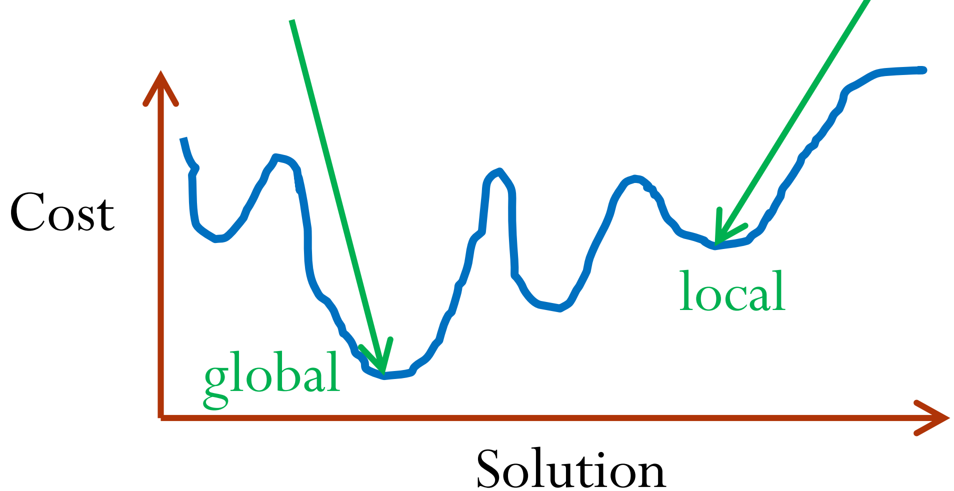
Example: Best vs “Good Enough”

cd \ ab	00	01	11	10
00	1	1		
01	1	1	1	1
11			1	1
10	1	1	1	1

Best

cd \ ab	00	01	11	10
00	1	1		
01	1	1	1	1
11			1	1
10	1	1	1	1

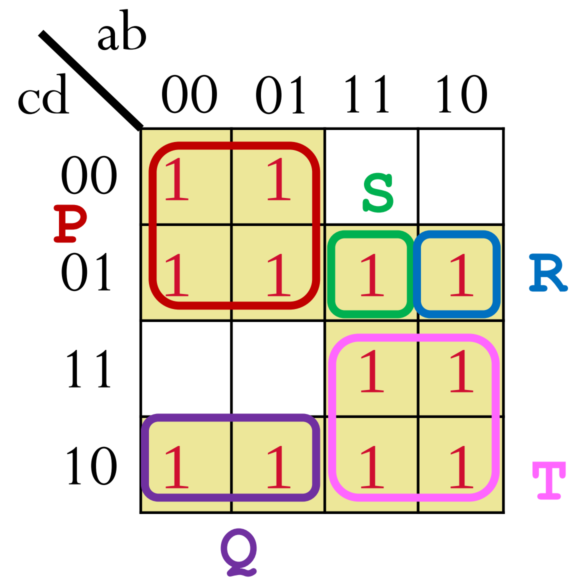
Good



Heuristic Method: Example

Assume start with a cube list

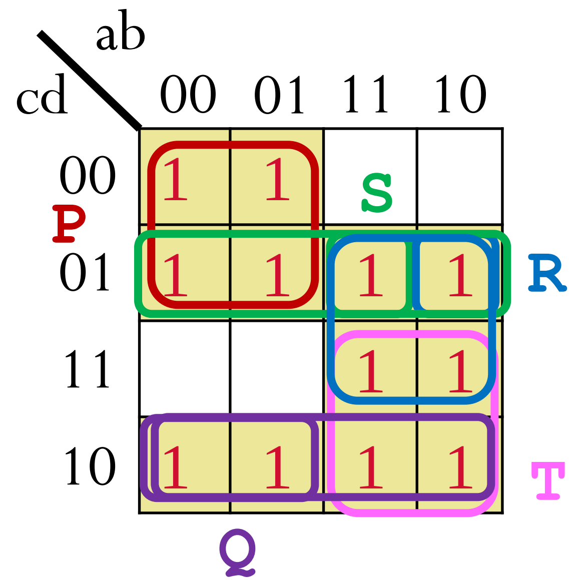
abcd	F	label
0-0-	1	P
0-10	1	Q
1001	1	R
1101	1	S
1-1-	1	T



- Each row defines a **product** (**cube**)
  - It is an **implicant** that lets $F = 1$.
  - Might not be **prime**, but it surely covers all the 1's.

Next Step: Expand Each Cube to be Prime

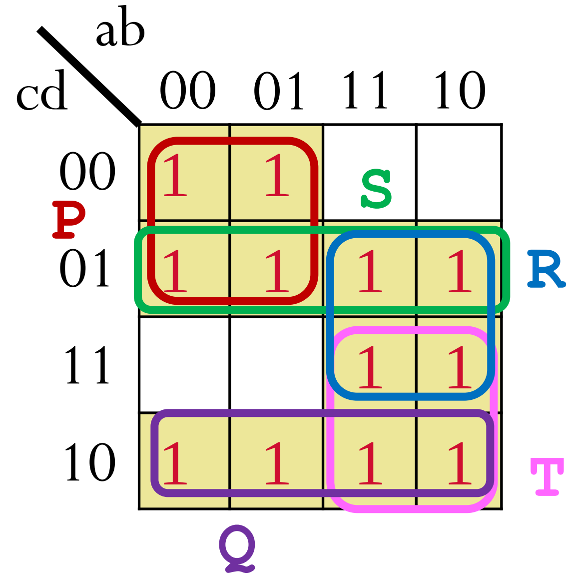
- "Expand" is a **heuristic**, done one cube at a time.
- Make each cube **as big as possible**.
  - 3 of our cubes have now been grown: Q, R, and S.
  - Might have **<u>different</u>** ways to do this for any specific cube…
    - E.g., cube S
- This new solution is a **prime cover**.
- But it might **not be best**, we need do something further.



abcd	F	label
0-0-	1	P
--10	1	Q
1--1	1	R
--01	1	S
1-1-	1	T

Next Step: Remove Redundant Cubes

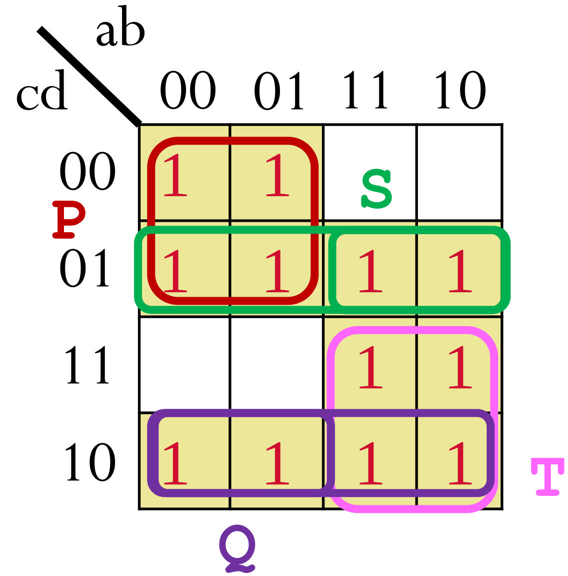
- "Irredundant" is a **heuristic**.
  - A cube is **redundant** if we can remove it, and all its 1's are **still** covered by **other** cubes in the rest of the cover.
  - Irredundant operation **removes** redundant cubes in our cover.
- Question: which cube is redundant?
- Assume we remove cube R
  - This new solution is a prime cover.
  - And it is technically "**minimal**" –cannot remove another cube without **uncovering** some 1s.
- But maybe we can still do better…



abcd	F	label
0-0-	1	P
--10	1	Q
--01	1	S
1-1-	1	T

Next Step: Reduce the Prime Cover

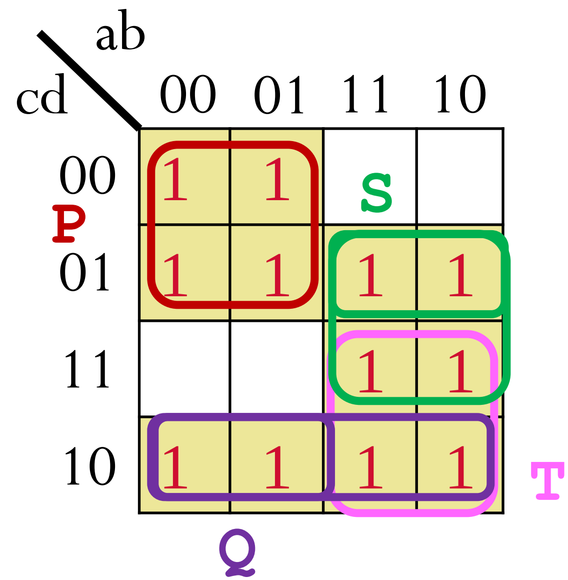
- "Reduce" is another **heuristic**.
  - Take each cube, "**shrink it**" as much as possible, but **do not uncover** any 1s.
    - In our example: shrink cubes Q and S.
  - These result cubes may **not** be prime; i.e. this is **not** necessarily a prime cover.
- Surprisingly, **essential** step!
  - This new solution has different **shape**.
  - **Big Idea**: When we expand it again, maybe we get a new, better solution.
  - So, maybe we can still do better…



abcd	F	label
0-0-	1	P
0-10	1	Q
1-01	1	S
1-1-	1	T

Next Step: Expand Cubes Again

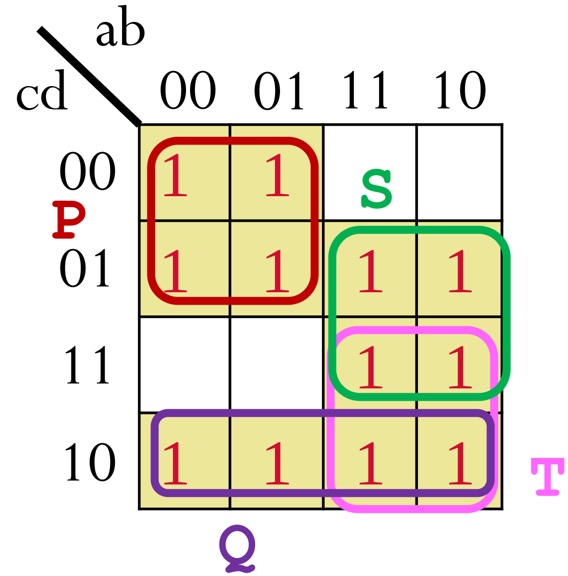
- Same "Expand" **heuristic**.
  - But it is starting from a **different cover**, so can get a **different** answer!
  - Take each cube and "**expand**" it to make it prime, and also…
  - …try to cover other cubes, to make them **redundant** (so we kill them later).
- In this example, we expand Q and S.



abcd	F	label
0-0-	1	P
--10	1	Q
1--1	1	S
1-1-	1	T

Next Step: Check Redundant Again

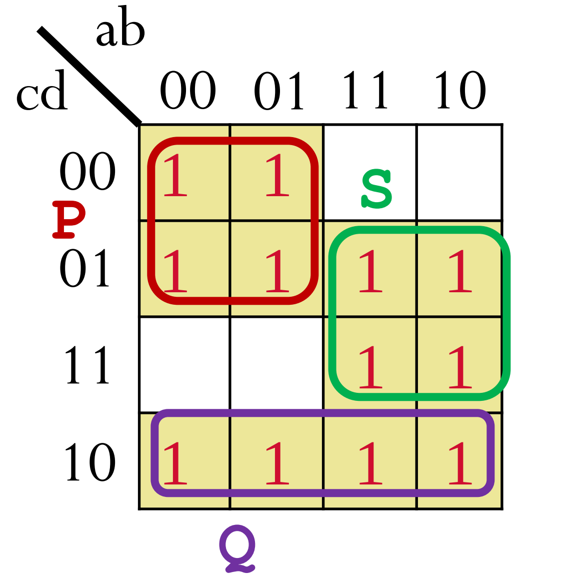
- Same "Irredundant" **heuristic**.
  - But it is starting from a **different cover**, so can get a **different** answer!
- In this example: we can kill another cube T, it is **redundant**.
  - After this, the cover is again **prime** and **irredundant**. Can't remove anything to make it better (smaller).



abcd	F	label
0-0-	1	P
--10	1	Q
1--1	1	S

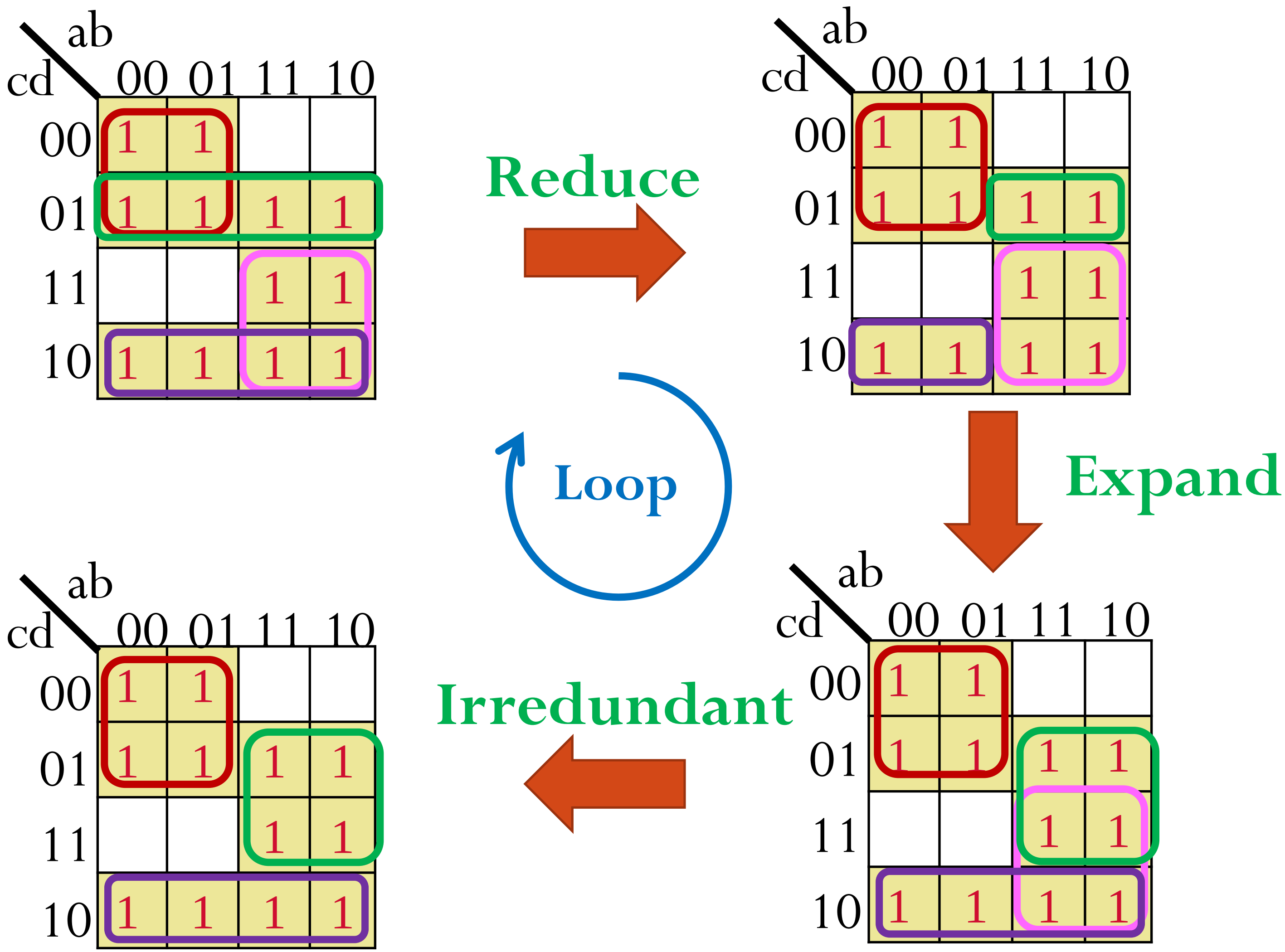
This Result Is Really Good!

- Got lucky: this is the **BEST** answer.
  - This will **not** generally happen.
  - But we can guarantee a **prime, irredundant, and minimal** solution.
  - And it turns out in practice, that this **iterative** improvement that "**reshapes**" the cover produces **excellent solutions**.



abcd	F	label
0-0-	1	P
--10	1	Q
1--1	1	S

Famous: Reduce-Expand-Irredundant Loop



Famous Tool: ESPRESSO

- For **two-level minimization**
- Started at IBM, finished at Berkeley.

- References:
  - Brayton, Hachtel, McMullen, Sangiovanni-Vincentelli, Logic Minimization Algorithms for VLSI Synthesis, Kluwer Academic Press, 1984.
  - Richard L. Rudell,, "Multiple-Valued Logic Minimization for PLA Synthesis", U. California Berkeley M.S. Thesis.

Outline

- Introduction
- Quine-McCluskey Algorithm
  - Phase I
  - Phase II
- Heuristic Method
  - Overview
  - Expand step

Lets Look (Briefly) At One Step: Expand

- What does "expand a cube" mean?
  - **Remove variables** from cube.

$xy\bar{z}\bar{w}$: **1100**

zw \ xy	00	01	11	10
00	1		1	1
01		1	1	1
11			1	
10			1	

Remove $y\bar{w}$

$x\bar{z}$: **1-0-**

zw \ xy	00	01	11	10
00	1		1	1
01		1	1	1
11			1	
10			1	

Remove $\bar{z}\bar{w}$

zw \ xy	00	01	11	10
00	1		1	1
01		1	1	1
11			1	
10			1	

xy: **11--**

Expand: Transform into a Covering Problem

- Here is the most basic **Covering Problem**:
  - Given matrix of R rows and C columns. Matrix has 1s and 0s in it. (Only show the 1s.)
  - Choose **smallest set of rows** so that, using only these rows, **every column** has **at least** a single 1 in them – i.e., every column is **"covered" by the selected rows**.
  - Very good **heuristics** to get decent, fast solutions

	C1	C2	C3	C4	C5
R1		1			
R2	1	1	1		
R3			1		1
R4		1		1	1

Expand: the Blocking Matrix

- Expand = a **Covering Problem** on the **Blocking Matrix**
  - **First**: Given function F, build a **cube cover** of the **0s** in F (called the **OFF Set**).
  - **Why**: We need to know what our cube **cannot touch** when it expands.
  - **How**: **URP Complement** of the starting cover of the function! (this is exactly our Programming Assignment #2.)

yz \ wx	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	1	1	0	1

$$F = \bar{w}\bar{y}\bar{z} + xz + \bar{x}y\bar{z} + \bar{w}xy\bar{z}$$

URP Complement

OFF Set: $\bar{F} = \bar{x}z + wx\bar{z} + w\bar{y}\bar{z}$

Assume we want to expand $\bar{w}xy\bar{z}$

Next: Build the Blocking Matrix

- Blocking matrix is a binary matrix structured as follows:
  - One row for **each variable** in the cube you are trying to expand.
  - One column for **each cube** in the cover of the **OFF set**.
    - May have **many** columns.

yz \ wx	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	1	1	0	1

To expand $\bar{w}xy\bar{z}$

OFF Set: $\bar{F} = \bar{x}z + wx\bar{z} + w\bar{y}\bar{z}$

	$\bar{x}z$	$wx\bar{z}$	$w\bar{y}\bar{z}$
$\bar{w}$			
x			
y			
$\bar{z}$			

What to Fill in the Blocking Matrix?

- If the variable in the cube to be expanded (**row**) ≠ **polarity** of variable in the cube of the OFF cover (**column**), put a "**1**".
- If the variable in the cube to be expanded (**row**) = **polarity** of variable in the cube of the OFF cover (**column**), put a "**0**".
- If the variable in the cube to be expanded (**row**) **does not show** in the cube of the OFF cover (**column**), i.e., **don't care**, put a "**0**".

yz \ wx	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	1	1	0	1

To expand $\bar{w}xy\bar{z}$

OFF Set: $\bar{F} = \bar{x}z + wx\bar{z} + w\bar{y}\bar{z}$

	$\bar{x}z$	$wx\bar{z}$	$w\bar{y}\bar{z}$
$\bar{w}$	0	1	1
x	1	0	0
y	0	0	1
$\bar{z}$	1	0	0

What does “1” in Blocking Matrix Mean?

- **<u>Claim</u>**: if all the row variables you have kept **cover each column**, then **product** of these variables is a **legal cube expansion**.
  - Why?
    - For any cube $c \in \overline{F}$, these is a <u>kept</u> literal l let $l \cdot c = 0$
    - $\Pi(kept\ literal) \cdot \overline{F} = 0 \Rightarrow \Pi(kept\ literal) \subseteq F$

yz \ wx	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	1	1	0	1

To expand $\overline{w}xy\bar{z}$

OFF Set: $\bar{F} = \bar{x}z + wx\bar{z} + w\bar{y}\bar{z}$

	$\bar{x}z$	$wx\bar{z}$	$w\bar{y}\bar{z}$
$\overline{w}$		1	1
x	1		
y			1
$\bar{z}$	1		

Keep rows $\overline{w}$ and x

A cover!

What does “1” in Blocking Matrix Mean?

- **Claim**: if all the row variables you have kept **cover each column**, then **product** of these variables is a **legal cube expansion**.

yz \ wx	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	1	1	0	1

To expand $\overline{w}xy\bar{z}$

OFF Set: $\bar{F} = \bar{x}z + wx\bar{z} + w\bar{y}\bar{z}$

	$\bar{x}z$	$wx\bar{z}$	$w\bar{y}\bar{z}$
$\overline{w}$		1	1
x	1		
y			1
$\bar{z}$	1		

Another example: keep rows $\overline{w}$ and $\bar{z}$

A cover!

What does "1" in Blocking Matrix Mean?

- **<u>On the other hand</u>**, if the set of row variables you have kept is **<u>not</u>** a **cover of all columns**, then **product** of these variables is **<u>not</u>** an **legal cube expansion**.
  - Because it may touch some cubes in the **OFF set**.

OFF Set: $\bar{F} = \bar{x}z + wx\bar{z} + w\bar{y}\bar{z}$

yz \ wx	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	1	1	0	1

To expand $\bar{w}xy\bar{z}$

	$\bar{x}z$	$wx\bar{z}$	$w\bar{y}\bar{z}$
$\bar{w}$		1	1
x	1		
y			1
$\bar{z}$	1		

Keep rows x and y

Not a cover!

Indeed, the expansion touches $wx\bar{z}$

How to Expand to A Prime Implicant?

- **Keep as few variables as possible** from the initial cube, without touch any cubes in the **OFF set**.
- **Equivalent**: Find **smallest** set of rows that covers each column.
  - This is the **covering problem**. (Can be solved fast by heuristic.)

yz \ wx	00	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	1	1	0
10	1	1	0	1

To expand $\bar{w}xy\bar{z}$

OFF Set: $\bar{F} = \bar{x}z + wx\bar{z} + w\bar{y}\bar{z}$

	$\bar{x}z$	$wx\bar{z}$	$w\bar{y}\bar{z}$
$\bar{w}$		1	1
x	1		
y			1
$\bar{z}$	1		

Either
$\bar{w}$ and x,
or
$\bar{w}$ and $\bar{z}$

ESPRESSO: Collection of Elegant Heuristics

- **Reduce-Expand-Irredundant loop**
  - **Reduce**: Rank cubes in a clever order and reduce them individually.
  - **Expand**: Rank cubes in the opposite of the above clever order and expand each individually as a covering problem.
  - **Irredundant**: A clever recursive algorithm + a clever covering problem.
  - And a bunch of other interesting steps we did not mention…

Other Thing Can Do

- Minimize **several** functions at the **same time**.
  - Each function will be reduced to a 2-level form.
  - But some product terms (AND gates) will be **shared**.
  - This means: make this AND product once in hardware, connect its output to many OR gates to sum the product into other functions. Can save a lot of hardware this way.
- Handle conventional **Don’t Cares**.
  - Can specify a row of the truth table as being a “**Don’t Care**”.
  - Means the hardware can make a 1 or a 0 as output for this input – you don’t care.
  - Let algorithm choose 0 vs 1 output to make better, smaller circuit.

How Well Does All This Work?

- Fabulous: Very **fast**, very **robust**
- Where does ESPRESSO spend its time?
  - Complement 14% (big if there are lots of cubes in cover)
  - Expand 29% (depends on of size of complement)
  - Irredundant 12%
  - Reduce 8%
  - Essentials 13% (some primes must be in answer; find them first)
  - Various optimizations 22% (special case optimizations)
- How fast?
  - Usually < 5 reduce-expand-irredundant iterations;
    often converges in just 1-2.
  - Thousands of cubes, tens of thousands of literals: ≪ 1 CPU second.

Summary

- 2-level logic synthesis uses **heuristics** to find good solutions.
  - Not "best", but instead "good enough".
  - Minimal (not minimum), prime, irredundant.
  - Famous idea: iterative improvement – **reduce-expand-irredundant loop**.
  - All done with PCN cube lists, covering matrices, and recursive ideas.
- But, not every piece of logic is implemented in 2-level form.
- Next: **Multi-level logic**