Iterative coordinates

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Abstract

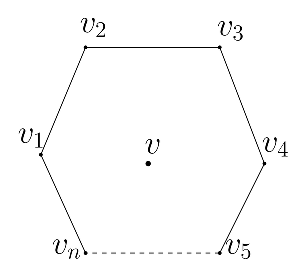
We present a modification of mean value coordinates that is based on the observation that the mean value coordinates of some point v inside a polygon can be negative if the central projection of the polygon onto the unit circle around v folds over. By iteratively smoothing the projected polygon and carrying over this smoothing procedure to the barycentric coordinates of v, these fold-overs as well as the negative coordinate values and shape deformation artefacts gradually disappear, and are guaranteed to completely vanish after a finite number of iterations.

Generalized barycentric coordinates (GBCs)

Given a polygon Ω with n vertices $v_1, v_2, \ldots, v_n$ and any $v \in \Omega$, find coordinates $\boldsymbol{\lambda}(v) = [\lambda_1(v), \lambda_2(v), \ldots, \lambda_n(v)]^\top$ such that

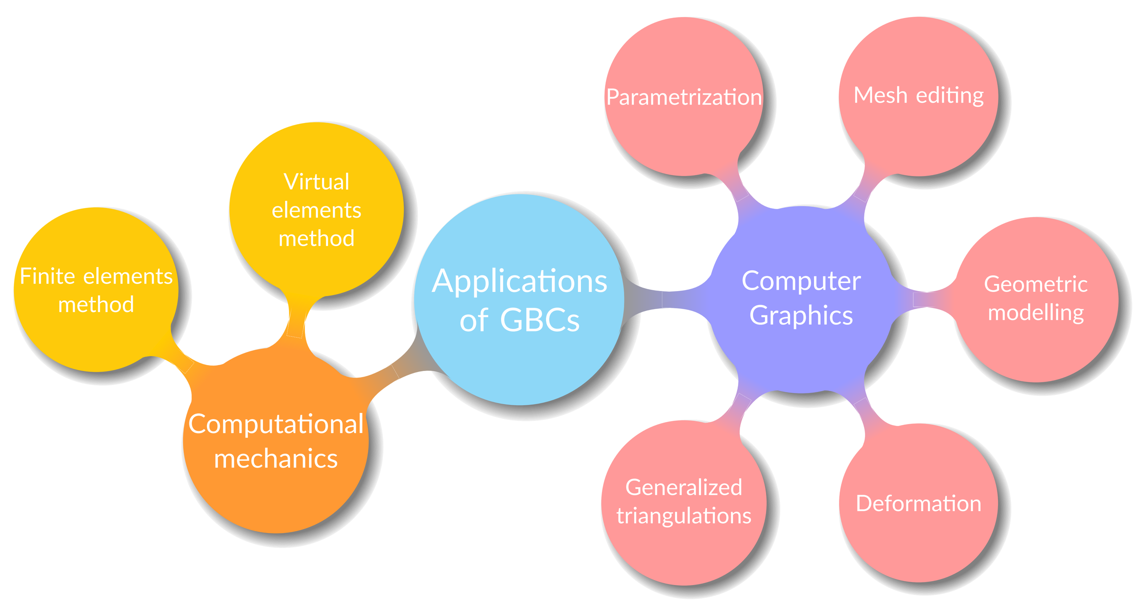
$$\sum_{i=1}^n \lambda_i = 1,$$

$$\sum_{i=1}^n \lambda_i v_i = v.$$



$\boldsymbol{\lambda}$ are generalized barycentric coordinates of v with respect to the polygon Ω.

- When the $n > 3$, such $\boldsymbol{\lambda}$ are usually not unique.



All these applications share a common data interpolation task. Given $f_1, \ldots, f_n \in \mathbb{R}^m$ at the vertices $v_1, \ldots, v_n$ of a polytope $\Omega \subset \mathbb{R}^d$ with $n \geq d+1$, these application utilize the *barycentric interpolant*

$$f : \Omega \to \mathbb{R}^m, \qquad f(v) = \sum_{i=1}^n \lambda_i(v) f_i,$$

where the function $\lambda_i : \Omega \to \mathbb{R}, i = 1, \ldots, n$ are *generalized barycentric coordinates* with most or even all of the following properties:

- Partition of unity: $\sum_{i=1}^n \lambda_i(v) = 1$ for $v \in \Omega$;
- Linear reproduction: $\sum_{i=1}^n \lambda_i(v) v_i = v$ for $v \in \Omega$;
- Lagrange property: $\lambda_i(v_j) = \delta_{i,j}$ for $i, j = 1, \ldots, n$;
- Non-negativity: $\lambda_i(v) \geq 0$ for $v \in \Omega$ and $i = 1, \ldots, n$;
- Smoothness: $\lambda_1(v), \ldots, \lambda_n(v)$ vary smoothly with $v \in \Omega$;
- Linearity on the edges: λ_i is linear on the edges of Ω.

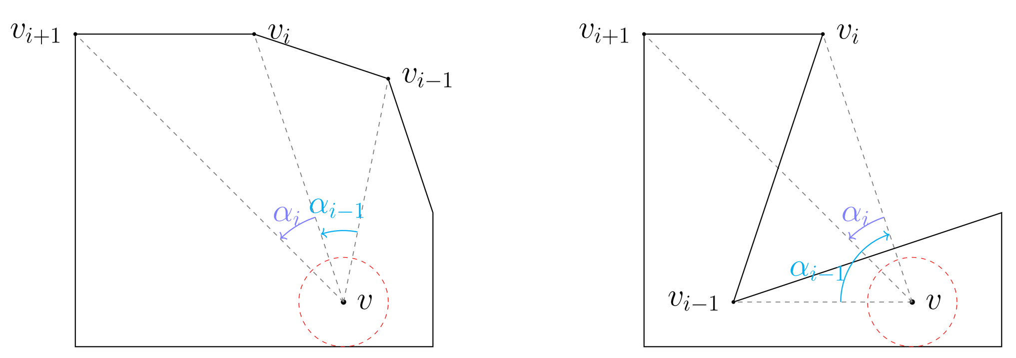
Mean value coordinates (MVCs)

In 2003, inspired by the mean value theorem for harmonic functions, Floater [1] derived *mean value coordinates* (MVCs), which are defined as

$$w_i(v) = \frac{\tan(\frac{\alpha_{i-1}}{2}) + \tan(\frac{\alpha_i}{2})}{\|v_i - v\|}, \tag{1}$$

$$\lambda_i(v) = w_i(v) \Big/ \sum_{j=1}^n w_j(v), \tag{2}$$

where $\alpha_i \in (-\pi, \pi]$ and $\|v_i - v\|$ denote the signed angle from edge $\overline{vv_i}$ to $\overline{vv_{i+1}}$ and the length of the segment $\overline{vv_i}$ respectively, see below.



It follows from (1) and (2) that MVCs may possess negative values, if $\alpha_{i-1} + \alpha_i < 0$, which may occur for concave polygons (above right), but they are always positive inside convex polygons.

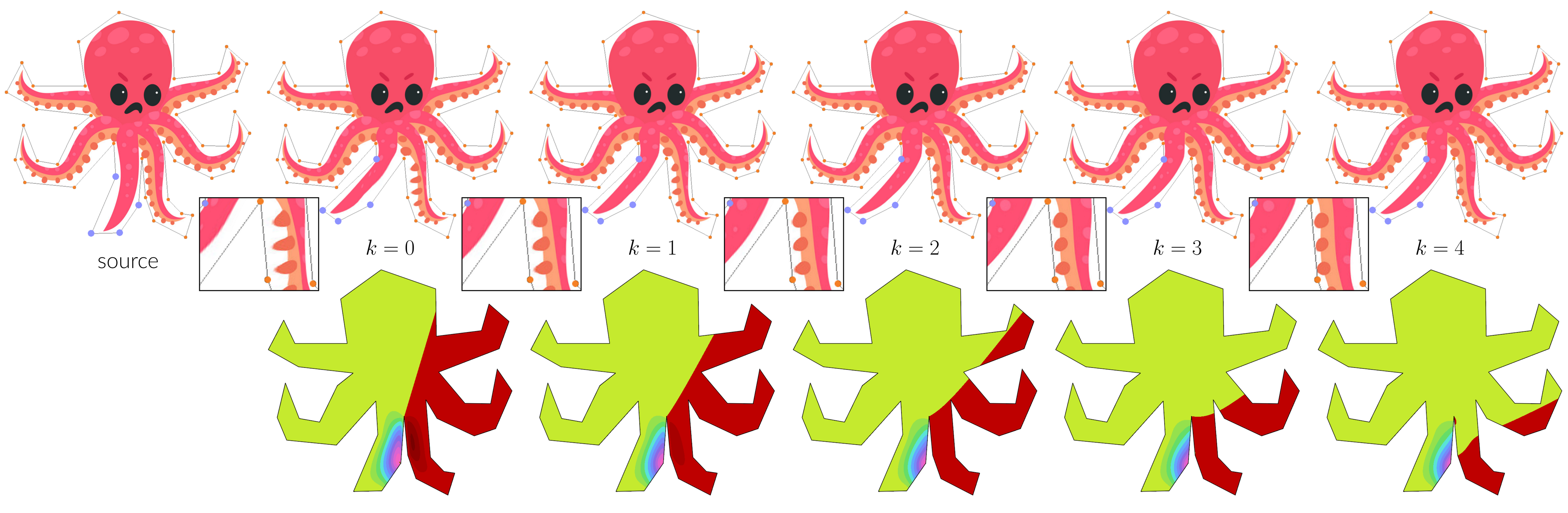


Figure 1: Deformation of a source image (left), obtained by moving four vertices (blue) of the control polygon, using iterative coordinates with respect to the source control polygon (top row). The deformation artefacts (see close-ups) are caused by the negative values (red) of the indicated coordinate function (bottom row), which are relatively large for mean value coordinates (i.e., $k = 0$), but quickly become negligible as the number of iterations increases (see $k = 1, 2, 3, 4$). Function values are visualized using the colour bar in Figure 4.

Iterative coordinates

We found a novel and easy-to-implement construction of *generalized barycentric coordinates* that is based on mean value coordinates and an iterative process. These *iterative coordinates* are guaranteed to be positive after a sufficient number of iterations.

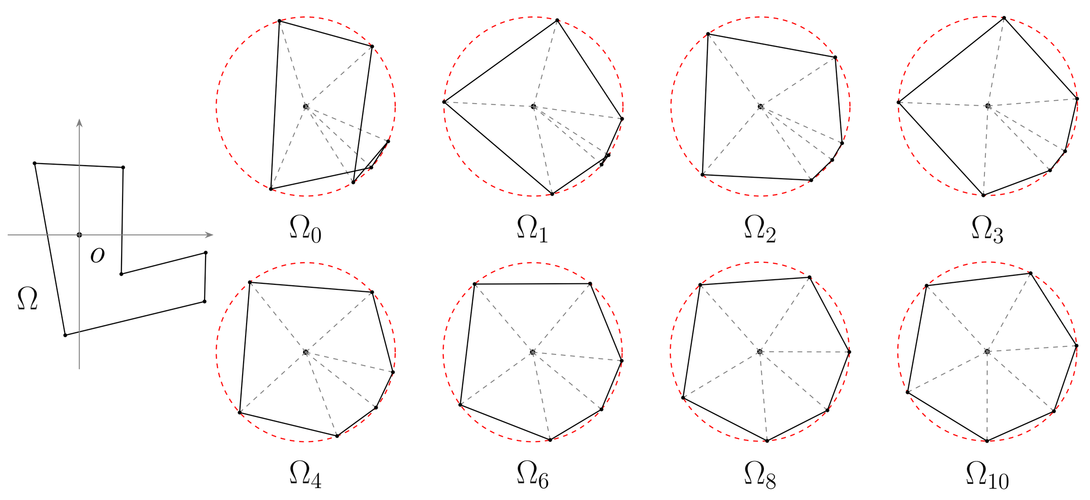


Figure 2: Illustration of the iterative process.

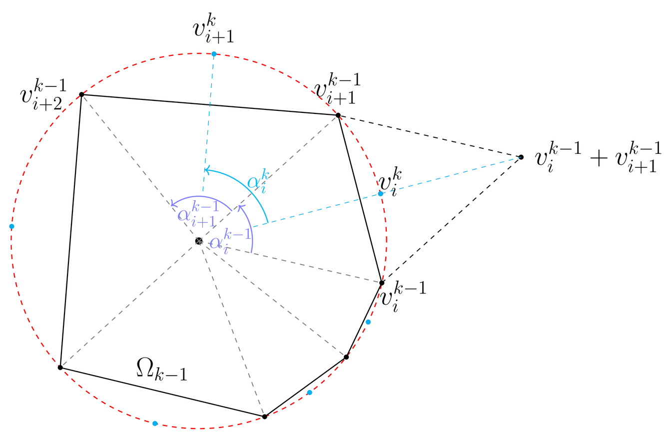


Figure 3: Iteration rules (untangle).

Given a simple polygon Ω with vertices $\boldsymbol{v} = [v_1, \ldots, v_n]$ and a point v inside the polygon, we first shift the polygon Ω so that the point v becomes the origin O. Then we project the polygon Ω onto the unit circle centred at O and get a new polygon Ω_0 with vertices $\boldsymbol{v}^0$ on the unit circle. Next, we iteratively "smooth" the polygon Ω_0 as follows. Given Ω_{k-1} with vertices $\boldsymbol{v}^{k-1}$, we update the polygon by projecting the midpoints between every two adjacent points onto the unit circle as the new points of Ω_k, see Figure 3 and Figure 2. Defining the transformation matrix $\mathring{\mathbf{T}} = \operatorname{diag}\{1/r_1, 1/r_2, \ldots, 1/r_n\}$ and

$$\mathbf{T}_k = \begin{bmatrix} 1 & & & 1 \\ 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1/r_1^k & & & \\ & 1/r_2^k & & \\ & & \ddots & \\ & & & 1/r_n^k \end{bmatrix}, \tag{3}$$

where $r_i = \|v_i\|$ and $r_i^k = \|v_i^k + v_{i+1}^k\|$.

- Projection: $\boldsymbol{v}^0 = \boldsymbol{v}\mathring{\mathbf{T}}$
- Untangle: $\boldsymbol{v}^k = \boldsymbol{v}^{k-1}\mathbf{T}_{k-1}, \; k = 1, 2, \ldots$

$$\Longrightarrow \boldsymbol{v}^k = \boldsymbol{v}\underbrace{\mathring{\mathbf{T}}\mathbf{T}_0\mathbf{T}_1 \ldots \mathbf{T}_{k-1}}_{\mathbf{T}} = \boldsymbol{v}\mathbf{T}$$

After k iterations, we find the mean value coordinates $\hat{\boldsymbol{w}}$ of the o with respect to Ω_k. That is

$$\boldsymbol{w}^k = \mathbf{T}\hat{\boldsymbol{w}},$$

$$\lambda_i^k = w_i^k \Big/ \sum_j w_j^k.$$

Experiments and results

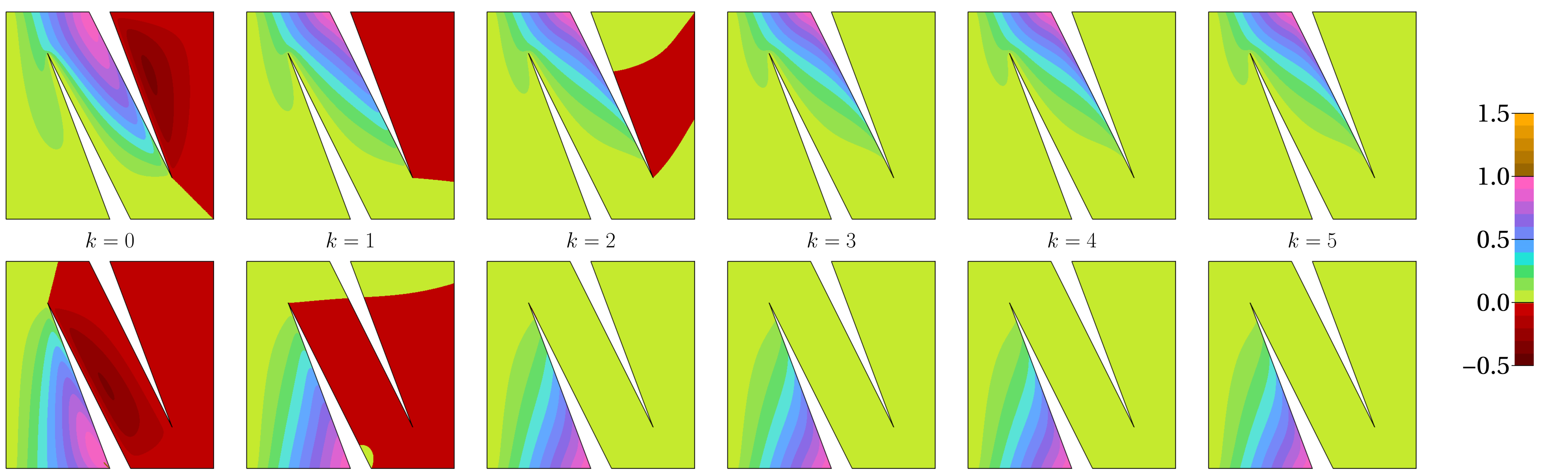


Figure 4: Comparison of iterative coordinates of two vertices (row-wise) for different values of k for a concave polygon.

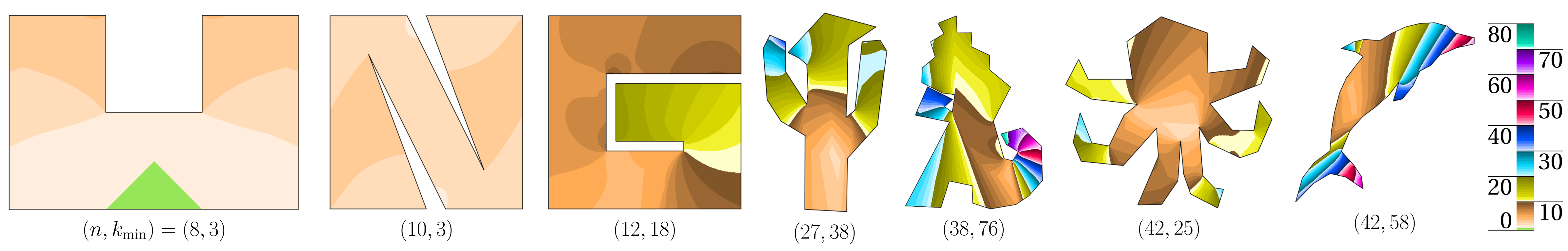


Figure 5: Examples of concave polygons with different numbers of n vertices, for which the iterative coordinates are non-negative after the stated number of $k_{\min}$ iterations (determined by regularly sampling the polygon's bounding square and computing the coordinates for the interior sample points in double precision at a resolution of $10^3 \times 10^3$ for the whole square and an effective resolution of $10^6 \times 10^6$ in the regions where the largest number of iterations is needed). The colour-coding refers to the number of iterations required to get positive coordinates at the individual interior points.

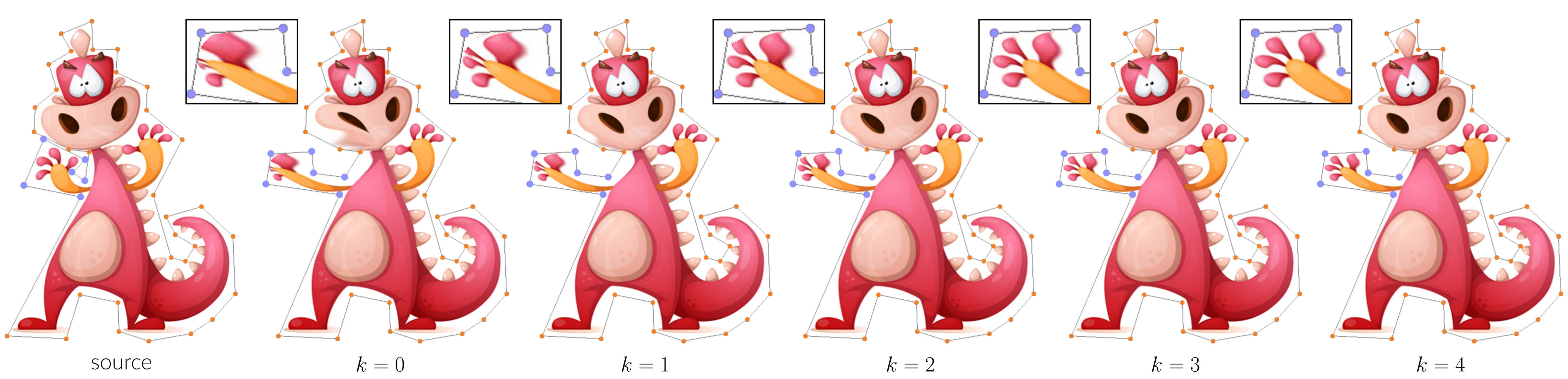


Figure 6: Deformation of a source image (left), obtained by moving six vertices (blue) of the control polygon. The deformation based on mean value coordinates ($k = 0$) exhibits severe artefacts, caused by negative coordinate values. Using iterative coordinates, these deformation artefacts gradually disappear as the number of iterations increases ($k = 1, 2, 3, 4$).

Limitations

- Iterative coordinates do not have a simple closed form.
- We have not yet determined the minimum number of iterations K required to guarantee non-negativity.

Acknowledgements

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- SNSF: 200021-188577

References

[1] Michael S. Floater.
Mean value coordinates.
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