COMP0014 Tutorial 1

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Basic Linear Algebra

Vector Space

A vector space over a field (e.g., $\mathbb{R}$ or $\mathbb{C}$) is a set V of elements, or 'vectors', together with two binary operators.

- *vector addition* denoted for $v_1, v_2, \in V$ by $v_1 + v_2$, where $v_1 + v_2 \in V$, i.e., a vector space is closed under addition.
- *scalar multiplication* denoted for $\lambda \in \mathbb{R}$ and $v \in V$ by λv, where $\lambda v \in V$, so that the vector space is closed under scalar multiplication.

Vector spaces satisfy the following 8 rules:

- Addition is commutative, i.e. for all $v_1, v_2 \in V$
$$v_1 + v_2 = v_2 + v_1 \tag{1}$$
- Addition is associative, i.e. for all $v_1, v_2, v_3 \in V$
$$v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \tag{2}$$
- There exists a unique element $0 \in V$, called the null or zero vector, such that for all $v \in V$
$$v + 0 = v \tag{3}$$
- For all $v \in V$ there exists an additive negative or inverse vector $v' \in V$ such that
$$v + v' = 0 \tag{4}$$
- Scalar multiplication is distributive over scalar addition, i.e. for all $\lambda, \mu \in \mathbb{R}$, and $v \in V$
$$(\lambda + \mu)v = \lambda v + \mu v \tag{5}$$
- Scalar multiplication is distributive over vector addition, i.e. for all $\lambda \in \mathbb{R}$ and $v_1, v_2 \in V$
$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2 \tag{6}$$
- Scalar multiplication of vectors is 'associative', i.e. for all $\lambda, \mu \in \mathbb{R}$ and $v \in V$
$$\lambda(\mu v) = (\lambda\mu)v \tag{7}$$
- Scalar multiplication has an identity element, i.e. for all $v \in V$
$$1 \cdot v = v \tag{8}$$
where 1 is the multiplicative identity in $\mathbb{R}$.

Spanning Sets, Linear Independence, Bases

First consider 2-dimensional space, $\mathbb{R}^2$, an origin O, and two non-zero and non-parallel vectors v_1 and v_2. Then any vector $v \in \mathbb{R}^2$, we have

$$v = \lambda v_1 + \mu v_2 \tag{9}$$

for scalars $\lambda, \mu \in \mathbb{R}$. We say that the set $\{v_1, v_2\}$ **spans** the set of vectors lying in $\mathbb{R}^2$.

Definition: Spanning set. We say that $S = \{v_1, \dots, v_n\}$ spans a vector space V if for all $v \in V$, v can be expressed as a linear combination of the vectors in S, i.e. for all $v \in V$

$$v = \sum_{i=1}^{n} \lambda_i v_i \tag{10}$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. In such cases, we say that S spans V.

Definition: Linear independence A set of vectors $S = \{v_1, \dots, v_n\}$ is said to be a linearly independent set if

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \quad \Rightarrow \quad \lambda_i = 0, \quad i = 1, \dots, n \tag{11}$$

Definition: Basis We say that the set $S = \{v_1, \dots, v_n\}$ is a *basis* for a vector space if S is a spanning set and S is linearly independent.