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Stability of Dynamical Systems

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MONOGRAPH SERIES ON NONLINEAR SCIENCE AND COMPLEXITY

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Preface

The main purpose of developing stability theory is to examine the dynamic responses of a system to disturbances as time approaches infinity. It has been and still is the subject of intense investigations due to its intrinsic interest and its relevance to all practical systems in engineering, finance, natural science and social science. Lyapunov stability theory, one celebrated theory, is the foundation of stability analyses for dynamic systems that are mathematically described by ordinary differential equations (ODE). Inspired by numerous applications and new emerging fields, it has been significantly developed and extended to systems that are modeled using difference equations (DE), differential-difference equations (DDE), functional differential equations (FDE), integral-differential equations (IDE), partial differential equations (PDE), and stochastic differential equations (SDE). For instance, interest in automatic control starting in the 1950s has generated the theories of global and absolute stability. Theory describing the co-existence and sustainability of ecological systems originated from interest in bio-systems analysis beginning in the 1970s. Interest in artificial neural networks beginning in the 1980s has stimulated the solution of nonlinear equations by using electronic circuit analogues.

The evolution of stability theory has been very rapid and extensive. Major developments are scattered throughout an array of scientific journals, making it often difficult to discover what the real advances are, especially for a researcher new to the field or a practitioner using the results in various applicable areas. Therefore, it appears necessary to have monographs on topics of current interest to both researchers and practitioners in the field. The present monograph is intended to provide some state-of-the-art expositions of major advances in fundamental stability theories and methods for dynamical systems of ODE and DDE types and in limit cycle, normal form and Hopf bifurcation control of nonlinear dynamic systems.

The present monograph comes mainly from our research results and teaching of graduate students in the stability of dynamical systems. Chapter 1 is the introduction where we define various stabilities mathematically, illustrate their relations using examples and discuss the main mathematical tools for stability analyses (e.g., Lyapunov functions, K-class functions, Dini derivatives, differential and integral inequalities and matrices). In Chapter 2, we re-visit the stability of linear systems with constant coefficients, present a new method for solving the Lyapunov matrix equation and discuss our geometrical method for stability analyses. Chapter 3 describes the stability of linear systems with variable coefficients.

Particularly, we first develop relations between the stabilities of homogeneous and nonhomogeneous systems, and relations between Cauchy matrix properties and various stabilities. We then discuss the robust stability, analytical expressions of Cauchy matrix solutions for some linear systems and the Floquet–Lyapunov theory for linear systems with periodic coefficients. Finally, we present the truncated Cauchy matrix and partial variable stability. In Chapter 4, we present the Lyapunov stability theory by using a modern approach that employs the K-class function and Dini derivative. The necessary and sufficient conditions are systematically developed for stability, uniform stability, uniformly asymptotic stability, exponential stability, and instability. We present classical Lyapunov theorems of stability and their inverse theorems together to illustrate the universality of the Lyapunov direct method. Also developed in Chapter 4 are some new sufficient conditions for stability, asymptotic stability, and instability. This chapter ends with a brief summary constructing Lyapunov functions. Chapter 5 presents a major extension and development of the Lyapunov direct method, including the LaSalle invariant principle, theory of comparability, robust stability, practical stability, Lipschitz stability, asymptotic equivalence, conditional stability, partial variable stability, stability and the boundedness of sets. In Chapter 5, we also apply the Lyapunov function to study the classical Lagrange stability, Lagrange asymptotic stability and Lagrange exponential stability.

Chapter 6 is devoted to the stability of nonlinear systems with separable variables. The topics covered include linear and nonlinear Lyapunov functions, the global stability of autonomous and nonautonomous systems, transformation into systems with separable variables, and partial variable stability. This chapter provides the methods and tools for examining the absolute stability of nonlinear control systems in Chapter 9 and the stability of neural networks in Chapter 10. Chapter 7 describes the iteration method that uses the convergence of iteration for stability analyses and avoids the difficulty encountered in constructing Lyapunov functions. Particularly, we discuss iteration methods of Picard and Gauss–Seidel types and their applications in examining the extreme stability and the stationary oscillation, in improving the freezing coefficient method, and in investigating the robust stability of interval systems. Dynamical systems with temporal delay are often modeled by differential-difference equations (DDE). The stability of such systems is discussed in Chapter 8. We first present the Lyapunov functional method and the Lyapunov function method with the Razumikhin technique for the stability analyses of time-delaying nonlinear differential equations and DDE with separable variables. We then apply the eigenvalue method and M matrix theory to develop an algebraic method for modeling the stability of linear systems with constant coefficients and constant time-delays. Finally, we use the iteration method in Chapter 7 to examine the stability of neutral DDE systems with temporal delays. Chapter 9 covers the absolute stability of Lurie control systems. The topics contain some algebraic sufficient conditions for stability, and the necessary and sufficient conditions for the absolute stability of direct, indirect, critical and time-delaying Lurie control systems and for the absolute stability of Lurie control systems with multiple nonlinear controls or with feedback loops. Also discussed in this chapter is the application of Lurie theory in chaos synchronization. Chapter 10 focuses on stabilities (Lyapunov, globally asymptotic, globally exponential) and exponential periodicity of various neural networks (Hopfield with and without time-delay, Rosko bidirectional associative memory, cellular, generalized). In Chapter 11, we present the computational methods of normal form and limit cycle, the control of Hopf bifurcations and their engineering applications.

We acknowledge with gratitude the support received from the Huazhong University of Science and Technology (Department of Control Science and Engineering), the University of Hong Kong (Department of Mechanical Engineering) and the University of Western Ontario (Department of Applied Mathematics). The support of our research program by the Natural Science Foundation of China (NSFC 60274007 and 60474011), the Research Grant Council of the Hong Kong Special Administration Region of China (RGC HKU7086/00E and HKU7049/06P) and the Natural Sciences and Engineering Research Council of Canada (NSERC R2686A02) is also greatly appreciated. We are very grateful to Dr. Zhen Chen and Mr. Fei Xu who typed part of the manuscript. And, of course, we owe special thanks to our respective families for their tolerance of the obsession and the late nights that seemed necessary to bring this monograph to completion. Looking ahead, we will appreciate it very much if users will write to call our attention to the imperfections that may have slipped into the final version.

Xiaoxin Liao, Liqiu Wang, Pei Yu
London, Canada
December 2006

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Contents

Chapter 1

Fundamental Concepts and Mathematical Tools

In this chapter, we first present some definitions, examples, equivalent relations and geometric explanation of Lyapunov function and wedge function (K-class function). Lyapunov function method is a classical but still powerful tool for stability analysis, while wedge function approach is a modern tool in simplifying and unifying the proofs of many stability and boundedness theorems. Then, we introduce the Dini derivative of Lyapunov function. The Hurwitz condition, the Sylvester condition and the M matrix condition in unified and simplified forms are also discussed. Finally, we will discuss mathematical definitions and geometrical explanations of various stability and attraction concepts. Several examples are given to show the concepts and their relations.

Good mathematical tools are always useful and necessary in solving problems. It would be helpful to know the basic concepts presented in this chapter in order to understand well the materials given in the following chapters.

The contents presented in this chapter are based on the following sources: [153,163,419] for Section 1.1, [98,151,234,298,300] for Section 1.2, [234,151] for Section 1.3, [234,418] for Section 1.4, [163,419] for Section 1.5, [284] for Section 1.6, [98,151,233,298,300] for Section 1.7, and [234] for Section 1.8.

1.1. Fundamental theorems of ordinary differential equations

As a preliminary, first of all, we list some principal theorems of ordinary differential equations without proofs. In fact, stability theory is based on these theorems or, exactly, is established by extending these theorems. For example, stability in Lyapunov sense is just an extension of the concept that a solution is continuously dependent on its initial value by extending the finite interval to an infinite one.

Consider the following equations:

$$\frac{dx_i}{dt} = g_i(t, x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n, \tag{1.1.1}$$

where $t \in I := (t_1, t_2)$, $t_1 \geqslant -\infty$, $t_2 \leqslant +\infty$, the state vector $x = (x_1, x_2, \ldots, x_n)^T \in \Omega \subset R^n$, $g_i \in C[I \times \Omega, R^1]$, i.e., $I \times \Omega$ is a domain defined in $I \times R^n$ and R^1 is the image of the continuous function g_i $(i = 1, 2, \ldots, n)$, Ω is n-dimensional subset of R^n, $0 \in \Omega$. System (1.1.1) can be rewritten in the vector form:

$$\frac{dx}{dt} = g(t, x), \quad g = (g_1, \ldots, g_n)^T. \tag{1.1.2}$$

Assume that g_i satisfies the Lipschitz condition, i.e., $\forall x, y \in \Omega$, $\forall t \in I$, there exists a constant $L > 0$ such that

$$\big|g_i(t, x) - g_i(t, y)\big| \leqslant L \sum_{j=1}^{n} |x_j - y_j|.$$

Obviously, if $|\frac{\partial g_i(t, x_1, \ldots, x_n)}{\partial x_j}| \leqslant k_{ij} = \text{constant}$, $i, j = 1, 2, \ldots, n$, on $I \times \Omega$, then the Lipschitz condition is satisfied.

THEOREM 1.1.1 *(Existence and uniqueness theorem). If* $g(t, x) = (g_1(t, x), \ldots, g_n(t, x))^T$ *satisfies the Lipschitz condition, then* $\forall (t_0, x_0) \in I \times \Omega$, $\exists t^* > 0$, *there exists a unique solution* $x(t, t_0, x_0)$ *which satisfies the differential equation* (1.1.2) *with the initial conditions*:

$$x(t_0, t_0, x_0) = x_0, \tag{1.1.3}$$

$$\frac{dx(t, t_0, x_0)}{dt} = g\big(t, x(t, t_0, x_0)\big), \tag{1.1.4}$$

on the interval $[t_0 - t^*, t_0 + t^*]$.

THEOREM 1.1.2 *(Continuity and differentiability theorems for initial value problem). Suppose that the conditions of Theorem* 1.1.1 *are satisfied, and that two solutions of* (1.1.2) $x^{(1)}(t) := x(t, t_0, x_0^{(1)})$, $x^{(2)}(t) := x(t, t_0, x_0^{(2)})$ *defined on* $[t_0, t_1] \times \Omega$. *Then* $\forall \varepsilon > 0, \exists \delta > 0$, $\|x_0^{(1)} - x_0^{(2)}\| < \delta$ *implies* $\|x^{(1)}(t, t_0, x_0) - x^{(2)}(t, t_0, x_0)\| < \varepsilon$. *The continuity of* $\frac{\partial g_i}{\partial x_j}$ $(i, j = 1, 2, \ldots, n)$ *implies the continuity of* $\frac{\partial x_i(t, t_0, x_0)}{\partial x_{0j}}$ $(i, j = 1, 2, \ldots, n)$.

In the following, we consider the differential equations with parameters:

$$\frac{dx}{dt} = g(t, x, \mu),$$

where $x \in \Omega$, $t \in I$ and $\mu \in [\mu_1, \mu_2]$ is a parameter vector.

THEOREM 1.1.3 *(Continuity and differentiability of solution with respect to parameter). Suppose that* $g(t, x, \mu) \in C[I \times \Omega \times [\mu_1, \mu_2], R^n]$, *g satisfies the Lipschitz condition for any values of parameter* $\mu \in [\mu_1, \mu_2]$. *Then,*

(1) $\forall t_0 \in I, x_0 \in \Omega, \mu_0 \in [\mu_1, \mu_2]$ *there exist constants* $\rho > 0, a > 0$ *such that when* $|\mu - \mu_0| \leqslant \rho$, *the solution of* (1.1.2) $x(t) := x(t, t_0, x_0, \mu)$, *defined on* $[t_0 - a, t_0 + a]$ *continuously depends on* μ.
(2) g_i *being analytic with respect to all variables implies that* $x(t) := x(t, t_0, x_0, \mu)$ *is also analytic with respect to* μ.
(3) *the continuous differentiability of* g_i *with respect to* $x_1, x_2, \ldots, x_n$ *and* μ *implies the continuous differentiability of* $x(t) := x(t, t_0, x_0, \mu)$ *with respect to* μ.

EXAMPLE 1.1.4. Consider a second-order linear system:

$$\frac{d^2x}{dx^2} + \lambda \frac{dx}{dt} + x = 0. \tag{1.1.5}$$

When $\lambda = 0$, equation (1.1.5) has a family of periodic solutions:

$$\begin{cases} x(t) = A \sin(t + \alpha), \\ \dot{x}(t) = A \cos(t + \alpha), \end{cases} \tag{1.1.6}$$

where A and α are constants. Eliminating t in (1.1.6) yields the equation of orbit, $\dot{x}^2 + x^2 = A^2$, which represents a family of circles when A is varied. When $0 < \lambda \ll 1$, according to Theorem 1.1.2, the solution orbit of system (1.1.6) approximates the solution of (1.1.5), as shown in Figure 1.1.1.

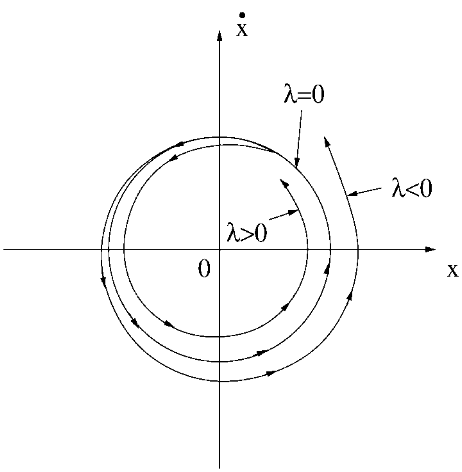


Figure 1.1.1. Illustration of continuity on parameter.

1.2. Lyapunov function

Suppose that the function $W(x) \in C[\Omega, R^1]$, i.e., $W : \Omega \to R^1$ is continuous, $W(0) = 0$; $V(t,x) \in C[I \times \Omega, R^1]$, i.e., $V(t,x) : I \times \Omega \to R^1$ is continuous and $V(t,0) \equiv 0$.

DEFINITION 1.2.1. The function $W(x)$ is said to be positive definite if

$$W(x) \begin{cases} > 0 & \text{for } x \in \Omega,\ x \neq 0, \\ = 0 & \text{for } x = 0. \end{cases}$$

$W(x)$ is said to be positive semi-definite if $W(x) \geqslant 0$ for $x \in \Omega$. The function $W(x)$ is said to be negative definite if $W(x)$ is positive definite. $W(x)$ is said to be negative semi-definite, if $W(x) \leqslant 0$. The positive definite and negative definite functions are called definite sign functions. The positive or negative semi-definite functions are called constant sign functions.

DEFINITION 1.2.2. The function $V(t,x) \in C[I \times \Omega, R^1]$ (or $W(x) \in C[\Omega, R^1]$) is said varying if there exist $t_1, t_2 \in I$ and $x_1, x_2 \in \Omega$ such that $V(t_1, x_1) > 0$, $V(t_2, x_2) < 0$ ($W(x_1) > 0$, $W(x_2) < 0$).

EXAMPLE 1.2.3. $W(x_1, x_2) = 3x_1^2 + 2x_2^2 + 2x_1x_2$ is positive definite.

EXAMPLE 1.2.4. $W(x_1, x_2) = x_1^2 + x_2^2 + 2x_1x_2 = (x_1 + x_2)^2$ is positive semi-definite.

EXAMPLE 1.2.5. $W(x_1, x_2) = x_1^2 + x_2^2 - 3x_1x_2$ is a variable sign function.

EXAMPLE 1.2.6. $V(t, x_1, x_2) = x_1^2 \sin t + x_2^2 \cos t$ is a variable sign function.

DEFINITION 1.2.7. The function $V(t,x)$ is said to be positive definite, if there exists a positive definite function $W(x)$ such that $V(t,x) \geqslant W(x)$ and $V(t,0) \equiv 0$. The function $V(t,x)$ is said to be negative definite, if $-V(t,x)$ is positive definite. The function $V(t,x) \in C[I \times \Omega, R^1]$ is said to be positive semi-definite if $V(t,x) \geqslant 0$. $V(t,x)$ is negative semi-definite if $V(t,x) \leqslant 0$.

The meaning of Definition 1.2.7 is depicted in Figure 1.2.1.

EXAMPLE 1.2.8. $V(t, x_1, x_2) = (2 + e^{-t})(x_1^2 + x_2^2 + x_1x_2)$ is positive definite, because

$$V(t, x_1, x_2) = \bigl(2 + e^{-t}\bigr)\bigl(x_1^2 + x_2^2 + x_1x_2\bigr) \geqslant x_1^2 + x_2^2 + x_1x_2 := W(x_1, x_2),$$

where $W(x_1, x_2)$ is positive definite, and $V(t,0) = 0$.

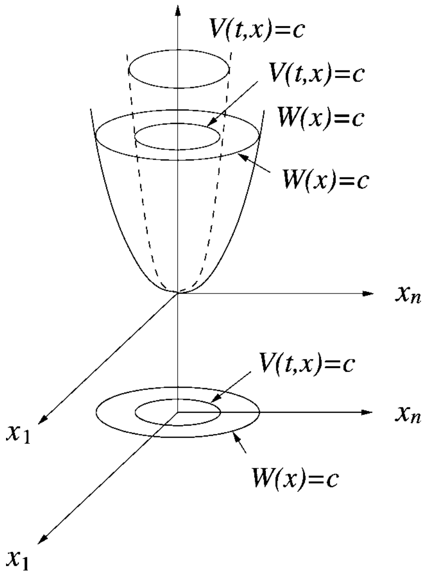


Figure 1.2.1. Geometric demonstration of time-varying positive definite function.

EXAMPLE 1.2.9. $V(t, x_1, x_2) = (e^{-t})(x_1^2 + \frac{3}{5}x_1x_2 + x_2^2)$ is positive semi-definite, since there does not exist a positive definite function $W(x)$ such that $V(t, x_1, x_2) \geqslant W(x)$.

DEFINITION 1.2.10. The function $W(x) \in C[R^n, R^1]$ is said to be positive definite and radially unbounded if $W(x)$ is positive definite and $x \to \infty$ implies $W(x) \to +\infty$.

DEFINITION 1.2.11. The function $V(t, x) \in C[I \times R^n, R^1]$ is said to be positive definite and radially unbounded if there exists a positive definite and radially unbounded function $W_2(x)$ such that $V(t, x) \geqslant W_2(x)$. The function $V(t, x)$ is said to have infinitesimal upper bound if there exists a positive definite function $W_1(x)$ such that $|V(t, x)| \leqslant W_1(x)$.

EXAMPLE 1.2.12.

$$\begin{aligned} W(x_1, x_2) &= a^2x_1^2 + b^2x_2^2 + abx_1x_2\cos(x_1 + x_2) \\ &\geqslant \frac{1}{2}a^2x_1^2 + \frac{1}{2}b^2x_2^2 + \frac{1}{2}a^2x_1^2 + \frac{1}{2}b^2x_2^2 - |ab||x_1|x_2| \\ &= \frac{1}{2}a^2x_1^2 + \frac{1}{2}b^2x_2^2 + \frac{1}{2}\big(|ax_1| - |bx_2|\big)^2 \end{aligned}$$

$$\geqslant \frac{1}{2}a^2x_1^2 + \frac{1}{2}b^2x_2^2 \to +\infty \quad \text{as } x_1^2 + x_2^2 \to +\infty,$$
$$W(0,0) = 0,$$

so $W(x_1, x_2)$ is positive definite and radially unbounded function.

EXAMPLE 1.2.13. $V(t, x_1, x_2) = \frac{2t}{1+t^2}x_1^2 + x_2^2 \sin t \leqslant |x_1^2| + |x_2|^2 = W(x)$, thus $V(t, x_1, x_2)$ is a function with infinitesimal upper bound.

The geometric interpretation for positive definite function $V(t, x) \geqslant W(x)$ is shown in Figure 1.2.1. $V(t, x)$ is a family of hypersurfaces in R^{n+1}, with parameter t, which locates over the fixed hypersurface $W(x)$, i.e., the family of level curves $V(t, x) = c$ changed with respect to t is included in $W(x) = c$.

The geometric interpretation of positive definite function with infinitesimal upper bound

$$W_1(x) \leqslant V(t, x) \leqslant W_2(x)$$

is illustrated in Figure 1.2.2.

Assume $W(x)$ is positive definite on $\|x\| \leqslant H$. The structure of $W(x) = c$ may be very complex, and may be not closed.

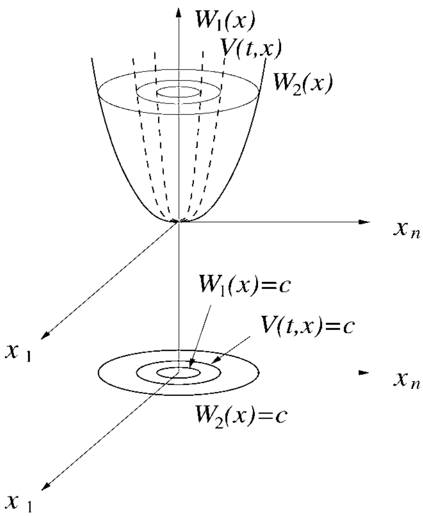


Figure 1.2.2. Geometric demonstration of time-varying positive definite function with infinitesimal upper bound.

EXAMPLE 1.2.14. Consider

$$W(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2}.$$

When $0 < c < 1$, $W(x_1, x_2) = c$ is a close curve; but when $c \geqslant 1$, $W(x_1, x_2) = c$ is not closed. In fact, when $c \geqslant 1$,

$W(x_1, 0) = \frac{x_1^2}{1+x_1^2} = c$ has no finite solution for x_1;

$W(0, x_2) = \frac{x_2^2}{1+x_2^2} = c$ has no finite solution for x_2.

So in the direction x_1 $(x_2 = 0)$ or x_1 $(x_2 = 0)$, $W(x_1, x_2) = c$ is not closed. However, when $0 < c < 1$, let $x_2 = kx_1$, where $k \neq 0$ is any real number, the equation

$$\frac{kx_1^2}{1+k^2x_1^2} + \frac{x_1^2}{1+x_1^2} = c$$

has finite solution x_1, therefore the curves $W(x_1x_2) = c$ and the straight lines $x_1 = kx_1$ have finite intersection points. Similarly, $W(x_1, x_2) = c$ and $x_1 = kx_2 (k \neq 0)$ have finite intersection points. Thus, $W(x_1x_2) = c$ $(0 < c < 1)$ is a closed curve (see Figure 1.2.3).

1.3. *K*-class function

In this section, we introduce K-class functions and discuss the relation between K-class function and positive definite function.

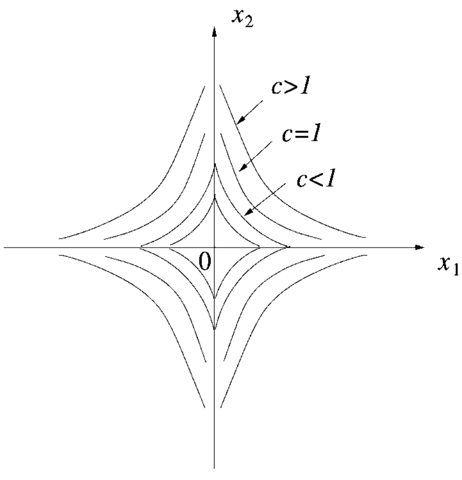


Figure 1.2.3. $V = c$ is a closed curve in the neighborhood of the origin.

DEFINITION 1.3.1. If a function $\varphi \in [R^+, R^+]$, where $R^+ := [0, +\infty)$, or $\varphi \in C[[0, h], R^+]$ is monotonically strictly increasing, and $\varphi(0) = 0$, we call φ a Wedge function, or simply call it a K-class function, denoted by $\varphi \in K$.

DEFINITION 1.3.2. If $\varphi \in [R^+, R^+]$ is a K-class function and $\lim_{r\to+\infty} \varphi(r) = +\infty$, then $\varphi(r)$ is called a radially unbounded K-class function, denoted by $\varphi \in KR$.

Among the positive definite functions and the K-class functions, some essential equivalent relations exist.

THEOREM 1.3.3. *Let $\Omega := \{x, \ \|x\| \leqslant h\}$. For a given arbitrarily positive definite function $W(x) \in [\Omega, R^1]$, there exist two functions $\varphi_1, \varphi_2 \in K$ such that*

$$\varphi_1\big(\|x\|\big) \leqslant W(x) \leqslant \varphi_2\big(\|x\|\big). \tag{1.3.1}$$

PROOF. For any $h > 0$, we prove that (1.3.1) holds for $\|x\| \leqslant h$. Let

$$\varphi(r) = \inf_{r\leqslant\|x\|\leqslant h} W(x).$$

Obviously, we have $\varphi(0) = 0$, $\varphi(r) > 0$ for $r > 0$ and $\varphi(r)$ is a monotone nondecreasing function on $[0, h]$. Now we prove that $\varphi(r)$ is continuous. Since $W(x)$ is continuous, $\forall\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}\varphi(r_2) - \varphi(r_1) &:= \inf_{r_2\leqslant\|x\|\leqslant h} W(x) - \inf_{r_1\leqslant\|x\|\leqslant k} W(x)\\ &= \inf_{r_2\leqslant\|x\|\leqslant h} W(x) - W(x_0)\\ &\leqslant W(x_1) - W(x_0)\\ &\leqslant \varepsilon \quad \text{when } \|x_1 - x_0\| \leqslant r_2 - r_1 \leqslant \delta(\varepsilon),\end{aligned}$$

where we take $x_1 = x_0$ when $x_0 \in D_2 := \{x \mid r_2 \leqslant \|x\| \leqslant h\}$.

When $x_0 \in D_1 := \{x|r_1 \leqslant \|x\| \leqslant h\}$, take the intersection point of the line $\overline{Ox_0}$ and $\|x\| = r_2$, as shown in Figure 1.3.1. Let $\varphi_1(r) := \frac{r\varphi(r)}{R} \leqslant \varphi(r)$. Evidently, we have $\varphi_1(0) = 0$, and if $0 \leqslant r_1 < r_2 \leqslant h$, we get

$$\varphi_1(r_1) = \frac{r_1\varphi(r_1)}{h} \leqslant \frac{r_1\varphi(r_2)}{h} < \frac{r_2\varphi(r_2)}{h} = \varphi_1(r_2).$$

Thus, $\varphi_1(r)$ is strictly monotone increasing and hence $\varphi_1 \in K$. Let

$$\varphi(r) := \max_{\|x\|\leqslant r} W(x).$$

Then it follows that $\varphi(0) = 0$. By the same method, we can prove that $\varphi(r)$ is monotone nondecreasing and continuous. Choosing $\varphi_2(r) := \varphi(r) + kr$ $(k > 0)$,

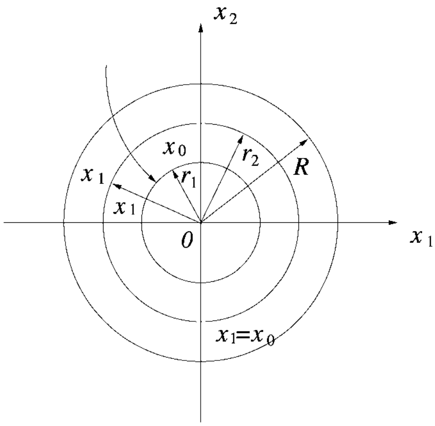


Figure 1.3.1. The relation between positive definite function and K-class function.

we have

$$\varphi_2(r_1) = \psi(r_1) + kr_1 \leqslant \psi(r_2) + kr_1 < \psi(r_2) + kr_2 = \varphi_2(r_2).$$

Hence, $\varphi_2(r)$ is strictly monotone increasing and $\varphi_2(r) \in K$. The above results show that

$$\begin{aligned}\varphi_1\big(\|x\|\big) \leqslant \varphi\big(\|x\|\big) &:= \inf_{\|x\| \leqslant \|\xi\| \leqslant h} W(\xi) \leqslant W(x) \\ &\leqslant \max_{\|\xi\| \leqslant \|x\|} W(\xi) := \psi\big(\|x\|\big) \\ &\leqslant \varphi_2\big(\|x\|\big).\end{aligned}$$

Thus,

$$\varphi\big(\|x\|\big) \leqslant W(x) \leqslant \varphi\big(\|x\|\big).$$

The proof is complete. □

By a similar procedure, we can prove the following theorem.

THEOREM 1.3.4. *For an arbitrarily given positive definite and radially unbounded function $W(x) \in C[R^n, R^1]$, there must exist two functions $\varphi_1(r)$, $\varphi_2(r)$ $\in KR$ such that*

$$\varphi_1\big(\|x\|\big) \leqslant W(x) \leqslant \varphi_2\big(\|x\|\big).$$

Consequently, without loss of generality, we usually replace the positive definite functions (or the radially unbounded and positive definite functions) by the K-class functions (or radially unbounded K-class functions).

1.4. Dini derivative

Let $I := [t_0, +\infty]$, $f(t) \in C[I, R^1]$. For any $t \in I$ the following four derivatives:

$$
\begin{aligned}
D^+ f(t) &:= \overline{\lim_{h\to 0^+}} \frac{1}{h}\big(f(t+h) - f(t)\big) \\
&= \lim_{h\to 0^+} \sup \frac{1}{h}\big(f(t+h) - f(t)\big), \qquad (1.4.1)\\
D_+ f(t) &:= \underline{\lim_{h\to 0^+}} \frac{1}{h}\big(f(t+h) - f(t)\big) \\
&= \lim_{h\to 0^+} \inf \frac{1}{h}\big(f(t+h) - f(t)\big), \qquad (1.4.2)\\
D^- f(t) &:= \overline{\lim_{h\to 0^-}} \frac{1}{h}\big(f(t+h) - f(t)\big) \\
&= \lim_{h\to 0^-} \sup \frac{1}{h}\big(f(t+h) - f(t)\big), \qquad (1.4.3)\\
D_- f(t) &:= \underline{\lim_{h\to 0^-}} \frac{1}{h}\big(f(t+h) - f(t)\big) \\
&= \lim_{h\to 0^-} \inf \frac{1}{h}\big(f(t+h) - f(t)\big), \qquad (1.4.4)
\end{aligned}
$$

are respectively called right-upper, right-lower, left-upper and left-lower derivatives of $f(t)$. They are all called Dini derivatives.

In some cases, the Dini derivative may become $\pm\infty$, otherwise, there always exists finite Dini derivative. In particular, when $f(t)$ satisfies the local Lipschitz condition, the four Dini derivatives are finite. Moreover, the standard derivative of $f(t)$ exists if and only if the four Dini derivatives are equal.

For a continuous function, the relation between the monotonicity and the definite sign of the Dini derivative is as follows.

THEOREM 1.4.1. *If $f(t) \in C[I, R^1]$, the necessary and sufficient condition for $f(t)$ being monotone nondecreasing on I is $D^+ f(t) \geqslant 0$ for $t \in I$.*

PROOF. The *necessity* is obvious because $t_2 > t_1$ implies $f(t_2) \geqslant f(t_1)$.

Now we prove the *sufficiency*. First, suppose $D^+ f(t) > 0$ on I. If there are two points $\beta \in I$ and $\alpha < \beta$ such that $f(\alpha) > f(\beta)$, then there exists μ satisfying $f(\alpha) > \mu > f(\beta)$ and some point $t \in [\alpha, \beta]$ such that $f(t) > \mu$. Let ξ be the supremum of these points. Then $\xi \in [\alpha, \beta]$ and the continuity of $f(t)$ leads to $f(\xi) = \mu$. Therefore, for $t \in [\xi, \beta]$, it follows that

$$\frac{f(t) - f(\xi)}{t - \xi} < 0. \qquad (1.4.5)$$

Hence, we have $D^+ f(\xi) \leqslant 0$, which contradicts the hypotheses. Thus $f(t)$ is monotone nondecreasing.

Next assume that $D^+ f(t) \geqslant 0$. Then for any $\xi > 0$, we get

$$D^+(f(t) + \xi t) = D^+ f(t) + \xi \geqslant \xi > 0. \tag{1.4.6}$$

As a consequence, $f(t) + \xi t$ is monotone nondecreasing since ξ is arbitrary. So $f(t)$ is monotone nondecreasing on I.

The theorem is proved. □

REMARK 1.4.2. If we replace $D^+ f(t) \geqslant 0$ by $D_+ f(t) \geqslant 0$, then the sufficient condition of Theorem 1.4.1 still holds because the latter implies the former. Similarly, if we replace $D^+ f(t) \geqslant 0$ by $D^- f(t) \geqslant 0$, then it is sufficient to change the supremum of the points satisfying $f(t) \geqslant \mu$ to the impious of the points satisfying $f(t) < \mu$. We may further intensify $D^- f(t) \geqslant 0$ to be $D_- f(t) \geqslant 0$, and thus any of the four derivatives is not less than zero, each of which implies that $f(t)$ is monotone nondecreasing.

In the following, we consider the Dini derivative of a function along the solution of a differential equation.

Consider a system of differential equations, given by

$$\frac{dx}{dt} = f(t, x), \tag{1.4.7}$$

where $f(t, x) \in C[I \times R^n, R^n]$.

THEOREM 1.4.3. *(See [418].) Suppose that $V(t, x) \in C[I \times \Omega, R^1]$, where $\Omega \subset R^n$, Ω is a neighborhood containing the origin and $V(t, x)$ satisfies the local Lipschitz condition on x with respect to t, i.e.,*

$$\bigl|V(t, x) - V(t, y)\bigr| \leqslant L\|x - y\|.$$

Then the right-upper derivative and the right-lower derivative of $V(t, x)$ along the solution $x(t)$ of (1.4.7) have the following forms:

$$D^+ V\bigl(t, x(t)\bigr)\big|_{(1.4.7)} = \overline{\lim_{t \to +\infty}} \frac{1}{h}\bigl\{V(t + h), x + hf(t, x) - V(t, x)\bigr\}, \tag{1.4.8}$$

$$D_+ V\bigl(t, x(t)\bigr)\big|_{(1.4.7)} = \underline{\lim_{t \to +\infty}} \frac{1}{h}\bigl\{V(t + h, x + hf(t, x)) - V(t, x)\bigr\}. \tag{1.4.9}$$

PROOF. Assume that the solution $x(t)$ stays in the region $I \times \Omega$. For $(t, x) \in I \times \Omega$, $(t + h, x + hf(t, x)) \in U$, $(t + h, x(t + h)) \in U$. Let L be the Lipschitz constant of $V(t, x)$ in $I \times \Omega$. Making use of the Taylor expansion and the Lipschitz condition, we obtain

$$\begin{aligned} &V\big(t+h, x(t+h)\big) - V\big(t, x(t)\big) \\ &\quad = V\big(t+h, x+hf(t,x)+h\varepsilon\big) - V(t,x) \\ &\quad < V\big(t+h, x+hf(t,x)\big) + Lh|\varepsilon| - V(t,x), \end{aligned} \tag{1.4.10}$$

where $\varepsilon \to 0$ as $h \to +0$. Hence,

$$\begin{aligned} D^+V\big(t, x(t)\big)\big|_{(1.4.7)} &:= \overline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x(t+h)\big) - V\big(t, x(t)\big)\big] \\ &\leqslant \overline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x+hf(t,x)\big) + Lh|\varepsilon| - V(t,x)\big] \\ &= \overline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x+hf(t,x)\big) - V(t,x)\big]. \end{aligned} \tag{1.4.11}$$

On the other hand,

$$\begin{aligned} &V\big(t+h, x(t+h)\big) - V\big(t, x(t)\big) \\ &\quad = V\big(t+h, x+hf(t,x)+h\varepsilon\big) - V(t,x) \\ &\quad \geqslant V\big(t+h, x+hf(t,x) - Lh|\varepsilon|\big) - V(t,x). \end{aligned} \tag{1.4.12}$$

Thus,

$$\begin{aligned} D^+V\big(t, x(t)\big)\big|_{(1.4.7)} &:= \overline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x(t+h)\big) - V\big(t, x(t)\big)\big] \\ &\geqslant \overline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x+hf(t,x)\big) - V(t,x)\big]. \end{aligned} \tag{1.4.13}$$

Combining (1.4.11) with (1.4.13), we have

$$D^+V\big(t, x(t)\big)\big|_{(1.4.7)} = \overline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x+hf(t,x)\big) - V(t,x)\big].$$

Thus (1.4.8) is true. The proof of (1.4.9) goes along the same line. Therefore, we have

$$D_+V\big(t, x(t)\big)\big|_{(1.4.7)} = \underline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x+hf(t,x)\big) - V(t,x)\big].$$

If $V(t, x)$ has a continuous partial derivative with respect to the first variable, then along the solution $x(t)$ of (1.4.7) we have

$$\frac{dV}{dt}\bigg|_{(1.4.7)} = D^+V\big(t, x(t)\big)\big|_{(1.4.7)} = D_+V\big(t, x(t)\big)\big|_{(1.4.7)}$$

$$= D^- V\big(t, x(t)\big)\big|_{(1.4.7)} = D_- V\big(t, x(t)\big)\big|_{(1.4.7)}$$
$$= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x) = \frac{\partial V}{\partial t} + \operatorname{grad} V \cdot f(t,x).$$

By Theorem 1.4.1, $V(t, x(t))$ is nondecreasing [nonincreasing] along the solution of (1.4.7) if and only if

$$D^+ V\big(t, x(t)\big)\big|_{(1.4.7)} \geqslant 0 \quad \big[D^+ V\big(t, x(t)\big)\big|_{(1.4.7)} \leqslant 0\big].$$

This completes the proof of Theorem 1.4.3. □

The significance of Theorem 1.4.3 lies in the fact that one does not need to know the solution while calculating the Dini derivative of $V(t,x)$ along the solution of (1.4.7).

1.5. Differential and integral inequalities

In this section we present some differential and integral inequalities, which are important in dealing with stability.

THEOREM 1.5.1. *Suppose that the function $\varphi(t)$ is continuous on $\tau \leqslant t < b$, and the right-lower Dini derivative $D_+\varphi(t)$ exists and satisfies the differential inequality*

$$D_+\varphi(t) \leqslant F\big(t, \varphi(t)\big), \quad \varphi(\tau) = \xi, \tag{1.5.1}$$

where $F(t,x) \in C[I \times \Omega, R^1]$, $(t, \varphi(t)) \in I \times \Omega$. If $x = \Phi(t)$ is the maximum right solution on $[\tau, b)$ for differential equations:

$$\begin{cases} \frac{dx}{dt} = F(t,x), \\ x(t_0) = \eta \geqslant \varphi(\tau) = \xi, \end{cases} \tag{1.5.2}$$

then $\varphi(t) \leqslant \Phi(t)$ $(\tau \leqslant t < b)$.

THEOREM 1.5.2. *Assume that the function f(t,x) is continuous on $\bar{R} = \{(t,x) | |t - \tau| \leqslant a - |x - \xi| \leqslant b\}$ and nondecreasing for x, $x = \varphi(t)$ is continuous, and when $|t - \tau| \leqslant a$, $(t, \varphi(t)) \in \bar{R}$, $\varphi(t)$ satisfies the integral inequality*:

$$\begin{cases} \varphi(t) \leqslant \xi + \int_\tau^t f\big(s, \varphi(s)\big)\, ds, \quad \tau \leqslant t \leqslant \tau + h, \\ \varphi(\tau) \leqslant \xi, \end{cases} \tag{1.5.3}$$

and $\Phi(t)$ satisfies the differential equation on the interval $\tau \leqslant t \leqslant \tau + h$:

$$\begin{cases} \frac{dx}{dt} = f(t,x), \\ x(\tau) = \xi. \end{cases} \tag{1.5.4}$$

Then we have the following inequality on $t \in [\tau, \tau + h]$:

$$\varphi(t) \leqslant \Phi(t), \tag{1.5.5}$$

where

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \max_{t,x \in \bar{R}} \left| f(t,x) \right|.$$

According to Theorem 1.5.2, one can derive the well-known Gronwall–Bellman inequality.

COROLLARY 1.5.3 *(Gronwall–Bellman inequality). Suppose that* $g(t)$ *and* $u(t)$ *are continuous, nonnegative real functions, and* c *is a nonnegative real constant. Then* $\forall t \in [t_0, t_1]$,

$$u(t) \leqslant c + \int_{t_0}^{t} g(\xi) u(\xi)\, d\xi \tag{1.5.6}$$

implies that the inequality

$$u(t) \leqslant c e^{\int_{t_0}^{t} g(\xi)\, d\xi} \tag{1.5.7}$$

is true.

PROOF. Consider the equation

$$\begin{cases} \frac{dV(t)}{dt} = g(t)V(t), \\ V(t_0) = c, \end{cases} \tag{1.5.8}$$

which has the solution

$$V(t) = c e^{\int_{t_0}^{t} g(\xi)\, d(\xi)}.$$

We have $u(t) \leqslant V(t) = c e^{\int_{t_0}^{t} g(\xi)\, d\xi}$ by Theorem 1.5.2. □

Next, we introduce two comparison theorems.

THEOREM 1.5.4 *(First comparison theorem). Let* $f(t,x)$ *and* $F(t,x)$ *be continuous scalar functions on* G*, satisfying the inequality*

$$f(t,x) < F(t,x), \quad (t,x) \in G. \tag{1.5.9}$$

Then $x = \varphi(t)$, $y = \Phi(t)$ *respectively stand for the solution of the ordinary differential equations*:

$$\begin{cases} \frac{dx}{dt} = f(t,x), \\ x(\tau) = \xi; \end{cases} \tag{1.5.10}$$

$$\begin{cases} \frac{dx}{dt} = F(t, x), \\ x(\tau) = \xi. \end{cases} \tag{1.5.11}$$

Then we have the following results:

(1) $\varphi(t) < \Phi(t)$, *when* $t > \tau$ *and* t *belongs to the common existence interval.*
(2) $\varphi(t) > \Phi(t)$, *when* $t < \tau$ *and* t *belongs to the common existence interval.*

PROOF. Let $g(t) = \Phi(t) - \varphi(t)$. Since

$$g(\tau) = \Phi(\tau) - \varphi(\tau) = \xi - \xi = 0,$$

$g'(\tau) = \Phi'(\tau) - \varphi'(\tau) = F(\tau, \xi) - f(\tau, \xi) > 0$. Therefore, when $0 < t - \tau \ll 1$, $g(t) > 0$ holds.

If in the common existence interval, there is $t > \tau$ such that

$$\varphi(t) \geqslant \Phi(t). \tag{1.5.12}$$

Let the infimum of these t be α, which satisfies (1.5.12). Thus, $\tau < \alpha$, $g(a) = 0$, $g(t) > 0$ $(\tau < t < \alpha)$. Therefore, $g'(\alpha) \leqslant 0$. Otherwise,

$$g'(\alpha) = \Phi'(\alpha) - \varphi'(\alpha) = F\big(\alpha, \Phi(\alpha) - f(\alpha), \varphi(\alpha)\big) > 0.$$

Since $g(\alpha) = 0$, we have $\Phi(\alpha) = \varphi(\alpha)$ which is a contradiction. Thus conclusion (1) holds. By using the same method we can prove that conclusion (2) is also true. $\square$

Similarly we can prove the following theorem.

THEOREM 1.5.5 *(Second comparison theorem). Suppose that* $f(t, x)$ *and* $F(t, x)$ *are continuous on* G*, and satisfy the inequality*

$$f(t, x) \leqslant F(t, x). \tag{1.5.13}$$

Let $(\tau, \xi) \in G$*, and* $x = \varphi(t)$ *and* $x = \Phi(t)$ *be respectively the solutions of the differential equations*:

$$\begin{cases} \frac{dx}{dt} = f(t, x), \\ x(\tau) = \xi \end{cases} \tag{1.5.14}$$

and

$$\begin{cases} \frac{dx}{dt} = F(t, x), \\ x(\tau) = \xi, \end{cases}$$

on $[a, b]$*. Then, the following conclusions hold*:

(1) $\varphi(t) \leqslant \Phi(t)$ *when* $\tau \leqslant t < b$;
(2) $\varphi(t) \geqslant \Phi(t)$ *when* $a < t \leqslant \tau$.

1.6. A unified simple condition for stable matrix, p.d. matrix and M matrix

The Hurwitz stability, positive definite (p.d.) or negative definite (n.d.) property of a symmetric matrix and the M matrix property are often applied in stability analysis. We now present a unified simple condition for Hurwitz stable matrix, positive definite matrix and M matrix. Then, based on the established condition, a simplified method is proposed, which is easy to be used in determining these matrices.

Let $A \in R^{n\times n}$ be a real matrix, with

$$\det(A) = \det|\lambda E - A| = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0. \tag{1.6.1}$$

If $a_i > 0$ $(i = 1, 2, \ldots, n)$, then $f(\lambda)$ is Hurwitz stable if and only if

$$\begin{aligned}
&\Delta_1 = a_1 > 0,\\
&\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0,\\
&\quad\vdots\\
&\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \vdots & & & & \\ & & & a_{n-1} & a_{n-2} \\ a_{2n-1} & a_{2n-2} & \cdots & 0 & a_n \end{vmatrix} = \Delta_{n-1}a_n > 0,
\end{aligned}$$

where $a_s = 0$ for $s < 0$ or $s > n$.

If the following quadratic form with real constant coefficients:

$$f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j := x^TAx \quad \left(A = A^T\right) \tag{1.6.2}$$

is positive definite, negative definite, positive semi-definite, and negative semi-definite, we simply call A is positive definite, negative definite, positive semi-definite, and negative semi-definite, respectively.

It is well known by Sylvester conditions that when $A = A^T$, A is positive definite if and only if

$$\Delta_1 = a_{11} > 0, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \ldots,$$

$$\Delta_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0.$$

A being negative definite is equivalent to

$$\tilde{\Delta}_1 = a_{11} < 0, \quad \tilde{\Delta}_2 = (-1)^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \ldots,$$

$$\tilde{\Delta}_n = (-1)^n \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n-1} & \cdots & \cdots & a_{nn} \end{vmatrix} > 0.$$

The sufficient and necessary conditions for $A = A^T$ to be positive semi-definite are

$$\Delta_1 = a_{11} \geqslant 0, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geqslant 0, \quad \ldots,$$

$$\Delta_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \geqslant 0.$$

A is negative semi-definite if and only if

$$\tilde{\Delta}_1 = (-a_{11}) \geqslant 0, \quad \tilde{\Delta}_2 = (-1)^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geqslant 0, \quad \ldots,$$

$$\tilde{\Delta}_n = (-1)^n \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \geqslant 0.$$

DEFINITION 1.6.1. The matrix $A(a_{ij})_{n\times n}$ is called a nonsingular M matrix (simply called M matrix), if

(1)

$$a_{ii} > 0, \quad i = 1, 2, \ldots, n, \quad a_{ij} \leqslant 0, \quad i \neq j,\ i, j = 1, 2, \ldots, n;$$

(2)

$$\Delta_i = \begin{vmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \\ a_{i1} & \cdots & a_{ii} \end{vmatrix} > 0, \quad i = 1, 2, \ldots, n.$$

For M matrix, there are many equivalent conditions. The main equivalent conditions are

(1) $a_{ii} > 0$ $(i = 1, \ldots, n)$, $a_{ij} \leqslant 0$ $(i \neq j,\ i, j = 1, \ldots, n)$ and $A^{-1} \geqslant 0$, i.e., A^{-1} is a nonnegative matrix;
(2) $a_{ii} > 0$ $(i = 1, 2, \ldots, n)$, $a_{ij} \leqslant 0$ $(i \neq j,\ i, j = 1, \ldots, n)$, $-A$ is a Hurwitz matrix;

(3) $a_{ii} > 0$ $(i = 1, 2, \dots, n)$, $a_{ij} \leqslant 0$ $(i \neq j,\ i, j = 1, 2, \dots, n)$ and there exist n positive constants c_j $(j = 1, 2, \dots, n)$ such that

$$\sum_{j=1}^{n} c_j a_{ij} > 0, \quad i = 1, 2, \dots, n;$$

(4) $a_{ii} > 0$ $(i = 1, 2, \dots, n)$, $a_{ij} \leqslant 0$ $(i \neq j,\ i, j = 1, 2, \dots, n)$ and there exist n positive constants d_i $(i = 1, 2, \dots, n)$ such that

$$\sum_{i=1}^{n} d_i a_{ij} > 0, \quad j = 1, 2, \dots, n;$$

(5) $a_{ii} > 0$ $(i = 1, 2, \dots, n)$, $a_{ij} \leqslant 0$ $(i \neq j,\ i, j = 1, 2, \dots, n)$ and the spectral radius of matrix

$$B := (1 - \delta_{ij}) \left\| \left(\frac{a_{ij}}{a_{ii}} \right)_{n \times n} \right\|$$

is smaller than 1. That is, $\rho(B) < 1$, namely, the norm of all eigenvalues of G are smaller than 1, where δ_{ij} is the Kronecker delta function.

The conditions given in Definition 1.6.1 for M matrix are more convenient to use than any other equivalent conditions.

In the following, we present a unified simple method for checking the sign of determinants.

DEFINITION 1.6.2. A transform of the determinant is said to be isogeny sign transform, if any column or row of the determinant is multiplied by a positive constant, or any column or row of the determinant is multiplied by any arbitrary number plus other columns or rows. A transform of determinant is said to be complete isogeny sign transform if every major subdeterminant is isogeny sign.

This complete isogeny sign transform is denoted by ψ, therefore, by several complete isogeny sign transforms ψ, we have the following form

$$|A| := \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} \xrightarrow[\psi_1]{} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{vmatrix} := |B|$$

or

$$|A| \xrightarrow[\psi_2]{} \begin{vmatrix} c_{11} & 0 & \cdots & 0 \\ c_{21} & c_{22} & & \vdots \\ \vdots & & \ddots & \\ c_{m} & \cdots & \cdots & c_{nn} \end{vmatrix} := |C|,$$

where ψ_1, ψ_2 are complete isogeny sign transforms.

THEOREM 1.6.3.

(1) *Let $A = A^T \in R^{n\times n}$. Choose a complete isogeny sign transform ψ_1 or ψ_2 such that*

$$|A| \xrightarrow[\psi_1]{} |B|, \quad or \quad |A| \xrightarrow[\psi_2]{} |C|,$$

then the matrix A is positive definite (positive semi-definite) if and only if $b_{ii} > 0$ or $c_{ii} > 0$ ($b_{ii} \geqslant 0$ or $c_{ii} \geqslant 0$), $i = 1, 2, \ldots, n$.

A is negative definite (semi-negative definite) if and only if $b_{ii} < 0$ or $c_{ii} < 0$ ($b_{ii} \leqslant 0$ or $c_{ii} \leqslant 0$), $i = 1, 2, \ldots n$.

A has variable signs if and only if there exist $b_{ii} > 0, b_{jj} < 0$ (or $c_{ii} > 0, c_{jj} < 0$), $i, j \in (1, \ldots, n)$.

(2) *Assume that $a_{ii} > 0$, $a_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$, and there are complete isogeny sign transforms ψ_1, ψ_2 such that*

$$|A| \xrightarrow[\psi_1]{} |B|, \quad or \quad |A| \xrightarrow[\psi_2]{} |C|,$$

then A is a M matrix if and only if $b_{ii} > 0$ or $c_{ii} > 0$, $i = 1, 2, \ldots, n$.

(3) *Let $f(\lambda) = \det|\lambda E_n - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, $a_i > 0$, $i = 0, 1, \ldots, n-1$. Then A is a Hurwitz matrix if and only if there exist ψ_1, ψ_2 such that*

$$\Delta_{n-1} := \begin{vmatrix} a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & & \vdots \\ \vdots & & & a_{n-2} \\ a_{2n-3} & \cdots & \cdots & a_{n-1} \end{vmatrix} \xrightarrow[\psi_1]{} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n-1} \\ & b_{22} & \cdots & b_{2n-1} \\ 0 & & \ddots & \vdots \\ & & & b_{n-1n-1} \end{vmatrix},$$

or

$$\Delta_{n-1} \xrightarrow[\psi_2]{} \begin{vmatrix} c_{11} & 0 & \cdots & 0 \\ c_{21} & c_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ c_{n-11} & & & c_{n-1n-1} \end{vmatrix},$$

$$b_{ii} > 0 \text{ or } c_{ii} > 0, \quad i = 1, \ldots, n-1.$$

PROOF. We only prove the case of $A = A^T$ with a positive definite A. Proofs for other cases are similar and thus omitted. Since A is positive definite if and only if

$$\Delta_1 = a_{11} > 0, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \ldots,$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} > 0.$$

By $a_{11} > 0$, we have $b_{11} > 0$, and then

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \quad \text{implies} \quad \begin{vmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{vmatrix} > 0,$$

so $b_{22} > 0$. By mathematical induction we can prove that $b_{ii} > 0$ $(i = 1, \ldots, n)$.

Now, if $b_{ii} > 0$ $(i = 1, 2, \ldots, n)$, then

$$\tilde{\Delta}_{11} = b_{11} > 0, \quad \tilde{\Delta}_{22} = \begin{vmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{vmatrix} > 0, \quad \ldots,$$

$$\tilde{\Delta}_n = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & & \vdots \\ & & \ddots & \vdots \\ & & & b_{nn} \end{vmatrix} > 0.$$

According to the property of the complete isogeny sign transform ψ, we have

$$\Delta_1 = a_{11} > 0, \quad \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \ldots,$$

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} > 0.$$

The proof is complete. □

EXAMPLE 1.6.4. Prove that

$$A = \begin{bmatrix} 4 & -1 & -2 & -3 \\ -1 & 3 & -2 & -1 \\ -\frac{1}{2} & -1 & 4 & 0 \\ 0 & -1 & -1 & 5 \end{bmatrix}$$

is an M matrix. Obviously, A satisfies that $a_{ii} > 0$, $a_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, 2, 3, 4$. Thus,

$$|A| := \begin{vmatrix} 4 & -1 & -2 & -3 \\ -1 & 3 & -2 & -1 \\ -\frac{1}{2} & -1 & 4 & 0 \\ 0 & -1 & -1 & 5 \end{vmatrix} \xrightarrow[4r_3+r_2]{2r_3-r_2}$$

$$= \begin{vmatrix} 4 & * & * & * \\ 0 & 11 & -10 & -7 \\ 0 & -5 & 10 & 1 \\ 0 & -1 & -1 & 5 \end{vmatrix} \xrightarrow[11r_4+r_2]{r_3-5r_4} \begin{vmatrix} 4 & * & * & * \\ 0 & 11 & * & * \\ 0 & 0 & 15 & -24 \\ 0 & 0 & -21 & 48 \end{vmatrix} \xrightarrow[\frac{2}{24}c_4]{\frac{1}{3}c_3}$$

$$\begin{vmatrix} 4 & * & * & * \\ 0 & 11 & * & * \\ 0 & 0 & 5 & -1 \\ 0 & 0 & -7 & 2 \end{vmatrix} \xrightarrow{c_3+\frac{7}{2}c_4} \begin{vmatrix} 4 & * & * & * \\ 0 & 11 & * & * \\ 0 & 0 & \frac{3}{2} & * \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$:= |B|$$

and so $b_{11} = 4 > 0$, $b_{22} = 11 > 0$, $b_{33} = \frac{3}{2} > 0$, $b_{44} = 2 > 0$. Hence, A is an M matrix. Here, $*$ denotes an element which does not affect the result.

1.7. Definition of Lyapunov stability

The stability considered in this section is in the Lyapunov sense. For convenience, in the following we will simply call it stability. We consider physical systems that can be described by the following ordinary differential equation:

$$\frac{dy}{dt} = g(t, y), \tag{1.7.1}$$

where $\Omega \subset R^n$, $0 \in \Omega$, $g \in C[I \times \Omega, R^n]$. Assume that the solution of Cauchy problem of (1.7.1) is unique. Let

$$y := (y_1, y_2, \ldots, y_n)^T,$$
$$g(t, y) := \big(g_1(t, y), \ldots, g_n(t, y)\big)^T.$$

Suppose that $\bar{y} = \varphi(t)$ is a particular solution of (1.7.1). To study the properties of solutions of (1.7.1) in the neighborhood of the solution $\bar{y}(t)$, we substitute the transformation

$$x = y - \varphi(t)$$

into system (1.7.1) to obtain a system for the new variable x:

$$\frac{dx}{dt} = g\big(t, x + \varphi(t)\big) - g\big(t, \varphi(t)\big) := f(t, x). \tag{1.7.2}$$

Following Lyapunov, we call $x = 0$ or $y = \varphi(t)$ the unperturbed motion or unperturbed trajectory, and call equation (1.7.2) the equation of perturbed motion. Thus, the solution $y = \varphi(t)$ of (1.7.1) corresponds to the zero solution $x = 0$ of (1.7.2). Hence, we only study the stability of the zero solution $x = 0$ of (1.7.2). Suppose $f \in C[I \times \Omega, R^n]$, and the solution of Cauchy problem is uniquely determined. $f(t, x) = 0$ if $x = 0$, $x(t, t_0, x_0)$ represents the perturbed solution which satisfies the initial condition $x(t_0) = x_0$.

$x(t, t_0, x_0)$ is a function of variables t, t_0, x_0. Generally, we consider t_0, x_0 as parameters. Therefore, when we consider the asymptotic behavior of $x(t, t_0, x_0)$ with respect to t, we must investigate whether or not the asymptotic behavior uniformly depends on t_0 or x_0. Hence, there are many stability definitions.

DEFINITION 1.7.1. The zero solution $x = 0$ of (1.7.2) is said to be stable, if $\forall \varepsilon > 0, \forall t_0 \in I, \exists \delta$ such that $\forall x_0$, $\|x_0\| < \delta(\varepsilon, t_0)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for $t \geqslant t_0$. The zero solution $x = 0$ of (1.7.2) is said to be unstable (complete unstable), if $\exists \varepsilon_0, \exists t_0, \forall \delta > 0, \exists x_0 (\forall x_0)$, $\|x_0\| < \delta$, but $\exists t_1 \geqslant t_0$ such that $\|x(t_1, t_0, x_0)\| \geqslant \varepsilon_0$.

DEFINITION 1.7.2. The zero solution $x = 0$ of (1.7.2) is said to be uniformly stable with respect to t_0, if $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ ($\delta(\varepsilon)$ is independent of t_0) such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for $t \geqslant t_0$.

DEFINITION 1.7.3. The zero solution $x = 0$ of (1.7.2) is said to be attractive, if $\forall t_0 \in I$, $\forall \varepsilon > 0$, $\exists \sigma(t_0) > 0$, $\exists T(\varepsilon, t_0, x_0) > 0$, $\|x_0\| < \sigma(t_0)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$, for $t \geqslant t_0 + T$, i.e.,

$$\lim_{t \to +\infty} x(t, t_0, x_0) = 0.$$

The above definitions of different stabilities are illustrated in Figures 1.7.1–1.7.4.

DEFINITION 1.7.4. The zero solution $x = 0$ of (1.7.2) is said to be uniformly attractive with respect to x_0, if the T in Definition 1.7.3 is independent of x_0, i.e., $\forall x_0$, $\|x_0\| < \sigma(t_0)$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for $t \geqslant t_0 + T(\varepsilon, t_0)$. $x = 0$ is also said to be equi-attractive.

DEFINITION 1.7.5. The zero solution $x = 0$ of (1.7.2) is said to be uniformly attractive with respect to t_0, x_0, if $x = 0$ is equi-attractive, σ does not depend on

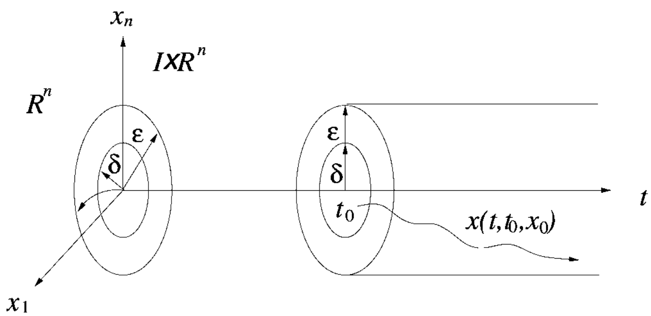


Figure 1.7.1. Stability.

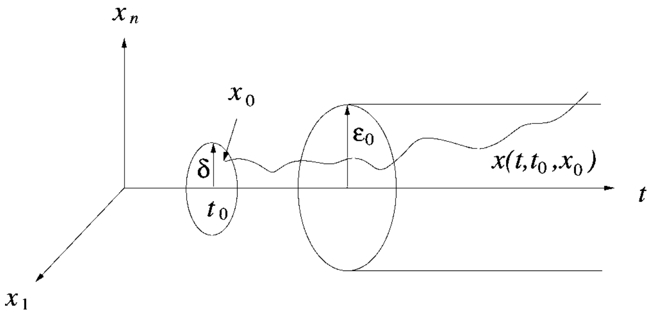


Figure 1.7.2. Instability.

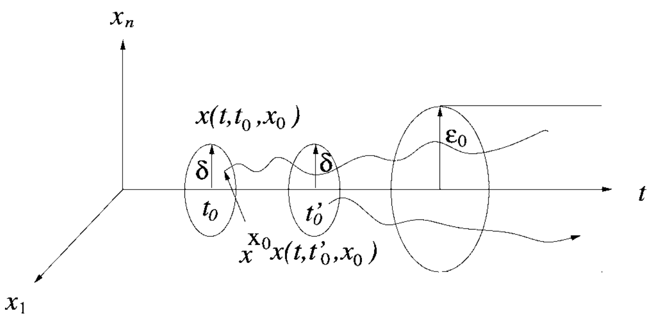


Figure 1.7.3. Uniform stability.

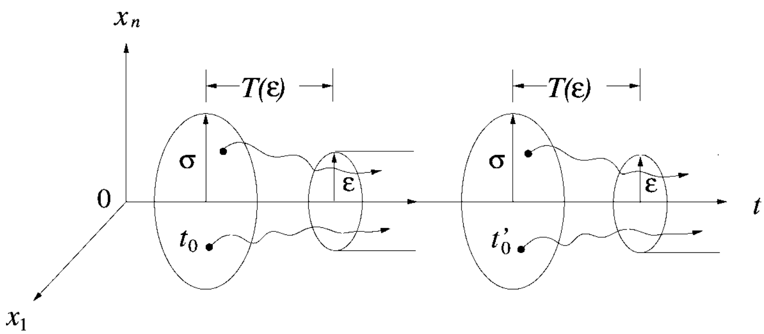


Figure 1.7.4. Uniform attractivity.

t_0, and T does not depend on x_0 and t_0 i.e., $\|x(t, t_0, x_0)\| < \varepsilon$ for $\|x_0\| \leqslant \sigma$ and $t \geqslant t_0 + T(\varepsilon)$.

The region $\Omega := \{\|x\| < \sigma\}$ of R^n is said to lie in the region of attraction of the point $x = 0$ (at $t = t_0$), or simply called a region of attraction.

DEFINITION 1.7.6. The zero solution $x = 0$ of (1.7.2) is respectively said to be asymptotically stable, equi-asymptotically stable, quasi-uniformly asymptotically stable, globally asymptotically stable, globally quasi-uniformly asymptotically stable, if

(1) the zero solution $x = 0$ of (1.7.2) is stable;
(2) $x = 0$ is attractive, equi-attractive, uniformly attractive, globally attractive, globally uniformly attractive, respectively.

DEFINITION 1.7.7. The zero solution $x = 0$ of (1.7.2) is said to be uniformly asymptotically stable, globally uniformly asymptotically stable, respectively, if

(1) $x = 0$ is uniformly stable;
(2) $x = 0$ is uniformly attractive, globally uniformly attractive and all solutions are uniformly bounded (i.e., $\forall r > 0$, $\exists B(r)$ such that when $\|x_0\| < r$, $\|x(t, t_0, x_0)\| < B(r)$ for $t \geqslant t_0$) respectively.

DEFINITION 1.7.8. The zero solution $x = 0$ is said to have exponential stability, if $\forall \varepsilon > 0$, $\exists \lambda > 0$, $\exists \delta(\varepsilon)$, $\forall t_0 \in I$, $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| \leqslant \varepsilon e^{-\lambda(t-t_0)}$ $(t \geqslant t_0)$.

DEFINITION 1.7.9. The zero solution $x = 0$ is said to have globally exponential stability, if $\forall \delta > 0$, $\exists \lambda > 0$, $\exists k(\delta) > 0$, when $\|x_0\| < \delta$, we have

$$\left\|x(t, t_0, x_0)\right\| \leqslant k(\delta) e^{-\lambda(t-t_0)}.$$

From the above definitions, we can find the relation between the stability and attraction, which is discussed in the next section.

1.8. Some examples of stability relation

EXAMPLE 1.8.1. This example shows uniform stability but not asymptotic stability. Consider

$$\begin{cases} \frac{dx_1}{dt} = -x_2, \\ \frac{dx_2}{dt} = x_1. \end{cases} \tag{1.8.1}$$

The general solution of (1.8.1) is

$$x_1(t) = x_1(t_0)\cos(t - t_0) - x_2(t_0)\sin(t - t_0),$$
$$x_2(t) = x_1(t_0)\sin(t - t_0) + x_2(t_0)\cos(t - t_0),$$

or $x_1^2(t) + x_2^2(t) = x_1^2(t_0) + x_2^2(t_0)$. $\forall\varepsilon > 0$, take $\delta = \delta(\varepsilon) = \varepsilon$. When $0 < x_1^2(t_0) + x_2^2(t_0) < \delta$, we have

$$x_1^2(t) + x_2^2(t) < \delta = \varepsilon.$$

Hence the zero solution of (1.8.1) is uniformly stable, but

$$\lim_{t\to\infty} \big(x_1^2(t) + x_2^2(t)\big) = x_1^2(t_0) + x_2^2(t_0) \neq 0.$$

This means that the zero solution is not attractive. Therefore, it is not asymptotically stable.

EXAMPLE 1.8.2. This example, given by

$$\begin{cases} \frac{dx_1}{dt} = f(x_1) + x_2, \\ \frac{dx_2}{dt} = -x_1, \end{cases} \tag{1.8.2}$$

shows that a system can be attractive but unstable. Here,

$$f(x_1) = \begin{cases} -4x_1, & \text{for } x_1 > 0, \\ 2x_1, & \text{for } -1 \leqslant x_1 \leqslant 0, \\ -x_1 - 3, & \text{for } x_1 < -1. \end{cases}$$

When $x_1 > 0$, we can rewrite (1.8.2) as:

$$\begin{cases} \frac{dx_1}{dt} = -4x_1 + x_2, \\ \frac{dx_2}{dt} = -x_1. \end{cases} \tag{1.8.3}$$

Equation (1.8.3) has general solution:

$$\begin{cases} x_1(t) = c_1(2-\sqrt{3})e^{(-2+\sqrt{3})t} + c_2(2+\sqrt{3})e^{(-2-\sqrt{3})t}, \\ x_2(t) = c_1 e^{(-2+\sqrt{3})t} + c_2 e^{(-2-\sqrt{3})t}. \end{cases}$$

When $-1 \leqslant x_1 \leqslant 0$, (1.8.2) becomes

$$\begin{cases} \frac{dx_1}{dt} = 2x_1 + x_2, \\ \frac{dx_2}{dt} = -x_1, \end{cases} \tag{1.8.4}$$

which has general solution:

$$\begin{cases} x_1(t) = c_1 e^t + c_2 t e^t, \\ x_2(t) = (-c_1 + c_2)e^t - c_2 t e^t. \end{cases}$$

When $x_1 < -1$, (1.8.2) is

$$\begin{cases} \frac{dx_1}{dt} = -x_1 + x_2 - 3, \\ \frac{dx_2}{dt} = -x_1, \end{cases} \tag{1.8.5}$$

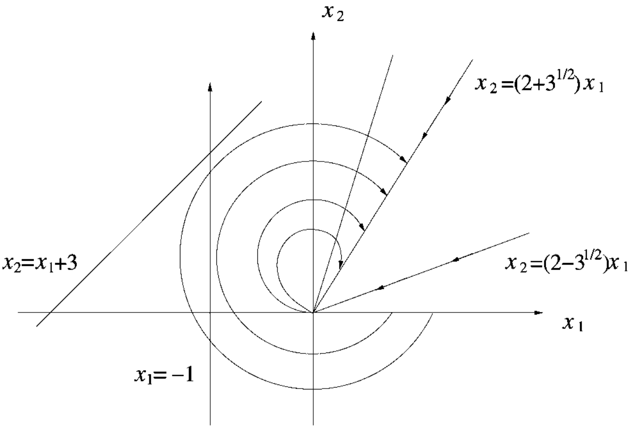


Figure 1.8.1. The relation between different stabilities.

which has general solution:

$$
\begin{aligned}
x_1(t) &= \frac{1}{2}e^{-\frac{1}{2}t}\left(c_1\left(\cos\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right) \\
&\quad + \frac{1}{2}e^{-\frac{1}{2}t}\left(c_2(\sin\frac{\sqrt{3}}{2}t) - \sqrt{3}\cos\frac{\sqrt{3}}{2}t\right), \\
x_2(t) &= c_1e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + c_2e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t + 3.
\end{aligned}
$$

On the phase plane, asymptotic behavior of the trajectories are shown in Figure 1.8.1.

For every solution $x_1(t)$, $x_2(t)$, the following asymptotic behavior holds

$$\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} x_2(t) = 0,$$

so the zero solution is attractive.

But on the other hand, the solution with the initial condition: $x_1(0) = -1$, $x_2(0) = 1$ is $x_1(t) = -e^t$, $x_2(t) = e^t$. Thus, when $t \leqslant 0$, $-1 \leqslant x_1 \leqslant 0$, and

$$
\begin{aligned}
&\lim_{t\to-\infty} x_1(t) = \lim_{t\to-\infty} \left(-e^t\right) = 0, \\
&\lim_{t\to-\infty} x_2(t) = \lim_{t\to-\infty} e^t = 0.
\end{aligned}
$$

EXAMPLE 1.8.3. This example regards a system that is asymptotically stable, but not uniformity asymptotically stable. The equation is given by

$$\frac{dx}{dt} = -\frac{x}{t+1}, \tag{1.8.6}$$

which has the general solution:

$$x(t, t_0, x_0) = x_0 \frac{t_0 + 1}{t + 1}.$$

$\forall \varepsilon > 0$, take $\delta = \varepsilon$. When $|x_0| < \delta$, $|x(t, t_0, x_0) \leqslant |x_0| < \delta = \varepsilon$ for $t \geqslant t_0$ holds, so the zero solution is uniformly stable, and $\lim_{t_0 \to +\infty} x(t, t_0, x_0) = 0$. Hence, the zero solution is asymptotically stable. But $\forall T > 0$, we can choose $t_0 = t - T$, i.e., for $t = t_0 + T$, we obtain

$$x(t, t_0, x_0) = x_0 \frac{t_0 + 1}{t_0 + T + 1} \to x^0 \neq 0 \quad \text{as } t \to +\infty.$$

This means that the zero solution is not uniformly attractive, and therefore, the zero solution is not uniformly asymptotically stable.

EXAMPLE 1.8.4. This example regards a system that is equi-asymptotically stable, but not uniformly asymptotically stable. Consider

$$\frac{dx}{dt} = (t \sin t - \cos t - 2)x. \tag{1.8.7}$$

Its general solution is $x(t, t_0, x_0) = x_0 e^{-2(t-t_0) - t\cos t + t_0 \cos t_0}$. $\forall \varepsilon > 0$, take $\delta = \varepsilon e^{-(t-t_0)}$. When $|x_0| < \delta$, we have

$$\left|x(t, t_0, x_0)\right| \leqslant |x_0| e^{2t_0} e^{-(t-t_0)} < \delta e^{2t_0} = \varepsilon.$$

Thus, the zero solution is stable. However, $\forall \varepsilon > 0$, if we take $\delta = \varepsilon$, $T = 2t_0$, then when $|x_0| < \delta$, $t \geqslant t_0 + T$, we have

$$\left|x(t, t_0, x_0)\right| \leqslant |x_0| e^{-2(t-t_0)+t+t_0} = |x_0| e^{2t_0} e^{-(t-t_0)} < \delta e^{2t_0} = \varepsilon_0.$$

Since T does not depend on x_0, the zero solution is equi-attractive. Hence it is equi-asymptotically stable. But, when $x_0 \neq 0$,

$$\left|x\left(2n\pi + \frac{\pi}{\alpha}, 2n\pi, x_0\right)\right| = |x_0| e^{(2n-1)\pi} \to +\infty \quad (n \to \infty).$$

Therefore, the zero solution is not uniformly stable.

EXAMPLE 1.8.5. A system with uniformly asymptotic stability but not exponential stability.

$$\frac{dx}{dt} = -x^3. \tag{1.8.8}$$

The general solution can be written as $x(t, t_0, x_0) = x_0[1 + 2x_0^2(t - t_0)]^{-\frac{1}{2}}$. $\forall \varepsilon > 0$, take $\delta = \varepsilon$. So when $|x_0| < \delta$,

$$\left|x(t, t_0, x_0)\right| < \varepsilon,$$

and $\forall \varepsilon > 0$, $\exists T(\varepsilon) > 0$, $\exists \sigma > 0$, when $|x_0| < \sigma$, $t \geqslant t_0 + T(\varepsilon)$,

$$\left|x(t, t_0, x_0)\right| \leqslant |x_0| \frac{1}{\sqrt{x_0^2(t - t_0)}} = \frac{1}{\sqrt{t - t_0}} < \frac{1}{\sqrt{T}} < \varepsilon.$$

Obviously, it can be seen from the above general solution that the zero solution is globally uniformly asymptotically stable, but not exponentially stable.

EXAMPLE 1.8.6. A system with asymptotic stability but not equi-asymptotic stability.

$$\begin{cases} \frac{dr}{dt} = \frac{\dot{g}(t,\varphi)}{g(t,\varphi)}, \\ \frac{d\varphi}{dt} = 0, \end{cases} \tag{1.8.9}$$

where

$$g(t, \varphi) = \frac{\cos^4 \varphi}{\cos^4 \varphi + (1 - t\cos^2 \varphi)^2} + \frac{1}{1 + \cos^4 \varphi} \frac{1}{1 + t^2}.$$

Directly integrating equation (1.8.9) yields the general solution:

$$\begin{cases} r(t) = r(t, t_0, r_0) = r_0 \frac{g(t,\varphi_0)}{g(t_0,\varphi_0)}, \\ \varphi(t) = \varphi(t, t_0, r_0) = \varphi_0. \end{cases}$$

Hence,

$$r(t) = r_0 \frac{g(t, \varphi_0)}{g(t_0, \varphi_0)} \leqslant r_0 \frac{2}{g(t_0, \varphi_0)} \quad \forall \varepsilon > 0.$$

Take $\delta = \varepsilon g(t_0, \varphi_0)/k$, then when $r_0 < \delta$, we have

$$\left|g(t)\right| \leqslant r_0 \frac{2}{g(t_0, \varphi_0)} < \delta \frac{2}{g(t_0, \varphi_0)} = \varepsilon,$$

so the zero solution is stable.

From the general solution, we can see that the zero solution is attractive, but is not equi-attractive. In fact, take

$$t_0 = \frac{1}{2\cos^2 \varphi_0}, \qquad t_1 = \frac{1}{\cos^2 \varphi_0},$$

and let $\varphi_0 \to k\pi + \frac{\pi}{2}$ and $k \to \infty$, then $t_1 \to +\infty$. In this case,

$$r(t_1) = r_0 \frac{g(t_1, \varphi_0)}{g(t_0, \varphi_0)} \to \infty \quad \text{when } \varphi_0 \to k\pi + \frac{\pi}{2}.$$

Thus, the zero solution is not equi-attractive, i.e., it is not equi-asymptotically stable.

EXAMPLE 1.8.7. A system showing exponential stability but not globally asymptotic stability. The equation is

$$\frac{dx}{dt} = -x + x^2, \tag{1.8.10}$$

which has general solution

$$x(t, t_0, x_0) = \frac{x_0 e^{-(t-t_0)}}{x_0 e^{-(t-t_0)} - x_0 + 1}.$$

Consider a region $\Omega_0 := \{x \| x\| \leqslant r_0 < 1\}$. $\forall \varepsilon > 0$, take $\delta = \min\{r_0, (1 - r_0)\varepsilon\}$. Then, when $t_0 \in [t_0 + \infty)$, $|x_0| < \delta$, for all $t \geqslant t_0$ we have

$$\begin{aligned}
\left|x(t, t_0, x_0)\right| &= \frac{x_0 e^{-(t-t_0)}}{1 - x_0} < \frac{\delta}{1 - r_0} e^{-(t-t_0)} \\
&= \varepsilon e^{-(t-t_0)} \leqslant \varepsilon e^{-(t-t_0)} \quad \text{for } 0 \leqslant x_0 \leqslant r_0, \\
\left|x(t, t_0, x_0)\right| &= \frac{|x_0| e^{-(t-t_0)}}{1 - x_0(1 - e^{-(t-t_0)})} \leqslant |x_0| e^{-(t-t_0)} \\
&< \varepsilon e^{-(t-t_0)} \leqslant \varepsilon \quad \text{for } -r_0 \leqslant x_0 \leqslant 0.
\end{aligned}$$

From the above expression, we easily find that the zero solution is exponentially stable. But if we take $t = t_0$, $x_0 = 1$, then the solution $x(t, t_0, x_0) \equiv 1$ (as $t \to +\infty$). Hence, the zero solution is not globally stable.

EXAMPLE 1.8.8. A system with equi-asymptotic stability but not globally asymptotic stability and not quasi-uniformly asymptotic stability.

Consider the differential equation:

$$\frac{dx}{dt} = \begin{cases} -\frac{x}{t}, & \text{for } -1 \leqslant xt \leqslant 1, \\ \frac{x}{t} - \frac{2}{t^2}, & \text{for } xt > 1, \\ \frac{x}{t} + \frac{2}{t^2}, & \text{for } xt < -1. \end{cases} \tag{1.8.11}$$

Its general solution is

$$x(t, t_0, x_0) = \begin{cases} \frac{t_0, x_0}{t}, & \text{for } -1 \leqslant xt \leqslant 1, \\ \frac{1}{t} + t\left(\frac{x_0}{t_0} - \frac{1}{t_0^2}\right), & \text{for } xt > 1, \\ -\frac{1}{t} + t\left(\frac{x_0}{t_0} - \frac{1}{t_0^2}\right), & \text{for } xt < -1. \end{cases}$$

$\forall \varepsilon > 0$ ($\varepsilon < 1$), $\forall t_0 > 0$, choose $\delta = \min(1/t_0, \varepsilon)$. Then $|x_0| < \delta$ implies

$$\left|x(t_1, t_0, x_0)\right| = \left|\frac{t_0 x_0}{t}\right| \leqslant |x_0| < \varepsilon.$$

Hence the zero solution of (1.8.11) is stable. On the other hand, $\forall\varepsilon > 0$ ($\varepsilon < 1$), $\forall t_0 > 0$, take $\eta(t_0) = \frac{1}{t_0}$, $T(\varepsilon, t_0) = \frac{t_0\eta(t_0)}{\varepsilon}$. Then, when $|x_0| < \eta(t_0)$, we have

$$\left|x(t, t_0, x_0)\right| = \left|\frac{t_0 x_0}{t}\right| = \frac{t_0|x_0|}{t} \leqslant \frac{t_0|x_0|}{t_0 + T(\varepsilon, t_0)} < \frac{b_0\eta(t_0)}{T(\varepsilon, t_0)} = \varepsilon$$

for all $t \geqslant t_0 + T(\varepsilon, t_0)$. So the zero solution is equi-attractive.

Let $\varepsilon = \frac{1}{3}$, $\delta > 0$. Then, $\exists t_0 > 0$ and x_0 ($|x_0| < \delta$) such that $t_0 x_0 > 1$. Taking $t_1 = \frac{1}{3}\max(t_0, \frac{t_1^2}{x_0 t_0 - 1})$ results in

$$x(t_1, t_0, x_0) = \frac{1}{t_1} + t_1\left(\frac{x_0}{t_0} - \frac{1}{t_0^2}\right) \geqslant t_1\frac{t_0 x_0 - 1}{t_1^2} \geqslant \frac{1}{3}.$$

This means that the zero solution is not uniformly stable.

Choose $\varepsilon_1 = \frac{1}{2}$, $\eta > 0$, $\exists t_0 > 0$ and x_0, $|x_0| < \eta$ such that $t_0 x_0 > 1$. Take $T \geqslant \max(\frac{1}{1}\frac{t_0^2}{t_0 x_0 - 1} - t_0, 0)$. Then, when $t \geqslant t_0 + T$,

$$\left|x(t, t_0, x_0)\right| = \left|\frac{1}{t} + t\left(\frac{x_0}{t_0} - \frac{1}{t_0^2}\right)\right| > t\frac{t_0 x_0 - 1}{t_0^2} \geqslant (t_0 + T)\frac{t_0 x_0 - 1}{t_0^2} \geqslant \frac{1}{2}.$$

Therefore, the zero solution is not uniformly attractive.

Finally, let $t_0 x_0 > 1$. Then

$$x(t, t_0, x_0) = \frac{1}{t} + t\left(\frac{x_0}{t_0} - \frac{1}{t_0^2}\right) = \frac{1}{t} + t\left(\frac{t_0 x_0 - 1}{t_0^2}\right) \nrightarrow \quad \text{as } t \to \infty.$$

So the zero solution is not globally attractive.

The zero solution of (1.8.11) is only equi-asymptotically stable but not globally asymptotically stable and nor quasi-uniformly asymptotically stable, as shown in Figure 1.8.2.

The above eight examples show the difference in various stability definitions. However, for certain specific systems such as time independent systems, periodic systems and linear systems, there still exist some equivalent relations between some stability definitions.

Consider the periodic system:

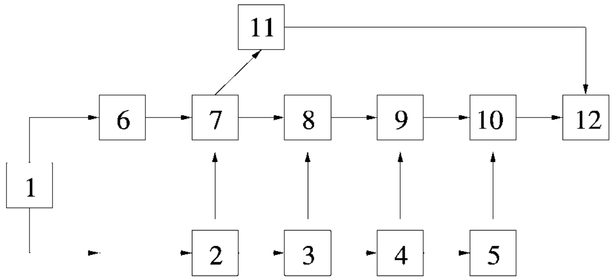
$$\frac{dx}{dt} = f(t, x), \quad f(t, 0) = 0, \tag{1.8.12}$$

$f \in C[I \times R^n]$. Assure that the solution of (1.8.12) is unique for Cauchy problem.

THEOREM 1.8.9. *If $\exists$ a period $\omega > 0$ such that*

$$f(t + \omega, x) \equiv f(t, x),$$

then the zero solution of (1.8.12) is stable, if and only if it is uniformly stable.



(1) G.E.S.	(2) G.U.A.S.	(3) G.q.U.A.S.
(4) G.Eq.A.S.	(5) G.A.S.	(6) E.S.
(7) U.A.S.	(8) q.U.A.S.	(9) Eq.A.S.
(10) A.S.	(11) U.S.	(12) S.

Figure 1.8.2. The relation of various stabilities.

PROOF. We only need to prove that the stability implies the uniform stability. $\forall \varepsilon > 0$, $\exists \delta_1(\varepsilon, \omega) > 0$, when $\|x_0\| < \delta_1(\varepsilon, \omega)$, $\|x(t, \omega, x_0)\| < \varepsilon$ holds for all $t \geqslant \omega$. Let $\|x(t_0, \omega, x_0)\| \leqslant \delta_2$ $(0 \leqslant t_0 \leqslant \omega, \|x_0 < \delta_1)$, $\delta = \min(\delta_1, \delta_2)$. Then, when $\|x_0\| < \delta$, $\forall t_0 \in [0, \omega]$, for all $t \geqslant t_0$, we have $\|x(t, t_0, x_0)\| < \varepsilon$, $\forall \tilde{t_0} \in [t_0 + \infty)$. Let $\tilde{t_0} = m\omega + t_0$, $\tilde{t} = m\omega + t$, where $m = [\frac{\tilde{t_0}}{\omega}]$ denotes the largest integer part. Suppose that $\xi(\tilde{t}, \tilde{t_0}, x_0)$ is an arbitrary solution of (1.8.12), set $x(t + m\omega, t_0 + m\omega, x_0) := \xi(\tilde{t}, \tilde{t_0}, x_0)$.

Since $f(t + \omega, x) \equiv f(t, x)$ implies that $x(t, t_0, x_0)$ is also a solution of (1.8.12), and $x(t, t_0, x_0) \equiv x(t + m\omega, t_0 + m\omega, x_0) := \xi(\tilde{t}, \tilde{t_0}, x_0)$, thus when $\|x_0\| < \xi, t \in [0, +\infty)$, $\|\xi(\tilde{t}, \tilde{t_0}, x_0)\| = \|x(t, t_0, x_0)\|$ holds. So the zero solution of (1.8.12) is uniformly stable. □

COROLLARY 1.8.10. *If system (1.8.12) is autonomous, i.e., $f(t, x) \equiv f(x)$, then the stability and uniform stability of the zero solution of (1.8.12) are equivalent, since the autonomous system is a specific case of periodic systems.*

THEOREM 1.8.11. *For periodic system (1.8.12), its zero solution is asymptotically stable if and only if it is uniformly asymptotically stable.*

PROOF. We only need to prove that the attraction of the zero solution implies the uniform attraction. By Theorem 1.8.9, $\forall \varepsilon > 0$, $\exists \sigma(t_0) > 0$, $\exists T(\varepsilon, \sigma, t_0)$, when $t > t_0 + T(\varepsilon, \sigma, t_0)$, $\|x_0\| < \sigma(t_0)$, we have

$$\|x_0\| < \left\|\sigma(t_0)\right\| < \varepsilon.$$

For an arbitrary solution $\xi(\tilde t, \tilde t_0, x_0)$, let $\tilde t_0 = m\omega + t_0$, $\tilde t = m\omega + t$ and $x(t + m\omega, t_0 + m\omega, x_0) := \xi(\tilde t, \tilde t_0, x_0)$ owing to the uniqueness of solution for (1.8.12) and $f(t+\omega, x) \equiv f(t)$. Therefore,

$$x(t, t_0, x_0) \equiv x(t + m\omega, t_0 + m\omega, x_0) = \xi\big(\tilde t, \tilde t_0, x_0\big).$$

Thus, when

$$t + m\omega - t_0 - m\omega = t - t_0 > T(\varepsilon, \sigma, t_0),$$
$$\big\|\xi(\tilde t, \tilde t_0, x_0)\big\| = \big\|x(t, t_0, x_0)\big\| < \varepsilon,$$

i.e., the zero solution of (1.8.12) is uniformly attractive.

The proof is complete. □

COROLLARY 1.8.12. *If* (1.8.12) *is an autonomous system, i.e.,* $f(t,x) \equiv f(x)$, *then the asymptotic stability and uniformly asymptotic stability of the zero solution are equivalent.*

THEOREM 1.8.13. *In* (1.8.12), *if* $f(t,x) = A(t)x$, *then the zero solution of* (1.8.11) *has the following stability equivalent relations*:

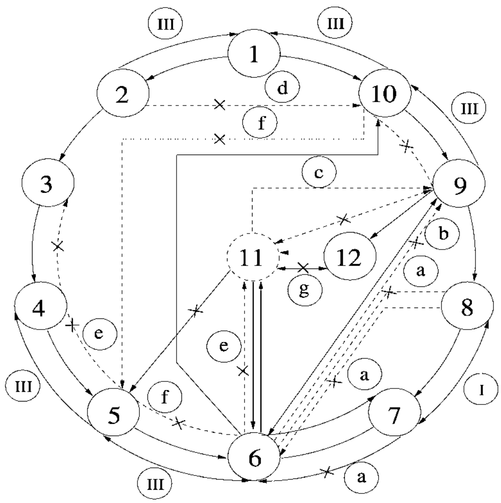


Figure 1.8.3. The equivalent relations of different stabilities: ⓐ Ex. 1 ⓑ Ex. 3 ⓒ Ex. 4 ⓓ Ex. 5 ⓔ Ex. 6 ⓕ Ex. 7 ⓖ Ex. 8 Ⓘ Auto. or periodic system (Theorem 1.8.1) Ⓘⓘ Periodic system (Theorem 1.8.2) ⓘⓘⓘ linear system (Theorem 1.8.3) ① globally exponential stability ② globally uniformly asymptotic stability ③ globally quasi-uniformly asymptotic stability ④ globally equi-asymptotic stability ⑤ globally asymptotic stability ⑥ asymptotic stability ⑦ stability ⑧ uniform stability ⑨ uniformly asymptotic stability ⑩ exponential stability ⑪ equi-asymptotic stability ⑫ quasi-uniform stability $\longrightarrow$ imply $\overset{ⓐ}{\longrightarrow}$ imply under condition ⓐ $--\times--$ not imply.

(1) *local asymptotic stability and global asymptotic stability are equivalent*;
(2) *asymptotic stability equivalent to equi-asymptotic stability*;
(3) *uniformly asymptotic stability and exponential stability are the same*;
(4) *if* $A(t) = A$ (*a constant matrix*)*, then asymptotic stability and exponential stability are the same.*

The proof for Theorem 1.8.13 will be given in Chapter 2. The equivalent relations of different stabilities are shown in Figure 1.8.3.

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Chapter 2

Linear Systems with Constant Coefficients

The solutions to linear differential equations with constant coefficients can be expressed as linear combination of the product of a polynomial and an exponential function, i.e., linear combination of the terms $P_i(x)e^{\lambda_i t}$. Solving a linear differential equation with constant coefficients can be transformed to solve a linear algebraic problem, more precisely, an eigenvalue-eigenvector problem. Therefore, there are many classical results and methods. Moreover, many practical approaches are also available in computer software.

In this chapter, we elaborate various criteria of stability for linear system of differential equations with constant coefficients. These results and methods are practical for engineers, especially, for those who are concerned with automatic control.

The materials presented in this chapter are mainly taken from [342,343] for Section 2.1, [234] for Section 2.2, [342] for Section 2.3, [396] for Section 2.4, and [395] for Section 2.5.

2.1. NASCs for stability and asymptotic stability

In this section, we consider the necessary and sufficient conditions (NASCs) for stability and asymptotic stability. Consider the following n-dimensional linear differential equations with real constant coefficients:

$$\frac{dx}{dt} = Ax, \quad x = (x_1, \dots, x_n)^T,\ A = (a_{ij})_{n\times n} \in R^{n\times n}. \tag{2.1.1}$$

Let $\lambda(A)$ be the eigenvalue of $A(a_{ij})_{n\times n}$.

DEFINITION 2.1.1. If all eigenvalues of matrix $A(a_{ij})_{n\times n}$ are located on the open left side of complex plane, i.e., $\operatorname{Re}\lambda_i(A) < 0$, $i = 1, 2, \dots, n$, then A is said to be Hurwitz stable. If all eigenvalues of A lie on the closed left side of complex plane, i.e., $\operatorname{Re}\lambda_i(A) \leqslant 0$ $(i = 1, \dots, n)$ and if $\operatorname{Re}\lambda_{j_0}(A) = 0$, λ_{j_0} only correspond to simple elementary divisor of A, then A is said to be quasi-stable.

THEOREM 2.1.2. *The zero solution of systems* (2.1.1) *is asymptotically stable, if and only if A is Hurwitz stable*; *the zero solution of* (2.1.1) *is stable if and only if A is quasi-stable.*

PROOF. The general solution of (2.1.1) can be expressed as

$$x(t,t_0,x_0) = e^{A(t-t_0)}x_0 = K(t,t_0)x_0, \tag{2.1.2}$$

where $K(t,t_0) = e^{A(t-t_0)}$ is called Cauchy matrix solution or standard fundamental solution matrix.

Let $A = SJS^{-1}$, J is a Jordan canonical form. Then,

$$\begin{aligned}
e^{A(t-t_0)} &= e^{SJS^{-1}(t-t_0)} = Se^{J(t-t_0)}S^{-1},\\
e^{J(t-t_0)} &= \operatorname{diag}\bigl(e^{J_1(t-t_0)}, e^{J_2(t-t_0)}, \dots, e^{J_r(t-t_0)}\bigr),\\
e^{J_j(t-t_0)} &= \begin{bmatrix}
1 & (t-t_0) & \frac{(t-t_0)^2}{2!} & \cdots & \frac{(t-t_0)^{n_j-1}}{(n_j-1):}\\
0 & 1 & t-t_0 & & \vdots\\
\vdots & & 1 & & \vdots\\
 & & & 1 & (t-t_0)\\
0 & \cdots & 0 & 0 & 1
\end{bmatrix}_{n_j\times n_j} e^{\lambda_j(t-t_0)},\\
\sum_{i=1}^{r} n_i &= n.
\end{aligned}$$

One can easily show that the stability of zero solution of system (2.1.1) is determined by the boundedness of $e^{A(t-t_0)}$, or the boundedness of $e^{J(t-t_0)}$ or the boundedness of all $e^{J_i(t-t_0)}$ $(i = 1,2,\dots,r)$, i.e., $\operatorname{Re}\lambda_j \leqslant 0$, and when $\operatorname{Re}\lambda_j = 0$, $n_j = 1$, i.e., A is quasi-stable. Asymptotic stability of the zero solution of system (2.1.1) is given by

$$\lim_{t\to+\infty} e^{A(t-t_0)} = 0,$$

or

$$\lim_{t\to+\infty} e^{J(t-t_0)} = 0,$$

which is equivalent to

$$\lim_{t\to+\infty} e^{J_i(t-t_0)} \quad (i = 1,2,\dots,n_r).$$

That is,

$$\lim_{t\to+\infty}\begin{bmatrix} 1 & t-t_0 & \frac{t-t_0}{2!} & \cdots & \frac{(t-t_0)^{n_j-1}}{(n_j-1)!} \\ 0 & 1 & & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \\ \vdots & \vdots & & 1 & t-t_0 \\ 0 & 0 & & & 1 \end{bmatrix} e^{\lambda_j(t-t_0)} = 0$$

for all $j = 1, 2, \ldots, r$,

and thus $\operatorname{Re}\lambda_j < 0$ $(j = 1, 2, \ldots, n)$ implying that A is a Hurwitz matrix.

The proof is complete. □

Let

$$f_n(\lambda) := \det(\lambda E_n - A) = a_0 + a_1\lambda + \cdots + \lambda^n, \tag{2.1.3}$$

where

$$a_0 = (-1)^n \det|A|, \quad a_1 = (-1)\sum_{i=1}^{n} a_{ii},$$

$$a_2 = (-1)^2 \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \quad \ldots, \quad a_n = 1,$$

$$M_f := \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & & \vdots \\ \vdots & & & & \vdots \\ & & & a_{n-1} & a_{n-2} \\ a_{2n-1} & a_{2n-2} & \cdots & 0 & a_n \end{bmatrix},$$

where $a_s = 0$ when $s < 0$ or $s > n$.

THEOREM 2.1.3. *Suppose $a_i > 0$, $i = 1, \ldots, n$. A is a Hurwitz matrix if and only if*

$$\Delta_1 = a_1 > 0,\ \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0,\ \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0,\ \ldots,$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & & \vdots \\ \vdots & \vdots & & a_{n-2} \\ a_{2n-1} & a_{2n-2} & & a_n \end{vmatrix} = \Delta_{n-1}a_n > 0,\ a_n = 1.$$

This is a well-known result, and the proof can be found in [98]. If A is a Hurwitz matrix, we write $f_n(\lambda) \in H$.

Next we study the quasi-stability of A.

We still consider system (2.1.3). If $\operatorname{Re}\lambda(A) \leqslant 0$, the polynomial $f_n(\lambda)$ is called Routh polynomial and denoted by $f_n(\lambda) \in qH$. Obviously, A is quasi-stable only if $f_n(\lambda) \in qH$, and it is easy to prove that $f(\lambda) \in qH$ only when $a_i \geqslant 0$ $(i = 1, \dots, n-1)$. One may determine whether $f_n(x) \in qH$ by using Hurwitz criterion. Demidovich [98] stated that $f_n(\lambda) \in qH$ if and only if

$$\Delta_1 = a_1 \geqslant 0,$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} \geqslant 0, \quad \dots, \quad \Delta_n = \begin{vmatrix} a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & & \vdots \\ \vdots & & & a_{n-2} \\ a_{2n-1} & a_{2n-2} & & a_n \end{vmatrix} \geqslant 0,$$

but without giving proof. Later, a counter-example [408] shows that the above conditions are not sufficient. Now we discuss this issue further.

LEMMA 2.1.4. *$f_n(\lambda) \in qH$ if and only if $\forall\varepsilon > 0$, $g_\varepsilon(\lambda) := f_n(\lambda + \varepsilon) \in H$.*

PROOF. *Necessity*. Consider a fixed ε and $\forall\varepsilon > 0$. Let $\alpha = c + id$ (c, d are real numbers) be any zero point of $g_\varepsilon(\lambda) = f_n(\lambda + \varepsilon)$, then $f_n((c + \varepsilon) + id) = f_n(\alpha + \varepsilon) = g_\varepsilon(\alpha) = 0$. Thus, $c + \varepsilon + id$ is a zero point of $f_n(\lambda)$. Since $f_n(\lambda) \in qH$, $c + \varepsilon \leqslant 0$, $\varepsilon > 0$, so $c < 0$. This mean that $g_\varepsilon(\lambda) = f_n(\lambda + \varepsilon) \in H$.

Sufficiency. Let $\lambda_0 = \xi + i\eta$ be any zero point of $f_n(\lambda)$.$\forall\varepsilon > 0$ ($\varepsilon \ll 1$), due to $g_\varepsilon(\lambda_0 - \varepsilon) = f_n((\lambda_0 - \varepsilon) + \varepsilon) = f_n(\lambda_0) = 0$, $\lambda_0 - \varepsilon$ is a zero point of $g_\varepsilon(\lambda)$ and $g_\varepsilon(\lambda) \in H$. It implies $\operatorname{Re}(\lambda_0 - \varepsilon) < 0$. Since ε is arbitrary, $\operatorname{Re}\lambda_0 \leqslant 0$. i.e., $f_n(\lambda) \in qH$.

Lemma 2.1.4 is proved. □

Let $g_\varepsilon = f_n(\lambda + \varepsilon) = \lambda^n + \tilde{a}_{n-1}\lambda^{n-1} + \cdots + \tilde{a}_1(\varepsilon)\lambda + \tilde{a}_0(\varepsilon)$. Corresponding to g_ε, we can verify whether the matrix $M_{g\varepsilon}$ is Hurwitz. Here,

$$M_{g\varepsilon} = \begin{bmatrix} \tilde{a}_1(\varepsilon) & \tilde{a}_0(\varepsilon) & 0 & \cdots & 0 \\ \tilde{a}_3(\varepsilon) & \tilde{a}_2(\varepsilon) & \tilde{a}_1(\varepsilon) & \cdots & 0 \\ \vdots & \vdots & & & \\ \tilde{a}_{2n-1}(\varepsilon) & \tilde{a}_{2n-2}(\varepsilon) & \cdots & \cdots & \tilde{a}_n(\varepsilon) \end{bmatrix}$$

in which $\tilde{a}_s(\varepsilon) = 0$ for $s > n$ or $s < 0$.

COROLLARY 2.1.5. *$g_\varepsilon \in H$ if and only if all major subdeterminants of $M_{g\varepsilon}$ are greater than zero.*

Since $\tilde{a}_i(\varepsilon)$ is a polynomial of ε, it is difficult to check the conditions of Corollary 2.1.5. In the following, we give an equivalent and simple condition.

It is well known that for an mth-degree polynomial with real coefficients, given by

$$g(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0,$$

if $a_0 = a_1 = \cdots = a_{k-1} = 0$, $a_k \neq 0$, then when $0 < \varepsilon \ll 1$, $g(\varepsilon)a_k > 0$. Since $g(\varepsilon) = (a_m\varepsilon^{m-k} + \cdots + a_k)\varepsilon^k = a_k\varepsilon^k[1 + 0(1)]$ (as $\varepsilon \to 0$). Thus $g(\varepsilon)a_k = a_k^2\varepsilon^k[1 + 0(1)] > 0$ as $\varepsilon \to 0$.

According to this fact, we can ignore higher order terms of ε in every term of $M_{g\varepsilon}$, which is denoted as $M_{H\varepsilon}$ Thus, we obtain a new criterion for Hurwitz method. That is, all major subdeterminants of $M_{g\varepsilon}$ are greater than zero if and only if all major subdeterminants of $M_{H\varepsilon}$ are greater than zero.

EXAMPLE 2.1.6. Verify $f_5(z) = 1 + z^2 + z^5 \notin qH$.

Let

$$\begin{aligned} g_\varepsilon(z) &= f(z+\varepsilon) \\ &= 1 + (z+\varepsilon)^2 + (z+\varepsilon)^5 \\ &= z^5 + 5\varepsilon z^4 + 10z^2 z^3 + \big(10z^3 + 1\big)z^2 + \big(5\varepsilon^4 + 2\varepsilon\big)z + \big(\varepsilon^5 + \varepsilon^2 + 1\big), \end{aligned}$$

$$M_{g\varepsilon} = \begin{bmatrix} 5\varepsilon^4 + 2\varepsilon & \varepsilon^5 + \varepsilon^2 + 1 & 0 & 0 & 0 \\ 10\varepsilon^2 & 10\varepsilon^3 + 1 & 5\varepsilon^4 + 2\varepsilon & \varepsilon^5 + \varepsilon^2 + 1 & 0 \\ 1 & 5\varepsilon & 10\varepsilon^2 & 10\varepsilon^3 + 1 & 5\varepsilon^4 + 2\varepsilon \\ 0 & 0 & 1 & 5\varepsilon & 10\varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_{H\varepsilon} = \begin{bmatrix} 2\varepsilon & 1 & 0 & 0 & 0 \\ 10\varepsilon^2 & 1 & 2\varepsilon & 1 & 0 \\ 1 & 5\varepsilon & 10\varepsilon^2 & 1 & 2\varepsilon \\ 0 & 0 & 1 & 5\varepsilon & 10\varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we compute the major subdeterminants of $M_{H\varepsilon}$ as follows:

$$\begin{aligned} \Delta_1 &= 2\varepsilon > 0 \\ \Delta_2 &= \begin{vmatrix} 2\varepsilon & 1 \\ 10\varepsilon^2 & 1 \end{vmatrix} = 2\varepsilon\big[1 + 0(1)\big] > 0 \quad \text{when } 0 < \varepsilon \ll 1, \\ \Delta_3 &= \begin{vmatrix} 2\varepsilon & 1 & 0 \\ 10\varepsilon^2 & 1 & 2\varepsilon \\ 1 & 5\varepsilon & 10\varepsilon^2 \end{vmatrix} = 2\varepsilon\big[1 + 0(1)\big] > 0 \quad \text{when } 0 < \varepsilon \ll 1, \\ &\ \ \vdots \end{aligned}$$

$$\Delta_n = 5\varepsilon\Delta_3 - \begin{vmatrix} 2\varepsilon & 1 & 0 \\ 10\varepsilon^2 & 1 & 1 \\ 1 & 5\varepsilon & 1 \end{vmatrix} = -\big[1 + 0(1)\big] < 0 \quad \text{when } 0 < \varepsilon \ll 1,$$

so $f(z) \notin qH$.

LEMMA 2.1.7. *Let λ_0 be an eigenvalue of real matrix $A(a_{ij})_{n\times n}$. Suppose that its algebraic multiplicity is S, then λ_0 only corresponds to simple elementary divisor of $A(a_{ij})_{n\times n}$, if and only if* $\operatorname{rank}(\lambda_0 I - A) = n - s$.

PROOF. Let C^n be an n-dimensional compound vector space. Let $V_{\lambda_0} = \{\beta \mid \beta \in c^n,\ A\beta = \lambda_0\beta\}$ be eigensubspace with λ_0, $\tau = \dim V_{\lambda_0} = n - \operatorname{rank}(\lambda_0 I - A)$ is called the geometric dimension. So λ_0 only corresponds to simple elementary divisor of $A(a_{ij})_{n\times n}$ if and only if $s = \tau$, i.e., if and only if $n - \operatorname{rank}(\lambda_0 I - A) = s$ or $\operatorname{rank}(\lambda_0 I - A) = n - s$. □

LEMMA 2.1.8. *Let the order of Jordan block,*

$$J = \begin{bmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & a & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & a \end{bmatrix}_{m\times m}$$

be m. Then $m = 1$ if and only if

$$\operatorname{rank}(\alpha I_n - J) = \operatorname{rank}(\alpha I_n - J)^2. \tag{2.1.4}$$

PROOF. *Necessity*. When $m = 1$, $aI - J = 0$, so $(aI - J)^2 = 0$, i.e., $\operatorname{rank}(aI - J) = \operatorname{rank}(\alpha I - J)^2$.

Sufficiency. If $m > 1$, when

$$J - aI_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & 0 \\ \vdots & & & & 1 \\ 0 & \cdots & \cdots & & 0 \end{bmatrix}_{m\times m},$$

$$(J - aI_m)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \\ & & & \ddots & 0 \\ & & & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}_{m\times m}.$$

Obviously, $\operatorname{rank}(J - aI_m) > \operatorname{rank}(J - aI_m)^2$, which contradicts (2.1.4), so $m = 1$.

The proof is complete. □

COROLLARY 2.1.9. *Let* $J = \mathrm{diag}(J_1, \ldots, J_r)$ *and* λ_0 *be an eigenvalue of* J. *Then,* λ_0 *only corresponds to simple elementary divisor if and only if* $\mathrm{rank}(\lambda_0 I - J) = \mathrm{rank}(\lambda_0 I - J)^2$.

PROOF. Since $J^2 = \mathrm{diag}(J_i^2, \ldots, J_r^2)$, by Lemma 2.1.8 we know that the conclusion is true. □

COROLLARY 2.1.10. *Let* λ_0 *be an eigenvalue of* $A_{n\times n}$. *Then* λ_0 *only corresponds to simple elementary divisor if and only if*

$$\mathrm{rank}(\lambda_0 I - A) = \mathrm{rank}(\lambda_0 I - A)^2. \tag{2.1.5}$$

PROOF. By Jordan theorem, there exists an invertible matrix T such that

$$T^{-1}AT = \begin{bmatrix} J_1 & 0 \\ 0 & J_r \end{bmatrix} := J,$$

where J_k $(k = 1, \ldots, r)$ are Jordan blocks. This indicates that J has the same elementary divisor. λ_0 corresponds to simple elementary divisor of J if and only if λ_0 corresponds to simple elementary divisor of A. But since

$$\lambda_0 I = \lambda_0 I - T^{-1}(\lambda_0 I - A)T,$$
$$(\lambda_0 I - J)^2 = T^{-1}(\lambda_0 I - A)^2 T,$$

$\mathrm{rank}(\lambda_0 I - J) = \mathrm{rank}(\lambda_0 I - J)^2$ if and only if $\mathrm{rank}(\lambda_0 I - A) = \mathrm{rank}(\lambda_0 I - A)^2$. □

COROLLARY 2.1.11. *A has no pure imaginary eigenvalues or A has pure imaginary eigenvalues* (*including zero*) *which only correspond to simple elementary divisor of A if and only if* $\mathrm{rank}(i\omega I - A) = \mathrm{rank}(i\omega I - A)^2$ *for all* $\omega \in R^1$.

Based on the above lemmas and corollaries, we obtain the following theorems for $A_{n\times n}$ being quasi-stable.

THEOREM 2.1.12. *A is quasi-stable if and only if* $f_n(\lambda) \in qH$ *and* $\mathrm{rank}(i\omega I - A) = \mathrm{rank}(i\omega I - A)^2$.

THEOREM 2.1.13. *A is quasi-stable if and only if* $f_n(\lambda) \in qH$ *and for every real number* ω, *the linear algebraic equation* $(i\omega I - A)^2 x = 0$ *and* $(i\omega I - A)x = 0$ *have the same solution.*

THEOREM 2.1.14. *If* $f_n(\lambda) \in qH$ *and*

$$\left(f_n(\lambda), \frac{df_n(\lambda)}{d\lambda}\right) = 1, \tag{2.1.6}$$

i.e., $f_n(\lambda)$ *and* $\frac{df_n(\lambda)}{d\lambda}$ *have no common factors, then A is quasi-stable.*

PROOF. Equation (2.1.6) means that A has no multiple eigenvalues, so all eigenvalues correspond to simple elementary divisor of A. Thus, $f_n(\lambda) \in qH$ implies A being quasi-stable. □

Let ω be a real number and $f_n(i\omega) = U_n(\omega) + iV_n(\omega)$, where $U_n(\omega)$, $V_n(\omega)$ are polynomials with real coefficients. Then we have the following result.

COROLLARY 2.1.15. *If $f_n(\lambda) \in qH$ and*

$$\left(U_n(\omega), \frac{dU_n(\omega)}{d\omega}\right) = 1 = \left(V_n(\omega), \frac{dV_n(\omega)}{d\omega}\right), \tag{2.1.7}$$

then A is quasi-stable.

PROOF. It is easy to see from (2.1.7) that the pure imaginary eigenvalues of $f_n(\lambda)$ are all mono-roots. Hence, they are only correspond to simple elementary divisor and $f_n(\lambda) \in qH$. Therefore, A is quasi-stable. □

EXAMPLE 2.1.16. Consider quasi-stability of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Let $f_3(\lambda) = |\lambda I_3 - A| = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$.

(1) When $a_0^2 + a_1^2 \neq 0$, A is quasi-stable if and only if $f_3(\lambda) \in qH$.
(2) When $a_0^2 + a_1^2 = 0$, $a_2 \neq 0$ A is quasi-stable if and only if $f_3(\lambda) \in qH$ and $\operatorname{rank}(A) = 1$.
(3) When $a_0 = a_1 = a_2 = 0$, A is quasi-stable if and only if $A = 0$.

PROOF. (1) *Necessity* is obvious. *Sufficiency*. When $a_0 \neq 0$, by $f_3(\lambda) \in qH$ we know that $f_3(\lambda)$ has three negative real roots, or has one negative real root and two complex conjugate roots with nonpositive real parts. Hence all of them are mono-roots; when $a_0 = 0$, but $a_1 \neq 0$, $f_3(\lambda) \in qH$, $a_2^2 - 4a_1 > 0$ implies that $f_3(\lambda)$ has three nonequal, nonpositive real roots; $a_2^2 - 4a_1 < 0$ implies that $f_3(\lambda)$ has three different nonpositive real roots. While $a_2^2 - 4a_1 = 0$ implies that $f_3(\lambda)$ has three nonpositive real roots which are all mono-roots. Thus, A is quasi-stable. $f_3(\lambda) \in qH$ is obvious.

(2) When $a_1 = a_0 = 0$, $a_2 \neq 0$, $f_3(\lambda) = \lambda^2(a_2 + \lambda)$ has two multiroots $\lambda = 0$ and $\lambda = -a_2 < 0$.

If A is quasi-stable, then $f_3(\lambda) \in qH$ and $\lambda = 0$ corresponds to simple elementary divisor of A and the algebraic multiplicity number of $\lambda = 0$ is 2. By

Lemma 2.1.7 we know that

$$\dim V_0 = 3 - 2 = 1 = \operatorname{rank}(OI - A_{3\times 3}) = \operatorname{rank}(-A_{3\times 3})$$
$$= \operatorname{rank}(A_{3\times 3}).$$

On the other hand, if $f_3(\lambda) \in qH$, then $a_2 > 0$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = -a_2$ are three roots of $f_3(\lambda) = 0$. Then by $\operatorname{rank}(A) = 1$ and Lemma 2.1.1 we know that $\lambda = 0$ only corresponds to linear elementary divisor. Thus, A is quasi-stable.

(3) $f_3(\lambda) = \lambda^3$ has three multiroot $\lambda = 0$, which only corresponds to linear elementary divisor if and only if $\dim V_0 = 3 - 3 = 0 = \operatorname{rank}(A)$ and therefore if and only if $A = 0$. □

2.2. Sufficient conditions of Hurwitz matrix

By using the necessary and sufficient condition, theoretically, to check whether a matrix A is Hurwitz or a quasi-stable matrix becomes an algebraic problem. However, it becomes very involved when n is large, since it is very difficult to expand $\det(\lambda I_n - A)$. Thus, we need some simple sufficient conditions in practical designs. In this section, we present various algebraic sufficient conditions that guarantee A to be a Hurwitz matrix.

LEMMA 2.2.1. *Let $B(b_{ij})_{r\times r}$ be a real matrix, $b_{ij} \geqslant 0$ $(i \neq j)$. $g(t, t_0, c)$ and $z(t, t_0, c)$ are respectively, the solutions of*

$$\begin{cases} \frac{dy}{dt} \leqslant By + f(t), \\ y(t_0) = c, \end{cases} \tag{2.2.1}$$

and

$$\begin{cases} \frac{dz}{dt} = Bz + f(t), \\ z(t_0) = c, \end{cases} \tag{2.2.2}$$

where $f(t) = (f_1(t), \dots, f_n(t))^T \in C[I, R^n]$, $z, y \in R^n$. Then $y(t, t_0, c) \leqslant z(t, t_0, c)$, i.e., $y_i(t, t_0, c) \leqslant z_i(t, t_0, c)$ $\forall t \geqslant t_0$.

PROOF. Let $W(t) := z(t) - y(t)$, $U(t) := \frac{dW}{dt} - BW \geqslant 0$. Then, $W(t)$ satisfies the differential equation:

$$\begin{cases} \frac{dW}{dt} = BW + U(t), \\ W(t_0) = 0. \end{cases} \tag{2.2.3}$$

Obviously, one can choose a constant $k > 0$ such that $(B + kI) \geqslant 0$ (i.e., every element is nonnegative). Rewrite (2.2.3) as

$$\begin{cases} \frac{dW}{dt} = (-kI_n + B + kI_n)W + U(t), \\ W(t_0) = 0. \end{cases} \tag{2.2.4}$$

The solution of (2.2.4) is the continuous solution of the following integral equation:

$$W(t) = \int_{t_0}^{t} \left[e^{-kI_n(t-t_1)}[B + kI_n]W(t_1) + U(t_1) \right] dt_1. \tag{2.2.5}$$

By Picard step-by-step integration we find the solution of (2.2.5) from (2.2.4):

$$\left.\frac{dW}{dt}\right|_{t=t_0} = U(t_0) \geqslant 0.$$

One can take zero degree approximation $W^{(0)}(t) \geqslant 0$. Since $(B + kI_n) \geqslant 0$, so $W^{(1)}(t) \geqslant 0$. By the method of mathematical induction, we can prove that $W^{(m)}(t) \geqslant 0$ hold for all m. Thus,

$$\lim_{m\to\infty} W^{(m)}(t) = W(t) \geqslant 0,$$

i.e., $y(t, t_0, c) \leqslant z(t, t_0, c)$. □

Now, we rewrite (2.1.1) as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = \left(\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A_{rr} \end{bmatrix} + \begin{bmatrix} 0 & A_{12} & \cdots & A_{1r} \\ A_{21} & 0 & \cdots & A_{2r} \\ \vdots & & & \\ A_{r0} & \cdots & \cdots & 0 \end{bmatrix} \right) \times \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix}, \tag{2.2.6}$$

where $x_i \in R^{n_i}$, A_{ii} and A_{ij} are $n_i \times n_i$ and $n_i \times n_j$ matrices, respectively, satisfying

$$\sum_{i=1}^{r} n_i = n.$$

Consider the isolated subsystem:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix}. \tag{2.2.7}$$

THEOREM 2.2.2. *If there exist constants $M_i \geqslant 1$, $\alpha_i > 0$ such that*

$$\left\|e^{A_{ii}(t-t_0)}\right\| \leqslant M_i e^{-\alpha_i(t-t_0)}, \quad i = 1, 2, \ldots, r, \tag{2.2.8}$$

then the $r \times r$ matrix $B = (-\delta_{ij}\alpha_i + (1-\delta_{ij})M_i\|A_{ij}\|)_{r\times r}$ being a Hurwitz matrix implies that A is a Hurwitz matrix.

PROOF. We can write the solution of (2.2.6) as

$$x_i(t, t_0, x_0) = e^{A_{ii}(t-t_0)}x_{0_i} + \int_{t_0}^{t} e^{A_{ii}(t-t_1)} \sum_{j=1}^{r}(1-\delta_{ij})A_{ij}x_j(t_1, t_0, x_0)\,dt_1,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.2.9}$$

Further, we have

$$\begin{aligned} \left\|x_i(t, t_0, x_0)\right\| \leqslant{}& M_i\|x_{i0}\|e^{-\alpha_i(t-t_0)} + \int_{t_0}^{t} M_i e^{-\alpha_i(t-t_1)} \\ &\times \sum_{j=1}^{r}(1-\delta_{ij})\|A_{ij}\|\left\|x_j(t_1, t_0, x_0)\right\| dt_1, \quad i = 1, 2, \ldots, r. \end{aligned}$$

Let

$$y_i(t, t_0, y_0) := \int_{t_0}^{t} M_i e^{-\alpha_i(t-t_0)} \sum_{j=1}^{r}(1-\delta_{ij})\|A_{ij}\|\left\|x_j(t_1, t_0, x_0)\right\| dt_1,$$

then $y_i(t, t_0, y_0)$ satisfies

$$\begin{aligned} \frac{dy_i}{dt} \leqslant{}& -\alpha_i y_i + \sum_{j=1}^{r} M_i\|A_{ij}\|(1-\delta_{ij})y_i \\ &+ \sum_{j=1}^{r} M_i\|x_{j0}\|(1-\delta_{ij})e^{-\alpha_j(t-t_0)}\|A_{ij}\|M_j, \\ y_i(t_0) ={}& 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2.2.10}$$

Consider a system of comparison equations for (2.2.10):

$$\frac{dz_i}{dt} = -\alpha_i z_i + \sum_{j=1}^{r} M_i\|A_{ij}\|(1-\delta_{ij})z_i$$

$$
\begin{aligned}
&+\sum_{i=1}^{r} M_i \|x_{j0}\|(1-\delta_{ij})e^{-\alpha_i(t-t_0)}\|A_{ij}\|M_j,\\
&z_i(t_0)=0, \quad i=1,2,\ldots,n.
\end{aligned} \tag{2.2.11}
$$

Rewrite (2.2.11) as a vector form:

$$
\begin{cases} \frac{dz}{dt} = B(b_{ij})z + f(t),\\ z(t_0)=0, \end{cases} \tag{2.2.12}
$$

where

$$
\begin{aligned}
f(t) := \Bigg(&\sum_{j=2}^{r} M_1\|x_{j:}\|\|A_{1j}\|M_j e^{-\alpha_1(t-t_0)}, \ldots,\\
&\sum_{j=1}^{r-1} M_r\|x_{j0}\|\|A_{rj}\|M_j e^{-\alpha_r(t-t_0)}\Bigg)^T.
\end{aligned}
$$

Since B is a Hurwitz matrix, there exists constants $h \geqslant 1$, $\beta > 0$ such that

$$
\left\|e^{B(t-t_0)}\right\| \leqslant he^{-\beta(t-t_0)}.
$$

The solution of (2.2.11) has the form

$$
z(t,t_0,z_0) = \int_{t_0}^{t} e^{B(t-t_1)} f(t_1)\, dt_1. \tag{2.2.13}
$$

Thus, we have

$$
\left\|z(t)\right\| \leqslant \int_{t_0}^{t} he^{-\beta(t-t_1)} f(t_1)\, dt_1 = he^{-\beta t}\int_{t_0}^{t} e^{\beta t_1}\left\|f(t_1)\right\| dt_1.
$$

Further, with the fact

$$
\lim_{t\to\infty}\left\|f(t)\right\| = 0
$$

and using the L'Hospital rule, we obtain

$$
\lim_{t\to+\infty} h\int_{t_0}^{t} e^{-\beta(t-t_1)}\left\|f(t_1)\right\| dt_1 = \lim_{t\to+\infty} h\frac{e^{\beta t}\|f(t\|)}{\beta e^{\beta t}} = 0.
$$

Hence, we have

$$
\begin{aligned}
\left\|x_i(t,t_0,x_0)\right\| &\leqslant M_i\|x_{i0}\|e^{-\alpha_i(t-t_0)} + y_i(t,t_0,y_0)\\
&\leqslant M_i\|x_{i0}\|e^{-\alpha_i(t-t_0)} + z_i(t,t_0,z_0)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant M_i \|x_{i0}\| e^{-\alpha_i (t-t_0)} + \left\| z_i(t, t_0, z_0) \right\| \\
&\leqslant M_i \|x_{i0}\| e^{-\alpha_i (t-t_0)} + \int_{t_0}^{t} h e^{-\beta(t-t_1)} \left\| f(t_1) \right\| dt_1 \\
&\to 0 \quad \text{as } t \to +\infty, \ i = 1, 2, \ldots, n.
\end{aligned}
$$

This means that A is a Hurwitz matrix. □

EXAMPLE 2.2.3. Prove that the matrix

$$
A = \begin{bmatrix} -6 & 3 & 1/8 & -1/8 \\ -5 & 2 & 1/7 & -1/7 \\ 1/2 & 1 & -4 & 1 \\ 1 & 1/2 & 1 & -4 \end{bmatrix}
$$

is a Hurwitz matrix.

PROOF. Take

$$
\begin{aligned}
&A_{11} = \begin{bmatrix} -6 & 3 \\ -5 & 2 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}, \\
&A_{12} = \begin{bmatrix} 1/8 & -1/8 \\ 1/7 & -1/7 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 1/2 & 1 \\ 1 & 1/2 \end{bmatrix}.
\end{aligned}
$$

Then, $\lambda_1(A_{11}) = \lambda_1 = -3$, $\lambda_2(A_{11}) = \lambda_2 = -1$, $\lambda_1(A_{22}) = \tilde{\lambda}_1 = -5$, $\lambda_2(A_{22}) = \tilde{\lambda}_2 = -3$. Further, choose $t_0 = 0$. Then,

$$
\begin{aligned}
&e^{A_{11}(t)} = \begin{bmatrix} \frac{5}{2}e^{-3t} - \frac{3}{2}e^{-t} & -\frac{3}{2}e^{-3t} + \frac{3}{2}e^{-t} \\ \frac{5}{2}e^{-3t} - \frac{5}{2}e^{-t} & -\frac{3}{2}e^{-3t} + \frac{5}{2}e^{-t} \end{bmatrix}, \\
&e^{A_{22}(t)} = \begin{bmatrix} \frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-5t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} \end{bmatrix}, \\
&\left\| e^{A_{11}t} \right\|_m \leqslant 3e^{-t}, \quad \left\| e^{A_{22}t} \right\|_m \leqslant \frac{3}{2}e^{-3t}, \\
&\|A_{12}\|_m = \frac{2}{7}, \quad \|A_{21}\|_m = \frac{3}{2}, \\
&M_1 = 3, \quad M_2 = \frac{3}{2}, \quad \alpha_1 = 1, \quad \alpha_2 = 3.
\end{aligned}
$$

Thus, $B = \begin{bmatrix} -1 & 6/7 \\ 9/4 & -3 \end{bmatrix}$ is a Hurwitz matrix, and so is A. Here,

$$
\|A\|_m := \max_{1 \leqslant i \leqslant 4} \sum_{j=1}^{4} |a_{ij}|.
$$

□

In the following, for $A(a_{ij})_{2\times 2}$, we can directly give a formula for $e^{A(t-t_0)}$. Suppose $a_{11}a_{22} - a_{12}a_{12} \neq 0$.

Let

$$f_2(\lambda) = \det(\lambda I_2 - A) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$
$$\Delta := (a_{11} + a_{12})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = (a_{11} - a_{12})^2 + 4a_{12}a_{21}.$$

Then, we have the following formulas:

(1) when $\Delta > 0$,

$$e^{At} = \begin{bmatrix} -\frac{a_{22}-a_{11}-\sqrt{\Delta}}{2\sqrt{\Delta}} e^{\lambda_1 t} + \frac{a_{22}-a_{11}-\sqrt{\Delta}}{2\sqrt{\Delta}} e^{\lambda_2 t} & \frac{a_{12}}{\sqrt{\Delta}}(e^{\lambda_1 t} - e^{\lambda_2 t}) \\ \frac{a_{12}}{\sqrt{\Delta}}(e^{\lambda_1 t} - e^{\lambda_2 t}) & \frac{a_{22}-a_{11}+\sqrt{\Delta}}{2\sqrt{\Delta}} e^{\lambda_1 t} - \frac{a_{22}-a_{11}-\sqrt{\Delta}}{2\sqrt{\Delta}} e^{\lambda_2 t} \end{bmatrix};$$

(2) when $\Delta = 0$, $a_{12} \neq 0$,

$$e^{At} = \begin{bmatrix} 1 - \frac{a_{22}-a_{11}}{2} t & a_{22} t \\ -\frac{(a_{22}-a_{11})^2}{4a_{12}} t & 1 + \frac{a_{22}-a_{11}}{2} t \end{bmatrix} e^{\frac{a_{11}+a_{22}}{2} t};$$

(3) when $\Delta < 0$,

$$e^{At} = \begin{bmatrix} \cos\beta t - \frac{a_{22}-a_{11}}{2\beta} \sin\beta t & \frac{a_{12}}{\beta} \sin\beta t \\ \frac{a_{21}}{\beta} \sin\beta t & \cos\beta t + \frac{a_{22}-a_{11}}{2\beta} \sin\beta t \end{bmatrix} e^{\alpha t},$$

where $\beta = \frac{\sqrt{-\Delta}}{\alpha}$, $\alpha = \frac{a_{11}+a_{22}}{2}$.

COROLLARY 2.2.4. *If $a_{ii} < 0$ $(i = 1, 2, \dots, n)$, and matrix $a_{ij}\delta_{ij} + (1 - \delta_{ij}|a_{ij}|)_{n\times n}$ is a Hurwitz matrix, then $A = (a_{ij})_{n\times n}$ is a Hurwitz matrix.*

PROOF. Take $A_{ii} = a_{ii}$, $\|e^{A_{ii}t}\| = |e^{a_{ii}t}| = e^{-a_{ii}t}$, $M_i = 1$, $-\alpha_i = a_{ii}$. Then, the conditions in Theorem 2.2.2 are satisfied. Recall that the conditions of Corollary 2.2.4 are equivalent to condition (6) of M matrix (Definition 1.6.1). □

COROLLARY 2.2.5. *Assume $a_{ii} < 0$ $(i = 1, \dots, n)$. Then any of the following six conditions implies that A is a Hurwitz matrix*:

(1) $$\max_{1\leqslant j\leqslant n} \sum_{\substack{i=1\\ i\neq j}}^{n} \left|\frac{a_{ij}}{a_{jj}}\right| < 1;$$

(2) $$\max_{1\leqslant i\leqslant n} \sum_{\substack{j=1\\ j\neq i}}^{n} \left|\frac{a_{ij}}{a_{jj}}\right| < 1;$$

(3) $$\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(\left|\frac{a_{ij}}{a_{jj}}\right|\right)^2 < 1;$$

(4) $\displaystyle\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|\mu_i + \sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right| := \mu_j < 1, \quad j = 1, 2, \ldots, n;$

(5) $\displaystyle v^{(j)} = \sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|v^{(i)} + \max_{j+1\leqslant i\leqslant n}\left|\frac{a_{ij}}{a_{jj}}\right|, \quad j = 1, 2, \ldots, n, \quad \sum_{j=1}^{n} v^{(j)} < 1;$

(6) $\displaystyle j\left(\sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|^2\sigma_i^2 + \sum_{i=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right|^2\right) := \sigma_j^2, \quad j = 1, 2, \ldots, n,$

$\displaystyle\sum_{j=1}^{n}\sigma_j^2 = \sigma^2 < 1.$

PROOF. Any of the conditions (1), (2) and (3) implies $\rho(B) \leqslant \|B\| < 1$. By the equivalent conditions (5), (6) of M matrix (Definition 1.6.1), we know that the conclusion is true.

Consider the following nonhomogeneous linear algebraic equations:

$$\eta_j = \sum_{\substack{i=1\\ i\neq j}}^{n}\left|\frac{a_{ij}}{a_{jj}}\right|\eta_i + \omega_j \quad (\omega_j = \text{constant} > 0,\ j = 1, 2, \ldots, n). \tag{2.2.14}$$

By Gauss–Seidel iteration:

$$\eta_j^{(m)} = \sum_{i=1}^{j-1}\left|\frac{a_{ij}}{a_{jj}}\right|\eta_i^{(m)} + \sum_{j=j+1}^{n}\left|\frac{a_{ij}}{a_{jj}}\right|\eta_i^{(m-1)} + \omega_j, \tag{2.2.15}$$

one can show that any condition of (4), (5) and (6) is a sufficient condition for the convergence of (2.2.15). So (2.2.14) has unique solution. Take $\eta_j^{(0)} \geqslant 0$ ($j = 1, 2, \ldots, n$). Owing to $\omega_j > 0$, $|\frac{a_{ij}}{a_{jj}}| \geqslant 0$ $(i, j = 1, 2, \ldots, n,\ i \neq j)$,

$$\eta_j^{(m)} \geqslant \omega_j > 0 \quad (m = 1, 2, \ldots,\ j = 1, 2, \ldots, n).$$

Thus,

$$\tilde{\eta}_j := \lim_{m\to\infty}\eta_j^{(m)} \geqslant \omega_j > 0.$$

Further, we have

$$a_{jj}\tilde{\eta}_j + \sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}|\eta_i < 0.$$

By using the equivalent condition, A is a Hurwitz matrix. □

EXAMPLE 2.2.6. Check whether matrix

$$A = \begin{bmatrix} -8 & 1 & -2 & 2 \\ 6 & -6 & 0 & 1 \\ -4 & 6 & -8 & 0 \\ 12 & 0 & 0 & -10 \end{bmatrix}$$

is a Hurwitz matrix.

Obviously, the diagonal elements of A, $a_{ii} < 0$, $i = 1, 2, 3, 4$. Thus,

$$\mu_1 = \frac{1}{8} + \frac{2}{8} + \frac{2}{8} = \frac{5}{8} < 1,$$
$$\mu_2 = \frac{5}{8} \times 1 + \frac{1}{6} = \frac{19}{24} < 1,$$
$$\mu_3 = \frac{5}{8} \times \frac{1}{2} + \frac{19}{24} \times \frac{4}{3} = \frac{5}{16} + \frac{57}{96} = \frac{87}{96} < 1,$$
$$\mu_4 = \frac{12}{10} \times \frac{5}{8} = \frac{8}{16} < 1.$$

Hence, A satisfies condition (4) in Corollary 2.2.5. Therefore A is a Hurwitz matrix.

THEOREM 2.2.7. *If the following conditions*:

(1) $a_{ii} < 0$ $(i = 1, 2, \ldots, n)$ *and* $\det A \neq 0$; *and*
(2) *there exist constants* $c_i > 0$ $(i = 1, 2, \ldots, n)$ *such that*

$$c_j a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |c_i||a_{ij}| \leqslant 0, \qquad j = 1, 2, \ldots, n;$$

are satisfied, then A is a Hurwitz matrix.

PROOF. Let $C = \operatorname{diag}(c_1, c_2, \ldots, c_n)$. By the transformation $y = Cx$, the equation $\frac{dx}{dt} = Ax$ becomes

$$\frac{dy}{dt} = CAC^{-1}y := By.$$

$\det B = \det A \neq 0$ and the eigenvalues of the matrices A and B are the same. By the Gershgorin's circular disc theorem, all eigenvalues are on the following circular disc:

$$|\lambda - a_{jj}| \leqslant \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{|c_i a_{ij}|}{c_j} \leqslant |a_{jj}|.$$

The above inequality takes equality if and only if when $\lambda = 0$. But $\lambda = 0$ is not the eigenvalue of B. Thus,

$$|\lambda - a_{jj}| < |a_{jj}|,$$

i.e., $\operatorname{Re}\lambda(A) < 0$, so A is a Hurwitz matrix. □

THEOREM 2.2.8. *If there exist constants $\xi_i > 0$ $(i = 1, \ldots, n)$ such that the matrix $B := (\xi_i a_{ij} + \xi_j a_{ji})$ is negative definite, then A is a Hurwitz matrix.*

PROOF. Multiplying $2\xi_i x_i$ on both sides of the following equation:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j \tag{2.2.16}$$

yields

$$\xi_i \frac{dx_i^2}{dt} = \sum_{j=1}^{n} 2\xi_i a_{ij} x_i x_j. \tag{2.2.17}$$

Further, we have

$$\begin{aligned}\frac{d}{dt}\left(\sum_{i=1}^{n} \xi_i x_i^2\right) &= (x_1, x_2, \ldots, x_n) B (x_1, \ldots, x_n)^T \leqslant \lambda_m \sum_{i=1}^{n} x_i^2 \\ &\leqslant \frac{\lambda_m}{\max_{1\leqslant i\leqslant n} |\xi_i|} \sum_{i=1}^{n} \xi_i x_i^2,\end{aligned} \tag{2.2.18}$$

where λ_m is the maximum eigenvalue of B. Let

$$-\delta = \frac{\lambda_m}{\max_{1\leqslant i\leqslant n} |\xi_i|} < 0.$$

Thus, integrating (2.2.18) results in

$$\sum_{i=1}^{n} \xi_i x_i^2(t, t_0, x_0) \leqslant \sum_{i=1}^{n} \xi_i x_i^2(t_0, t_0, x_0) e^{-\delta(t-t_0)}.$$

So

$$\sum_{i=1}^{n} \xi_i x_i^2(t, t_0, x_0) \leqslant \frac{1}{\min_{1\leqslant i\leqslant n} \xi_i} \sum_{i=1}^{n} \xi_i x_i^2(t_0, t_0, x_0) e^{-\delta(t-t_0)}, \tag{2.2.19}$$

which shows that the zero solution of (2.2.16) is exponentially stable. Hence, A is a Hurwitz matrix. □

COROLLARY 2.2.9. *If any of the following conditions*:

(1) *there exist constant* $\xi_i > 0$, $i = 1, 2, \ldots, n$, *such that*

$$\sum_{\substack{i=1\\i\neq j}}^{n} |\xi_i a_{ij} + \xi_j a_{ji}| \leqslant -2\xi_i a_{ii}, \quad i = 1, 2, \ldots, n;$$

(2)

$$\sum_{\substack{j=1\\j\neq i}}^{n} |a_{ii}a_{ij} + a_{jj}a_{ji}| \leqslant 2a_{ii}^2 \quad \textit{and} \quad a_{ii} < 0, \quad i = 1, 2, \ldots, n;$$

(3)

$$\sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij} + a_{ji}| \leqslant -2a_{ii};$$

holds, then A *is a Hurwitz matrix.*

PROOF. It is easy to verify that any of the conditions (1), (2) and (3) implies that B is negative. □

EXAMPLE 2.2.10. Prove that

$$A = \begin{bmatrix} -4 & 3 & -6 & -3 & 2 \\ -3 & -5 & 3 & -4 & 3 \\ 5 & -4 & -4 & -5 & 2 \\ 3 & 5 & 4 & -5 & 1 \\ -2 & -3 & -1 & -2 & -5 \end{bmatrix}$$

is a Hurwitz matrix.

PROOF. Since $a_{ii} < 0$, $i = 1, 2, 3, 4, 5$, and

$$\sum_{\substack{j=2\\j\neq 1}}^{5} |a_{11}a_{1j} + a_{jj}a_{j1}| = 12 < 32 = 2a_{11}^2,$$

$$\sum_{\substack{j=1\\j\neq 2}}^{5} |a_{22}a_{2j} + a_{jj}a_{j2}| = 9 < 50 = 2a_{22}^2,$$

$$\sum_{\substack{j=1\\j\neq 3}}^{5} |a_{33}a_{3j} + a_{jj}a_{j3}| = 8 < 32 = 2a_{33}^2,$$

$$\sum_{\substack{j=1\\ j\neq 4}}^{5} |a_{44}a_{4j} + a_{jj}a_{j4}| = 13 < 50 = 2a_{44}^2,$$

$$\sum_{\substack{j=1\\ j\neq i=5}}^{4} |a_{55}a_{5j} + a_{jj}a_{j5}| = 10 < 50 = 2a_{55}^2,$$

condition (2) of Corollary 2.2.9 is satisfied. So A is a Hurwitz matrix. □

DEFINITION 2.2.11. If there exists a constant $h > 0$ such that

$$\max_{1\leqslant i\leqslant n} \operatorname{Re}\lambda_i(A) + h < 0,$$

A is called stable with degree h. By the above sufficient conditions of Hurwitz matrix A, we can obtain the degree of stability of A.

THEOREM 2.2.12.

(1) *If the conditions in Theorem 2.2.2 hold under the elements α_i of B replaced by $\alpha_i + h$ $(h > 0)$, then A has the degree of stability h.*
(2) *If any of the conditions in Corollary 2.2.4 or Corollary 2.2.5 holds under the element a_{ii} of $A(a_{ij})$ replaced by $a_{ii} + h$ $(h > 0)$, then A has degree of stability h.*
(3) *If the condition in Theorem 2.2.7 or Theorem 2.2.8 or Corollary 2.2.9 holds under the element a_{ii} of $A(a_{ij})$ replaced by $a_{ii} + h$ $(h > 0)$, then A has stability degree h.*

PROOF. Let $A = (a_{ij})_{n\times n}$, $\tilde{A} = ((a_{ij} + h)\delta_{ij} + (1 - \delta_{ij})a_{ij})$. Then, $\operatorname{Re}\lambda(\tilde{A}) = \operatorname{Re}\lambda(A) + h$, and any of the above conditions implies $\operatorname{Re}\lambda(\tilde{A}) < 0$. So $\operatorname{Re}\lambda(A) + h < 0$, i.e., $\operatorname{Re}\lambda(A) < -h < 0$. Hence, A has stability of degree h. □

2.3. A new method for solving Lyapunov matrix equation: $BA + A^T B = C$

In this section we introduce a new method for solving Lyapunov matrix equation.

LEMMA 2.3.1. *If there exists a symmetric, positive definite matrix $B(b_{ij})_{n\times n}$ such that the matrix $C := BA + A^T B$ is negative definite, then $A(a_{ij})_{n\times n}$ is a Hurwitz matrix.*

PROOF. Let

$$\underline{\lambda}(C) = \min_{1\leqslant i\leqslant n} \lambda_i(C), \qquad \bar{\lambda}(C) = \max_{1\leqslant i\leqslant n} \lambda_i(C),$$

be respectively the minimal and maximum eigenvalues of the symmetric matrix C. Consider function $V = x^T Bx$. Then

$$\min_{1\leqslant i\leqslant 0} \lambda_i(B)x^T x = \underline{\lambda}(B)x^T x \leqslant x^T Bx \leqslant \bar{\lambda}(B)x^T x = \max_{1\leqslant i\leqslant n} \lambda_i(B)x^T x$$

is true. Computing the derivative of $V = x^T Bx$ along the solution of (2.1.1), we have

$$\begin{aligned}
\frac{dV}{dt} &= x^T(BA + A^T B)x \\
&= x^T Cx \leqslant \bar{\lambda}(C)x^T x \\
&= \bar{\lambda}(C)\frac{\bar{\lambda}(B)}{\bar{\lambda}(B)}x^T x \\
&= \frac{\bar{\lambda}(C)}{\bar{\lambda}(B)}\bar{\lambda}(B)x^T x \leqslant \frac{\bar{\lambda}(C)}{\bar{\lambda}(B)}V(t).
\end{aligned} \tag{2.3.1}$$

Integrating (2.3.1) yields

$$V(t, t_0, x_0) \leqslant V(t_0)e^{(\bar{\lambda}(C)/\bar{\lambda}(B)(t-t_0)}.$$

Thus,

$$\underline{\lambda}(B)\sum_{i=1}^{n} x_i^2(t, t_0, x_0) \leqslant V(t, t_0, x_0) \leqslant V(x_0)e^{(\bar{\lambda}(C)/\bar{\lambda}(B)(t-t_0)}.$$

Then, we obtain

$$\sum_{i=1}^{n} x_i^2(t, t_0, x_0) \leqslant V(t_0)/\underline{\lambda}(B)e^{(\bar{\lambda}(C)/\bar{\lambda}(B)(t-t_0)}, \tag{2.3.2}$$

which means that A is a Hurwitz matrix.

$$BA + A^T B = C \tag{2.3.3}$$

is called Lyapunov matrix equation. □

For a given negative definite, symmetric matrix, it is interesting to know whether the Lyapunov matrix equation (2.3.3) has a symmetric, positive matrix solution $X = B$ or not. This problem has been studied by many scholars.

Now we introduce a new method. By using Kronecker product and pulling linear operator, we may directly transform a Lyapunov matrix equation (2.3.3) to a linear algebraic equation:

$$My = b. \tag{2.3.4}$$

Thus, we can use a simple method to solve (2.3.3).

DEFINITION 2.3.2. Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{j\times l}$. Then,

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & \cdots & \cdots & a_{mn}B \end{bmatrix}$$

is called a Kronecker product of A and B.

DEFINITION 2.3.3. Let $A = (a_{ij})_{m\times n}$,

$$\overrightarrow{A} := (a_{11}, a_{12}, \ldots a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, \ldots a_{m1}, \ldots, a_{mn})^T$$

is called a pulling linear operator.

From Definitions 2.3.2 and 2.3.3 one can easily prove that

(1) the pulling linear operator is linear operator, i.e., $\overrightarrow{A+B} = \overrightarrow{A} + \overrightarrow{B}$, $\overrightarrow{kA} = k\overrightarrow{A}$;
(2) if $A = (a_{ij})_{m\times m}$, $B = (b_{ij})_{n\times n}$, then

$$\det(A \otimes B) = (\det A)^m (\det B)^n; \tag{2.3.5}$$

(3) $A \otimes B$ has inverse matrix if and only if both A and B have inverse matrices, and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{2.3.6}$$

LEMMA 2.3.4. *Let $A = (a_{ij})_{m\times n}$. E_{ij} denotes $m \times n$ matrix with only (i, j) element as* 1 *and all other elements as* 0*. Let*

$$e_i = (\underbrace{0, \ldots, 0, 1}_{i}, \underbrace{0, \ldots, 0}_{n-i})^T_{1\times m},$$

where 1 *is located at the ith position. Then,*

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}, \tag{2.3.7}$$

$$Ae_i = (a_{1i}, a_{2i}, \ldots, a_{mi})^T, \tag{2.3.8}$$

$$e_i^T A = (a_{i1}, a_{i2}, \ldots, a_{in}), \tag{2.3.9}$$

$$E_{ij} = e_i e_j^T, \tag{2.3.10}$$

$$E_{ij} = e_i \otimes e_j. \tag{2.3.11}$$

PROOF. It can be shown by a direct computation. □

LEMMA 2.3.5. *Let A, B, C be $n \times m$, $m \times s$, $s \times t$ matrices, respectively. Then,*

$$\overrightarrow{ABC} = \big(A \otimes C^T\big)\vec{B}. \tag{2.3.12}$$

PROOF. First prove

$$\overrightarrow{AE_{ij}C} = \big(A \otimes C^T\big)\overrightarrow{E_{ij}}. \tag{2.3.13}$$

Since

$$\overrightarrow{AE_{ij}C} \overset{\text{by (2.3.10)}}{=} \overrightarrow{Ae_i e_j^r C} \underset{\text{by (2.3.4)}}{\overset{\text{by (2.3.8)}}{=}} \overrightarrow{(a_{1i}, a_{2i}, \ldots, a_{ni})^T (c_{j1}, \ldots, c_{jt})}$$
$$= \begin{pmatrix} A_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix} \otimes \begin{pmatrix} c_{j1} \\ \vdots \\ c_{jt} \end{pmatrix}, \tag{2.3.14}$$

we have

$$\big(A \otimes C^T\big)\overrightarrow{E_{ij}} = \big(A \otimes C^T\big)(e_i \otimes e_j) = Ae_i \otimes C^T e_j$$
$$= \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} \otimes \big(e_j^T C\big)^T = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} \otimes \begin{pmatrix} c_{j1} \\ \vdots \\ c_{jt} \end{pmatrix}. \tag{2.3.15}$$

Thus, (2.3.12) holds, and then

$$\overrightarrow{ABC} = \overrightarrow{A\left(\sum_{i=1}^{m}\sum_{j=1}^{s} b_{ij} E_{ij}\right)C} = \sum_{i=1}^{m}\sum_{j=1}^{s} b_{ij}\overrightarrow{AE_{ij}C}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}\big(A \otimes C^T\big)\overrightarrow{E_{ij}}$$
$$= \big(A \otimes C^T\big)\sum_{i=1}^{m}\sum_{j=1}^{s} b_{ij}\overrightarrow{E_{ij}}$$
$$= \big(A \otimes C^T\big)\vec{B}.$$

□

THEOREM 2.3.6. *Let A, C be $n \times n$ real matrices. Then, the following four propositions are equivalent*:

(1) *The Lyapunov matrix equation*:

$$A^T X + XA = C \tag{2.3.16}$$

has unique matrix solution $X = B$.

(2) *The linear equation*

$$\left(A^T \otimes I + I \otimes A^T\right)\vec{x} = \vec{C}$$

has unique solution, where I is $n \times n$ identity matrix.

(3) $\operatorname{rank}(A^T \otimes I + I \otimes A^T) = n^2$ *or* $\det(A^T \otimes I + I \otimes A^T) \neq 0$.

(4)

$$\prod_{i,j=1}^{r} (\lambda_i + \lambda_j) \neq 0,$$

where λ_i and λ_j are eigenvalues of A.

PROOF. Since $A^T X + XA = C \iff \overrightarrow{A^T X + XA} = \vec{C}$, and

$$\overrightarrow{A^T X + XA} = \overrightarrow{A^T X I} + \overrightarrow{I X A} = \left(A^T \otimes I + I \otimes A^T\right)\vec{X},$$

(1) $\iff$ (2), (2) $\iff$ (3) are proved.

The remaining is to prove (3) $\iff$ (4), i.e.,

$$\det\left(A^T \otimes I + I \otimes A^T\right) = \prod_{i,j=1}^{n} (\lambda_i + \lambda_j).$$

By using the Jordan theorem, there exists an $n \times n$ inverse matrix S such that

$$S^{-1} A^T S = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

$$\begin{aligned}
&(S \otimes S)^{-1}\left(A^T \otimes I + I \otimes A^T\right)(S \otimes S) \\
&\quad = \left(S^{-1} \otimes S^{-1}\right)\left(A^T \otimes I\right)(S \otimes S) + \left(S^{-1} \otimes S^{-1}\right)\left(I \otimes A^T\right)(S \otimes S) \\
&\quad = S^{-1} A^T S \otimes S^{-1} I S + \left(S^{-1} I S\right) \otimes \left(S^{-1} A^T S\right) \\
&\quad = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \\
&\qquad + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \otimes \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}
\end{aligned}$$

$$
= \begin{bmatrix}
\lambda_1 & * & \cdots & \cdots & \cdots & \cdots & * \\
0 & \ddots & & & & & \vdots \\
\vdots & & \lambda_1 & & & & \vdots \\
\vdots & & & \ddots & & & \vdots \\
\vdots & & & & \lambda_n & & \vdots \\
\vdots & & & & & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_n
\end{bmatrix}
+ \begin{bmatrix}
\lambda_1 & * & \cdots & \cdots & \cdots & \cdots & * \\
0 & \ddots & & & & & \vdots \\
\vdots & & \lambda_n & & & & \vdots \\
\vdots & & & \ddots & & & \vdots \\
\vdots & & & & \lambda_n & & \vdots \\
\vdots & & & & & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_n
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\lambda_1+\lambda_1 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & \lambda_1+\lambda_2 & & & & & & & & \vdots \\
\vdots & & \lambda_1+\lambda_n & & & & & & & \vdots \\
\vdots & & & \lambda_2+\lambda_1 & & & & & & \vdots \\
\vdots & & & & \cdots & & & & & \vdots \\
\vdots & & & & & \lambda_2+\lambda_n & & & & \vdots \\
\vdots & & & & & & \ddots & & & \vdots \\
\vdots & & & & & & & \lambda_n+\lambda_1 & & \vdots \\
\vdots & & & & & & & & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_n+\lambda_n
\end{bmatrix}.
$$

Therefore,

$$
\det\bigl(A^T \otimes I + I \otimes A^T\bigr) = \det(S \otimes S)^{-1}\bigl(A^T \otimes I + I \otimes A^T\bigr),
$$

$$
(S \otimes S) = \prod_{i,j=1}^{n} (\lambda_i + \lambda_j).
$$

The proof of Theorem 2.3.6 is complete. $\square$

THEOREM 2.3.7.

(1) *The Lyapunov matrix equation* (2.3.16) *has infinite number of solutions if and only if*

$$\operatorname{rank}\big(A^T \otimes I + I \otimes A^T, \vec{C}\big) = \operatorname{rank}\big(A^T \otimes I + I \otimes A^T\big) < n^2.$$

(2) $A^T X + XA = C$ *has no solution if and only if*

$$\operatorname{rank}\big(A^T \otimes I + I \otimes A^T, C\big) > \operatorname{rank}\big(A^T \otimes I, I \otimes A^T\big).$$

PROOF. A similar method to Theorem 2.3.6 can be used here and thus omitted. □

THEOREM 2.3.8. *If A is a Hurwitz matrix, then*

(1) *the solution of the Lyapunov matrix equation* (2.3.16) *is unique*;
(2) *for a given arbitrary symmetric negative matrix C, there exists a unique symmetric positive matrix B satisfying $A^T B + BA = -C$ and*

$$V(x) = x^T Bx = \frac{1}{\det \Delta} \begin{vmatrix} 0 & X \\ \vec{C}^T & \Delta \end{vmatrix},$$

where

$$\begin{aligned} \Delta &= A^T \otimes I + I \otimes A^T, \\ X &= (X_1, X_2, \ldots, X_n), \\ X_1 &= \big(x_1^2, 2x_1x_2, \ldots, 2x_1x_n\big), \\ X_2 &= \big(0, x_2^2, \ldots, 2x_2x_n\big), \\ &\cdots \\ X_n &= \big(0, \ldots, 0, x_n^2\big). \end{aligned}$$

PROOF. (1) Since A is a Hurwitz matrix,

$$\prod_{i,j=1}^{n} (\lambda_i + \lambda_j) \neq 0.$$

By using condition (2) of Theorem 2.3.7, for any symmetric negative matrix, Lyapunov matrix equation:

$$A^T X + XA = -C \tag{2.3.17}$$

has unique solution $X = B$, i.e.,

$$A^T B + BA = -C.$$

By $C = C^T$, we have $A^T B^T + B^T A = -C$. This means that B^T is also a solution. Thus $B = B^T$, and let

$$B_1 := \int_0^\infty e^{tA^T} C e^{tA} dt,$$

and then B_1 is symmetric positive definite.

(2) By uniqueness, $B_1 = B$. Let $B = (b_{ij})_{n\times n}$. Form (2.3.17) we have

$$\left(A^T \otimes I + I \otimes A^T\right)\vec{B} = -\vec{C}. \tag{2.3.18}$$

Since $\det \Delta = \det(A^T \otimes I + I \otimes A^T) \neq 0$, by the Gramer law, the solution of (2.3.18) can be written as

$$b_{ij} = \frac{\det \Delta_{ij}}{\det \Delta} \quad (i, j = 1, 2, \dots, n), \tag{2.3.19}$$

where Δ_{ij} is formed from Δ with its jth column replaced by $\vec{C}$. Hence,

$$V = x^T B x = \sum_{i,j=1}^n b_{ij} x_i x_j.$$

On the other hand,

$$V = \sum_{i,j=1}^n b_{ij} x_i x_i = \frac{1}{\det} \begin{vmatrix} 0 & X \\ \vec{C} & \Delta \end{vmatrix}. \tag{2.3.20}$$

□

EXAMPLE 2.3.9. Consider system

$$\begin{cases} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2, \\ \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2, \end{cases}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is symmetric, positive definite,

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Then, by (2.3.20) we have

$$V = \frac{1}{\det \varDelta} \begin{vmatrix} 0 & x_1^2 & 2x_1x_2 & 0 & x_2^2 \\ c_{11} & 2a_{11} & a_{21} & a_{21} & 0 \\ c_{12} & a_{12} & a_{11} + a_{22} & 0 & a_{21} \\ c_{21} & c_{12} & 0 & a_{11} + a_{22} & a_{21} \\ c_{22} & 0 & a_{12} & a_{12} & 2a_{22} \end{vmatrix},$$

where

$$\varDelta = \begin{bmatrix} 2a_{11} & a_{21} & 0 & 0 \\ a_{12} & a_{11} + a_{22} & 0 & a_{21} \\ a_{12} & 0 & a_{11} + a_{22} & a_{21} \\ 0 & a_{12} & a_{12} & 2a_{22} \end{bmatrix}.$$

EXAMPLE 2.3.10.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$C = \begin{bmatrix} c_{11} & c_{11} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = C^T.$$

Then

$$V = \frac{1}{\det \varDelta}$$

$$\times \begin{vmatrix} 0 & x_1^2 & 2x_1x_2 & 2x_1x_3 & 0 & x_2^2 & 2x_2x_3 & 0 & 0 & x_2^3 \\ c_{11} & 2a_{11} & a_{21} & a_{31} & a_{21} & 0 & 0 & a_{31} & 0 & 0 \\ c_{12} & c_{12} & a_{11}+a_{22} & a_{32} & 0 & a_{21} & 0 & 0 & a_{31} & 0 \\ c_{13} & a_{13} & a_{23} & a_{21}+a_{33} & 0 & 0 & a_{21} & 0 & 0 & a_{31} \\ c_{21} & a_{12} & 0 & 0 & a_{11}+a_{22} & a_{21} & a_{31} & a_{32} & 0 & 0 \\ c_{22} & 0 & a_{12} & 0 & a_{12} & 2a_{22} & a_{32} & 0 & a_{32} & 0 \\ c_{23} & 0 & 0 & a_{12} & a_{13} & a_{23} & a_{22}+a_{33} & 0 & 0 & a_{32} \\ c_{31} & a_{13} & 0 & 0 & a_{23} & 0 & 0 & a_{11}+a_{33} & a_{21} & a_{31} \\ c_{32} & 0 & a_{13} & 0 & 0 & a_{23} & 0 & a_{12} & a_{22}+a_{33} & a_{32} \\ c_{33} & 0 & 0 & a_{13} & 0 & 0 & a_{23} & a_{13} & a_{23} & a_{33} \end{vmatrix},$$

where

$$\varDelta = A^T \otimes I + I \otimes A^T.$$

2.4. A simple geometrical NASC for Hurwitz matrix

We again consider

$$f_n(\lambda) = \det(\lambda I - A) := \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0. \tag{2.4.1}$$

According to Cauchy's argument principle, the following geometrical method for judging Hurwitz matrix was obtained [326].

THEOREM 2.4.1. *Suppose that $f_n(\lambda)$ has no pure imaginary roots. Then, A is a Hurwitz matrix if and only if when ω varies from* 0 *to* $+\infty$*, the argument of $f(i\omega)$ increases to $n\varphi = \frac{\pi}{2}$, along the counter clock-wise direction.*

The proof of this theorem is very complex and checking the condition is difficult [98], because it contains an infinite interval $[0, +\infty)$.

Wang [396] derived a simple sufficient and necessary condition for $f_n(\lambda)$ being stable, using the boundedness and conjugate character of the zero points of $f_n(\lambda)$ with real coefficients.

LEMMA 2.4.2. *All zero points of $f_n(\lambda)$ lie inside the following circle*:

$$|\lambda| < \rho := 1 + \max_{0 \leqslant i \leqslant n-1} |a_i|.$$

PROOF. $\forall \lambda_0$, let

$$|\lambda_0| \geqslant 1 + \max_{0 \leqslant i \leqslant n-1} |a_i|.$$

Then, we have

$$\begin{aligned}
\left|f(\lambda_0)\right| &\geqslant \left|\lambda_0^n\right| - \left[|a_{n-1}|\left|\lambda_0^{n-1}\right| + \cdots + |a_0|\right] \\
&\leqslant \left|\lambda_0^n\right| - \max_{0 \leqslant i \leqslant n-1} |a_i|\left[1 + |\lambda_0| + \cdots + \left|\lambda_0^{n-1}\right|\right] \\
&= \left|\lambda_0^n\right| - \max_{0 \leqslant i \leqslant n-1} |a_i|\left[\frac{|\lambda_0^n| - 1}{|\lambda_0| - 1}\right] \\
&\geqslant \left|\lambda_0^n\right| - \max_{0 \leqslant i \leqslant n-1} |a_i|\frac{|\lambda_0^n| - 1}{\max_{0 \leqslant i \leqslant n-1} |a_i|} \\
&\geqslant 1 \neq 0.
\end{aligned}$$

So all zero points of $f_n(\lambda)$ lie inside the circle:

$$|\lambda| < \rho = 1 + \max_{0 \leqslant i \leqslant n-1} |a_i|.$$

Now we construct a circle S, centered at $(0, 0)$ with radius

$$\rho = 1 + \max_{0 \leqslant i \leqslant n-1} |a_i|.$$

Then, by Lemma 2.3.1 all zero points of $f_n(\lambda)$ lie inside this circle. □

Next, consider a quarter of circle $\overparen{A(\rho, 0)\, B(0, i\rho)\, 0}$, as shown in Figure 2.4.1. Let

$$\varphi_i = \mathop{\Delta}_{\widehat{ABO}} \operatorname{Arg} f_n(\lambda)$$

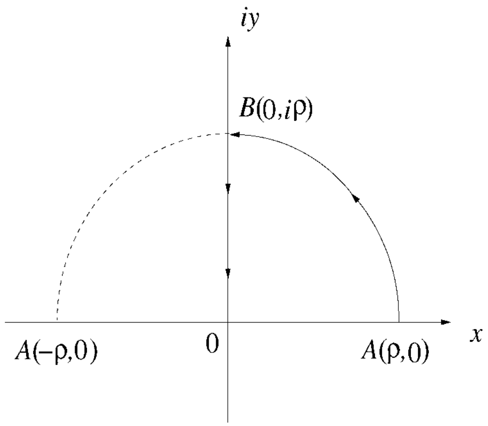

Figure 2.4.1. The polynomial variable varied on the one-quarter circle $\widehat{AOB}$, centered at the origin with radius ρ.

denote the increment of argument φ when λ is varied along $\widehat{AB}$ and imaginary axis $\overrightarrow{B0}$ from A to 0.

THEOREM 2.4.3. *Let $f_n(i\omega) \neq 0$, $\omega \in [0, B]$. Then*

$$\Phi := \underset{\widehat{ABO}}{\Delta} \operatorname{Arg} f_n(\lambda) = k\pi$$

if and only if $k = 0$. A is a Hurwitz matrix if and only if $0 < k \leqslant n$ and $f_n(\lambda)$ has k zero points with positive real parts.

PROOF. Let $f_n(\lambda)$ have $2p$ complex zero points z_j, $\bar{z}_j$ $(j = 1, 2, \dots, p)$ and q real zero points x_r $(r = 1, 2, \dots, q)$. Every zero is counted with multiplicity. Then $2p + q = n$.

By Lemma 2.3.1, $|z_i| < \rho$, $|x_r| < \rho$, $i = 1, \dots, p$, $r = 1, 2, \dots, q$. Decompose $f_n(\lambda)$ into linear factors:

$$f_n(\lambda) = \prod_{j=1}^{p} (\lambda - z_j)(\lambda - \bar{z}_j) \prod_{k=1}^{q} (\lambda - r_k).$$

Hence,

$$\begin{aligned}\Phi &:= \underset{\widehat{ABO}}{\Delta} \operatorname{Arg} f_n(\lambda) \\ &= \sum_{j=1}^{P} \underset{\widehat{ABO}}{\Delta} \big[\operatorname{Arg}(\lambda - z_j)\big] + \sum_{j=1}^{P} \underset{\widehat{ABO}}{\Delta} \big[\operatorname{Arg}(\lambda - \bar{z}_j)\big]\end{aligned}$$

$$+\sum_{k=1}^{q} \underset{\widehat{ABO}}{\Delta}\big[\operatorname{Arg}(\lambda - r_k)\big],$$

where $\operatorname{Arg}\lambda = \arg\lambda + 2k\pi$ $(k = 1, 2, \ldots)$ is a multi-valued function in $(-\pi < \arg\lambda \leqslant \pi)$. When $\lambda = \rho$, let $\operatorname{Arg}\lambda = \arg\lambda$.

Let r_{k_0} be any real negative number. Consider the variable of the argument of $\lambda - r_{k_0}$ along $\widehat{ABO}$ (see Figure 2.4.2),

$$\underset{\widehat{ABO}}{\Delta} \operatorname{Arg}(\lambda - r_{k_0}) = \underset{\widehat{AB}}{\Delta} \operatorname{Arg}(\lambda - r_{k_0}) + \underset{\widehat{BO}}{\Delta} \operatorname{Arg}(\lambda - r_{k_0}) = 0 - 0 = 0.$$

Let z_{j0} and $\bar{z}_{j0}$ are two conjugate complex zero points with negative real part. By the symmetry, one can prove that the triangle $\Delta A z_{jO} O \cong \Delta A \bar{z}_{jO} O$. Thus, $|\angle A z_{jO} O| = |\angle A \bar{z}_{j0} O|$, and also

$$\begin{aligned}
&\underset{\widehat{ABO}}{\Delta} \operatorname{Arg}(\lambda - z_{j0}) + \underset{\widehat{ABO}}{\Delta} \operatorname{Arg}(\lambda - \bar{z}_{j0}) \\
&\quad = +|\angle A z_{j0} B| - |\angle B z_{j0} O| + +|\angle A \bar{z}_{j0} B| - |\angle B \bar{z}_{j0} O| \\
&\quad = -|\angle A z_{j0} O| + |\angle A \bar{z}_j O| \\
&\quad = -\theta + \theta = 0,
\end{aligned}$$

which is shown in Figure 2.4.3.

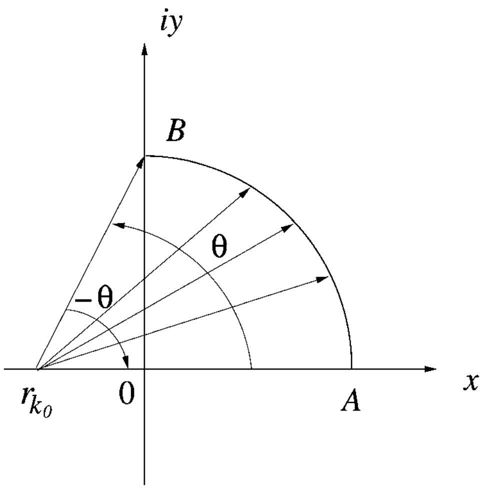

Figure 2.4.2. The case having negative real root x_{k_0}.

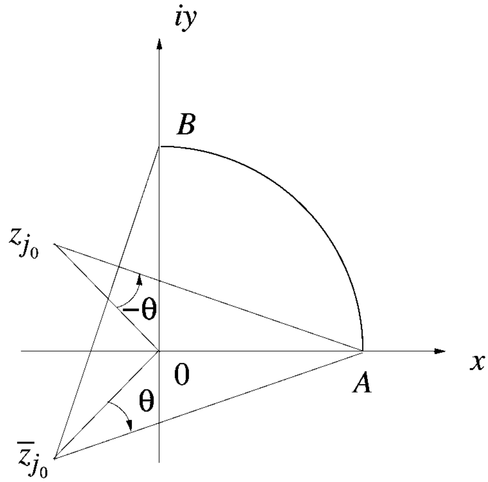

Figure 2.4.3. The case having a complex conjugate pair with negative real part.

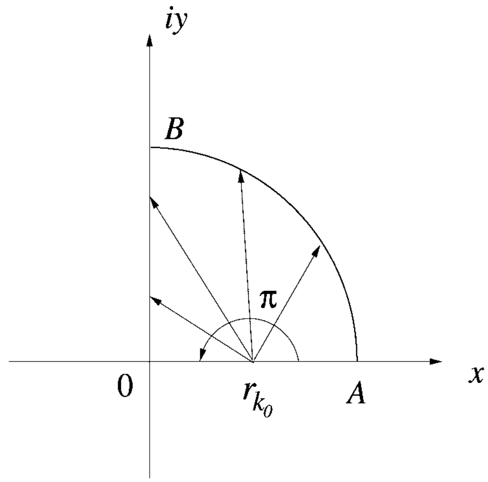

Figure 2.4.4. The case having positive real roots.

Let $r_{k_0} > 0$ be any positive zero point of $f_n(\lambda)$. Consider variation of argument $\lambda - r_{k_0}$ along $\widehat{ABO}$ (see Figure 2.4.4) to obtain

$$\underset{\widehat{ABO}}{\Delta} \operatorname{Arg}(\lambda - r_{k_0}) = \pi.$$

Let $z_{j_1}, \bar{z}_{j_1}$ be any two zero points with positive real parts of $f_n(\lambda)$. Then, $Az_{i_1}O \cong A\bar{z}_{j_1}O$, and also $|\angle Az_{j_1}O| = |\angle \bar{z}_{ji}O|$. Further

$$\underset{\widehat{ABO}}{\Delta} \operatorname{Arg}(A - z_{j_1}) + \underset{\widehat{ABO}}{\Delta} \operatorname{Arg}(\lambda - \bar{z}_{j_1}) = 2\pi - |\angle A\bar{z}_{j_1}O| + |\angle A\bar{z}_{j_1}O|$$
$$= 2\pi.$$

Based on the above analysis, we obtain

$$\underset{\widehat{ABO}}{\Delta} (\lambda - z_j)(\lambda - \bar{z}_j) = \begin{cases} 2\pi, & \text{when } \operatorname{Re} z_j > 0, \\ 0, & \text{when } \operatorname{Re} z_j < 0, \end{cases}$$

$$\underset{\widehat{ABO}}{\Delta} (\lambda - r_k) = \begin{cases} \pi, & \text{when } r_k > 0, \\ 0, & \text{when } r_k < 0. \end{cases}$$

Thus, A is a Hurwitz matrix if and only if

$$\Phi = \underset{\widehat{ABO}}{\Delta} \operatorname{Arg} f_n(\lambda) = k\pi, \quad k = 0,$$

and $f_n(\lambda)$ has k zero points with positive real parts if $0 < k \leqslant n$. This case is depicted in Figure 2.4.5. □

Let

$$\psi := \underset{\widehat{OBA_1}}{\Delta} \operatorname{Arg} f_n(\lambda).$$

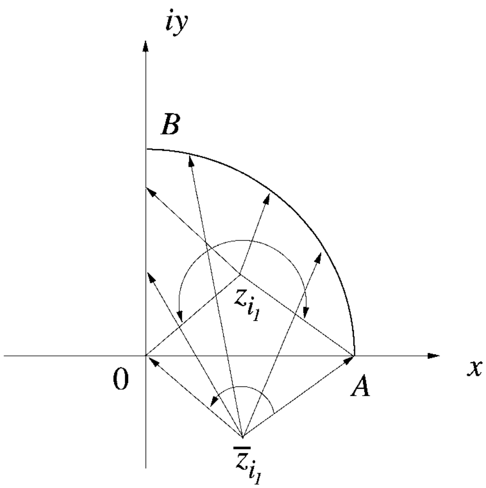

Figure 2.4.5. The case having a complex conjugate pair with positive real part.

By inspecting the increment of argument when λ is varied along $\widehat{OB}$, $\widehat{AB_1}$ from O to A_1, we have the following theorem.

THEOREM 2.4.4. *Let $f_n(i\omega) \neq 0$, where $\omega \in [0, B]$.*

$$\psi = \underset{\widehat{OBA_1}}{\Delta} \operatorname{Arg} f_n(\lambda) = m\pi,$$

if and only if $m = n$. A is a Hurwitz matrix if and only if $m < n$, $f_n(\lambda)$ has $n - m$ zero points with positive real parts.

One can follow the proof of Theorem 2.4.3 to prove Theorem 2.4.4.

REMARK 2.4.5. To check $f_n(i\omega) \neq 0$, $\omega \in [0, B]$, one only needs to check $(u(\omega), v(\omega)) = 1$, where $f_n(i\omega) = u(\omega) + iv(\omega)$. Later, we will use the division algorithm to check this condition.

This method can be generalized to obtain the degree of stability of A. Moving the half circle (see Figure 2.4.6) $A(p, 0)$-$B(0, ip)$-$A(-p, 0)$ to left by h, we can construct a new circle, centered at O' with the radius $r = |\widehat{O'A}|$.

THEOREM 2.4.6. *Suppose that $f_n(\lambda)$ has no zero points on $O'B^*$. Let*

$$\Phi := \underset{\widehat{AB^*O}}{\Delta} \operatorname{Arg} f_n(\lambda) = k_1\pi$$

(i.e., A has stability degree h), then $\operatorname{Re}\lambda(A_{n\times n}) < -h < 0$ *if and only if $k_1 = 0$.*

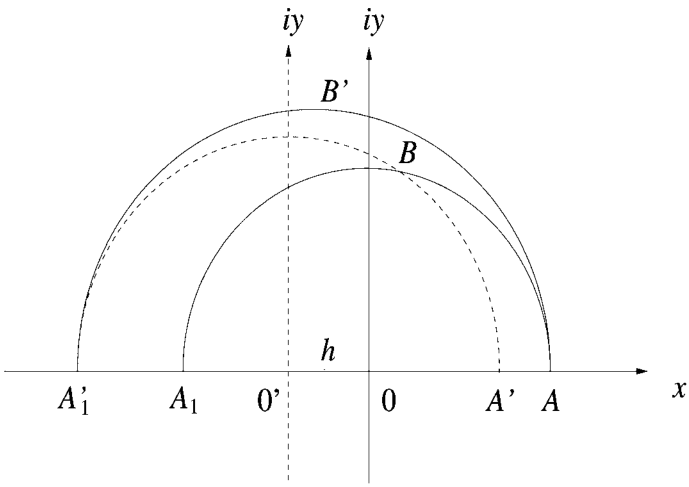

Figure 2.4.6. Demonstration of the degree of stability.

THEOREM 2.4.7. *Assume that $f_n(\lambda)$ has no zero points on $O'B^*$. Let*

$$\psi := \underset{\widehat{OB^*A_1'}}{\Delta} f_n(\lambda) = m_1\pi,$$

then A has stability degree $h > 0$ if and only if $m_1 = n$.

EXAMPLE 2.4.8. Discuss the stability of $f_3(z) = z^3 + 3z^2 + 4z + 2.1$. Take $\rho = 5$, then

$$f_1(\theta) = 5^3\cos 3\theta + 3\times 5^2\cos 2\theta + 20\cos\theta + 2.1 := u,$$
$$f_2(\theta) = 5^3\sin 3\theta + 3\times 5^2\cos 2\theta + 20\sin\theta + 2.1 := v,$$
$$\tilde{f}_1(\omega) = -3\omega^2 + 2.1 := u,$$
$$\tilde{f}_2(\omega) = -\omega^3 + 4 := v.$$

θ°	0	10	20	30	40	50	60	70	80	90				
f_1	$+$	$+$	$+$	$+\surd$	$-$	$-$	$-$	$-$	$-$	$-$				
f_2	$0\surd$	$+$	$+$	$+$	$+$	$+$	$+$	$+\surd$	$-$	$-$				
ω	5	4.6	4.2	3.8	3.4	3	2.6	2.2	1.8	1.4	1.0	0.6	0.2	0
$\tilde{f}_1$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-\surd$	$+$	$+$	$+$
$\tilde{f}_2$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-\surd$	$+$	$+$	$+$	$+$	$+\surd$	0

where $\surd$ denotes varying sign variables. So the characteristic curve of $f_3(z)$, as depicted in Figure 2.4.7, shows $k = 0$. So $f_3(z)$ is Hurwitz stable.

EXAMPLE 2.4.9. Check the stability of

$$f_4(z) = z^4 + 3z^3 + 5z^2 + 4z + 2.$$

Taking $\rho = 6$ results in

$$f_1 = 6^4\cos 4\theta + 3\times 6^3\cos 3\theta + 5\times 6^2\cos 2\theta + 4\times 6\cos\theta + 2 := u,$$
$$f_2 = 6^4\sin 4\theta + 3\times 6^3\sin 3\theta + 5\times 6^2\sin 2\theta + 4\times 6\sin\theta + 2 := u,$$
$$\tilde{f}_1 = \omega^4 - 5\omega^2 + 2 := u,$$
$$\tilde{f}_2 = -3\omega^3 + 4\omega := v.$$

θ°	0	10	20	30	40	50	60	70	80	90			
f_1	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-\surd$	$+$	$+$			
f_2	0	$+$	$+$	$+$	$+$	$+\surd$	$-$	$-$	$-$	$-$			
ω	6	5.5	5	4.5	4	3.5	3	2.5	2	1.5	1	0.5	0
$\tilde{f}_1$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+\surd$	$-$	$-$	$-\surd$	$+$	$+$
$\tilde{f}_v$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-\surd$	$+$	$+\surd$	0

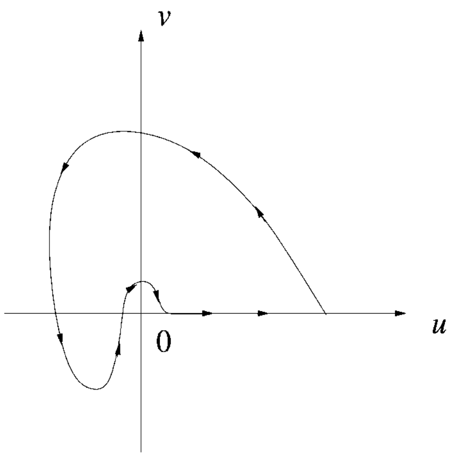

Figure 2.4.7. Characteristic curve for Example 2.4.8.

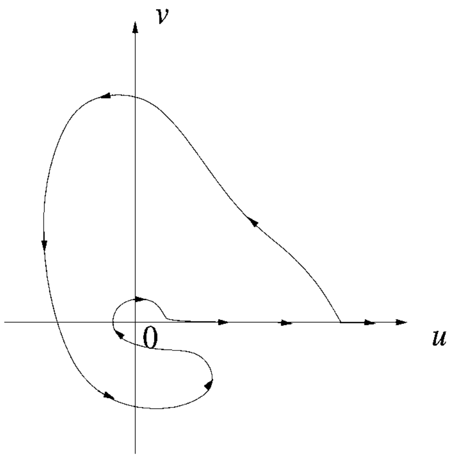

Figure 2.4.8. The characteristic curve for Example 2.4.9.

the characteristic curve $f_4(\xi)$ (see Figure 2.4.8) shows $k = 0$. Hence, $f_4(z)$ is stable.

2.5. The geometry method for the stability of linear control systems

Frequency-domain approach of classical control theory, the Nyquist criterion, is a method widely used in control field. It has sound physical background and

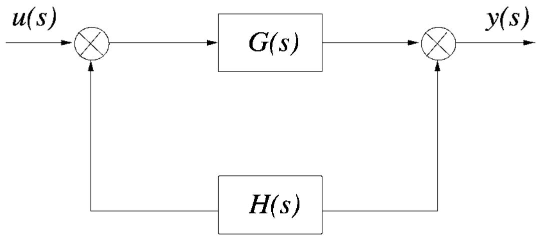

Figure 2.5.1. Illustration of input–output.

better geometrical explanation. Moreover, it is effective for both open-loop and closed-loop control systems. Especially, it can be used to determine the stability of closed-loop feedback systems according to the features of the open-loop transfer function. However, this method requires constructing a so-called Nyquist characteristic curve of a rational function on $(-\infty, +\infty)$. For infinite interval, it is very hard to construct such characteristic curves in practice. Using the important information of the real-coefficients, such as the boundedness and conjugacy of the zero and the apices, an improved geometrical criterion was developed, with which the domain of the frequency characteristic curve is changed to $[0, \frac{\pi}{2}]$ and $[0, \rho]$. So calculation is significantly simplified. Moreover, a new geometrical necessary and sufficient condition is obtained. The details of the improved geometrical criterion are introduced as follows.

Consider a canonical linear system with feedback control, as shown in Figure 2.5.1, where $G(s)$ is a transfer function of forward loop, and $H(s)$ is the transfer function of the feedback loop. $G(s)H(s)$ is called system transfer function or open-loop transfer function.

$$W(s) = \frac{Y(s)}{u(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{2.5.1}$$

is called the transfer function of this control system, where

$$G(s) := \frac{N_1(s)}{D_1(s)}, \quad H(s) := \frac{N_2(s)}{D_2(s)}.$$

Let

$$\begin{aligned}
f(s) &:= D_1(s)D_2(s) := (s - p_1)(s - p_2)\cdots(s - p_m) \\
&=: b_m + b_{m-1}s + \cdots + s^m, \\
F(s) &:= 1 + G(s)H(s) \\
&:= \frac{D_1(s)D_2(s) + N_1(s)N_2(s)}{D_1(s)D_2(s)} \\
&:= \frac{(s - s_1)(s - s_2)\cdots(s - s_n)}{(s - p_1)(s - p_2)\cdots(s - p_m)}
\end{aligned}$$

$$:= \frac{a_n + a_{n-1}s + \cdots + s^n}{b_m + b_{m-1}s + \cdots + s^m}$$
$$:= \frac{F_1(s)}{f(s)},$$

where s_i, ρ_i denote the zero points and poles of $F(s)$, respectively. Let

$$\rho_1 = 1 + \max_{1\leqslant i\leqslant n} |a_i|, \quad \rho_2 = 1 + \max_{1\leqslant i\leqslant m} |b_i|, \quad \rho = \max(\rho_1, \rho_2).$$

From Lemma 2.4.2, $|s_i| < \rho_1$ $(i = 1, \ldots, m)$,

$$|s_j| < \rho_2, \quad j = 1, 2, \ldots, n.$$

Following the method in Section 2.4 we construct a circle S on the complex plane, centered at (0, 0) with the radius $\rho = \max(\rho_1, \rho_2)$.

THEOREM 2.5.1. *Let $f(i\omega) \neq 0$, $\omega \in [0, \rho]$. Then, $f(s)$ has k zero points with positive real parts if and only if*

$$\Phi := \mathop{\Delta}_{\widehat{ABO}} \operatorname{Arg} f(s) = k\pi \leqslant m\pi.$$

Hence the open-loop control system is stable if and only if $k = 0$.

The proof is similar to proof of Theorem 2.4.3. If in Figure 2.5.2, let $\rho = \rho_2$, then for the closed-loop control system, we have a similar result.

THEOREM 2.5.2. *If $F_1(i\omega) \neq 0$, $\omega \in [0, \rho_2]$ and $(F_i(s), f(s)) = 1$. Then, $F_1(s)$ has zero points with positive real parts if and only if*

$$\psi := \mathop{\Delta}_{\widehat{ABO}} \operatorname{Arg} F_1(s) = l\pi \leqslant n\pi.$$

Hence the closed-loop control system is stable if and only if $l = 0$.

If in Figure 2.5.2, $\rho = \max(\rho_1, \rho_2)$, by Theorems 2.4.1 and 2.4.3, we further obtain the following theorem.

THEOREM 2.5.3. *Suppose that $F(s)$ has no zero points and poles on line (OB) and $F(s)$ has q poles with $\operatorname{Re} s > 0$. Then, $F(s)$ has p zero points with $\operatorname{Re} s > 0$ if and only if*

$$k = \frac{1}{\pi} \mathop{\Delta}_{\widehat{ABC}} \operatorname{Arg} F(s) = p - q.$$

Hence the closed-loop control system is stable if and only if $k = -q$ $(p = 0)$. The open-loop control system and closed-loop control system are stable if and only if $p = q = k = 0$.

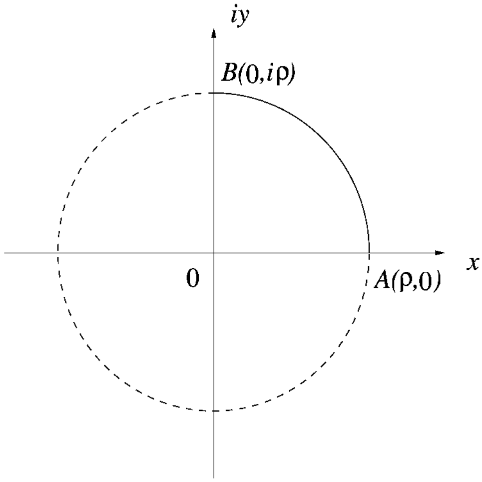

Figure 2.5.2. The polynomial variable varied on the one-quarter circle *AOB*, centered at the origin with radius ρ.

Therefore, according to Theorems 2.5.1 and 2.5.3, we have the following program to check the stability of control systems.

(1) For an open-loop control system with transfer function

$$G(s)H(s) = \frac{N_1(s)N_2(s)}{D_1(s)D_2(s)},$$

by Theorem 2.5.1 (take $\rho = \rho_1$ in Figure 2.5.2), let

$$\begin{aligned} f\big(\rho_1 e^{i\theta}\big) &:= D_1\big(\rho_1 e^{i\theta}\big)D_2\big(\rho_2 e^{i\theta}\big) \\ &:= U_1(\rho_1, \theta) + iV_1(\rho_1, \theta), \\ f(i\omega) &= D_1(i\omega)D_2(i\omega) \\ &:= U_2(\omega) + iV_2(\omega), \\ 0 &\leqslant \theta_1 < \theta_2 < \cdots < \theta_k = \frac{\pi}{2}, \\ \rho_1 &\geqslant \omega_1 > \omega_2 > \cdots > \omega_e = 0. \end{aligned}$$

Compute $U_1(\rho_1, \theta_i)$, $V_1(\rho_1, \theta_i)$, $U_2(\omega_j)$, $V_2(\omega_j)$ ($i = 1, 2, \ldots, k,\ j = 1, \ldots, l$) and plot the characteristic curve $f(z)$, using

$$\Phi = \underset{\widehat{ABO}}{\Delta} \operatorname{Arg} f(z) = k\pi,$$

to obtain stability.

(2) For a closed-loop control system with the transfer function

$$\frac{G(s)}{1+G(s)H(s)},$$

by Theorem 2.5.2 the stability (take $\rho=\rho_2$ in the figure) can be obtained by letting $F_1(s) := D_1(s)D_2(s) + N_1(s)N_2(s)$, computing

$$\Phi := \underset{\widehat{ABO}}{\Delta} \operatorname{Arg} f(z) = l\pi,$$

and using l to determine stability.

(3) For the above closed-loop control system, by Theorem 2.5.3 one can study the stability. Let $F(s) = 1 + G(s)H(s) = \frac{F_1(s)}{f(s)}$,

$$F\left(\rho e^{i\theta}\right) := U_3(\rho,\theta) + iV_3(\rho,\theta) \quad \left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right),$$
$$F(i\omega) := U_4(i,\omega) + iV_4(i\omega) \quad (0 \leqslant \omega \leqslant \rho).$$

Take $0 \leqslant \theta_1 < \theta_2 < \cdots < \theta_k = \frac{\pi}{2}$, $\rho \geqslant \omega_1 > \omega_2 > \cdots > \omega_k = 0$.

Let $\tilde{U}_3$, $\tilde{V}_3$, $\tilde{U}_4$, $\tilde{V}_4$ be numerators of U_3, V_3, U_4, V_4, respectively. Since their denominators are all positive, we only need to compute the signs of $\tilde{U}_3$, $\tilde{V}_3$, $\tilde{U}_4$, $\tilde{V}_4$ at θ_i, ρ_j, $i = 1,\dots,k$, $j = 1,2,\dots,l$.

Using the signs, plot the qualitative characteristic curve and obtain $p = k + q = \frac{1}{\pi}\Delta \operatorname{Arg} F(s) + q$. Then use p to determine the stability.

EXAMPLE 2.5.4. Consider an open-loop control system with the transfer function

$$\frac{N_1(s)}{D_1(s)}\frac{N_2(s)}{D_2(s)}.$$

When

$$f(s) = D_1(s)D_2(s) = s^3 + 3s^2 + 4s + 2.1,$$
$$f_1(\theta) = 5^3\cos 3\theta + 3\times 5^2\cos 2\theta + 20\cos\theta + 2.1 := u_1,$$
$$f_2(\theta) = 5^3\sin 3\theta + 3\times 5^2\sin 2\theta + 20\sin\theta + 2.1 := v_1,$$
$$\tilde{f}_1(\omega) = -3\omega^2 + 2.1 := u_2,$$
$$\tilde{f}_2(\omega) = -\omega^3 + 4\omega := v_2,$$

compute the sign

$$\left|\begin{array}{lllllllll}
\theta & 0^\circ & 20^\circ & 40^\circ & 60^\circ & 80^\circ & 90^\circ & & \\
U_1 & + & +\surd & - & - & - & - & & \\
V_1 & 0\surd & + & + & +\surd & - & - & & \\
\omega & 5 & 4.2 & 3.4 & 2.6 & 1.8 & 1.0 & 0.6 & 0 \\
u_2 & - & - & - & - & - & -\surd & + & + \\
V_2 & - & - & - & -\surd & + & + & + & \surd 0
\end{array}\right|,$$

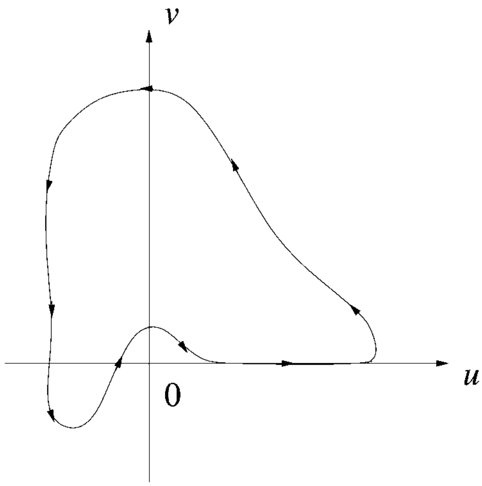

Figure 2.5.3. The characteristic curve for Example 2.5.4.

where $\surd$ denotes varying sign variables. The qualitative characteristic curve is shown in Figure 2.5.3.

By Theorem 2.5.1, $k = 0$ and thus the open-loop system is stable.

EXAMPLE 2.5.5. Consider the stability of a closed-loop system with

$$F(z) = \frac{z^3 - 5z^2 + 8z - 6.1}{z^3 - 4z^2 + 5z - 2}.$$

Since $z^3 - 4z^2 + 5z - 2 = (z-1)(z-1)(z-2)$, $F(z)$ has three poles with $\operatorname{Re} s > 0$ (i.e., $q = 3$). Let

$$\begin{aligned}
\rho &= \max[9, 6] = 9,\\
F\left(\rho e^{i\theta}\right) &:= U_3(\rho, \theta) + i V_3(\rho, \theta),\\
F(i\omega) &:= U_4(\omega) + i V_4(\omega).
\end{aligned}$$

One then obtains

$$\begin{aligned}
\tilde{U}_3 := \Big\{ &\Big[\left(\rho^3 \cos 3\theta - 4\rho^2 \cos 2\theta + 5\rho \cos\theta - 2\right)\\
&\times \left(\rho^3 \cos 3\theta - 5\rho^2 \cos 2\theta + 8\rho \cos\theta - 6.1\right)\Big]\\
&+ \Big[\left(\rho^3 \sin 3\theta - 5\rho^2 \sin 2\theta + 8\rho \sin\theta\right)\\
&\times \left(\rho^3 \sin 3\theta - 4\rho^2 \sin 2\theta + 5\rho \sin\theta\right)\Big]\Big\},
\end{aligned}$$

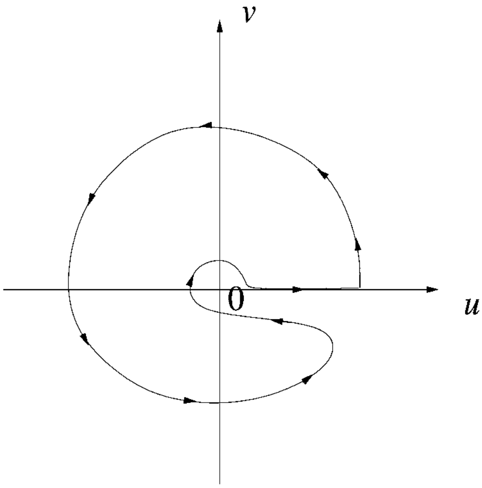

Figure 2.5.4. The characteristic curve for Example 2.5.5.

$$\begin{aligned}
\tilde{V}_3 &= \{[(\rho^3\cos 3\theta - 4\rho^2\cos 2\theta + 5\rho\cos\theta - 2) \\
&\quad \times (\rho^3\sin 3\theta - 5\rho^2\sin 2\theta + 8\rho\sin\theta)] \\
&\quad + [(\rho^3\cos 3\theta - 5\rho^2\cos 2\theta + 8\rho\cos\theta - 6.1) \\
&\quad \times (\rho^3\sin 3\theta - 4\rho^2\sin 2\theta + 5\rho\sin\theta)]\}, \\
\tilde{U}_4 &= \omega^6 + 7\omega^4 + 5.6\omega^2 + 12.2, \\
\tilde{V}_4 &= \omega^5 + 2.9\omega^3 + 14.5\omega.
\end{aligned}$$

Thus,

$$\left|\begin{array}{ccccccc}
Q & 0^\circ & 20^\circ & 40^\circ & 60^\circ & 80^\circ & 90^\circ \\
\tilde{U}_3 & + & + & + & + & + & + \\
\tilde{V}_3 & 0\surd & - & \surd + & + & + & + \\
\omega & 9 & 7 & 5 & 2 & 0 & \\
\tilde{U}_4 & + & + & + & + & + & \\
\tilde{V}_4 & + & + & + & +\surd & 0 &
\end{array}\right|,$$

so $k = 0$, $p = q = 3$.

The qualitative characteristic curve is shown in Figure 2.5.4. By Theorem 2.5.3 the closed-loop control system is unstable.

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Chapter 3

Time-Varying Linear Systems

The stability problem of linear systems with constant coefficients can be transformed to a problem of eigenvalues of a matrix. However, the eigenvalue method fails in solving linear systems with time-varying coefficients. Nevertheless, all solutions of an n-dimensional linear system with time-varying coefficients form an n-dimensional linear space. Therefore, the standard fundamental solution matrix, i.e., the Cauchy matrix solution plays a very important role in determining the behavior of solutions. In this chapter, we concentrate on the Cauchy matrix solution and discuss various properties of solutions for time-varying linear systems.

Materials are mainly selected from [98,151,234] for Section 3.1, [98,151,234] for Section 3.2, [153,234] for Section 3.3, [98,400,459] for Section 3.4, [459] for Section 3.5, [98,455] for Section 3.6 and [248,283] for Section 3.7.

3.1. Stabilities between homogeneous and nonhomogeneous systems

Consider the n-dimensional nonhomogeneous time-varying linear differential equations:

$$\frac{dx}{dt} = A(t)x + f(x), \tag{3.1.1}$$

and the corresponding homogeneous time-varying linear differential equations:

$$\frac{dx}{dt} = A(t)x, \tag{3.1.2}$$

where

$$\begin{aligned} x &= (x_1, \dots, x_n)^T, \\ A(t) &= \big(a_{ij}(t)\big)_{n\times n} \in C\big[I, R^{n\times n}\big], \\ f(t) &= \big(f_1(t), \dots, f_n(t)\big)^T \in C\big[I, R^n\big]. \end{aligned}$$

It is well known that if x, y are any two solutions of (3.1.2), $\alpha x + \beta y$ is also a solution of (3.1.2) $\forall \alpha,\ \beta \in R^1$. If x and y are solutions of (3.1.1), then $x - y$ is also solution of (3.1.2). Thus, n linearly independent solutions of (3.1.2) form the basis of solution space of (3.1.2).

Let $X(t) = (x_{ij}(t))_{n\times n}$ be fundamental solution matrix of (3.1.2). Then, $K(t, t_0) := X(t)X^{-1}(t_0)$ is called standard fundamental solution matrix or Cauchy matrix solution. The general solution of (3.1.2) can be expressed as

$$x(t, t_0, x_0) = K(t, t_0)x_0. \tag{3.1.3}$$

By using the Lagrange formula of the variation of constants, the general solution of (3.1.1) can be written as

$$y(t, t_0, y_0) := y(t) = K(t, t_0)y(t_0) + \int_{t_0}^{t} K(t, t_1)f(t_1)\,dt_1. \tag{3.1.4}$$

DEFINITION 3.1.1. If all solutions of (3.1.1) have same stability, then systems (3.1.1) is said to stable with this class of stability.

THEOREM 3.1.2. $\forall f(t) \in C[I, R^n]$, *system* (3.1.1) *has certain class of stability if and only if the zero solution of* (3.1.2) *has the same type of stability.*

PROOF. The general solutions of (3.1.1) and (3.1.2) can be expressed respectively as

$$y(t, t_0, y_0) = K(t, t_0)C + \int_{t_0}^{t} K(t, t_1)f(t_1)\,dt_1, \tag{3.1.5}$$

$$x(t, t_0, x_0) = K(t, t_0)C. \tag{3.1.6}$$

Let

$$\xi(t, t_0, 0) = K(t, t_0)0,$$
$$x(t, t_0, x_0) = K(t, t_0)x_0,$$

denote respectively the zero solution of (3.1.2) and any perturbed solution. Then

$$\eta(t, t_0, \eta_0) = K(t, t_0)C + \int_{t_0}^{t} K(t, t_1)f(t_1)\,dt_1$$

and

$$y(t, t_0, y_0) = K(t, t_0)(C + x_0) + \int_{t_0}^{t} K(t, t_1)f(t_1)\,dt_1$$

respectively represent the solution without perturbation and the corresponding perturbed solution. Obviously, for any fixed C, x_0 and $x_0 + C$ are one to one. Thus,

$$\left\|x(t, t_0, x_0) - \xi(t, t_0, 0)\right\| = \left\|K(t, t_0)x_0\right\| \tag{3.1.7}$$

and

$$\left\|y(t, t_0, y_0) - \eta(t, t_0, \eta_0)\right\| = \left\|K(t, t_0)x_0\right\| \tag{3.1.8}$$

have the same expression. Therefore, they have the same stability. □

COROLLARY 3.1.3. *Systems (3.1.1) has certain type of stability if and only if any solution of the system has the same stability.*

PROOF. By Definition 3.1.1, a system with certain type of stability implies that every solution of the system has the same stability. Now assume that any solution $\eta(t)$ of (3.1.1) is stable.

Assume $x(t) = y(t) - \eta(t)$. Then, from Theorem 3.1.2 we know that the zero solution of (3.1.2) has the same stability. Again, according to Theorem 3.1.2, system (3.1.1) has the same stability. □

REMARK 3.1.4. There exists a homeomorphism between the solutions of (3.1.1) and (3.1.2), as illustrated in Figures 3.1.1 and 3.1.2.

COROLLARY 3.1.5. *Systems (3.1.1) has certain type of stability if and only if system (3.1.2) has the same stability, i.e., if and only if the zero solution of system (3.1.2) has the same stability.*

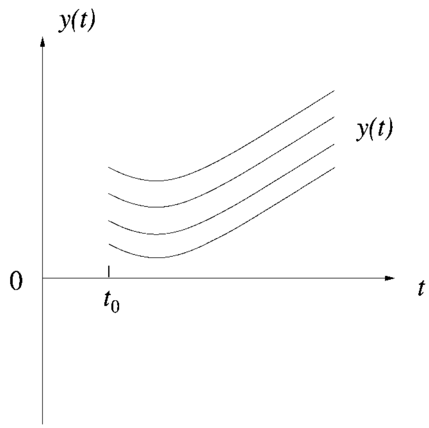

Figure 3.1.1. The solution of the nonhomogeneous equation (3.1.1).

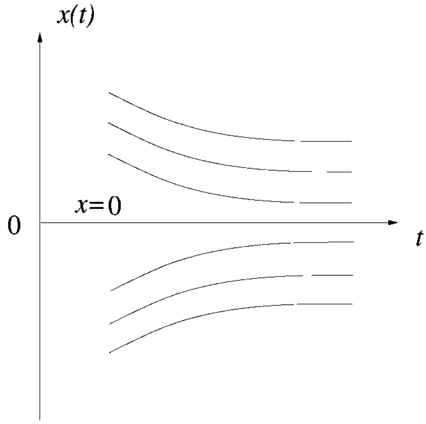

Figure 3.1.2. The solution of the homogeneous equation (3.1.2).

PROOF. In Theorem 3.1.2 take $f(t) = 0$ and then the conclusion follows. □

REMARK 3.1.6. Theorem 3.1.2 means that for linear stability of system (3.1.1) or (3.1.2), we only need to study the zero solution of (3.1.2). For example, in modern control theory a linear system takes the general form:

$$\begin{cases} \frac{dx}{dt} = A(t)x(t) + B(t)U(t), \\ y(t) = C(t)x(t) + D(t)U(t), \end{cases} \tag{3.1.9}$$

where $U(t)$ is the input function, $y(t)$ is the output function. $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are matrices with corresponding dimensions. To study the stability of (3.1.9), we only need to consider the stability of the zero solution of this system.

3.2. Equivalent condition for the stability of linear systems

THEOREM 3.2.1. *The zero solution of* (3.1.2) *is stable if and only if the Cauchy matrix solution* $K(t, t_0)$ $(t \geqslant t_0)$ *of* (3.1.2) *is bounded.*

PROOF. *Sufficiency*. Suppose any perturbed solution of the zero solution is given by

$$x(t, t_0, x_0) = K(t, t_0)x(t_0), \tag{3.2.1}$$

and there exists a constant $M(t_0)$ such that

$$\|K(t, t_0)\| \leqslant M(t_0).$$

$\forall \varepsilon > 0$, $t \geqslant t_0$, take $\delta(\varepsilon, t_0) = \frac{\varepsilon}{M(t_0)}$. Then for $\|x(t_0)\| \leqslant \delta$, we have

$$\left\|x(t, t_0, x_0)\right\| \leqslant \left\|K(t, t_0)\right\| \left\|x(t_0)\right\| < \frac{M(t_0)}{M(t_0)} \varepsilon = \varepsilon.$$

Necessity. Since $\forall \varepsilon > 0$, $\exists \delta(\varepsilon, t_0) > 0$ such that when $\|x(t_0)\| < \delta$,

$$\left\|x(t, t_0, x_0)\right\| = \left\|K(t, t_0)x(t_0)\right\| < \varepsilon$$

is true.

Now, take

$$x(t_0) = \frac{\delta}{2}(\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{n-k})^T := \frac{\delta}{2} e_k.$$

Then, the kth column, $x(k)$, of $K(t, t_0)$ can be expressed as

$$x^{(k)}(t, t_0, x_0) = \frac{2}{\delta} K(t, t_0)x(t_0). \tag{3.2.2}$$

$\|x(t_0)\| < \frac{\delta}{2} < \delta$ implies that

$$\left\|x^{(k)}(t, t_0, x_0)\right\| = \frac{2}{\delta} \left\|x(t, t_0, x_0)\right\| \leqslant \frac{2\varepsilon}{\delta} := M.$$

Thus every column of $K(t, t_0)$ is bounded.

Theorem 3.2.1 is proved. □

COROLLARY 3.2.2. *If system (3.1.2) is stable, then all solutions of the system are either bounded or unbounded.*

THEOREM 3.2.3. *The necessary and sufficient conditions* (*NASC*) *for any solution of (3.1.2) being stable is that $K(t, t_0)$ is bounded.*

Following the proof of Theorem 3.2.1 one can prove this theorem.

THEOREM 3.2.4. *The NASC for the zero solution of (3.2.2) to be asymptotically stable is that the zero solution of (3.1.2) is attractive.*

PROOF. *Necessity* is obvious.

For *sufficiency*, suppose that the zero solution of (3.1.2) is attractive. Then, $\forall t_0 \in I$, $\exists \sigma(t_0)$ such that $\|x_0\| \leqslant \sigma(t_0)$, so we have $x(t, t_0, x_0) = K(t, t_0)x_0 \to 0$ as $t \to +\infty$ by (3.2.1) and thus $x^{(k)}(t, t_0) = K(t, t_0)e_k \to 0$ as $t \to +\infty$. So $x^{(k)}(t, t_0)$ is bounded, and further $K(t, t_0)$ is bounded. By Theorem 3.2.1 we know that the zero solution of (3.1.2) is stable.

The proof is complete. □

Following the proof of Theorems 3.2.1 and 3.2.3, one can easily prove the following theorem.

THEOREM 3.2.5. *The uniform attraction of the zero solution of (3.1.2) is equivalent to uniformly asymptotic stability of the zero solution of (3.1.2) and uniform boundedness of $K(t, t_0)$.*

COROLLARY 3.2.6. *The asymptotic stability of the zero solution of (3.1.2) is equivalent to its globally asymptotic stability.*

PROOF. $\forall x_0 \in R^n$ since $x(t, t_0, x_0) = K(t, t_0)x_0 \to 0$ as $t \to +\infty$, the conclusion is true. □

COROLLARY 3.2.7. *The uniformly asymptotic stability of the zero solution of (3.1.2) is equivalent to its globally uniformly asymptotic stability.*

THEOREM 3.2.8. *The zero solution of (3.1.2) is asymptotically stable if and only if $K(t, t_0) \to 0$ uniformly holds as $(t - t_0) \to +\infty$, and $K(t, t_0)$ is uniformly bounded for $t \geqslant t_0$.*

PROOF. *Sufficiency*. Since $K(t, t_0) \to 0$ as $t - t_0 \to +\infty$ and $K(t, t_0)$ $(t \geqslant t_0)$ is uniformly bounded, there exists a constant M such that

$$\left\|K(t, t_0)\right\| \leqslant M, \quad t \geqslant t_0. \tag{3.2.3}$$

By Theorem 3.2.1 we know that the zero solution is stable. So according to $x(t, t_0, x_0) = K(t, t_0)x_0$, $x(t, t_0, x_0) \to 0$ uniformly as $(t - t_0) \to 0$.

For *necessity*, simply follow the proof of Corollary 3.2.2 and Theorem 3.2.3. $x(t, t_0, x_0) \to 0$ as $(t - t_0) \to 0$ uniformly for t_0, the uniform boundedness of $x(t, t_0, x_0)$ implies that $K(t, t_0) \to 0$ uniformly holds for t_0, and thus $K(t, t_0)$ is uniformly bounded. □

COROLLARY 3.2.9. *The zero solution of (3.1.2) is asymptotically stable if and only if $K(t, t_0) \to 0$ as $t \to +\infty$.*

THEOREM 3.2.10. *The zero solution of (3.1.2) is attractive if and only if it is quasi-attractive, namely, asymptotic stability is equivalent to quasi-asymptotic stability.*

PROOF. We only need to prove that attraction of the zero solution of (3.1.2) implies its quasi-attraction.

Since the attraction of the zero solution of (3.1.2) implies $K(t, t_0) \to 0$ as $t \to +\infty$, $\exists \sigma(t_0) > 0$, $\forall \varepsilon > 0$, $\exists T(t_0, \varepsilon, \sigma)$ such that $\|K(t, t_0)\| < \frac{\varepsilon}{\sigma}$ when

$t \geqslant t_0 + T$. Therefore, when $\|x(t_0)\| \leqslant \sigma(t_0)$, we have

$$\big\|x(t, t_0, x_0)\big\| \leqslant \big\|K(t, t_0)\big\| \big\|x(t_0)\big\| < \frac{\varepsilon}{\sigma(t_0)}\sigma(t_0) = \varepsilon.$$

This means that the zero solution is quasi-attractive.

The proof is complete. □

THEOREM 3.2.11. *The zero solution of (3.1.2) is uniformly asymptotically stable if and only if it is exponentially stable.*

PROOF. One only needs to prove that the asymptotic stability of the zero solution of (3.1.2) implies its exponential stability.

$\forall \varepsilon > 0$ $(0 < \varepsilon < 1)$ $\exists \tau(\varepsilon) > 0$, when $t \geqslant t_0 + T$, $\|K(t, t_0)\| < \varepsilon$ holds. Owing to the uniform stability of the zero solution, there exists a constant $M > 0$ such that

$$\big\|K(t, t_0)\big\| < M, \quad t_0 \leqslant t \leqslant t_1 + T.$$

Assume that

$$n\tau \leqslant t - t_0 \leqslant (n + 1)\tau \quad (n = 0, 1, 2, \ldots).$$

Then, from the property of the Cauchy matrix solution $K(t, t_0)$, we have

$$K(t, t_0) = K(t, t_1)K(t_1, t_0) \quad (t_0 \leqslant t_1 \leqslant t).$$

Therefore,

$$\begin{aligned} K(t, t_0) = {}& K(t, n\tau + t_0) \\ & \times K\big(n\tau + t_0, (n - 1)\tau + t_0\big) \cdots K(\tau + t_0, t_0), \end{aligned} \tag{3.2.4}$$

and thus

$$\begin{aligned} \big\|K(t, t_0)\big\| &\leqslant \big\|K(t, n\tau + t_0)\big\| \\ &\quad \times \big\|K\big(n\tau + t_0, (n - 1)\tau + t_0\big)\big\| \cdots \big\|K(\tau + t_0, t_0)\big\| \\ &\leqslant Me^{\lambda\tau}e^{-(n+1)\lambda\tau} \\ &\leqslant Ne^{-\lambda(t-t_0)}, \end{aligned} \tag{3.2.5}$$

where $N = Me^{\lambda\tau}$. This implies that the zero solution of (3.1.2) is exponentially stable. □

THEOREM 3.2.12. *The zero solution of (3.1.2) is exponentially stable if and only if the Cauchy matrix solution $K(t, t_0)$ satisfies*

$$\big\|K(t, t_0)\big\| \leqslant Me^{-\alpha(t-t_0)},$$

where $M \geqslant 1$, $\alpha > 0$ are constants.

PROOF. *Sufficiency*. Since any solution $x(t, t_0, x_0)$ of (3.1.2) can be expressed as

$$x(t, t_0, x_0) = K(t, t_0)x_0,$$

we have

$$\|x(t, t_0, x_0)\| \leqslant \|K(t, t_0)x_0\| \leqslant \|K(t, t_0)\| \|x_0\| \leqslant M\|x_0\|e^{-\alpha(t-t_0)}.$$

This means that the zero solution of (3.1.2) is exponentially stable.

Necessity. Following the proof of the necessity for Theorem 3.2.1, we have

$$x(t, t_0, x_0) = K(t, t_0)x(t_0)$$

which yields

$$\|x(t, t_0, x_0)\| \leqslant \|K(t, t_0)x(t_0)\| \leqslant M(x_0)e^{-\alpha(t-t_0)}.$$

Now, take

$$x(t_0)\frac{\delta}{2}(\underbrace{0, 0, \dots, 0}_{k-1}, 1, \underbrace{0, 0, \dots, 0}_{n-k})^T := \frac{\delta}{2}e_k.$$

Then, the kth column $x(k)$ of $K(t, t_0)$ can be expressed by

$$x^{(k)}(t, t_0, x_0) = \frac{2}{\sigma}K(t, t_0)x(t_0),$$

which implies that

$$\begin{aligned}\|x^{(k)}(t, t_0, x_0)\| &\leqslant \frac{2}{\sigma}\|K(t, t_0)x(t_0)\| \leqslant \frac{2}{\sigma}M(x_0)e^{-\alpha(t-t_0)}\\ &:= \tilde{M}(x_0)e^{-\alpha(t-t_0)}.\end{aligned}$$

Hence, $\|K(t, t_0)\| \leqslant N(x_0)e^{-\alpha(t-t_0)}$, where $N(x_0)$ is a constant. $\square$

3.3. Robust stability of linear systems

Consider a definite linear system

$$\frac{dx}{dt} = A(t)x, \tag{3.3.1}$$

and an indefinite linear system

$$\frac{dx}{dt} = A(t)x + B(t)x, \tag{3.3.2}$$

where $x \in R^n$, $A(t)$ is a known $n \times n$ continuous matrix function and $B(t)$ an unknown $n \times n$ continuous matrix function. One only knows some boundedness of $B(t)$. We will discuss the equivalent problem of stabilities between systems (3.3.1) and (3.3.2), so that one can use the simple and definite system (3.1.1) to study the complex and indefinite system (3.3.2).

THEOREM 3.3.1. *If the following condition is satisfied*:

$$\int_0^{+\infty} \|B(t)\| \, dt := N < \infty,$$

then the uniform stability of the zero solutions of (3.3.1) *and* (3.3.2) *are equivalent.*

PROOF. Let the zero solution of (3.3.1) be uniformly stable. By Theorem 3.2.1, there exists a constant $M > 0$ such that

$$\|K(t, t_0)\| \leqslant M (t \geqslant t_0).$$

However, the general solution of (3.3.2) can be written as

$$x(t, t_0, x_0) = K(t, t_0)x_0 + \int_{t_0}^{t} K(t, t_1)B(t_1)x(t_1) \, dt_1,$$

and so

$$\begin{aligned}
\|x(t, t_0, x_0)\| &\leqslant \|K(t, t_0)\| \|x_0\| + \int_{t_0}^{t} \|K(t, t_1)\| \|B(t_1)\| \|x(t_1)\| \, dt_1 \\
&\leqslant M\|x_0\| + \int_{t_0}^{t} M\|B(t_1)\| \|x(t_1)\| \, dt_1.
\end{aligned}$$

By using the Gronwall–Bellman inequity, we have the estimation

$$\|x(t, t_0, x_0)\| \leqslant M\|x_0\|e^{M \int_{t_0}^{t} \|B(t_1)\| \, dt} \leqslant M\|x_0\|e^{MN}.$$

Thus, $\forall \varepsilon > 0$ take $\delta(\varepsilon) = \frac{\varepsilon}{Me^{NM}}$. Then, when $\|x_0\| < \delta$ we obtain

$$\|x(t, t_0, x_0)\| \leqslant |\delta|Me^{NM} = \varepsilon,$$

which implies that the zero solution of (3.3.2) is uniformly stable.

On the other hand, (2.2.8) can be rewritten as

$$\begin{aligned}
\frac{dx}{dt} &= A(t)x = \big(A(t) + B(t)\big)x - B(t)x \\
&:= \tilde{A}(t)x + \tilde{B}(t)x,
\end{aligned} \tag{3.3.3}$$

where $\tilde{A}(t) := A(t) + B(t)$, $\tilde{B}(t) := -B(t)$, implying that

$$\int_0^{+\infty} \|\tilde{B}(t)\| \, dt = \int_0^{+\infty} \|B(t)\| \, dt = N < \infty.$$

Therefore, the uniform stability of the zero solution of (3.3.3) implies the uniform stability of the zero solution of (3.3.2).

The proof is completed. □

THEOREM 3.3.2. *If there exist constant* $M > 0$ *and* $0 < r \ll 1$ *such that the following estimation*

$$\int_{t_0}^{t} \|B(t_1)\| \, dt_1 \leqslant r(t - t_0) + M$$

is valid, then the exponential stabilities of the zero solutions of (3.3.1) *and* (3.3.2) *are equivalent.*

PROOF. Assume that the zero solution of (3.3.1) is exponentially stable, then there exist constants $M_1 > 0$ and $\alpha_1 > 0$ such that the Cauchy matrix solution of (3.3.1), $K(t, t_0)$, admits the estimation:

$$\|K(t, t_0)\| \leqslant M_1 e^{-\alpha_1 (t - t_0)}. \tag{3.3.4}$$

Let $r < \frac{\alpha_1}{M_1}$. Then, the general solution of (3.3.2) can be expressed as

$$x(t, t_0, x_0) = K(t, t_0)x_0 + \int_{t_0}^{t} K(t, t_1)B(t_1)x(t_1)\, dt_1.$$

Thus, we obtain

$$\begin{aligned} \|x(t, t_0, x_0)\| &< \|K(t, t_0)\| \|x_0\| + \int_{t_0}^{t} \|K(t, t_1)\| \|B(t_1)\| \|x(t_1)\| \, dt_1 \\ &\leqslant M_1 e^{-\alpha_1 (t - t_0)} \|x_0\| + \int_{t_0}^{t} M_1 e^{-\alpha_1 (t - t_1)} \|B(t_1)\| \|x(t_1)\| \, dt_1. \end{aligned}$$

By using the Gronwall–Bellman inequity, we have the following estimation:

$$\begin{aligned} \|x(t)\| e^{\alpha_1 t} &\leqslant M_1 e^{\alpha_1 t_0} \|x_0\| e^{\int_{t_0}^{t} M_1 \|B(t_1)\| \, dt_1} \\ \|x(t)\| &\leqslant M_1 e^{-\alpha_1 (t - t_0)} \|x_0\| e^{M_1 r (t - t_0) + M_1 M} \\ &= M_1 e^{M_1 M} \|x_0\| e^{-(\alpha_1 - M_1 r)(t - t_0)}. \end{aligned} \tag{3.3.5}$$

This estimation shows that the zero solution of (3.3.2) is exponentially stable, where $r < \alpha_1 / M_1$. On the other hand, system (3.3.1) can be rewritten as

$$\frac{dx}{dt} = A(t)x = \big(A(t) + B(t) - B(t)\big)x := \big(\tilde{A}(t) + \tilde{B}(t)\big)x,$$

where $\tilde{A}(t) = A(t) + B(t)$, $\tilde{B}(t) = -B(t)$.

However,

$$\int_{0}^{t} \|\tilde{B}(t)\|\, d\tau = \int_{t_0}^{t} \|B(\tau)\|\, d\tau \leqslant r(t - t_0) + M.$$

From the above results, we know that the exponential stability of the zero solution of (3.3.2) implies the exponential stability of the zero solution of (3.3.1).

The proof is complete. □

COROLLARY 3.3.3. *If*

$$\lim_{t\to+\infty} B(t) = 0,$$

then the conclusion of Theorem 3.3.2 holds.

PROOF. Since

$$\lim_{t\to+\infty} B(t) = 0$$

implies that

$$\lim_{t\to+\infty} \|B(t)\| = 0,$$

so $\forall\varepsilon > 0$, $\exists T$ when $t \geqslant t_0 + T$, we have

$$\|B(t)\| < \varepsilon$$

and

$$\begin{aligned}\int_{t_0}^{t} \|B(t_1)\|\, dt_1 &= \int_{t_0}^{t_0+T} \|B(t_1)\|\, dt_1 + \int_{t_0+T}^{t} \|B(t_1)\|\, dt_1 \\ &\leqslant M + \varepsilon(t - T - t_0) \leqslant M + \varepsilon(t - t_0).\end{aligned} \tag{3.3.6}$$

Therefore, the conditions of Theorem 3.3.2 are satisfied.

This proves the corollary. □

The commonly used method in nonlinear systems is linearization, which is based on the following theorem. Consider the nonlinear system:

$$\frac{dx}{dt} = A(t)x + f(t, x), \tag{3.3.7}$$

where $f(t, x) \in C[I \times R^n, R^n]$, $f(t, x) \equiv x$ if and only if $x = 0$.

THEOREM 3.3.4. $\forall\varepsilon > 0$, $\exists\sigma(\varepsilon)$, *when* $x \in D := \{x\|x\| < \sigma\}$, $t \in [t_0 + \infty)$, *the estimation*

$$\|f(t,x)\| < \varepsilon\|x\|$$

holds. Then, the exponential stability of the zero solution of (3.3.1) *implies the exponential stability of the zero solution of* (3.3.7).

PROOF. Suppose that the zero solution of (3.3.1) be exponentially stable. Then, there exist constant $M \geqslant 1$ and $\alpha > 0$ such that

$$\|K(t,x_0)\| \leqslant Me^{-\alpha(t-t_0)},$$

where $K(t,t_0)$ is the Cauchy matrix solution of (3.3.1). Take $\sigma(\varepsilon) > 0$ such that when $\|x\| < \sigma(\varepsilon)$, $\|f(t,x)\| < \varepsilon\|x\|$ is true. Then, the general solution of (3.3.7) is

$$x(t) = K(t,t_0)x(t_0) + \int_{t_0}^{t} K(t,t_1)f\big(t_1, x(t_1)\big)\,dt_1.$$

Take $\|x(t_0)\| < \sigma$. By the continuity of solution, there exists a constant $\delta > 0$ such that $\|x(t,t_0,x_0)\| < \sigma$. Hence, when $t \in [t_0, t_0+\delta]$, we have

$$\|x(t,t_0,x_0)\| \leqslant Me^{-\alpha(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} \varepsilon Me^{-\alpha(t-t_1)}\|x(t_1)\|\,dt_1.$$

Furthermore, we obtain

$$\|x(t)\|e^{\alpha t} \leqslant Me^{\alpha t_0}\|x(t_0)\|e^{\varepsilon M(t-t_0)},$$

i.e.,

$$\begin{aligned}\|x(t)\| &\leqslant M\|x(t_0)\|e^{-\alpha(t-t_0)+\varepsilon M(t-t_0)}\\ &= M\|x(t_0)\|e^{-(\alpha-\varepsilon M)(t-t_0)}, \quad t \in [t_0, t_0+\delta].\end{aligned} \tag{3.3.8}$$

Since $\alpha - \varepsilon M > 0$, when $\|x(t_0)\| < \frac{\sigma}{M}$, (3.3.8) holds for $t \geqslant t_0$.

Theorem 3.3.4 is proved. □

COROLLARY 3.3.5. *If* $\frac{\|f(t,x)\|}{\|x\|} \to 0$ *holds uniformly as* $\|x\| \to 0$ *for t, and* $f(t,x) \in C[I, R^n, R^n])$, *then the conditions of* Theorem 3.3.4 *are satisfied.*

EXAMPLE 3.3.6. If A is a Hurwitz matrix, then the stability of the zero solutions of the system

$$\frac{dx}{dt} = \big(Ae^t + p(t)B\big)x \tag{3.3.9}$$

and the system

$$\frac{dx}{dt} = Ax \tag{3.3.10}$$

is equivalent, where $p(t)$ is any nth-degree polynomial of t, or any continuous function satisfying $\frac{p(t)}{e^t} \to 0$ as $t \to +\infty$.

PROOF. Introduce a transform $\tau = e^t$ to (3.3.9) to obtain

$$\frac{dx}{d\tau} = \big(A + f(\tau)B\big)x, \tag{3.3.11}$$

where $f(\tau) = \frac{p(t)}{e^t} = \frac{p(\ln\tau)}{\tau}$.

Since $\tau \to +\infty \Longleftrightarrow t \to +\infty$, the two systems (3.3.9) and (3.3.11) have the same stability. If replacing τ in (3.3.11) by t, then (3.3.11) becomes

$$\frac{dx}{dt} = \big(A + f(t)B\big)x. \tag{3.3.12}$$

However, $f(t)B \to 0$ as $t \to +\infty$, so the conditions of Corollary 3.3.3 are satisfied. □

For the general nonlinear system

$$\frac{dx}{dt} = f(t, x), \tag{3.3.13}$$

where $f(t,x) \in C[I \times R^n, R^n]$, $f(t,x) \equiv x \Longleftrightarrow x = 0$. One can rewrite (3.3.13) as

$$\frac{dx}{dt} = A(t)x + g(t, x), \tag{3.3.14}$$

where $A(t) = \frac{\partial f}{\partial t}|_{x=0}$ is Jacobi matrix of f and $g(t,x)$ represents higher order terms. Then, the corresponding linear system is

$$\frac{dx}{dt} = A(t)x, \tag{3.3.15}$$

which can be used to analyze the stability of the zero solution for (3.3.13). This is the Lyapunov first approximation theory.

COROLLARY 3.3.7. *If $\forall\varepsilon > 0$, $\exists\delta > 0$ when $\|x\| < \delta$ it admits $\|g(t,x)\| \leqslant \varepsilon\|x\|$, then the exponential stability of the zero solution of (3.3.15) implies the exponential stability of the zero solution of (3.3.14).*

COROLLARY 3.3.8. *If $A(t) = A$, A is a Hurwitz matrix and other conditions of Corollary 3.3.7 hold, then the zero solution of (3.3.14) is exponentially stable.*

3.4. The expression of Cauchy matrix solution

It is well know that for linear system (3.1.2), the Cauchy matrix solution $K(t, t_0)$ plays a key role, since it determines all properties of the general solution of system (3.1.2). However, except for the cases of $A(t)$ being constant matrix, diagonal matrix, or trigonometric matrix, $K(t, t_0)$ cannot be expressed in finite integral form of the elements of $A(t)$. For some certain specific form $A(t)$, we can find the expression of the Cauchy matrix solution $K(t, t_0)$ or its estimation, and obtain the algebraic criteria for stability.

THEOREM 3.4.1. *If the following conditions are satisfied*:

(1) $A(t) \in [I, R^{n^2}]$;
(2) $A(t) \int_\tau^t A(t_1)\, dt_1 \equiv \int_\tau^t A(t_1)\, dt_1 A(t)\ \forall(\tau, t) \in [t_0 + \infty)$;
(3) $\bar{A} := \lim_{t\to+\infty} \frac{1}{t} \int_{t_0}^t A(t_1)\, dt_1$ *exists and is a Hurwitz matrix*;

then

(1) *the Cauchy matrix solution* $K(t, t_0)$ *of* (3.1.2) *can be expressed as*

$$K(t, t_0) := \Omega(t) := e^{\int_{t_0}^t A(t_1)\, dt_1}; \tag{3.4.1}$$

(2) *the zero solution of* (3.1.2) *is exponentially stable.*

PROOF.

(1) Obviously, condition (2) implies that

$$A(t) e^{\int_{t_0}^t A(t_1)\, dt_1} = e^{\int_{t_0}^t A(t_1)\, dt_1} A(t).$$

Since $\Omega(t_0) = I_n$, and

$$\frac{d\Omega(t)}{dt} = e^{\int_{t_0}^t A(t_1)\, dt_1} A(t) = A(t) e^{\int_{t_0}^t A(t_1)\, dt_1} = A(t)\Omega(t),$$

$K(t, t_0) = \Omega(t) = e^{\int_{t_0}^t A(t_1)\, dt_1}$ is true.

(2) With the condition

$$A(t) \int_s^t A(t_1)\, dt_1 = \int_s^t A(t_1)\, dt_1 A(t), \tag{3.4.2}$$

computing the derivative of (3.4.2) with respect to variable s, we obtain $A(t)A(s) = A(s)A(t)$. Hence, we have

$$\int_{t_0}^t A(t_1)\, dt_1 \frac{1}{s} \int_{t_0}^s A(t_2)\, dt_2 = \frac{1}{s} \int_{t_0}^t dt_1 \int_{t_0}^s A(t_1)A(t_2)\, dt_2$$

$$
\begin{aligned}
&= \frac{1}{s}\int_{t_0}^{t} dt_1 \int_{t_0}^{s} A(t_2)A(t_1)\,dt_2 \\
&= \frac{1}{s}\int_{t_0}^{s} A(t_2)\,dt_2 \int_{t_0}^{t} A(t_1)\,dt_1. \qquad (3.4.3)
\end{aligned}
$$

Let $s \to +\infty$, the limit is

$$
\int_{t_0}^{t} A(t_1)\,dt_1 \bar{A} = \bar{A}\int_{t_0}^{t} A(t_1)\,dt_1.
$$

Suppose $\frac{1}{t}\int_{t_0}^{t} A(t_1)\,dt_1 = \bar{A} + B(t)$, where $B(t) \to 0$ as $t \to +\infty$. However,

$$
\bar{A}B(t) = \bar{A}\left[\frac{1}{t}\int_{t_0}^{t} A(t_1)\,dt_1 - \bar{A}\right] = B(t)\bar{A}. \qquad (3.4.4)
$$

So the zero solution of (3.1.2) can be expressed as

$$
\begin{aligned}
x(t, t_0, x_0) &= e^{\int_{t_0}^{t} A(t_1)\,dt_1} x(t_0) = e^{t\bar{A}+tB(t)} x(t_0) \\
&= e^{t\bar{A}} e^{tB(t)} x(t_0). \qquad (3.4.5)
\end{aligned}
$$

Let

$$
\max_{i\leqslant j\leqslant n} \operatorname{Re}\lambda_j(\bar{A}) = \alpha < 0.
$$

Choose $\varepsilon > 0$ such that $\alpha + 2\varepsilon < 0$, and $T \gg 1$ such that $t \geqslant T > 0$. Suppose $\|B(t)\| < \varepsilon$ holds. From (3.4.5) we know that there exists a constant $M > 0$ such that

$$
\begin{aligned}
\|x(t, t_0, x_0)\| &\leqslant \|e^{t\bar{A}}\| \|e^{tB(t)}\| \|x(t_0)\| \leqslant M e^{(\alpha+\varepsilon)t} e^{t\|B(t)\|} \|x(t_0)\| \\
&\leqslant M\|x(t_0)\| e^{(\alpha+2\varepsilon)t} \quad (t \geqslant T), \qquad (3.4.6)
\end{aligned}
$$

which, due to $\alpha + 2\varepsilon < 0$, implies that the zero solution of (3.1.2) is exponentially stable.

□

THEOREM 3.4.2. *If system (3.1.2) satisfies the following conditions*:

(1)

$$
A(t)\int_{t_0}^{t} A(t_1)\,dt_1 - \int_{t_0}^{t} A(t_1)\,dt_1 A(t) := K^{(1)}(t) \neq 0,
$$

but

$$K^{(1)}(t)\int_{t_0}^{t} A(t_1)\,dt_1 - \int_{t_0}^{t} A(t_1)\,dt_1 K^{(1)}(t) := K^{(2)}(t) = 0;$$

(2) $\lim_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t} A(t_1)\,dt_1 =: \bar{A}$ *exists and is a Hurwitz matrix*;

(3) $K^{(1)}(t) \to 0$ *as* $t \to +\infty$;

then the zero solution of (3.1.2) *is exponentially stable.*

PROOF. If condition (1) is satisfied, then the Cauchy matrix solution of (3.1.2) is

$$X(t) = e^{\int_{t_0}^{t} A(t_1)\,dt_1} Y(t), \tag{3.4.7}$$

where $Y(t)$ is the Cauchy matrix solution of

$$\frac{dy}{dt} = K^{(1)}(t)y. \tag{3.4.8}$$

By condition (2), let $\int_{t_0}^{t} A(t_1)\,dt_1 = \bar{A}t + tB(t)$, where $B(t) \to 0$ as $t \to +\infty$. Then, (3.4.7) can be expressed as $X(t) = e^{\bar{A}t}e^{tB(t)}Y(t)$. Furthermore, suppose

$$\max_{1\leqslant j\leqslant n} \operatorname{Re}\lambda_j(A) = \alpha < 0.$$

Therefore, there exists $\varepsilon > 0$ such that $\alpha + 3\varepsilon < 0$, $K^{(1)}(t) \to 0$ (as $t \to +\infty$) implies that

$$\|y(t, t_0, y_0)\| \leqslant \|y(t_0)\| e^{\int_{t_0}^{t} \|K^{(1)}(t_1)\|\,dt_1} \leqslant M_2 e^{\varepsilon t} \quad (M_2 = \text{const}),$$

and $B(t) \to 0$ as $t \to +\infty$ implies that there exists $T > 0$ such that when $t > T$,

$$\|B(t)\| < \varepsilon$$

holds. Therefore, when $t > T$, we obtain

$$\begin{aligned}\|X(t)\| \leqslant M_1 e^{(\alpha+\varepsilon)t} e^{t\|B(t)\|}\|Y(t)\| &\leqslant M_1 e^{(\alpha+2\varepsilon)t} M_2 e^{\varepsilon t}\\ &:= M e^{(\alpha+3\varepsilon)t},\end{aligned} \tag{3.4.9}$$

where $M_1 =$ constant and $M = M_1 M_2$. (3.4.9) indicates that the zero solution of (3.1.2) is exponentially stable. □

THEOREM 3.4.3. *(See [459].) If* (3.1.2) *satisfies the following conditions*:

(1) $A(t) \in C^1[I, R^{n\times n}]$ *and there exists constant matrix* $A_1 \in R^{n\times n}$ *such that*

$$A_1 A(t) - A(t) A_1 = \dot{A}; \tag{3.4.10}$$

(2) *let*

$$A_2 := e^{-A_1 t_0}\big[A(t_0) - A_1\big]e^{A_1 t_0},$$

$\alpha = \max_{1\leqslant j\leqslant n} \operatorname{Re}\lambda_j(A_1)$, $\beta = \max_{1\leqslant j\leqslant n} \operatorname{Re}\lambda_j(A_2)$, *and* $\alpha + \beta < 0$;

then:

(a) *the Cauchy matrix solution of* (3.1.2) *can be expressed as*

$$K(t, t_0) = e^{A_1 t} e^{A_2(t-t_0)} e^{-A_1 t}; \tag{3.4.11}$$

(b) *the zero solution of* (3.1.2) *is exponentially stable.*

PROOF. (1) Multiplying respectively $e^{-A_1(t-t_0)}$ and $e^{A_1(t-t_0)}$ to the left-hand and right-hand sides of (3.4.10) yields

$$\frac{d}{dt}\big[e^{-A_1(t-t_0)}A(t)e^{A_1(t-t_0)}\big] = 0. \tag{3.4.12}$$

Integrating (3.4.12) results in $A(t) = e^{A_1(t-t_0)}A(t_0)e^{-A_1(t-t_0)}$. Take the transform $x(t) = e^{A_1(t)}y(t)$ into (3.1.2) yields

$$A_1 e^{A_1(t)}y(t) + e^{A_1 t}\dot{y}(t) = A(t)e^{A_1 t}y(t),$$

or

$$\dot{y}(t) = e^{-A_1 t_0}\big[A(t_0) - A_1\big]e^{A_1 t_0}y(t) = A_2 y(t).$$

So

$$y(t) = e^{A_2(t-t_0)}y(t_0) = e^{A_2(t-t_0)}e^{-A_1 t_0}x(t_0).$$

(2) Since $K(t, t_0) = e^{A_1 t}e^{A_2(t-t_0)}e^{-A_1 t_0}$, we have $\|K(t, t_0)\| \leqslant Me^{(\alpha+\beta)(t-t_0)}$, implying that the zero solution of (3.1.2) is exponentially stable, because $\alpha + \beta < 0$. □

THEOREM 3.4.4. *(See [400].) If system* (3.1.2) *satisfies the following conditions*:

(1) $A(t) \in C^1[I, R^{n\times n}]$, *there exists a constant matrix* A_1 *and function* $V(t) \in C^1[I, R]$, $V(t) \neq 0$ *such that*

$$A_1 A(t) - A(t)A_1 = \frac{d}{dt}\left(\frac{A(t)}{V(t)}\right), \tag{3.4.13}$$

and $\int_{t_0}^{+\infty} V(t)\,dt = +\infty$;

(2) $A_2 = P(t_0) - A_1$, *where*

$$P(t_0) = \lim_{t \to t_0} \frac{A(t)}{V(t)},$$

and

$$\alpha = \max_{1 \leqslant j \leqslant n} \operatorname{Re} \lambda_j(A_1), \quad \beta = \max_{1 \leqslant j \leqslant n} \operatorname{Re} \lambda_j(A_2)$$

satisfying $\alpha + \beta < 0$;

then the Cauchy matrix solution of (3.1.2) *is*

$$K(t, t_0) = e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} e^{A_2 \int_{t_0}^{t} V(t_1)\, dt_1}, \tag{3.4.14}$$

and the zero solution of (3.1.2) *is asymptotically stable.*

PROOF. Multiplying respectively

$$e^{-A_1 \int_{t_0}^{t} V(t_1)\, dt_1} \quad \text{and} \quad e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1}$$

to the left-hand and right-hand sides of (3.4.13) results in

$$\frac{d}{dt}\left[e^{-A_1 \int_{t_0}^{t} V(t_1)\, dt_1} \frac{A(t)}{V(t)} e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1}\right] = 0. \tag{3.4.15}$$

Then, integrating (3.4.15) gives

$$A(t) = V(t) e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} P(t_0) e^{-A_1 \int_{t_0}^{t} V(t_1)\, dt_1}. \tag{3.4.16}$$

Introduce the transform

$$x(t) = e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} y(t)$$

into (3.1.2) yields

$$\begin{aligned} A_1 V(t) e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} y(t) + e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} \frac{dy(t)}{dt} &= A(t) e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} y(t) \\ &= V(t)\big[P(t_0) - A_1\big] y(t) := V(t) A_2 y(t). \end{aligned}$$

So $y(t) = e^{A_2 \int_{t_0}^{t} V(t_1)\, dt_1} y(t_0)$.

Thus the Cauchy matrix solution of (3.1.2) is

$$K(t, t_0) = e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} e^{A_2 \int_{t_0}^{t} V(t_1)\, dt_1}.$$

Furthermore, we have

$$\begin{aligned} \big\| K(t, t_0) \big\| &\leqslant \big\| e^{A_1 \int_{t_0}^{t} V(t_1)\, dt_1} \big\| \big\| e^{A_2 \int_{t_0}^{t} V(t_1)\, dt_1} \big\| \\ &\leqslant M_1 e^{(\alpha+\varepsilon) \int_{t_0}^{t} V(t_1)\, dt_1} M_2 e^{(\beta+\delta) \int_{t_0}^{t} V(t_1)\, dt_1} = M_1 M_2 e^{(\alpha+\beta+\varepsilon+\delta) \int_{t_0}^{t} V(t_1)\, dt_1}, \end{aligned}$$

where M_1, M_2 are positive constants. Take $T \gg 1$. When $t \geqslant T$, for $0 < \varepsilon \ll 1$, $0 < \delta \ll 1$, we have $\alpha + \beta + \varepsilon + \delta < 0$, due to $\int_{t_0}^{t} V(t_1)\,dt_1 \to +\infty$.

This shows that the zero solution of (3.1.2) is asymptotically stable. □

EXAMPLE 3.4.5. Consider the stability of following system

$$\begin{cases} \frac{dx_1}{dt} = -x_1 - e^{2t}x_2, \\ \frac{dx_2}{dt} = e^{-2t}x_1 - 4x_2, \end{cases}$$

where $A(t) = \begin{bmatrix} -1 & -e^{2t} \\ e^{-2t} & -4 \end{bmatrix}$. Take $A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. Then, we have $A_1 A(t) - A(t) A_1 = A(t)$. Choose $t_0 = 0$, $A_2 = \begin{bmatrix} -4 & -1 \\ 1 & -5 \end{bmatrix}$. Since

$$\alpha = \max_{1 \leqslant j \leqslant 2} \operatorname{Re} \lambda_j(A_1) = 3,$$

$$\beta = \max_{1 \leqslant j \leqslant 2} \operatorname{Re} \lambda_j(A_2) = -\frac{9}{2},$$

$$\alpha + \beta < 0,$$

by Theorem 3.4.3 we can conclude that the zero solution of this system is exponentially stable.

3.5. Linear systems with periodic coefficients

Consider the following linear systems with periodic coefficients:

$$\frac{dx}{dt} = A(t)x, \tag{3.5.1}$$

where $A(t) = (a_{ij}(t)) \in C[I, R^n]$, $x \in R^n$, $A(t+T) \equiv A(t)$ and $T > 0$ is the period.

Floquet–Lyapunov theory shows that the Cauchy matrix solution of (3.5.1) can be expressed as

$$X(t) = F(x)e^{Kt}, \tag{3.5.2}$$

where $F(t)$ is a periodic, continuous nonsingular matrix and

$$K = \frac{1}{T} \ln X(T). \tag{3.5.3}$$

Therefore, the zero solution of (3.5.1) is exponentially stable if and only if K is Hurwitz; the zero solution of (3.5.1) is stable if and only if K is quasi-stable.

It follows from (3.5.3) that K is a Hurwitz matrix $\Longleftrightarrow$ the spectral radius of matrix $X(T)$, $\rho(X(T))$ is less than one, i.e., $\rho(X(T)) < 1$.

K is quasi-stable $\Longleftrightarrow$ $\rho(X(T)) \leqslant 1$ and the eigenvalue λ_0, $|\lambda_0| = 1$, only corresponds to simple elementary divisor of $X(T)$. However, computing the eigenvalues of $X(T)$ is very difficult because one needs to compute the unknown Cauchy matrix $X(T)$.

For some specific systems, we may be able to derive the formula for $X(T)$, and then get the stability conditions.

First, we use an elemental method to derive the Floquet–Lyapunov theory.

Let $X(t)$ be the Cauchy matrix solution of (3.5.1). Due to $A(t+T) \equiv A(t)$, it is easy to prove that $X(t+T)$ is the fundamental solution matrix of (3.5.1). Hence, $X(t+T)$ can be expressed by $X(t)$, i.e.,

$$X(t+T) = X(t)C. \tag{3.5.4}$$

Let $t = 0$, then $X(T) = C$. So

$$\begin{aligned}&X(t+T) = X(t)X(T),\\ &\det X(t+T) = \det X(t)\det\bigl(X(T)\bigr) \neq 0.\end{aligned}$$

Let $kT \leqslant t < (k+1)T$, $t = [\frac{t}{T}]T + (\frac{t}{\omega}) := kT + t_1$, $k = [\frac{t}{T}]$, $t_1 = (\frac{t}{\omega})$. Thus, we have

$$\begin{aligned}X(t) &= X(kT + t_1)\\ &= X\bigl((k-1)T + t_1 + T\bigr)\\ &= X\bigl((k-1)T + t_1\bigr)X(T) = \cdots\\ &= X(t_1)X^k(T),\end{aligned} \tag{3.5.5}$$

where $X(t_1)$ is nonzero and is bounded. Equation (3.5.5) shows that

$$\lim_{t\to+\infty} X(t) = 0$$

if and only if

$$\lim_{k\to+\infty} X^k(T) = 0.$$

$X(t)$ is bounded $\Longleftrightarrow$ $X^k(T)$ is bounded $\Longleftrightarrow$ $\rho(X(T)) \leqslant 1$, and the λ satisfying $\lambda(X(T)) = 1$ only corresponds to simple elementary divisor of $X(T)$.

This is main result of Floquet–Lyapunov theory. In the following, for some specific linear periodic systems of (3.5.1), we give the Cauchy matrix's expression and the stability criteria.

THEOREM 3.5.1. *If the following conditions are satisfied*:

(1) $A(t)\int_{t_0}^t A(\tau)\,d\tau = \int_{t_0}^t A(\tau)\,d\tau\, A(t)$;

(2) $B(T) - B(0) := \int_0^T A(t)\,dt$ *is a Hurwitz matrix*;

then the zero solution of (3.5.1) *is exponentially stable.*

PROOF. According to Theorem 3.4.1, the Cauchy matrix solution of (3.5.1) can be repressed as $X(t) = e^{\int_{t_0}^t A(\tau)\,d\tau}$.

Let $t \in [kT, (k+1)T]$, i.e., $t = kT + t_1$. Then, we have

$$X(t) = X(t_1)X^k(T) = X(t_1)e^{kB(T)-B(0)}. \tag{3.5.6}$$

By condition (2) and (3.5.6), there exist constants $M > 0$ and $\alpha > 0$ such that

$$\|x(t)\| \leqslant Me^{-\frac{\alpha kT}{T}} = Me^{\frac{\alpha}{T}t_1}e^{-\frac{\alpha}{T}t} := M^* e^{-\frac{\alpha}{T}t}, \tag{3.5.7}$$

which means that the zero solution of (3.5.1) is exponentially stable. □

THEOREM 3.5.2. *Assume that*

(1) $W(t) = A(t)\int_{t_0}^t A(\tau)\,d\tau - \int_{t_0}^t A(\tau)\,d\tau A(t) \not\equiv 0$;
(2) $W(t)\int_{t_0}^t W(\tau)\,d\tau - \int_{t_0}^t W(\tau)\,d\tau W(t) \equiv 0$;
(3) $A(t)W(t) - W(t)A(t) \equiv 0$, $B_1(T) - B_1(0) = \int_0^T (A(\tau) + \frac{1}{2}W(t))\,dt$ *is a Hurwitz matrix*;

then the zero solution of (3.5.1) *is exponentially stable.*

PROOF. According to Theorem 3.4.2, the Cauchy matrix solution of (3.5.1) can be written as

$$X(t) = e^{\int_0^t A(\tau)\,d\tau}Y(t), \tag{3.5.8}$$

where $Y(t)$ is the Cauchy matrix solution of the following system

$$\frac{dy}{dt} = W(t)y. \tag{3.5.9}$$

By conditions (2) and (3) and the conclusion of Theorem 3.4.2, we have

$$X(t) = e^{\int_0^t A(\tau)\,dt}e^{\int_0^t W(\tau)\,d\tau} = e^{\int_0^t (A(\tau)+W(\tau))\,d\tau}, \tag{3.5.10}$$

where $W(t)$ is a matrix with period T. The rest of the proof follows the proof of Theorem 3.4.2. □

THEOREM 3.5.3. *If the conditions* (1) *and* (2) *of Theorem* 3.5.2 *are satisfied, and*

$$B(T) - B(0) := \int_0^T A(t_1)\,dt_1$$

and

$$\tilde{B}(T) - \tilde{B}(0) := \int_0^T \omega(t_1)\,dt_1$$

are Hurwitz matrices, then the zero solution is exponentially stable.

PROOF. By using (3.5.10), the Cauchy matrix solution of (3.5.1) is

$$X(t) = e^{\int_0^t A(t_1)\,dt_1} e^{\int_0^t W(t_1)\,dt_1}.$$

Let

$$\operatorname{Re}\lambda\big(B(T) - B(0)\big) < -\alpha_1 < 0,$$
$$\operatorname{Re}\lambda\big(\tilde{B}(T) - \tilde{B}(0)\big) < -\alpha_2 < 0.$$

Then, there exists a constant $M > 0$ such that

$$\big\|X(t)\big\| \leqslant Me^{-\frac{\alpha_1}{T}t} e^{-\frac{\alpha_2}{T}t} = Me^{-[\frac{\alpha_1}{T}+\frac{\alpha_2}{T}]t} = Me^{-\frac{\alpha_1+\alpha_2}{T}t},$$

and so the conclusion is true. □

THEOREM 3.5.4. *Suppose that*

(1) *there exists a constant matrix* A_1 *such that*

$$A_1 A(t) - A(t) A_1 = \frac{dA(t)}{dt}; \tag{3.5.11}$$

(2) $A_1 A_2 = A_2 A_1$, *where* $A_2 := e^{-A_1 t_0}[A(t_0) - A_1]e^{A_1 t_0}$;
(3) $A_1 + A_2$ *is a Hurwitz matrix*;

then the zero solution of (3.5.1) *is exponentially stable.*

PROOF. Multiplying respectively $e^{-A_1(t-t_0)}$ and $e^{A_1(t-t_0)}$ to the left-hand and right-hand sides of (3.5.11) we obtain

$$e^{-A_1(t-t_0)} A_1 A(t) e^{A_1(t-t_0)} - e^{-A_1(t-t_0)} A(t) A_1 e^{A_1(t-t_0)}$$
$$= e^{-A_1(t-t_0)} \frac{dA(t)}{dt} e^{A_1(t-t_0)}, \tag{3.5.12}$$

i.e.,

$$\frac{d}{dt}\big[e^{-A_1(t-t_0)} A(t) e^{A_1(t-t_0)}\big] = 0. \tag{3.5.13}$$

Integrate (3.5.13) to obtain

$$A(t) = e^{A_1(t-t_0)} A(t_0) e^{-A_1(t-t_0)}.$$

Hence, (3.5.1) can be rewritten as

$$\frac{dx}{dt} = e^{A_1(t-t_0)} A(t_0) e^{-A_1(t-t_0)} x. \tag{3.5.14}$$

Using the transformation

$$x(t) = e^{A_1 t} y(t),$$

we get

$$\frac{dy}{dt} = e^{-A_1 t_0}\big[A(t_0) - A_1\big] e^{A_1 t_0} y := A_2 y,$$
$$y(t) = e^{A_2(t-t_0)} y(t_0).$$

Finally, we obtain

$$X(t) = e^{A_1 t} e^{A_2(t-t_0)} e^{-A_1(t_0)} x(t_0),$$

which indicates that (3.5.1) has the following Cauchy matrix solution:

$$X(t) = e^{A_1 t} e^{A_2 t} = e^{(A_1+A_2)t}.$$

So there exist constants $M > 0$ and $\alpha > 0$ such that $\|X(t)\| \leqslant Me^{-\alpha t}$. This implies that the zero solution of (3.5.1) is exponentially stable. □

THEOREM 3.5.5. *Assume that*

(1) *the conditions* (1) *and* (2) *of Theorem* 3.5.3 *are satisfied*;
(2) A_1 *and* $A_2 = e^{-A_1 t_0}[A(t_0) - A_1]e^{A_1 t_0}$ *are Hurwitz matrices*;

then the zero solution of (3.5.1) *is exponentially stable.*

PROOF. Following the proof of Theorem 3.5.4, we have the Cauchy matrix solution of (3.5.1), $X(t) = e^{A_1 t} e^{A_2 t}$. There exist constants $M_1 > 0$, $M_2 > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\big\|X(t)\big\| \leqslant M_1 e^{-\alpha_1 t} M_2 e^{-\alpha_2 t} = M_1 M_2 e^{-(\alpha_1+\alpha_2)t}.$$

So the condition is true. □

EXAMPLE 3.5.6. Discuss the stability of the following periodic system

$$\begin{cases} \frac{dx_1}{dt} = (a + \cos bt)x_1 - \frac{1}{2}x_2, \\ \frac{dx_2}{dt} = \frac{1}{2}x_1 + (a + \cos bt)x_2. \end{cases} \tag{3.5.15}$$

Obviously, the coefficient matrix $A(t)$ satisfies

$$A(t)\int_{t_0}^{t} A(t_1)\,dt_1 = \int_{t_0}^{t} A(t_1)\,dt_1 A(t)$$

and

$$\int_0^{\frac{\alpha\pi}{|b|}} A(t_1)\,dt = B\left(\frac{\alpha\pi}{|b|}\right) - B(0) = \begin{bmatrix} a\frac{\alpha\pi}{|b|} & -\frac{1}{\alpha}\frac{\alpha\pi}{|b|} \\ -\frac{1}{\alpha}\frac{\alpha\pi}{|b|} & a\frac{\alpha\pi}{|b|} \end{bmatrix}.$$

$B(\frac{\alpha\pi}{|b|}) - B(0)$ is a Hurwitz matrix if and only if $\alpha < -\frac{1}{2}$.

$B(\frac{\alpha\pi}{|b|}) - B(0)$ is quasi-stable if and only if $\alpha = -\frac{1}{2}$.

Hence, the zero solution of system (3.5.15) is exponentially stable if and only if $\alpha < -\frac{1}{2}$; and is stable if and only if $\alpha = -\frac{1}{2}$.

3.6. Spectral estimation for linear systems

Let $x^{(k)}(t)$ be an arbitrary solution of (3.1.2). Then,

$$\alpha := \overline{\lim_{t\to\infty}} \frac{1}{t} \ln\|x^{(k)}(t)\|$$

is called eigenexponent of $x^{(k)}$. All eigenexponents form a set called the spectral of systems (3.1.2).

Lyapunov proved that if $\|A(t)\| \leqslant c < \infty$, then for every solution $x(t) \neq 0$, the eigenexponent is infinite.

Let $x(t)$ be any solution of (3.1.2), we have the following estimation:

$$\|x(t_0)\| e^{\int_{t_0}^t \lambda(t_1)\,dt_1} \leqslant \|x(t)\| \leqslant \|x(t_0)\| e^{\int_{t_0}^t \Lambda(t_1)\,dt_1} \tag{3.6.1}$$

where $\|\cdot\|$ denotes the Euclidean norm, $\lambda(t)$ and $\Lambda(t)$ are respectively the minimum and maximum eigenvalues of the matrix

$$A^H = \frac{1}{2}\left[A(t) + A^T(t)\right]. \tag{3.6.2}$$

Hence, from (3.6.2) one can obtain the interval of the spectral, $[l, L]$, where

$$l = \underline{\lim_{t\to\infty}} \frac{1}{t} \int_{t_0}^{t} \lambda(t_1)\,dt_1,$$

$$L = \overline{\lim_{t\to\infty}}\, \frac{1}{t}\int_{t_0}^{t} \Lambda(t_1)\, dt_1.$$

Further, it follows that

(1) if $L < 0$, then the zero solution of (3.1.2) is exponentially stable;
(2) if $l > 0$, then the zero solution of (3.1.2) is unstable.

Next, we present a general result.

DEFINITION 3.6.1. The matrix $L(t) = (\ell_{ij}(t))_{n\times n} \in C[I, R^{n\times n}]$ is called Lyapunov matrix, if

(1) $\|\dot{L}(t)\| < \infty$ and $\|L(t)\| < \infty$,
(2) $|\det L(t)| > m > 0$;

and $y = L(t)x$ is called Lyapunov transformation.

THEOREM 3.6.2. *Take a Lyapunov transformation*

$$g = L(t)x := \operatorname{diag}\bigl(\ell_{11}(t), \ldots, \ell_{nn}(t)\bigr)x.$$

Then, any solution of (3.1.2) *has the estimation*:

$$\begin{aligned}
&\bigl\|L(t)\bigr\|^{-1}\bigl\|L(t_0)x(t_0)\bigr\|e^{\int_{t_0}^{t}\tilde{\lambda}(t_1)\,dt_1}\\
&\quad \leqslant \bigl\|x(t)\bigr\| \leqslant \bigl\|L^{-1}(t)\bigr\|\bigl\|L(t_0)x(t_0)\bigr\|e^{\int_{t_0}^{t}\tilde{\Lambda}(t_1)\,dt_1},
\end{aligned} \tag{3.6.3}$$

where $\|\cdot\|$ *denotes the Euclidean norm, and* $\tilde{\lambda}(t)$ *and* $\tilde{\Lambda}(t)$ *are respectively minimum and maximum eigenvalues of the matrix*

$$\tilde{A}^H := \frac{1}{2}\bigl[\tilde{A}(t) + \tilde{A}^T(t)\bigr], \tag{3.6.4}$$

where

$$\tilde{A} := \bigl(L(t)A(t)L^{-1}(t) - L(t)\dot{L}^{-1}(t)\bigr). \tag{3.6.5}$$

PROOF. Since $L(t)$ is a Lyapunov transformation, the inverse transformation $L^{-1}(t)$ exists. Let $x(t) = L^{-1}(t)y$. Then,

$$\frac{dy}{dt} = \bigl(L(t)A(t)L^{-1}(t) - \dot{L}(t)L^{-1}(t)\bigr)y := \tilde{A}(t)y. \tag{3.6.6}$$

If $y = (y_1, \ldots, y_n)^T$ is a solution of (3.3.6), Then, $y^* = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$ is the solution of

$$\frac{dy^*}{dt} = y^* A^T. \tag{3.6.7}$$

As a result, we have

$$\frac{d\|y\|^2}{dt} = \frac{dy^*y}{dt} = y^*\frac{dy}{dt} + \frac{dy^*}{dt}y = y^*\tilde{A}y + y^*\tilde{A}^T y$$
$$= y^*\big(\tilde{A}(t) + \tilde{A}^T(t)\big)y = 2y^*\tilde{A}^H y. \tag{3.6.8}$$

But since $\tilde{\lambda}(t)y^*y \leqslant y^*A^H y \leqslant \tilde{\Lambda}(t)y^*y$, we obtain

$$2\tilde{\lambda}(t)\|y(t)\|^2 \leqslant \frac{d\|y\|^2}{dt} \leqslant 2\tilde{\Lambda}(t)\|y(t)\|^2,$$
$$\|y(t_0)\|^2 e^{2\int_{t_0}^t \tilde{\lambda}(t_1)\,dt_1} \leqslant \|y(t)\|^2 \leqslant \|y(t_0)\|^2 e^{2\int_{t_0}^t \tilde{\Lambda}(t_1)\,dt_1},$$
$$\|y(t_0)\| e^{\int_{t_0}^t \tilde{\lambda}(t_1)\,dt_1} \leqslant \|y(t)\| \leqslant \|y(t_0)\| e^{\int_{t_0}^t \tilde{\Lambda}(t_1)\,dt_1}. \tag{3.6.9}$$

Furthermore, we have

$$y(t) = L(t)x(t),$$
$$\|y(t)\| \leqslant \|L(t)\|\,\|x(t)\|,$$
$$\|x(t)\| \leqslant \|L^{-1}(t)\|\,\|y(t)\|,$$

which yields

$$\|L(t)\|^{-1}\|y(t_0)\| e^{\int_{t_0}^t \tilde{\lambda}(t_1)\,dt_1}$$
$$= \|L(t)\|^{-1}\|L(t_0)\|\,\|x(t_0)\| e^{\int_{t_0}^t \tilde{\lambda}(t_1)\,dt_1} \leqslant \|x(t)\|, \tag{3.6.10}$$

and

$$\|x(t)\| \leqslant \|L^{-1}(t)\|\,\|L(t_0)x(t_0)\| e^{\int_{t_0}^t \tilde{\Lambda}(t_1)\,dt_1}. \tag{3.6.11}$$

Combining (3.6.10) and (3.6.11) shows that (3.6.3) is true.

The proof is completed. □

COROLLARY 3.6.3. *The spectral of systems* (3.1.2) *is distributed on interval* $[\underline{l}, \overline{L}]$, *where*

$$\underline{l} = \varliminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \tilde{\lambda}(t_1)\,dt_1,$$

$$\tilde{L} = \varlimsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \tilde{\Lambda}(t_1)\,dt_1.$$

If $\tilde{L} < 0$, *the zero solution of* (3.1.2) *is exponentially stable*; *and if* $\underline{l} > 0$, *the zero solution of* (3.1.2) *is unstable.*

PROOF. Since $L(t)$ and $L^{-1}(t)$ are bounded, we have

$$\lim_{t\to+\infty}\frac{1}{t}\ln\|L(t)\|=0,\qquad \lim_{t\to+\infty}\frac{1}{t}\ln\|L^{-1}(t)\|=0.$$

Hence,

$$\underline{\lim}_{t\to+\infty}\frac{1}{t}\ln\|L(t)\|^{-1}\|L(t_0)x(t_0)\|e^{\int_{t_0}^t\tilde{\lambda}(t_1)\,dt_1}=\underline{\lim}_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\tilde{\lambda}(t_1)\,dt_1,$$

$$\overline{\lim}_{t\to+\infty}\frac{1}{t}\ln\|L^{-1}(t)\|\left\|L(t_0)x(t_0)e^{\int_{t_0}^t\tilde{\lambda}(t_1)\,dt_1}\right\|=\overline{\lim}_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\tilde{\lambda}(t_1)\,dt_1,$$

$$\begin{aligned}\tilde{l}:=\underline{\lim}_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\tilde{\Lambda}(t_1)\,dt_1&\leqslant\overline{\lim}_{t\to+\infty}\frac{1}{t}\ln\|x(t)\|\\&\leqslant\overline{\lim}_{t\to+\infty}\frac{1}{t}\int_{t_0}^t\tilde{\Lambda}(t_1)\,dt_1:=\tilde{L}.\end{aligned}$$

Thus, the conclusion is true. □

COROLLARY 3.6.4. *If $L(t)=L=\operatorname{diag}(L_{11},\dots,L_{nn})$ is a constant matrix, then $\tilde{A}=(LA(t)L^{-1})$. If $L(t)=I_n$, then $l=\underline{l}$, $L=\overline{L}$.*

EXAMPLE 3.6.5. Consider the stability of the system

$$\begin{cases}\frac{dx_1}{dt}=-\left(1+\frac{1}{t}\right)x_1+81(1+\sin t)x_2,\\ \frac{dx_2}{dt}=-(1+\sin t)x_1-\left(1+\frac{1}{t}\right)x_2.\end{cases}\tag{3.6.12}$$

It is easy to obtain that

$$\begin{aligned}A(t)&=\begin{bmatrix}-1-\frac{1}{t}&81(1+\sin t)\\-(1+\sin t)&-(1+\frac{1}{t})\end{bmatrix},\\ \frac{1}{2}\left[A(t)+A^T(t)\right]&=\begin{bmatrix}-1-\frac{1}{t}&40(1+\sin t)\\40(1+\sin t)&-1-\frac{1}{t}\end{bmatrix},\\ \lambda&=\pm40(1+\sin t)-1-\frac{1}{t},\\ \lambda(t)&=-41-\frac{1}{t}-40\sin t,\\ \Lambda(t)&=39-\frac{1}{t}+40\sin t,\end{aligned}$$

$$l = \underline{\lim}_{t\to\infty} \int_{t_0}^{t} \tilde{\lambda}(t_1)\, dt_1 = -40,$$

$$L = \overline{\lim}_{t\to\infty} \int_{t_0}^{t} \Lambda(t_1)\, dt = 39.$$

Since we only know that the spectral of (3.6.12) is distributed on interval $[-40, 39]$, we cannot determine the stability of the system.

Now using Theorem 3.6.2, let $L = \operatorname{diag}(1, 9)$, $L^{-1} = \operatorname{diag}(1, \frac{1}{9})$. Then, we have

$$\begin{aligned}
L &= \operatorname{diag}(1, 9), \quad L^{-1} = \operatorname{diag}\left(1, \frac{1}{9}\right), \\
\tilde{A} &= \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} -1 - \frac{1}{t} & 81(1 + \sin t) \\ -(1 + \sin t) & -1 - \frac{1}{t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \\
&= \begin{bmatrix} -1 - \frac{1}{t} & 9(1 + \sin t) \\ -9(1 + \sin t) & -1 - \frac{1}{t} \end{bmatrix}, \\
\frac{1}{2}\left[\tilde{A} + \tilde{A}^T\right] &= \begin{bmatrix} -1 - \frac{1}{t} & 0 \\ 0 & -1 - \frac{1}{t} \end{bmatrix}, \\
\tilde{\lambda}(t) &= -1 - \frac{1}{t}, \quad \tilde{\Lambda}(t) = -1 - \frac{1}{t}.
\end{aligned}$$

Thus,

$$\underline{l} = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \tilde{\lambda}(t_1)\, dt_1 = -1,$$

$$\tilde{L} = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \tilde{\Lambda}(t_1)\, dt_1 = -1,$$

implying that the spectral of the system is -1. So the zero solution of the system is exponentially stable.

3.7. Partial variable stability of linear systems

Partial variable stability theory is very useful and is still an active area. For stationary linear systems, many results on partial stability have been obtained. For time-varying linear systems, however, general theory of partial variable stability has not been established.

In this section, we give a series of necessary and sufficient conditions for partial variable stability of linear time-varying systems.

We rewrite (3.1.1) and (3.1.2) as

$$\begin{cases} \frac{dy}{dt} = A_{11}(t)y + A_{12}(t)z + f_I(t), \\ \frac{dz}{dt} = A_{21}(t)y + A_{22}(t)z + f_{II}(t), \end{cases} \tag{3.7.1}$$

$$\begin{cases} \frac{dy}{dt} = A_{11}(t)y + A_{12}(t)z, \\ \frac{dz}{dt} = A_{21}(t)y + A_{22}(t)z. \end{cases} \tag{3.7.2}$$

Let

$$\begin{aligned} y &:= (x_1, \dots, x_m)^T, \\ z &:= (x_{m+1}, \dots, x_n)^T, \\ y_{(n)} &:= (x_1, \dots, x_m, \overbrace{0, \dots, 0}^{n-m})^T, \quad 1 \leqslant m \leqslant n, \\ A_{11}(t) &= \big(a_{ij}(t_1)\big)_{m\times m}, \quad i \leqslant i, j \leqslant m, \\ A_{12}(t) &= \big(a_{ij}(t)\big)_{m\times(n-m)}, \quad 1 \leqslant i \leqslant m,\ m+1 \leqslant j \leqslant n, \\ A_{21}(t) &= \big(a_{ij}(t)\big)_{(n-m)\times n}, \quad m+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m, \\ A_{22}(t) &= \big(a_{ij}(t)\big)_{(n-m)\times(n-m)}, \quad m+1 \leqslant i,\ j \leqslant n, \\ f_I(t) &= \big(f_1(t), f_2(t), \dots, f_m(t_1)\big)^T, \\ f_{II}(t) &= \big(f_{m+1}(t), \dots, f_n(t)\big)^T, \\ f(t) &= \big(f_I(t), f_{II}(t)\big)^T. \end{aligned}$$

$K(t, t_0)$ is the Cauchy matrix solution of (3.7.2), $k_m(t, t_0) := E_m K(t, t_0)$ is called cut matrix of $K(t, t_0)$, where $E_m = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$, I_m is $m \times m$ unit matrix.

DEFINITION 3.7.1. The zero solution of (3.7.2) is said to be stable with respect to variable y, if $\forall \varepsilon > 0$, $\forall t_0$, $\exists \delta(t_0, \varepsilon)$, $\forall x_0 \in S_\delta := \{x, \|x\| < \delta\}$ such that the following condition

$$\big\|y(t, t_0, x_0)\big\| < \varepsilon, \quad t \geqslant t_0,$$

holds. If $\delta(t_0, \varepsilon) = \delta(\varepsilon)$, i.e., δ is independent of t_0, then the zero solution is said to be uniformly stable with respect to variable y.

DEFINITION 3.7.2. The zero solution of (3.7.2) is said to be attractive with respect to variable y, if $\forall t_0 \in I$, $\exists \sigma(t_0)$, $\forall \varepsilon > 0$, $\forall x_0 \in S_{\sigma(t_0)} = \{x, \|x\| \leqslant \sigma(t_0)\}$, $\exists T(t_0, x_0, \varepsilon)$ such that when $t \geqslant t_0 + T$,

$$\big\|y(t, t_0, x_0)\big\| < \varepsilon.$$

If $\sigma(t_0) = \sigma$, $T(t_0, x_0, \varepsilon) = T(\varepsilon)$, then the zero solution of (3.7.2) is said to be uniformly attractive with respect to variable y.

DEFINITION 3.7.3. The zero solution of (3.7.2) is said to be asymptotically stable with respect to variable y, if $x = 0$ is stable and attractive with respect to variable y.

Similarly we can define uniformly asymptotic stability, equi-asymptotic stability, globally asymptotic stability, exponential stability with respect to variable y.

THEOREM 3.7.4. $\forall f(t) \in C[I, R^n]$, *system* (3.7.1) *has certain class of stability with respect to variable y if and only if the zero solution of* (3.1.2) *has the same stability with respect to variable y.*

PROOF. Any solution $x(t)$ of (3.7.1) corresponding to $y_{(n)}(t)$ can be expressed as

$$y_{(n)}(t) = E_m X(t) = K_m(t, t_0)x(t_0) + \int_{t_0}^{t} K_m(t, \tau) f(\tau)\, d\tau. \tag{3.7.3}$$

Then any perturbed solution $\tilde{x}(t)$ of $x(t)$ corresponding to $\hat{y}_{(n)}(t)$ can be expressed as

$$\hat{y}_{(n)}(t) = K_m(t, t_0)\hat{x}(t_0) + \int_{t_0}^{t} K_m(t, \tau) f(\tau)\, d\tau. \tag{3.7.4}$$

Hence, one can obtain

$$y_{(n)}(t) - \tilde{y}_n(t) = K_m(t, t_0)\big(x(t_0) - \tilde{x}(t_0)\big). \tag{3.7.5}$$

Let any perturbed solution for the zero solution of (3.7.2) corresponding to $y_{(n)}(t)$ be $\eta(t)$. Obviously, $\eta_{(n)}(t) = K_m(t, t_0)\eta(t_0)$.

Let $\eta(t_0) = x(t_0) - \tilde{x}(t_0)$, $\eta_{(n)}(t_0) = y_{(n)}(t_0) - \tilde{y}_n(t_0)$. Then,

$$\eta_{(n)}(t) = K_m(t, t_0)\big(x(t_0) - \tilde{x}(t_0)\big). \tag{3.7.6}$$

Equations (3.7.4) and (3.7.6) imply that the conclusion is true. □

COROLLARY 3.7.5. *Any solution of* (3.7.2) *has certain type of stability with respect to partial variable y if and only if the zero solution of* (3.7.2) *has the same stability with respect to variable y.*

THEOREM 3.7.6. *The zero solution of* (3.7.2) *is stable* (*uniformly stable*) *if and only if $K_m(t, t_0)$ is bounded* (*uniformly bounded*).

PROOF. *Sufficiency*. Any perturbed solution $x(t)$ of the zero solution corresponding to $y_{(n)}$ can be written as

$$y_{(n)} = K_m(t, t_0)x(t_0), \tag{3.7.7}$$

where $K_m(t, t_0)$ is bounded (uniformly bounded). So there exists a constant $M(t_0)$ ($M > 0$) such that $\|K_m(t, t_0)\| \leqslant M(t_0)$ ($\|K_m(t, t_0)\| \leqslant M$) for $t \geqslant t_0$. $\forall \varepsilon > 0$, take $\delta(\varepsilon, t_0) = \frac{\varepsilon}{M(t_0)}[\delta(\varepsilon) = \frac{\varepsilon}{M}]$. Then, when $\|x(t_0)\| < \delta$, we have

$$\|y(t)\| = \|y_{(n)}(t)\| = \|K_m(t, t_0)\| \|x(t_0)\| < \varepsilon, \quad t \geqslant t_0. \tag{3.7.8}$$

Hence the zero solution of (3.7.2) is stable (uniformly stable) with respect to variable y.

Necessity. Suppose the zero solution of (3.7.2) is stable (uniformly stable) with respect to y. Then, for ε_0, $\exists\delta(\varepsilon_0, t_0) > 0$ ($\delta(\varepsilon_0) > 0$) such that

$$\begin{aligned} &\|x(t_0)\| < \delta, \\ &\|y(t)\| = \|y_{(n)}(t)\| = \|K_m(t, t_0)x(t_0)\| < \varepsilon_0 \quad (t \geqslant t_0). \end{aligned}$$

Take

$$x(t_0) = \frac{\delta}{2}(\overbrace{0, \ldots, 0}^{k-1}, 1, \overbrace{0, \ldots, 0}^{n-k})^T := \frac{\delta}{2}e_k, \quad 1 \leqslant k \leqslant n. \tag{3.7.9}$$

Then, the kth column of $K_m(t, t_0)$ can be expressed as

$$\big(x_{1k}(t), \ldots, x_{mk}(t), 0, \ldots, 0\big)^T = \frac{2}{\delta}K_m(t, t_0)x(t_0), \quad k = 1, 2, \ldots, n,$$

and $\|x(t_0)\| = \frac{\delta}{2} < \delta$. So

$$\big\|\big(x_{1k}(t), \ldots, x_{mk}(t), 0, \ldots, 0\big)^t\big\| \leqslant 2\frac{\varepsilon_0}{\delta} := M, \quad k = 1, 2, \ldots, n.$$

This implies that $K_m(t, t_0)$ is bounded (uniformly bounded).

The proof is complete. □

THEOREM 3.7.7. *The zero solution of* (3.7.2) *is asymptotically stable with respect to variable y if and only if it is attract with respect to variable y.*

PROOF. *Necessity* is obvious.

For *sufficiency*, by the conditions, $\forall t_0 \in I$, $\exists\sigma(t_0) > 0$ such that when $\|x(t_0)\| \leqslant \sigma(t_0)$, $\|y(t)\| = \|y_{(n)}(t)\| = \|K_m(t, t_0)x(t_0)\| \to 0$ as $t \to +\infty$. Take

$$x(t_0) = (\overbrace{0, \ldots, 0}^{k-1}, 1, \overbrace{0, \ldots, 0}^{n-k})^T, \quad \frac{\sigma(t_0)}{2} := \frac{\sigma(t_0)}{2}e_k.$$

Then, we have

$$\left(x_{1k}(t), \ldots, x_{mk}(t), 0, \ldots, 0\right)^T = \frac{2}{\sigma(t_0)} K_m(t, t_0) \frac{\sigma(t_0)}{2} e_k \to 0 \quad \text{as} \quad t \to +\infty.$$

Hence when $t \geqslant t_0$, $K_m(t, t_0)$ is bounded. According to Theorem 3.7.2, the conclusion is true. □

THEOREM 3.7.8. *The zero solution of* (3.7.2) *is uniformly asymptotically stable with respect to variable y, if and only if it is uniformly attractive with respect to variable y and $K_m(t, t_0)$, $t \geqslant t_0$, is uniformly bounded.*

PROOF. Since the zero solution of (3.7.2) is uniformly asymptotically stable with respect to variable y, it is uniformly stable and uniformly attractive with respect to y. By Theorem 3.7.7, the uniform stability of the zero solution with respect to y is equivalent to the uniform boundedness of $K_m(t, t_0)$, $t \geqslant t_0$. So the conclusion is true. □

THEOREM 3.7.9. *The zero solution of* (3.7.2) *is asymptotically stable* (*uniformly asymptotically stable*) *if and only if $K_m(t, t_0) \to 0$ as $t \to \infty$ ($K_m(t, t_0) \rightrightarrows 0$ as $t - t_0 \to +\infty$ uniformly holds for t_0).*

PROOF. *Sufficiency* is obvious.

Necessity. By Theorems 3.7.6 and 3.7.7, we only need to prove the conditions of Theorem 3.7.9 are necessary for the zero solution to be attract (uniformly attractive) with respect to variable y. But this is just the conclusion of Theorem 3.7.6. □

THEOREM 3.7.10. *The zero solution of* (3.7.2) *is asymptotically stable* (*uniformly asymptotically stable*) *with respect to variable y if and only if it is globally asymptotically stable* (*globally uniformly asymptotically stable*).

The proof of the theorem is left to readers as an exercise.

THEOREM 3.7.11. *The zero solution of* (3.7.2) *is asymptotically stable with respect to y if and only if it is equi-asymptotically stable with respect to y.*

PROOF. We only need to prove that the zero solution being attractive with respect to y implies the equi-attractive of the zero solution with respect to y.

In fact, by Theorems 3.7.7 and 3.7.8, we have

$$K_m(t, t_0) \to 0 \quad \text{as } t \to +\infty.$$

Then, $\forall t_0 \in I$, $\forall \alpha > 0$, $\forall \varepsilon > 0$, $\exists T(t, \sigma, \varepsilon) > 0$ such that $\|K_m(t, t_0)\| < \varepsilon/\sigma$ for $t > T$. Therefore, when $\|x(t_0)\| < \sigma$, $t > T$, we obtain

$$\|y(t)\| = \|y_{(n)}\| = \|K_m(t, t_0)x(t_0)\| < \varepsilon.$$

This implies that the zero solution of (3.7.2) is equi-attractive with respect to variable y. □

THEOREM 3.7.12. *The zero solution of* (3.7.2) *is exponentially stable with respect to variable y if and only if there exist constants $M \geqslant 1$ and $\alpha > 0$ such that*

$$\|K_m(t, t_0)\| \leqslant Me^{-\alpha(t-t_0)}, \quad t \geqslant t_0. \tag{3.7.10}$$

PROOF. *Sufficiency* is obvious.

Necessity. If the zero solution of (3.7.2) is exponentially stable with respect to y, then there exist constants $M \geqslant 1$ and $\alpha > 0$ such that $\forall t_0 \in I$, $\forall x_0 \in R^n$ the following expression

$$\|y(t)\| = \|y_{(n)}\| = \|K_m(t, t_0)x(t_0)\| \leqslant M\|x(t_0)\|e^{-\alpha(t-t_0)} \quad \forall t \geqslant t_0,$$

holds. Hence, we have

$$\|K_m(t, t_0)\| \leqslant \sup_{\|x(t_0)\|=1} \|K_m(t, t_0)x(t_0)\| \leqslant Me^{-\alpha(t-t_0)}, \quad t \geqslant t_0.$$

The proof is complete. □

Generally, computing $K_m(t, t_0)$ is difficult, but sometimes estimating $\|K_m(t, t_0)\|$ is possible.

EXAMPLE 3.7.13. Discuses the stability of the zero solution with respect to x_1 for the following system

$$\begin{cases} \frac{dx_1}{dt} = -2x_1 + \frac{1}{t+1}e^{-rt}x_2, \\ \frac{dx_2}{dt} = \frac{-10}{1+t}e^{-rt}x_1 + x_2, \end{cases} \tag{3.7.11}$$

where $r \geqslant 1$ is a constant.

Let $(\xi_1(t), \xi_2(t))^T$, $(\eta_1(t), \eta_2(t))^T$ be solutions of (3.7.11) satisfying the initial conditions $\xi_1(t_0) = 1$, $\xi_2(t_0) = 0$ and $\eta_1(t) = 0$, $\eta_2(t_0) = 1$ respectively. Then, we have

$$K_1(t, t_0) = \begin{bmatrix} e^{-2(t-t_0)} + \int_{t_0}^t e^{-2(t-\tau)}\frac{1}{1+\tau}e^{-r\tau}\xi_2(\tau)\,d\tau & \int_{t_0}^t e^{-2(t-\tau)}\frac{1}{1+\tau}e^{-r\tau}\eta_2(\tau)\,d\tau \\ 0 & 0 \end{bmatrix}.$$

For any solution $(x_1(t), x_2(t))^T$, by Theorem 3.7.12 the estimation is given by

$$\left|x_2(t)\right| \leqslant \left|x_1(t)\right| + \left|x_2(t)\right| \leqslant M\left(\left|x_1(t)\right| + \left|x_2(t_0)\right|\right)e^{t-t_0},$$

where $M > 0$ is a constant. So when $t \geqslant t_0 > 0$, we have

$$\left\|K_1(t, t_0)\right\| \leqslant Ne^{-2(t-t_0)} + NMe^{-t_0} \int_{t_0}^{t} e^{-2(t-\tau)} \frac{1}{\tau+1} e^{-\tau r} e^{\tau}\, d\tau,$$

where $N > 0$ is a constant.

When $r = 1$ it is easy to show that

$$NMe^{-t_0} \int_{t_0}^{t} e^{-2(t-\tau)} \frac{1}{\tau+1} e^{-\tau} e^{\tau}\, d\tau \to 0, \quad \text{as} \quad t \to +\infty.$$

Therefore, $K_1(t, t_0) \to 0$ as $t \to +\infty$, implying that the zero solution of (3.7.11) is asymptotically stable with respect to variable x_1.

When $r > 1$, one can prove that there exist constants $c > 0$ and $\varepsilon > 0$ such that

$$NMe^{-t_0} \int_{t_0}^{t} \frac{1}{\tau+1} e^{-2(t-\tau)} e^{-r\tau} e^{\tau}\, d\tau \leqslant ce^{-\varepsilon(t-t_0)} \quad \forall\, t \geqslant t_0.$$

Hence, $\|K_1(t, t_0)\| \leqslant Le^{-\beta(t-t_0)}$, where $\beta = \min(2, \varepsilon)$ and L is a constant. This shows that the zero solution of the system is exponentially stable with respect to variable x_1.

Chapter 4

Lyapunov Direct Method

It is well know that Lyapunov direct method plays the key role in the stability study of dynamical systems. Historically, Lyapunov presented four celebrated original theorems on stability, asymptotic stability and instability, which are now called the principal theorems of stability which are fundamental to stability of dynamical systems.

Many researchers have extensively studied the Lyapunov direct method, in order to explore the effectiveness of the principal theorems. While such work was difficulty to perform, many good results were achieved. In particular, it was clarified that almost all of the principal theorems have their inverses. This motivated the development of stability theory. On the other hand, researchers have also extended and improved the principal theorems, and investigated how to constructing Lyapunov functions. It had took a long time and hard work from the presentation of Lyapunov principal theorems to establishing the inverse theorems.

In this chapter, we first give an example to show the Lyapunov direct method geometrically. Then, we present Lyapunov principal theorems and their extended versions together with their inverse theorems, since some of the inverse theorems cannot be found in the existing literature or they are not expressed in detail. The advantage of this treatment is that the panorama could be clarified easily.

From the view point of applications, inverse theorems are not perfect, since, in their proofs (i.e., the necessary part of the theorems), the Lyapunov functions are constructed based on the solutions of the equations. By this reason, we will present the extensions of various stability theorems for applications. Furthermore, Lyapunov direct method is not merely limited to stability study in the Lyapunov sense. The results of using the Lyapunov direct method to other applications will be discussed in Chapter 5.

The main results presented in this chapter are chosen from [234] for Section 4.1, [298,330,331] for Section 4.2, [280,299,331,332] for Section 4.3, [98] for Section 4.4, [292] for Section 4.5, [292] for Section 4.6, [76] for Section 4.7 and [234] for Section 4.8.

4.1. Geometrical illustration of Lyapunov direct method

First, we illustrate the geometrical meaning of the Lyapunov direct method by using a two-dimensional autonomous systems, described by:

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, x_2), \\ \frac{dx_2}{dt} = f_2(x_1, x_2), \end{cases} \tag{4.1.1}$$

where $f_1, f_2 \in C[I, R^2]$ satisfying $f_1(0,0) = f_2(0,0) = 0$, and assume that the solution of (4.1.1) is unique.

Let $V(x) = V(x_1, x_2) \in K$ and $V(x) \in C^1[R^2, R^1]$. The solution $x(t) = (x_1(t), x_2(t))^T$ is unknown or finding solution is very difficult, but assume that its derivative satisfies

$$\big(\dot{x}_1(t), \dot{x}_2(t)\big) = \big(f_1(x_1, x_2), f_2(x_1, x_2)\big).$$

If we substitute the solution $x(t)$ into function $V(t)$, we have $V(t) := V(x(t))$. Then the stability, asymptotic stability and instability can be normally described as

"Be around the origin and goes to the origin $x_1 = x_2 = 0$",
"Does not leave the origin" and
"Leaves the origin",

which are equivalent to $V(x(t))$ being nonincreasing, decreasing and increasing, respectively, i.e.,

$$\frac{dV(x(t))}{dt} \leqslant 0, \quad \frac{dV(x(t))}{dt} < 0 \quad \text{and} \quad \frac{dV(x(t))}{dt} > 0.$$

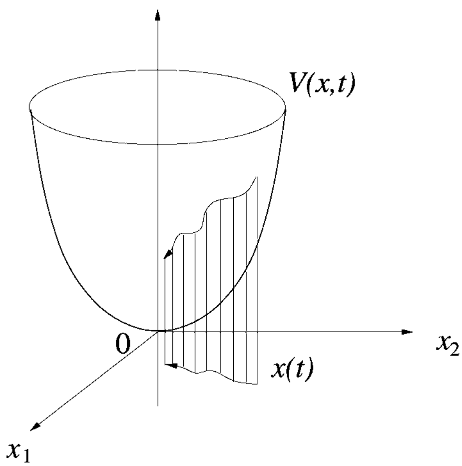

Figure 4.1.1. Geometric expression of the Lyapunov direct method.

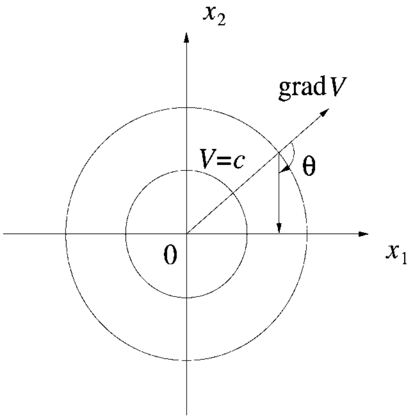

Figure 4.1.2. Geometric expression of the Lyapunov function method.

This is shown in Figure 4.1.1.

$$\frac{dV}{dt} = \sum_{i=1}^{2} \frac{\partial V}{\partial x_i}\frac{\partial V}{dt} = \sum_{i=1}^{2} \frac{\partial V}{\partial x_i} f_i(x_1, x_2) = \text{grad}\, V \bullet f$$
$$\begin{cases} < 0, & \text{when } \theta > \frac{\pi}{2}, \\ = 0, & \text{when } \theta = \frac{\pi}{2}, \\ > 0, & \text{when } \theta < \frac{\pi}{2}, \end{cases}$$

where θ is the angle between the directions of grad V and the vector f. (See Figure 4.1.2.)

However, the last expression is independent of the solution $x(t)$, but only depends on the function $V(x)$ and the known vector $f(x)$. This is the original, geometrical idea of the Lyapunov direct method.

4.2. NASCs for stability and uniform stability

Consider the general n-dimensional nonautonomous system:

$$\frac{dx}{dt} = f(t, x) \tag{4.2.1}$$

where $x = (x_1, \ldots, x_n)^T \in R^n$, $f = (f_1, f_2, \ldots, f_n)^T \in C[I \times R^n, R^n]$ which assures the uniqueness of the solution of (4.2.1) and $f(t, 0) \equiv 0$.

THEOREM 4.2.1. *(See [298,330].) The necessary and sufficient condition* (*NASC*) *for the zero solution of system,* (4.2.1) *being stable is that there exists* *a positive definite function* $V(t,x) \in C[I \times G_H, R^1]$ *such that along the solution of* (4.2.1),

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \leqslant 0 \tag{4.2.2}$$

holds, where $G_H := \{(t,x),\ t \geqslant t_0,\ \|x\| < H = constant\}$.

PROOF. *Sufficiency* (see [298]). Since $V(t,x)$ is positive definite, there exists $\varphi(\|x\|) \in K$ such that

$$V(t,x) \geqslant \varphi\big(\|x\|\big). \tag{4.2.3}$$

$\forall \varepsilon > 0$ $(0 < \varepsilon < H)$, $\exists \delta(t_0,\varepsilon) > 0$ such that $V(t_0,0) = 0$ and $V(t_0,x) \geqslant 0$ is continuous, and when $\|x_0\| < \delta(t_0,\varepsilon)$ we have

$$V(t_0,x_0) < \varphi(\varepsilon). \tag{4.2.4}$$

Equations (4.2.3) and (4.2.4) imply that

$$V\big(t, x(t,t_0,x_0)\big) \leqslant V(t_0,x_0) < \varphi(\varepsilon) \quad (t \geqslant t_0).$$

Further,

$$\varphi\big(\big\|x(t,t_0,x_0)\big\|\big) \leqslant V\big(t, x(t_0,x_0)\big) \leqslant V(t_0,x_0) \leqslant \varphi(\varepsilon) \quad (t \geqslant t_0).$$

It follows from $\varphi \in K$ that

$$\big\|x(t,t_0,x_0)\big\| < \varepsilon \quad (t \geqslant t_0),$$

i.e., the zero solution of (4.2.1) is stable.

Necessity. Let $x(t,t_0,a)$ be a solution of (4.2.1). By uniqueness of solution, we have $a(t_0,t,x) \equiv a$ (see Figure 4.2.1).

Let

$$V(t,x) = \big(1 + e^{-t}\big)\big\|a(t_0,t,x)\big\|^2. \tag{4.2.5}$$

When t and x vary on a same integral curve, $a(t_0,t,x)$ is not varied. But when t and x vary on different integral curves, $a(t_0,t_0,x)$ takes different values. By the continuity theorem on the initial value, we know that $V(t,x)$ of (4.2.5) is continuous.

(1) Prove that $V(t,x)$ is positive definite. $\forall \varepsilon > 0$, $\exists \delta(t_0,\varepsilon) > 0$, when $\|x\| < \delta$,

$$\big\|x(t,t_0,a)\big\| < \varepsilon \quad (t \geqslant t_0).$$

Thus, for $\varepsilon \leqslant \|x\| \leqslant H$, we have

$$\big\|a(t_0,t,x)\big\| = \|a\| \geqslant \delta > 0. \tag{4.2.6}$$

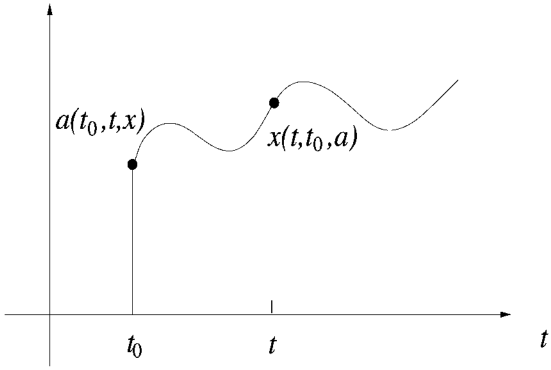

Figure 4.2.1. The relation between $a(t_0, t, x)$ and $x(t, t_0, a)$.

Hence, when $\varepsilon \leqslant \|x\| \leqslant H$,

$$V(t, x) \geqslant \left\|a(t_0, t, x)\right\|^2 = \|a\|^2 \geqslant \delta^2 := \eta > 0.$$

Now let $\varepsilon_1 = \frac{H}{2}$, $\varepsilon_2 = \frac{H}{3}$, $\ldots$, $\varepsilon_n = \frac{H}{n+1}$. Then, we obtain corresponding values $\eta_1 > \eta_2 > \cdots > \eta_n$ such that

$$V(t, x) > \eta_n$$

in the interval $\varepsilon_n = \frac{H}{n+1} \leqslant \|x\| \leqslant \frac{H}{n} = \varepsilon_{n-1}$.

Construct a function

$$W(x) := \eta_{n+1} + \frac{n(n+1)}{H}(\eta_n - \eta_{n+1})\left(\|x\| - \frac{H}{n+1}\right),$$

see Figure 4.2.2 for the geometric meaning of this function.

Then obviously, we have

$$\begin{aligned} W(x) &\geqslant \eta_{n+1} + \frac{n(n+1)}{H}(\eta_n - \eta_{n+1})\left(\frac{H}{n+1} - \frac{H}{n+1}\right) \\ &= \eta_{n+1} > 0 \quad \text{for } \frac{H}{n+1} \leqslant \|x\| \leqslant \frac{H}{n}, \end{aligned} \tag{4.2.7}$$

$$\begin{aligned} W(x) &\leqslant \eta_{n+1} + \frac{n(n+1)}{H}(\eta_n - \eta_{n+1})\left(\frac{H}{n} - \frac{H}{n+1}\right) \\ &= \eta_{n+1} + \frac{n(n+1)}{H}(\eta_n - \eta_{n+1})\frac{H}{n(n+1)} \\ &= \eta_{n+1} + \eta_n - \eta_{n+1} = \eta_n \quad \text{for } \frac{H}{n+1} \leqslant \|x\| \leqslant \frac{H}{n}. \end{aligned} \tag{4.2.8}$$

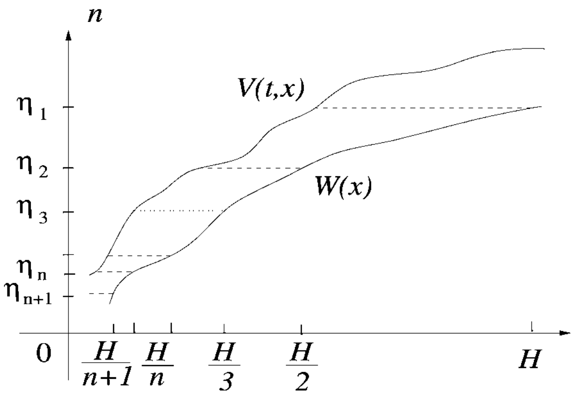

Figure 4.2.2. The geometric meaning of function $W(x)$.

Therefore, $V(t,x) \geqslant \eta_n \geqslant W(x)$. Since $W(0^+) \leqslant V(t,0) = 0$, we can definite $W(0) := 0$. This implies that $V(t,x)$ is positive definite.

(2) Along an arbitrary solution $x(t,t_0,a)$ of (4.2.1), we have

$$\begin{aligned} V(t) &:= V\big(t, x(t,t_0,a)\big) := \big(1+e^{-t}\big)\big\|a\big(t_0,t,x(t,t_0,a)\big)\big\|^2 \\ &= \big(1+e^{-t}\big)\big\|a(t_0,t_0,a)\big\| = \big(1+e^{-t}\big)\|a\|^2. \end{aligned} \tag{4.2.9}$$

So

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} = -e^{-t}\big\|a\big(t_0,t_0,x(t,t_0,a)\big)\big\|^2 \leqslant 0. \tag{4.2.10}$$

The proof of Theorem 4.2.1 is complete. □

EXAMPLE 4.2.2. (See [234].) If the definition of positive finite for $V(t,x)$ is changed to $V(t,x) > 0$ $(x \neq 0)$, $V(t,0) = 0$, then the conclusion of Theorem 4.2.1 does not hold. For example, consider

$$\begin{cases} \frac{dx_1}{dt} = \frac{1}{2}x_1, \\ \frac{dx_2}{dt} = \frac{1}{2}x_2. \end{cases} \tag{4.2.11}$$

The general solution is

$$\begin{cases} x_1 = x_1^{(0)} e^{\frac{1}{2}(t-t_0)}, \\ x_2 = x_2^{(0)} e^{\frac{1}{2}(t-t_0)}. \end{cases} \tag{4.2.12}$$

Obviously, the zero solution is unstable. But if we construct the function

$$V(t,x) = \big(x_1^2 + x_2^2\big)e^{-2t},$$

then $V(t,0)=0$, $V(t,x)>0$. For $x\neq 0$

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial t}+\frac{\partial V}{\partial t_1}\frac{dx_1}{dt}+\frac{\partial V}{\partial t_2}\frac{dx_2}{dt}\\ &= -2e^{-2t}\left(x_1^2+x_2^2\right)+e^{-2t}\left(x_1^2+x_2^2\right)\\ &= -e^{-2t}\left(x_1^2+x_2^2\right)\\ &\leqslant 0.\end{aligned}$$

Let $V(t,x)=(x_1^2+x_2^2)e^{-2t}=c$, i.e.,

$$x_1^2+x_2^2=c\,e^{2t}. \tag{4.2.13}$$

It is easy to find that the equivalent solution curve (4.2.13) leaves the origin with a rate e^{2t}, but the solution (4.2.12) leaves the origin with a rate $e^{\frac{1}{2}t}$. Hence, $\frac{dV}{dt}\leqslant 0$ still holds. This shows that the definition of positive definite for $V(t,x)$ cannot be changed to $V(t,x)>0$, $x\neq 0$.

THEOREM 4.2.3. *The zero solution of* (4.2.1) *is uniformly stable if and only if there exists* $V(t,x)\in C[G_H,R^1]$ *with infinitesimal upper bound such that*

$$D^+V(t,x)\big|_{(4.2.1)}\leqslant 0. \tag{4.2.14}$$

PROOF. *Sufficiency*. By the given condition, $\exists\varphi_1,\varphi_2\in K$ such that

$$\varphi_1\left(\|x\|\right)\leqslant V(t,x)\leqslant\varphi_2\left(\|x\|\right),$$

Then, $\forall\varepsilon>0$ ($\varepsilon<H$) such that by taking $\delta=\varphi_2^{-1}(\varphi_1(\varepsilon))$, i.e., $\varepsilon=\varphi_1^{-1}(\varphi_2(\delta))$, we have

$$\varphi_1\left(\left\|x(t,x_0,x_0)\right\|\right)\leqslant V\left(t,x(t,t_0,x_0)\right)\leqslant V(t_0,x_0)\leqslant\varphi_2\left(\|x_0\|\right)<\varphi_2(\delta).$$

Then, when $\|x_0\|<\delta$, it follows that.

$$\left\|x(t,t_0,x_0)\right\|<\varphi_1^{-1}\left(\varphi_2(\delta)\right)=\varepsilon\quad(t\geqslant t_0).$$

However, $\delta=\varphi_2^{-1}(\varphi_1(\varepsilon))=\delta(\varepsilon)$ is independent of t_0. So the zero solution is uniformly stable.

Necessity. Choose

$$V(t,x):=\left(1+e^{-t}\right)\inf_{t_0\leqslant\tau\leqslant t}\left\|p(\tau,t,x)\right\|^2. \tag{4.2.15}$$

(1) Obviously, $V(t,x)$ is continuous and $V(t,x)\leqslant 2\|p(t,t,x)\|^2=2\|x\|^2$. So $V(t,x)$ is infinitesimally upper bounded.

(2) Prove that $V(t,x)$ is positive definite. Since the zero solution of (4.2.1) is uniformly stable, $\forall\varepsilon>0$, $\exists\delta(\varepsilon)$, when $\|a\|<\sigma$, for all $\tau\geqslant t_0$ and $t\geqslant\tau$, we

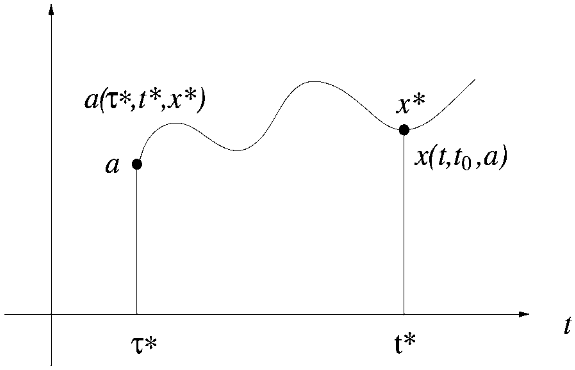

Figure 4.2.3. The relation between $a(\tau^*, t^*, x^*)$ and $x(t, t_0, a)$.

have

$$\|x(t, \tau, a)\| < \varepsilon. \tag{4.2.16}$$

Thus, when $\varepsilon \leqslant \|x\| \leqslant H$, for all $t \geqslant \tau \geqslant t_0$, we have

$$\|p(\tau, t, x)\| \geqslant \delta > 0. \tag{4.2.17}$$

Otherwise, for certain x^*, τ^*, t^*, $\varepsilon \leqslant \|x^*\| \leqslant H$, $t_0 \leqslant \tau^* \leqslant t^*$, we have (see Figure 4.2.3)

$$\|p(\tau^*, t^*, x^*)\| < \delta.$$

Let $a = p(\tau^*, t^*, x^*)$. Then, we have

$$x^* = p(t^*, \tau^*, a).$$

By uniform stability, when $\|a\| = \|p(\tau^*, t^*, x^*)\| < \delta$, $p(t, \tau^*, a)\| < \varepsilon$ holds for all $t \geqslant \tau^*$. Particularly, when $t = t^* > \tau^*$, $\|p(t^*, \tau^*, a)\| = \|x^*\| < \varepsilon$, which contradicts that $\|x^*\| \geqslant \varepsilon$. Thus, (4.2.16) is true.

Following the proof of Theorem 4.2.1, one can construct positive definite function $W(x)$ such that $V(t, x) \geqslant W(x)$. Therefore, $V(t, x)$ is positive definite.

(3) Now prove $D^+V(t, x)|_{(4.2.1)} \leqslant 0$. Since $x = p(t, t_0, a)$ along an arbitrary solution of (4.2.1), we have

$$\begin{aligned} V(t) &:= V\big(t, p(t, t_0, a)\big) \\ &= \big(1 + e^{-t}\big) \inf_{t_0 \leqslant \tau \leqslant t} \big\|p\big(t, t, p(t, t_0, a)\big)\big\| \\ &= \big(1 + e^{-t}\big) \inf_{t_0 \leqslant \tau \leqslant t} \big\|p(\tau, t_0, a)\big\|, \end{aligned}$$

showing that $V(t)$ is a monotone increasing function of t. As a result, $D^+V(t,x)|_{(4.2.1)} \leqslant 0$ is true.

The proof of Theorem 4.2.3 is complete. □

4.3. NASCs for uniformly asymptotic and equi-asymptotic stabilities

We again consider system (4.2.1).

THEOREM 4.3.1. *The zero solution of* (4.2.1) *is uniformly asymptotically stable if and only if there exists a positive definite function with infinitesimal upper bounded,* $V(t,x) \in C^1[G_H, R^1]$*, such that along the solutions of* (4.2.1)*,*

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} \tag{4.3.1}$$

is negative definite, where G_H *is defined in Theorem* 4.2.1.

PROOF. *Sufficiency*. Obviously, the conditions in Theorem 4.3.1 imply the conditions of Theorem 4.2.2, so the zero solution of (4.2.1) is uniformly stable. Thus, we only need to prove that the zero solution of (4.2.1) is uniformly attractive. By the given conditions, there exist $\varphi_1, \varphi_2, \varphi_3 \in K$ such that

$$\varphi_1(\|x\|) \leqslant V(t,x) \leqslant \varphi_2(\|x\|),$$

and

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} \leqslant -\varphi_3(\|x\|) \leqslant -\varphi_3(\varphi_2^{-1}V(t)) < 0,$$

i.e.,

$$\int_{V(t_0)}^{V(t)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))} \leqslant -(t-t_0),$$

or

$$\int_{V(t)}^{V(t_0)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))} \geqslant t-t_0. \tag{4.3.2}$$

Hence, $V(t) := V(t, x(t,t_0,x_0))$, $\forall \varepsilon > 0$ $(\varepsilon < H)$. Using $\varphi_1(\|x(t)\|) \leqslant V(t) := V(t,x(t))$ and $V(t_0) \leqslant \varphi_2(\|x_0\|) \leqslant \varphi_2(H)$, we obtain

$$\int_{\varphi_1(\|x(t)\|)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))}$$

$$
\begin{aligned}
&= \int_{\varphi_1(\|x(t)\|)}^{\varphi_1(\varepsilon)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))} + \int_{\varphi_1(\varepsilon)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))} \\
&\geqslant \int_{V(t)}^{V(t_0)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))} \geqslant t - t_0. \qquad (4.3.3)
\end{aligned}
$$

Take

$$
T = T(\varepsilon, H) > \int_{\varphi_1(\varepsilon)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))}. \qquad (4.3.4)
$$

Then, when $t \geqslant t_0 + t$, it follows that

$$
\begin{aligned}
\int_{\varphi_1(\|x(t)\|)}^{\varphi_1(\varepsilon)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))} &\geqslant t - t_0 - \int_{\varphi_1(\varepsilon)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}V(t))} \\
&> t - t_0 - T \geqslant 0.
\end{aligned}
$$

Hence, we can obtain

$$
\varphi_1\big(\|x(t)\|\big) < \varphi_1(\varepsilon) \quad \big(t \geqslant t_0 + T(\varepsilon, H)\big),
$$

i.e., $\|x(t)\| < \varepsilon$. Since $T = T(\varepsilon, H)$ is independent of t_0 and x_0, this means that the zero solution of (4.2.1) is uniformly attractive. This completes the proof of sufficiency.

We need the following lemma for proving necessity.

LEMMA 4.3.2. *Let $\varphi(\tau)$ be a continuous decreasing function for $\tau \geqslant 0$ and* $\lim_{t\to+\infty} \varphi(\tau) = 0$, *$\xi(\tau)$ be continuous nondecreasing positive function for $\tau \geqslant$* 0. *Then, there exists continuous increasing function $G(r)$ with $G(0) = 0$ such that for arbitrary positive continuous function $\varphi^*(\tau)$ $(\tau \geqslant 0)$ which satisfies $\varphi^*(\tau) \leqslant \varphi(\tau)$, the inequality*

$$
\int_0^{+\infty} G\big(\varphi^*(\tau)\big)\xi(\tau)\, d\tau < 1 \qquad (4.3.5)
$$

holds.

The function $\phi(t)$ is shown in Figure 4.3.1.

PROOF. As Figure 4.3.1 shows, for function $\varphi(t)$ we can choose a sequence $\{t_n\}$ such that when $t \geqslant t_n$,

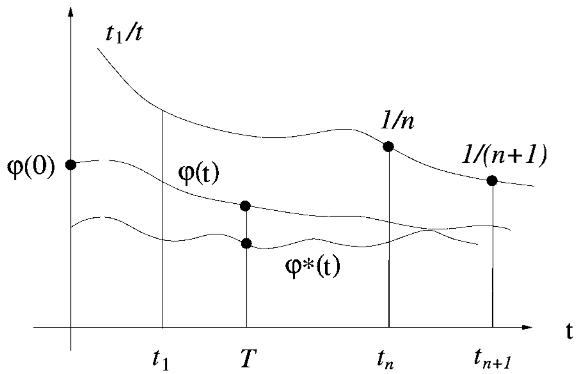

Figure 4.3.1. Function $\phi(t)$.

$$\varphi(t) \leqslant \frac{1}{n+1} \tag{4.3.6}$$

holds, where $t_n > 1$, $t_{n+1} > t_n + 1$.

Construct a function $\eta(t)$ such that $\eta(t_n) = \frac{1}{n}$ and $\eta(t)$ is linear in $[t_n, t_{n+1}]$. However, $\eta(t) = (\frac{t_1}{t})^p$ for $0 < t < t_1$, where p is the maximum positive integer such that

$$\dot{\eta}(t_1 - 0) < \dot{\eta}(t_1 + 0).$$

Obviously, when $t \geqslant t_1$, $\varphi(t) < \eta(t)$ and

$$\lim_{t\to\infty} \eta(t) = \lim_{t\to\infty} \varphi(t) = 0.$$

Let $G(0) = 0$,

$$G(r) = \begin{cases} \frac{e^{-\eta^{-1}(r)}}{\xi(\eta^{-1}(r))} & \text{for } r > 0, \\ 0 & \text{for } r = 0, \end{cases}$$

where η^{-1} is the inverse function of η and η^{-1} is decreasing, and $\varphi^*(\tau) \leqslant \varphi(\tau)$ holds for all $t \geqslant t_1$. Hence,

$$\begin{aligned} &\eta^{-1}\big(\varphi^*(\tau)\big) \geqslant \eta^{-1}\big(\varphi(\tau)\big) > \eta^{-1}\big(\eta(\tau)\big) = \tau, \\ &e^{-\eta^{-1}(\varphi^*(\tau))} < e^{-\tau}, \\ &\xi\big(\eta^{-1}\big(\varphi^*(\tau)\big)\big) \geqslant \xi(\tau). \end{aligned}$$

Thus, we have estimation:

$$G\big(\varphi^*(\tau)\big) = \frac{e^{-\eta^{-1}(\varphi^*(\tau))}}{\xi(\eta^{-1}(\varphi^*(\tau)))} < \frac{e^{-\tau}}{\xi(\tau)}. \tag{4.3.7}$$

Integrating the above inequality from 0 to $+\infty$ yields

$$\int_0^{+\infty} G\big(\varphi^*(\tau)\big)\xi(\tau)\,d\tau < \int_0^{+\infty} \frac{e^{-\tau}}{\xi(\tau)}\xi(\tau)\,d\tau = \int_0^{+\infty} e^{-\tau}\,d\tau = 1. \tag{4.3.8}$$

Lemma 4.3.2 is proved. □

Now we prove the *Necessity* for Theorem 4.3.1. By the uniformly asymptotic stability of the zero solution of (4.2.1), we can prove that there exist monotone increasing function $\varphi(\tau)$, with $\varphi(0) = 0$ and positive continuous decreasing function $\sigma(\tau)$ with

$$\lim_{t\to+\infty} \sigma(\tau) = 0,$$

such that the solution of (4.2.1) satisfies

$$\big\| P(t_0+\tau, t_0, a)\big\| \leqslant \varphi\big(\|a\|\big)\sigma(\tau) \tag{4.3.9}$$

in $\|a\| \leqslant H$. Take $\varphi(\tau) = \varphi(H)\sigma(\tau)$, $\xi(\tau) \equiv 1$. According to Lemma 4.3.2 there exists a continuous increasing function $G(r)$ with $G(0) = 0$, which is positive definite in the interval $0 \leqslant r \leqslant \varphi(0) = \varphi(H)\sigma(0)$.

Let $g(\tau) = G^2(\tau)$. By the property of functions $\varphi(\tau)$ and $\sigma(\tau)$, we know that

$$\begin{aligned} g\big(\varphi\big(\|a\|\big)\sigma(\tau-t)\big) &= \big[g\big(\varphi\big(\|a\|\big)\sigma(\tau-t)\big)\big]^{1/2}\big[g\big(\varphi\big(\|a\|\big)\sigma(\tau-t)\big)\big]^{1/2} \\ &\leqslant \big[g\big(\varphi\big(\|a\|\big)\sigma(0)\big)\big]^{1/2}\big[g\big(\varphi\big(\|a\|\big)\sigma(\tau-t)\big)\big]^{1/2}. \end{aligned} \tag{4.3.10}$$

Now, we define

$$V(t,x) := \int_t^{\infty} g\big(\big\|p(\tau,t,x)\big\|\big)\,d\tau, \tag{4.3.11}$$

and then obtain

$$\begin{aligned} V(t,x) &\leqslant \big[g\big(\varphi\big(\|x\|\big)\sigma(0)\big)\big]^{1/2}\int_t^{\infty}\big[g\big(\varphi(H)\sigma(\tau-t)\big)\big]^{1/2}\,d\tau \\ &= G\big(\varphi\big(\|x\|\big)\sigma(0)\big)\int_t^{\infty} G\big(\varphi(H)\sigma(\tau-t)\big)\,d\tau \\ &= G\big(\varphi\big(\|x\|\big)\sigma(0)\big)\int_0^{\infty} G\big(\varphi(H)\sigma(\tau)\big)\,d\tau \\ &< G\big(\varphi\big(\|x\|\big)\sigma(0)\big). \end{aligned}$$

Thus, $V(t,x)$ exists, is continuous and has an infinitesimal upper bound. So $V(t,x) \geqslant 0$ and

$$\frac{dV}{dt}\bigg|_{(4.2.1)} = -G^2\big(\|p(t,t,x)\|\big) = -G^2\big(\|x\|\big).$$

By Theorem 4.2.3, there exists a positive definite function with infinitesimal upper bound $W(t,x)$ satisfying

$$\varphi_1\big(\|x\|\big) \leqslant W(t,x) \leqslant \varphi_2\big(\|x\|\big) \quad (\varphi_1, \varphi_2 \in K),$$
$$\frac{dW}{dt}\bigg|_{(4.2.1)} \leqslant 0.$$

Defining $U(t,x) = V(t,x) + W(t,x)$, we have

$$\varphi_1\big(\|x\|\big) \leqslant W(t,x) \leqslant U(t,x) \leqslant G\big(\varphi\big(\|x\|\big)\sigma(0)\big) + \varphi_2\big(\|x\|\big).$$

Thus, $U(t,x)$ is positive definite and has infinitesimal upper bound, and

$$\begin{aligned}\frac{dU(x)}{dt}\bigg|_{(4.2.11)} &= \frac{dV(t,x)}{dt}\bigg|_{(4.2.1)} + \frac{dW(x,x)}{dt}\bigg|_{(4.2.1)} \\ &\leqslant \frac{dV(t,x)}{dt}\bigg|_{(4.2.1)} = -G^2\big(\|x\|\big).\end{aligned}$$

Hence, $U(t,x)$ satisfies the conditions of Theorem 4.3.1. □

REMARK 4.3.3. Lyapunov only proved that the conditions of Theorem 4.3.1 imply that the zero solution of (4.2.1) is asymptotically stable. We have proved that the conditions of Theorem 4.3.1 is sufficient for the zero solution of (4.2.1) to be uniformly asymptotically stable.

EXAMPLE 4.3.4. Massera [308] gave the following illustrative example to show that the infinitesimal upper bound condition is important for asymptotic stability.

Let $g \in C^1[I,R]$. Then, as shown in Figure 4.3.2, $g(n) = 1$ $(n = 1,2,\ldots)$. Consider a differential equation

$$\frac{dx}{dt} = \frac{g'(t)}{g(t)}x, \tag{4.3.12}$$

which has general solution $x(t) = \frac{g(t)}{g(t_0)}$, and $x(t_0)$ does not tend to zero as $t \to \infty$. So the zero solution is not asymptotically stable.

We construct a Lyapunov function:

$$V(t,x) = \frac{x^2}{g^2(t)}\left[3 - \int_0^t g^2(s)\,ds\right].$$

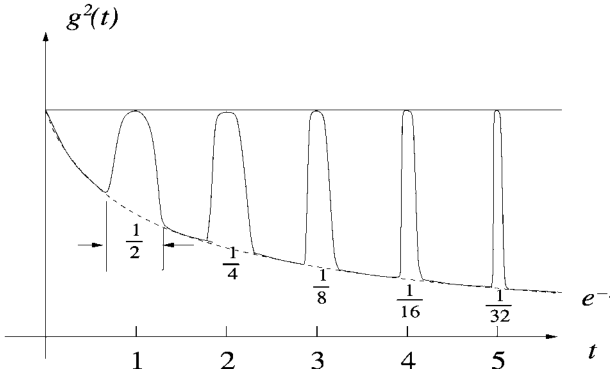

Figure 4.3.2. Massera function.

Since

$$\int_0^{+\infty} g^2(s)\,ds < \int_0^{+\infty} e^{-s}\,ds + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{\frac{1}{2}}{1-\frac{1}{2}} = 2,$$

$V(t,x) \geqslant \frac{x^2}{g^2(t)} \geqslant x^2 > 0$ for $x \neq 0$. Thus, $V(t,x)$ is positive definite and

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} \\ &= \frac{2 \times [3 - \int_0^t g^2(s)\,ds]}{g^2(t)} \cdot \frac{g'(t)}{g(t)} x \\ &\quad + \frac{-x^2 g^4(t) - x^2[3 - \int_0^t g^2(s)\,ds] 2g(t)g'(t)}{g^4(t)} \\ &= -x^2 < 0 \quad (x \neq 0).\end{aligned}$$

But there does not exist a positive definite $W(x)$ such that $V(t,x) \leqslant W(x)$. This illustrates that the infinitesimal upper bound condition is important.

EXAMPLE 4.3.5. We use this example to show that the condition of Theorem 4.3.1 is not necessary, i.e., Lyapunov asymptotic stability theorem does not have an inverse counterpart.

Consider the differential equation:

$$\frac{dx}{dt} = -\frac{x}{t+1}. \tag{4.3.13}$$

Its general solution is

$$x(t, t_0, x_0) = \frac{1+t_0}{1+t} x_0 \to 0 \quad \text{as } t \to +\infty.$$

$\forall \varepsilon > 0$, take $\delta = \varepsilon$. Then for $|x_0| < \delta$ we obtain $|x(t, t_0, x_0)| < |x_0| < \varepsilon$ for $t > t_0$. So the zero solution is asymptotically stable, but there does not exist a positive definite function $V(t, x)$ with infinitesimal upper bound such that $\frac{dV}{dt}$ is negative definite.

In fact, if otherwise, then $\forall \varepsilon > 0$, when $\|x\| \leqslant \delta$, we have $0 \leqslant V(t, x) \leqslant \varepsilon$ for all $t \geqslant t_0$, and $\frac{dV}{dt}$ is negative definite. So there exists a positive definite function $W(x)$ such that

$$\left.\frac{dV}{dt}\right|_{(4.3.13)} \leqslant -W(x).$$

Let

$$l = \inf_{\frac{\delta}{2} \leqslant |x| \leqslant H} W(x),$$

and take $|x_0| = \delta$. Then

$$\begin{aligned} V\big(t_1, V(t_1)\big) &= V(t_0, x_0) + \int_{t_0}^{t} \frac{dV}{dt}\, dt \\ &\leqslant V(t_0, x_0) - l(t_1 - t_0) \\ &= V(t_0, x_0) - l(1 + t_0) < 0 \quad (\text{taking } t_1 = 2t_0 + 1) \text{ for } t_0 \gg 1. \end{aligned}$$

But on the other hand, when $t_1 = 2t_0 + 1$, we have

$$x(t_1, t_0, x_0) = \frac{1+t_0}{1+t} x_0 = \frac{1+t_0}{2(1+t_0)} x_0 = \frac{x_0}{2}.$$

Thus,

$$\big\|x(t_1, t_0, x_0)\big\| = \frac{\delta}{2} \neq 0.$$

Hence $V(t_1, x(t_1, t_0, x_0)) > 0$, leading to a contradiction. This means that the conclusion is true.

REMARK 4.3.6. Example 4.3.5 shows that if one only assumes $V(t, x)$ is positive definite and $\frac{dV}{dt}$ is negative definite one still cannot assure the zero solution being asymptotically stable. But if adding the infinitesimal upper bound condition, then not only the asymptotic stability but also the uniformly asymptotic stability of the zero solution are valid. Therefore, for asymptotic stability, the required conditions of Theorem 4.3.1 may be relaxed.

THEOREM 4.3.7. *Suppose $f(t,x) \in C^1[G_H, R^n]$. Then the zero solution of (4.2.1) is equi-asymptotically stable if and only if there exists positive definite $V(t,x) \in C^1[G_H, R^1]$ such that $\frac{dV}{dt}|_{(4.2.1)} \leqslant 0$, and for any $\eta > 0$ and all $(t, x(t))$, when $V(t,x) < \eta$ holds uniformly, we have*

$$\lim_{t\to\infty} \|x(t)\| = 0. \tag{4.3.14}$$

PROOF. *Necessity*. Let $x(t) = x(t, t_0, x_0)$ be solution of (4.2.1), and s a fixed positive number. Construct a function

$$V(t,x) = \|x(t_0, t, x)\|^2 \big(1 + e^{-t}\big). \tag{4.3.15}$$

Then, $f(t,x) \in C^1[G_H, R^n]$ implies $V(t,x) \in C^1[G_H, R^n]$. So the zero solution of (4.2.1) is uniformly stable. Then, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$, when $\|x\| > \varepsilon$ the following inequality holds:

$$\|x(t_0, t, x)\| \geqslant \delta(\varepsilon).$$

So $V(t,x) \geqslant \delta^2(\varepsilon)$ ($\|x\| \geqslant \varepsilon$), i.e., $V(t,x)$ is positive definite.

Because of the equi-attraction of the zero solution, there exists $\delta_0 = \delta(x_0) > 0$ such that $\|x_0\| = \|x(t_0, t, x)\| \leqslant \delta(t_0)$. Thus, $\forall \varepsilon > 0$, there exists $T = T(\varepsilon, t_0)$ such that

$$\|x(t, t_0, x_0)\| < \varepsilon \quad (t \geqslant t_0 + T).$$

Take $\eta = \delta(t_0)$. Then, when $V(t,x) < \eta$, we have

$$\|x\big(t_0, t, x(t)\big)\|^2 = \frac{V(t, x(t))}{1 + e^{-t}} < \frac{\eta}{1 + e^{-t}} < \eta = \delta(t_0).$$

Therefore, when $t \geqslant t_0 + T$, $\|x(t)\| < \varepsilon$, i.e., for all $(t, x(t))$ which satisfy $V(t, x(t)) < \eta$ uniformly, we have

$$\lim_{t\to+\infty} x(t) = 0.$$

On the other hand,

$$V\big(t, x(t)\big) = \|x\big(t_0, x(t)\big)\|^2 \big(1 + e^{-t}\big) = \|x_0\|^2 \big(1 + e^{-t}\big),$$

so

$$\begin{aligned}\frac{dV(t, x(t))}{dt} &= -e^{-t}\|x_0\|_E^2 = -e^{-t}\|x\big(t_0, t, x(t)\big)\|_E^2 \\ &= \frac{-e^{-t}}{1 + e^{-t}} V\big(t, x(t)\big) \leqslant 0.\end{aligned}$$

The necessity is proved.

Sufficiency. If the condition is satisfied, obviously the zero solution is stable. When $V(t_0, x_0) < \eta$, $\|x_0\| \leqslant \delta(\eta)$ holds owing to

$$\frac{dV}{dt}\bigg|_{(4.2.1)} \leqslant 0.$$

Thus, one obtains $V(t, x(t)) \leqslant V(t_0, x_0) < \eta$. So

$$\lim_{t\to+\infty} x(t) = 0$$

uniformly holds for x_0. This means that the zero solution is equi-asymptotically stable.

The proof is complete. □

THEOREM 4.3.8. *Assume $f(t,x) \in C^1[G_H, R^n]$. Then the zero solution of (4.2.1) is asymptotically stable if and only if there exists a positive definite $V(t,x) \in C^1[G_H, R^1]$ such that*

$$\frac{dV}{dt}\bigg|_{(4.2.1)} \leqslant 0,$$

and for any $\eta > 0$ and for all $(t, x(t))$ which satisfy $V(t,x) < \eta$,

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

PROOF. The proof of Theorem 4.3.8 is almost the same as the proof of Theorem 4.3.7, so is omitted. □

4.4. NASCs of exponential stability and instability

THEOREM 4.4.1. *Let $f(t,x) \in C[I \times R^n, R^n]$, $f(t,0) \equiv 0$ and f satisfy the Lipschitz condition for x. Then the zero solution of (4.2.1) is globally exponentially stable if and only if there exists $V(t,x) \in C^1[I \times R^n, R^1]$ such that*

(1) $\|x\| \leqslant V(t,x) \leqslant K(\alpha)\|x\|$, $x \in S_\alpha := \{x\colon \|x\| \leqslant \alpha\}$;
(2) $\frac{dV}{dt}|_{(4.2.1)} \leqslant -qcV(t,x)$, *where* $0 < q < 1$, $c > 0$, *q, c are constants.*

PROOF. *Sufficiency*. $\forall\alpha > 0$, when $x_0 \in S_\alpha$, let $x(t) := x(t, t_0, x_0)$. By condition (2), we have

$$\frac{d}{dt}V\big(t, x(t)\big) \leqslant -cqV\big(t, x(t)\big). \tag{4.4.1}$$

Consider the comparison equation:

$$\frac{du}{dt} = -cqu. \tag{4.4.2}$$

Let $u_0 = V(t_0, x_0)$. Then,

$$u(t, t_0, u_0) = u_0 e^{-cq(t-t_0)}.$$

By comparison theorem we obtain

$$V\big(t, x(t)\big) \leqslant u_0 e^{-cq(t-t_0)} = V(t_0, x_0) e^{-cq(t-t_0)}, \quad t \geqslant t_0.$$

By conditions (1), we have

$$\begin{aligned}\big\|x(t)\big\| \leqslant V\big(t, x(t)\big) &\leqslant K(\alpha)\|x_0\| e^{-cq(t-t_0)} \\ &:= K(\alpha)\|x_0\| e^{-\lambda(t-t_0))} \quad (\lambda = cq > 0),\end{aligned}$$

i.e.,

$$\big\|x(t, t_0, x_0)\big\| \leqslant K(\alpha)\|x_0\| e^{-\lambda(t-t_0)} \quad (t \geqslant t_0).$$

So the zero solution of (4.2.1) is globally exponentially stable.

Necessity. Let the zero solution of (4.2.1) be globally experientially stable. Then there exists constant $c > 0$ such that $\forall \alpha > 0$, $\exists K(\alpha) > 0$, when $x_0 \in S_\alpha$,

$$\big\|x(t, t_0, x_0)\big\| \leqslant K(\alpha)\|x_0\| e^{-c(t-t_0)} \tag{4.4.3}$$

holds. For $0 < q < 1$, define a function:

$$V(t, x) := \sup_{\tau \geqslant 0} \big\|x(t+\tau, t, x)\big\| e^{cq\tau}.$$

Then, $\forall x \in S_\alpha$, we have:

(1)

$$\begin{aligned}\|x\| \leqslant V(t, x) &\leqslant \sup_{\tau \geqslant 0} K(\alpha)\|x\| e^{-c\tau} e^{cq\tau} \\ &= K(\alpha)\|x\| \sup_{\tau \geqslant 0} e^{-(1-q)c\tau} \leqslant K(\alpha)\|x\|,\end{aligned}$$

i.e.,

$$\|x\| \leqslant V(t, x) \leqslant K(\alpha)\|x\|. \tag{4.4.4}$$

(2) Let $x^* = x(t+h, t, x)$. It follows that

$$\begin{aligned}V\big(t+h, t, x^*\big) &= \sup_{\tau \geqslant 0} \big\|x\big(t+h+\tau, t+h, x^*\big)\big\| e^{cq\tau} \\ &= \sup_{\tau \geqslant 0} \big\|x(t+h+\tau, t, x)\big\| e^{cq\tau} \\ &\leqslant \sup_{\tau \geqslant 0} \big\|x(t+h, t, x)\big\| e^{cq\tau} \cdot e^{-cqh} \\ &= V(t, x) e^{-cqh}.\end{aligned}$$

Furthermore, we can obtain

$$\frac{V(t+h,x^*)-V(t,x)}{h}\leqslant V(t,x)\frac{e^{-cqh}-1}{h}.$$

Thus, we have

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(4.2.1)}&=\lim_{h\to 0^+}\frac{V(t+h,x^*)-V(t,x)}{h}\\&\leqslant\lim_{h\to 0^+}V(t,x)\frac{e^{-cqh}-1}{h}\\&=V(t,x)\lim_{h\to 0^+}\frac{e^{-cqh}}{h}\\&=-cqV(t,x),\end{aligned}$$

i.e.,

$$\frac{dV}{dt}\bigg|_{(4.2.1)}\leqslant -cqV(t,x).$$

The proof is complete. □

In the following, we present the NASC of instability [331,332].

THEOREM 4.4.2. *Let $f(t,x)\in C^1[G_H,R^n]$. Then the zero solution is unstable if and only if the following conditions hold*:

(1) *when t is fixed, $V(t,x)$ and $\frac{dV}{dt}|_{(4.2.1)}$ are positive definite*;
(2) *$\forall\varepsilon>0$, there exist $\alpha>0$, $T=T(\varepsilon)>0$ and $\|x_0\|=\alpha$ such that $V(T,x_0)<\varepsilon$.*

PROOF. *Necessity*. Assume that the zero solution of (4.2.1) is unstable and $x(t):=x(t,0,x_0)$ is an arbitrary solution. Construct a function

$$V(t,x)=\left\|x(t,0,x_0)\right\|^2\frac{1+2t}{1+t}.$$

Since $f(t,x)\in C^1$, $V(t,x)\in C^1$ and $x=0$ is the solution. Hence, for an arbitrarily fixed t, when $x\neq 0$, we have

$$V(t,x)\geqslant\left\|x(0,t,x)\right\|^2>0.$$

So when t is fixed, $V(t,x)$ is positive definite. Furthermore, since

$$\frac{dV}{dt}\bigg|_{(4.2.1)}=\frac{d}{dt}\left(\left\|x\big(0,t,x(t)\big)\right\|^2\frac{1+2t}{1+t}\right)$$

$$= \|x_0\|^2 \frac{d}{dt}\left(\frac{1+2t}{1+t}\right)$$
$$= \frac{1}{(1+t)(1+2t)} V\big(t, x(t)\big),$$

$\frac{dV}{dt}|_{(4.2.1)}$ is positive definite for a fixed t.

By the instability of the zero solution we know that for any $\varepsilon > 0$, there exist $\alpha > 0$, $T > 0$ and $\|x_0\| < \frac{\varepsilon}{2}$ such that

$$\big\|x(T, 0, x_0)\big\|^2 = a.$$

Taking $\tilde{x}_0 = x(T, 0, x_0)$, we obtain

$$V(T, \tilde{x}_0) = \big\|x\big(0, T, x(T, 0, x_0)\big)\big\|^2 \frac{1+2T}{1+T} = \|x_0\|^2 \frac{1+2T}{1+T} < \frac{\varepsilon}{2} \cdot 2 = \varepsilon.$$

The necessity is proved.

Sufficiency. Suppose that the conclusion were not true, i.e., the zero solution of (4.2.1) is stable. Let $V(t, x) \in C^1[G_H, R^1]$, which satisfies the conditions (1) and (2). Then, $\forall a > 0$, there exists $\delta = \delta(\alpha) > 0$ such that

$$\big\|x(t, t_0, x_0)\big\|^2 < a \quad (t \geqslant t_0) \quad \text{when } \|x_0\| < \delta,$$

because $V(t, x)$ is positive definite for a fixed t. Then, there exists $l > 0$ such that

$$V(t_0, x) \geqslant l \quad \text{when } \|x\| \geqslant \delta.$$

We take $0 < \varepsilon < l$. By the given condition, $\exists x$ and $T \geqslant t_0$ such that

$$V(T, x) < \varepsilon \quad \text{when } \|x\| = 2.$$

Let $x_T := x(T, t_0, x_0)$, $\|x_T\|^2 = \alpha$. Then $x_0 = x(t_0, T, x_T)$, which means that $\|x_0\| > \delta$. Otherwise, if $\|x_0\| \leqslant \delta$, then we have

$$\alpha = \|x_T\|^2 = \big\|x(T, t_0, x_0)\big\|^2 < \alpha.$$

This is impossible. So $V(t_0, x_0) \geqslant l > \varepsilon > 0$. By $\frac{dV}{dt}|_{(4.2.1)} > 0$,

$$\varepsilon \leqslant l \leqslant V(t_0, x_0) \leqslant V\big(T, x(T, t_0, x_0)\big) < \varepsilon.$$

This is a contradiction. Therefore, the zero solution of (4.2.1) is unstable. □

4.5. Sufficient conditions for stability

In this section, we present some sufficient conditions for stability.

THEOREM 4.5.1. *If there exist a positive definite function $V(t,x) \in C^1[G_H, R^1]$ with infinitesimal upper bound and a negative definite function $V_1(t,x) \in C[G_H, R^1]$ such that in any fixed region $0 < \lambda \leqslant \|x\| \leqslant \mu \leqslant H$, $\forall \delta > 0$, $\exists t^*(\delta)$, when $t \geq t^*$,*

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} < V_1(t,x) + \delta. \tag{4.5.1}$$

Then the zero solution of (4.2.1) is stable.

PROOF. By the given conditions of Theorem 4.5.1, we know that there exist functions $\varphi_1, \varphi_2, \varphi_3 \in K$ such that

$$\begin{aligned} 0 < \varphi_1\big(\|x\|\big) \leqslant V(t,x) \leqslant \varphi_2\big(\|x\|\big) \quad \text{for } x \neq 0, \\ V_1(t,x) \leqslant -\varphi_3\big(\|x\|\big). \end{aligned}$$

$\forall \varepsilon > 0$, let

$$\varepsilon_1 := \varphi_2^{-1}\big(\varphi_1(\varepsilon)\big), \qquad \lambda := \inf_{\|x\| \geqslant \frac{\varepsilon_1}{2}} \varphi_3\big(\|x\|\big),$$

and take $\delta = \frac{\lambda}{2}$. From the given conditions of the theorem we know that when $\frac{\varepsilon_1}{2} \leqslant \|x\| \leqslant H$, for this $\delta > 0$, $\exists t^*(\delta)$, where $t \geqslant t^*$, we have

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(4.2.1)} &< V_1\big(t, x(t)\big) + \delta \\ &\leqslant -\varphi_3\big(\big\|x(t)\big\|\big) + \delta \\ &\leqslant -\inf_{\|x\| \geqslant \frac{\varepsilon_1}{2}} \varphi_3\big(\big\|x(t)\big\|\big) + \delta \\ &= -\frac{\lambda}{2} < 0. \end{aligned}$$

Therefore, when $\|x(t^*, t_0, x_0)\| := \|x(t^*)\| \leqslant \varepsilon_1$ for $t \geqslant t^*$, we have

$$\varphi_1\big(\big\|x(t)\big\|\big) \leqslant V\big(t, x(t)\big) < V\big(t^*, x\big(t^*\big)\big) \leqslant \varphi_2\big(\big\|x\big(t^*\big)\big\|\big) \leqslant \varphi_2(\varepsilon_1).$$

Furthermore, we can show that $\|x(t)\| \leqslant \varphi_1^{-1}(\varphi_2(\varepsilon_1)) = \varepsilon$ holds in a finite interval $[t_0, t^*]$. Then, by the continuity of Theorem 1.1.2 we only need $\|x_0\| \leqslant \eta \ll 1$. Thus, $\|x(t, t_0, x_0)\| \leqslant \varepsilon_1$, $t \geqslant t_0$, and so the zero solution of (4.2.1) is stable. □

REMARK 4.5.2. Theorem 4.5.1 is a generalization of the following Malkin's theorem [300].

MALKIN'S THEOREM. *(See [300].) If there exists a positive definite $V(t,x)$ with infinitesimal upper bound and negative definite $V_1(t,x)$ such that*

$$\lim_{t\to\infty}\left(\frac{dV}{dt} - V_1\right) = 0, \tag{4.5.2}$$

then the zero solution of (4.2.1) is stable.

Obviously, (4.5.2) holds if and only if $\forall\delta > 0$, $\exists t^*(\delta)$, when $t \geqslant t^*$,

$$V_1(t,x) - \delta < \frac{dV}{dt} < V_1(t,x) + \delta. \tag{4.5.3}$$

The following example removes the restriction on the left-hand side of (4.5.3) and admits $\frac{dV}{dt}$ being variable.

EXAMPLE 4.5.3. Consider the system

$$\begin{aligned}&\frac{dx_i}{dt} = \left(\frac{1}{1+t} + \sin t - |\sin t|\right)\left(\sum_{j=1}^{n} a_{ij}x_j\right) - x_i^{2n-1},\\ &\quad i = 1, 2, \ldots, n,\end{aligned} \tag{4.5.4}$$

where a_{ij} $(i, j = 1, \ldots, n)$ are constants, and the general quadratic form

$$x^T Ax = \sum_{i,j=1}^{n} a_{ij}x_i x_j$$

is positive semi-definite, then the zero solution of (4.5.4) is stable.

In fact, choosing the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{n} x_i^2,$$

and evaluating the derivative along the solution of (4.5.4) yields

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(4.5.4)} &= \sum_{i=1}^{n} x_i\dot{x}_i\\ &= \sum_{i=1}^{n}\left[\left(\frac{1}{1+t} + \sin t - |\sin t|\right)\right]\sum_{j=1}^{n}(a_{ij}x_i x_j - x_i^{2n}\\ &= \left(\frac{1}{1+t} + \sin t - |\sin t|\right)x^T Ax - \sum_{i=1}^{n} x_i^{2n}.\end{aligned} \tag{4.5.5}$$

Let

$$V_1 = -\sum_{i=1}^{n} x_i^{2n}.$$

Obviously, V is positive definite with infinitesimal upper bound and V_1 is negative definite, but

$$\frac{dV}{dt} = V_1 + \left(\frac{1}{1+t} + \sin t - |\sin t|\right)x^T Ax \tag{4.5.6}$$

in any region $\lambda \leqslant \|x\| \leqslant \mu$, satisfying

$$\lim_{t\to+\infty}\left(\frac{dV}{dt} - V_1\right) \neq 0.$$

But the conditions of Theorem 4.5.1 are satisfied, because $\frac{dV}{dt} - V_1 \leqslant \frac{1}{1+t}x^T Ax$ and $x^T Ax$ is bounded. So the zero solution of (4.5.4) is stable.

THEOREM 4.5.4. *(See [411].) Assume that*

(1) $h(t) \in C[I, R^+]$, $g(t) \in [I, R^1]$. $\frac{1}{h(s)}$ *and* $g(s)$ *are integral functions on I, and* $\int_{t_0}^{+\infty} g(t)\,dt$ *is convergent*;
(2) *there exist* $V(t,x) \in C[G_H, R^1]$ *with* $V(t,0) \equiv 0$, $\varphi_1(\|x\| \leqslant V(t,x))$, *when* $T \gg 1$, $\varphi_1 \in K$, *such that for arbitrary* $\lambda \in (0,\rho)$ $(0 < \rho \ll 1)$,

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} \leqslant g(t)h\big(V(t,x)\big) \tag{4.5.7}$$

holds for $\forall t \geqslant T$, $0 < \lambda \leqslant V \leqslant \rho$;

then the zero solution of (4.2.1) *is stable.*

PROOF. Given $\varepsilon > 0$ such that $\varepsilon_1 := \varphi_1(\varepsilon) \in (0,\rho)$. Let

$$\lambda = \frac{1}{2}\varepsilon_1, \qquad H(s) = \int_{t_0}^{s} \frac{dt}{h(t)} \quad (0 \leqslant t_0 \ll 1).$$

$h(t) \geqslant 0$ implies $H(\varepsilon_1) - H(\lambda) > 0$ by condition (1). Therefore, we know that $\exists T > 0$ such that

$$\left|\int_{t'}^{t''} g(t)\,dt\right| < H(\varepsilon_1) - H(\lambda) \quad \forall t', t'' \in [T, \infty). \tag{4.5.8}$$

First, we prove that there exists $\delta_1 < \lambda$ such that when $V_T = V(T, x(T)) < \delta_1$,

$$V(t) := V\big(t, x(t)\big) < \varepsilon_1 = \varphi_1(\varepsilon) \quad \text{for } t \geqslant T. \tag{4.5.9}$$

Otherwise, we have $t_2 > t_1 \geqslant T$ such that $\forall t \in [t_1, t_2]$,

$$V(t_1) = \lambda \leqslant V(t) \leqslant V(t_2) = \varepsilon_1. \tag{4.5.10}$$

From (4.5.7), (4.5.9) and (4.5.10) we obtain

$$\begin{aligned} H(\varepsilon_1) = H\big(V(t_2)\big) &\leqslant H\big(V(t_1)\big) + \left| \int_{t_1}^{t_2} g(t)\, dt \right| \\ &\leqslant H(\lambda) + \big(H(\varepsilon_1) - H(\lambda)\big) \\ &= H(\varepsilon_1). \end{aligned}$$

Therefore (4.5.9) is valid. So we have

$$\varphi_1\big(\big\|x(t, T, x_T)\big\|\big) \leqslant V\big(t, x(t, T, x_T)\big) < \varphi_1(\varepsilon),$$

i.e., when $V_T < \delta_1$,

$$\big\|x(t, T, x_T)\big\| < \varepsilon \quad \forall t \geqslant T. \tag{4.5.11}$$

Since $V(t, x)$ is continuous, $V(t, 0) \equiv 0$, for the above-chosen δ_1, there exists $\delta_2 < \varepsilon$, when $\|x_T\| < \delta_2$,

$$0 < V(t, x_T) < \delta_1.$$

Using Theorem 1.1.2, for δ_2, $\exists \delta$, when $\|x_0\| < \delta < r$, we have

$$\big\|x(t, t_0, x_0)\big\| < \delta_2 < \varepsilon \quad \forall t \in [t_0, T]. \tag{4.5.12}$$

Combining (4.5.11) and (4.5.12) shows that the zero solution of (4.2.1) is stable.

Theorem 4.5.4 is proved. □

THEOREM 4.5.5. *If the conditions in Theorem 4.5.4 hold and, in addition, there exists $\varphi_2(\|x\|) \in K$ such that*

$$V(t, x) \leqslant \varphi_2\big(\|x\|\big),$$

then the zero solution of (4.2.1) is uniformly stable.

By using the same method used for proving Theorem 4.5.4, one can easily prove Theorem 4.5.5.

EXAMPLE 4.5.6. Consider the system:

$$\frac{dx_1}{dt} = \frac{1}{1+t^2}x_1\left(e^{x_1^2+x_2^2}+1\right) - (\sin t)x_2,$$
$$\frac{dx_2}{dt} = (\sin t)x_1 + \frac{1}{1+t_2}x_2\left(e^{x_1^2+x_2^2}+1\right). \tag{4.5.13}$$

Take $V(x_1, x_2) = x_1^2 + x_2^2$. Then,

$$\frac{dV}{dt} = \frac{2}{1+t^2}\left(Ve^V + V\right).$$

Let

$$g(t) = \frac{2}{1+t^2}, \quad \varphi_1 = \varphi_2 = x_1^2 + x_2^2, \quad h(v) = Ve^V + V,$$

$\int_{t_0}^{+\infty} g(t)\,dt$ is convergent, and $\frac{1}{h(t)}$ and $g(t)$ are integrated in $[t_0, +\infty)$.

According to Theorem 4.5.5, the zero solution of (4.5.13) is uniformly stable.

THEOREM 4.5.7. *Suppose that there exist positive definite function $V(t,x) \in C^1[G_H, R^1]$, $W(x) \in C^1[G_H, R^1]$ and $\theta(t) \in [I, R^1]$ such that*

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} \leqslant 0, \qquad \left.\frac{dU}{dt}\right|_{(4.2.1)} \geqslant 0,$$

where $U(t,x) := V(t,x) - \theta(t)W(t)$, $\theta(t)$ is a monotonic increasing function, and

$$\theta(t_0) = 1, \qquad \lim_{t\to+\infty} \theta(t) = +\infty.$$

Then the zero solution of (4.2.1) is uniformly stable and is attractive.

PROOF. (1) If $V(t,x) - \theta(t)W(x) = 0$, then

$$W\big(x(t)\big) = \frac{V(t,x(t))}{\theta(t)}, \qquad \left.\frac{dV(t,x)}{dt}\right|_{(4.2.1)} \leqslant 0.$$

So $V(t, x(t))$ is monotonically decreasing. But $\theta(t)$ is a monotone increasing function, so it follows that

$$\left.\frac{dW(x(t))}{dt}\right|_{(4.2.1)} \leqslant 0.$$

This implies that the zero solution of (4.2.1) is stable. Owing to $W(x)$ being positive definite, there exists $\varphi \in K$ such that

$$\varphi\big(\|x(t)\|\big) \leqslant W\big(x(t)\big) = \frac{V(t,x(t))}{\theta(t)} \to 0 \quad \text{as } t \to +\infty.$$

Thus, $\|x(t)\| \to 0$ as $t \to +\infty$, i.e., the zero solution of (4.2.1) is attractive.

(2) If $V(t,x) - \theta(t)W(t) = U(t,x) \neq 0$, $\frac{dU}{dt}|_{(4.2.1)} \geqslant 0$. Let

$$\tilde{V}(t,x) := V(t,x) - U(t,x) = \theta(t)W(x).$$

Then,

$$\tilde{V}(t,x) - \theta(t)W(t) \equiv 0,$$

and so

$$\frac{d\tilde{V}}{dt}\bigg|_{(4.2.1)} = \frac{dV}{dt}\bigg|_{(4.2.1)} - \frac{dU}{dt}\bigg|_{(4.2.1)} \leqslant 0.$$

By the proof of (1) we know that the zero solution of (4.2.1) is stable and attractive.

The theorem is proved. □

EXAMPLE 4.5.8. Discuss the stability of the zero solution of the equation:

$$\frac{dx}{dt} = -\frac{x}{1+t}. \tag{4.5.14}$$

We choose the positive definite function $V(t,x) = (1+t)x^2$ with no infinitesimal upper bound. Let $W(x) = x^2$ and $\theta(t) = 1+t$. Then, $U(t,x) = V(t,x) - \theta W(x) \equiv 0$, and

$$\frac{dV}{dt}\bigg|_{(4.5.1)} = -x^2, \qquad \frac{dU}{dt} \geqslant 0.$$

So the zero solution of (4.5.14) is uniformly stable and attractive.

In the following, we consider the n-dimensional autonomous system:

$$\frac{dx}{dt} = f(x), \quad f(0) = 0, \tag{4.5.15}$$

where

$$x = (x_1, x_2, \ldots, x_n)^T \in R^n, \qquad f = (f_1, f_2, \ldots, f_n)^T \in C\big[\Omega_H, R^n\big],$$
$$\Omega_H := \big\{x\colon \|x\| \leqslant H\big\}.$$

THEOREM 4.5.9. *(See [215].) Assume that*

(1) *there exist two positive number series* $\{r_k\}$, $\{\eta_k\}$ *with*

$$\lim_{k\to\infty} r_k = 0, \quad \eta_k < r_k;$$

(2) *there exists a positive definite function* $V(x) \in C^1[\Omega_H, R^1]$ *such that*

$$\left.\frac{dV}{dt}\right|_{(4.5.15)} \leqslant 0$$
$$\bigl(x \in D_k := \{x \mid r_k \geqslant V(t) \geqslant r_k - \eta_k > 0,\ k = 1, 2, \ldots\}\bigr).$$

Then the zero solution of (4.5.15) is stable.

PROOF. The theorem is illustrated in Figure 4.5.1. $\forall \varepsilon > 0$, $\varepsilon < H$, let $l = \inf_{\|x\|=\varepsilon} V(t) > 0$. Since $\lim_{k\leftarrow\infty} r_k = 0$, there exists $K > 0$, when $k > K$, $r_k < l$ holds. Take $\tilde{K} = K + 1$. Due to $\lim_{x\to 0} V(x) = 0$, for $r_k > 0$, $\exists\delta$, $0 < \delta < \varepsilon$, when $\|x\| < \delta$, we have

$$V(x) < r_{\tilde{k}} - \eta_{\tilde{k}}.$$

Now, we prove that for all $t \geqslant t_0$, when $\|x_0\| < \delta$,

$$\bigl\|x(t, t_0, x_0)\bigr\| < \varepsilon, \quad t \geqslant t_0.$$

Otherwise, when $0 < t - t_0 \ll 1$, $\|x(t, t_0, x_0)\| < \varepsilon$ holds, but $\exists t^* > t_0$ such that $\|x(t^*, t_0, x_0)\| = \varepsilon$. By the mean value theorem of continuous function, there must exist $t_2 > t_1 > t_0$ such that

$$V\bigl(x(t_1, t_0, x_0)\bigr) = r_{\tilde{k}} - \eta_{\tilde{k}},$$
$$V\bigl(x(t_2, t_0, x_0)\bigr) = r_{\tilde{k}}.$$

Thus,

$$V\bigl(x(t_1, t_0, x_0)\bigr) < V\bigl(x(t_2, t_0, x_0)\bigr).$$

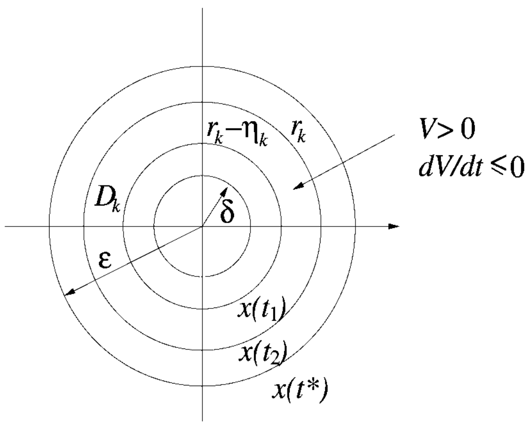

Figure 4.5.1. Geometric expression of Theorem 4.5.9.

However, on the other hand, since $\frac{dV}{dt}|_{(4.5.15)} \leqslant 0$ for $r_{\tilde{k}} - \eta_{\tilde{k}} \leqslant V(t) \leqslant r_{\tilde{k}}$, $V(x(t, t_0, x_0))$ is decreasing, implying that $V(x(t_2, t_0, x_0)) \leqslant V(t_1, t_0, x_0)$. Thus the zero solution of (4.5.15) is stable. □

REMARK 4.5.10. Theorem 4.5.9 is also valid for the case of a variable sign function $\frac{dV}{dt}$.

EXAMPLE 4.5.11. Analyze the stability of the zero solution of the system:

$$\begin{aligned}
\frac{dx}{dt} &= \begin{cases} -y + x(x^2+y^2)\sin(\frac{1}{x^2+y^2}) & \text{when } x^2+y^2 \neq 0, \\ -y & \text{when } x^2+y^2 = 0, \end{cases} \\
\frac{dy}{dt} &= \begin{cases} x + y(x^2+y^2)\sin(\frac{1}{x^2+y^2}) & \text{when } x^2+y^2 \neq 0, \\ x & \text{when } x^2+y^2 = 0. \end{cases}
\end{aligned} \tag{4.5.16}$$

Construct a positive definite Lyapunov function $V = (x^2 + y^2)$, so

$$\left.\frac{dV}{dt}\right|_{(4.5.16)} = 2\big(x^2+y^2\big)^2 \sin\left(\frac{1}{x^2+y^2}\right) \quad \big(\text{when } x^2+y^2 \neq 0\big).$$

Taking

$$\begin{aligned}
r_k &= \frac{1}{2k\pi + \frac{3\pi}{2}} > 0, \\
\eta_k &= \frac{\pi/4}{(2k\pi + \frac{3\pi}{2})(2k\pi + \frac{3\pi}{2} + \frac{\pi}{4})} > 0 \quad (k = 1, 2, \ldots),
\end{aligned}$$

we have

$$\lim_{k\to\infty} r_k = 0, \qquad \left.\frac{dV}{dt}\right|_{(4.5.16)} < 0$$

when $r_k - \eta_k \leqslant x^2 + y^2 \leqslant r_k$. All the conditions of Theorem 4.5.9 are satisfied, and so the zero solution of (4.5.16) is stable.

REMARK 4.5.12. Let $x = r\cos\theta$, $y = r\sin\theta$. Then, system (4.5.16) can be rewritten as

$$\begin{cases} \frac{dr}{dt} = \begin{cases} r^3 \sin\frac{1}{r^2} & \text{when } r \neq 0, \\ 0 \end{cases} \\ \frac{d\theta}{dt} = 1. \end{cases} \tag{4.5.17}$$

Obviously, system (4.5.17) has a sequence of closed orbits, i.e., it has infinite number of periodic solutions. Other trajectories are family of spirals.

EXAMPLE 4.5.13. Study the stability of the zero solution of the system:

$$\begin{cases} \frac{dx}{dt} = -x(x^2+y^2)\left[1-\cos\ln(x^2+y^2)-\sin\ln(x^2+y^2)\right] \\ \frac{dy}{dt} = -y(x^2+y^2)\left[1-\cos\ln(x^2+y^2)-\sin\ln(x^2+y^2)\right] \end{cases}$$
$$\text{when } x^2+y^2 \neq 0; \tag{4.5.18}$$
$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{cases} \quad \text{when } x^2+y^2 = 0.$$

Choose a positive Lyapunov function $V = x^2+y^2$. Then,

$$\left.\frac{dV}{dt}\right|_{(4.5.17)} = -2(x^2+y^2)^2\left[1-\sqrt{2}\sin\left(\frac{\pi}{4}+\ln(x^2+y^2)\right)\right].$$

Taking $r_k = e^{-2k\pi-\frac{\pi}{4}} > 0$, $\eta_k = e^{-2k\pi}(e^{-\frac{\pi}{4}} - e^{-\frac{\pi}{2}})$ results in

$$\lim_{k\to\infty} e^{-2k\pi-\frac{\pi}{4}} = 0,$$

and so $\frac{dV}{dt}|_{(4.5.17)} < 0$ when $e^{-2k\pi-\pi/2} \leqslant x^2+y^2 \leqslant e^{-2k\pi-\pi/4}$, implying that the zero solution of (4.5.17) is stable.

4.6. Sufficient conditions for asymptotic stability

In this section, we present some sufficient conditions for asymptotic stability.

THEOREM 4.6.1. *If there exist functions $V(t,x) \in C[R_H, R^1]$, negative semi-definite function $\theta(t,x) \in C[R_H, R^1]$ and positive definite function $W(x) \in C[\Omega_H, R^1]$, such that*

(1) $V(t,x) - \theta(t,x)W(x) \geqslant 0$;
(2) $\theta(t,x) \to +\infty$ *as* $t \to +\infty$ *holds uniformly for* x;
(3) $\frac{dV}{dt}|_{(4.2.1)} \leqslant 0$;

then the zero solution of (4.2.1) is asymptotically stable.

PROOF. Condition (2) implies that $\theta(t,x) \geqslant 1$ holds uniformly for x when $t \gg 1$. So $V(t,x) \geqslant W(x)$, i.e., $V(t,x)$ is positive definite. By conditions (3) we know that the zero solution is stable.

Since $W(x)$ is positive definite, there exists $\varphi \in K$ such that $W(x) \geqslant \varphi(\|x\|)$. By condition (3), $V(t,x(t))$ is monotone decreasing and bounded. Hence, there exists a constant $M > 0$ such that

$$M \geqslant V(t,x(t)) \geqslant \theta(t,x(t))W(x(t)) \geqslant \theta(t,x(t))\varphi(\|x(t)\|).$$

Therefore,

$$\varphi\big(\|x(t)\|\big) \leqslant \frac{M}{\theta(t, x(t))}.$$

$\forall\varepsilon > 0$, $\varphi(\varepsilon) > 0$, there exists T, when $t > T$,

$$\theta\big(t, x(t)\big) > \frac{M}{\varphi(\varepsilon)},$$

i.e., $\varphi(\varepsilon) > \frac{M}{\theta(t,x(t))}$. Thus, $\varphi(\|x(t)\|) < \varphi(\varepsilon)$, and so

$$\big\|x(t)\big\| < \varepsilon,$$

i.e.,

$$\lim_{t\to\infty} x(t) = 0.$$

This implies that the zero solution of (4.2.1) is asymptotically stable.

The proof is complete. □

REMARK 4.6.2. Theorem 4.6.1 is a generation of Theorem 4.5.7.

EXAMPLE 4.6.3. Consider the asymptotic stability of the solution of the system

$$\frac{dx}{dt} = -\frac{x}{t + \sin x} \tag{4.6.1}$$

in $[t_0, +\infty)$.

Choose $V(t, x) = (t + \sin x)x^2$, $\theta(t, x) = t + \sin x$ $(t_0 \geqslant 2)$. Then,

$$\begin{aligned}\frac{dV}{dt} &= 2(t + \sin x)x\left(\frac{-x}{t + \sin x}\right) + x^2\left(1 - \cos x \cdot \frac{x}{t + \sin x}\right)\\ &= -2x^2 + x^2\left(1 - \frac{x\cos x}{t + \sin x}\right) = -x^2\left(1 + \frac{x\cos x}{1 + \sin x}\right)\\ &\leqslant 0 \quad \big(\|x\| \ll 1\big).\end{aligned}$$

Hence, when $\|x\| \leqslant H \ll 2$, $t \geqslant t_0 \gg 1$, the conditions of Theorem 4.6.1 are satisfied. Therefore, the zero solution of (4.6.1) is asymptotically stable.

THEOREM 4.6.4. *Assume that (4.2.1) satisfies the following conditions: $f(t, x)$ is bounded for $t \geqslant 0$, $\|x\| \leqslant H$, and there exists a positive definite function $V(t, x) \in C^1[G_H, R^1]$ such that $\frac{dV}{dt}|_{(4.2.1)}$ is negative definite, then the zero solution of (4.2.1) is asymptotically stable.*

PROOF. Obviously, the conditions of Theorem 4.6.4 imply that the zero solution of (4.2.1) is stable. Thus, we only need to prove that the zero solution of (4.2.1) is attractive.

Otherwise, there exists an infinite sequence $\{t_m\}$, where $t_m \to +\infty$ as $m \to \infty$ such that for certain $\varepsilon > 0$, $\|x(t_m, t_0, x_0)\| \geqslant \varepsilon$ holds. By the boundedness of $f(t, x)$, there exists a constant $k > 0$ such that

$$\left|\dot{x}(t)\right| < k.$$

So in the close interval $[t_m - \frac{\varepsilon}{2k}, t_m + \frac{\varepsilon}{2k}]$, we have

$$x(t) = x(t_m) + \dot{x}(\xi)(t - t_m) \quad \text{where } \xi \in \left[t_m - \frac{\varepsilon}{2k}, t - t_m + \frac{\varepsilon}{2k}\right],$$

$$\left|x(t)\right| \geqslant \varepsilon - k\left(\frac{\varepsilon}{2k}\right) = \frac{\varepsilon}{2}.$$

Thus, in these intervals, there exists a constant $c > 0$ such that $\frac{dV}{dt} \leqslant -c$. Without loss of generality, suppose that $t_1 - \frac{\varepsilon}{2k} > t_0$ and these intervals do not intersect. Then, we obtain

$$\begin{aligned} V\left(t_m + \frac{\varepsilon}{2k}\right) - V(t_0) &= \int_{t_0}^{t_m + \frac{\varepsilon}{2k}} \frac{dV}{dt}\, dt \leqslant \sum_{k=1}^{m} \int_{t_k - \frac{\varepsilon}{2k}}^{t_k + \frac{\varepsilon}{2k}} \frac{dV}{dt}\, dt \\ &\leqslant -cm\frac{\varepsilon}{k} \to -\infty \quad \text{as } m \to \infty, \end{aligned}$$

which contradicts the fact that $V(t, x)$ is positive definite. So the zero solution of (4.2.1) is asymptotically stable. □

THEOREM 4.6.5. *Consider (4.2.1) for $t \geqslant 0$ and assume that $\|x\| \leqslant H$ satisfies the following conditions*:

(1) *there exists positive definite function $W(x)$ such that*

$$\left.\frac{dW}{dt}\right|_{(4.2.1)} = \sum_{i=1}^{n} \frac{\partial W}{\partial x_i} f_i(t, x) = \left(\frac{\partial W}{\partial x} \cdot f(t, x)\right) \tag{4.6.2}$$

has upper bound or lower bound;

(2) *there exists a positive definite function $V(t, x)$ such that $\frac{dV}{dt}|_{(4.2.1)}$ is negative*;

then the zero solution of (4.2.1) is asymptotically stable.

PROOF. The condition (2) in Theorem 4.6.5 implies that the zero solution of (4.2.1) is stable. So we only need to prove the attraction of zero solution of (4.2.1).

Consider the case of $(\frac{\partial W}{\partial x}, f(t,x))$ with the upper bound. By the method of reducing into a contradiction, suppose that the zero solution is not attractive. Then, there exists an infinite sequence $\{t_m\}$, $t_m \to +\infty$ as $m \to \infty$, such that for certain $\varepsilon > 0$,

$$\left\| x(t_m, t_0, x_0) \right\| \geqslant \varepsilon. \tag{4.6.3}$$

By condition (1) of the theorem, there exists $\delta > 0$ such that

$$W\big(x(t_m, t_0, x_0)\big) \geqslant \delta \tag{4.6.4}$$

holds for all natural numbers, and there exists a constant $k > 0$ such that

$$\frac{dW(x(t))}{dt} = \left(\frac{\partial W}{\partial x}, f(t,x)\right) < k. \tag{4.6.5}$$

Next, we prove that $\forall t \in [t_m - \frac{\delta}{2k}, t_m]$,

$$W\big(x(t)\big) \geqslant \frac{\delta}{2}. \tag{4.6.6}$$

In fact, otherwise, there exists $\bar{t} \in [t_m - \frac{\delta}{2k}, t_m]$ such that

$$W\big(x(\tilde{t})\big) < \frac{\delta}{2}.$$

By the mean value theorem, there exists $t^* \in (\bar{t}, t_m)$ such that

$$\frac{dW(x(t^*))}{dt} = \frac{W(x(t_m)) - W(x(\tilde{t}_0))}{t_m - t} \geqslant \frac{\delta - \frac{\delta}{2}}{\frac{\delta}{2k}} = k, \tag{4.6.7}$$

which is a contradiction with (4.6.5). So (4.6.6) is true. By the property of $W(x)$, there exists $\eta > 0$ such that

$$\left\| x(t) \right\| \geqslant \eta > 0$$

holds for all natural numbers and for all $t \in [t_m - \frac{\delta}{2k}, t_m]$. Since $\frac{dV}{dt}|_{(4.2.1)}$ is negative definite, there exists a constant $c > 0$ such that

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} \leqslant -c$$

is true for all $t \in [t_m - \frac{\delta}{2k}, t_m]$.

Without loss of generality, let $t_1 - \frac{\delta}{2k} > t_0$. Suppose that the above intervals do not intersect. Then,

$$V(t_m) - V(t_0) = \int_{t_0}^{t_m} \frac{dV}{dt}\, dt \leqslant \sum_{i=1}^{m} \int_{t_i - \frac{\delta}{2k}}^{t_i} \frac{dV}{dt}\, dt$$

$$\leqslant -cm\frac{\delta}{2k} \to -\infty \quad \text{as } m \to \infty,$$

which contradicts that $V(t,x)$ is positive definite. So the zero solution is asymptotically stable.

For the case of $(\frac{\partial W}{\partial x}, f(t,x))$ having a lower bound, the proof is similar.

Suppose that there exists $k > 0$ such that

$$\frac{dW(x(t))}{dt} = \left(\frac{\partial W}{\partial x}, f(t,x)\right) > -k. \tag{4.6.8}$$

Using (4.6.4) and (4.6.8) one can obtain

$$W\bigl(x(t)\bigr) \geqslant \frac{\delta}{2} \tag{4.6.9}$$

for $\forall t \in [t_m, t_m + \frac{\delta}{2k}]$. Otherwise, $\exists \tilde{t} \in [t_m, t_m + \frac{\delta}{2k}]$ such that

$$W\bigl(x(\bar{t})\bigr) < \frac{\delta}{2}.$$

By the mean value theorem, $\exists t^* \in (t_m, \tilde{t})$ such that

$$\begin{aligned}\frac{dW(x(t^*))}{dt} &= \frac{W(\tilde{t}) - W(t_m)}{\tilde{t} - t_m} = -\frac{W(t_m) - W(\tilde{t})}{\tilde{t} - t_m}\\ &\leqslant -\frac{\delta - \frac{\delta}{2}}{\frac{\delta}{2k}} = -k.\end{aligned} \tag{4.6.10}$$

This contradicts that $W(x(t)) > -k$. So $W(x(t)) \geqslant \frac{\delta}{2}$. Then, we only need to replace $[t_m - \frac{\delta}{2k}, t_m]$ by $[t_m, t_m + \frac{\delta}{2k}]$. Similar to the case of $(\frac{\partial W}{\partial x}, f(t,x))$ with an upper bound, we obtain

$$\bigl\|x(t)\bigr\| \geqslant \eta > 0, \qquad \left.\frac{dV}{dt}\right|_{(4.2.1)} \leqslant -c,$$

$$V(t_m) - V(t_0) = \int_{t_0}^{t_m} \frac{dV}{dt}\,dt \leqslant \sum_{i=1}^{m} \int_{t_i}^{t_i + \frac{\delta}{2k}} \frac{dV}{dt}\,dt \leqslant -cm\frac{\delta}{2k} \to -\infty$$

$\forall t \in [t_m, t_m + \frac{\delta}{2k}]$. This is impossible, because $V(t,x)$ is positive definite. So the zero solution is asymptotically stable □

REMARK 4.6.6. Theorem 4.6.5 includes Theorem 4.6.4 as a special case.

EXAMPLE 4.6.7. Consider the asymptotic stability of the zero solution of system

$$\begin{cases} \frac{dx}{dt} = \frac{-x}{1+t} + t^2 y^{2k-1} - t x^{2r-1} + \frac{1}{t} x y^{2k}, \\ \frac{dy}{dt} = \frac{y}{1+t} - t^2 x^{2k-1} - t y^{2r-1} + \frac{1}{t} x^{2k} y, \end{cases} \tag{4.6.11}$$

where $0 < \tau < t < +\infty$, $\tau =$ constant, r and k are natural numbers.

PROOF. To prove that the zero solution of (4.6.11) is asymptotically stable, choose a positive definite Lyapunov function:

$$V(t, x, y) = (1 + t)\big(x^{2k} + y^{2k}\big) \geqslant x^{2k} + y^{2k} \tag{4.6.12}$$

with no infinitesimal upper bound when

$$x^{2k} + y^{2k} \leqslant \alpha^2 := \frac{\tau}{2(1 + \tau)}.$$

From

$$\begin{aligned}
&4k\left(1 + \frac{1}{\tau}\right)x^{2k}y^{2k} \\
&\quad = -2k\left(1 + \frac{1}{\tau}\right)\big(x^{2k} - y^{2k}\big)^2 + 2k\left(1 + \frac{1}{\tau}\right)\big(x^{4k} + y^{4k}\big) \\
&\quad \leqslant -2k\left(1 + \frac{1}{\tau}\right)\big(x^{2k} - y^{2k}\big)^2 + k\big(x^{2k} + y^{2k}\big),
\end{aligned} \tag{4.6.13}$$

we obtain

$$\begin{aligned}
&\frac{dV}{dt}\bigg|_{(4.6.11)} = \big(x^{2k} + y^{2k}\big) + 2k(1 + t)\big(x^{2k-1}\dot{x} + y^{2k-1}\dot{y}\big) \\
&\quad = \big(x^{2k} + y^{2k}\big) + 2k(1 + t) \\
&\qquad \times \left[-\frac{x^{2k} + y^{2k}}{1 + t} - t\big(x^{2(k+r-1)} + y^{2(k+r-1)}\big) + \frac{1}{t}x^{2k}y^{2k}\right] \\
&\quad < (1 - 2k)\big(x^{2k} + y^{2k}\big) - 2k\tau(1 + \tau)\big[x^{2(k+r-1)} \\
&\qquad + y^{2(k+r-1)}\big] + 4k\left(1 + \frac{1}{\tau}\right)x^{2k}y^{2k} \\
&\quad \leqslant -(2k - 1 - k)\big(x^{2k} + y^{2k}\big) \\
&\qquad - 2k\tau(1 + \tau)\big[x^{2(k+r-1)} + y^{2(k+r-1)}\big].
\end{aligned} \tag{4.6.14}$$

So $\frac{dV}{dt}|_{(4.6.11)}$ is negative definite.

Let $W(x) = x^{2k} + y^{2k}$. $W(x)$ is positive definite and $\frac{\partial W}{\partial x} = 2k(x^{2k-1}, y^{2k-1})$. Thus,

$$\begin{aligned}
&\left(\frac{\partial W}{\partial x}, f(t, x)\right) = 2k\big(x^{2k-1}\dot{x} + y^{2k-1}\dot{y}\big) \\
&\quad = 2k\left[-\frac{x^{2k} + y^{2k}}{1 + t} - t\big(x^{2(k+r-1)} + y^{2(k+r-1)}\big) + \frac{2}{t}x^{2k}y^{2k}\right]
\end{aligned}$$

$$\leqslant \frac{4k}{\tau}x^{2k}y^{2k} \leqslant \frac{2k}{\tau}\big(x^{2k}+y^{2k}\big)^2 \leqslant \frac{2k}{\tau}\alpha^4,$$

indicating that $(\frac{\partial W}{\partial x}, f(t,x))$ has an upper bound, according to Theorem 4.6.4. Hence, the zero solution is asymptotically stable. □

For autonomous systems, the Lyapunov asymptotic stability theorem has been generalized [21,192].

Consider the n-dimensional autonomous system (4.5.15).

DEFINITION 4.6.8. The set $E = \{x \mid x(t,t_0,x_0), t \geqslant t_0\}$ is called positive half trajectory with $x(t_0,t_0,x_0) = x_0$; when $x_0 \neq 0$, E is called nontrivial positive half trajectory; $x^* \in \Omega_H$ is called ω-limit point of positive half trajectory, if there exists a sequence $\{t_k\}$ ($t_k \to +\infty$ as $k \to \infty$) such that

$$x^* = \lim_{t_k\to\infty} x(t_k,t_0,x_0). \tag{4.6.15}$$

The set consisting of ω-limit point is denoted by $\Omega(t_0)$.

For example, if $x = 0$ is asymptotically stable, then $x = 0$ is an ω-limit point of all positive half trajectory $x(t,t_0,x_0)$, x_0 in attractive basin of $x = 0$.

LEMMA 4.6.9. *Let x^* be ω limit point of $x(t,t_0,x_0)$. Then, all the points of positive half trajectories $x(t,t_0,x^*)$ are ω limit points of positive half trajectory. Therefore, Ω is constituted by full positive half trajectories.*

PROOF. Since there exists a sequence $\{t_n\} \to \infty$ as $n \to \infty$ such that

$$x^* = \lim_{n\to\infty} x(t_n,t_0,x_0).$$

Let $x(\tau,t_0,x^*)$ be an arbitrary point on the trajectory $x(t,t_0,x^*)$. By the Group's property of autonomous systems, we have

$$x(t_n+\tau,t_0,x_0) = x\big(\tau,t_0,x(t_n,t_0,x_0)\big).$$

Furthermore,

$$\lim_{n\to\infty} x(t_n+\tau,t_0,x_0) = \lim_{n\to\infty} x\big(\tau,t_0,x(t_n,t_0,x_0)\big) = x\big(\tau,t_0,x^*\big)$$

holds, i.e., $x(\tau,t_0,x^*)$ is also an ω-limit point of $x(t,t_0,x^*)$.

Lemma 4.6.9 is proved. □

THEOREM 4.6.10. *If there exists positive definite function $V(x(t)) \in C^1[D,R^1]$ such that*

$$\left.\frac{dV}{dt}\right|_{(4.5.15)} \leqslant 0, \tag{4.6.16}$$

then the set $M := \{x | \frac{dV}{dt}|_{(4.5.15)} = 0,\ x \in D\}$ *excludes* $x = 0$*, i.e., it does not include all positive half trajectories which are nonzero. Then, the zero solution of* (4.5.15) *is asymptotically stable.*

PROOF. Obviously, the conditions in Theorem 4.6.10 imply the conditions of Lyapunov stability theorem (see Theorem 4.4.2) for autonomous cases. So the zero solution is stable. $\forall \varepsilon > 0$ $(0 < \varepsilon < H)$, $\exists \delta(\varepsilon)$, when $\|x_0\| < \delta$, $\|x(t, t_0, x_0)\| < \varepsilon < H$ holds. By the Weierstrass accumulation principle we know that the ω limit point set $\Omega(t_0)$ is not empty and so $\Omega(t_0)$ is bounded.

Now prove $\Omega(t_0) = \{0\}$. Otherwise, there exists $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to +\infty} x(t_n, t_0, x_0) = x^* \neq 0.$$

By the property of positive definiteness of $V(x(t))$ and negative semi-definiteness of $\frac{dV}{dt}$, $V(x(t, t_0, x_0))$ is monotone decreasing, negative, and continuous. Hence, we have

$$\lim_{t \to +\infty} V\big(x(t, t_0, x_0)\big) = V\big(x^*\big) > 0. \tag{4.6.17}$$

Consider the trajectory $x(t, t_0, x^*)$. Since

$$\left.\frac{dV}{dt}\right|_{(4.5.15)} \leqslant 0,$$

we have

$$V\big(x\big(t, t_0, x^*\big)\big) \leqslant V\big(x^*\big).$$

If for all $t \geqslant t_0$, $V(x(t, t_0, x_0^*)) \equiv V(x_0^*)$ is true, then $\frac{dV}{dt}|_{(4.5.15)} \equiv 0$. Then, there is no trivial full positive half trajectory

$$x\big(t, t_0, x^*\big) \subset M.$$

This is impossible. Thus, there exists $t_1 \geqslant t_0$ such that

$$V\big(x\big(t_1, t_0, x^*\big)\big) < V\big(x^*\big).$$

Lemma 4.6.9 shows that $\forall t_1 > t_0$, $x(t_1, t_0, x^*)$ is an ω-limit point of $x(t, t_0, x_0)$. Hence, there exists $\{t_n^*\}$, $t_n^* \to t_1$ as $n \to \infty$ such that

$$\lim_{n \to \infty} x\big(t_n^*, t_0, x_0\big) = x\big(t_1, t_0, x^*\big).$$

It follows that

$$\lim_{n \to \infty} V\big(x\big(t_n^*, t_0, x_0\big)\big) = V\big(x\big(t_1, t_0, x^*\big)\big) < V\big(x^*\big), \tag{4.6.18}$$

which contradicts (4.6.17), so $\Omega = \{0\}$, i.e.,

$$\overline{\lim_{t\to\infty}}\, x(t, t_0, x_0) = \underline{\lim_{t\to\infty}}\, x(t, t_0, x_0) = 0 = \lim_{t\to\infty} x(t, t_0, x_0).$$

The proof of Theorem 4.6.10 is complete. □

EXAMPLE 4.6.11. Consider the stability of the equilibrium for a single pendulum system with damping:

$$\frac{d^2\varphi}{dt^2} + \frac{H}{m}\frac{d\varphi}{dt} + \frac{g}{l}\sin\varphi = 0, \tag{4.6.19}$$

where $H > 0$ is the damping coefficient, m is the mass of the pendulum, l is the length of the pendulum, and g is the gravity constant. $\theta = 0$ is an equilibrium, as shown in Figure 4.6.1.

Solution: Let $x = \varphi$, $y = \frac{d\varphi}{dt}$. Then, equation (4.6.19) can be rewritten as

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\frac{g}{l}\sin x - \frac{H}{m} y. \end{cases} \tag{4.6.20}$$

Choose a Lyapunov function:

$$V = \frac{1}{2}y^2 + \frac{g}{l}(1 - \cos x). \tag{4.6.21}$$

When $0 < x^2 + y^2 \leqslant 1$, V is positive definite, and

$$\left.\frac{dV}{dt}\right|_{(4.6.20)} = \frac{\partial V}{\partial x} y + \frac{\partial V}{\partial y}\left[-\frac{g}{l}\sin x - \frac{H}{m} y\right] = -\frac{H}{m} y^2 \leqslant 0.$$

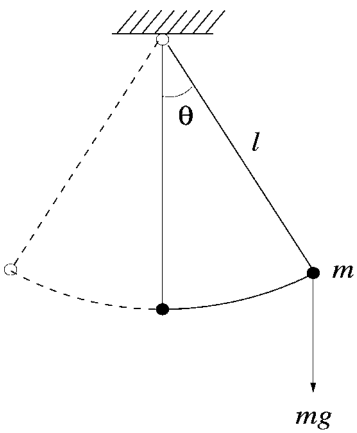

Figure 4.6.1. Single pendulum.

Letting $\frac{dV}{dt} = 0$ yields

$$M = \{y \mid y = 0\}.$$

But $y = 0$ does not include full nonzero positive half trajectories. So the equilibrium $\theta = 0$ of (4.6.21) is asymptotically stable.

THEOREM 4.6.12. *(See [21].) Assume that there exists $V(x) \in C^1[R^n, R^1]$, which is positive definite and radially unbounded, and*

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} \leqslant 0.$$

Then, the set $M = \{x | \frac{dV}{dt}|_{(4.2.1)} = 0\}$ excludes $x = 0$, i.e., it does not include full nonzero positive half trajectories. Then, the zero solution of (4.2.1) is globally asymptotically stable.

PROOF. Obviously, the conditions of Theorem 4.6.12 imply that the zero solution of (4.2.1) is stable $\forall x_0 \in R^n$ and $V(x(t, t_0, x_0)) \leqslant V(x_0)$ holds, because $\frac{dV}{dt}|_{(4.2.1)} \leqslant 0$.

Since V is a positive definite and radially unbounded function, $\forall M > 0$, $\exists R$ such that $V(x) \geqslant R$ and $V(x) \geqslant M$ holds. If $V(x) \leqslant V(x_0)$, $\exists r > 0$ such that $\|x\| < r$, then $x(t, t_0, x_0)$ is in a compact set of R^n.

Following the proof of Theorem 4.6.10 one can show that

$$\lim_{t \to \infty} x(t, t_0, x_0) = 0.$$

The proof of Theorem 4.6.12 is complete. □

EXAMPLE 4.6.13. Prove that the zero solution of the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = f(z) - az, \\ \frac{dz}{dt} = -g(x) - by, \end{cases} \tag{4.6.22}$$

is globally asymptotically stable, where $f(0) = g(0) = 0$, $a > 0$, $b > 0$ are constants. $g(x) \in C^1$ and $f(z) \in C$ satisfy

(1) $xg(x) > 0$ for $x \neq 0$;
(2) $zf(z) > 0$ for $z \neq 0$;
(3) $\dot{g}(x) < ab$ for $x \neq 0$;
(4) $V(t, x, z) = a \int_0^x g(\xi)\, d\xi + yg(x) + \int_0^z f(\eta)\, d\eta + \frac{b}{2} y^2 \to +\infty$ as $(x^2 + y^2 + z^2) \to \infty$.

PROOF. Let

$$
\begin{aligned}
M(x, y) &= a \int_0^x g(\xi)\, d\xi + yg(x) + \frac{b}{2} y^2 \\
&:= aG(x) + yg(x) + bY(y) \\
&= \frac{[\alpha\sqrt{b}y(y) + \frac{1}{\sqrt{b}} yg(x)]^2}{4Y(g)} + \frac{4aG(x)Y(y) - \frac{1}{b} y^2 g^2(x)}{4Y(g)},
\end{aligned}
$$

and

$$
\begin{aligned}
U(x, y) &:= 4aG(x)Y(y) - \frac{1}{b} y^2 g^2(x) \\
&= 4 \int_0^x g(x) \left\{ \int_0^y \left[a - \frac{y'(t)}{b} \right] y\, dy \right\} dx.
\end{aligned}
$$

By conditions (1) and (3) we know that $U(x, y)$ is positive definite. This implies that $M(t, y)$ is positive definite. Hence, $V(x, y, z)$ is a positive definite and radially unbounded function, and

$$
\left.\frac{dV}{dt}\right|_{(4.6.22)} = y^2 \big[g'(x) - ab \big] \leqslant 0.
$$

The set

$$
M = \left\{ x, y, z, \left.\frac{dV}{dt}\right|_{(4.6.22)} = 0 \right\} = \{y, y = 0\}
$$

does not include full nonzero positive half trajectory. Hence, the zero solution of (4.6.22) is globally asymptotically stable. $\square$

For general nonautonomous systems the above theorem is not true.

EXAMPLE 4.6.14. Consider the system

$$
\begin{cases}
\dfrac{dx_1}{dt} = \dfrac{-3}{1+t^2} x_1 + \dfrac{1}{1+t^2} x^2, \\
\dfrac{dx_2}{dt} = \dfrac{-1}{1+t^2} x_1 - \dfrac{1}{1+t^2} x^2.
\end{cases}
\tag{4.6.23}
$$

Taking $V = \frac{1}{2}(x_1^2 + x_2^2)$ we have

$$
\left.\frac{dV}{dt}\right|_{(4.6.23)} = -\frac{3}{1+t^2} x_1^2 - \frac{1}{1+t^2} x_2^2 \leqslant 0.
$$

$M := \{(t,x) \mid \frac{dV}{dt}|_{(4.6.23)} = 0\}$ only includes $\{x_1 = x_2 = 0\}$, but the equation admits a special solution:

$$x_1(t) = x_2(t) = e^{-2\int_{t_0}^{t} \frac{d\xi}{1+\xi^2}} = e^{-2[\arctan(t) - \arctan(t_0)]}.$$

This special solution does not tend to zero as $t \to +\infty$. So the zero solution of (4.6.23) is not asymptotically stable.

We again consider (4.2.1) but let (4.2.1) be a periodic system, i.e., $\exists \tau > 0$ such that $f(t+\tau, x) \equiv f(t,x)$.

THEOREM 4.6.15. *(See [19].) Assume that there exists $V(t,x) \in C^1[G_H, R^1]$ with $V(t+\tau, x) \equiv V(t,x)$ such that*

(1) *$V(t,x)$ is positive definite, i.e., there exists $\varphi \in K$ such that $V(t,x) \geqslant \varphi(\|x\|)$;*
(2) $\frac{dV}{dt}|_{(4.2.1)} \leqslant 0$;
(3) $M := \{(t,x) \in G_H \cap \frac{dV}{dt}|_{(4.2.1)} = 0\}$ *does not include nonzero full positive half trajectory*;

then the zero solution of (4.2.1) is uniformly asymptotically stable.

PROOF. The conditions in the theorem imply that the zero solution is stable. But system (4.2.1) is periodic, so the stability is uniform, i.e.,

$$\forall \alpha > 0,\ \exists \delta(\alpha) > 0,\ \text{when } \|x_0\| < \delta,\ \left\|x(t,t_0,x_0)\right\| \leqslant \alpha.$$

Now, we prove that the zero solution is uniformly attractive. Since $\|x(t,t_0,x_0)\| \leqslant \alpha$, the sequence

$$x_0^{(k)} := x(t_0 + k\tau, t_0, x_0) \tag{4.6.24}$$

has limit point x_0^*, i.e., there exists a convergent subsequence, still denoted by $\{x_0^{\{k\}}\}$, such that

$$\lim_{k\to\infty} x_0^{(k)} = x_0^*.$$

First, we prove that $x_0^* = 0$. Otherwise, suppose $x_0^* \neq 0$. Consider the solution $x(t,t_0,x_0^*)$. By condition (3) we know that there exists $t^* > t_0$ such that

$$\frac{dV(t^*, x(t^*, t_0, x_0^*))}{dt} < 0. \tag{4.6.25}$$

Furthermore, for all $\tilde{t} \geqslant t^*$ we have

$$V\big(t^*, x\big(t^*, t_0, x_0^*\big)\big) < V\big(t_0, x_0^*\big). \tag{4.6.26}$$

According to periodic propriety, we obtain

$$\begin{aligned}
x\big(t^*, t_0, x_0^{(k)}\big) &\equiv x\big(t^* + k\tau, t_0 + k\tau, x_0^{(k)}\big) \\
&\equiv x\big(t^* + k\tau, t_0 + k\tau, x(t_0 + k\tau, x_0, x_0)\big) \\
&\equiv x\big(t^* + k\tau, t_0, x_0\big).
\end{aligned}$$

Without loss of generality, let $t^* = t_0 + 2r\tau$. Using the periodic property of $V(t, x)$ with respect t we have

$$\begin{aligned}
V\big(t_0, x^*\big) &= \lim_{\kappa \to \infty} V\big(t^* + \kappa\tau, x\big(t^* + k\tau, t_0, x_0\big)\big) \\
&= \lim_{\kappa \to +\infty} V\big(t^* + k\tau, x(t^* + k\tau, t_0, x_0)\big) \\
&= \lim_{\kappa \to \infty} V\big(t^*, x\big(t^*, t_0, x_0^{(k)}\big)\big) \\
&= V\big(t^*, x\big(t^*, t_0, x_0^*\big)\big) \\
&< V\big(t_0, x_0^*\big).
\end{aligned} \tag{4.6.27}$$

This is a contradiction, and therefore $x_0^* = 0$.

Next, we prove that $x_0^* = 0$ implies $\lim_{t\to\infty} x(t, t_0, x_0) = 0$. Otherwise, there exists $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{t_n \to \infty} x(t_n, t_0, x_0) = \tilde{x} \neq 0.$$

Therefore, on one hand, we have

$$\begin{aligned}
\lim_{t_n \to \infty} V\big(t_n, x(t_n, t_0, x_0)\big) &= \lim_{t \to \infty} V\big(t, x(t, t_0, x_0)\big) \\
&= \lim_{k \to \infty} V\big(t_0 + k\tau, x(t_0 + k\tau, t_0, x_0)\big) \\
&= V\big(t_0, x_0^*\big) = V(t_0, 0) = 0,
\end{aligned} \tag{4.6.28}$$

but on the other hand, we have

$$\begin{aligned}
\lim_{t_n \to \infty} V\big(t_n, x(t_n, t_0, x_0)\big) &\geqslant \lim_{t_n \to \infty} \varphi\big(\big\|x(t_n, t_0, x_0)\big\|\big) \\
&= \varphi\big(\|\tilde{x}\|\big) > 0.
\end{aligned} \tag{4.6.29}$$

This is a contradiction. Therefore $x_0^* = 0$, implying

$$\lim_{t \to \infty} x(t, t_0, x_0) = 0.$$

So the zero solution is attractive and is uniformly attractive by periodic propriety. Therefore, the zero solution is uniformly asymptotically stable. □

4.7. Sufficient conditions for instability

In this section, we introduce some sufficient conditions for instability.

THEOREM 4.7.1. *If there exists* $V(t,x) \in C^1[G_H, R^1]$ *with* $V(t,0) = 0$ *such that*

(1) *in any neighborhood of the origin of* R^n, *there exists a region such that* $V > 0$ $(t \geqslant t_0)$;
(2) *in region* $V > 0$, $V(t,x)$ *is bounded*;
(3) $\frac{dV}{dt}\big|_{(4.2.1)}$ *is positive definite in* $V > 0$, *i.e.,* $\forall \varepsilon > 0$, $\exists l > 0$ *such that when* $V \geqslant \varepsilon > 0$,

$$\frac{dV}{dt}\bigg|_{(4.2.1)} \geqslant l > 0 \quad \forall\, t \geqslant t_0 \tag{4.7.1}$$

holds;

then the zero solution of (4.2.1) *is unstable.*

PROOF. Choose $\varepsilon > 0$, $0 < \varepsilon < H$. $\forall \delta > 0$, there exist $x_0 \in S_\delta := \{x,\ \|x\| < \delta\}$ and $t_1 > t_0$ such that $\|x(t_1, t_0, x_0)\| \geqslant \varepsilon$. Take x_0 in the region $V > 0$, $x_0 \in S_\delta$, such that

$$V(t_0, x_0) > 0.$$

By condition (1), this is possible. Condition (2) implies that $V(t, x(t, t_0, x_0)) \geqslant V(t_0, x_0) > 0$ for $t \geqslant t_0$. If

$$\big\|x\big(t, x(t, t_0, x_0)\big)\big\| \geqslant V(t_0, x_0) > 0,$$

then due to condition (3) the trajectory $x(t, t_0, x_0)$ stays in $V > 0$ $\forall t \geq t_0$. So there exists $l > 0$ such that

$$\frac{dV(t, x(t))}{dt}\bigg|_{(4.2.1)} \geqslant l > 0.$$

Thus, by condition (2), we have

$$\begin{aligned} V\big(t, x(t)\big) &= V(t_0, x_0) + \int_{t_0}^{t} \frac{dV}{dt}\, dt \\ &\geqslant V(t_0, x_0) + l(t - t_0) \to +\infty \quad \text{as } t \to +\infty. \end{aligned} \tag{4.7.2}$$

This is impossible, therefore the zero solution of (4.2.1) is unstable. □

In the following we give two Lyapunov instability criteria, which are special cases of Theorem 4.7.1.

COROLLARY 4.7.2. *(See [298].) If there exists $V(t,x) \in C^1[G_H, R^1]$ with $V(t,0) \equiv 0$ such that*

(1) *condition* (2) *in Theorem* 4.7.1 *holds*;
(2) *$V(t,x)$ has infinitesimal upper bound*;
(3) $\frac{dV}{dt}|_{(4.2.1)}$ *is negative definite*;

then the zero solution of (4.2.1) *is unstable.*

PROOF. By condition (2), there exists positive definite function $W(x)$ such that

$$\left|V(t,x)\right| \leqslant W(x).$$

So $\forall \varepsilon > 0$, $\exists \delta(\varepsilon)$, when $\|x\| < \delta$, we have

$$\left|V(t,x)\right| < \varepsilon \quad \forall t > t_0.$$

Hence, on the intersection set of certain neighborhood of the origin O_δ and $V > 0$, $V(t,x)$ is bounded, implying that the condition (2) in Theorem 4.7.1 holds.

Condition (3) implies that there exists a positive definite $W(x)$ such that $\frac{dV}{dt}|_{(4.2.1)} \geqslant W(x)$. So $\forall \varepsilon^* > 0$ on $V \geqslant \varepsilon^*$ and $\|x\| \geqslant \delta$ one can take

$$l = \inf_{\delta \leqslant \|x\| \leqslant H} W(x).$$

Thus, $\frac{dV}{dt} \geqslant l > 0$, i.e., $\frac{dV}{dt}$ is positive definite in $V > 0$. Thus, all conditions of Theorem 4.7.1 are satisfied. Hence, the zero solution of (4.2.1) is unstable. □

COROLLARY 4.7.3. *(See [298].) If there exists $V(t,x) \in C^1[G_H, R^1]$ such that*

(1) *the condition* (1) *of Theorem* 4.7.1 *holds*;
(2) *$V(t,x)$ is bounded in G_H*;
(3)

$$\left.\frac{dV}{dt}\right|_{(4.7.2)} = \lambda V + W(t,x), \tag{4.7.3}$$

where $\lambda > 0$, $W(t,x) \geqslant 0$;

then the zero solution of (4.2.1) *is unstable.*

PROOF. Obviously, conditions (1) and (2) imply conditions (1) and (2) in Theorem 4.7.1. $\forall \epsilon > 0$, take $l(\varepsilon) = \lambda\varepsilon$. Then for $V \geqslant \varepsilon$,

$$\frac{dV}{dt} \geqslant \lambda\varepsilon > 0.$$

Hence, condition (3) in Theorem 4.7.1 is satisfied. So the conclusion is true. □

THEOREM 4.7.4. *Assume that there exists* $V(t,x) \in C^1[G_H, R^1]$ *with* $V(t,0) = 0$ *such that*

(1) *in any neighborhood* $B_\delta := \{x, \|x\| < \delta \leqslant H\}$ *of the origin there exists region in which* $V > 0$;
(2) *in the region* $V > 0$, V *is bounded and*

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} = \xi(t)K(V) + U(t,x) \geqslant 0, \tag{4.7.4}$$

where $\xi(t)$ *is integrable in any bounded interval* I *and* $\int_{t_0}^{+\infty} \xi(t)\,dt = +\infty$, $K(V)$ *is a continuous function and when* $V > 0$, $K(V) > 0$ *and* $U(t,x) \geqslant 0$;

then the zero solution of (4.2.1) *is unstable.*

PROOF. According to condition (1), $\forall \delta > 0$, one can take (t_0, x_0) such that $\|x_0\| < \delta$, $V(t_0, x_0) = \alpha > 0$.

We now prove that for certain h that is $0 < h < H$, there must exist $t_1 > t_0$ such that $\|x(t_1, t_0, x_0)\| > h$. By condition (2),

$$V\big(t, x(t)\big) \geqslant V\big(t_0, x(t_0)\big) = \alpha > 0$$

holds. So $x(t, t_0, x_0)$ stays in $V > 0$ $\forall t \geq t_0$.

Let $\|x(t, t_0, x_0)\| \leqslant h < H$ for $t \geqslant t_0$. By condition (2), we obtain

$$\left.\frac{dV}{dt}\right|_{(4.2.1)} \geqslant \xi(t)K(V).$$

Hence,

$$\int_{V(t_0,x_0)}^{V(t,x(t))} \frac{dV}{K(V)} \geqslant \int_{t_0}^{t} \xi(t)\,dt \to +\infty \quad \text{as } t \to +\infty. \tag{4.7.5}$$

This implies $V(t, x(t)) \to +\infty$ as $t \to +\infty$, which contradicts that V is bounded in $V > 0$. Thus, $\exists t_1$ such that $\|x(t_1, t_0, x_0)\| > h$, i.e., the zero solution is unstable. □

EXAMPLE 4.7.5. Consider the stability of the zero solution of the system:

$$\begin{cases} \frac{dx}{dt} = \frac{2}{t+1}(x^2 + xy)e^{\cos(x+y)} + ye^{\sin t}, \\ \frac{dy}{dt} = xe^{\sin t} + y^2 e^{\cos(x+y)} \quad (t \geqslant 0). \end{cases} \tag{4.7.6}$$

Let $V(x,y) = x + y$. Then in the positive definite region $D := \{t \geqslant 0,\ x + y > 0\}$, as shown in Figure 4.7.1, one can show that

$$\left.\frac{dV}{dt}\right|_{(4.7.6)} = \frac{dx}{dt} + \frac{dy}{dt}$$

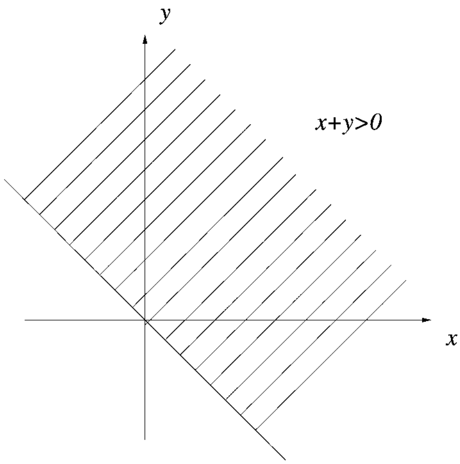

Figure 4.7.1. Illustration for Example 4.7.5.

$$
\begin{aligned}
&= \frac{2}{t+1}\big(x^2+xy\big)e^{\cos(x+y)} + ye^{\sin t} + xe^{\sin t} + y^2 e^{\cos(x+y)} \\
&= \frac{1}{1+t}(x+y)^2 e^{\cos(x+y)} + \frac{x^2}{t+1}e^{\cos(x+y)} \\
&\quad + (x+y)e^{\sin t} + y^2 e^{\cos(x+y)}\left(1-\frac{1}{t+1}\right).
\end{aligned}
$$

$$\xi(t) := \frac{1}{t+1} > 0,$$

$$\int_0^{+\infty} \frac{dt}{1+t} = +\infty, \qquad K(V) = V^2 e^{\cos V} = (x+y)^2 e^{\cos(x+y)} > 0,$$

$$
\begin{aligned}
U(t,x,y) &= y^2\left(1-\frac{1}{t+1}\right)e^{\cos(x+y)} + (x+y)e^{\sin t} \\
&\quad + \frac{1}{1+t}e^{\cos(x+y)} \geqslant 0.
\end{aligned}
$$

According to Theorem 4.7.4 we know that the zero solution of (4.7.6) is unstable.

EXAMPLE 4.7.6. Analyze the stability of the zero solution of the system:

$$
\begin{cases}
\frac{dx}{dt} = x\arctan(y-x^2) + x^3\sin y - \frac{1}{t+1}x^3, \\
\frac{dy}{dt} = (4+t)y\arctan(y-x^2) + 2x^2 y \quad (t \geqslant 0).
\end{cases} \tag{4.7.7}
$$

Let $V(x,y) = y - x^2$. Then $D := \{t, x, y \mid t \geqslant 0,\ y > x^2\}$ is a positive definite region of $V(x,y)$.

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(4.7.7)} &= (4+t)y\arctan\bigl(y-x^2\bigr) + 2x^2y\\ &\quad - 2x\left[x\arctan\bigl(y-x^2\bigr) + x^3\sin y - \frac{1}{t+1}x^3\right]\\ &= 2\bigl(y-x^2\bigr)\arctan\bigl(y-x^2\bigr) + (2+t)y\arctan\bigl(y-x^2\bigr)\\ &\quad + 2x^2y - 2x^4\sin y + \frac{2}{t+1}x^4 \geqslant 0\end{aligned}$$

in D.

$$\xi(t) := 2 > 0, \qquad \int_0^{+\infty} \xi(t)\,dt = +\infty,$$

$$\begin{aligned}K(V) &= V\arctan V = \bigl(y-x^2\bigr)\arctan\bigl(y-x^2\bigr) > 0,\\ U(t,x,y) &= (2+t)y\arctan\bigl(y-x^2\bigr) + 2x^2y - 2x^4\sin y + \frac{2}{t+1}x^4\\ &> (2+t)y\arctan\bigl(y-x^2\bigr) + 2x^4(1-\sin y) + \frac{2}{1+t}x^4 > 0.\end{aligned}$$

So the zero solution of (4.7.7) is unstable.

THEOREM 4.7.7. *Suppose that there exist $V(t,x) \in C^1[G_H, R^1]$ with $V(t,0) \equiv 0$ and $\varphi(\|x\|) \in K$ such that*

(1) *in any neighborhood of the origin and I, there exist t and x such that $V(t,x) > 0$;*
(2) *in region $V(t,x) > 0$,*

$$\frac{dV}{dt}\bigg|_{(4.2.1)} \geqslant \eta\bigl(V(t,x),t\bigr) \geqslant 0, \quad 0 \leqslant V(t,x) \leqslant \varphi\bigl(\|x\|\bigr) \tag{4.7.8}$$

holds, where $\eta(V,t) \geqslant 0$, when $\alpha_1 > \alpha_2$, $\eta(\alpha_1,t) \geqslant \eta(\alpha_2,t)$ and for any $\alpha > 0$, $\int_{t_0}^{t} \eta(\alpha,t)\,dt$ is not bounded;

then the zero solution of (4.2.1) is unstable.

PROOF. We wish to prove that for certain $\varepsilon > 0$ $(\varepsilon < H)$, $\forall\delta > 0$ $(\delta < \varepsilon)$, $\exists x_0 \in B_\delta := \{x,\ \|x\| < \delta\}$ and $t_1 > t_0$ such that $\|x(t_1,t_0,x_0)\| \geqslant \varepsilon$.

Otherwise, let $\|x(t,t_0,x_0)\| < \varepsilon$, $t \geqslant t_0$. By condition (1) we know that $\exists t_1 \geqslant t_0$ and $x_1 \in B_\delta$ such that $V(t_1,x_1) > 0$. The continuity of $V(t,x)$ implies that there exists a neighborhood $D(t_1,x_1)$ of (t_1,x_1) such that when

$(t,x) \in D(t,x) \subset B_\varepsilon = \{x \mid \|x\| < \varepsilon\}$, $0 < V(t,x) \leqslant \varphi(\|x\|)$, $x(t,t_1,x_1) \in B_\varepsilon$ $(t \geqslant t_0)$ holds.

By the condition (2), we have

$$\dot{V}_{(4.2.1)}\big(t, x(t,t_1,x_1)\big) \geqslant \eta\big(V(t,x),t\big) \geqslant 0, \tag{4.7.9}$$

so when $t \geqslant t_1 \geqslant t_0$,

$$V\big(t, x(t,t_1,x_1)\big) \geqslant V(t_1,x_1) > 0.$$

Integrating (4.7.9) yields

$$\begin{aligned} V\big(t, x(t,t_1,x_1)\big) - V(t_1,x_1) &\geqslant \int_{t_1}^{t} \eta\big(V\big(s, x(s,t_1,x_1)\big), s\big)\, ds \\ &\geqslant \int_{t_1}^{t} \eta\big(V(t_1,x_1), s\big)\, ds \quad (t \geqslant t_0). \end{aligned} \tag{4.7.10}$$

Therefore, we have

$$V\big(t, x(t,t_1,x_1)\big) \geqslant V\big(t, x(t,t_1,x_1)\big) - V(t_1,x_1) \geqslant \int_{t_1}^{t} \eta\big(V(t_1,x_1), s\big)\, ds.$$

Furthermore, we have

$$\varphi(H) \geqslant \varphi\big(\|x(t)\|\big) \geqslant V\big(t, x(t,t_1,x_1)\big) \geqslant \int_{t_1}^{t} \eta\big(V(t_1,x_1), s\big)\, ds,$$

which contradicts that $\int_{t_1}^{t} \eta(V(t_1,x_1),s)\, ds$ is unbound. So the zero solution of (4.2.1) is unstable. □

For autonomous system (4.5.15) Karsovskii [192] generated the Lyapunov instability theorem.

LEMMA 4.7.8. *If there exists function $V(x) \in C^1[G_H, R^1]$ with a lower bound such that*

$$\frac{dV}{dt}\bigg|_{(4.5.15)} \leqslant 0, \tag{4.7.11}$$

and any trajectory $x(t,t_0,x_0)$ does not leave G_H when $t \to +\infty$. Then, the ω-limit set Ω is located on certain hypersurface $V = V_0 = \text{constant}$.

PROOF. Let $x_0 \in G_H$ and $y \in \Omega(t_0)$. Then, there exist $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that

$$y = \lim_{n\to+\infty} x(t_n, t_0, x_0),$$

owing to $\frac{dV}{dt}|_{(4.5.15)} \leqslant 0$.

The function $V(x(t_n, t_0, x_0))$ is descending with a lower bound, so

$$V_0 = \lim_{n\to+\infty} V\big(x(t_n, t_0, x_0)\big) \tag{4.7.12}$$

by using the continuity of $V(t, x)$. Thus,

$$V(y) = \lim_{n\to+\infty} V\big(x(t_n, t_0, x_0)\big).$$

Therefore, for any sequence $\{\tilde{t}_n\}$, $\tilde{t}_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to+\infty} V\big(x(t_n, t_0, x_0)\big) = V_0,$$

i.e., for any $y \in \Omega$, we have $V(y) = V_0$. □

THEOREM 4.7.9. *If the following conditions are satisfied*:

(1) *there exists a function $V(x) \in C^1[G_H, R^1]$ with $V(0) = 0$, and in any neighborhood of the origin there exists x_0 such that $V(x_0) > 0$;*
(2) *$\frac{dV}{dt}|_{(4.5.15)} \geqslant 0$ and $M := \{x, \frac{dV}{dt}|_{(4.5.15)} = 0\}$ does not include full nonzero positive half trajectories*;

then the zero solution of (4.5.15) is unstable.

PROOF. By the conditions in Theorem 4.7.9, for some fixed $\varepsilon > 0$, $\forall\delta > 0$ ($\delta < \varepsilon$), there exists $x_0 \in B_\delta = \{x \mid \|x\| < \delta\}$ such that $V(x_0) = V_0 > 0$. The continuity of $V(x)$ and $V(0) = 0$ imply that there exists $\eta > 0$, $0 < \eta < \delta$, when $\|x\| < \eta$

$$\big|V(x)\big| < V_0$$

holds.

Next, we prove that there exists $t_1 > t_0$ such that

$$\big\|x(t_1)\big\| := \big\|x(t_1, t_0, x_0)\big\| \geqslant \varepsilon.$$

Otherwise, let $\|x(t)\| < \varepsilon$ ($t \geqslant t_0$). By $\frac{dV}{dt} \geqslant 0$ one obtains

$$V\big(x(t)\big) \geqslant V(x_0) > 0 \quad (t \geqslant t_0).$$

Hence, $\eta \leqslant \|x(t)\| < \varepsilon$. Since the ω limit set is not empty, $\forall y \in \Omega(x_0)$, there exist $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to+\infty} x(t_n, t_0, x_0) = y$$

owing to $\frac{dV(x(t))}{dt} \geqslant 0$ and $V(x(t, t_0, x_0))$ is increasing and has an upper bound. Thus,

$$\lim_{t \to +\infty} V\big(x(t, t_0, x_0)\big) = c > 0$$

by using the continuity of $V(x)$. Thus, $\forall y \in \Omega(x_0)$, $V(y) \equiv c$ and on $\Omega(t_0)$, $\frac{dV}{dt} = 0$. But $\Omega(t_0)$ consists of full positive half trajectories. This is a contradiction. Therefore the zero solution of (4.5.15) is unstable. □

EXAMPLE 4.7.10. Consider the stability of the zero solution for the system:

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -x_1 + 3(1 + x_2)^2 x_2. \end{cases} \tag{4.7.13}$$

Solution: Choose

$$V = x_1^2 + x_2^2, \qquad \frac{dV}{dt}\bigg|_{(4.7.13)} = 6\big(1 + x_2^2\big)x_2^2 \geqslant 0.$$

Letting $\frac{dV}{dt} = 0$ yields $x_2 = 0$. Substitute $x_2(t) = 0$ into

$$\frac{dx_2}{dt} = -x_1 + 3(1 + x_2)^2 x_2$$

to obtain $x_1 = 0$. Hence, $\frac{dV}{dt}$ only includes $x_1 = x_2 = 0$. So the zero solution is unstable.

We again consider (4.2.1) as a periodic system, i.e., $\exists \tau > 0$ such that $f(t + \tau, x) \equiv f(t, x)$.

THEOREM 4.7.11. *If there exists $V(t, x) \in C^1[G_H, R^1]$ with $V(t + \tau, x) \equiv V(t, x)$, $V(t, 0) \equiv 0$ such that*

(1) *in any neighborhood of the region $B_\delta := \{x, \|x\| < \delta\}$ there exists a region $V > 0$;*
(2) *in the region $V > 0$, $\frac{dV}{dt}|_{(4.2.1)} \geqslant 0$;*
(3) *the set $M = \{x \mid \frac{dV}{dt}|_{(4.2.1)} = 0 \cap V(t, x) > 0\}$ does not include full nonzero positive half trajectories*;

then the zero solution of (4.2.1) *is unstable.*

PROOF. By condition (1), there exists some $\varepsilon_0 > 0$ in $\|x\| \leqslant \varepsilon_0$ such that $V(t, x) > 0$, and $\forall \delta(\delta < \varepsilon_0)$, $\exists x_0 \in V > 0$ and $\|x_0\| < \delta$. Let $V(t_0, x_0) = V_0 > 0$. We want to show that there exists $t_1 > t_0$ such that

$$\big\|x(t_1, t_0, x_0)\big\| > \varepsilon_0.$$

Otherwise, let $\|x(t,t_0,x_0)\| \leqslant \varepsilon_0$. Condition (2) implies

$$V\big(t, x(t,t_0,x_0)\big) \geqslant V(t_0,x_0) = V_0 > 0.$$

Thus, $x_0^{(k)} := (t_0 + k\tau, t_0, x_0)$ have limiting points. Without loss of generality, let $x_0^{(k)} \to x_0^*$ as $k \to \infty$. Owing to that $V(t,x(t,t_0,x_0))$ is continuous, monotone increasing and bounded, there exists the limit

$$\lim_{t\to+\infty} V\big(t, x(t,t_0,x_0)\big) := V_\infty. \tag{4.7.14}$$

Since $V(t+\tau,x) \equiv V(t,x)$, the above limit can be obtained:

$$\begin{aligned}
\lim_{t\to+\infty} V\big(t, x(t,t_0,x_0)\big) &= \lim_{k\to+\infty} V\big(t_0 + k\tau, x(t_0+k\tau,t_0,x_0)\big)\\
&= \lim_{k\to+\infty} V\big(t_0, x(t_0+k\tau,t_0,x_0)\big)\\
&= \lim_{k\to+\infty} V\big(t_0, x^{(k)}\big)\\
&= V\big(t_0, x^*\big) = V_\infty.
\end{aligned} \tag{4.7.15}$$

But on the other hand, consider the solution $x(t,t_0,x_0^*)$. By condition (3), $\exists t^* > t_0$,

$$\frac{dV(t^*, x(t^*,t_0,x_0^*))}{dt} > 0,$$

and

$$V\big(t^*, x\big(t^*,t_0,x_0^*\big)\big) > V\big(t_0,x_0^*\big).$$

Owing to $f(t+\tau,x) \equiv f(t,x)$ and $V(t+\tau,x) \equiv V(t,x)$ we have

$$\begin{aligned}
x\big(t^*,t_0,x_0^{(k)}\big) &= x\big(t^* + k\tau, t_0 + k\tau, x_0^{(k)}\big)\\
&= x\big(t^* + k\tau, t_0 + k\tau, x(t_0+k\tau,t_0,x_0)\big)\\
&= x\big(t^* + k\tau, t_0, x_0\big)
\end{aligned} \tag{4.7.16}$$

and

$$\begin{aligned}
V\big(t_0,x_0^*\big) &= \lim_{k\to\infty} V\big(t^*+k\tau, x\big(t^*+k\tau,t_0,x_0\big)\big)\\
&= \lim_{k\to\infty} V\big(t^*, x\big(t^*,t_0,x_0^{(k)}\big)\big)\\
&= V\big(t^*, x\big(t^*,t_0,x_0^*\big)\big)\\
&> V\big(t_0,x_0^*\big).
\end{aligned} \tag{4.7.17}$$

This is impossible. Hence, the zero solution of (4.2.1) is unstable. □

EXAMPLE 4.7.12. Consider a third-order nonlinear equation:

$$\dddot{x} + f_1(x, \dot{x})\ddot{x} + f_2(x, \dot{x})\dot{x} + f_3(x) = 0, \tag{4.7.18}$$

where f_1, f_2 and f_3 are continuous differentiable functions of x and $\dot{x}$ and $f_2(0, 0) = f_3(0) = 0$.

To analyze the stability of the zero solution, let $\dot{x} = y$ and $\ddot{x} = z$. Then, the above equation can be rewritten as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -f_3(x) - f_2(x, y) - f_1(x, y,)z. \end{cases} \tag{4.7.19}$$

Let $D_1 = \{x, y, z \mid y \geqslant x \geqslant 0, z \geqslant 0\}$ and D_2 be any small neighborhood of the origin. In $D_1 \cap D_2$ the following conditions are satisfied:

(1) $f_1(x, y) < 0$;
(2) $xf_3(x) > 0$ $(x \neq 0)$, $yf_2(x, y) > 0$ $\forall y \neq 0$;
(3) $\frac{\partial f_2(x,y)}{\partial x} \geqslant 0$, $f_3'(x) > 0$.

Then the zero solution of (4.7.18) is unstable.

PROOF. Choose a Lyapunov function:

$$V(x, y, z) = yf_3(x) + \frac{1}{2}z^2 + \int_0^y f_2(x, y)\, dy.$$

By condition (2),

$$\int_0^y f_2(x, y)\, dy > 0 \quad \forall y \neq 0,$$

in the region defined by $y \geqslant x$ and $yf_3(x) \geqslant 0$. So $V(x, y, z)$ is positive definite in $D_1 \cap D_2$. By condition (3) we have

$$\left.\frac{dV}{dt}\right|_{(4.7.19)} = y^2 f_3'(x) - f_1(x, y)z^2 + y \int_0^y \frac{\partial f_2}{\partial x}\, dy \geqslant 0.$$

Letting $\frac{dV}{dt}|_{(4.7.19)} = 0$ yields $y = z = 0$ and $x = c$. If $x = 0$, then $\frac{dV}{dt}|_{(4.7.19)}$ only includes $x = y = z = 0$. If $x = c \neq 0$, one can choose $x = C \bar{\in} D_1 \cap D_2$. So the zero solution of (4.7.19) is unstable. □

EXAMPLE 4.7.13. Again consider system (4.7.19), but now let

$$\tilde{D}_1 := \{x, y, z | z \geqslant x|\},$$

and $\tilde{D}_2$ be any small neighborhood of the origin. In $\tilde{D}_1 \cap \tilde{D}_2$, the following conditions are satisfied:

(1) $f_1(x, y) > a > 0$;
(2) $yf_2(x, y) < 0$ $(y \neq 0, xf_3(x) > 0,\ x \neq 0)$;
(3) $\frac{\partial f_1(x,y)}{\partial x} y \geqslant 0$.

Then the zero solution of (4.7.19) is unstable.

PROOF. Construct a Lyapunov function:

$$V(t, y, z) = \int_0^x f_3(x)\,dx + yz + \int_0^y yf_1(x, y)\,dy.$$

By condition (2) we know that $\int_0^x f_3(x)\,dx > 0$ $(x \neq 0)$ in the region defined by $z \geqslant x$ and $yz > 0$. So $V(x, y, z)$ is positive definite, and

$$\left.\frac{dV}{dt}\right|_{(4.7.19)} = z^2 - yf_2(x, y) + y \int_0^y \frac{\partial f_1}{\partial x} y\,dy \geqslant 0.$$

Letting $\frac{dv}{dt}|_{(4.7.19)} = 0$, we obtain $y = z = 0$ and $x = c$. So the zero solution of (4.7.19) is unstable. □

4.8. Summary of constructing Lyapunov functions

After the fundamental Lyapunov stability theory was established, further studies were carried out by many scholars. Most of the basic Lyapunov theorems are invertible. So theoretically, stability implies the existence of Lyapunov function. To seek suitable Lyapunov function to perform stability analysis is of interest, but is difficult because there is no general rule for constructing Lyapunov functions. In most of situations, it relies on one's experience and skill.

The success of constructing Lyapunov functions may depends on practical background. For example, for some physical models, the V function has a clear physical meaning. The kinetic and potential energies can be combined to construct a V function for a conservative mechanical system. Also, one can employ linear analogy method to have a similar V function for nonlinear differential equations.

There are two fundamental methods for constructing Lyapunov functions, while they cannot ensure success.

By the first essential method, one tries to construct a positive definite function and calculate the derivative $\frac{dV}{dt}$ along the solution of the system. If the conditions of the system ensure $\frac{dV}{dt}$ to be negative definite or negative semi-definite, one can get asymptotic stability or stability of the system. Otherwise, no result of stability can be obtained, and one has to try the other approaches. This method is used by almost all books on stability. The good candidates of Lyapunov functions may be (1) quadratic form; (2) sum of quadratic terms; (3) combination of absolute values; (4) quadratic form plus nonlinear integrals, etc.

The second method is first to assume that $\frac{dV}{dt}$ is negative definite or negative semi-definite, and then obtain V by integration. At the same time check the positive definiteness of V. If V is positive definite, one can determine asymptotic stability or stability of the system Otherwise, there is no conclusion. By this method, one can generalize the gradient-method, the variable gradient method, the integral method and the energy measure method to develop new methods.

The third method is called differential variant method, that is, to construct V and $\frac{dV}{dt}$ simultaneously.

Since energy function is usually employed in the analysis of neural network, which is relevant to the second method. We will introduce the gradient method in detail below.

Consider the n-dimensional nonlinear autonomous system:

$$\frac{dx}{dt} = f(x), \tag{4.8.1}$$

where $x \in R^n$, $f \in C[R^n, R^n]$ and $f(0) = 0$ and satisfies the Lipschitz condition. Consider a function $V = V(x)$. The derivative of V along the solution of system (4.8.1) is

$$\left.\frac{dV}{dt}\right|_{(4.8.1)} = (\mathrm{grad}\, V)^T \cdot f(x), \tag{4.8.2}$$

where $\mathrm{grad}\, V = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})^T$ is the gradient of V. First, let $\mathrm{grad}\, V = W$, where W is a given vector and satisfies

$$\frac{\partial W_i}{\partial x_j} = \frac{\partial W_j}{\partial x_i} \quad (i, j = 1, \dots, n). \tag{4.8.3}$$

Then, V is solved as the integral of grad V, i.e.,

$$V = \int_0^x (\mathrm{grad}\, V)^T dx = \int_0^x W^T\, dx, \tag{4.8.4}$$

where the upper limit of the integral is an arbitrary point in the phase space. From equation (4.8.3), we know that the integral (4.8.4) is independent of the integration path. Thus, we have

$$V = \int_0^{x_1} W_1(\tau_1, 0, \dots, 0)\, d\tau_1 + \int_0^{x_2} W_2(x_1, \tau_2, 0, \dots, 0)\, d\tau_2$$

$$+\cdots+\int_0^{x_n} W_n(x_1,\ldots,x_{n-1},\tau_n)\,d\tau_n. \tag{4.8.5}$$

The problem is now reduced into the problem of determining the gradient:

$$\operatorname{grad} V = W.$$

The key step is to choose $\frac{dV}{dt}$ such that the V solved from (4.8.5) is positive definite or positive semi-definite. If so, the equilibrium $x=0$ of system (4.8.1) is stable.

According to the choice of the gradient $\operatorname{grad} V = W$, the method can be generalized to the variable gradient method, the integrand method and the energy measure method. In the following, we discuss the variable gradient method.

By this method, one can seek the energy function for a neural by assuming that $\operatorname{grad} V = Bx$, where $B=(b_{ij})$ is to be determined. b_{ij} can be a constant or a function of x and $b_{ij}(x)=b_{ji}(x)$.

Choosing suitable b_{ij} such that W satisfies the general restrictive condition (4.8.3) and $x^T B^T f(x)$ is positive definite or positive semi-definite as well. For example, if $f(x)=A(x)x$, then (4.8.1) becomes

$$\dot{x} = x^T B^T A(x)x, \tag{4.8.6}$$

where $H(x)=B^T A(x)=(h_{ij})$.

For general quadratic form $H(x)$, the following condition is taken

$$\begin{cases} h_{ii}<0 \quad \text{or} \quad h_{ii}\leqslant 0, \\ h_{ij}+h_{ji}=0, \quad i,j=1,\ldots,n, \end{cases} \tag{4.8.7}$$

to ensure that $\dot{V}$ is negative definite or negative semi-definite. Thus, h_{ij} should be selected such that (4.8.7) and (4.8.3) are satisfied.

If B is given, by integral (4.8.5), we can calculate V and check whether it is positive definite or positive semi-definite. One could adjust b_{ij} untill it satisfies the conditions.

EXAMPLE 4.8.1. Consider the system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_2^2 - x_2. \end{cases} \tag{4.8.8}$$

Assume that

$$W = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By (4.8.8) we have $A=\left[\begin{smallmatrix} 0 & 1 \\ -x_1^2 & -1 \end{smallmatrix}\right]$. Then, it follows from $H=B^T A$ and (4.8.7) that

$$
\begin{aligned}
&h_{11} = -b_{21}x_1^2, \qquad h_{22} = b_{12} - b_{22} < 0, \\
&k_{12} + h_{21} = b_{11} - b_{12} - b_{22}x_1^2 = 0.
\end{aligned} \tag{4.8.9}
$$

Simplifying (4.8.9) yields

$$
b_{21} > 0, \qquad b_{22} > b_{12}, \qquad b_{11} = b_{21} + b_{22}x_1^2.
$$

Let b_{ij} $(i \neq j)$ be constants. Choosing $b_{12} = 1$, by using the general restrictive condition (4.8.3), we have $b_{12} = b_{21}$ and then $b_{21} = 1$. Let $b_{21} = 2$, we have $b_{11} = 1 + 2x_1^2$ and then obtain that

$$
W = \begin{bmatrix} 1 + 2x_1^2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1^3 + x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix},
$$

and

$$
\dot{V} = W^T f(x) = \begin{pmatrix} 2x_1^3 + x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}^T \begin{pmatrix} x_2 \\ x_1^3 - x_2 \end{pmatrix} = -x_1^4 - x_2^2
$$

is negative definite. Finally, performing the integration (4.8.5) we obtain

$$
\begin{aligned}
V &= \int_0^{x_1} \left(2\tau_1^2 + \tau_1 + 0\right) d\tau_1 + \int_0^{x_2} (x_1 + 2\tau_2)\, d\tau_2 \\
&= \frac{1}{2}x_1^4 + \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 \\
&= \left(\frac{1}{2}x_1 + x_2\right)^2 + \frac{1}{4}x_1^2 + \frac{1}{2}x_1^4.
\end{aligned}
$$

Obviously, V is a positive definite and radially unbounded. Furthermore, it can be verified that $\frac{dV}{dt}\big|_{(4.8.8)}$ is negative definite. Then, the zero solution of system (4.8.8) is globally asymptotically stable.

Variable gradient method is a kind of exploring and assembling method if V cannot be obtained by using one approach. This certainly does not mean that a suitable V does not exist.

The technique to select b_{ij} could yield different V functions and could give different attractive basins.

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Chapter 5

Development of Lyapunov Direct Method

Based on energy functions, Lyapunov direct method has its general significance in both theoretical development and applications. This method can be used to study not only the stability in the sense of Lyapunov, but also the asymptotic behaviors of dynamical systems without solving equations. In this chapter, we introduce typical extensions of Lyapunov direct method, such as LaSalle invariant principle, comparison principle, Lagrange stability, robust boundedness, practical stability, partial variable stability, asymptotic equivalence, conditional stability and set stability, etc.

The materials given in this chapter are mainly chosen from [222] for Section 5.1, [163] for Section 5.2, [151] for Section 5.3, [151,234] for Section 5.4, [265] for Section 5.5, [299,153] for Section 5.6, [225] for Section 5.7, [127] for Section 5.8, [268] for Section 5.9, [98] for Section 5.10, [333] for Section 5.11 and [234] for Section 5.12.

5.1. LaSalle's invariant principle

In 1960, the well-known American mathematician LaSalle discovered the relation between Lyapunov function and Birkhoff limit set. He extended Lyapunov direct method and gave a uniform concept of Lyapunov's theory. He considered the limit position of a motion as an asymptotic behavior. Moreover, he pointed out that based on a suitable Lyapunov function, especially by utilizing the invariance of the limit set, one could obtain the information of the limit set. This idea is now called "invariance principal". By this principal, LaSalle presented the essential theory for the stability of motion of dynamical systems.

In the following, for the dynamical systems composed of ordinary differential equations, we introduce the LaSalle invariant principle.

Consider the n-dimensional autonomous system, described by

$$\frac{dx}{dt} = f(x), \qquad f(0) = 0, \tag{5.1.1}$$

where $x = (x_1, \ldots, x_n)^T \in R^n$, $f = (f_1, f_2, \ldots, f_n)^T \in C[R^n, R^n]$. Assume that the solution of (5.1.1) is unique.

DEFINITION 5.1.1. The set $M \in R^n$ is called positive invariant set of solution of (5.1.1), if $\forall x_0 \in M$, the trajectory $x(t, t_0, x_0) \subset M (t \geqslant t_0)$; and $x(t, t_0, x_0) \to M$ as $t \to \infty$ if there exist point $p \in M$ and $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $\|x(t_n, t_0, x_0) - p\| \to 0$.

LEMMA 5.1.2. *If $x(t, t_0, x_0)$ is bounded for all $t \geqslant t_0$, then the ω-limit set $\Omega(x_0)$ has the following proprieties*:

(1) $\Omega(x_0)$ *is not empty*;
(2) $\Omega(x_0)$ *is compact* (*i.e., bounded and closed*);
(3) $\Omega(x_0)$ *is invariant on the set of the trajectories of system* (5.1.1);
(4) $x(t, t_0, x_0) \to \Omega(x_0)$ *when* $t \to +\infty$.

PROOF. (1) By the Weierstrass accumulation principle, we know that there exist $t_n \to +\infty$ $(n \to +\infty)$ such that

$$\lim_{t_n \to +\infty} x(t_n) := \lim_{t_n \to +\infty} x(t_n, t_0, x_0) = x^* \in \Omega(x_0),$$

so $\Omega(x_0)$ is not empty.

(2) $\forall \{p_n\} \in \Omega(x_0)$ satisfying $p_n \to p$ (as $n \to \infty$), we prove $p \in \Omega(x_0)$. $\forall \varepsilon > 0$, $\exists \eta_0$ such that $p_{n_0} \in \Omega(x_0)$ and $\|p_{n_0} - p\| < \frac{\varepsilon}{2}$. Hence, there exist $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, when $n \gg 1$ it holds

$$\left\|x(t_n) - p_{n_0}\right\| < \frac{\varepsilon}{2}.$$

Thus, $\|x(t_n) - p\| \leqslant \|x(t_n) - P_{n_0}\| + \|P_{n_0} - P\| < \varepsilon$ by arbitrary property of ε. This means that $p \in \Omega(x_0)$ and $\Omega(x_0)$ is a compact set.

(3) $\forall p \in \Omega(t_0)$, there exist $t_n \to \infty$ as $n \to +\infty$ such that

$$\lim_{n \to +\infty} x(t_n, t_0, x_0) = p.$$

If using $t = t_0 = 0$, $x(0) = x_0$ in $x(t, t_0, p)$, then by the uniqueness of solution, we have

$$x(t + t_n, 0, x_0) = x\bigl(t, 0, x(t_n, 0, x_0)\bigr),$$

and

$$\lim_{n \to +\infty} x(t + t_n, 0, x_0) = x(t, 0, p).$$

Therefore, the solution $x(t, 0, p) \subset \Omega(x_0)$ is positively invariant.

(4) Now, we prove $x(t, 0, x_0) \to \Omega(x_0)$. If otherwise, let $t \to +\infty$, and assume that $x(t)$ does not move towards $\Omega(x_0)$. Then, there exists $\varepsilon_0 > 0$, $\forall T > 0$,

$\exists t^* > T$, such that $\forall p \in \Omega(x_0)$, we have

$$\left\| x(t^*, 0, x_0) - p \right\| \geqslant \varepsilon.$$

So there exist $\{t_n^*\}$, $t_n \to +\infty$ $(n \to \infty)$, such that for all $p \in \Omega(x_0)$ it holds

$$\left\| x(t_n^*, 0, x_0) - p \right\| \geqslant \varepsilon.$$

But $x(t, 0, x_0)$ is bounded for $t \geqslant 0$. By Weierstrass accumulation principle, there exists a limiting point $x^* \in \Omega(x_0)$ such that $x(t_n^*) \to x^*$. This contradiction shows that $x(t, 0, x_0) \to \Omega(x_0)$ as $t \to +\infty$. □

THEOREM 5.1.3 *(LaSalle invariant principle [219]). Suppose D is a compact set. $\forall x_0 \in \mathcal{D}$, $x(t, t_0, x_0) \in \mathcal{D}$, i.e., D is a positive invariant set, and there exists $V(x) \in C^1[\mathcal{D}, R]$ such that*

$$\left.\frac{dV}{dt}\right|_{(5.1.1)} \leqslant 0.$$

Let $E = \{x \mid \frac{dV}{dt}|_{(5.1.1)} = 0, x \in \mathcal{D}\}$. $M \subset E$ is the biggest invariant set. Then, $x(t, t_0, x_0) \to M$ as $t \to +\infty$. Particularly, if $M = \{0\}$, then the zero solution of (5.1.1) is asymptotically stable.

PROOF. Let $x_0 \in \mathcal{D}$, so $x(t, t_0, x_0) \in \mathcal{D}$ by condition. Denote the ω-limit set of $x(t, t_0, x_0)$ by $\Omega(x_0)$. Since $\frac{dV}{dt}|_{(5.1.1)} \leqslant 0$, $V(x(t, t_0, x_0))$ is monotone decreasing and $V(x(t, t_0, x_0))$ is continuous on compact $\mathcal{D}$. So we have the lower bound:

$$\lim_{t \to +\infty} V\big(x(t, t_0, x_0)\big) := V_\infty.$$

Thus, on $\Omega(x_0)$ it holds

$$V(t) \equiv V_\infty = C, \qquad \left.\frac{dV(x(t))}{dt}\right|_{(5.5.1)} = 0,$$

where $x(t) \in \Omega$. However, $\Omega \subset E$, so $x(t, t_0, x_0) \to \Omega$ implies $x(t, 0, x_0) \to M$.

Theorem 5.1.3 is proved. □

REMARK 5.1.4. The basic idea of Theorem 5.1.3 is, based on the LaSalle invariant principle, to use the properties of the Lyapunov function $V(x)$ and $\frac{dV}{dt} \leqslant 0$ to determine the position of the largest invariant set,

$$\Omega \subset M \subset E \subset D.$$

In some applications, given a Lyapunov function $V(x)$, and at the some time, one can also find $\mathcal{D}$. Define $\mathcal{D}$ as

$$\mathcal{D} := \big\{x \mid V(x) \leqslant l\big\}.$$

If f is bounded, and in $\frac{dV}{dt}|_{(5.1.1)} \leqslant 0$, $\forall x_0 \in \mathcal{D}$, $x(t, t_0, x_0) \subset D$. Then, we have the following theorem.

THEOREM 5.1.5. *Let* $\mathcal{D} := \{x \mid V(x) \leqslant l\}$ *be bounded.* $V(x) \in C^1[\mathcal{D}, R^1]$ *and* $\frac{dV}{dt}|_{(5.1.1)} \leqslant 0$, *then* $\forall x_0 \in \mathcal{D}$, $x(t, t_0, x_0) \in \mathcal{D}$ *and* $x(t, t_0, x_0) \to M \subset E := \{x \mid \frac{dV}{dt}|_{(5.1.1)} = 0\}$.

The proof is similar to that for Theorem 5.1.3 and thus omitted.

EXAMPLE 5.1.6. Consider the stability of the zero solution of the equation:

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx + x^2 = 0 \quad (a > 0,\ b > 0). \tag{5.1.2}$$

We first rewrite (5.1.2) as

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -bx_1 - ax_2 - x_1^2. \end{cases} \tag{5.1.3}$$

Choose the function:

$$V(x_1, x_2) = \frac{1}{2}bx_1^2 + \frac{1}{2}x_2^2 + \frac{1}{3}x_1^3,$$

and then construct the following bounded and closed region $\mathcal{D}$, as shown in Figure 5.1.1.

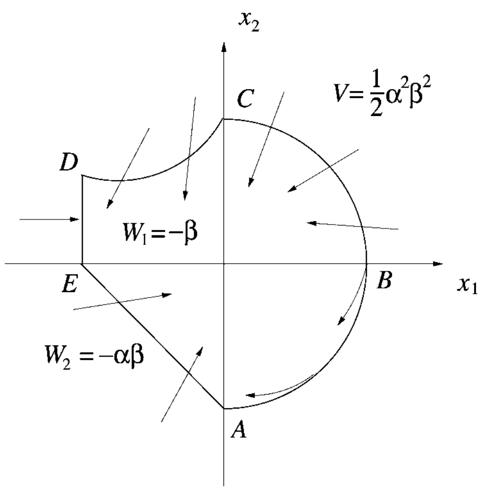

Figure 5.1.1. Construction of set D for Example 5.1.6.

Take

$$V \leqslant \frac{1}{2}\alpha^2\beta^2, \quad W_1 = x_1 \geqslant -\beta, \quad W_2 = x_2 + ax_1 \geqslant -\alpha\beta, \quad \beta > 0.$$

Then, we prove that $\forall x_0 \in \mathcal{D}$, when $t \geqslant t_0$, $x(t, t_0, x_0) \subset \mathcal{D}$. Otherwise, the trajectory $x(t, t_0, x_0)$ will leave $\mathcal{D}$. Then, it crosses over the curve $\widehat{ABCD}$ or the lines $\overline{AE}$, $\overline{DE}$ at certain $t = t_1$ to $R^n/\mathcal{D}$. But $\frac{dV}{dt}|_{(5.1.3)} = -ax_2^2 < 0$, $x_2 \neq 0$ for (x_1, x_2) on $\widehat{ABCD}$, so $x(t, t_0, x_0)$ cannot move from D to $R^n/\mathcal{D}$, crossing the curve $\widehat{ABCD}$. Further,

$$\frac{dW_1}{dt}\bigg|_{(5.1.3)} = \frac{dx_1}{dt}\bigg|_{x_1=-\beta} = x_2 > 0 \quad (x_2 \neq 0, x_1 \text{ on } \overline{DE}).$$

So $x(t, t_0, x_0)$ cannot move from $\mathcal{D}$ to $R^n/\mathcal{D}$ crossing line $\overline{DE}$.

On the line $\overline{AE}$, $x_1 \leqslant 0$ and $-x_1 = \beta + \frac{x_2}{a} < \beta$ ($x_2 \neq 0$) when $0 < \beta < b$. Thus, $b + x_1 > 0$ ($x_2 \neq 0$). Hence, $\frac{dW_2}{dt}|_{(5.1.3)} = -x_1(b + x_1) > 0$ ($x_2 \neq 0$) for (x_1, x_2) on line $\overline{AE}$. So $x(t, t_0, x_0)$ cannot cross the line $\overline{AE}$ moving from $\mathcal{D}$ to $R^n/\mathcal{D}$.

Therefore,

$$\frac{dV}{dt} = ax_2^2, \qquad E = \{x_2 = 0\}, \qquad M = \{0, 0\},$$

i.e., when $0 < \beta < b$, $x(t, t_0, x_0) \to 0$ as $t \to \infty$. As a result, the zero solution of (5.1.3) is asymptotically stable.

REMARK 5.1.7. Compare LaSalle invariant principle with the Barabashin–Krasovskii Theorem 4.6.10, the former, theoretically, is more general and it does not request $V(x) \geqslant 0$, but only requires $\frac{dV}{dt} \leqslant 0$. If $M = \{0\}$, then the absorbing area $\mathcal{D}$ is given by invariant principle, yet generally, the structure of the maximum invariant set may be very complicated, and the Barabashin–Krasovskii Theorem 4.6.10 is much convenient in verifying the conditions.

5.2. Comparability theory

The basic theorems of the Lyapunov direct method and some generalization presented in Chapter 4 can solve many problems in practice, but some problems are still very hard to solve. Among other methods, the comparability method (or called comparability theory) is the most useful theory and is widely used in applications.

First, we use the following simple example to explain the basic idea of this method.

EXAMPLE 5.2.1. Consider the following system:

$$\begin{cases} \frac{dx_1}{dt} = (-3 + 8\sin t)x_1 + \frac{4}{5}(\sin t)x_2, \\ \frac{dx_2}{dt} = \frac{6}{5}(\cos t)x_1 + (-3 + 8\sin t)x_2. \end{cases} \tag{5.2.1}$$

If we choose the Lyapunov function, given by

$$V = \frac{1}{2}\left(x_1^2 + x_2^2\right).$$

Then,

$$\begin{aligned} \frac{dV}{dt}\bigg|_{(5.2.1)} &= (-3 + 8\sin t)x_1^2 + \frac{4}{5}(\sin t)x_1x_2 \\ &\quad + \frac{6}{5}(\cos t)x_1x_2 + (-3 + 8\sin t)x_2^2. \end{aligned} \tag{5.2.2}$$

Obviously, the sign of $\frac{dV}{dt}|_{(5.2.1)}$ can be positive or negative. So we cannot use the theorems in Chapter 4 to determine the stability of the zero solution of (5.2.1).

Now we change (5.2.2) to the following inequality:

$$\begin{aligned} \frac{dV}{dt}\bigg|_{(5.2.1)} &\leqslant (-3 + 8\sin t)x_1^2 + x_1^2 + x_2^2 + (-3 + 8\sin t)x_2^2 \\ &= (-2 + 8\sin t)x_1^2 + (-2 + 8\sin t)x_2^2 \\ &\leqslant 2(-2 + 8\sin t)V. \end{aligned} \tag{5.2.3}$$

Then, we have

$$V\big(t, x(t)\big) \leqslant V(t_0, x_0)e^{2\int_{t_0}^{t}(-2+8\sin s)\,ds} \to 0 \quad \text{as } t \to +\infty. \tag{5.2.4}$$

From (5.2.4), we can conclude that the zero solution of (5.2.1) is asymptotically stable. However, note that

$$V(t_0, x_0)e^{2\int_{t_0}^{t}(-2+8\sin s)\,ds}$$

is actually just the solution of the following differential equation:

$$\begin{cases} \frac{dU}{dt} = 2(-2 + 8\sin t)U, \\ U(t_0) = V(t_0, x_0). \end{cases} \tag{5.2.5}$$

This example tells us that combining the Lyapunov function with comparison principle can lead to a more general conclusion on stability.

In the following, we present the general comparison principle. Consider the general n-dimensional nonautonomous system:

$$\frac{dx}{dt} = f(t, x), \tag{5.2.6}$$

where $x \in R^n$, $f = (f_1, f_2, \dots, f_n)^T \in [I \times R^n, R^n]$, which assures the uniqueness of solution of (5.2.6) and $f(t, 0) \equiv 0$. For comparison, consider a scale differential equation:

$$\frac{dU}{dt} = g(t, U), \tag{5.2.7}$$

where $g \in C[I \times R^+, R^1]$, $g(t, 0) \equiv 0$ if and only if $U = 0$.

THEOREM 5.2.2. *If there exists positive definite function $V(t, x) \in C[I \times R^n, R^+]$, which satisfies the Lipschitz condition for x, and $V(t, 0) \equiv 0$, further,*

$$D^+V(t, x)|_{(5.2.6)} \leqslant g(t, V),$$

then the following conclusions hold:

(1) *the stability of the zero solution of (5.2.7) implies the stability of the zero solution of (5.2.6)*;
(2) *if V has infinitesimal upper bound, then the uniform stability of the zero solution of (5.2.7) implies the uniform stability of the zero solution of (5.2.6)*;
(3) *the asymptotic stability of the zero solution of (5.2.7) implies the asymptotic stability of (5.2.6)*;
(4) *if V has infinitesimal upper bound, then the uniform asymptotic stability of the zero solution of (5.2.7) implies the uniformly asymptotic stability of the zero solution of (5.2.6)*;
(5) *if there exist constants $a > 0$, $b > 0$ such that*

$$a\|x\|^b \leqslant V(t, x), \tag{5.2.8}$$

and V has infinitesimal upper bound, then the exponential stability of the zero solution of (5.2.7) implies the exponential stability of the zero solution of (5.2.6);
(6) *if there exist φ, $\psi \in KR$ such that*

$$\varphi\big(\|x\|\big) \leqslant V(t, x) \leqslant \psi\big(\|x\|\big), \tag{5.2.9}$$

then the globally uniformly asymptotic stability of the zero solution of (5.2.7) implies the same type stability of the zero solution of (5.2.6).

PROOF. (1) Since $V(t, x)$ is positive definite, there exists $\phi(\|x\|) \in K$ such that $V(t, x) \geqslant \varphi(\|x\|)$. Further, since the zero solution of (5.2.7) is stable, $\forall \varepsilon > 0$, $\forall t_0 \in I$, $\exists \delta^*(t_0, \varepsilon) > 0$, when $0 < U_0 < \delta^*$, we have

$$U(t, t_0, U_0) < \varphi(\varepsilon).$$

By the continuity of $V(t, x)$ and $V(t, 0) \equiv 0$, for the above $\delta^* > 0$, $\exists \delta(t_0, \varepsilon) > 0$ such that when $\|x_0\| < \delta$, it holds

$$0 < V(t_0, x_0) < \delta^*.$$

Let $V(t) := V(t, x(t, t_0, x_0))$. Then, we have

$$\begin{cases} D^+V(t)|_{(5.2.6)} \leqslant g\big(t, V(t)\big), \\ V(t_0, x_0) := V_0. \end{cases} \tag{5.2.10}$$

Consider the comparison equation:

$$\begin{cases} \frac{dU}{dt} = g\big(t, U(t)\big), \\ U(t_0) = U_0 := V_0. \end{cases} \tag{5.2.11}$$

By Theorem 1.5.1 we obtain

$$\varphi\big(\|x(t)\|\big) \leqslant V\big(t, x(t)\big) \leqslant U(t, t_0, V_0) < \varphi(\varepsilon),$$

i.e., $\|x(t)\| < \varepsilon$. So the zero solution of (5.2.6) is stable.

(2) Owing to $V(t, x)$ with infinitesimal upper bound, there exist $\varphi, \psi \in K$ such that

$$\varphi\big(\|x\|\big) \leqslant V(t, x) \leqslant \psi\big(\|x\|\big).$$

$\forall \varepsilon > 0$, $\varphi(\varepsilon) > 0$, for this $\varphi(\varepsilon)$, $\exists \delta(\varepsilon)$ such that when $\|x_0\| < \delta$, we have

$$U_0 := V(t_0, x_0) \leqslant \psi\big(\|x_0\|\big) < \psi\big(\delta(\varepsilon)\big),$$
$$U(t, t_0, U_0) < \varphi(\varepsilon).$$

With a similar method used in the proof of (1) we can show that

$$\varphi\big(\big\|x(t)\big\|\big) \leqslant V\big(t, x(t)\big) \leqslant U(t, t_0, U_0) < \varphi(\varepsilon), \tag{5.2.12}$$

which implies $\|x(t)\| < \varepsilon$ when $\|x_0\| < \varepsilon$. Thus, the zero solution of (5.2.6) is uniformly stable.

(3) Choose $\sigma(t_0) > 0$. When $\|U_0\| < \sigma(t_0)$, from (5.2.10), (5.2.11) and Theorem 1.5.1, we have

$$\varphi\big(\big\|x(t)\big\|\big) \leqslant V\big(t, x(t)\big) \leqslant U(t, t_0, U_0) \to 0 \quad \text{as } t \to +\infty.$$

So the zero solution of (5.2.6) is asymptotically stable.

(4) Obviously, the conditions assure that the zero solution of (5.2.6) is uniformly stable. Since the zero solution of (5.2.7) is uniformly asymptotically stable, $\forall \varepsilon > 0$, $\forall t_0 \in I$, $\exists \eta > 0$ and $T(\varepsilon) > 0$ such that when $0 < U_0 < \eta$, $t \geqslant t_0 + T(\varepsilon)$, it holds

$$0 < U(t, t_0, U_0) < \varphi(\varepsilon).$$

Further, choose δ_0, U_0, x_0 such that

$$U_0 = V(t_0, x_0) \leqslant \psi\big(\|x_0\|\big) < \psi(\delta_0) \leqslant \eta.$$

Thus, by (5.2.10), (5.2.12) and Theorem 1.5.1 we have

$$\varphi\big(\big\|x(t)\big\|\big) \leqslant V\big(t, x(t)\big) \leqslant U(t, t_0, U_0) < \varphi(\varepsilon).$$

Therefore, $\|x(t)\| < \varepsilon$ when $t \geqslant t_0 + T(\varepsilon)$, i.e., the zero solution of (5.2.6) is uniformly asymptotically stable.

(5) According to the exponential stability of the zero solution of (5.2.7), there exists constant $\alpha > 0$, $\forall\varepsilon > 0$, $\exists\eta(\varepsilon) > 0$, when $0 < U_0 < \eta(\varepsilon)$ we have

$$U(t, t_0, U_0) \leqslant \varepsilon e^{-\alpha(t-t_0)} \quad \forall t \geqslant t_0.$$

Take $U_0 = V(t_0, x_0) \leqslant \psi(\|x_0\|) < \psi(\delta(\varepsilon)) \leqslant \eta(\varepsilon)$. Then, by (5.2.10), (5.2.12) and Theorem 1.5.1 we obtain

$$a\big\|x(t, t_0, x_0)\big\|^b \leqslant V\big(t, x(t)\big) \leqslant U(t, t_0, U_0) \leqslant \varepsilon e^{-\alpha(t-t_0)},$$

i.e.,

$$\big\|x(t, t_0, x_0)\big\| \leqslant \left(\frac{\varepsilon}{a}\right)^{1/b} e^{-\frac{a}{b}(t-t_0)}.$$

So the zero solution of (5.2.6) is exponentially stable.

(6) In this case, the conditions and conclusion of (2) hold. Thus, the zero solution of (5.2.6) is uniformly stable. Then, we need to prove the uniform boundedness of all solutions of (5.2.6). $\forall r > 0$, for $\varphi(r) > 0$, when $0 \leqslant U_0 < \varphi(r)$, $\exists\tilde{\beta}(r)$ such that

$$U(t, t_0, U_0) < \tilde{\beta}(r).$$

So when $\|x_0\| < r$,

$$U_0 = V(t_0, x_0) \leqslant \psi\big(\|x_0\|\big) < \psi(r),$$

It follows from (5.2.10), (5.2.12) and Theorem 1.5.1 that

$$\varphi\big(\big\|x(t, t_0, x_0)\big\|\big) \leqslant V\big(t, x(t, t_0, x_0)\big) \leqslant U(t, t_0, U_0) < \tilde{\beta}(r),$$

which can be rewritten as

$$\big\|x(t, t_0, x_0)\big\| < \varphi^{-1}\big(\tilde{\beta}(r)\big) := \beta(r).$$

This mean that all solutions of (5.2.6) are uniformly bounded.

Next, we prove the globally uniform attraction of the zero solution of (5.2.6). $\forall\varepsilon, \alpha > 0, t_0 \in I$ then for $\varphi(\varepsilon) > 0$, $\psi(\alpha) > 0$ and fixed ε, α, there exists $T(\varepsilon, \alpha) > 0$ such that when $0 < U_0 < \psi(\alpha)$, $t \geqslant t_0 + T(\varepsilon, \alpha)$, it holds

$$U(t, t_0, U_0) < \varphi(\varepsilon).$$

Take $U_0 = V(t_0, x_0) \leqslant \psi(\|x_0\|) < \psi(\alpha)$. By (5.2.10), (5.2.12) and Theorem 1.5.1 we get

$$\varphi\big(\big\|x(t, t_0, x_0)\big\|\big) \leqslant V\big(t, x(t)\big) \leqslant U(t, t_0, U_0) < \varphi(\varepsilon).$$

It follows that

$$\|x(t, t_0, x_0)\| < \varepsilon$$

which indicates that the zero solution of (5.2.6) is globally, uniformly and asymptotically stable.

The proof of Theorem 5.2.2 is complete. □

THEOREM 5.2.3. *Let $f(t, x) \in C[I \times B_\sigma, R^n]$, $g(t, u) \in C[I \times B_\delta, R^1]$, and there exists positive definite $V(t, x) \in C[I \times B_\sigma, R^1]$, with infinitesimal upper bound, such that*

$$D_+V|_{(5.2.6)} \geqslant g(t, V), \quad (t, x) \in I \times B_\delta,$$

then the instability of the zero solution of (5.2.7) implies the instability of the zero solution of (5.2.6).

PROOF. Since $V(t, x)$ admits infinitesimal upper bound, there exists $\varphi \in K$ such that

$$\varphi(\|x\|) \geqslant V(t, x).$$

By the instability of the zero solution of (5.2.7), $\exists \varepsilon_0$, $\forall \delta > 0$, $\forall t_0 \in I$, $\exists t_1 > t_0$, $\exists U_0 > 0$, $U_0 < \delta$ such that

$$U(t_1, t_0, U_0) \geqslant \varphi(\varepsilon_0).$$

Take $U_0 = V(t_0, x_0)$. Applying (5.2.10), (5.2.12) and Theorem 1.5.1 yields

$$\varphi(\|x(t_1)\|) \geqslant V(t_1, x(t_1)) \geqslant U(t_1, t_0, U_0) \geqslant \varphi(\varepsilon_0).$$

Hence, $\|x(t_1)\| \geqslant \varepsilon_0$, i.e., the zero solution of (5.2.6) is unstable. □

REMARK 5.2.4. The conclusion of the comparison Theorem 5.2.2 can include the sufficient condition of Lyapunov stability Theorem 4.2.1 as particular. The great advantage of the comparison method does not require the sign invariant of $\frac{dV}{dt}$. However, this does not imply the Lyapunov asymptotic stability Theorem 4.3.1 and Barabashin–Krasovskii Theorem 4.6.10 which require $\frac{dV}{dt} \leqslant 0$. By comparison theorems, one can only obtain the conclusion of stability. The failure of the comparison method is usually due to the request that the expression has to be given in a function form $g(t, V)$ for variables t, V. Sometimes this is very difficult.

Figure 5.2.1 shows the idea of the comparison method.

We summarize various cases as follows:

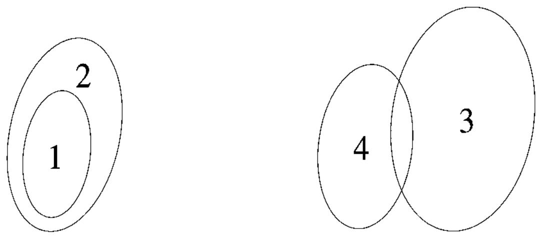

Figure 5.2.1. Illustration of the comparison method with respect to the Lyapunov direct method.

(1) when $\frac{dV}{dt} \leqslant 0$, the stability of the zero solution of the system can be determined;
(2) the comparison method can be used to determine stability of the system;
(3) the comparison method can be used to determine asymptotic stability of the system;
(4) when $\frac{dV}{dt} \leqslant 0$, the asymptotic stability of the system can be determined.

5.3. Lagrange stability

In this section, we study another type of stability—Lagrange stability. We can still use the Lyapunov direct method to analyze this stability.

Consider again the n-dimensional nonautonomous system:

$$\frac{dx}{dt} = f(t, x), \tag{5.3.1}$$

where $f(t, x) \in C[I \times R^n, R^n]$. Assume that the solution of (5.3.1) is unique.

DEFINITION 5.3.1. If every solution $x(t, t_0, x_0)$ of (5.3.1) is bounded, i.e., there exists a constant $\beta(t_0, x_0) > 0$ such that

$$\|x(t, t_0, x_0)\| \leqslant \beta(t_0, x_0) \quad \forall x_0 \in R^n,$$

then the solution $x(t, t_0, x_0)$ is said to be Lagrange stable, or bounded.

DEFINITION 5.3.2. If $\forall \alpha > 0$, $\forall t_0 \in I$, there exists $\beta(t_0, \alpha) > 0$ such that $\forall x_0 \in S_\alpha := \{x, \|x\| \leqslant \alpha\}$ it holds

$$\|x(t, t_0, x_0)\| \leqslant \beta(t_0, \alpha) \quad (t \geqslant t_0),$$

then the solution of (5.3.1) is said to be equi-Lagrange stable or equi-bounded. If the above $\beta(t_0, \alpha) = \beta(\alpha)$, independent of t_0, then the solution of (5.3.1) is said to be uniformly Lagrange stable or uniformly bounded.

THEOREM 5.3.3. *The solution of (5.3.1) is Lagrange stable if and only if there exists* $V(t,x) \in C[I \times R^n, R^1]$ *such that*

(1) $V(t,x) \geqslant \varphi(\|x\|)$ *for* $\varphi(\|x\|) \in KR$;
(2) *for every solution* $x(t,t_0,x_0)$, $V(t,x(t,t_0,x_0))$ *is not an increasing function of* t.

PROOF. *Sufficiency*. When the conditions are satisfied, we have

$$\varphi\big(\|x(t,t_0,x_0)\|\big) \leqslant V\big(t,x(t,t_0,x_0)\big) \leqslant V(t_0,x_0),$$

so

$$\|x(t,t_0,x_0)\| \leqslant \varphi^{-1}\big(V(t_0,x_0)\big) := \beta(t_0,x_0) \quad (t \geqslant t_0),$$

i.e., the solution $x(t,t_0,x_0)$ is Lagrange stable.

Necessity. Let an arbitrary solution $x(t,t_0,x_0)$ be bounded on $[t_0,+\infty)$. Let

$$V(t,x) = \sup_{\tau\geqslant 0}\|x(t+\tau,t,x_0)\|^2 \geqslant \|x(t,t,x)\|^2 = \|x\|^2 := \varphi\big(\|x\|\big).$$

Obviously, $\varphi(\|x\|) \in KR$. So condition (1) is satisfied.

Next, $\forall t_1,t_2 \in I$, let $t_0 < t_1 < t_2$. By the uniqueness of the solution, $x(t,t_2,x(t_2,t_0,x_0))$ is a continuation of $x(t,t_1,x(t_1,t_0,x_0))$, so we have

$$\begin{aligned}
&V(t_1,x(t_1,t_0,x_0))\\
&\quad= \sup_{\tau\geqslant 0}\big\|x\big(t_1+\tau,t_1,x(t_1,t_0,x_0)\big)\big\|^2\\
&\quad= \max\Big[\sup_{0\leqslant\tau\leqslant t_2-t_1}\big\|x\big(t_1+\tau,t_1,x(t_1,t_0,x_0)\big)\big\|^2,\\
&\qquad\qquad \sup_{\tau\geqslant 0}\big\|x\big(t_2+\tau,t_2,x(t_2,t_0,x_0)\big)\big\|^2\Big]\\
&\quad\geqslant \sup_{\tau\geqslant 0}\big\|x\big(t_2+\tau,t_2,x(t_2,t_0,x_0)\big)\big\|^2\\
&\quad= V\big(t_2,x(t_2,t_0,x_0)\big).
\end{aligned}$$

This means that condition (2) is satisfied. □

REMARK 5.3.4. Condition (2) in Theorem 5.3.3 is very hard to verify. If $V(t,x) \in C^1[I \times R^n, R]$, then condition (2), as a sufficient condition, can be replaced by $D^+V(t,x)|_{(5.3.1)} \leqslant 0$.

EXAMPLE 5.3.5. Discuss the Lagrange stability for the following system:

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)f(x) = 0, \tag{5.3.2}$$

where $p(t) \in C[I, R]$, $q(t) \in C^1[I, R]$, $f(x) \in C[R, R]$, satisfying

(1) $0 < q(t) \leqslant M = \text{constant}$;
(2) $p(t) \geqslant -\frac{q'(t)}{2q(t)}$;
(3) $\int_0^{\pm\infty} f(x)\,dx = +\infty$.

Then, an arbitrary solution $x(t)$ and its derivative $\dot{x}(t)$ are bounded.

PROOF. Rewrite (5.3.2) as

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -p(t)y - q(t)f(x). \end{cases} \tag{5.3.3}$$

Let

$$V(t, x, y) = \int_0^x f(\xi)\,d\xi + \frac{y^2}{2q(t)}.$$

According to condition (1), we have

$$V(t, x, y) \geqslant \int_0^x f(\xi)\,d\xi + \frac{y^2}{2M} = W(x, y) \to +\infty$$
$$\text{when } x^2 + y^2 \to +\infty.$$

So there exists $\varphi \in KR$ such that

$$V(t, x, y) \geqslant \varphi\big(x^2 + y^2\big)$$

and

$$\begin{aligned} D^+V\big(t, x(t), y(t)\big)\big|_{(5.3.3)} &= f\big(x(t)\big)y(t) + \frac{y(t)}{q(t)}\frac{dy}{dt} - \frac{y^2(t)\dot{q}(t)}{2q^2(t)} \\ &= f\big(x(t)\big)y(t) - \frac{y(t)}{q(t)}\big\{p(t)y(t) + q(t)f\big(x(t)\big)\big\} - \frac{y^2(t)\dot{q}(t)}{2q^2(t)} \\ &= -\frac{y^2(t)}{q(t)}\left[p(t) + \frac{\dot{q}(t)}{2q(t)}\right] \\ &\leqslant 0, \end{aligned}$$

indicating that $x(t)$ and $y(t) = \dot{x}(t)$ are bounded on I. □

THEOREM 5.3.6. *The solution of (5.3.1) is equi-Lagrange stable if and only if there exists $V(t, x) \in C[I \times R^n, R^1]$ such that*

(1) $V(t, x) \geqslant \varphi(\|x\|)$ *for* $\varphi \in KR$;

(2) *for every solution* $x(t, t_0, x_0)$, $V(t, x(t, t_0, x_0))$ *is not increasing*;
(3) $\forall \alpha > 0, \exists \beta(t, \alpha) > 0, \forall x \in S_\alpha := \{x \mid \|x\| \leqslant \alpha\}$, $V(t, x) \leqslant \beta(t, \alpha)$ *holds.*

PROOF. *Sufficiency*. Choose a Lyapunov function $V(t, x)$ which satisfies the conditions of Theorem 5.3.6. By condition (3), $\forall \alpha > 0$, $\forall x_0 \in S_\alpha := \{x_0 \mid \|x_0\| \leqslant \alpha\}$, $\exists \beta(t_0, \alpha) > 0$ such that

$$V(t_0, x_0) < \varphi\big(\beta(t_0, \alpha)\big).$$

Then, by the conditions (1) and (2), we have

$$\varphi\big(\|x(t)\|\big) \leqslant V\big(t, x(t)\big) \leqslant V\big(t_0, x(t_0)\big) \leqslant \varphi\big(\beta(t_0, \alpha)\big),$$

so

$$\|x(t)\| \leqslant \varphi^{-1}\big(\beta(t_0, \alpha)\big) = \beta(t_0, \alpha),$$

i.e., the solution of (5.3.1) is equi-Lagrange stable or equi-bounded.

Necessity. Suppose the solution of (5.3.1) is equi-Lagrange stable. Let

$$V(t, x) = \sup_{\tau \geqslant 0} \big\|x(t + \tau, t, x)\big\|^2.$$

Then, for any fixed t, on any compact set $\|x\| \leqslant \alpha$, $V(t, x)$ is bounded, i.e., condition (3) holds and

$$V(t, x) \geqslant \|x\|^2 := \varphi\big(\|x\|\big) \in KR.$$

So condition (1) is true.

Following the proof of Theorem 5.3.3, it is easy to prove that $V(t, x(t, t_0, x_0))$ is a monotonically nonincreasing function. Thus, condition (2) holds.

The proof of Theorem 5.3.6 is complete. □

COROLLARY 5.3.7. *If there exists* $V(t, x) \in C[I \times R^n, R]$ *such that*

(1) $V(t, x) \geqslant \varphi(\|x\|)$ *for* $\varphi \in KR$;
(2) $D^+ V(t, x)|_{(5.3.1)} \leqslant 0$;

then the solution of (5.3.2) *is equi-Lagrange stable.*

PROOF. Choose a Lyapunov function $V(t, x)$, which satisfies the conditions of Corollary 5.3.7. Since $V(t_0, x)$ is continuous, $\forall t_0 \in I$, on any compact set, $V(t_0, x)$ is bounded. Thus, $\forall \alpha > 0$, $\forall x_0 \in S_\alpha := \{x \mid \|x\| \leqslant \alpha\}$, $\exists \beta(t_0, \alpha) > 0$ such that

$$V(t_0, x) \leqslant \varphi\big(\beta(t_0, \alpha)\big).$$

By the conditions we have

$$\varphi\big(\|x(t)\|\big) \leqslant V\big(t, x(t, t_0, x_0)\big) \leqslant V(t_0, x_0) < \varphi\big(\beta(t_0, \alpha)\big),$$

so

$$\|x(t)\| \leqslant \beta(t_0, \alpha),$$

i.e., the solution of (5.3.1) is equi-Lagrange stable. □

EXAMPLE 5.3.8. Consider the system

$$\begin{cases} \frac{dx}{dt} = 0, \\ \frac{dy}{dt} = -z|x|, \\ \frac{dz}{dt} = yx^2, \end{cases} \tag{5.3.4}$$

and study the Lagrange stability of the solution of the system, with the initial condition $(0, x_0, y_0, z_0)$. Obviously, if $x_0 = 0$, then $x(t) = 0$, $y(t) = y_0$, $z(t) = z_0$; if $x_0 \neq 0$, then the solution is given by

$$\begin{cases} x = x_0, \\ y = y_0 \cos\sqrt{|x_0|^3}t - \frac{z_0}{\sqrt{|x_0|}} \sin\sqrt{|x_0|^3}t, \\ z = y_0\sqrt{|x_0|} \sin\sqrt{|x_0|^3}t + z_0 \cos\sqrt{|x_0|^3}t. \end{cases}$$

So every solution is bounded. But when $|x_0| \ll 1$, $|y| \gg 1$, it indicates that the solution of (5.3.4) is not equi-Lagrange stable.

REMARK 5.3.9. The condition $V(t, x) \in C[I \times R^n, R^1]$ in Theorem 5.3.3 can be changed to $V(t, x) \in C[I \times \Omega^H, R^1]$, where $\Omega^H = \{x \mid \|x\| \geqslant H\}$.

THEOREM 5.3.10. *If $f(t, x)$ satisfies the Lipschitz condition for x, i.e.,*

$$\big\|f(t, x) - f(x, y)\big\| \leqslant L\|x - y\|,$$

then $\forall x, y \in S_\alpha = \{x, \|x\| \leqslant \alpha\}$, the solution of (5.3.1) is uniformly Lagrange stable if and only if there exists $V(t, x) \in C[I \times R^n, R]$ such that

(1) $\varphi_1(\|x\|) \leqslant V(t, x) \leqslant \varphi_2(\|x\|)$ *for* $\varphi_1, \varphi_2 \in KR$;
(2) $D^+V(t, x)|_{(5.3.1)} \leqslant 0$.

PROOF. *Sufficiency.* $\forall \alpha > 0$, choose $\beta(\alpha)$ such that $\varphi_2(\alpha) < \varphi_1(\beta)$. Therefore, $\forall x_0 \in S_\alpha := \{x \mid \|x\| \leqslant \alpha\}$, by the conditions (1) and (2), we have

$$\varphi_1\big(\|x(t)\|\big) \leqslant V\big(t, x(t, t_0, x_0)\big) \leqslant V(t_0, x_0) \leqslant \varphi_2\big(\|x_0\|\big) \leqslant \varphi_2(\alpha) < \varphi_1(\beta).$$

So $\|x(t, t_0, x_0)\| < \beta$ where β is independent of t_0 and x_0. As a result, the solution of (5.3.1) is uniformly Lagrange stable.

Necessity. (1) Let

$$V(t,x) := \left(1+e^{-t}\right) \inf_{t_0 \leqslant \tau \leqslant t} \left\| x(\tau,t,x) \right\|^2.$$

Obviously, $V(t,x)$ is continuous and

$$V(t,x) \leqslant 2\left\| x(t,t,x) \right\|^2 = 2\|x\|^2 := \varphi_2\left(\|x\|\right).$$

Following the proof of Theorem 4.2.1, one can show that $V(t,x)$ is positive definite. So there exists $\varphi_1(\|x\|)$ such that

$$\varphi_1\left(\|x\|\right) \leqslant V(t,x).$$

Hence, condition (1) holds.

(2) Along the arbitrary solution $x(t,t_0,a)$, we have

$$\begin{aligned} V(t_2) := V\left(t_2, x(t_2,t_0,\alpha)\right) &:= \left(1+e^{-t_2}\right) \inf_{t_0 \leqslant \tau \leqslant t_2} \left\| x\left(\tau, t_2, x(t_2,t_0,\alpha)\right) \right\| \\ &= \left(1+e^{-t_2}\right) \inf_{t_0 \leqslant \tau \leqslant t_2} \left\| x(\tau,t_0,\alpha) \right\| \\ &\leqslant \left(1+e^{-t_1}\right) \inf_{t_0 \leqslant \tau \leqslant t_1} \left\| x(\tau,t_0,\alpha) \right\| \\ &= V(t_1). \end{aligned}$$

Thus, $V(t)$ is a continuously monotone decreasing function. Thus, we have

$$D^+V(t)|_{(5.3.1)} \leqslant 0,$$

and so condition (2) is satisfied.

The theorem is proved. □

EXAMPLE 5.3.11. Study the uniform Lagrange stability of the following system:

$$\frac{d^2x}{dt^2} + f(x,\dot{x})\frac{dx}{dt} + g(x) = p(t). \tag{5.3.5}$$

Assume that

(1) $f(x,y)$, $g(x)$ are continuous with respect to all variables;
(2) $p(t)$ is continuous on I and $\int_0^{+\infty} |p(t)|\,dt < +\infty$;
(3) $f(x,y) \geqslant 0$ for all x, y;
(4) $G(x) := \int_0^x g(u)\,du > 0$ $\forall x \neq 0$, and $G(x) \to +\infty$ as $x \to \infty$.

Then, the solution $x(t)$ and its derivative $\dot{x}(t)$ are uniformly bounded.

PROOF. Rewrite (5.3.5) as

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -f(x,y)y - g(x) + p(t). \end{cases} \tag{5.3.6}$$

Now take

$$V(t,x,y)=\sqrt{y^2+2G(x)}-\int_0^t \bigl|p(s)\bigr|\,ds.$$

When $x^2+y^2\geqslant k^2\gg 1$, it holds

$$\begin{aligned}\varphi_1\bigl(x^2+y^2\bigr):=\frac{1}{2}\sqrt{y^2+2G(x)}\leqslant V(t,x,y)\leqslant \sqrt{y^2+2G(x)}\\ :=\varphi_2\bigl(x^2+y^2\bigr)\end{aligned}$$

and

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(5.3.6)}&=\frac{1}{\sqrt{y^2+2G(t)}}\bigl\{g(x)y+y\bigl(-f(x,y)y-g(x)+p(t)\bigr)\bigr\}\\ &\quad-\bigl|p(t)\bigr|\leqslant 0.\end{aligned}$$

So all conditions of Theorem 5.3.10 are satisfied. This implies that the solution $x(t)$ and its derivative $\dot{x}(t)$ are uniformly bounded. □

In the following, we introduce a theorem which is convenience in for certain applications. Consider the system:

$$\begin{cases}\frac{dx}{dt}=F(t,x,y),\\ \frac{dy}{dt}=G(t,x,y),\end{cases}\tag{5.3.7}$$

where $x\in R^n$, $y\in R^m$, $F(t,x,y)\in C[I\times R^n\times R^m,R^n]$, $G(t,x,y)\in C[I\times R^n\times R^m,R^m]$.

THEOREM 5.3.12. *(See [418].) If there exists $V(t,x,y)\in C[I\times\Omega^H,R^1]$ such that*

(1) $V(t,x,y)\to+\infty$ *as* $\|y\|\to+\infty$ *holds uniformly for* (t,x);
(2) $V(t,x,y)\leqslant b(\|x\|,\|y\|)$, *where* $b(r,s)$ *is continuous positive function*;
(3) $\frac{dV}{dt}|_{(5.3.7)}\leqslant 0$ *where* $\Omega^H:=\{x,y\mid \|x\|^2+\|y\|^2\geqslant H>0\}$.

Further, for every $M>0$, there exists $W(t,x,y)\in C[I\times\Omega^M,R]$ such that

(4) $W(t,x,y)\Rightarrow+\infty$ *as* $\|x\|\to+\infty$ *holds for* (x,y);
(5) $W(t,x,y)\leqslant c(\|x\|)$, *where* $c(r)$ *is continuous positive function*;
(6) $\frac{dW}{dt}|_{(5.3.7)}\leqslant 0$, *where* $\Omega^M:=\{x,y,\ \|x\|\geqslant K(M),\ \|y\|\leqslant M\}$, $K\gg 1$.

Then the solution of (5.3.6) is uniformly Lagrange stable.

PROOF. Let $x(t):=x(t,t_0,x_0,y_0)$, $y(t):=y(t,t_0,x_0,y_0)$ be solutions of (5.3.6) with the initial values $\|x_0\|+\|y_0\|\leqslant\alpha$ $(\alpha>H)$. Choose $\beta(\alpha)\gg 1$

such that

$$l = \sup_{\substack{\|x\|+\|y\|=\alpha \\ t\in I}} V(t,x,y) \leqslant \sup_{\|x\|+\|y\|=\alpha} b\big(\|x\|,\|y\|\big) < \inf_{\substack{\|y\|=\beta \\ t\in I}} V(t,x,y).$$

This is possible under the conditions (1) and (2). By condition (3) we know that $\|x_0\|^2 + \|y_0\|^2 \leqslant \alpha^2$ which implies

$$\big\|y(t,t_0,x_0,y_0)\big\| < \beta(\alpha). \tag{5.3.8}$$

Otherwise, there exists $t_1 > t_0$ such that

$$\big\|y(t_1,t_0,x_0,y_0)\big\| = \beta(\alpha).$$

On one band, it holds

$$V\big(t_1,x(t_1),y(t_1)\big) \leqslant V(t_0,x_0,y_0) \leqslant l;$$

while on the other hand, we have

$$V\big(t_1,x(t_1),y(t_1)\big) \geqslant \inf_{\|y\|=\beta} V\big(t,x(t),y(t)\big) = L > l.$$

This means that (5.3.8) is true.

Now consider $W(t,x,y) \in C[I\times \Omega^{k_1(\beta)}, R^1]$, where $\Omega^{K_1(\beta)} = \{x, y \mid \|x\| \geqslant K_1(\beta), \|y\| \leqslant \beta\}$. Let $\alpha^* = \max\{\alpha, K_1(\beta)\}$ and $r \gg 1$ such that

$$\sup_{\substack{t\in I \\ \|x\|=\alpha^*,\ \|y\|\leqslant\beta}} \big\{W(t,x,y)\big\} < \inf_{\substack{t\in I \\ \|x\|=\gamma,\ \|y\|\leqslant\beta}} \big\{W(t,x,y)\big\}.$$

This is possible under condition (4). From condition (6) we obtain $\|x(t)\| < \gamma(\alpha)$. For all $t \geqslant t_0$,

$$\big\|x(t)\big\| < \gamma(\alpha), \qquad \big\|y(t)\big\| < \beta(\alpha).$$

Therefore, the solution of (5.3.8) is uniformly Lagrange stable.

Theorem 5.3.12 is proved. □

EXAMPLE 5.3.13. Consider the Lagrange stability of the solution of the equation:

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = p(t). \tag{5.3.9}$$

Assume that

(1) $f(x) \in C[R^1, R^1]$ and $F(x) := \int_0^x f(u)\,du \to +\infty$ as $x \to \pm\infty$;
(2) $g(x) \in C[R^1, R^1]$ and $xg(x) > 0$ for $|x| > q \gg 1$;
(3) $p(t) \in C[I, R^1]$ and $p(t) := \int_0^t p(\xi)\,d\xi$ is bounded.

Then, the solution and its derivative are uniformly bounded.

PROOF. Consider the equivalent system of (5.3.9), given by

$$\begin{cases} \frac{dx}{dt} = y - F(x) + p(t), \\ \frac{dy}{dt} = -g(x). \end{cases} \tag{5.3.10}$$

Choose constants $\alpha > 0$, $b > 0$. Let

$$G(x) = \int_0^x g(u)\,du, \qquad U(x, y) = G(x) + \frac{y^2}{2}.$$

Let

$$V(t, x, y) = \begin{cases} U(x, y) & (x \geqslant a, |y| < \infty), \\ U(x, y) - x + a & (|x| \leqslant a, y \geqslant b), \\ U(x, y) + 2a & (x \leqslant -a, y \geqslant b), \\ U(x, y) + \frac{2a}{b} y & (x \leqslant -a, |y| < b), \\ U(x, y) - 2a & (x \leqslant -a, y \leqslant -b), \\ U(x, y) + x - a & (|x| \leqslant a, y \leqslant -b). \end{cases}$$

It is easy to see that $V(t, x, y)$ satisfies the conditions (1), (2) and (3) of Theorem 5.3.12. For appropriate k_1, on $|x| > k_1(M)$ and $|y| \leqslant M$ define a function $W(t, x, y) := |x|$, which satisfies the conditions (3), (4) and (5). Since $y(t)$, $F(x(t))$ and $|p(t)|$ are bounded, the solution and its derivative are thus uniformly bounded, i.e., the solution of (5.3.9) is uniformly Lagrange stable. □

REMARK 5.3.14. This example provides a good idea, that is, applying different Lyapunov functions for different regions of state variables, and then summarizing the results solves the global boundedness of the system.

5.4. Lagrange asymptotic stability

Consider the general n-dimensional nonautonomous system:

$$\frac{dx}{dt} = f(t, x), \tag{5.4.1}$$

where $f(t, x) \in C[I \times R^n, R^n]$, $f(t, x)$ assumes the uniqueness of the solution of (5.4.1).

DEFINITION 5.4.1. If there exists a constant $B > 0$ such that every solution $x(t, t_0, x_0)$ of (5.4.1) satisfies $\overline{\lim}_{h \to +\infty} \|x(t, t_0, x_0)\| \leqslant B$, i.e., there exists a constant $T(t_0, x_0) > 0$, when $t \geqslant t_0 + T(t_0, x_0)$ it holds $\|x(t, t_0, x_0)\| < B$. Then, (5.4.1) is said to be Lagrange asymptotically stable; or the system is called a dissipative system with the limit bound B.

DEFINITION 5.4.2. If there exists a constant $B > 0$, $\forall \alpha > 0$, $\forall x_0 \in S_\alpha := \{x,\ \|x\| \leqslant \alpha\}$ such that $\overline{\lim}_{h\to+\infty} \|x(t, t_0, x_0)\| \leqslant B$ holds uniformly for any $x_0 \in S_\alpha$, i.e., $\exists T(t_0, \alpha) > 0$ such that

$$\|x(t, t_0, x_0)\| < B$$

when $t \geqslant t_0 + T(t_0, \alpha)$. Then, (5.4.1) is said to be an equi-dissipative system with limit bound B; or system (5.4.1) is said to be equi-Lagrange asymptotically stable.

DEFINITION 5.4.3. If in Definition 4.5.2 $T(t_0, \alpha) = T(\alpha)$ independent of t_0, then (5.4.1) is an uniformly dissipative system with limit bound B; or system (5.4.1) is said to be a uniformly Lagrange asymptotically stable. Here, $X(t) = (x(t), y(t), z(t))$.

REMARK 5.4.4. The dissipation propriety of a system is merely the attractive propriety of the set $S_B := \{x \mid \|x\| \leqslant B\}$.

THEOREM 5.4.5. *System (5.4.1) is uniformly Lagrange asymptotically stable for bound B if and only if there exists $V(t, x) \in C[I \times \Omega^H]$ such that on $[I \times \Omega^H]$ it holds*:

(1) $\varphi_1(\|x\|) \leqslant V(t, x) \leqslant \varphi_2(\|x\|)$ *for* $\varphi_1, \varphi_2 \in K$;
(2) $D^+V(t, x)|_{(5.4.1)} \leqslant -\psi(\|x\|)$ *for* $\psi \in K$, *where* $\Omega^H := \{x \mid \|x\| \geqslant H\}$, $0 < H < B$.

PROOF. *Sufficiency*. $\forall x_0 \in S_\alpha := \{x\colon \|x\| \leqslant \alpha\}$, $\alpha \geqslant H$, choose $\beta > B$ such that $\varphi_2(\alpha) < \varphi_1(\beta)$. Then, condition (2) of the theorem implies that

$$\begin{aligned}\varphi_1\big(\|x(t, t_0, x_0)\|\big) &\leqslant V\big(t, x(t, t_0, x_0)\big) \leqslant V(t_0, x_0)\\ &\leqslant \varphi_2\big(\|x(t, t_0, x_0)\|\big) \leqslant \varphi_2(\alpha) < \varphi_1(\beta).\end{aligned} \tag{5.4.2}$$

Hence,

$$\|x(t, t_0, x_0)\| < \beta \quad (t \geqslant t_0),$$

i.e., the solution of (5.4.1) is uniformly Lagrange stable

Choose $B > H$, $\forall \alpha > B$, $\forall x_0 \in S_\alpha = \{x \mid \|x\| \leqslant \alpha\}$ there must exist $t_1 > t_0$ such that

$$\|x(t_1, t_0, x_0)\| < B.$$

If otherwise for all $t \geqslant t_0$ it holds

$$B \leqslant \|x(t, t_0, x_0)\| \leqslant \beta.$$

Let $c^* = c(\beta) = \inf_{B \leqslant \|x\| \leqslant \beta} c(\|x\|) > 0$, then

$$V\big(t, x(t, t_0, x_0)\big) \leqslant V(t_0, x_0) - c^*(t - t_0) \to -\infty. \tag{5.4.3}$$

This is imposable, and so $\|x(t_1, t_0, x_0)\| < B$. Again choose B^* satisfying $B < B^* \leqslant \alpha$. Let $T(\alpha) = \frac{\varphi_2(\alpha) - \varphi_1(B^*)}{c^*} > 0$. Then, we have

$$\begin{aligned}
\varphi_1\big(\|x(t, t_0, x_0)\|\big) &\leqslant V\big(t, x(t, t_0, x_0)\big) \\
&\leqslant V(t_0, x_0) - c^* \frac{\varphi_2(\alpha) - \varphi_1(B^*)}{c^*} \\
&\leqslant \varphi_2\big(\|x_0\|\big) - \varphi_2(\alpha) + \varphi_1\big(B^*\big) \\
&< \varphi_2(\alpha) - \varphi_2(\alpha) + \varphi_1\big(B^*\big) \\
&= \varphi_1\big(B^*\big)
\end{aligned}$$

for $t \geqslant t_0 + T(\alpha)$. Hence, $\|x(t, t_0, x_0)\| \leqslant B^*$. This shows that system (5.4.1) is uniformly Lagrange asymptotically stable with the limit bound B.

Necessity. Suppose system (5.4.1) is uniformly Lagrange asymptotically stable with the limit bound B. Let $M_B = \{x\colon \|x\| \leqslant B\}$ and $d(p(t_0 + \tau, t_0, x_0), M_B)$ represent the distance from point $p(t_0 + \tau, t_0, x_0)$ to set M_B. Then, one can apply Lemma 4.3.2 to show that there exist monotone increasing function $\psi(\tau)$ with $\psi(0) = 0$ and positive continuous decreasing function $\sigma(t) \to 0$ (as $t \to +\infty$) such that

$$d\big(p(t_0 + \tau, t_0, x_0), M_B\big) \leqslant \psi\big(d(x_0, M_B)\big)\sigma(\tau).$$

Take $\varphi(\tau) = \psi(\alpha - \beta)\sigma(\tau)$, $\xi(\tau) \equiv 1$. Then, there exists continuous increasing function $G(r)$ defined on $0 \leqslant r \leqslant \varphi(0) = \psi(\alpha - \beta)\sigma(0)$ with $G(0) = 0$. Let $g(\tau) = G^2(\tau)$. By the property of ψ and σ we know that

$$\begin{aligned}
&g\big(\psi\big(d(x_0, M_B)\big)\sigma(\tau - t)\big) \\
&\quad = \big[g\big(\psi\big(d(x_0, M_B)\big)\sigma(\tau - t)\big)\big]^{\frac{1}{2}}\big[g\big(\psi\big(d(x_0, M_B)\big)\sigma(\tau - t)\big)\big]^{\frac{1}{2}} \\
&\quad \leqslant \big[g\big(\psi\big(d(x_0, M_B)\big)\sigma(0)\big)\big]^{\frac{1}{2}}\big[g\big(\psi(\alpha - \beta), \sigma(\tau - t)\big)\big]^{\frac{1}{2}},
\end{aligned} \tag{5.4.4}$$

when $\tau \geqslant t$, $d(x_0, M_B) < \alpha - \beta$. Now we define the function:

$$V(t, x) = \int_t^{\infty} g\big(d\big(p(t_0 + \tau, t_0, x_0), M_B\big)\big)\, d\tau \geqslant 0. \tag{5.4.5}$$

Then,

$$V(t, x) \leqslant \big[g\big(\psi\big(d(x, M_B)\big)\sigma(0)\big)\big]^{\frac{1}{2}} \int_t^{\infty} g\big(\psi(\alpha - \beta), \sigma(\tau - t)\big)^{\frac{1}{2}}\, d\tau$$

$$
\begin{aligned}
&= G\big(\psi\big(d(x, M_B)\sigma(0)\big)\big) \int_0^\infty G\big(\psi(\alpha-\beta), \sigma(\tau)\big)\, d\tau \\
&< G\big(\psi\big(d(x, M_B)\sigma(0)\big)\big) := \tilde{\varphi}_2\big(\|x\|\big), \\
\frac{dV}{dt}\bigg|_{(5.4.1)} &= -G^2\big(d(x, M_B)\big) := c\big(\|x\|\big).
\end{aligned}
$$

According to Theorem 5.3.10, there exists $W(t,x) \in C[I \times R^n, R^1]$ such that

(1) $\varphi_1(\|x\|) \leqslant W(t,x) \leqslant \varphi_2(\|x\|)$ for $\varphi_1, \varphi_2 \in KR$;
(2) $D^+W(t,x)|_{(5.4.1)} \leqslant 0$.

Let $U(t,x) := V(t,x) + W(t,x)$. Then, we have

$$
\begin{aligned}
\varphi_1\big(\|x\|\big) \leqslant W(t,x) \leqslant U(t,x) &\leqslant \varphi_2\big(\|x\|\big) + b\big(\|x\|\big) := \tilde{\varphi}_2\big(\|x\|\big), \\
D^+U(t,x)|_{(5.4.1)} &\leqslant D^+V(t,x)|_{(5.4.1)} + D^+W(t,x)|_{(5.4.1)} \\
&\leqslant -c\big(\|x\|\big),
\end{aligned} \tag{5.4.6}
$$

so all the conditions are satisfied. The necessity is proved, and the proof is complete. □

5.5. Lagrange exponential stability of the Lorenz system

As we know, so far not much work has been done on the Lagrange globally exponential stability and even very little has been discussed on this topic in the literature. In this section, we introduce our new results on the Lagrange globally exponential stability of the well-known Lorenz chaotic system. The method given in this section may help study other chaotic systems.

Since Lorenz discovered the Lorenz chaotic attractor in 1963, extensive studies have been given to the well-known Lorenz system [294]:

$$
\begin{aligned}
\frac{dx}{dt} &= a(y-x), \\
\frac{dy}{dt} &= cx - y - xz, \\
\frac{dz}{dt} &= xy - bz,
\end{aligned} \tag{5.5.1}
$$

where a, b and c are parameters. The typical parameter values for system (5.5.1) to exhibit a chaotic attractor are: $a = 10$, $b = 8/3$, $c = 28$. The Lorenz system has played a fundamental role in the area of nonlinear science and chaotic dynamics. Although everyone believes the existence of the Lorenz attractor, no rigorous mathematical proof has been given so far. This problem has been listed as one of the fundamental mathematical problems, proposed by Smale [364] for the 21st century. This problem is extensively discussed with the aid of numerical computation. It has been realized that it is extremely difficult to obtain the information of the chaotic attractor directly from the differential equation (5.5.1). Most of the results in the literature are computer simulations. Even based on computation of Lyapunov exponents of the system, one needs to assume the system being bounded in order to conclude that the system is chaotic. Therefore, the study of the globally attractive set of the Lorenz system is not only theoretically significant, but also practically important.

Chen and Lü [69] proposed the following Lorenz family:

$$\begin{aligned}
\frac{dx}{dt} &= (25\alpha + 10)(y - x) := a_\alpha (y - x), \\
\frac{dy}{dt} &= (28 - 35\alpha)x - xz + (29\alpha - 1)y := d_\alpha x - xz + c_\alpha y, \\
\frac{dz}{dt} &= xy - \frac{\alpha + 8}{3} y := xy - b_\alpha z,
\end{aligned} \tag{5.5.2}$$

where $\alpha \in [0, 1/29)$. In the following, we consider the globally exponentially Lagrange stability of systems (5.5.2) and (5.5.1).

Up to now, the concept of globally exponentially attractive set has not been formally proposed in the literature for studying the bounds of chaotic attractors. Thus, the convergent speed of trajectories from outside of the globally attractive set to the boundary of the set is unknown. In this section, we propose the concept of globally exponentially attractive set and apply it to obtain the exponential estimation of such set. Our results contain various existing results on the globally exponentially attractive set as special cases.

DEFINITION 5.5.1. If there exists a positive number $L_\lambda > 0$, $r_\lambda > 0$ and a generalized positive definite and radially unbounded Lyapunov function for system (5.5.2) such that for $\forall X_0 \in R^3$, it holds

$$V_\lambda\big(X(t)\big) - L_\lambda \leqslant \big(V_\lambda(X_0) - L_\lambda\big) e^{-r_\lambda (t - t_0)}$$

when $V_\lambda(X(t)) > L_\lambda$, $t \geqslant t_0$, then system (5.5.2) is said to be globally exponentially Lagrange stable.

THEOREM 5.5.2. *Define*

$$L_\lambda = \frac{b_\alpha^2 (\lambda a_\alpha + c_\alpha)^2}{8(b_\alpha - d_\alpha) d_\alpha}.$$

Then, we have estimation for the globally exponentially attractive set of system (5.5.2), given by

$$V_\lambda\big(X(t)\big) - L_\lambda \leqslant \big(V_\lambda(X_0) - L_\lambda\big) e^{-2d_\lambda (t - t_0)}.$$

In particular, the set

$$\begin{aligned}\Omega_\lambda &= \{X \mid V_\lambda(X) \leqslant L_\lambda\} \\ &= \left\{X \,\middle|\, \lambda x^2 + y^2 + (z - \lambda a_\lambda - c_\lambda)^2 \leqslant \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)d_\alpha}\right\}\end{aligned}$$

is the globally attractive set of (5.5.2).

PROOF. Let $f(z) = -(b_\alpha - d_\alpha)z^2 + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)z$. Then, setting $f'(z) = -2(b_\alpha - d_\alpha)z + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)$ zero yields

$$z_0 = \frac{(b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)}{2(b_\alpha - d_\alpha)}.$$

since $b_\alpha > 2 > d_\alpha$, $0 < d_\alpha \leqslant 1$, it follows that $z_0 > 0$ and $f''(z_0) = -2(b_\alpha - d_\alpha) < 0$. Thus,

$$\sup_{z\in R} f(z) = f(z_0) = \frac{[(b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)]^2}{4(b_\alpha - d_\alpha)}.$$

Then, using the facts that $a_\alpha > 1$ and $0 < d_\alpha \leqslant 1$, we obtain

$$\begin{aligned}\left.\frac{dV_\lambda}{dt}\right|_{(5.5.2)} &= \lambda x^2 = y^2 + (z - \lambda a_\alpha - c_\alpha)^2 \\ &= -\lambda x^2 - d_\alpha y^2 - d_\alpha z^2 + b_\alpha(\lambda a_\alpha + c_\alpha)z \\ &= -\lambda x^2 - d_\alpha y^2 - d_\alpha z^2 + 2d_\alpha(\lambda a_\alpha + c_\alpha)z \\ &\quad - (b_\alpha - d_\alpha)z^2 + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)z \\ &\leqslant -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + d_\alpha(\lambda a_\alpha + c_\alpha)^2 + f(z) \\ &\leqslant -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + d_\alpha(\lambda a_\alpha + c_\alpha)^2 + f(z_0) \\ &= -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)} \\ &\leqslant -\lambda x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + 2d_\alpha L_\alpha \\ &\leqslant -2d_\alpha V_\lambda + 2d_\alpha L_\lambda \\ &\leqslant 0 \quad \text{when } V_\lambda \geqslant L_\lambda.\end{aligned} \tag{5.5.3}$$

By comparison theorem and integrating both sides of (5.5.3) yields

$$\begin{aligned}V_\lambda\big(X(t)\big) &\leqslant V_\lambda(X_0)e^{-2d_\alpha(t-t_0)} + \int_{t_0}^{t} e^{-2d_\alpha(t-\tau)}2d_\alpha L_\lambda\, d\tau \\ &= V_\lambda(X_0)e^{-2d_\alpha(t-t_0)} + L_\lambda\big(1 - e^{-2d_\alpha(t-t_0)}\big).\end{aligned} \tag{5.5.4}$$

So, if $V_\lambda(X(t)) > L_\lambda,\ t \geqslant t_0$, we have the following estimation for the globally exponentially attractive set:

$$V_\lambda\big(X(t)\big) - L_\lambda \leqslant \big(V_\lambda(X_0) - L_\lambda\big)e^{-2d_\lambda(t-t_0)}.$$

By the definition, taking the upper limit on both sides of the above inequality results in

$$\overline{\lim_{h\to\infty}}\, V_\lambda\big(X(t)\big) \leqslant L_\lambda,$$

namely, the set

$$\begin{aligned}\Omega_\lambda &= \big\{X \mid V_\lambda(X) \leqslant L_\lambda\big\}\\ &= \left\{X \mid \lambda x^2 + y^2 + (z - \lambda a_\lambda - c_\lambda)^2 \leqslant \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)d_\alpha}\right\}\end{aligned}$$

is a globally exponentially attractive set, i.e., system (5.5.2) is Lagrange globally exponentially stable. □

THEOREM 5.5.3. *Let*

$$V_0 = \frac{1}{2}\big[y^2 + (z - c_\alpha)^2\big] \quad \text{and} \quad L_0 = \frac{b_\alpha^2 c_\alpha^2}{8(b_\alpha - d_\alpha)d_\alpha}.$$

Then, an estimation of the globally exponentially attractive set of the interval Lorenz system (5.5.2) is

$$\begin{cases} V_0(X(t)) - L_0 \leqslant (V_0(X_0) - L_0)e^{-2d_\alpha(t-t_0)} \\ \quad \leqslant (V_0(X_0) - L_0)e^{-\min(2d_\alpha, a_\alpha)(t-t_0)}, \\ x^2(t) - 2L_0 \leqslant (x_0^2 - 2L_0)e^{-a_\alpha(t-t_0)} \leqslant (x_0^2 - 2L_0)e^{-\min(2d_\alpha, a_\alpha)(t-t_0)}. \end{cases}$$

Especially, the set

$$\Omega_0 = \left\{X \left| \begin{matrix} V_0(X) \leqslant L_0 \\ x^2 \leqslant 2L_0 \end{matrix}\right.\right\} = \left\{X \left| \begin{matrix} y^2 + (z - c)^2 \leqslant \dfrac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} \\ x^2 \leqslant \dfrac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} \end{matrix}\right.\right\} \tag{5.5.5}$$

is the globally attractive and positive invariant set of (5.5.2), where $X = (y, z)$, and so system (5.5.2) is Lagrange globally exponentially stable.

PROOF. Setting $\lambda = 0$ in Theorem 5.5.2, we analogously obtain the estimation for the globally exponentially attractive set with respect to the variables y and z:

$$V_0\big(X(t)\big) - L_0 \leqslant \big(V_0(X_0) - L_0\big)e^{-2d_\alpha(t-t_0)}. \tag{5.5.6}$$

Then, taking upper limit on both sides of (5.5.6) leads to

$$\overline{\lim_{h\to\infty}} V_0\big(X(t)\big) \leqslant L_0,$$

i.e.,

$$\overline{\lim_{h\to\infty}} \big(y^2(t) + \big(z(t) - c_\alpha\big)^2\big) \leqslant \frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} = 2L_0.$$

So, the estimation of the ultimate bound for y is

$$y^2 \leqslant 2L_0.$$

Next, for the first equation of system (5.5.2), we construct the positive definite and radially unbounded Lyapunov function:

$$V = \frac{1}{2}x^2.$$

Then,

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(5.5.2)} &= -a_\alpha x^2 + a_\alpha xy \leqslant -a_\alpha x^2 + a_\alpha |x||y| \\ &= -a_\alpha x^2 + \frac{1}{2}a_\alpha x^2 + a_\alpha L_0 = -a_\alpha V + a_\alpha L_0.\end{aligned}$$

Hence,

$$V\big(X(t)\big) - L_0 \leqslant \big(V(x_0) - L_0\big)e^{-a_\alpha(t-t_0)}, \tag{5.5.7}$$

i.e.,

$$x^2(t) - \frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} \leqslant \left(x_0^2 - \frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha}\right)e^{-a_\alpha(t-t_0)}.$$

Therefore, the ultimate bound is given by the upper limit

$$\overline{\lim_{h\to\infty}} x^2(t) \leqslant \frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} = 2L_0.$$

This implies that Ω_0 is a globally attractive and positive invariant set of (5.5.2). □

Now, we return to system (5.5.1), but assume that the system parameters are defined as $a > 0$, $b > 1$, $c > 0$. For convenience, we call such system (5.5.1) the infinite interval Lorenz system family. Comparing with (5.5.2), though system (5.5.1) has one less parameter (d), its parameter values are unbounded and analysis is different from that of system (5.5.2). For certain values of the parameters, system (5.5.1) may be not chaotic. Here, we consider the globally exponentially attractive set of (5.5.1), regardless whether it is chaotic or not.

THEOREM 5.5.4. *Let*

$$\bar{V}_\lambda = \frac{1}{2}\big[\lambda x^2 + y^2 + (z - \lambda a - c)^2\big],$$

$$\bar{L}_\lambda^{(1)} = \frac{(\lambda a + c)^2 b^2}{8(b-1)}, \quad \bar{L}_\lambda^{(2)} = \frac{(\lambda a + c)^2}{2}, \quad \bar{L}_\lambda^{(3)} = \frac{(\lambda a + c)^2 b^2}{8a(b-a)}.$$

Then, the globally exponentially attractive and positive invariant sets of the infinite interval Lorenz system (5.5.1) are given by

$$\bar{V}_\lambda\big(X(t)\big) - L_\lambda^{(1)} \leqslant \big(\bar{V}_\lambda(X_0) - L_\lambda^{(1)}\big)e^{-2(t-t_0)} \quad \text{when } a \geqslant 1,\ b \geqslant 2,$$

$$\bar{V}_\lambda\big(X(t)\big) - L_\lambda^{(2)} \leqslant \big(\bar{V}_\lambda(X_0) - L_\lambda^{(2)}\big)e^{-2b(t-t_0)} \quad \text{when } a > \frac{b}{2},\ b < 2,$$

$$\bar{V}_\lambda\big(X(t)\big) - L_\lambda^{(3)} \leqslant \big(\bar{V}_\lambda(X_0) - L_\lambda^{(3)}\big)e^{-2a(t-t_0)} \quad \text{when } 0 < a < 1,\ b \geqslant 2a.$$

Especially, the sets

$$\begin{aligned} \Omega_\lambda^{(k)} &= \big\{X \mid \bar{V}_\lambda(X) \leqslant L_\lambda^{(k)}\big\} \\ &= \big\{X \mid \lambda x^2 + y^2 + (z - \lambda a - c)^2 \leqslant 2L_\lambda^{(k)}\big\}, \quad k = 1, 2, 3, \end{aligned}$$

are the estimations of the globally exponentially attractive and positive invariant sets of system (5.5.1).

PROOF. Take

$$\bar{V}_\lambda = \frac{1}{2}\big[\lambda x^2 + y^2 + (z - \lambda a - c)^2\big].$$

(1) When $a \geqslant 1$, $b \geqslant 2$, analogous to the proof of (5.5.3), we have

$$\left.\frac{d\bar{V}_\lambda}{dt}\right|_{(5.5.1)} \leqslant -\lambda x^2 - y^2 - (z - \lambda a - c)^2 + 2L_\lambda^{(1)} = -2\bar{V}_\lambda + 2L_\lambda^{(1)},$$

which yields

$$\bar{V}_\lambda\big(X(t)\big) - L_\lambda^{(1)} \leqslant \big(\bar{V}_\lambda(X_0) - L_\lambda^{(1)}\big)e^{-2(t-t_0)}. \tag{5.5.8}$$

(2) When $a > \frac{b}{2}$, $b < 2$, we obtain

$$\begin{aligned} \left.\frac{d\bar{V}_\lambda}{dt}\right|_{(5.5.1)} &\leqslant -\lambda a x^2 - y^2 - bz^2 + b(\lambda a + c)z \\ &\leqslant -\lambda\frac{b}{2}x^2 - \frac{b}{2}y^2 - \frac{b}{2}z^2 + \frac{b}{2}2(\lambda a + c)z \\ &\leqslant -\lambda\frac{b}{2}x^2 - \frac{b}{2}y^2 - \frac{b}{2}(z - \lambda a - c)^2 + \frac{b}{2}(\lambda a + c)^2 \\ &\leqslant -\frac{b}{2}\big(2\bar{V}_\lambda - 2L_\lambda^{(2)}\big) = -b\big(\bar{V}_\lambda - L_\lambda^{(2)}\big). \end{aligned}$$

Thus,

$$\bar{V}_\lambda\big(X(t)\big) - L_\lambda^{(1)} \leqslant \big(\bar{V}_\lambda(X_0) - L_\lambda^{(1)}\big)e^{-b(t-t_0)}. \tag{5.5.9}$$

(3) When $a < 1, b \geqslant 2a$, we have

$$\begin{aligned}
\frac{d\bar{V}_\lambda}{dt}\bigg|_{(5.5.1)} &\leqslant -\lambda a x^2 - y^2 - bz^2 + b(\lambda a + c)z \\
&\leqslant -\lambda a x^2 - a y^2 - a z^2 + 2a(\lambda a + c)z - a(\lambda a + c)^2 \\
&\quad + (a-b)z^2 + (b-2a)(\lambda a + c)z + a(\lambda a + c)^2 \\
&\leqslant -a\big[\lambda x^2 + y^2 + (z - \lambda a - c)^2\big] \\
&\quad + (a-b)z^2 + (b-2a)(\lambda a + c)z + a(\lambda a + c)^2 \\
&\leqslant -2a\big(\bar{V}_\lambda - L_\lambda^{(3)}\big).
\end{aligned}$$

Hence,

$$\bar{V}_\lambda\big(X(t)\big) - L_\lambda^{(3)} \leqslant \big(\bar{V}_\lambda(X_0) - L_\lambda^{(3)}\big)e^{-2a(t-t_0)}. \tag{5.5.10}$$

Further, taking upper limit on both sides of (5.5.8), (5.5.9) and (5.5.10) yields

$$\begin{aligned}
&\overline{\lim_{h\to\infty}}\big(\lambda x^2 + y^2(t) + \big(z(t) - \lambda a - c\big)^2\big) \\
&\leqslant \begin{cases} \frac{(\lambda a+c)^2 b^2}{4(b-1)} & \text{when } a \geqslant 1, b \geqslant 2, \\ (\lambda a + c)^2 & \text{when } a > \frac{b}{2},\ b < 2, \\ \frac{(\lambda a+c)^2 b^2}{4a(b-a)} & \text{when } a < 1,\ b > 2a. \end{cases}
\end{aligned} \tag{5.5.11}$$

□

THEOREM 5.5.5. *Let*

$$V_0 = \frac{1}{2}\big[y^2 + (z-c)^2\big] \quad \textit{and} \quad L_0 = \frac{b^2c^2}{8(b-1)}.$$

Then, the estimation of the globally exponentially attractive set of the infinite interval Lorenz system (5.5.1) is

$$\begin{cases} \bar{V}_0(X(t)) - L_0^{(1)} \leqslant (\bar{V}_0(X_0) - L_0^{(1)})e^{-2b(t-t_0)} \\ \quad \leqslant (\bar{V}_0(X_0) - \bar{L}_0)e^{-\min(2b,a)(t-t_0)}, \\ x^2(t) - L_0 \leqslant (x_0^2 - \bar{L}_0)e^{-a(t-t_0)} \leqslant (x_0^2 - \bar{L}_0)e^{-\min(2b,a_\alpha)(t-t_0)}. \end{cases}$$

Especially, the set

$$\bar{\Omega}_0 = \left\{ X \,\middle|\, \begin{array}{c} y^2 + (z-c)^2 \leqslant \dfrac{b^2c^2}{4(b-1)d_\alpha} \\ x^2 \leqslant \dfrac{b^2c^2}{4(b-1)d_\alpha} \end{array} \right\} \tag{5.5.12}$$

is the globally attractive and positive invariant set of (5.5.1).

PROOF. Similar to the proofs for (5.5.7) and (5.5.9), differentiating $\bar{V}_0$ with respect to time t and using the second and third equations of system (5.5.1) lead to the conclusion. The details are omitted here for simplicity. □

Equilibrium points, periodic and almost-periodic solutions are all positive invariant sets. Therefore, as a direct application of the results obtained in this section, we have the following theorem.

THEOREM 5.5.6. *Outside the globally attractive sets of the interval Lorenz systems* (5.5.1) *and* (5.5.2), *there are no bounded positive invariant sets that do not intersect the globally attractive sets.*

PROOF. By contradiction, suppose Ω is the globally attractive set of (5.5.1) and there is a bounded positive invariant set Q outside the set Ω, and $\Omega \cap Q = \Phi$ (empty set). Thus, we have

$$\inf_{\substack{X \in \Omega \\ \bar{X} \in Q}} \|X - \bar{X}\| > 0.$$

By the definition of positive invariant set, we have $X(t, t_0, X_0) \in Q$ for $X_0 \in Q$ and $t \geqslant t_0$. Hence,

$$\inf_{\substack{X \in \Omega \\ X(t,t_0,X_0) \in Q \\ t \geqslant t_0}} \left\| X - X(t, t_0, X_0) \right\| > 0.$$

On the other hand, since Ω is the globally attractive set, we have $X(t, t_0, X_0) \to \Omega$ as $t \to +\infty$. This implies that

$$\inf_{\substack{X \in \Omega \\ X(t,t_0,X_0) \in Q \\ t \geqslant t_0}} \left\| X - X(t, t_0, X_0) \right\| = 0,$$

leading to a contradiction. □

5.6. Robust stability under disturbance of system structure

It is well known that both the Lyapunov stability and the Lagrange stability are defined in the sense of disturbance on initial conditions. In this section, we study the robust stability of nonlinear systems, which admit simultaneously disturbances on the structure of the system and the initial conditions.

Consider the general nonlinear system without disturbance:

$$\frac{dx}{dt} = f(t, x), \tag{5.6.1}$$

where $f(t, x) \equiv x$ if and only if $x = 0$, $f \in C[I \times S_H, R^n]$, and

$$S_H := \{x \mid \|x\| \leqslant H\}.$$

At the same time, we consider the system under disturbance on the structure of the system:

$$\frac{dy}{dt} = f(t, y) + g(t, y), \quad g \in C\big[I \times S_H, R^n\big]. \tag{5.6.2}$$

DEFINITION 5.6.1. The zero solution of systems (5.6.1) is said to be robust stable in the sense of Lyapunov under disturbance of structure, if $\forall \varepsilon > 0$, $\exists \delta_1(\varepsilon) > 0$, $\delta_2(\varepsilon) > 0$, when $\|g(t, y)\| \leqslant \delta_1$, $\|y_0\| \leqslant \delta_2$, the solution $y(t, t_0, y_0)$ of (5.6.2) satisfies

$$\big\|y(t, t_0, y_0)\big\| < \varepsilon \quad (t \geqslant t_0). \tag{5.6.3}$$

LEMMA 5.6.2. *If there exists a function $V(t, x) \in C[I \times S_H, R]$ such that*

(1) $\|x\| \leqslant V(t, x) \leqslant K(\alpha)\|x\|$ $\forall x \in S_\alpha \subset S_H$, $K(\alpha) = constant$;
(2) $D^+V(t, x)|_{(5.6.1)} \leqslant -cV(t, x)$ $(c > 0)$;

then the zero solution of system (5.6.1) is exponentially stable.

The proof is similar to that of Theorem 4.4.1, and is thus omitted.

THEOREM 5.6.3. *If there exists a function $V(t, x) \in C[I \times S_H, R^1]$, satisfying*

(1) $\varphi_1(\|x\|) \leqslant V(t, x) \leqslant \varphi_2(\|x\|)$;
(2) $D^+V(t, x)|_{(5.6.1)} \leqslant -cV(t, x)$;
(3) $\|V(t, x) - V(t, y)\| \leqslant k\|x - y\|$.

Then the zero solution of system (5.6.1) is robust stable in the sense of Lyapunov under disturbance of structure.

PROOF. $\forall \varepsilon > 0$, take $\delta_1(\varepsilon) < \varepsilon$ such that $\varphi_2(\delta_1) < \varphi_1(\varepsilon)$, and choose $0 < \delta \ll 1$ such that

$$C\varphi_1(\delta_1) - k\delta > 0 \quad (\delta < \delta_1).$$

In the following, we prove that when $\|g(t, y)\| < \delta$, $\|y_0\| < \delta_1$ it holds

$$\left\| y(t, t_0, y_0) \right\| < \varepsilon \quad (t \geqslant t_0).$$

If otherwise, suppose that there exist $t_1, t_2, t_1 < t_2$, such that

$$\left\| y(t_1, t_0, y_0) \right\| = \delta_1,$$
$$\left\| y(t_2, t_0, y_0) \right\| = \varepsilon,$$

and for $t \in (t_1, t_2)$ it holds

$$\delta_1 < \left\| y(t, t_0, y_0) \right\| < \varepsilon.$$

However, on other hand, for $t \in [t_1, t_2]$ we have

$$\begin{aligned} D^+ V\bigl(t, y(t, t_0, y_0)\bigr)\big|_{(5.6.2)} &\leqslant -cV\bigl(t, y(t, t_0, y_0)\bigr) + k\bigl|g\bigl(t, y(t, t_0, y_0)\bigr)\bigr| \\ &\leqslant -c\varphi_1(\delta_1) + k\delta \\ &\leqslant 0. \end{aligned}$$

This implies that

$$\varphi_1(\varepsilon) \leqslant V\bigl(t_2, y(t_2, t_0, y_0)\bigr) \leqslant V\bigl(t_1, y(t_1, t_0, y_0)\bigr) \leqslant \varphi_2(\delta_1) < \varphi_1(\varepsilon),$$

leading to a contradiction. Hence, for all $t \geqslant t_0$, we have

$$\left\| y(t, t_0, y_0) \right\| < \varepsilon,$$

i.e., the zero solution of systems (5.6.1) is robust stable in the sense of Lyapunov under disturbance of structure. □

For linear systems, exponential stability implies robust stability in the sense of Definition 5.6.1, but this not generally true for nonlinear systems.

THEOREM 5.6.4. *Consider the linear system*:

$$\frac{dx(t)}{dt} = A(t)x, \tag{5.6.4}$$

where $A(t) = (a_{ij}(t))_{n\times n}$ *is a continuous matrix function. The robust stability of the zero solution of* (5.6.4) *implies its exponential stability.*

PROOF. Since the zero solution of (5.6.4) is robust stable in the sense of Lyapunov under disturbance of structure, in the region $\|x\| \leqslant 1$, for $\varepsilon = \frac{1}{2}$ there exists $\delta > 0$ such that for the disturbed system (5.6.4):

$$\frac{dy}{dt} = A(t)y + \delta y, \tag{5.6.5}$$

the solution of (5.6.5) $y(t, t_0, y_0)$ (when $\|y_0\| < \delta$) satisfies

$$\left\| y(t, t_0, y_0) \right\| < \frac{1}{2}.$$

But the solution $x(t, t_0, x_0)$ of (5.6.4) and the solution $y(t, t_0, y_0)$ of (5.6.5) have the following relation:

$$y(t, t_0, y_0) = x(t, t_0, x_0) e^{\delta(t-t_0)},$$

i.e., $\|x(t, t_0, x_0)\| \leqslant \frac{1}{2} e^{-\delta(t-t_0)}$. Hence, the zero solution of (5.6.4) is exponentially stable. □

DEFINITION 5.6.5. The solution of (5.6.1) is said to be robust stable in the sense of Lagrange under disturbance of structure, if $\forall \alpha > 0$, there exist constants $\beta(\alpha) > 0$ and $r(\alpha) > 0$ such that

$$\left\| g(t, y) \right\| < r(\alpha),$$

and further if $y_0 \in S_\alpha$, then for all $t \geqslant t_0$ it holds

$$\left\| y(t, t_0, y_0) \right\| < \beta(\alpha),$$

where $y(t, t_0, y_0)$ is the solution of (5.6.2).

THEOREM 5.6.6. *If there exists $V(t, x) \in C[I \times S^k, R]$ such that on $I \times S^k$ it holds*

(1) $\varphi_1(\|x\|) \leqslant V(t, x) \leqslant \varphi_2(\|x\|)$ *for* $\varphi_1, \varphi_2 \in KR$;
(2) *on any compact $k \leqslant \alpha \leqslant x \leqslant \beta$,*

$$\left| V(t, x) - V\big(t, x^*\big) \right| \leqslant K(\alpha, \beta) \left\| x - x^* \right\|, \quad K(\alpha, \beta) = \textit{constant};$$

(3) $D^+ V(t, x)|_{(5.6.1)} \leqslant -\lambda(\|x\|)$, *$C(r)$ is a continuous positive function.*

Then the solution of (5.6.1) *is robust stable in the sense of Lagrange under disturbance of structure.*

PROOF. $\forall \alpha > k > 0$, take $\beta(\alpha) > \alpha > 0$ such that $\varphi_1(\beta) > \varphi_2(\alpha)$. By the conditions (2) and (3), we have

$$\left| V(t, x) - V\big(t, x^*\big) \right| \leqslant K(\alpha, \beta) \left\| x - x^* \right\|,$$

$$D^+V(t,x)|_{(5.6.1)} \leqslant -\lambda(\alpha).$$

So on $0 \leqslant t < +\infty$, $\alpha \leqslant \|x\| \leqslant \beta$, we have

$$\begin{aligned} D^+V(t,y)|_{(5.6.2)} &\leqslant D^+V(t,y)|_{(5.6.1)} + K(\alpha,\beta)\big\|g(t,y)\big\| \\ &\leqslant -\lambda(\alpha) + K(\alpha,\beta)\big\|g(t,y)\big\|. \end{aligned}$$

Thus, we can choose $r(\alpha) > 0$ such that

$$\gamma(\alpha) \leqslant \frac{\lambda(\alpha)}{K(\alpha,\beta)}$$

on the region defined by $0 \leqslant t < +\infty$, $\alpha \leqslant \|x\| \leqslant \beta$. If

$$\big\|g(t,y)\big\| \leqslant \gamma(\alpha),$$

then $D^+V(t,y)|_{(5.6.2)} \leqslant 0$. Therefore, if $g_0 \in S_\alpha$, we have

$$\varphi_1\big(\big\|y(t)\big\|\big) \leqslant V\big(t, y(t,t_0,y_0)\big) \leqslant V(t_0,y_0) \leqslant \varphi_2\big(\|y_0\|\big) \leqslant \varphi_2(\alpha) \leqslant \varphi_2(\beta),$$

i.e., $\|y(t)\| \leqslant \beta$. So the conclusion is true. □

DEFINITION 5.6.7. For a given function $f(r) > 0$, system (5.6.1) is said to be robust dissipative under $f(r)$ degree disturbance of structure, if there exist two constants $\beta > 0$ and $\alpha > 0$ such that for $\|y\| \geqslant \beta$ it holds

$$\big\|g(t,y)\big\| < \alpha f\big(\|y\|\big)$$

and

$$\overline{\lim_{t\to\infty}}\big\|y(t,t_0,y_0)\big\| < \beta,$$

where $y(t,t_0,y_0)$ is a solution of (5.6.2).

THEOREM 5.6.8. *Suppose the conditions of Theorem 5.6.4 are satisfied. Let $L(r)$ be the Lipschitz constant, i.e.,*

$$\big|V(t,x) - V\big(t,x^*\big)\big| \leqslant L(r)\big\|x - x^*\big\| \quad \forall x, x^* \in S_r,$$

and $L(r)f(r) = 0(c(r))$ as $r \to \infty$. Then, the system (5.6.1) is robust dissipative under $f(r)$ degree structure disturbance.

PROOF. According to the conditions, we can choose $\alpha > 0$ such that

$$\alpha \leqslant \frac{c(r)}{2L(r)f(r)}$$

when $r \geqslant k$ (k is a constant). For this α, if $\|g(t,y)\| \leqslant \alpha f(\|y\|)$, then

$$D^+V(t,y)|_{(5.6.2)} \leqslant D^+V(t,y)|_{(5.6.1)} + L\big(\|y\|\big)\big\|g(t,y)\big\|$$

$$\leqslant -c(\|y\|) + \alpha L(\|y\|) f(\|y\|)$$
$$\leqslant -\frac{1}{2}c(\|y\|) \quad \text{when } \|y\| \geqslant k.$$

The above inequality indicates that system (5.6.1) is robust dissipative under $f(r)$ degree structure disturbance. □

REMARK 5.6.9. In the stability study under structural disturbance, the disturbing function $g(t, x)$ is unknown except for it being bounded by δ_1. If more information about $g(t, x)$ is known, we could obtain stronger stability results.

5.7. Practical stability

In the definition of Lyapunov stability, ε is arbitrary and $\delta(\varepsilon)$ is only requested to exist no matter how small it is. However, in practice if δ is too small, requiring the initial disturbance not to exceed δ is impossible. Similarly, the discrepancy in the actual running state and the ideal state is impossible or not necessary to be infinitely small. Therefore, real systems admit to run within a given discrepancy bound. For example, many air crafts and missiles work in such situation. Thus, practical stability was necessary and developed. In the following, we briefly introduce the practical stability according to Lefschetz [225].

Consider two n-dimensional nonautonomous systems, given by

$$\frac{dx}{dt} = f(t, x), \quad f(t, 0) \equiv 0, \quad f \in C[I \times S_H, R^n], \tag{5.7.1}$$
$$\frac{dx}{dt} = f(t, x) + p(t, x) \quad (p \in C[I \times S_H, R^n]). \tag{5.7.2}$$

DEFINITION 5.7.1. For pre-defined positive number δ, and two sets Q and Q_0, where $Q_0 \subset Q \subset S_H$, if $\forall x_0 \in Q_0$, $t_0 \geqslant 0$ and $\forall p(t, x) \in P := \{p(t, x) \mid \|p(t, x)\| \leqslant \delta\}$, the solution of (5.7.2) $x(t, t_0, x_0) \subset Q(t \geqslant t_0)$. Then the zero solution of (5.7.1) is said to be practically stable with respect to δ, Q and Q_0.

The practical stability is correlated with δ, and the set $\{Q, Q_0, Q\}$ is the permissive state set, Q_0 is the initial state set. See the illustration given in Figure 5.7.1. Before studying the practical stability, we must consider:

(1) the dimension of Q needed to ensure the system to perform normally;
(2) the admitted disturbance range (the value of δ);
(3) the deviation to be controlled by the initial condition (the dimension of Q_0).

Note that the practical stability and the Lyapunov stability do not include each other.

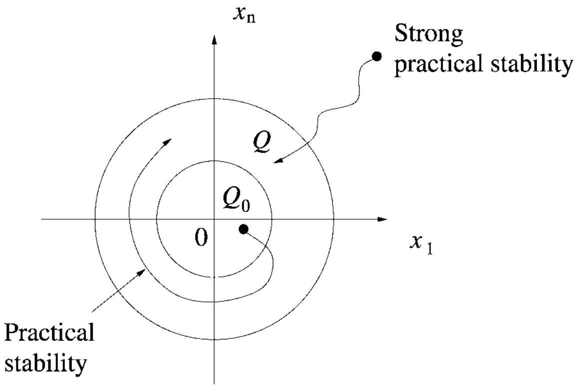

Figure 5.7.1. Practical stability.

THEOREM 5.7.2. *Let Q_0 be a compact set of $R^n, 0 \in Q_0$. If there exists $V(t,x) \in C[I \times R^n, R]$ such that for all $x \in Q_0^c$ it holds*:

$$D^+V(t,x)|_{(5.7.1)} \leqslant 0, \tag{5.7.3}$$

and $\forall x_1 \in Q_0, x_2 \in Q^c, \forall t_2 \geqslant t \geqslant 0$, *we have the inequality*:

$$V(t_1,x_1) < V(t_2,x_2), \tag{5.7.4}$$

where Q_0^c and Q^c are respectively the complementary sets of Q_0 and Q, then $\forall x_0 \in Q_0$, the solution $x(t,t_0,x_0) \subset Q$, $t \geqslant t_0$, i.e., the zero solution of (5.7.1) is practically stable with respect to δ, Q and Q_0.

PROOF. $\forall x_0 \in Q_0$, consider the solution $x(t,t_0,x_0)$. If at certain $T > t_0$, $x(T,t_0,x_0) \subset Q_0^C$, then there exist $t_1, t_0 < t_1 < T$ such that $x(t,t_0,x_0) \subset Q_0^c$, and when $t_1 < t \leqslant T$, $x(t_1,t_0,x_0) \in Q_0$. Hence, we have

$$V\big(t_1, x(t_1,t_0,x_0)\big) < V\big(T, x(T,t_0,x_0)\big). \tag{5.7.5}$$

However, on the other hand, for $t_1 < t \leqslant T$, by condition (5.7.3), we obtain

$$D^+V(t,x)|_{(5.7.1)} \leqslant 0.$$

It follows that

$$V\big(t, x(t,t_0,x_0)\big) \geqslant V\big(T, x(T,t_0,x_0)\big).$$

Hence,

$$V\big(t_1, x(t_1,t_0,x_0)\big) = \lim_{t\to t_1} V\big(t, x(t)\big) \geqslant V\big(T, x(T)\big). \tag{5.7.6}$$

The inequalities (5.7.5) and (5.7.6) show a contradiction. So $x(t,t_0,x_0) \subset Q$, i.e., the zero solution of (5.7.1) is practically stable with respect to δ, Q and Q_0. $\square$

DEFINITION 5.7.3. The zero solution of (5.7.1) is said to be practically stable in finite time interval $[t_0, t_0 + T]$ with respect to δ, Q and Q_0, if $\forall p \in P := \{p(t,x) \mid \|p(t,x)\| \leqslant \delta\}$, $\forall x_0 \in Q_0, x(t, t_0, x_0) \subset Q$ when $t \in [t_0, t_0 + T]$.

THEOREM 5.7.4. *If there exists $V(t,x) \in C[I \times R^n, R]$ such that $V \leqslant l_0$ when $x \in Q_0$, and $V \geqslant l$ when $x \in Q^c$. Further, $\frac{dV}{dt}|_{(5.7.2)} \leqslant (l - l_0)/T$ when $x \in Q_0^c$. Then, $\forall x_0 \in Q_0, x(t, t_0, x_0) \subset Q$ (when $t \in [t_0, t_0 + T]$), i.e., the zero solution of (5.7.1) is practically stable in finite time interval $[t_0, t_0 + T]$, with respect to δ, Q and Q_0.*

PROOF. Let $V(t, x(t)) := V(t, x(t, t_0, x_0))$, and $x_0 \in Q_0$ when $t > t_0$, and $x(t) \in Q_0^c$ when $t \in [t_0, t_0 + T]$. Then, we have

$$\begin{aligned} V\big(t, x(t)\big) - V\big(t_0, x(t_0)\big) &= V\big(\xi, x(\xi)\big)(t - t_0) \\ &\leqslant \frac{l - l_0}{T}(t - t_0) \\ &\leqslant \frac{l - l_0}{T} T = l - l_0. \end{aligned}$$

Hence, $V(t, x(t)) \leqslant l - l_0 + l_0 = l$. Thus, $x(t, t_0, x_0) \subset Q$, i.e., the conclusion of theorem is true. $\square$

DEFINITION 5.7.5. The zero solution of (5.7.1) is said to be practically strongly stable with respect to δ, Q and Q_0, if it is practically stable with respect to δ, Q and Q_0, and $\forall x_0 \in Q_0$, there exists $T > 0$ such that the solution $x(t, t_0, x_0)$ of (5.7.2) satisfies

$$x(t, t_0, x_0) \subset Q \quad \text{when } t \geqslant t_0 + T.$$

That is, the system (5.7.1) is robust dissipative.

THEOREM 5.7.6. *If there exists $V(t,x) \in C[I \times R^n, R^n]$ such that $V(t,x) \to +\infty$ when $\|x\| \to \infty$ and $\frac{dV}{dt}|_{(5.7.2)} \leqslant -\varepsilon < 0$ for all $x \in Q_0$ and for all $p \in P$, and $V(t,x) < V(t,y)$ for all $x \in Q_0, y \in Q^c$, then the zero solution of (5.7.1) is practically strongly stable with respect to δ, Q and Q_0.*

PROOF. First, we prove that the zero solution of (5.7.1) is practically stable with respect to δ, Q and Q_0. Suppose $\forall x_0 \in Q_0$, then at certain time $T > t_0$, $x(T, t_0, x_0) \subset Q^c$, there exist t_1, $t_0 < t_1 < T$ such that $x(t_1, t_0, x_0) \in Q_0$, but $x(T, t_0, x_0) \subset Q^c$, and for all $t \in [t_1, T]$, $x(t, t_0, x_0) \subset Q_0^c$. Hence,

$$V\big(t_1, x(t_1, t_0, x_0)\big) < V\big(T, x(T, t_0, x_0)\big),$$

which is impossible because $V(t, x(t, t_0, x_0))$ is monotone decreasing function of t. So the zero solution of (5.7.1) is practically stable with respect to δ, Q and Q_0.

Next, we prove the dissipation, i.e., to prove that $\forall x_0 \in Q^c$ there exists $T(t_0, x_0)$ such that

$$x(t, t_0, x_0) \subset Q \quad \forall\, t \geqslant t_0 + T(t_0, x_0).$$

If otherwise, suppose for all $t \geqslant t_0$ we have

$$x(t, t_0, x_0) \subset Q^c.$$

Since $\frac{dV}{dt} \leqslant -\varepsilon < 0$, it follows that

$$V\big(t, x(t, t_0, x_0)\big) \leqslant V(t_0, x_0) - \varepsilon(t - t_0) \to -\infty \quad \text{as } t \to +\infty.$$

Therefore, we can take $x_0 \in Q^c$. Let $y(t) := x(t, t_0, x_0)$. Then,

$$C := V(t_0, x_0) < V\big(t, y(t)\big) \to -\infty \quad (t \to \infty),$$

leading to a contradiction. So the system (5.7.2) is robust dissipative. □

5.8. Lipschitz stability

Lipschitz stability is a new stability developed in recent years. It is often used to analyze nonlinear problems. It is also useful in studying boundedness of solutions and existence of periodic solutions. Application of this stability has been extended from the dynamical systems defined by ordinary differential equation to those described by functional differential equations or other types of equations. In this chapter, we introduce the basic concepts of the Lipschitz stability theory.

Consider the n-dimensional system:

$$\frac{dx}{dt} = f(t, x), \tag{5.8.1}$$

where $f(t, x) \in C[I \times R^n, R^n]$, $f(t, 0) \equiv 0$. Assume that the solution of (5.8.1) is unique. Take $x(t, t_0, x_0)$ as a solution of (5.8.1), and then define the variational system of (5.8.1) as

$$\frac{dz}{dt} = f_x\big(t, x(t, t_0, x_0)\big)z, \tag{5.8.2}$$

where f_x denotes the Jacobi matrix $\big(\frac{\partial f_i}{\partial x_j}\big)_{n \times n}$.

DEFINITION 5.8.1.

(1) The zero solution of (5.8.1) is said to be uniformly Lipschitz stable, if there exist constants $M > 0$, $\delta > 0$ such that $\|x_0\| < \delta$ $(t \geqslant t_0 \geqslant 0)$ implies $\|x(t, t_0, x_0)\| \leqslant M\|x_0\|$, where M is the Lipschitz constant.

(2) The zero solution of (5.8.1) is said to be globally uniformly Lipschitz stable, if $\delta = +\infty$ in (1).

(3) The zero solution of (5.8.1) is said to be uniformly variational Lipschitz stable, if there exit constants $M > 0, \delta > 0$ such that when $\|x_0\| < \delta, t \geqslant t_0 \geqslant 0$, $\|K(t, t_0, x_0)\| \leqslant M$, where $K(t, t_0, x_0) = \frac{\partial}{\partial x_0}(x(t, t_0, x_0))$.
(4) The zero solution of (5.8.1) is said to be uniformly variational globally Lipschitz stable, if $\delta = +\infty$ in (3).

The above definitions show that the uniform Lipschitz stability of the zero solution of (5.8.1) implies the uniform Lyapunov stability of the zero solution of (5.8.1), but the reverse is generally not true.

THEOREM 5.8.2. *Let the zero solution of variational equation of* (5.8.1) *be*

$$\frac{dz}{dt} = f_x(t, 0)z, \tag{5.8.3}$$

then the uniform Lipschitz stability of the zero solution of (5.8.1) *implies the uniform Lipschitz stability of the zero solution of* (5.8.3).

PROOF. Let the zero solution of (5.8.1) be uniformly Lipschitz stable. Then, there exist $\alpha > 0, \delta > 0$ such that when $\|x_0\| \leqslant \delta$, $\|x(t, t_0, x_0)\| \leqslant \alpha\|x_0\|$ for $t \geqslant t_0$. Now take

$$x_{0_i} = e_i h, \quad h \leqslant \alpha, \quad e_i = (\overbrace{0, \ldots, 1}^{i}, \overbrace{0, \ldots, 0}^{n-i})^T.$$

Then, we have

$$\left\| \frac{\partial x(t, t_0, 0)}{\partial x_{0_i}} \right\| = \left\| \lim_{h \to 0} \frac{x(t, t_0, x_0) - x(t, t_0, 0)}{h} \right\| \leqslant \lim_{h \to 0} \frac{\alpha \|x_0\|}{h} = \alpha.$$

On the other hand,

$$\|K(t, t_0)\| = \|K(t, t_0, 0)\| = \left\| \frac{\partial x(t, t_0, 0)}{\partial x_0} \right\| \leqslant \alpha,$$

where $K(t, t_0)$ is the Cauchy matrix solution of (5.8.3). So the zero solution of (5.8.1) is uniformly Lipschitz stable. $\square$

EXAMPLE 5.8.3. Consider a nonlinear system:

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -x_1^3. \end{cases} \tag{5.8.4}$$

We choose the positive definite Lyapunov function:

$$V(x_1, x_2) = \frac{1}{2}x_1^4 + x_2^2.$$

Then

$$\left.\frac{dV}{dt}\right|_{(5.8.4)} = 2x_1^3 x_2 - 2x_1^3 x_2 = 0.$$

So the zero solution of (5.8.4) is Lyapunov uniformly stable. However, the zero solution of (5.8.4) corresponds to the variational system:

$$\begin{cases} \frac{dy_1}{dt} = y_2, \\ \frac{dy_2}{dt} = 0, \end{cases} \tag{5.8.5}$$

which has the general solution:

$$\begin{cases} y_1 = y_2^{(0)}(t - t_0), \\ y_2 = y_2^{(0)}. \end{cases}$$

Obviously, the zero solution of (5.8.4) is Lipschitz unstable.

This example shows that for certain nonlinear systems, the Lipschitz stability is stronger than the Lyapunov stability. But for the linear system:

$$\frac{dx}{dt} = A(t)x, \tag{5.8.6}$$

the general solution can be expressed as

$$x(t, t_0, x_0) = K(t, t_0)x_0,$$

where $K(t, t_0)$ is the Cauchy matrix solution. It is well known that the Lipschitz stability and the Lyapunov stability of the zero solution of (5.8.6) are equivalent with respect to the boundedness of $K(t, t_0)$. Hence, for linear systems, the Lyapunov stability and Lipschitz stability of the zero solution are equivalent.

In the following, we present several Lipschitz stability theorems.

THEOREM 5.8.4. *Suppose that $f(t, x)$ satisfies uniformly local Lipschitz condition with respect to t. Then the zero solution of (5.8.1) is uniformly Lipschitz stable if and only if there exists function $V(t, x) \in [I \times S_\delta, R]$ such that on $U \times S_\delta$ the following conditions hold*:

(1) $\|x\| \leqslant V(t, x) \leqslant L\|x\|,\ L = const > 0$;
(2) $|V(t, x) - V(t, y)| \leqslant \|x - y\|$;
(3) $D^+V(t, x|_{(5.8.1)}) \leqslant 0$.

PROOF. *Sufficiency*. By the conditions, we have

$$\|x(t, t_0, x_0)\| \leqslant V\big(t, x(t, t_0, x_0)\big) \leqslant V(t_0, x_0) \leqslant L\|x_0\| \quad \text{when } \|x_0\| \leqslant \delta.$$

So the sufficiency is true.

Necessity. Let $V(t,x) := \sup_{\tau \geqslant 0} \|x(t+\tau,t,x)\|(1+e^{-\tau-t})$. Then,

$$\begin{aligned}\|x\| := \big\|x(t,t,x)\big\| &\leqslant \big\|x(t,t,x)\big\|\big(1+e^{-t}\big) \leqslant V(t,x)\\ &\leqslant M \sup_{\tau\geqslant 0} \|x\|\big(1+e^{-\tau-t}\big) \leqslant 2M\|x\| := L\|x\|.\end{aligned}$$

It follows that condition (1) is true.

According to the Lipschitz propriety of $f(t,x)$, it is easy to prove that there exists $K = K(M,\delta)$ such that

$$\begin{aligned}&\big\|x(t+\tau,t,x) - x(t+\tau,t,y)\big\| \leqslant e^{k\tau}\|x-y\|\\ &\quad \text{for } \|x\| \leqslant \delta,\ \|y\| \leqslant \delta,\ \tau \geqslant 0.\end{aligned}$$

Due to the Lipschitz stability of the zero solution of (5.8.1), we have

$$\begin{aligned}&\big\|x(t+\tau,t,x)\big\| \leqslant M\|x\| \leqslant M\delta,\\ &\big\|x(t+\tau,t,y)\big\| \leqslant M\|y\| \leqslant M\delta.\end{aligned}$$

Thus,

$$\sup_{\tau\geqslant 0}\big\|x(t+\tau,t,x)\big\|\big(1+e^{-\tau-t}\big) \leqslant \sup_{\tau\geqslant 0}\big(e^T + e^{T-(t+\tau)}\big)\|x\|,$$

where $T = \ln M$. When $t < T$, choose τ such that $0 \leqslant t+\tau \leqslant T$. When $t \geqslant T$, take $\tau = 0$. Then, we have

$$\begin{aligned}\big|V(t,x) - V(t,y)\big| &\leqslant \sup_{\tau}\big\{\big\|x(t+\tau,t,x) - x(t+\tau,t,y)\big\|\big(1+e^{-t-\tau}\big)\big\}\\ &\leqslant \sup_{\tau} e^{k}\|x-y\|\big(1+e^{-t-\tau}\big)\\ &\leqslant L\|x-y\|,\end{aligned}$$

which leads to condition (2). Further,

$$\begin{aligned}D^+V(t,x)|_{(5.8.1)} &\leqslant \varlimsup_{h\to 0^+}\frac{1}{h}\big\{V\big(t+h,x(t+h,t,x)\big) - V(t,x)\big\}\\ &= \varlimsup_{h\to 0^+}\frac{1}{h}\Big\{\sup_{\tau\geqslant 0}\big\|x(t+h+\tau,t,x)\big\|\big(1+e^{-t-\tau-h}\big)\\ &\qquad - \sup_{\tau\geqslant 0}\big\|x(t+\tau,t,x)\big\|\big(1+e^{-t-\tau}\big)\Big\}\\ &= \varlimsup_{h\to 0^+}\frac{1}{h}\Big\{\sup_{\tau\geqslant h}\big\|x(t+\tau,t,x)\big\|\big(1+e^{-t-\tau}\big)\\ &\qquad - \sup_{t\geqslant 0}\big\|x(t+\tau,t,x)\big\|\big(1+e^{-t-\tau}\big)\Big\}\\ &\leqslant 0,\end{aligned}$$

i.e., condition (3) holds.

To prove the continuity of $V(t,x)$, we have

$$\begin{aligned}
&\big|V(t+h,x+y)-V(t,x)\big| \\
&\quad\leqslant \big|V(t+h,x+y)-V\big(t+h,x(t+h,t,x+y)\big)\big| \\
&\qquad+\big|V\big(t+h,x(t+h,t,x+y)\big)-V(t,x+y)\big| \\
&\qquad+\big|V(t,x+y)-V(t,x)\big|. \qquad (5.8.7)
\end{aligned}$$

By the Lipschitz propriety of $V(t,x)$ and the continuity of the solution $x(t,t_0,x_0)$, the first term and third term in (5.8.7) are infinitesimal, and the second term is also infinitesimal by the method of proving D^+V (see Theorem 5.8.2). So $V(t,x)$ is continuous.

The proof of Theorem 5.8.4 is complete. □

THEOREM 5.8.5. *If there exist function $V(t,x)\in C[G_H,R]$ and $\varphi_1,\varphi_2\in K$ such that*

$$\begin{aligned}
&\varphi_1\big(\|x\|\big)\leqslant V(t,x)\leqslant\varphi_2\big(\|x\|\big)\quad (t\geqslant t_0),\\
&\varlimsup_{s\to 0^+}\frac{\varphi_1^{-1}(\varphi_2(s))}{s}\leqslant M=const\quad (M\geqslant 1),\\
&\frac{dV}{dt}\bigg|_{(5.8.1)}\leqslant 0\quad (t\geqslant t_0),
\end{aligned}$$

then the zero solution of (5.8.1) is uniformly Lipschitz stable.

PROOF. Since by the conditions, one easily obtain:

$$\varphi_1\big(\big\|x(t,t_0,x_0)\big\|\big)\leqslant V\big(t,x(t,t_0,x_0)\big)\leqslant V(t_0,x_0)\leqslant\varphi_2\big(\|x_0\|\big),$$

so $\|x(t,t_0,x_0)\|\leqslant\varphi_1^{-1}\varphi_2(\|x_0\|)\leqslant M\|M_0\|$. □

THEOREM 5.8.6. *If there exist constants $L>0$, $\beta>1$ and nondecreasing positive function $\alpha(t)$ such that*

(1) $\alpha(t)\|x\|^2\leqslant x^TG(t)x\leqslant L\alpha(t)\|x\|^2$;
(2) $\beta\|G(t)F(t,x)\|\leqslant\|x\|$,

where

$$\begin{aligned}
&F(t,x)=f(t,x)-f_x(t,0)x,\\
&G(t)=\int_t^\infty\psi^T(s,t)\psi(s,t)\,ds,
\end{aligned}$$

in which $\psi(t,t_0)$ is the Cauchy matrix solution of (5.8.3), then the zero solution of (5.8.1) is uniformly Lipschitz stable.

PROOF. Choose $V(t,x) = x^T G(t)x$. Obviously, $G(t)$ is a symmetric positive matrix. Then,

$$\frac{dV}{dt}\bigg|_{(5.8.1)} = x^T \dot{G}(t)x + 2x^T G(t) f_x(t,0)x + 2x^T G(t)F(t,x),$$

where $f(t,x) = f_x(t,0)x + F(t,x) = \frac{\partial f(t,0)}{\partial x}x + F(t,x)$. Owing to

$$\frac{\partial \psi(s,t)}{\partial t} = -\psi(s,t) f_x(t,0),$$

$$\dot{G}(t) = -I_n + \int_t^\infty \left[\frac{\partial \psi^T(s,t)}{\partial t}\psi(s,t) + \psi^T(s,t)\frac{\partial \psi(s,t)}{\partial t}\right] ds,$$

one can further obtain

$$G'(t) = -I_n - f_x^T(t,0)G(t) - G(t) f_x(t,0)$$

and

$$\frac{dV}{dt}\bigg|_{(5.8.1)} = -x^T x + 2x^T G(t)F(t,x) \leqslant -x^T x + x^T x \leqslant 0.$$

Hence,

$$V\big(t, x(t,t_0,x_0)\big) \leqslant V(t_0,x_0).$$

Further, we have

$$\begin{aligned}
\alpha(t)\big\|x(t)\big\|^2 &= x^T(t)G(t)x(t) \\
&\leqslant x^T(t_0)G(t_0)x(t_0) \\
&\leqslant L\alpha(t_0)\big\|x(t_0)\big\|^2, \\
\big\|x(t)\big\|^2 &\leqslant \alpha^{-1}(t)\alpha(t_0)L\big\|x(t_0)\big\|^2 \\
&\leqslant \alpha^{-1}(t_0)\alpha(t_0)L\|x_0\|^2 \\
&= L^2\|x_0\|^2, \\
\big\|x(t)\big\| &\leqslant L\|x_0\|,
\end{aligned}$$

implying that the zero solution of (5.8.1) is uniformly Lipschitz stable. Theorem 5.8.6 is proved. □

5.9. Asymptotic equivalence of two dynamical systems

The process of using a mathematical model to describe a real physical system is usually an approximation due to errors in measurements of experiments, model simplifications, or other factors. Hence, it is important to study the structural stability of systems under perturbations or the robust stability of systems (see [316, 361]).

In any study of the structural stability of dynamical systems under perturbations, the asymptotic equivalence of two systems is one of the most important concepts. It can be used to study the robustness of the unperturbed system or to explore whether the behavior of a complicated system can be determined by that of a simpler system. While there have been many studies of asymptotic equivalence in the literature, there are few in large-scale systems. The basic idea in studies of stability of large-scale systems seems to be to decompose the system into isolated subsystems and their connecting systems, and then to determine the stability of the original system by the asymptotic behavior of the subsystems. In general, however, the original system and the subsystems may not be asymptotically equivalent, which may produce misleading results.

In this section, we study asymptotic equivalence of certain large-scale systems and their isolated subsystems. The technique employed in this section is utilizing different stability degrees of the isolated subsystems of a given large-scale system to control the perturbations, in order to guarantee the stability of the large-scale system. Sufficient conditions are obtained and an example is given to illustrate the results.

Consider the following two systems

$$\frac{dx}{dt}=f(t,x), \tag{5.9.1}$$

$$\frac{dy}{dt}=g(t,y), \tag{5.9.2}$$

where $f(t,x), g(t,y):[t_0,\infty)\times R^n \to R^n$ are continuous. It is assumed that solutions of (5.9.1) and (5.9.2) with initial values exist and are unique for $t \geqslant t_0$. We are concerned with the asymptotic behavior of solutions of (5.9.1) and (5.9.2).

DEFINITION 5.9.1. Let $x(t,t_0,x_0)$ and $y(t,t_0,y_0)$ be solutions of (5.9.1) and (5.9.2) with initial values x_0 and y_0, respectively. If there is a homeomorphism which maps each $x(t,t_0,x_0)$ to $y(t,t_0,y_0)$ and

$$\lim_{t\to\infty}\left\|x(t,t_0,x_0)-y(t,t_0,y_0)\right\|=0,$$

where $\|\cdot\|$ is a suitable norm, then the two systems (5.9.1) and (5.9.2) are said to be asymptotically equivalent.

To obtain our main results stated later in this section we need the following lemma, which is a special case of the comparison principle in [153].

LEMMA 5.9.2. *Let $H(h_{ij}(t)) \in R^{r\times r}$ be continuous in (t_0, ∞), with $h_{ij}(t) \geqslant 0$, $i \neq j$, $h_{ij}(t)$ bounded, and let $f(t) \in C[t_0, \infty)$. Assume that $x \in R^n$ is a solution of the system*

$$\begin{cases} \frac{dx}{dt} \leqslant H(t)x + f(t), \\ x(t_0) = x_0, \end{cases}$$

and that $y \in R^n$ is a solution of the system

$$\begin{cases} \frac{dy}{dt} = H(t)y + f(t), \\ y(t_0) = y_0. \end{cases}$$

Then whenever $x_0 = y_0$,

$$x(t, t_0, x_0) \leqslant y(t, t_0, y_0), \quad t \geqslant t_0,$$

i.e.,

$$x_i(t, t_0, x_0) \leqslant y_i(t, t_0, y_0), \quad t \geqslant t_0,\ i = 1, 2, \dots, n.$$

Now, consider a large-scale dynamical system governed by the following system of equations

$$\frac{dy}{dt} = \operatorname{diag}\big(A_{11}(t), \dots, A_{rr}(t)\big)y + B\big(B_{ij}(t)\big)y, \tag{5.9.3}$$

and its associated isolated subsystems

$$\frac{dx}{dt} = \operatorname{diag}\big(A_{11}(t), \dots, A_{rr}(t)\big)x, \tag{5.9.4}$$

where

$$y = (y_1, \dots, y_r)^T, \qquad y_i = \big(y_1^{(i)}, \dots, y_{n_i}^{(i)}\big)^T \in R^{n_i}, \quad i = 1, \dots, r,$$
$$x = (x_1, \dots, x_r)^T, \qquad x_i = \big(x_1^{(i)}, \dots, x_{n_i}^{(i)}\big)^T \in R^{n_i}, \quad i = 1, \dots, r,$$

$A_{ii}(t) \in R^{n_i\times n_i}$ and $B_{ii}(t) \in R^{n_i\times n_i}$ are continuous, and $\sum_{i=1}^r n_i = n$.

THEOREM 5.9.3. *Let $P(t, t_0) = \operatorname{diag}(P_{11}(t, t_0), \dots, P_{rr}(t, t_0))$ be the fundamental matrix of (5.9.4) with $P(t_0, t_0) = I$, where I is the $n \times n$ identity matrix. Assume the following conditions are satisfied*:

(1)

$$\big\|P_{ii}(t, t_0)\big\| \leqslant M_i e^{-\alpha_i(t-t_0)}, \quad i = 1, 2, \dots, r-1,$$
$$\big\|P_{rr}(t, t_0)\big\| \leqslant M_r,$$

where M_i and α_i are constants;

$$\big\|B_{ij}(t)\big\| \leqslant L_{ij}, \quad i, j = 1, 2, \dots, r-1;$$

(2)

$$\int_{t_0}^{\infty} \|B_{rj}(t)\|\, dt < \infty, \quad \int_{t_0}^{\infty} \|B_{ir}(t)\|\, dt < \infty, \quad i, j = 1, 2, \ldots, r-1,$$

where L_{ij} are constants;

(3) *The matrix*

$$G := -\operatorname{diag}(\alpha_1, \ldots, \alpha_{r-1}) + \operatorname{diag}(M_1, \ldots, M_{r-1})(L_{ij}) \in R^{(r-1)\times(r-1)}$$

is stable, i.e., all eigenvalues of G have negative real part. Then system (5.9.3) and its isolated subsystems (5.9.4) are asymptotically equivalent.

PROOF. (I) First, we show that all solutions of (5.9.3) are bounded.

Set $y_i(t, t_0, y_0) = y_i(t)$ and $y_i(t) = y_{i0}$. The solutions of (5.9.3) can be written as

$$y_i(t) = P_{ii}(t, t_0) y_{i0} + \int_{t_0}^{t} P_{ii}(t, \tau) \sum_{j=1}^{r} B_{ij}(\tau) y_j(\tau)\, d\tau, \quad i = 1, 2, \ldots, r. \tag{5.9.5}$$

Then,

$$\begin{aligned}
\|y_i(t)\| &\leqslant M_i \|y_{i0}\| e^{-\alpha_i(t-t_0)} + M_i \sum_{j=1}^{r-1} \int_{t_0}^{t} e^{-\alpha_i(t-\tau)} L_{ij} \|y_j(\tau)\|\, d\tau \\
&\quad + M_i \int_{t_0}^{t} e^{-\alpha_i(t-\tau)} \|B_{ir}(\tau)\| \|y_r(\tau)\|\, d\tau, \quad i = 1, 2, \ldots, r-1, \\
\|y_r(t)\| &\leqslant M_r \|y_{r0}\| + M_r \sum_{j=1}^{r} \int_{t_0}^{t} \|B_{rj}(\tau)\| \|y_j(\tau)\|\, d\tau.
\end{aligned} \tag{5.9.6}$$

Define

$$\xi_i(t) = M_i \sum_{j=1}^{r} \int_{t_0}^{t} e^{-\alpha_i(t-\tau)} L_{ij} \|y_j(\tau)\|\, d\tau + M_i \int_{t_0}^{t} e^{-\alpha_i(t-\tau)} \|B_{ir}(\tau)\| \|y_r(\tau)\|\, d\tau, \quad i = 1, 2, \ldots, r-1,$$

$$\xi_r(t) = M_r \sum_{j=1}^{r} \int_{t_0}^{t} \left\| B_{rj}(\tau) \right\| \left\| y_j(\tau) \right\| d\tau. \tag{5.9.7}$$

Then,

$$\begin{aligned} \left\| y_i(t) \right\| &\leqslant M_i \| y_{i0} \| e^{-\alpha_i (t-t_0)} + \xi_i(t), \quad i = 1, 2, \ldots, r-1, \\ \left\| y_r(t) \right\| &\leqslant M_r \| y_{r0} \| + \xi_r(t), \end{aligned} \tag{5.9.8}$$

and hence,

$$\begin{cases} \frac{d\xi_i}{dt} \leqslant -\alpha_i \xi_i + M_i \sum_{j=1}^{r-1} L_{ij} \xi_j + M_i \| B_{ir}(t) \| \xi_r + f_i(t), \\ \quad i = 1, 2, \ldots, r-1, \\ \frac{d\xi_r}{dt} \leqslant M_r \sum_{j=1}^{r} \| B_{rj}(t) \| \xi_j + f_r(t), \end{cases} \tag{5.9.9}$$

where

$$\begin{cases} f_i(t) = M_r \sum_{j=1}^{r-1} M_j L_{ij} \| y_{j0} \| e^{-\alpha_j (t-t_0)} \\ \qquad\qquad + M_i M_r \| B_{ir}(t) \| \| y_{r0} \|, \\ f_i(t) = M_r \sum_{j=1}^{r-1} M_j \| B_{rj}(t) \| \| y_{j0} \| e^{-\alpha_j (t-t_0)} \\ \qquad\qquad + M_r^2 \| B_{rr}(t) \| \| y_{r0} \|. \end{cases} \tag{5.9.10}$$

Now consider the comparison system of (5.9.9):

$$\begin{cases} \frac{d\eta_i}{dt} = -\alpha_i \eta_i + M_i \sum_{j=1}^{r-1} L_{ij} \eta_j + M_i \| B_{ir}(t) \| \eta_r + f_i(t), \\ \quad i = 1, 2, \ldots, r-1, \\ \frac{d\eta_r}{dt} = M_r \sum_{j=1}^{r} \| B_{rj}(t) \| \eta_j + f_r(t). \end{cases} \tag{5.9.11}$$

Let W be the matrix defined by

$$W = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \in R^{n \times n}.$$

Then from condition (3), W has only a simple zero eigenvalue and other eigenvalues of W all have negative real part. Thus, the fundamental matrix $K(t, t_0)$ of the system

$$\frac{d\eta}{dt} = W\eta \tag{5.9.12}$$

is bounded for $t \geqslant t_0$, where $\eta = (\eta_1, \ldots, \eta_r)^T$.

Let $\tilde{K}(t, t_0)$ be the fundamental matrix of the homogeneous system associated with (5.9.11). Since $\| B_{rj}(t) \| \in L^1(t_0, \infty)$ and $\| B_{ir}(t) \| \in L^1(t_0, \infty)$, $i, j = 1, 2, \ldots, r$, it follows from condition 2) that $\tilde{K}(t, t_0)$ is also bounded.

The solutions of (5.9.11) can then be written as

$$\eta(t) = \tilde{K}(t, t_0)\eta(t_0) + \int_{t_0}^{t} \tilde{K}(t, \tau) f(\tau)\, d\tau, \tag{5.9.13}$$

where $f(t) = (f_1(t), \ldots, f_r(t))^T$ is defined in (5.9.10).

Since $f(t) \in L^1(t_0, \infty)$, $\eta(t)$ is bounded. In addition to (5.9.5) and (5.9.7), it follows from Lemma 5.9.2 that $y(t)$ is bounded.

(II) Then we show that $\lim_{t\to\infty} y_i(t, t_0, y_0) = 0$, $i = 1, 2, \ldots, r-1$.

Set $\tilde{\eta} = (\eta_1, \ldots, \eta_{r-1})^T$ and

$$\tilde{f}(t) = \big(M_1 \|B_{1r}(t)\| \eta_r(t), \ldots, M_{r-1} \|B_{r-1r}(t)\| \eta_r(t)\big)^T + \big(f_1(t), \ldots, f_{r-1}(t)\big)^T.$$

Since $\eta_r(t)$ is bounded and $\|B_{ir}(t)\| \in L^1(t_0, \infty)$, $i = 1, 2, \ldots, r$,

$$\int_0^\infty \|\tilde{f}(t)\|\, dt \leqslant k,$$

for some $k > 0$.

It then follows from (5.9.11) that $\tilde{\eta}$ satisfies the following nonhomogeneous equations:

$$\frac{d\tilde{\eta}}{dt} = G\tilde{\eta} + \tilde{f}(t). \tag{5.9.14}$$

Then,

$$\tilde{\eta}(t, t_0, \tilde{\eta}_0) = e^{G(t-t_0)}\tilde{\eta}_0 + \int_{t_0}^{t} e^{G(t-\tau)} \tilde{f}(\tau)\, d\tau. \tag{5.9.15}$$

Since G is stable from condition (3), there exist constants $c > 0$ and $\beta > 0$ such that

$$\|e^{G(t-\tau)}\| \leqslant ce^{-\beta(t-\tau)}.$$

Then, it follows that

$$\begin{aligned}
\|\tilde{\eta}(t, t_0, \tilde{\eta}_0)\| &\leqslant ce^{-\beta(t-\tau)}\|\tilde{\eta}_0\| + \int_{t_0}^{t} ce^{-\beta(t-\tau)} \|\tilde{f}(\tau)\|\, d\tau \\
&\leqslant ce^{-\beta(t-\tau)}\|\tilde{\eta}_0\| \int_{t_0}^{t/2} ce^{-\beta(t-\tau)} \|\tilde{f}(\tau)\|\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t/2}^{t} ce^{-\beta(t-\tau)}\|\tilde{f}(\tau)\|\,d\tau \\
&\leqslant ce^{-\beta(t-\tau)}\|\tilde{\eta}_0\| + \frac{ck}{2}e^{-\frac{\beta t}{2}} + c\int_{t/2}^{t}\|\tilde{f}(\tau)\|\,d\tau \\
&\to 0 \quad \text{as } t\to\infty,
\end{aligned}
\tag{5.9.16}
$$

which leads to $\xi_i(t,t_0,\xi_{i0}) \to 0$ and hence, $y_i(t,t_0,y_{i0}) \to 0$ as $t\to\infty$, for $i=1,2,\ldots,r-1$.

(III) We now construct a homeomorphic mapping between solutions of (5.9.3) and (5.9.4) as follows.

$$
\begin{aligned}
I_1 &= \operatorname{diag}\big(\overbrace{1,\ldots,1}^{n_1},0,\ldots,0\big) \in R^{n\times n}, \\
I_2 &= \operatorname{diag}\big(0,\ldots,0,\overbrace{1,\ldots,1}^{n_2},0,\ldots,0\big) \in R^{n\times n}, \\
&\vdots \\
I_r &= \operatorname{diag}\big(0,\ldots,0,\overbrace{1,\ldots,1}^{n_r}\big) \in R^{n\times n}.
\end{aligned}
\tag{5.9.17}
$$

Then the fundamental matrix of (5.9.4) can be written as

$$
P(t,t_0) = \sum_{i=1}^{r} P(t,t_0)I_i, \tag{5.9.18}
$$

which gives that

$$
\begin{aligned}
y(t) &= P(t,t_0)y_0 + \sum_{i=1}^{r}\int_{t_0}^{t} P(t,\tau)I_iB(\tau)y(\tau)\,d\tau \\
&= P(t,t_0)y_0 + \sum_{i=1}^{r-1}\int_{t_0}^{t} P(t,\tau)I_iB(\tau)y(\tau)\,d\tau \\
&\quad + \int_{t_0}^{\infty} P(t,\tau)I_rB(\tau)y(\tau)\,d\tau - \int_{t}^{\infty} P(t,\tau)I_rB(\tau)y(\tau)\,d\tau \\
&= P(t,t_0)y_0\left(y_0 + \int_{t_0}^{\infty} P(t_0,\tau)I_rB(\tau)y(\tau)\,d\tau\right)
\end{aligned}
$$

$$+\sum_{i=1}^{r-1}\int_{t_0}^{t} P(t,\tau)I_i B(\tau)y(\tau)\,d\tau - \int_{t}^{\infty} P(t,\tau)I_r B(\tau)y(\tau)\,d\tau. \tag{5.9.19}$$

Let $Y(t, t_0)$ be the transition matrix of (5.9.3) and

$$\begin{aligned} x_0 &= y_0 + \int_{t_0}^{\infty} P(t_0,\tau)I_r B(\tau)y(\tau)\,d\tau \\ &= \left(I + \int_{t_0}^{\infty} P(t_0,\tau)I_r B(\tau)Y(\tau,t_0)\,d\tau\right)y_0. \end{aligned} \tag{5.9.20}$$

it is clear from (5.9.20) that x_0 is a single-valued continuous function of y_0.

Denote

$$Z_0 = \int_{t_0}^{\infty} P(t_0,\tau)I_r B(\tau)Y(\tau,t_0)\,d\tau.$$

It follows from the boundedness of $y(t)$ and the absolute integrability of $\|B_{rj}(t)\|$, $i = 1, \ldots, t$, that $Z_0 \to 0$ as $t \to \infty$. Hence, t_0 can be chosen sufficiently large so that the matrix $I + Z_0$ is nonsingular. Then,

$$y_0 = (I + Z_0)^{-1} x_0,$$

which implies that y_0 is also a single-valued continuous function of x_0. Therefore, the mapping between the initial value spaces $R^n_{y_0}$ and $R^n_{x_0}$ given by (5.9.20) is a homeomorphism. From the existence and uniqueness of solutions of (5.9.3) and (5.9.4), this homeomorphic mapping is also a homeomorphism between the solutions of (5.9.3) and (5.9.4).

(IV) Finally, we show that

$$\sum_{t\to\infty} \left\|x(t,t_0,x_0) - y(t,t_0,y_0)\right\| = 0, \quad \text{where } x_0 = (I + Z_0)y_0.$$

Denote $y(t) = y(t, t_0, y_0)$ and $x(t) = x(t, t_0, x_0)$. Then it follows from (5.9.19) and (5.9.20) that

$$\begin{aligned} \left\|y(t) - x(t)\right\| \leqslant{}& \sum_{i=1}^{r-1}\int_{t_0}^{t} \left\|\sum_{i=1}^{r-1} P(t,\tau)I_i B_{ij}(\tau)y_j(\tau)\right\| d\tau \\ &+ \sum_{i=1}^{r-1}\int_{t_0}^{t} \left\|P(t,\tau)I_i B_{ir}(\tau)y_r(\tau)\right\| d\tau \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{r} \int_{t_0}^{\infty} \left\| P(t,\tau) I_r B_{rj}(\tau) y_j(\tau) \right\| d\tau \\
&\leqslant \sum_{i=1}^{r-1} M_i \int_{t_0}^{t} e^{-\alpha_i (t-t_0)} \sum_{j=1}^{r-1} \left\| B_{ij}(\tau) \right\| \left\| y_j(\tau) \right\| d\tau \\
&\quad + \sum_{i=1}^{r-1} M_i \int_{t_0}^{t} e^{-\alpha_i (t-t_0)} \left\| B_{ir}(\tau) \right\| \left\| y_r(\tau) \right\| d\tau \\
&\quad + \sum_{j=1}^{r} \int_{t_0}^{\infty} \left\| P_{rr}(t,\tau) \right\| \left\| B_{rj}(\tau) \right\| \left\| y_j(\tau) \right\| d\tau \\
&:= J_1 + J_2 + J_3.
\end{aligned}
$$

Straightforward calculations yield

$$
\lim_{t\to\infty} J_1 \lim_{t\to\infty} \sum_{i=1}^{r-1} \frac{1}{\alpha_i} M_i \sum_{j=1}^{r-1} \left\| B_{ij}(t) \right\| \left\| y_j(t) \right\| = 0,
$$

$$
\begin{aligned}
\lim_{t\to\infty} J_2 &= \sum_{i=1}^{r-1} M_i \lim_{t\to\infty} \int_{t_0}^{t/2} e^{-\alpha_i (t-\tau)} \left\| B_{ir}(t) \right\| \left\| y_r(t) \right\| d\tau \\
&\quad + \sum_{i=1}^{r-1} M_i \lim_{t\to\infty} \int_{t/2}^{t} e^{-\alpha_i (t-\tau)} \left\| B_{ir}(t) \right\| \left\| y_r(t) \right\| d\tau \\
&\leqslant \sum_{i=1}^{r-1} \left(\lim_{t\to\infty} c_1 e^{-\frac{\alpha_i t}{2}} + \lim_{t\to\infty} c_1 \int_{t/2}^{t} \left\| B_{ir}(t) \right\| d\tau \right) = 0,
\end{aligned}
$$

where c_1 is positive and sufficiently large, and

$$
\lim_{t\to\infty} J_3 = \sum_{i=1}^{r} \lim_{t\to\infty} \int_{t}^{\infty} \left\| P_{rr}(t,\tau) \right\| \left\| B_{rj}(\tau) \right\| \left\| y_r(\tau) \right\| d\tau = 0.
$$

The proof is complete. □

Next, we consider a nonlinear large-scale system governed by the following equation

$$
\frac{dy}{dt} = \operatorname{diag}\bigl(A_{11}(t), \ldots, A_{rr}(t)\bigr) y + f(t,y), \tag{5.9.21}
$$

where $f : [t_0, \infty) \times R^n \to R^n$ is continuous and $f(t, 0) \equiv 0$. Then, we have

THEOREM 5.9.4. *Assume that the condition* (1) *and* (3) *in* Theorem 5.9.3 *hold and that*

$$\| f_i(t, x) - f_i(t, y) \| \leqslant \sum_{j=1}^{r} \| B_{ij}(t) \| \|x_j - y_j\| \leqslant \sum_{j=1}^{r} L_{ij} \|x_j - y_j\|,$$

where

$$\int_{t_0}^{\infty} \| B_{rj}(t) \| \, dt < \infty, \qquad \int_{t_0}^{\infty} \| B_{ir}(t) \| \, dt < \infty, \quad i, j = 1, 2, \ldots, r.$$

Then the nonlinear system (5.9.21) *and* (5.9.4) *are asymptotically equivalent.*

The proof is similar to that of Theorem 5.9.3 and is omitted.

To end this section, we give an example to illustrate the significance of Theorem 5.9.3.

EXAMPLE 5.9.5. Consider the following system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} -2 + \sin t & \cos t & \frac{1}{1+t^2} \\ \cos 2t & -4 + 2\sin t & te^{-t} \\ \frac{\cos t}{1+t^2} & e^{-t} \sin t & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{5.9.22}$$

By rewriting (5.9.22) as

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{bmatrix} \sin t & \cos t & \frac{1}{1+t^2} \\ \cos 2t & 2\sin t & te^{-t} \\ \frac{\cos t}{1+t^2} & e^{-t} \sin t & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= \operatorname{diag}(A_{11}, A_{22}, A_{33}) y + B(B_{ij}) y, \end{aligned}$$

the isolated subsystems are

$$\begin{cases} \frac{dx_1}{dt} = -2x_1, \\ \frac{dx_2}{dt} = -4x_2, \\ \frac{dx_3}{dt} = 0. \end{cases} \tag{5.9.23}$$

A fundamental matrix of (5.9.23) is given by

$$P(t, t_0) = \begin{bmatrix} e^{-2(t-t_0)} & 0 & 0 \\ 0 & e^{-4(t-t_0)} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Set $M_i = 1, i = 1, 2, 3$, and $\alpha_1 = 2, \alpha_2 = 4$. Then condition (1) of Theorem 5.9.3 is satisfied.

Clearly,

$$\|B_{ij}(t)\| \leqslant L_{ij} = \begin{cases} 2 & \text{if } i = j = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Since, in addition,

$$\int_{t_0}^{\infty} \left(\frac{1}{1+t^2} + te^{-t} + \frac{|\cos t|}{1+t^2} + \left|e^{-t} \sin t\right| + e^{-t} \right) dt < \infty,$$

which indicates that condition (2) is satisfied.

Moreover, the matrix

$$G = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix} + \begin{bmatrix} M_1 L_{11} & M_1 L_{12} \\ M_2 L_{21} & M_2 L_{22} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

is stable, which satisfies condition (3). Then system (5.9.22) and its isolated subsystems (5.9.23) are asymptotically equivalent. As (5.9.23) has a two-dimensional exponentially stable manifold and a one-dimensional stable manifold, so does (5.9.22). This is difficult to obtain by using any other criterion in the literature.

5.10. Conditional stability

In the definitions of Lyapunov stability and asymptotic stability, it admits initial disturbance in the n-dimensional neighborhood of positive equilibrium. If the initial disturbance is restricted, we have the following conditional stability.

First, we give an example to illustrate the concept. Consider the system

$$\begin{cases} \frac{dx}{dt} = x, \\ \frac{dy}{dt} = -y. \end{cases} \tag{5.10.1}$$

The general solution of (5.10.1) is

$$\begin{cases} x(t, 0, x_0) = x_0 e^t, \\ y(t, 0, y_0) = y_0 e^{-t}. \end{cases}$$

Obviously, the zero solution of (5.10.1) is unstable. But if the initial value is chosen from the set $E = \{x_0 = 0\}$, then

$$x(t, 0, x_0) = 0,$$
$$y(t, 0, y_0) = y_0 e^{-t} \to 0 \quad \text{as } t \to +\infty.$$

In this case, the zero solution of systems (5.10.1) is said to be conditionally stable.

Next, consider general n-dimensional system:

$$\frac{dx}{dt} = f(t,x), \tag{5.10.2}$$

where $f \in C[I \times R^n, R^n]$, $f(t,0) \equiv 0$.

DEFINITION 5.10.1. The solution $\xi = \xi(t)$ of systems (5.10.2) is said to be conditionally stable, if there exists a k-dimensional subset $S_k(\xi(t_0)) \subset R^n$ $(1 \leqslant k < n)$ such that $\forall x(t_0) \in S_k(\xi(t_0))$, when $\|x(t_0) - \xi(t_0)\| < \delta(\varepsilon)$, we have

$$\left\|x(t,t_0,x_0) - \xi(t,t_0,\xi_0)\right\| < \varepsilon \quad (t \geqslant t_0),$$

where $x(t_0 = x_0)$, $\xi(t_0) = \xi_0$.

The solution $\xi = \xi(t)$ of (5.10.2) is said to be conditionally asymptotically stable, if it is conditionally stable and there exists $\sigma = \text{const} > 0$ such that

$$\left\|x(t_0) - \xi(t_0)\right\| < \sigma \quad \text{when } x(t_0) \in S_k,$$

and $\|x(t) - \xi(t)\| \to 0$ as $t \to +\infty$.

Consider a quasi-linear system:

$$\frac{dx}{dt} = Ax + \varphi(t,x), \quad A = (a_{ij})_{n\times n}, \quad \varphi \in C\bigl[I \times R^n, R^n\bigr]. \tag{5.10.3}$$

THEOREM 5.10.2. *Assume that A is a constant matrix having k eigenvalues with negative real part and*

$$\lim_{x\to 0} \frac{\varphi(t,x)}{x} = 0$$

uniformly holds with respect to t. Further,

$$\left\|\varphi(t,x') - \varphi(t,x)\right\| \leqslant L(\Delta)\left\|x - x'\right\|$$

for $t \geqslant 0$, $\|x'\| \leqslant \Delta$, $\|x\| \leqslant \Delta$, where $L = L(\Delta) \to 0$ as $\Delta \to 0$. Then the zero solution of (5.10.3) *is conditionally asymptotically stable with respect to certain k-dimensional subset S_k.*

PROOF. Without loss of generality, choose the nonsingular linear transformation:

$$x = Cy,$$

where C is an $n \times n$ real nonsingular matrix such that $C^{-1}AC = B = \operatorname{diag}(N, P)$, and

$$\operatorname{Re}\lambda_j(N) < 0 \quad (j = 1, 2, \ldots, k),$$
$$\operatorname{Re}\lambda_j(P) \geqslant 0 \quad (j = k+1, \ldots, n),$$

then system (5.10.3) is transformed to

$$\frac{dy}{dt} = By + \psi(t, y), \tag{5.10.4}$$

where $\psi(t, y) = C^{-1}\varphi(t, cy)$.

Let $L_1 = \|C^{-1}\|L$. It only needs $\|y\| < \Delta_1$, $\|y'\| < \Delta_1$, where $\Delta_1 = \Delta/\|C\|$. Obviously, it holds:

$$\left\|\psi(t, y') - \psi(t, y)\right\| \leqslant L_1\left\|y' - y\right\| \quad (t \geqslant 0).$$

Let $\beta < 0$, $|\beta| \ll 1$, $\alpha > 0$ be chosen such that $\alpha + |\beta| < \min[-\operatorname{Re}\lambda_j(N)]$ and $\alpha \geqslant -2\beta$. Then, we have

$$\begin{aligned} &\left\|e^{Nt}\right\| < Ke^{-(\alpha+|\beta|)t} \quad \text{for } t \geqslant 0,\\ &\left\|e^{pt}\right\| < Ke^{\beta t} \quad \text{for } t \leqslant 0, \end{aligned}$$

where K is a positive constant satisfying $K \gg 1$. Let

$$G(t) = \begin{cases} e^{Bt}\operatorname{diag}(E_k, 0) & \text{when } t \geqslant 0,\\ -e^{Bt}\operatorname{diag}(0, E_{n-k}) & \text{when } t \leqslant 0, \end{cases} \tag{5.10.5}$$

where E_k and E_{n-k} are kth- and $(n-k)$th-order unit matrices, respectively. Obviously,

$$G(t+0) - G(-0) = E_n.$$

From (5.10.5) and

$$e^{Bt} = \operatorname{diag}\bigl(e^{Nt}, e^{Pt}\bigr),$$

we obtain

$$\left\|G(t)\right\| \leqslant \begin{cases} Ke^{-(\alpha+|\beta|t)}, & t \geqslant 0,\\ Ke^{\beta t}, & t \leqslant 0. \end{cases} \tag{5.10.6}$$

On the other hand, we have

$$G(t) = BG(t) \quad (t \neq 0).$$

Consider the single integral equation:

$$y(t, a) = z(t)a + \int_0^\infty G(t-s)\psi\bigl(s, y(s, a)\bigr)\,ds,$$

where $z(t) = e^{Bt}\operatorname{diag}(E_k, 0) = \operatorname{diag}(e^{Nt,0})$, and a is a constant vector with zeros for the last $(n-k)$ components. We solve the integral equation by the iterated method:

$$\begin{cases} y_l(t,a) = Z(t)a + \int_0^\infty G(t-s)\psi(s, y_{l-1}(s,a))\,ds, \\ y_0(t,a) = 0, \quad l = 1, 2, \ldots. \end{cases} \tag{5.10.7}$$

Choose $|\Delta| \ll 1$ such that

$$L_1 < \frac{|B|}{\psi k},$$

and let $\|a\| < \frac{\Delta_1}{2k} = a_0$. Then for $t \geqslant 0$, it holds

$$\begin{aligned} \big\|y_1(t,a) - y_0(t,a)\big\| &\leqslant \big\|z(t)\big\|\,\|a\| \leqslant Ke^{-(\alpha+|\beta|)t}\|a\| \\ &\leqslant K\|a\|e^{-\alpha t}. \end{aligned} \tag{5.10.8}$$

Let

$$\big\|y_l(t,a) - y_{l-1}(t,a)\big\| \leqslant \frac{K}{2^{l-1}}\|\alpha\|e^{-\alpha t} \quad \text{when } t \geqslant 0, l \geqslant 1.$$

By using (5.10.6) we can deduce from (5.10.7) that

$$\begin{aligned} &\big\|y_{l+1}(t,a) - y_l(t,a)\big\| \\ &\quad\leqslant \int_0^\infty \big\|G(t-s)\big\|\,\big\|\psi\big(s, y_l(s,a)\big) - \psi\big(s, y_{l-1}(s,a)\big)\big\|\,ds \\ &\quad\leqslant \int_0^\infty Ke^{-(\alpha+|\beta|)(k-s)}\frac{|\beta|}{4k}\frac{K}{2^{p-1}}\|\alpha\|e^{-\alpha s}\,ds \\ &\qquad + \int_t^\infty Ke^{\beta(k-s)}\frac{|\beta|}{4k}\frac{K}{2^{l-1}}\|a\|e^{-\alpha s}\,ds \\ &\quad= \frac{|\beta|K}{2^{l+1}}\|a\|e^{-(\alpha+|\beta|)t}\frac{e^{|\beta|t}-1}{|\beta|} + \frac{|\beta|K}{2^{l+1}}\|a\|\frac{e^{-(\alpha+\beta)t}}{\alpha+\beta}e^{\beta t} \\ &\quad\leqslant \frac{K}{2^{l+1}}\|a\|e^{-\alpha t}\left(1 + \frac{|\beta|}{\alpha+\beta}\right) \\ &\quad= \frac{K}{2^{l+1}}\|a\|e^{-\alpha t}\frac{\alpha}{\alpha+\beta} \\ &\quad\leqslant \frac{K}{2^l}\|a\|e^{-\alpha t}, \quad \forall t \geqslant 0. \end{aligned}$$

Hence, all $y_l(t,a)$ are well defined and the inequity (5.10.8) holds for all integers l. So on the interval $[0, +\infty)$, $y_l(t,a) \to y(t,a)$, and the limit function $y(t,a)$ is continuous on certain neighborhood of t and a, where $0 \leqslant t < \infty$ and

$\|a\| < a_0$. Thus,

$$\lim_{l\to\infty} y_l(t,a) = z(t)a + \int_0^\infty G(t-s) \lim_{l\to\infty} \psi\big(s, y_{l-1}(s,a)\big)\, ds.$$

It follows that

$$y(t,a) = z(t)a + \int_0^\infty G(t-s)\psi\big(s, y(s,a)\big)\, ds, \tag{5.10.9}$$

i.e., the limit function $y(t,a)$ is the solution of the integral equation (5.10.9). Differentiating (5.10.9) with respect to t yields

$$\begin{aligned} y_t'(t,a) = Bz(t)a &+ \int_0^t BG(t-s)\psi\big(s, y(s,a)\big)\, ds \\ &+ \int_t^\infty BG(t-s)\psi\big(s, y(s,a)\big)\, ds \\ &+ \big(G(t-0) - G(-0)\big)\psi\big(t, y(t,a)\big), \end{aligned}$$

i.e., $y_t'(t,a) = By(t,a) + \psi(t, y(t,a))$. Thus, $y(t,a)$ is the solution of (5.10.4). Using estimation (5.10.8), we have

$$\begin{aligned} \big\|y(t,a)\big\| &\leqslant \big\|y_0(t,a)\big\| + \sum_{l=1}^\infty \big\|y_l(t,a) - y_{l-1}(t,a)\big\| \\ &\leqslant \sum_{l=1}^\infty \frac{k}{2^{l-1}}\|a\|e^{-\alpha t} = 2k\|a\|e^{-\alpha t}, \end{aligned} \tag{5.10.10}$$

which implies that

$$\lim_{t\to+\infty} y(t,a) = 0.$$

Therefore, $y(t,a)$ is the solution family of system (5.10.4) which continuously depends on $a_1, a_2, \ldots, a_k$. Let

$$\begin{aligned} &y_j(0,a) = y_j^{(0)}, \quad j = 1, 2, \ldots, k, \\ &y_j(0,a) = \left[\int_0^\infty G(-s)\psi\big(s, y(s,a)\big)\, ds\right]_j \quad (j = k+1, \ldots, n), \end{aligned} \tag{5.10.11}$$

where $[\int_0^{+\infty} G(-s)\psi(s, y(s,a))\,ds]_j$ denotes the jth component of the matrix $\int_0^{\infty} G(-s)\psi(s, y(s,a))\,ds$. So $y_j^{(0)} = [y(0,a)]_j$ satisfies the equations:

$$y_{k+j}^{(0)} = Q_j\big(y_j^{(0)}, \dots, y_k^{(0)}\big), \quad j = 1, 2, \dots, n-k,$$

which define certain k-dimensional manifold: S_k in R^n when $y_0 \in S_k$. Furthermore,

$$y(t, t_0, y_0) \to 0 \quad \text{as } t \to +\infty.$$

Since the transformation $x = Cy$ is nonsingular, the solution $x(t)$ of (5.10.3) have the some propriety. This completes the proof. □

COROLLARY 5.10.3. *Let the matrix A have k eigenvalues with negative real part and $(n-k)$ eigenvalues with positive real part, and $\psi(t,x)$ satisfy the Lipschitz condition*:

$$\big|\psi(t,x) - \psi\big(t, x^*\big)\big| \leqslant L\|x - x^*\|$$

when $0 < L \ll l$. Then there exist a k-dimensional manifold S_k^+ and an $(n-k)$-dimensional manifold S_{n-k}^- in certain neighborhood of $x = 0$ in R^n such that the solution $x(t)$ of (5.10.3) *holds the following limit relations*:
$x(t) \to 0$ *when* $t \to +\infty$ *if* $x(0) \in S_k^+$;
$x(t) \to 0$ *when* $t \to -\infty$ *if* $x(0) \in S_{n-k}^-$.

This is illustrated in Figure 5.10.1.

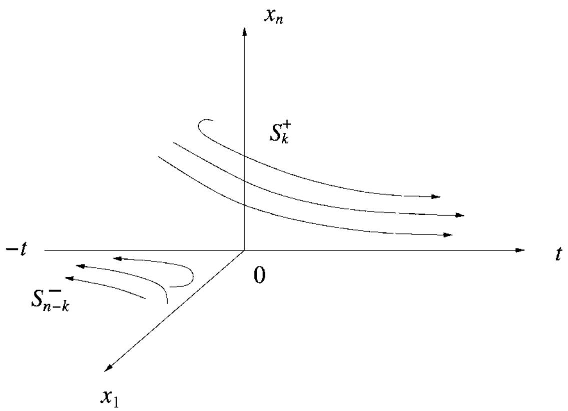

Figure 5.10.1. Stable and unstable manifolds.

5.11. Partial variable stability

Study of partial variable stability is demanded in practice. For instance, study of stability for a higher-order scaler equation can be transformed into the study of partial stability of a group of equations. Also, in some practical problems, one could only focus on some partial variables of the system. The reason for developing such technique may be due to some technical difficulty. For example, the remaining variables may be uncontrollable or unobservable. From the view point of methodology, one may use different methods to deal with different variables, and then synthesize them to solve the problem with respect to all system variables (see [418,357]). It is seen in Chapter 9 of this book that the key idea to analyze the absolute stability of a Lurie control system is to employ the partial variable stability.

We again consider the system:

$$\frac{dx}{dt} = f(t, x), \quad f \in C\big[I \times R^n, R^n\big]. \tag{5.11.1}$$

Let

$$\begin{aligned} y &= (x_1, \ldots, x_m)^T, \qquad z = (x_{m+1}, \ldots, x_n)^T, \\ x &= (y, z)^T, \qquad \|x\| := \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \\ \|y\| &:= \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}, \qquad \|z\| = \left(\sum_{i=m+1}^{n} x_i^2\right)^{1/2}. \end{aligned}$$

Then every solution of (5.11.1) is said to be an extension with respect to z, i.e., any solution $x(t)$ is defined for $t \geqslant t_0$ and $\|y(t)\| \leqslant H = \text{const}$.

DEFINITION 5.11.1. The zero solution of (5.11.1) is stable with respect to partial variable y, if $\forall \varepsilon > 0, \forall t_0 \in I, \exists \delta(t_0, \varepsilon), \forall x_0 \in S_\delta := \{x,\ \|x\| < \delta\}$ the following inequality holds:

$$\big\|y(t, t_0, x_0)\big\| < \varepsilon \quad (t \geqslant t_0).$$

Otherwise, the zero solution of the system is said to be unstable with respect to partial variable y. If $\delta(t_0, \varepsilon)$ is independent of t_0, then the zero solution of (5.11.1) is said to be uniformly stable with respect to partial variable y.

DEFINITION 5.11.2. The zero solution of (5.11.1) is said to be attractive with respect to partial variable y, if $\forall t_0 \in I, \exists \sigma(t_0) > 0, \forall \varepsilon > 0, \forall x_0 \in S_{\sigma(t_0)} = \{x\colon \|x\| \leqslant \sigma(t_0)\}, \exists T(t_0, x_0, \varepsilon)$ such that when $t \geqslant t_0 + T$ we have

$$\big\|y(t, t_0, x_0)\big\| < \varepsilon.$$

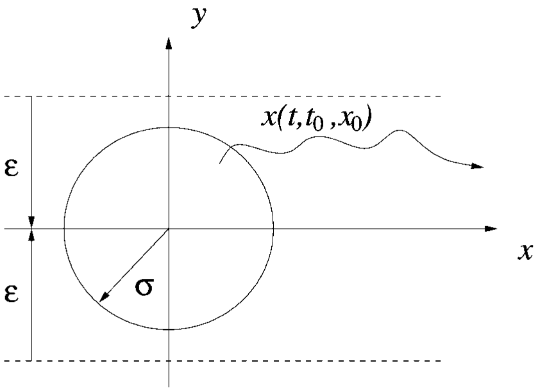

Figure 5.11.1. Partial variable stability.

If $\sigma(t_0) = \sigma,\ T(t_0, x_0, \varepsilon) = T(\varepsilon)$, independent of t_0 and x_0, then the zero solution of (5.11.1) is said to be uniformly attractive with respect to partial variable y.

DEFINITION 5.11.3. The zero solution of (5.11.1) is said to be asymptotically stable with respect to partial variable y, if it is stable and attractive with respect to y. The zero solution of (5.11.1) is said to be uniformly asymptotically stable with respect to y, if it is uniformly stable and uniformly attractive with respect to y.

Figures 5.11.1 and 5.11.2 geometrically illustrate partial variable stability in R^n space, and partial variable attractivity in R^n space, respectively.

In the following, we give the definitions of positive definite or negative definite with respect to partial variable y.

DEFINITION 5.11.4. The function $V(t,x) \in C[I \times R^n, R]$ is said to be positive definite (negative definite) with respect to partial variable y, if there exists $\varphi_1 \in K$ such that

$$V(t,x) \geqslant \varphi_1(\|y\|) \quad \big(V(t,x) \leqslant -\varphi_1(\|y\|)\big).$$

Following the definitions of infinitesimal upper bound and radially unboundedness of $V(t,x)$, we can give similar definitions for $V(t,x) \in C[I \times R^n, R^1]$ with respect to partial variable y.

THEOREM 5.11.5 *(Stability theorem with respect to y).*

(1) *Assume that there exists $V(t,x) \in C[I \times \Omega, R^1]$ such that $V(t,x) \geqslant \varphi(\|y\|)$ for $\varphi \in k$, and $D^+V(t,x)|_{(5.11.1)} \leqslant 0$, then the zero solution of (5.11.1) is stable with respect to partial variable y.*

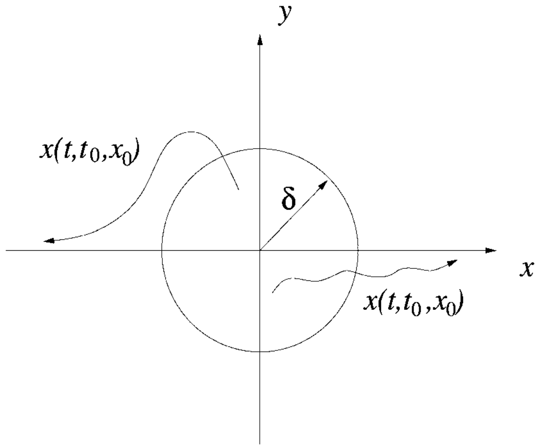

Figure 5.11.2. Partial variable attractivity.

(2) *If condition* (1) *is satisfied and* $V(t,x)$ *has infinitesimal upper bound, then the zero solution of* (5.11.1) *is uniformly stable with respect to partial variable* y.

PROOF. (1) $\forall \varepsilon > 0$, $\forall t_0 \in I$, $\exists \delta(t_0,\varepsilon) > 0$ such that $\forall x_0 \in S_\delta$ it holds:

$$V(t_0,x_0) < \varphi(\varepsilon).$$

Let $x(t) := x(t,t_0,x_0)$, $x_0 \in S_\delta$. By $D^+V(t,x)|_{(5.11.1)} \leqslant 0$, we have

$$\varphi\big(\|y(t,t_0,x_0)\|\big) \leqslant V\big(t,x(t,t_0,x_0)\big) \leqslant V(t_0,x_0) \leqslant \varphi(\varepsilon).$$

Hence, it follows that

$$\|y(t,t_0,x_0)\| < \varepsilon \quad (t \geqslant t_0).$$

So the zero solution of (5.11.1) is stable with respect to partial variable y.

(2) Since $\varphi_1(\|x\|) \leqslant V(t,x) \leqslant \varphi_2(\|x\|)$, $\forall \varepsilon > 0$, take $\delta(\varepsilon) := \varphi_2^{-1}(\varphi_1(\varepsilon))$. Then for $\|x_0\| < \delta$, we have

$$\begin{aligned}\varphi_1\big(\|y(t,t_0,x_0)\|\big) &\leqslant V\big(t,,x(t,t_0,x_0)\big) \leqslant V(t_0,x_0)\\ &\leqslant \varphi_2\big(\|x_0\|\big) \leqslant \varphi_2\big(\varphi_2^{-1}\big(\varphi_1(\varepsilon)\big)\big)\\ &= \varphi_1(\varepsilon).\end{aligned}$$

Therefore, $\|y(t,t_0,x_0)\| < \varepsilon$ $(t \geqslant t_0)$. Thus, the zero solution of (5.11.1) is uniformly stable with respect to the partial variable y.

The theorem is proved. □

THEOREM 5.11.6 *(Asymptotic stability theorem with respect to y). If there exists $V(t,x) \in C[I \times \Omega, R^1]$ such that*

$$\varphi_1\big(\|y\|\big) \leqslant V(t,x) \leqslant \varphi_2\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right) \quad \textit{for } m \leqslant k \leqslant n,$$

$$\left.\frac{dV}{dt}\right|_{(5.11.1)} \leqslant -\psi\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right),$$

then the zero solution of (5.11.1) *is asymptotically stable with respect to the partial variable y.*

PROOF. We only need to prove that the zero solution is attractive with respect to y, because the condition implies the stability with respect to y. By Theorem 5.11.5, $\exists\delta(t_0)$, $\forall x_0 \in S_\delta$ it holds:

$$\lim_{t\to\infty} V\big(t, x(t,t_0,x_0)\big) = 0.$$

Therefore,

$$\varphi_1\big(\big\|y(t)\big\|\big) \leqslant V\big(t, x(t,t_0,x_0)\big) \to 0 \quad \text{as } t \to +\infty. \tag{5.11.2}$$

Now we show that (5.11.2) is true by contradiction. Assume that there exists x_0, $\|x_0\| < \delta$ such that

$$V\big(t, x(t,t_0,x_0)\big) \nrightarrow 0 \quad \text{as } t \to +\infty.$$

Since

$$\frac{dV(t, x(t,t_0,x_0))}{dt} \leqslant 0$$

implies that

$$\lim_{t\to\infty} V\big(t, x(t,t_0,x_0)\big) = V_\infty,$$

hence, $V(t, x(t,t_0,x_0)) \geqslant V_\infty > 0$. By the condition we have

$$\left(\sum_{i=1}^{k} x_i^2(t,t_0,x_0)\right)^{1/2} \geqslant \varphi_2^{-1}(V_\infty) \quad \text{and} \quad \frac{dV}{dt} \leqslant -\psi\big(\varphi_2^{-1}(V_\infty)\big),$$

and further we obtain

$$0 \leqslant V\big(t, x(t,t_0,x_0)\big) \leqslant V(t_0,x_0) \leqslant -\psi\big(\varphi_2^{-1}(V_\infty)\big)(t-t_0) \to -\infty \\ \text{as } \to +\infty,$$

which is a contradiction. So (5.11.2) holds, implying that

$$\lim_{t\to\infty} \big\|y(t)\big\| = 0.$$

□

THEOREM 5.11.7 *(Uniformly asymptotic stability theorem with respect to y). Assume that there exists* $V(t, x) \in C[I \times \Omega, R^1]$ *such that*

(1) $\varphi_1(\|y\|) \leqslant V(t, x) \leqslant \varphi_2(\|x\|)$ *for* $\varphi_1, \varphi_2 \in K$;
(2) $D^+V(t, x)|_{(5.11.1)} \leqslant -\psi(\|x(t, t_0, x_0)\|)$ *for* $\psi \in K$.

Then the zero solution of (5.11.1) *is uniformly asymptotically stable with respect to partial variable y.*

PROOF. By the condition, $\forall\varepsilon > 0$, take $\delta(\varepsilon) = \varphi_2^{-1}(\varphi_1(\varepsilon))$ such that when $\|x_0\| < \delta$ we have

$$\varphi_1\big(\big\|y(t, t_0, x_0)\big\|\big) \leqslant V\big(t, x(t, t_0, x_0)\big) \leqslant V(t_0, x_0) \leqslant \varphi_2\big(\|x_0\|\big) \leqslant \varphi_1(\varepsilon),$$

so we obtain

$$\big\|y(t, t_0, x_0)\big\| < \varepsilon \quad (t \geqslant t_0).$$

Let $\Delta_0 = \delta(H)$, $T(\varepsilon) = \varphi_2(\Delta_0)/\psi(\delta(\varepsilon))$, $\|x_0\| < \Delta_0, t_0 \in I$. We prove that there exists $t^* \in (t_0, t_0 + T)$ such that

$$\big\|x\big(t^*, t_0, x_0\big)\big\| < \delta(\varepsilon). \tag{5.11.3}$$

If otherwise, assume that $\|x(t, t_0, x_0)\| \geqslant \delta(\varepsilon)$ for all $t \in (t_0, t_0 + T)$. Then by condition (2) it must hold:

$$\begin{aligned} 0 &\leqslant V\big(t_0 + t, x(t_0 + T, t_0, x_0)\big) \\ &\leqslant V(t_0, x_0) - \psi\big(\delta(\varepsilon)\big)T < \psi(\Delta_0) - \psi\big(\delta(\varepsilon)\big)T = 0, \end{aligned} \tag{5.11.4}$$

which is a contradiction, and so (5.11.3) is true.

Hence, $\|y(t, t_0, x_0)\| < \varepsilon$ when $t \geqslant t_0 + T \geqslant t^*$. This means that the conclusion of Theorem 5.11.7 holds. □

In the following, we discuss instability with respect to partial variable y. We only need to show that $\exists\delta > 0, \forall\delta_1 = 0, \exists x_0 \in S_{\delta_1}$ and $t_1 > t_0$ such that $\|y(t_1, t_0, x_0)\| \geqslant \delta_1$.

The function $U(t, x)$ is said to be positive definite in the region defined by $V(t, x) > 0$, where

$$V(t, x) \in C[I \times \Omega, R],$$

if $\forall\varepsilon > 0, \exists\delta(\varepsilon), \forall(t, x) \in G(t_0, x) := \{(t, x) \mid t \geqslant t_0, \|y\| \leqslant H, \|z\| < \infty\}$. Then $U(t, x) \geqslant \delta > 0$ holds when if $V(t, x) \geqslant \varepsilon$.

THEOREM 5.11.8 *(Instability theorem with respect to y). If there exists* $V(t, x) \in C[I \times \Omega, R^1]$ *for* $\forall t \in I, \forall\varepsilon > 0$, $\|x\| \leqslant \varepsilon$, *then there exists a region defined by* $V > 0$ *such that V is bounded on* $V > 0$, *and* $\frac{dV}{dt}|_{(5.11.1)}$ *is positive definite in the region* $V > 0$. *Then the zero solution of* (5.11.1) *is unstable.*

PROOF. By contradiction, if otherwise, assume that the zero solution of (5.11.1) is stable. Then $\forall\varepsilon > 0$ $(\varepsilon \leqslant H)$, $\forall t_0 \in I$, $\exists\delta(t_0, \varepsilon)$ such that when $\|x_0\| < \delta$, $t \geqslant t_0$ it holds:

$$\big\| y(t, t_0, x_0)\big\| < \varepsilon \leqslant H.$$

For given $\varepsilon_0 \leqslant H$, $t_0 \in I$, choose x_0, $\|x_0\| < \delta(t_0, \varepsilon)$, such that

$$V(t_0, x_0) = V_0 > 0.$$

Since $\frac{dV}{dt}|_{(5.11.1)}$ is positive definite in the region $V > 0$, there exists $l > 0$ for V_0 such that

$$\frac{dV(t, x(t))}{dt} \geqslant l > 0 \quad (t \geqslant t_0).$$

Hence,

$$V\big(t, x(t, t_0, x_0)\big) \geqslant l(t - t_0) + V(t_0, x_0) \to +\infty \quad \text{as } t \to +\infty.$$

This is a contradiction with the boundedness of $V > 0$. So the zero solution is unstable with respect to partial variable y. The proof of theorem is complete. □

In the following, we give some practically simple criteria of partial variable stability for nonlinear, time-varying dynamical systems.

Consider the following nonlinear, time-varying dynamical system:

$$\frac{dx}{dt} = A(t)x + g(t, x), \tag{5.11.5}$$

where

$$x = (x_1, \ldots, x_n)^T, \qquad A(t) = \big(a_{ij}(t)\big)_{n\times n} \in C\big[I, R^{n^2}\big],$$
$$g(t, x) \in C\big[I \times \Omega, R^n\big], \qquad \Omega = \big\{|\|x\| \leqslant H\big\},$$
$$g(t, 0) \equiv 0, \qquad g(t, x) = \big(g_1(t, x), \ldots, g_n(t, x)\big)^T,$$

and $\frac{\partial g(t,x)}{\partial x}$ exists.

We rewrite (5.11.5) as

$$\frac{dy}{dt} = A_{11}(t)y + A_{12}(t)z + g_I(t, x), \tag{5.11.5$_\mathrm{a}$}$$

$$\frac{dz}{dt} = A_{21}(t)y + A_{22}(t)z + g_{II}(t, x), \tag{5.11.5$_\mathrm{b}$}$$

where

$$y = (x_1, \ldots, x_m)^T, \qquad z = (x_{m+1}, \ldots, x_n)^T,$$

$$
\begin{aligned}
&A_{11}(t) = \big(a_{ij}(t)\big)_{m\times m}, \quad 1 \leqslant i, j \leqslant m,\\
&A_{12}(t) = \big(a_{ij}(t)\big)_{m\times(n-m)}, \quad 1 \leqslant i \leqslant m,\ m+1 \leqslant j \leqslant n,\\
&A_{21}(t) = \big(a_{ij}(t)\big)_{(n-m)\times n}, \quad m+1 \leqslant i \leqslant n,\ 1 \leqslant\ j \leqslant m,\\
&A_{22}(t) = \big(a_{ij}(t)\big)_{(n-m)\times(n-m)}, \quad m+1 \leqslant i,\ j \leqslant n;\\
&g_I(t,x) = \big(g_1(t,x), \dots, g_m(t,x)\big)^T,\\
&g_{II}(t,x) = \big(g_{m+1}(t,x), \dots, g_n(t,x)\big)^T,\\
&\big\|g_i(t,x)\big\| \leqslant \sum_{j=1}^{n} l_{ij}(t)|x_j| \quad (i = 1, \dots, m),\\
&\big\|g_{II}(t,x)\big\| \leqslant \sum_{j=1}^{n} l_{ij}(t)|x_j| \quad (i = m+1, \dots, n).
\end{aligned}
$$

Let

$$
\begin{aligned}
&L_{11}(t) = \big(l_{ij}(t)\big)_{m\times m}, \quad 1 \leqslant i,\ j \leqslant m,\\
&L_{12}(t) = \big(l_{ij}(t)\big)_{m\times(n-m)}, \quad 1 \leqslant i \leqslant m,\ m+1 \leqslant j \leqslant n,\\
&L_{21}(t) = \big(l_{ij}(t)\big)_{(n-m)\times m}, \quad m+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\\
&L_{22}(t) = \big(l_{ij}(t)\big)_{(n-m)\times(n-m)}, \quad m+1 \leqslant i,\ j \leqslant n.
\end{aligned}
$$

THEOREM 5.11.9. *If the following conditions are satisfied*:

(1)

$$
\int_{t_0}^{+\infty} \big(\big\|A_{11}(t)\big\| + \big\|L_{11}(t)\big\|\big)\,dt := M < +\infty,
$$

(2)

$$
\int_{t_0}^{+\infty} e^{\int_{t_0}^{t} \Lambda(\xi)\,d\xi} \big(\big\|A_{12}(t)\big\| + \big\|L_{12}(t)\big\|\big)\,dt := k < +\infty,
$$

then the zero solution of (5.11.5) is partially stable with respect to partial variable y, where $\Lambda(t) = \Lambda_1(t) + \Lambda_2(t)$, $\Lambda_1(t)$ is the maximal eigenvalue of $\frac{1}{2}(A(t) + A^T(t))$, and $\Lambda_2(t)$ is the maximal eigenvalue of $\frac{1}{2}(L(t) + L^T(t))$, where $A(t) = (a_{ij}(t))_{n\times n}$, $L(t) = (l_{ij}(t))_{n\times n}$.

PROOF. Choose the Lyapunov function:

$$
V = \frac{1}{2}x^2.
$$

Use $(x, \tilde{x})$ to denote the inner product of vectors x and $\tilde{x}$. Let $|x| = (|x_1|, \ldots, |x_n|)^T$. Then

$$\begin{aligned}
\frac{dV}{dt}\bigg|_{(5.11.5)} &= \left(\frac{dx}{dt}, x\right) \\
&= (Ax, x) + \big(g(t,x), x\big) \leqslant (Ax, x) + \big(L(t)|x|, |x|\big) \\
&\leqslant \frac{1}{2}x^T\big(A(t) + A^T(t)\big)x + \frac{1}{2}|x|^T\big(L(t) + L^T(t)\big)|x| \\
&\leqslant \Lambda_1(t)x^2 + \Lambda_2(t)x^2 \\
&= \Lambda(t)x^2 = 2\Lambda(t)V,
\end{aligned}$$

from which we obtain

$$V(t) \leqslant V(t_0)e^{\int_{t_0}^t 2\Lambda(\xi)\,d\xi}, \tag{5.11.6}$$

which implies that

$$\big\|x(t)\big\|^2 \leqslant \big\|x(t_0)\big\|^2 e^{\int_{t_0}^t 2\Lambda(\xi)\,d\xi}.$$

Thus,

$$\big\|y(t)\big\| \leqslant \big\|x(t)\big\| \leqslant \big\|x(t_0)\big\| e^{\int_{t_0}^t \Lambda(\xi)\,d\xi}, \tag{5.11.7}$$

$$\big\|z(t)\big\| \leqslant \big\|x(t)\big\| \leqslant \big\|x(t_0)\big\| e^{\int_{t_0}^t \Lambda(\xi)\,d\xi}. \tag{5.11.8}$$

Rewrite $(5.11.5)_{\mathrm{a}}$ as an integral equation:

$$y(t) = y(t_0) + \int_{t_0}^{t} A_{11}(\tau)y(\tau)\,d\tau + \int_{t_0}^{t} g_I\big(\tau, x(\tau)\big)\,d\tau. \tag{5.11.9}$$

Using (5.11.5) and (5.11.6) to estimate (5.11.9) yields

$$\begin{aligned}
\big\|y(t)\big\| \leqslant \big\|y(t_0)\big\| &+ \int_{t_0}^{t} \big\|A_{11}(\tau)\big\|\big\|y(\tau)\big\|\,d\tau + \int_{t_0}^{t} \big\|L_{11}(\tau)\big\|\big\|y(\tau)\big\|\,d\tau \\
&+ \int_{t_0}^{t} \big\|A_{12}(\tau)\big\|\big\|z(\tau)\big\|\,d\tau + \int_{t_0}^{t} \big\|L_{12}(\tau)\big\|\big\|z(\tau)\big\|\,d\tau \\
&\leqslant \|x_0\| + \|x_0\| \int_{t_0}^{t} \big(\big\|A_{12}(\tau)\big\| + \big\|L_{12}(\tau)\big\|\big)e^{\int_{t_0}^{\tau} \Lambda(\xi)\,d\xi\,d\tau}
\end{aligned}$$

$$+\int_{t_0}^{t}\big(\|A_{11}(\tau)\|+\|L_{11}(\tau)\|\big)\|y(\tau)\|\,d\tau$$

$$\leqslant \|x(t_0)\|(1+k)+\int_{t_0}^{t}\big(\|A_{11}(\tau)\|+\|L_{11}(\tau)\|\big)\|y(\tau)\|\,d\tau.$$

Now applying Gronwall–Bellman inequality, we finally obtain

$$\begin{aligned}\|y(t)\| &\leqslant \|x(t_0)\|(1+k)e^{\int_{t_0}^{t}(\|A_{11}(\tau)\|+\|L_{11}(\tau)\|)\,d\tau}\\ &\leqslant \|x(t_0)\|(1+k)e^{M},\end{aligned}\tag{5.11.10}$$

from which we know that the zero solution of (5.11.5) is partially stable with respect to partial variable y. □

THEOREM 5.11.10. *If the following conditions are satisfied*:

(1) *For the m-dimensional linear system*:

$$\frac{dy}{dt}=A_{11}(t)y,\tag{5.11.11}$$

its Cauchy matrix solution satisfies the following estimation:

$$\|k(t,t_0)\|\leqslant Me^{-\alpha(t-t_0)},$$

where M and α are positive constants;

(2)

$$e^{\int_{t_0}^{t}\Lambda(\xi)\,d\xi}\big(\|A_{12}(t)\|+\|L_{11}(t)\|+\|L_{12}(t)\|\big)\to 0\quad \text{as } t\to+\infty,$$

where $\Lambda(t)$ is defined in Theorem 5.11.9,

then the zero solution of (5.11.5) is partially asymptotically stable with respect to partial variable y.

PROOF. Based on equation $(5.11.5)_{\mathrm{a}}$, with the aid of the variation constant formula, we have

$$y(t)=k(t,t_0)y(t_0)+\int_{t_0}^{t}k(t,\tau)A_{12}(\tau)z(\tau)\,d\tau$$

$$+\int_{t_0}^{t}k(t,\tau)g_I\big(t,x(\tau)\big)\,d\tau.\tag{5.11.12}$$

Hence,

$$
\begin{aligned}
\|y(t)\| \leqslant{} & Me^{-\alpha(t-t_0)}\|x(t_0)\| \\
& + \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\|A_{12}(\tau)\|\|x(t_0)\|e^{\int_{t_0}^{\tau}\Lambda(\xi)\,d\xi}\,d\tau \\
& + \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\big(\|L_{11}(\tau)\| + \|L_{12}(\tau)\|\big)\|x(t_0)\|e^{\int_{t_0}^{\tau}\Lambda(\xi)\,d\xi}\,d\tau. \quad (5.11.13)
\end{aligned}
$$

Let

$$
\begin{aligned}
I_1(t) &= M\|x(t_0)\|e^{-\alpha(t-t_0)}, \\
I_2(t) &= \|x(t_0)\|\int_{t_0}^{t} Me^{-\alpha(t-\tau)}\big(\|A_{12}(\tau)\| + \|L_{11}(\tau)\| + \|L_{12}(\tau)\|\big)\,d\tau.
\end{aligned}
$$

Obviously, $I_1(t)$ and $I_2(t)$ are linearly dependent with respect to $\|x_0\|$, and

$$
\begin{aligned}
\lim_{t\to+\infty} I_1(t) &= 0, \\
\lim_{t\to+\infty} I_2(t) &= \lim_{t\to+\infty} M\|x(t_0)\|e^{-\alpha t} \\
&\quad \times \int_{t_0}^{t}\big(\|A_{12}(\tau)\| + \|L_{11}(\tau)\| + \|L_{12}(\tau)\|\big)e^{\alpha\tau}e^{\int_{t_0}^{\tau}\Lambda(\xi)\,d\xi}\,d\tau \\
&= M\|x(t_0)\| \\
&\quad \times \lim_{t\to+\infty}\frac{(\|A_{12}(t)\| + \|L_{11}(t)\| + \|L_{12}(t)\|)e^{\alpha t}e^{\int_{t_0}^{t}\Lambda(\xi)\,d\xi}}{\alpha e^{\alpha t}} \\
&= 0.
\end{aligned}
$$

Therefore, the zero solution of (5.11.5) is asymptotically stable with respect to partial variable y. □

THEOREM 5.11.11. *If the following conditions are satisfied*:

(1) *condition* (1) *of Theorem* 5.11.10;
(2) $e^{\int_{t_0}^{t}\Lambda(\xi)\,d\xi} \leqslant e^{\gamma(t-t_0)}$, *where γ is a constant, and $\Lambda(t)$ is defined as in Theorem* 5.11.9;
(3) $(\|A_{12}(t)\| + \|L_{11}(t)\| + \|L_{12}(t)\|) \leqslant \mu e^{\beta l}$,

where β, μ are positive constants and $\beta+\gamma < 0$, then the zero solution of (5.11.5) *is exponentially stable with respect to partial variable y.*

PROOF. Following Theorem 5.11.10, we can prove

$$\|z(t)\| \leqslant \|x(t)\| \leqslant \|x(t_0)\| \exp\left(\int_{t_0}^{t} \Lambda(\xi)\,d\xi\right) \leqslant \|x_0\| e^{\gamma(t-t_0)}, \quad (5.11.14)$$

$$\|y(t)\| \leqslant \|x(t)\| \leqslant \|x(t_0)\| \exp\left(\int_{t_0}^{t} \Lambda(\xi)\,d\xi\right) \leqslant \|x_0\| e^{\gamma(t-t_0)}. \quad (5.11.15)$$

Let

$$-\varepsilon = \beta + \gamma \quad \text{if } \alpha - \varepsilon > 0.$$

From $(5.11.5)_{\mathrm{a}}$ and the variation constant formula, we obtain

$$\begin{aligned}
y(t) &= K(t, t_0)y(t_0) + \int_{t_0}^{t} K(t,\tau)A_{12}(\tau)z(\tau)\,d\tau \\
&\quad + \int_{t_0}^{t} K(t,\tau)A_{12}(\tau)z(\tau)\,d\tau + \int_{t_0}^{t} K(t,\tau)g_I\big(\tau, x(\tau)\big)\,d\tau, \\
\|y(t)\| &\leqslant Me^{-\alpha(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\|A_{12}(\tau)\|\,\|z(\tau)\|\,d\tau \\
&\quad + \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\big(\|L_{11}(\tau)\|\,\|y(\tau)\| + \|L_{12}(\tau)\|\,\|z(\tau)\|\big)\,d\tau \\
&\leqslant Me^{-\alpha(t-\tau)}\|x(t_0)\| + M\|x(t_0)\| \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\|A_{12}(\tau)\|\,\|z(\tau)\|\,d\tau \\
&\quad + \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\big(\|L_{11}(\tau)\|\,\|y(\tau)\| + \|L_{12}(\tau)\|\,\|z(\tau)\|\big)\,d\tau \\
&\leqslant Me^{-\alpha(t-t_0)}\|x(t_0)\| + M\|x(t_0)\| \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\|A_{12}(\tau)\|e^{\gamma(t-t_0)}\,d\tau \\
&\quad + M\|x(t_0)\| \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\big(\|L_{11}(\tau)\| + \|L_{12}(\tau)\|\big)e^{\gamma(t-t_0)}\,d\tau
\end{aligned}$$

$$\begin{aligned}
&\leqslant Me^{-\alpha(t-t_0)}\|x_0\| + \mu M\big\|x(t_0)\big\|e^{-\gamma t_0}e^{-\alpha t}\int_{t_0}^{t} Me^{(\alpha+\beta+\gamma)\tau}\,d\tau \\
&= Me^{-\alpha(t-t_0)}\|x_0\| + \mu M\big\|x(t_0)\big\|e^{-\gamma t_0}e^{-\alpha t}\int_{t_0}^{t} e^{(\alpha-\varepsilon)\tau}\,d\tau \\
&= Me^{-\alpha(t-t_0)}\|x_0\| + \mu M\big\|x(t_0)\big\|e^{-\gamma t_0}e^{-\alpha t}\left[\frac{e^{(\alpha-\varepsilon)t}}{\alpha-\varepsilon} - \frac{e^{(\alpha-\varepsilon)t_0}}{\alpha-\varepsilon}\right] \\
&\leqslant M\|x_0\|e^{-\varepsilon(t-t_0)} + \frac{\mu Me^{-\gamma t_0}}{\alpha-\varepsilon}e^{\varepsilon t_0}\|x_0\|e^{-\varepsilon(t-t_0)} \\
&= \left[M\|x_0\| + \frac{\mu Me^{(-\gamma+\varepsilon)t_0}}{\alpha-\varepsilon}\|x_0\|\right]e^{-\varepsilon(t-t_0)} \\
&:= \tilde{M}\|x_0\|e^{-\varepsilon(t-t_0)},
\end{aligned} \tag{5.11.16}$$

where

$$\tilde{M} = M + \frac{\mu Me^{(-\gamma+\varepsilon)t_0}}{\alpha-\varepsilon}.$$

Then if $\alpha - \varepsilon < 0$, we have

$$\begin{aligned}
\big\|y(t)\big\| &\leqslant Me^{-\alpha(t-t_0)}\big\|x(t_0)\big\| + \mu M\big\|x(t_0)\big\|e^{-\gamma t_0}e^{-\alpha t}\left[\frac{e^{(\alpha-\varepsilon)t}}{\alpha-\varepsilon} - \frac{e^{(\alpha-\varepsilon)t_0}}{\alpha-\varepsilon}\right] \\
&\leqslant Me^{-\alpha(t-t_0)}\big\|x(t_0)\big\| + \mu M\big\|x(t_0)\big\|\frac{e^{(-\gamma-\varepsilon)t_0}}{\varepsilon-\alpha}e^{-\alpha(t-t_0)} \\
&:= M^*\big\|x(t_0)\big\|e^{-\varepsilon(t-t_0)},
\end{aligned} \tag{5.11.17}$$

where

$$M^* = M + \mu M\frac{e^{(-\gamma-\varepsilon)t_0}}{\varepsilon-\alpha}.$$

Finally, by (5.11.17) and (5.11.17), Theorem 5.11.11 is proved. □

5.12. Stability and boundedness of sets

The Lyapunov function method can be generalized to analyze the stability and boundedness of set. The concept of stability and boundedness of set is very general, which includes many other stabilities as particular, like the Lyapunov stability, Lagrange stability and partial variable stability.

Consider the general system:

$$\frac{dx}{dt} = f(t, x), \tag{5.12.1}$$

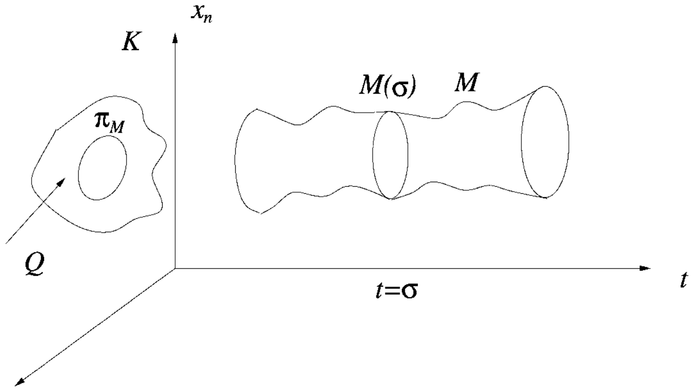

Figure 5.12.1. The set M in $I \times R^n$.

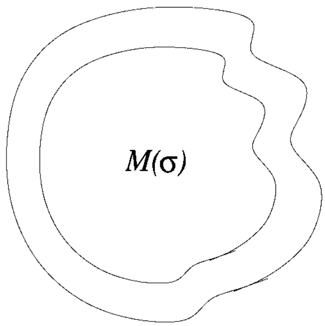

Figure 5.12.2. The projection of set M in R^n.

where $f(t,x) \in C[I \times R^n, R^n]$.

Consider a set M on $I \times R^n$, which can be unbounded or bounded. Let $M(\sigma) := \{x,\ (\sigma, x) \in M\}$ denote the intersect of hyperplane $t = \sigma$ and M, and Π_M stands for the projection of M on R^n, as shown in Figures 5.12.1 and 5.12.2.

If in R^n, there exists a set Q such that $Q \supset \Pi_M$, then M is bounded, and the minimum distance from x to M is expressed as $d(x, M)$,

$$d(x, M) := \inf_{y \in \Pi_M} \|x - y\|. \tag{5.12.2}$$

Let $M(\sigma, \varepsilon)$ be ε-neighborhood of $M(\sigma)$ in R^n, which denotes the distance of point of $M(\sigma, \varepsilon)$ to M without exceeding ε.

DEFINITION 5.12.1. The set M is said to be stable (uniformly stable) with respect to system (5.12.1) if $\forall \varepsilon > 0$, $\forall \alpha > 0$, $\forall t_0 \in I$ $\exists \delta(t_0, \varepsilon, \alpha) > 0$ ($\exists \delta(\varepsilon) > 0$)

such that

$$\forall x_0 \in S_\alpha := \{x \mid \|x\| \leqslant \alpha\},$$

when $d(x_0, M(t_0)) < \delta(t_0, \varepsilon, \alpha)$ $(< \delta(\varepsilon))$ it holds

$$d\big(x(t, t_0, x_0), M(t)\big) < \varepsilon \quad (t \geqslant t_0). \tag{5.12.3}$$

DEFINITION 5.12.2. The set M is said to be globally attractive with respect to system (5.12.1) if every solution $x(t, t_0, x_0) \to M$ as $t \to +\infty$. The set M is said to be globally uniformly attractive, if $\forall \varepsilon > 0, \forall \eta > 0, \forall \alpha > 0, \forall t_0 \in I$, there exists $T(\varepsilon, \eta) > 0$ such that when $x_0 \in S_\alpha := \{x \mid \|x\| \leqslant \alpha\}$ and

$$d\big(x_0, M(t_0)\big) \leqslant \eta, \tag{5.12.4}$$

it holds $d(x(t, t_0, x_0), M(t)) < \varepsilon$ $(t \geqslant t_0 + T)$.

DEFINITION 5.12.3. If the set is stable and attractive, then it is called asymptotically stable with respect to system (5.12.1).

DEFINITION 5.12.4. The set M is said to be uniformly bounded with respect to systems (5.12.1), if $\forall \eta > 0, \forall \alpha > 0, \forall t_0 \in I, \exists \beta(\eta) > 0$ such that $\forall x_0 \in R^n$ and when $d(x_0, M(t_0)) < \eta$, the following condition is satisfied:

$$d\big(x(t, t_0, x_0), M(t)\big) < \beta(\eta) \quad (t \geqslant t_0). \tag{5.12.5}$$

The concept of set is very general. Some examples are given below.

(1) Let $M(t) = \{I \times 0\}$. Then the stability, attraction of $M(t)$ are just, respectively, the stability and attraction of the zero solution in the sense of Lyapunov.
(2) Let $M(t) = \{I \times x(t, t_0, x_0)\}$, and assume that system (5.12.1) is an autonomous system. Then the stability and attraction are just the Lyapunov stability for the solution $x(t, t_0, x_0)$.
(3) Let $M(t) = \{I \times y = 0\}$. Then the stability or attraction of $M(t)$ is just the stability or attraction of the partial variable stability with respect to the partial variable y.
(4) Let $M(t) = I \times S_B := \{x \mid \|x\| \leqslant B\}$. Then the attraction of $M(t)$ is just the asymptotic stability of Lagrange.

THEOREM 5.12.5. *If there exists a function* $V(t, x) \in C[I \times D_H, R]$, $D_H := \{x \mid d(x, M(t)) \leqslant H\}$ *such that on* $I \times D_H$ *it holds*:

(1) $\varphi_1(d(x, M(t))) \leqslant V(t, x) \leqslant \varphi_2(d(x, M(t)))$ *for* $\varphi_1, \varphi_2 \in K$,
(2) $D^+V(t, x)|_{(5.12.1)} \leqslant 0$,

then the set M *is uniformly stable.*

PROOF. $\forall \varepsilon > 0$, $(\varepsilon < H)$, $\forall (t_0, x_0) \in I \times D_H$, let $x(t) = x(t, t_0, x_0)$, and take $\delta = \varphi_2^{-1}(\varphi_1(\varepsilon))$, i.e., $\varepsilon = \varphi_1^{-1}(\varphi_2(\delta))$. Then when $d(x_0, M(t_0)) < \delta$, it holds

$$\varphi_1\big(d\big(x(t), M(t)\big)\big) \leqslant V\big(t, x(t)\big) \leqslant V(t_0, x_0) \leqslant \varphi_2\big(d\big(x_0, M(t_0)\big)\big) < \varphi_2(\delta).$$

Therefore, we have

$$d\big(x(t), M(t)\big) < \varphi_1^{-1}\big(\varphi_2(\delta)\big) = \varepsilon \quad (t \geqslant t_0),$$

i.e., $M(t)$ is uniformly stable. □

THEOREM 5.12.6. *If there exists $V(t, x) \in C^{(1)}[I \times R^n, R^n]$ such that*

(1) $\varphi_1(d(x, M)) \leqslant V(t, x) \leqslant \varphi_2(d(x, M))$ *for* $\varphi_1, \varphi_2 \in KR$,
(2) $\frac{dV(t,x)}{dt}| \leqslant -\psi(d(x, M))$ *for* $\psi \in K$,

then the set M is uniformly, globally and asymptotically stable.

PROOF. First, since the conditions of Theorem 5.12.6 imply the conditions of Theorem 5.12.5. Thus the set is uniformly stable.

Next, we prove the global attraction. $\forall \varepsilon > 0, \forall \eta > 0, \forall \alpha > 0$, then $\forall t_0 \in I$, by condition (2) it holds:

$$\varphi_1\big(d\big(x(t), M(t)\big)\big) \leqslant V\big(t, x(t)\big) \leqslant \varphi_2\big(d\big(x(t), M(t)\big)\big).$$

Thus,

$$\begin{aligned}
&d\big(x(t), M(t)\big) \geqslant \varphi_2^{-1}\big(V\big(t, x(t)\big)\big),\\
&\frac{d}{dt}V\big(t, x(t)\big) \leqslant -\psi\big(\varphi_2^{-1}\big(V\big(t, x(t)\big)\big)\big) < 0,\\
&\frac{dV(t, x(t))}{\psi(\varphi_2^{-1}(V(t, x(t))))} \leqslant -dt,\\
&\int_{V(t_0,x(t_0))}^{V(t,x(t))} \frac{dV}{\psi(\varphi_2^{-1}(V(t, x(t))))} \leqslant -(t - t_0),\\
&\int_{V(t,x(t))}^{V(t_0,x(t_0))} \frac{dV}{\psi(\varphi_2^{-1}(V(t, x(t))))} \geqslant t - t_0.
\end{aligned} \tag{5.12.6}$$

Let

$$V(t_0) \leqslant \varphi_2\big(d\big(x_0, M(t_0)\big)\big) \leqslant \varphi_2(\eta).$$

Then, we have

$$t - t_0 \leqslant \int_{V(t,x(t))}^{V(t_0,x(t_0))} \frac{dV}{\psi[\varphi_2^{-1}(V(t,x(t)))]}$$
$$\leqslant \int_{\varphi_1(d(x(t),M(t)))}^{\varphi_1(\varepsilon)} \frac{dV}{\psi[\varphi_2^{-1}(V(t,x(t)))]}$$
$$+ \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{dV}{\psi[\varphi_2^{-1}(V(t,x(t)))]}. \tag{5.12.7}$$

Take

$$T = T(\varepsilon, \eta) > \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{dV}{\psi[\varphi_2^{-1}(V(t,x(t)))]}.$$

Obviously, when $t \geqslant t_0 + T$, the following holds:

$$\int_{\varphi_1(d(x(t),M(t)))}^{\varphi_1(\varepsilon)} \frac{dV}{\psi[\varphi_2^{-1}(V(t,x(t)))]} \geqslant t - t_0,$$
$$- \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{dV}{\psi[\varphi_2^{-1}(V(t,x(t)))]} \geqslant t - t_0 - T \geqslant 0.$$

So $\varphi_1(\varepsilon) > \varphi_1(d(x(t), M(t)))$, i.e.,

$$d\big(x(t), M(t)\big) < \varepsilon \quad \text{when } t \geqslant t_0 + T(\varepsilon, \eta).$$

Hence, M is globally, uniformly, asymptotically stable. □

Following the proof of Lagrange stability theorem, one can prove the following theorem.

THEOREM 5.12.7. *If there exists a function*

$$V(t,x) \in C\big[I \times D_H^c, R^1\big], \quad D_H^c := \big\{x \mid d\big(x, M(t)\big) \geqslant H\big\}$$

such that on $I \times D_H^c$ *it holds*:

(1) $\varphi_1(d(x, M(t))) \leqslant V(t,x) \leqslant \varphi_2(d(x, M(t)))$ *for* $\varphi_1, \varphi_2 \in KR$,
(2) $D^+V(t,x)|_{(5.12.1)} \leqslant 0$,

then the set M *is uniformly bounded with respect to system* (5.12.1).

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Chapter 6

Nonlinear Systems with Separate Variables

There is some special interest of nonlinear systems with separated variables in the field of control theory. The reason is that this kind of nonlinear systems is a natural extension of linear systems. For example, if a part of components of a linear system is changed into some appropriate nonlinear ones, the linear system is changed into a nonlinear system with separated variables. Also, some nonlinear systems can be transformed into this kind of systems by using certain invertible transformations. The typical examples can be found in ecological systems, Lurie control systems and some neural networks. Classical research for 2- and 3-dimensional systems was developed along with the Aizeman's conjecture.

In this chapter, we first introduce a linear form of V function method. Then, we present a general nonlinear form of V function method to obtain a general criterion of global stability for autonomous nonlinear systems with separated variables. By this method, we can study global stability of general systems with separated variables and nonautonomous systems with separated variables. Finally, we present some results on the properties of boundedness and dissipation for this kind of systems.

More details for the results presented in this chapter can be found in [228,229, 235,237,248,253,263,350].

6.1. Linear Lyapunov function method

Consider a nonlinear system with separated variables:

$$\frac{dx}{dt} = \left(\sum_{j=1}^{n} f_{1j}(x_j), \dots, \sum_{j=1}^{n} f_{nj}(x_j) \right)^T, \tag{6.1.1}$$

where $f_{ij}(x_j) \in C[R, R]$, $f_{ij}(0) = 0$, $i, j = 1, 2, \dots, n$. Suppose that the solution of the initial value problem (6.1.1) is unique.

We first introduce the results obtained in [228,229]. Let

$$\varphi_i(x_i) = \begin{cases} a_i & \text{when } x_i \geqslant 0, \\ -b_i & \text{when } x_i < 0, \end{cases} \qquad i = 1, 2, \dots, n, \tag{6.1.2}$$

where $a_i > 0$ and $b_i > 0$, $i = 1, 2, \ldots, n$, are constants.

THEOREM 6.1.1. *If there exist functions $\varphi_i(x_i)$, $i = 1, \ldots, n$, such that*

$$\sum_{i=1}^{n} \varphi_i(x_i) f_{ij}(x_j) < 0 \quad (x_j \neq 0,\ j = 1, 2, \ldots, n), \tag{6.1.3}$$

then the zero solution of system (6.1.1) is globally asymptotically stable.

PROOF. We employ the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \varphi_i(x_i) x_i, \tag{6.1.4}$$

which is positive definite and radially unbounded. Consider any nonzero solution

$$x(t) := x\bigl(t, t_0, x^{(0)}\bigr). \tag{6.1.5}$$

Now we prove $V(x(t)) < V(x^{(0)})$ for $t > t_0$, i.e., $V(x)$ is strictly monotone decreasing.

From system (6.1.1), we find that

$$\left.\frac{dx_i(t)}{dt}\right|_{t=t_0} = f_{i1}\bigl(x_1^{(0)}\bigr) + \cdots + f_{in}\bigl(x_n^{(0)}\bigr) := l_i \quad (i = 1, 2, \ldots, n),$$

that is,

$$\lim_{t \to t_0} \frac{x_i(t) - x_i^{(0)}}{t - t_0} = l_i.$$

Thus, $\forall \varepsilon > 0$, there exists $\delta > 0$, when $t_0 \leqslant t \leqslant t_0 + \delta$ we have

$$l_i - \varepsilon < \frac{x_i(t) - x_i^{(0)}}{t - t_0} < l_i + \varepsilon, \tag{6.1.6}$$

which is equivalent to

$$x_i(t) < x_i^{(0)} + (l_i + \varepsilon)(t - t_0), \tag{6.1.7}$$

$$x_i(t) > x_i^{(0)} + (l_i - \varepsilon)(t - t_0). \tag{6.1.8}$$

Then, from $\varphi_i(x_i(t)) = a_i > 0$ for $x_i(t) > 0$, and by (6.1.7), we drive that

$$\varphi_i\bigl(x_i(t)\bigr) x_i(t) < \varphi_i\bigl(x_i(t)\bigr) x_i^{(0)} + (l_i + \varepsilon)(t - t_0) \varphi_i\bigl(x_i(t)\bigr), \tag{6.1.9}$$

and

$$\varphi_i\bigl(x_i(t)\bigr) = -b_i < 0 \quad \text{for } x_i(t) < 0.$$

It follows from (6.1.8) that

$$\varphi_i\big(x_i(t)\big)x_i(t) < \varphi_i\big(x_i(t)\big)x_i^{(0)} + (l_i - \varepsilon)(t - t_0)\varphi_i\big(x_i(t)\big). \tag{6.1.10}$$

Let $\varepsilon_i = \pm\varepsilon$. Combining (6.1.9) and (6.1.10) yields

$$\varphi_i\big(x_i(t)\big)x_i(t) < \varphi_i\big(x_i(t)\big)x_i^{(0)} + (l_i + \varepsilon_i)(t - t_0)\varphi_i\big(x_i(t)\big). \tag{6.1.11}$$

Hence,

$$V\big(x(t)\big) < \sum_{i=1}^{n} \varphi_i\big(x_i(t)\big)x_i^{(0)} + \sum_{i=1}^{n} \varphi_i\big(x_i(t)\big)(l_i + \varepsilon_i)(t - t_0). \tag{6.1.12}$$

Since $x_i(t)$ is continuous with respect to t, when $x^{(0)} \neq 0$, we can choose $\delta > 0$ such that $x_i(t) \neq 0$, and when $t_0 \leqslant t \leqslant t_0 + \delta$, $x_j^{(0)} x_i(t) > 0$, i.e.,

$$\varphi_j\big(x_j(t)\big) = \varphi_j\big(x_j^{(0)}\big).$$

However,

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i\big(x_i(t)\big)l_i &= \sum_{i=1}^{n} \varphi_i\big(x_i(t)\big) \sum_{j=1}^{n} f_{ij}\big(x_j^{(0)}\big) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_i\big(x_i(t)\big) f_{ij}\big(x_j^{(0)}\big) \\
&= \sum_{j=1}^{n}\Big(\varphi_j\big(x_j^{(0)}\big) f_{jj}\big(x_j^{(0)}\big)\Big) + \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i\big(x_i(t)\big) f_{ij}\big(x_j^{(0)}\big) \\
&:= \sum_{j=1}^{n} -k_j\big(x_j^{(0)}\big).
\end{aligned} \tag{6.1.13}$$

As a result, it holds

$$V\big(x(t)\big) < \sum_{i=1}^{n} \varphi_i\big(x_i^{(0)}\big)x_i^{(0)} + (t - t_0)\left\{-\sum_{j=1}^{n} k_j\big(x_j^{(0)}\big) + \sum_{i=1}^{n} \varphi_i\big(x_i(t)\big)\varepsilon_i\right\},$$

where $k_i(x_i^{(0)}) > 0$ is a constant and $0 < \varepsilon_i \ll 1$, $\varphi_i(x_i(t))$ is bounded. So we can choose ε_i such that

$$-\sum_{j=1}^{n} k_j\big(x_j^{(0)}\big) + \sum_{i=1}^{n} \varphi_i\big(x_i(t)\big)\varepsilon_i < 0.$$

Thus, there exists $T > 0$ such that

$$V\big(x(t)\big) < V\big(x^{(0)}\big) \quad (t_0 < t \leqslant T). \tag{6.1.14}$$

Next, we prove that (6.1.14) is true for all $t > t_0$. If otherwise, there exists $t_2 > t_0$ such that

$$V\big(x(t)\big) < V\big(x^{(0)}\big) \quad (t_0 < t < t_2),$$
$$V\big(x(t_2)\big) = V\big(x^{(0)}\big).$$

Owing to that $V(x(t))$ is a continuous function of t, there exists $t_1 \in (t_0, t_2)$ such that

$$V\big(x(t_1)\big) \leqslant V\big(x(t)\big). \tag{6.1.15}$$

Let $x' = x(t_1, t_0, x^{(0)})$, $x(t, t_1, x') = x(t, t_0, x^{(0)})$. Then, we have

$$V\big(x\big(t, t_0, x^{(0)}\big)\big) = V\big(x(t, t_1, x')\big) < V(x'),$$
$$t_1 < t \leqslant t_1 + \delta_1,\ 0 < \delta_1 \ll 1. \tag{6.1.16}$$

Combining (6.1.15) and (6.1.16) leads to a contradiction. Therefore, (6.1.14) holds for all $t > t_0$.

Let

$$\lambda_m := \min_{1 \leqslant i \leqslant n} (a_i, b_i),$$
$$\lambda_M := \max_{1 \leqslant i \leqslant n} (a_i, b_i),$$
$$\|x\| := \sum_{i=1}^{n} |x_i|, \quad \forall \varepsilon > 0.$$

When

$$\big\|x^{(0)}\big\| \leqslant \delta(\varepsilon) := \frac{\lambda_m}{\lambda_M}\varepsilon,$$

we have

$$\big\|x(t)\big\| \leqslant \frac{1}{\lambda_m} V\big(x(t)\big) \leqslant \frac{1}{\lambda_m} V\big(x^{(0)}\big) \leqslant \frac{\lambda_M}{\lambda_m}\big\|x^{(0)}\big\| \leqslant \varepsilon \quad (t \geqslant t_0). \tag{6.1.17}$$

Hence, the zero solution of (6.1.1) is stable.

The limit

$$\lim_{t\to\infty} V\big(x(t)\big) = V_0 \geqslant 0$$

exists. Since the solution has a nonempty ω-limit set, it has an ω-limit orbit $\tilde{x}(t)$ for every fixed t. So we can find $\{t_k\}$, $t_k \to \infty$, such that

$$\lim_{k\to\infty} x(t_k) = \tilde{x}(t)$$

and

$$\lim_{k\to\infty} V\big(x(t_k)\big) = V\big(\tilde{x}(t)\big). \tag{6.1.18}$$

Thus, for all t it holds

$$V\big(\tilde{x}(t)\big) = V_0.$$

As a result, it has been verified that $\tilde{x}(t) \equiv 0$ and $V_0 = 0$. Since the function $V(x(t))$ is strictly monotone decreasing, this implies that

$$\lim_{t\to\infty} x(t) = 0.$$

The proof of Theorem 6.1.1 is complete. □

EXAMPLE 6.1.2. When $n = 2$, system (6.1.1) becomes

$$\begin{cases} \frac{dx_1}{dt} = f_{11}(x_1) + f_{12}(x_2), \\ \frac{dx_2}{dt} = f_{21}(x_1) + f_{22}(x_2). \end{cases}$$

We take

$$\begin{aligned} V(x_1, x_2) &= \varphi_1(x_1)x_1 + \varphi_2(x_2)x_2 \\ &= \begin{cases} a_1x_1 + a_2x_2 = c, & x_1 \geqslant 0,\ x_2 \geqslant 0, \\ -b_1x_1 + a_2x_2 = c, & x_1 < 0,\ x_2 \geqslant 0, \\ -b_1x_1 - b_2x_2 = c, & x_1 < 0,\ x_2 < 0, \\ a_1x_1 - b_2x_2 = c, & x_1 \geqslant 0,\ x_2 < 0. \end{cases} \end{aligned}$$

The geometric interpretation of V is shown as in Figure 6.1.1.

If system (6.1.1) is a linear system, described by

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, 2, \ldots, n, \tag{6.1.1$'$}$$

then we have a stronger result, since the asymptotic stability of a linear system is equivalent to the globally exponentially stability of the system.

COROLLARY 6.1.3. *Consider the linear system* (6.1.1)$'$. *If* $a_{jj} < 0$ ($j = 1, 2, \ldots, n$) *and there exist* $a_j > 0$, $b_j > 0$ ($j = 1, 2, \ldots, n$) *such that*

$$a_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i(x_i)a_{ij} < 0, \tag{6.1.19}$$

$$-b_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i(x_i)a_{ij} > 0, \tag{6.1.20}$$

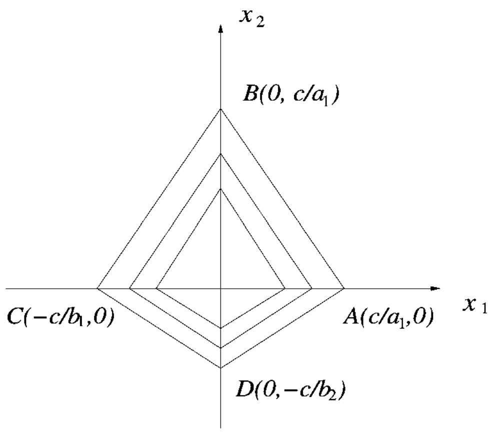

Figure 6.1.1. Linear Lyapunov functions.

then the zero solution of system (6.1.1)′ is globally exponentially stable.

PROOF. By (6.1.19) and (6.1.20), we have

$$a_j a_{jj} x_j + \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i(x_i) a_{ij} x_j < 0 \quad \text{for } x_j > 0,$$

$$-b_j a_{jj} x_j + \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i(x_i) a_{ij} x_j < 0 \quad \text{for } x_j < 0,$$

i.e., when $x_j \neq 0$, it holds

$$\sum_{j=1}^{n} \varphi_i(x_i) a_{ij} x_j < 0 \quad \text{for } j = 1, 2, \ldots, n.$$

Hence, the conditions in Theorem 6.1.1 are satisfied, and the conclusion holds. □

COROLLARY 6.1.4. *Suppose $a_{jj} < 0$, $j = 1, 2, \ldots, n$. Let*

$$a := \max_{j=2,\ldots,n} \left\{ \max_{i=1,2,\ldots,n-1} \left| \frac{n a_{ij}}{a_{jj}} \right|^{\frac{1}{j-i}} \right\} \quad (j > i),$$

$$\frac{1}{b} := \max_{j=1,2,\ldots,n-1} \left\{ \max_{i=2,\ldots,n} \left| \frac{a_{jj}}{n a_{ij}} \right|^{\frac{1}{i-j}} \right\} \quad (i > j).$$

If $b \geqslant a > 0$, then the zero solution of the system (6.1.1)′ is globally exponentially stable.

PROOF. From the conditions it follows that

$$a \geqslant \left|\frac{na_{ij}}{a_{jj}}\right|^{\frac{1}{j-i}}, \quad \text{i.e.,} \quad \frac{a^{j-i}}{n} \geqslant \left|\frac{a_{ij}}{a_{jj}}\right| \quad (j > i,\ j = 2, \dots, n),$$

$$\frac{1}{b} \geqslant \left|\frac{na_{ij}}{a_{jj}}\right|^{\frac{1}{i-j}}, \quad \text{i.e.,} \quad \frac{a^{j-i}}{n} \geqslant \left|\frac{a_{ji}}{a_{jj}}\right| \quad (i > j,\ j = 1, 2, \dots, n-1).$$

Taking $a_i = a^i$, $b_i = a^i$, $i = 1, 2, \dots, n$, we have

$$\begin{aligned}
a_j a_{jj} + \sum_{i \neq j}^{n} \varphi_i(x_i) a_{ij} &\leqslant a^j a_{jj} + \sum_{i=1, i\neq j}^{n} \left|\varphi_i(x_i)\right| |a_{ij}| \\
&= a^j |a_{jj}| \left\{-1 + \sum_{i=1, i\neq j}^{n} \frac{a^i}{a^j} \left|\frac{a_{ij}}{a_{jj}}\right| \right\} \\
&\leqslant a^j |a_{jj}| \left\{-1 + \sum_{i=1}^{j-1} a^{i-j} \frac{a^{j-i}}{n} + \sum_{i=j+1}^{n} a^{i-j} \frac{b^{j-i}}{n} \right\} \\
&\leqslant a^j |a_{jj}| \left\{-1 + \frac{j-1}{n} + \sum_{i=j+1}^{n} a^{i-j} \frac{a^{j-i}}{n} \right\} \\
&= a^j |a_{jj}| \left\{-1 + \frac{j-1}{n} + \frac{n-j}{n} \right\} \\
&= a^j |a_{ij}| \left(\frac{-1}{n}\right) \\
&< 0 \quad \text{for } a \leqslant b,\ i > j,\ a^{i-j} \leqslant b^{i-j}.
\end{aligned}$$

By the same argument, one can prove that

$$-b_j a_{jj} + \sum_{i \neq j}^{n} \varphi_j(x_i) a_{ij} > 0.$$

Thus, the conclusion of Corollary 6.1.4 is true. □

Let

$$|B| := \begin{vmatrix} a_{11} & |a_{21}| & \cdots & |a_{n1}| \\ |a_{12}| & a_{22} & \cdots & |a_{n2}| \\ \vdots & & & \\ |a_{1n}| & |a_{2n}| & \cdots & a_{nn} \end{vmatrix},$$

and $|B_{ij}|$ denote the complement minor of the element (i, j) in $|B|$. Then we have the following result.

COROLLARY 6.1.5. *If $|B||B_{ij}| < 0$, $j = 1, 2, \ldots, n$, then the zero solution of system (6.1.1)′ is globally exponentially stable.*

PROOF. $\forall\delta_1 > 0$, take $0 < \delta_i \ll 1$ $(i = 1, 2, \ldots, n)$ such that

$$\left(-\delta_1|B_{1s}| - \delta_2|B_{2s}| - \cdots - \delta_n|B_{nz}|\right)\left(-\delta_1|B_{1s}|\right) > 0.$$

Consider ξ_i $(i = 1, 2, \ldots, n)$ in the linear equations:

$$\begin{cases} \xi_1 a_{11} + \xi_2|a_{21}| + \cdots + \xi_n|a_{n1}| = -\delta_1, \\ \xi_1|a_{12}| + \xi_2 a_{22} + \cdots + \xi_n|a_{n2}| = -\delta_2, \\ \vdots \\ \xi_1|a_{1n}| + \xi_2|a_{2n}| + \cdots + \xi_n a_{nn} = -\delta_n. \end{cases} \tag{6.1.21}$$

The solution is given by

$$\xi_j = \frac{1}{|B|}\left(-\delta_1|B_{1j}| - \delta_2|B_{2j}| - \cdots - \delta_n|B_{nz}|\right) \quad (j = 1, 2, \ldots, n).$$

By the choice of δ_j $(j = 1, \ldots, n)$ and $|B||B_{ij}| < 0$ $(j = 1, 2, \ldots, n)$, we obtain

$$\xi_j a_{jj} + \sum_{\substack{i \neq j \\ i=1}}^{n} \varphi_i(x_i) a_{ij} \leqslant \xi_j a_{jj} + \sum_{\substack{i \neq j \\ i=1}}^{n} \xi_i|a_{ij}| = -\delta_j < 0.$$

So the conclusion is true. □

REMARK 6.1.6. Corollaries 6.1.3–6.1.5 are only applicable for linear systems. Actually, the conclusion is also true for nonlinear systems. We have the following result.

THEOREM 6.1.7. *If $x_j f_{jj}(x_j) < 0$ for $x_j \neq 0$, $j = 1, 2, \ldots, n$, and there exists a constant $\alpha > 0$ such that*

$$\left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)}\right| \leqslant \frac{a^{j-i}}{n} \quad (i, j = 1, 2, \ldots, n,\ i \neq j),$$

then the zero solution to system (6.1.1) is globally asymptotically stable.

PROOF. Let $a_i = b_i = a^i$, $i = 1, 2, \ldots, n$. When $x_j \neq 0$, we have

$$\sum_{i=1}^{n} \varphi_i(x_i) f_{ij}(x_j) = \varphi_j(x_j) f_{jj}(x_j) + \sum_{\substack{i=1 \\ i \neq j}}^{n} \varphi_i(x_i) f_{ij}(x_j)$$

$$
\begin{aligned}
&= \left|\varphi_j(x_j) f_{jj}(x_j)\right| \left\{ -1 + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\varphi_i(x_i) f_{ij}(x_j)}{|\varphi_j(x_j) f_{jj}(x_j)|} \right\} \\
&\leqslant \left|\varphi_j(x_j) f_{jj}(x_j)\right| \left\{ -1 + \sum_{\substack{i=1\\ i\neq j}}^{n} \left|\frac{\varphi_i(x_i)}{\varphi_j(x_j)}\right| \left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)}\right| \right\} \\
&\leqslant \left|\varphi_j(x_j) f_{jj}(x_j)\right| \left\{ -1 + \sum_{\substack{i=1\\ i\neq j}}^{n} a^{i-j} \frac{a^{j-i}}{n} \right\} \\
&= -n^{-1} \left|\varphi_j(x_j) f_{jj}(x_j)\right| \\
&< 0,
\end{aligned}
$$

which implies that the conditions in Theorem 6.1.1 are satisfied. Thus, the conclusion of Theorem 6.1.7 is true. □

Let

$$
\left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)}\right| \leqslant a_{ij} \quad (i, j = 1, 2, \ldots, n, i \neq j) \text{ for } x_j \neq 0,
$$

$$
|A^*| := \begin{vmatrix} -1 & a_{21} & \cdots & a_{n1} \\ a_{12} & -1 & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & \cdots & & -1 \end{vmatrix},
$$

where $|A^*_{ij}|$ denotes the complement minor of the element (i, j) in $|A^*|$.

THEOREM 6.1.8. *If $x_j f_{jj}(x_j) < 0$, $j = 1, 2, \ldots, n$, and $|A^*||A^*_{ij}| < 0$, $j = 1, 2, \ldots, n$, then the zero solution of system (6.1.1) is globally asymptotically stable.*

PROOF. Following the proof of Corollary 6.1.5, $\forall \delta_1 > 0$, choose $0 < \delta_i < 1$ such that

$$
a_j(-1) + \sum_{\substack{i=1\\ i\neq j}}^{n} a_i a_{ij} = -\delta_j, \quad j = 1, 2, \ldots, n,
$$

where

$$
a_i = \frac{1}{|A^*|}\left(-\delta_1 |A^*_{1i}| - \cdots - \delta_n |A^*_{ni}|\right) > 0, \quad i = 1, 2, \ldots, n.
$$

Take $b_i = a_i$ $(i = 1, 2, \ldots, n)$. Then $x_j \varphi_j(x_j) > 0$, $x_j f_{jj}(x_j) < 0$ and $x_j \neq 0$ imply

$$
\varphi_j(x_j) f_{jj}(x_j) < 0, \quad j = 1, 2, \ldots, n, \ x_j \neq 0.
$$

Thus, we have

$$
\begin{aligned}
\sum_{i=1}^{n} \varphi_i(x_i) f_{ij}(x_j) &= \varphi_j(x_j) f_{jj}(x_j) + \sum_{\substack{i=1\\ i\neq j}}^{n} \varphi_i(x_i) f_{ij}(x_j) \\
&\leqslant \left|\varphi_j(x_j) f_{jj}(x_j)\right| \left\{-1 + \sum_{\substack{i=1\\ i\neq j}}^{n} \left|\frac{\varphi_i(x_i)}{\varphi_j(x_j)}\right| \left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)}\right|\right\} \\
&\leqslant \left|\varphi_j(x_j) f_{jj}(x_j)\right| \left\{-1 + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{a_i}{a_j} a_{ij}\right\} \\
&= \frac{1}{a_j}\left|\varphi_j(x_j) f_{jj}(x_j)\right| \left\{a_j(-1) + \sum_{\substack{i=1\\ i\neq j}}^{n} a_i a_{ij}\right\} \\
&= \frac{1}{a_j}\left|\varphi_j(x_j) f_{jj}(x_j)\right| \{-\delta_j\} < 0 \quad (x_j \neq 0,\ j = 1, 2, \ldots, n),
\end{aligned}
$$

which means that the conditions of Theorem 6.1.1 are satisfied. Hence, the zero solution of system (6.1.1) is globally asymptotically stable. □

Note that the $2n$ constants a_i, b_i $(i = 1, 2, \ldots, n)$ need to be determined, but under some very weak conditions. Only n constants need to be determined. We have

THEOREM 6.1.9. *(See [234,235].) If the sign of $f_{ij}(x_j)$ $(i \neq j)$ is variable, then there exists a function*

$$
\varphi_i(x_i) = \begin{cases} a_i > 0, & x_i \geqslant 0, \\ -b_i < 0, & x_i < 0, \end{cases}
$$

such that

$$
\sum_{i=1}^{n} \varphi_i(x_i) f_{ij}(x_j) < 0, \quad j = 1, 2, \ldots, n \tag{6.1.22}
$$

if and only if there exist constants $c_i > 0$, $i = 1, 2, \ldots, n$, such that

$$
-c_j\left|f_{jj}(x_j)\right| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i\left|f_{ij}(x_j)\right| < 0, \quad j = 1, 2, \ldots, n. \tag{6.1.23}
$$

PROOF. Obviously, (6.1.22) implies (6.1.23) because one can take $b_i = a_i$.

Now suppose (6.1.23) is true. We rewrite (6.1.22) as

$$\varphi_j(x_j)f_{jj}(x_j) + \sum_{\substack{i=1\\i\neq j}}^{n} \varphi_i(x_i)f_{ij}(x_j) < 0. \tag{6.1.24}$$

Since $\varphi_i(x_i)$ $(i \neq j)$ are independent of $f_{ij}(x_j)$, let $a_{i_0} > b_{i_0}$ $(i_0 \neq j)$, then when $\varphi_{i_0}(x_{i_0}) = a_{i_0}$, (6.1.24) becomes

$$\varphi_j(x_j)f_{jj}(x_j) + \sum_{\substack{i=1\\i\neq j\\i\neq i_0}}^{n} \varphi_i(x_i)f_{ij}(x_j) + a_{i_0}f_{i_0 j}(x_j) < 0. \tag{6.1.25}$$

By the property of variable sign of $f_{i_0 j}(x_j)$, it must hold:

$$\varphi_j(x_j)f_{jj}(x_j) + \sum_{\substack{i=1\\i\neq i_0,\, i\neq j}}^{n} \varphi_i(x_i)f_{ij}(x_j) + a_{i_0}\left|f_{i_0,j}(x_j)\right| < 0.$$

For other cases when $i \neq j$, a similar argument yields

$$\varphi_j(x_j)f_{jj}(x_j) + \sum_{\substack{i=1\\i\neq j}}^{n} \max\{a_i, b_i\}\left|f_{ij}(x_j)\right| < 0,$$

i.e.,

$$a_j\varphi_j(x_j)f_{jj}(x_j) + \sum_{\substack{i=1\\i\neq j}}^{n} \max\{a_i, b_i\}\left|f_{ij}(x_j)\right| < 0 \quad (x_j \geqslant 0),$$

$$-b_j\varphi_j(x_j)f_{jj}(x_j) + \sum_{\substack{i=1\\i\neq j}}^{n} \max\{a_i, b_i\}\left|f_{ij}(x_j)\right| < 0 \quad (x_j \leqslant 0).$$

Let

$$c_i = \max\{a_i, b_i\} \quad (i = 1, 2, \ldots, n).$$

Then,

$$\begin{aligned}
&-\max\{a_j, b_j\}\left|f_{jj}(x_j)\right| + \sum_{\substack{i=1\\i\neq j}}^{n} \max\{a_i, b_i\}\left|f_{ij}(x_j)\right| \\
&\quad := -c_j\left|f_{jj}(x_j)\right| + \sum_{\substack{i=1\\i\neq j}}^{n} c_i\left|f_{ij}(x_j)\right| < 0 \quad (j = 1, 2, \ldots, n).
\end{aligned}$$

The proof of Theorem 6.1.9 is complete. □

In the following, we give some sufficient conditions for the existence of $c_i > 0$ $(i = 1, \ldots, n)$ in Theorem 6.1.9.

THEOREM 6.1.10. *Assume that*

(1) $x_j f_{jj}(x_j) < 0$ *for* $x_j \neq 0$;

(2) $\left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)}\right| \leqslant a_{ij}$;

(3) *the matrix*

$$A := \begin{vmatrix} 1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1 & \cdots & a_{2n} \\ \vdots & & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1 \end{vmatrix}$$

is an M matrix.

Then the zero solution of system (6.1.1) is globally asymptotically stable.

PROOF. Since A is an M matrix, A^T is also M matrix. $\forall \xi = (\xi_1, \ldots, \xi_n)^T > 0$, the linear equation for η:

$$A^T \eta = \xi, \quad \eta = (\eta_1, \ldots, \eta_n)^T$$

has positive solution $C = \eta = (A^T)^{-1}\xi > 0$. Take

$$\varphi_i(x_i) = -\operatorname{sign} f_{ii}(x_i) c_i.$$

Then,

$$\begin{aligned} \sum_{i=1}^n \varphi_i(x_i) \sum_{j=1}^n f_{ij}(x_i) &= \sum_{i=1}^n -\operatorname{sign} f_{ii}(x_i) c_i \sum_{j=1}^n f_{ij}(x_j) \\ &\leqslant \sum_{j=1}^n -c_j \left|f_{jj}(x_j)\right| + \sum_{\substack{i=1 \\ i\neq j}}^n c_i \left|f_{ij}(x_j)\right| \\ &\leqslant \sum_{j=1}^n \left[-c_j + \sum_{\substack{i=1 \\ i\neq j}}^n c_i a_{ij}\right] \left|f_{jj}(x_j)\right| \\ &= -\sum_{j=1}^n \xi_i \left|f_{jj}(x_j)\right| \\ &< 0 \quad \text{for } x \neq 0. \end{aligned}$$

So the conclusion holds. □

COROLLARY 6.1.11. *If* $a_{ij} \geqslant 0$ *and one of the following conditions holds, then* A *is an* M *matrix*:

(1) $\displaystyle\sum_{j=1}^{i-1} a_{ij}\mu_j + \sum_{j=i+1}^{n} a_{ij} := \mu_i < 1 \quad (i = 1, 2, \ldots, n);$

(2) $\displaystyle\rho_1 = \max_{2\leqslant j\leqslant n} a_{1j}, \quad \rho_2 = a_{21}\rho_1 + \max_{3\leqslant j\leqslant n} a_{2j}, \quad \ldots,$

$$\rho_n := \sum_{j=1}^{n-1} a_{nj}\rho_j, \quad \sum_{j=1}^{n} \rho_j < 1;$$

(3) $\displaystyle\left[\sum_{j=1}^{i} a_{ij}^2\sigma_j^2 + \sum_{j=i+1}^{n} a_{ij}^2\right] := \sigma_i^2, \quad \sum_{i=1}^{n} \sigma_i^2 < 1;$

(4) $\displaystyle\max_{1\leqslant i\leqslant n} \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} < 1;$

(5) $\displaystyle\max_{1\leqslant j\leqslant n} \sum_{\substack{i=1\\ i\neq j}}^{n} a_{ij} < 1;$

(6) $\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}^2 < 1;$

(7) $\displaystyle h_i := 1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{2}(a_{ij} + a_{ji}) > 0.$

The conclusion of Corollary 6.1.11 is obvious since each of the seven conditions implies the sufficient condition of Theorem 6.1.10.

6.2. General nonlinear Lyapunov function with separable variable

In this section, we generalize the method described in Section 6.1 to consider general nonlinear Lyapunov function with separable variables.

Consider the general nonlinear autonomous system:

$$\frac{dx}{dt} = f(x), \quad f(0) = 0, \tag{6.2.1}$$

where $x \in R^n$, $f(x) \in C[R^n, R^n]$, $f(x) = (f_1(x), \ldots, f_n(x))^T$. Assume that the solution of the initial value problem of (6.2.1) is unique.

DEFINITION 6.2.1. A function $\varphi(x) \in C[R, R]$ is said to have a discontinuous point of the first kind at x_0, if both the right limit

$$\lim_{x\to x_0^+} \varphi(x),$$

and the left limit

$$\lim_{x\to x_0^-} \varphi(x)$$

exist and are bounded.

THEOREM 6.2.2. *If there exist functions $\varphi_i(x) \in [R, R]$, $i = 1, 2, \ldots, n$, which are continuous or have only finite discontinuous points of the first kind, such that*

(1) $\varphi_i(x_i)x_i > 0$ *for* $x_i \neq 0$;
(2) $\int_0^{\pm\infty} \varphi_i(x_i)\,dx_i = +\infty$, $i = 1, 2, \ldots, n$;
(3) $G(x) := \sum_{i=1}^n \varphi_i(x_i \pm 0) f_i(x)$ *is negative definite*;

then the zero solution of system (6.2.1) is globally asymptotically stable.

PROOF. We construct a positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^n \int_0^{x_i} \varphi_i(x_i)\,dx_i. \tag{6.2.2}$$

We now prove that along any nonzero solution $x(t, \tau, \xi)$ of system (6.2.1), $V(x)$ is strictly monotone decreasing, i.e.,

$$V\big(x(t, \tau, \xi)\big) := \psi(t) < \psi(\tau) := V\big(x(\tau, \tau, \xi)\big) \quad \text{for } t > \tau. \tag{6.2.3}$$

First, we show that (6.2.3) holds when $0 < t - \tau \ll 1$. Rewrite (6.2.1) as an integral equation, and then by mean value theorem, we obtain

$$\begin{aligned} x_i(t, \tau, \xi) &= \xi_i + \int_\tau^t f_i\big(x(s, \tau, \xi)\big)\,ds = \xi_i + (t-\tau) f_i\big(x(t_i, \tau, \xi)\big) \\ &= \xi_i + (t-\tau) f_i(\xi) + (t-\tau)\big[f_i\big(x(t_i, \tau, \xi)\big) - f_i(\xi)\big] \end{aligned} \tag{6.2.4}$$

for $t_i \in [\tau, t]$, $i = 1, 2, \ldots, n$. Thus, $\forall \varepsilon_{i1} > 0$, $\varepsilon_{i2} > 0$, by the continuity of $f_i(x)$ and the solution $x(t, \tau, \xi)$, there exists $\delta_i > 0$, $i = 1, \ldots, n$, such that for $0 < t - \tau < \delta_i$ there is no discontinuous point of $\varphi_s(x_s)$ on $[\xi_i, \xi_i + (t -$

$\tau) f_i(x(t_i, \tau, \xi))]$ except ξ_i and at least there is an i such that $f_i(x(t_i, \tau, \xi)) \neq 0$. When $f_i(x(t_i, \tau, \xi)) > 0$, applying the mean value theorem once again yields

$$\int_0^{x_i} \varphi_i(x_i)\,dx_i = \int_0^{\xi_i - 0} \varphi_i(x_i)\,dx_i + \int_{\xi_i+0}^{\xi_i+(t-\tau)[f_i(x(t_i,\tau,\xi))-f_i(\xi)]} \varphi_i(x_i)\,dx_i$$

$$< \int_0^{\xi_i-0} \varphi_i(x_i)\,dx_i + (t-\tau)\big[f_i(\xi)\varphi_i\big(x_i(\bar{t}_i,\tau,\xi)\big) + \varepsilon_{i1}\big|\varphi_i\big(x_i(\bar{t}_i,\tau,\xi)\big)\big|\big]$$

$$= \int_0^{\xi_i-0} \varphi_i(x_i)\,dx_2 + (t-\tau)\big\{f_i(\xi)\big[\varphi_i(\varphi_i \pm 0) + \big(\varphi_i\big(x_i(\bar{t}_i,\tau,\xi)\big)\big)$$

$$- \varphi_i(\xi_i \pm 0)\big]\big\} + (t-\tau)\varepsilon_{i1}\big|\varphi_i\big(x_i(\bar{t}_i,\tau,\xi)\big)\big|$$

$$< \int_0^{\xi_i-0} \varphi_i(x_i)\,dx_i + (t-\tau) f_i(\xi)\varphi_i(\xi_i \pm 0) + (t-\tau)\varepsilon_{i2}\big|f_i(\xi)\big|$$

$$+ (t-\tau)\varepsilon_{i1}\big|\varphi_i\big(x_i(\bar{t}_i,\tau,\xi)\big)\big|, \quad \bar{t}_i \in [\tau, t],\ i = 1, 2, \ldots, n. \tag{6.2.5}$$

When $f_i(x(t_i, \tau, \xi)) < 0$, by using

$$\int_0^{x_i} \varphi_i(x_i)\,dx_i = \int_0^{\xi_i-0} \varphi_i(x_i)\,dx_i + \int_{\varphi_i+0}^{\xi_i+(t-\tau)[f_i(\xi)+f_i(x(t_i,\tau,\xi))-f_i(\xi)]} \varphi_i(x_i)\,dx_i,$$

we have a similar estimation as (6.2.5). Thus, we obtain

$$\begin{aligned}
V\big(x(t,\tau,\xi)\big) &= \sum_{i=1}^n \int_0^{x_i} \varphi_i(x_i)\,dx_i \\
&< \sum_{i=1}^n \int_0^{\xi_i} \varphi_i(x_i)\,dx_i + (t-\tau)\Bigg[\sum_{i=1}^n f_i(\xi)\varphi_i(\xi_i \pm 0) \\
&\quad + \sum_{i=1}^n \varepsilon_{i2}\big|f_i(\xi)\big| + \sum_{i=1}^n \varepsilon_{i1}\big|\varphi_i\big(x_i(\bar{t}_i,\tau,\xi)\big)\big|\Bigg] \\
&= V\big(x(\tau,\tau,\xi)\big) + (t-\tau)\Bigg[\sum_{i=1}^n f_i(\xi)\varphi_i(\xi \pm 0) \\
&\quad + \sum_{i=1}^n \varepsilon_{i2}\big|f_i(\xi)\big| + \sum_{i=1}^n \varepsilon_{i1}\big|\varphi_i\big(x_i(\bar{t}_i,\tau,\xi)\big)\big|\Bigg].
\end{aligned} \tag{6.2.6}$$

By condition (3) and the arbitrary of $\varepsilon_{i1}, \varepsilon_{i2}$, when $0 < t - \tau \ll 1$, (6.2.3) holds.

Second, we prove that (6.2.3) holds for all $t > \tau$. Otherwise, assume that there exists $\tau_1 > \tau$ such that $\psi(\tau_1) = \psi(\tau)$. Since $\psi(t)$ is a continuous function, there exists $t^* \in [\tau, \tau_1]$ such that

$$\psi(t^*) \leqslant \psi(t) \quad \forall t \in [\tau, \tau_1]. \tag{6.2.7}$$

Let $\xi^* = x(t^*, \tau, \xi)$. Then, $x(t, t^*, \xi^*) \equiv x(t, \tau, \xi)$ and it must have

$$\psi(t) = V\big(x(t, t^*, \xi^*)\big) < \psi(t^*) \quad \text{for } 0 < t - t^* \ll 1. \tag{6.2.8}$$

This is a contradiction. So (6.2.3) holds for all $t \geqslant t_0$.

Now by (6.2.3), we prove that the zero solution of system (6.1.1) is stable, and any solution is bounded. In fact, $\forall \varepsilon > 0$, let

$$l = \inf_{\|x\|=\varepsilon} V(x) > 0,$$

one can then choose $0 < \eta < \varepsilon$ such that

$$V_{\|\xi\|\leqslant\eta}\big(x(t, \tau, \xi)\big) < V_{\|\xi\|\leqslant\eta}\big(x(\tau, \tau, \xi)\big) < l \quad (t > \tau). \tag{6.2.9}$$

If there exists $t_1 > \tau$ such that

$$\big\|x(t_1, \tau, \xi)\big\| = \varepsilon,$$

then

$$\big|V\big(x(t_1, \tau, \xi)\big)\big| \geqslant l \tag{6.2.10}$$

Comparing (6.2.9) with (6.2.10) leads to a contradiction. Thus, the zero solution of system (6.2.1) is stable, and all solutions are bounded.

Finally, we prove

$$\lim_{t\to+\infty} x(t, \tau, \xi) = 0 \quad \forall \xi \in R^n.$$

Since $\psi(t)$ is a strictly monotone decreasing function, the limit

$$\lim_{t\to+\infty} V\big(x(t, \tau, \xi)\big) = V_0$$

exists.

The solution $x(t, \tau, \xi)$ has a nonempty ω-limit set. So it has an ω-limit orbit $\tilde{x}(t)$ for every fixed t. We can choose a fixed sequence $\{t_k\}$, $t_k \to +\infty$ as $k \to +\infty$, such that

$$\lim_{k\to+\infty} x(t_k, \tau, \xi) = \tilde{x}(t).$$

Hence,

$$\lim_{k\to+\infty} V\big(x(t_k, \tau, \xi)\big) = V\big(\tilde{x}(t)\big).$$

Thus, $V(\tilde{x}(t)) = V_0$ for all $t > \tau$. Therefore, $\tilde{x}(t) \equiv 0$ and $V_0 = 0$, i.e.,

$$\lim_{t\to+\infty} x(t,\tau,\xi) = 0.$$

The proof of Theorem 6.2.2 is complete. □

REMARK 6.2.3. If

$$f_i(x) = \sum_{j=1}^{n} f_{ij}(x_j)$$

and

$$\varphi_i(x_i) = \begin{cases} a_i, & x_i \geqslant 0, \\ -b_i, & x_i < 0, \end{cases} \quad i = 1, 2, \ldots, n,$$

then we can obtain all results presented in Section 6.1 as particular cases of Theorem 6.2.2.

THEOREM 6.2.4. *Let* $I\delta := \{x \mid \|x\| \leqslant \delta\}$. *If there exist* $\delta_i > 0$ *and functions* $\varphi_i(x_i) \in [I\delta, R]$, *which are continuous or only have finite discontinuous points of the first kind, such that*

(1) $\varphi_i(x_i)x_i \begin{cases} > 0, & i = 1, 2, \ldots, m-1, \\ < 0, & i = m, m+1, \ldots, n, \end{cases} \quad x_i \neq 0;$

(2) $G(x) := \sum_{i-1}^{n} \varphi_i(x_i \pm 0) f_i(x)$ *is negative definite*;

then the zero solution of system (6.2.1) is unstable.

PROOF. For a given $r > 0$ ($0 < r < \delta$) and $\forall\eta > 0$, we prove that one can choose $\xi \neq 0$ ($\|\xi\| \leqslant \eta$) such that the solution $x(t,\tau,\xi)$ of system (6.2.1) at a certain point $t_1 > \tau$ satisfies $\|x(t_1,\tau,\xi)\| \geqslant r$.

In fact, by employing the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} \varphi_i(x_i)\,dx_i \tag{6.2.11}$$

one can find that the set $E = \{x \mid V(x) < 0\}$ is not empty. Take $\xi \in E$ and $\|\xi\| \leqslant \eta$. Using the same method applied in (6.2.5) results in

$$\int_0^{x_i} \varphi_i(x_i)\,dx_i$$

$$
\begin{aligned}
&= \int_0^{\xi_i} \varphi_i(x_i)\,dx_i + \int_{\xi_i \pm 0}^{\xi_i + (t-\tau) f_i(\xi) + (t-\tau)[f_i(x(t_i,\tau,\xi)) - f_i(\xi)]} \varphi_i(x_i)\,dx_i \\
&= \int_0^{\xi_i} \varphi_i(x_i)\,dx_i + (t-\tau)\bigl\{ f_i(\xi)\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr) \\
&\quad + \bigl[f_i\bigl(x(t_i,\tau,\xi)\bigr) - f_i(\xi)\bigr]\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr)\bigr\} \\
&= \int_0^{\xi_i} \varphi_i(x_i)\,dx_i + (t-\tau)\bigl\{ f_i(\xi)\varphi_i(\xi_i \pm 0) + f_i(\xi)\bigl[\bigl(\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr)\bigr) \\
&\quad - \varphi_i(\xi_i \pm 0)\bigr] + \bigl[f_i\bigl(x(t_i,\tau,\xi)\bigr) - f_i(\xi)\bigr]\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr)\bigr\} \\
&\quad \text{for } i = 1,2,\ldots,n,\ t_i \in [\tau,t],\ \tilde{t}_i \in [\tau,t_i].
\end{aligned} \tag{6.2.12}
$$

Then it follows that

$$
\begin{aligned}
V\bigl(x(t,\tau,\xi)\bigr) &= \sum_{i=1}^n \int_0^{x_i} \varphi_i(x_i)\,dx_i \\
&= \sum_{i=1}^n \int_0^{\xi_i} \varphi_i(x_i)\,dx_i + (t-\tau)\left[\sum_{i=1}^n f_i(\xi)\varphi_i(\xi_i \pm 0)\right] \\
&\quad + (t-\tau)\left\{ \sum_{i=1}^n f_i(\xi)\bigl[\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr) - \varphi_i(\xi \pm 0)\bigr] \right. \\
&\quad \left. + \sum_{i=1}^n \bigl[f_i\bigl(x(t_i,\tau,\xi)\bigr) - f_i(\xi)\bigr]\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr) \right\} \\
&= V(\xi) + G(\xi)(t-\tau) \\
&\quad + (t-\tau)\left\{ \sum_{i=1}^n f_i(\xi)\bigl[\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr) - \varphi_i(\xi \pm 0)\bigr] \right. \\
&\quad \left. + \sum_{i=1}^n \bigl[f_i\bigl(x(t_i,\tau,\xi)\bigr) - f_i(\xi)\bigr]\varphi_i\bigl(x_i(\bar{t}_i,\tau,\xi)\bigr) \right\}.
\end{aligned} \tag{6.2.13}
$$

Let

$$
\mu = \inf_{x \in [x, V(x) \leqslant V(\xi) < 0]} \|x\|, \qquad \lambda = \inf_{\mu \leqslant \|x\| \leqslant H} \bigl|G(x)\bigr|.
$$

Then $\mu > 0$ and $\lambda > 0$. Note that

$$M_1 = \max_{\substack{1\leqslant i\leqslant n\\ \|x\|\leqslant H}} \left|f_i(x)\right|, \qquad M_2 = \sup_{\substack{1\leqslant i\leqslant n\\ |x_i|\leqslant\delta_i}} \left|\varphi\left(x_i(\bar{t}_i,\tau,s)\right)\right|,$$

and choose $0 < \delta \ll 1$ such that when $0 < t-\tau < \delta$, the conditions are satisfied:

$$\left|\varphi_i\left(x_i(\bar{t}_i,\tau,\xi)\right) - \varphi_i(\xi_1 \pm 0)\right| < \frac{\lambda}{3nM_1},$$
$$\left|f_i\left(x(t_i,\tau,\xi,\xi)\right) - f_i(\xi)\right| < \frac{\lambda}{3nM_2}.$$

Then by (6.2.13) we obtain

$$\begin{aligned} V\left(x(t,\tau,\xi)\right) &< V\left(x(\tau,\tau,\xi)\right) - (t-\tau)\lambda + \left(\frac{nM_1\lambda}{3nM_1} + \frac{nM_2\lambda}{3nM_2}\right)(t-\tau) \\ &= V\left(x(\tau,\tau,\xi)\right) - (t-\tau)\frac{\lambda}{3}. \end{aligned} \tag{6.2.14}$$

Choose $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ and $0 < t_i - t_{i-1} \ll 1$. Then it follows from (6.2.14) that

$$V\left(x(t_1,\tau,\xi)\right) < V\left(x(\tau,\tau,\xi)\right) - (t_1-\tau)\frac{\lambda}{3}, \tag*{$(6.2.15)_1$}$$

$$V\left(x(t_2,\tau,\xi)\right) < V\left(x(t_1,\tau,\xi)\right) - (t_2-t_1)\frac{\lambda}{3}, \tag*{$(6.2.15)_2$}$$

$$\vdots$$

$$V\left(x(t_n,\tau,\xi)\right) < V\left(x(t_{n-1},\tau,\xi)\right) - (t_n-t_{n-1})\frac{\lambda}{3}. \tag*{$(6.2.15)_n$}$$

From $(6.2.15)_1$–$(6.2.15)_n$ we get

$$V\left(x(t_n,\tau,\xi)\right) < V\left(x(\tau,\tau,\xi)\right) - (t_n-\tau)\frac{\lambda}{3}.$$

Since $t_n - \tau \gg 1$, $V(x(\tau,\tau,\xi)) = \text{const.}$, $|V(x(t_n,\tau,\xi))| \gg 1$, we have $\|x(t_n,\tau,\xi)\| \geqslant r$, i.e., the solution of (6.2.1) is unstable.

The proof is complete. □

As applications of Theorems 6.2.2 and 6.2.4, we again consider system (6.2.1).

THEOREM 6.2.5. *If system* (6.2.1) *satisfies the following conditions*:

(1) $f_{ii}(x_i)x_i < 0$ *for* $x_i \neq 0$;

(2) $\int_0^{\pm\infty} f_{ii}(x_i)\,dx_i = -\infty$;

(3) *there exist positive definite functions* $c_i(x_i) \in C[R^1, R^+]$, $i = 1, 2, \ldots, n$, *such that the matrix* $A(a_{ij}(x))$ *is negative definite*;

then the zero solution of system (6.2.1) *is globally asymptotically stable. Here,*

$$a_{ij}(x) = \begin{cases} -c_i(x_i), & i = j = 1, 2, \ldots, n, \\ -\frac{1}{2}\left(\frac{c_i(x_i) f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{c_j(x_j) f_{ji}(x_i)}{f_{ii}(x_i)}\right), & i \neq j,\ x_i x_j \neq 0, \\ & \quad i, j = 1, 2, \ldots, n. \end{cases}$$

PROOF. Let $\varphi_i(x_i) = -c_i(x_i) f_{ii}(x_i)$ $(i = 1, 2, \ldots, n)$. We prove that

$$G(t) := \sum_{i=1}^{n} \varphi_i(x_i) \sum_{j=1}^{n} f_{ij}(x_j)$$

is negative definite. $\forall x = \xi$, without loss the generality, let

$$\prod_{i=1}^{k} \xi_i \neq 0, \qquad \sum_{i=k+1}^{n} \xi_i^2 = 0 \quad (1 \leqslant k \leqslant n).$$

Then,

$$\begin{aligned} G(\xi) = {} & \sum_{i=1}^{k} \varphi_i(\xi_i) \sum_{j=1}^{k} f_{ij}(\xi_j) = -\sum_{i=1}^{k} c_i(\xi_i) f_{ii}^2(\xi_i) \\ & + \sum_{\substack{i,j=1 \\ i\neq j}}^{k} \left[\frac{-c_i(\xi_i) f_{ii}(\xi_i) f_{ij}(\xi_j)}{2} + \frac{-c_j(\xi_j) f_{jj}(\xi_j) f_{ji}(\xi_i)}{2}\right] \\ = {} & -\sum_{i=1}^{k} c_i(\xi_i) f_{ii}^2(\xi_i) \\ & - \sum_{\substack{i=1 \\ i\neq j}}^{k} \sum_{\substack{i=1 \\ i\neq j}}^{k} \left(\frac{c_i(\xi_i) f_{ij}(\xi_i)}{2 f_{jj}(\xi_j)} + \frac{c_j(\xi_j) f_{ji}(\xi_i)}{2 f_{ii}(\xi_i)}\right) f_{ii}(\xi_i) f_{jj}(\xi_j). \end{aligned}$$

By condition (3) we know that the general quadratic form

$$\begin{aligned} W(f_{11}, f_{22}, \ldots, f_{kk}) = {} & -\sum_{i=1}^{k} c_i(\xi_i) f_{ii}^2(x_i) \\ & + \sum_{\substack{i=1 \\ i\neq j}}^{k} \sum_{\substack{j=1 \\ i\neq j}}^{k} a_{ij}(\xi_i, \xi_j) f_{ii}(x_i) f_{jj}(x_j) \end{aligned}$$

is negative definite with respect to $f_{11}, f_{22}, \dots, f_{kk}$. By condition (1) and $f_{ii}(0) = 0$ $(i = 1, 2, \dots, n)$ we know that $W(0) = 0$, and

$$\sum_{i=1}^{k} f_{ii}^2 = 0.$$

By noticing that

$$W(f_{11}, \dots, f_{kk}) < 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} f_{ii}^2 \neq 0$$

and

$$\sum_{i=1}^{k} f_{ii}^2 = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} x_i^2 = 0,$$

W is negative definite with respect to $x_1, x_2, \dots, x_k$, and in particular, $G(\xi) = W(f_{11}(\xi_1), \dots, f_{kk}(\xi_k)) < 0$. Due to ξ being arbitrary and $G(0) = 0$, we know that $G(x)$ is negative definite. Hence, the zero solution of system (6.2.1) is globally asymptotically stable. $\square$

REMARK 6.2.6. If the elements of $A(a_{ij}(x))$ are changed to

$$a_{ij}(x) \begin{cases} = -c_i(x_i), & i = j, \\ \geqslant \frac{1}{2}\left|\frac{c_i(x_i) f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{c_j(x_j) f_{ji}(x_i)}{f_{ii}(x_i)}\right|, & i \neq j,\ x_i, x_j \neq 0, \end{cases}$$

then the conditions of Theorem 6.2.5 are still satisfied, and the conclusion holds. Thus, we can choose easier computing method, for example, we can use $a_{ij}(t) =$ constant $i \neq j$ to check the negative definiteness of matrix $A(a_{ij}(t))$ easier.

COROLLARY 6.2.7. *If the conditions* (1) *and* (2) *in Theorem* 6.2.2 *are satisfied, and in condition* (3) *there exist functions* $c_i(x_i) > \delta > 0$, $c_i(x_i) \in C[R, R]$, $i = 1, 2, \dots, n$, *and* $M_i^{(j)}(x_i, x_j)$, $M_j^{(i)}(x_i, x_j) \geqslant 0$ *such that*

$$\left|\frac{c_i(x_i) f_{ij}(x_i)}{2 f_{jj}(x_j)} + \frac{c_j(x_j) f_{ji}(x_i)}{2 f_{ii}(x_i)}\right| \leqslant M_i^{(j)}(x_i, x_j) M_j^{(i)}(x_i, x_j),$$

for $i \neq j$, $i, j = 1, 2, \dots, n$, $x_i, x_j \neq 0$, *and*

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(M_i^{(j)}(x_i, x_j)\right)^2 < c_i(x_i) \quad (i = 1, 2, \dots, n);$$

then the zero solution of system (6.2.1) *is globally asymptotically stable.*

PROOF. Let $\varphi_i(x_i) = -c_i(x_i)f_{ii}(x_i)$ $(i = 1, 2, \dots, n)$. $\forall x = \xi$, where

$$\prod_{i=1}^{k}\xi_i \neq 0, \quad \sum_{i=k+1}^{n}\xi_i^2 = 0 \quad (1 \leqslant k \leqslant n),$$

we have

$$\begin{aligned}
G(\xi) &= \sum_{i=1}^{k}\varphi_i(\xi_i)\sum_{j=1}^{k}f_{ij}(\xi_j) = -\sum_{i=1}^{k}c_i(\xi_i)f_{ii}(\xi_i)\sum_{j=1}^{k}f_{ij}(\xi_j) \\
&\leqslant -\sum_{i=1}^{k}c_i(\xi_i)f_{ii}^2(\xi_i) \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\left|\frac{-c_i(\xi_i)f_{ij}(\xi_j)}{2f_{jj}(\xi_j)} + \frac{-c_j(\xi_j)f_{ji}(\xi_i)}{2f_{ii}(\xi_i)}\right|\left|f_{ii}(\xi_i)f_{jj}(\xi_j)\right| \\
&\leqslant -\sum_{i=1}^{k}c_i(\xi_i)f_{ii}^2(\xi_i) \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\left|\frac{-c_i(\xi_i)f_{ij}(\xi_j)}{2f_{jj}(\xi_j)} + \frac{-c_j(\xi_j)f_{ji}(\xi_i)}{2f_{ii}(\xi_i)}\right|\left|f_{ii}(\xi_i)f_{jj}(\xi_j)\right| \\
&\leqslant -\sum_{i=1}^{k}c_i(\xi_i)f_{ii}^2(\xi_i) \\
&\quad + \sum_{\substack{i=1\\ i\neq j}}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}M_i^{(i)}(\xi_i,\xi_j)M_j^{(i)}(\xi_j,\xi_i)\left|f_{ii}(\xi_i)\right|\left|f_{jj}(\xi_j)\right| \\
&\leqslant -\sum_{i=1}^{k}c_i(\xi_i)f_{ii}^2(\xi_i) + \sum_{\substack{i=1\\ i\neq j}}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\left(M_i^{(j)}(\xi_i,\xi_j)\right)^2 f_{jj}^2(\xi_i) \\
&= \sum_{j=1}^{k}\left[-c_j(\xi_j) + \sum_{\substack{j=1\\ j\neq i}}^{k}M_i^{(j)}(\xi_i,\xi_j)^2\right]f_{jj}^2(\xi_j) < 0.
\end{aligned}$$

By the arbitrary of ξ, we know that $G(t)$ is negative definite. So the zero solution of system (6.2.1) is globally asymptotically stable. $\square$

REMARK 6.2.8. If take $c_i(x_1) = 1$, then

$$M_i^{(j)}(x_i, x_j) = M_j^{(i)}(x_j, x_i) = \sqrt{\frac{1}{2}\left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{f_{ji}(x_i)}{f_{ii}(x_i)}\right|},$$

and condition (3) of Corollary 6.2.7 becomes

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{f_{ji}(x_i)}{f_{ii}(x_i)}\right| < 2, \quad x_i x_j \neq 0.$$

THEOREM 6.2.9. *If in certain neighborhood of $x = 0$, system (6.1.1) satisfies*

(1)

$$f_{ii}(x_i)x_i < 0, \quad x_i \neq 0,\ i = 1, 2, \dots, m-1,$$
$$f_{ii}(x_i)x_i > 0, \quad x_i \neq 0,\ i = m, m+1, \dots, n;$$

(2) *there exist $c_i(x_i) \geqslant \delta > 0$ such that the matrix $A(a_{ij}(x))$ is negative definite, where*

$$a_{ij}(x) = \begin{cases} -c_i(x_i), & i = j = 1, 2, \dots, n \\ -\frac{1}{2}\left[\frac{c_i(x_i)f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{c_j(x_j)f_{ji}(x_i)}{f_{ii}(x_i)}\right], & i \neq j,\ x_i x_j \neq 0, \\ & \quad j = 1, 2, \dots, n; \end{cases}$$

then the zero solution of system (6.1.1) is unstable.

PROOF. One can follow the proof of Theorem 6.2.5 to prove this theorem. □

6.3. Systems which can be transformed to separable variable systems

In this section, we consider some systems which can be transformed to separable variables systems.

Consider the system:

$$\frac{dx_i}{dt} = F_i\left(\sum_{j=1}^{n} a_{ij}x_j\right), \quad i = 1, 2, \dots, n, \tag{6.3.1}$$

where $F_i(0) = 0$, $y_i F_i(y_i) > 0$ for $y_i \neq 0$ and $F_i(y_i) \in C[R^1, R^1]$. Assume that the solution of the Cauchy problem (6.3.1) is unique.

THEOREM 6.3.1. *If system (6.3.1) satisfies the following conditions*:

(1) $\int_0^{\pm\infty} F_j(y_j) = +\infty$;
(2) $a_{jj} < 0$, $j = 1, 2, \ldots, n$;
(3) *the matrix A is Lyapunov–Volterra stable, i.e., there exist constants $c_i > 0$, $i = 1, \ldots, n$, such that the matrix $B(b_{ij})_{n\times n}$ is negative definite, where*

$$b_{ij} = \begin{cases} c_i |a_{ii}|, & i = j = 1, 2, \ldots, n, \\ \frac{1}{2}[c_i a_{ij} + c_j a_{ji}], & i \neq j,\ i, j = 1, 2, \ldots, n; \end{cases}$$

then the zero solution of system (6.3.1) is globally asymptotically stable.

PROOF. Let us consider an auxiliary linear system:

$$\frac{dz_i}{dt} = \sum_{j=1}^{n} a_{ij} z_j, \quad i = 1, 2, \ldots, n. \tag{6.3.2}$$

By employing the positive definite and radially unbounded Lyapunov function

$$V(z) = \sum_{i=1}^{n} \int_0^{z_i} c_i z_i \, dz_i,$$

we obtain

$$\frac{dV(z)}{dt} = \sum_{i=1}^{n} -c_i |a_{ii}| z_i^2 + \sum_{\substack{i=1 \\ i\neq j}}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} \frac{1}{2}(c_i a_{ij} + c_j a_{ji}) z_i z_j < 0 \quad \text{for } z \neq 0,$$

showing that the zero solution of system (6.3.2) is globally asymptotically stable, and so $\det |A| \neq 0$.

By a nonsingular linear transformation:

$$y_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, n,$$

system (6.3.1) is transformed to

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} \frac{dx_j}{dt} = \sum_{j=1}^{n} a_{ij} F_j(y_j). \tag{6.3.3}$$

The stability of the zero solutions of systems (6.3.1) and (6.3.3) are equivalent, but system (6.3.3) has separable variables.

Let $\varphi_i(y_i) = c_i F_i(y_i)$. Then

$$G(y) := \sum_{i=1}^{n} \varphi_i(y_i) \sum_{j=1}^{n} a_{ij} F_j(y_j)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} c_i F_i(y_i) \sum_{j=1}^{n} a_{ij} F_j(y_j) \\
&= \sum_{i=1}^{n} c_i a_{ii} F_i^2(y_i) + \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} (c_i a_{ij} + c_j a_{ji}) F_i(y_i) F_j(y_j) \\
&= F^T(y) B F(y) < 0 \quad \text{for } F(y) \neq 0.
\end{aligned}
$$

By condition (3) and $y_m F(y_m) > 0$, $F_m(0) = 0$, we can apply the same method to prove that $G(y)$ is negative definite with respect to y.

Thus, the zero solution of system (6.3.1) is globally asymptotically stable. □

THEOREM 6.3.2. *Assume that*

(1) $\int_0^{\pm\infty} F_i(y_i)\, dy_i = +\infty$ $(i = 1, 2, \ldots, n)$;

(2)

$$
\begin{bmatrix}
-a_{11} & |a_{12}| & \cdots & |a_{1n}| \\
-|a_{12}| & -a_{22} & \cdots & |a_{2n}| \\
\vdots & & & \vdots \\
-|a_{nn}| & \cdots & & -a_{nn}
\end{bmatrix}
$$

is an M matrix.

Then the zero solution of system (6.3.1) *is globally asymptotically stable.*

PROOF. This conclusion is true simply because condition (2) implies the conditions (2) and (3) in Theorem 6.3.1. □

THEOREM 6.3.3. *If* $\det A \neq 0$, $a_{ii} \neq 0$, $i = 1, 2, \ldots, n$, *and there exist* $a_{i_0 i_0} > 0$, $c_i \neq 0$ $(i = 1, \ldots, n)$, $c_{i_0} > 0$ *such that the matrix* $\tilde{B}(\tilde{b}_{ij})$ *is negative definite, where*

$$
\tilde{b}_{ij} = \tilde{b}_{ji} = \begin{cases}
-c_i (\operatorname{sign} a_{ii}) a_{ii}, & i = j = 1, 2, \ldots, n, \\
-\frac{1}{2}\big[c_i (\operatorname{sign} a_{ii}) a_{ij} + c_j (\operatorname{sign} a_{jj}) a_{ji}\big], \\
\quad i \neq j,\ i, j = 1, \ldots, n,
\end{cases}
$$

then the zero solution of system (6.3.1) *is unstable.*

PROOF. Let $\varphi_i(y_i) = -c_i(\operatorname{sign} a_{ii}) F_i(y_i)$. Then,

$$
G(y) = \sum_{i=1}^{n} \varphi_i(y_i) \sum_{j=1}^{n} a_{ij} F_j(y_j)
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} c_i (\operatorname{sign} a_{ii}) F_i(y_i) \sum_{j=1}^{n} a_{ij} F_j(y_j) \\
&= -\sum_{i=1}^{n} c_i (\operatorname{sign} a_{ii}) a_{ii} F_i^2(y_j) \\
&\quad - \frac{1}{2} \sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{\substack{j=1\\ i\neq j}}^{n} \big[c_i (\operatorname{sign} a_{ii}) a_{ij} + c_j (\operatorname{sign} a_{jj}) a_{ji}\big] F_i(y_i) F_j(y_j) \\
&= -\sum_{i=1}^{n} c_i (\operatorname{sign} a_{ii}) a_{ii} F_i^2(y_i) + \sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{\substack{j=1\\ i\neq j}}^{n} \tilde{b}_{ij} F_i(y_i) F_j(y_j) \\
&< 0 \quad \text{for } y \neq 0.
\end{aligned}
$$

So the conclusion is true. □

THEOREM 6.3.4. *If there exist $a_{i_0 i_0} > 0$, and*

$$
\begin{bmatrix}
|a_{11}| & -|a_{12}| & \cdots & -|a_{1n}| \\
-|a_{21}| & |a_{22}| & & -|a_{2n}| \\
\vdots & & \ddots & \vdots \\
-|a_{nn}| & \cdots & \cdots & |a_{nn}|
\end{bmatrix}
$$

being an M matrix, then the zero solution of system (6.3.1) is unstable.

PROOF. One can follow Theorem 6.3.3 to prove this theorem. □

In the following, we consider the Volterra-type ecological systems, described by

$$
\frac{dx_i}{dt} = x_i \left(r_i + \sum_{j=1}^{n} a_{ij} x_j \right), \quad i = 1, 2, \ldots, n. \tag{6.3.4}
$$

This is the earliest mathematical model that describes the ecological system with n kinds of species, including animals, plants and microbes as nourishing elements.

Here, x_i is the amount (or density, or some other character) of the ith species, $\frac{dx_i}{dt}$ is the increasing rate of the whole ith species, and $\frac{dr_i}{dt}$ is the relative increasing rate (r_i is the inner increasing rate of x_i). If others do not exists, $r_i > 0$ and $r_i < 0$ denote the increasing rate and the mortality of the species, respectively. a_{ii} refers to the ith species's own increasing rate being curbed ($a_{ii} < 0$) or helped ($a_{ii} > 0$), or not influenced ($a_{ii} = 0$) by the limited food and the restrict environment; a_{ij} refers to the interaction between x_i and x_j, $a_{ij} > 0$, $a_{ij} = 0$ and $a_{ij} < 0$ mean that the jth species is helpful to, has no relation to, or retains the ith species.

According to the ecological meaning, the nonnegative conditions of the species, we confine $x_i \geqslant 0$, $i = 1, 2, \ldots, n$

Let $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $x_i = x_i^* > 0$, $i = 1, 2, \ldots, n$, denotes an equilibrium point. By the topological transformation

$$y_i = l_n \frac{x_i}{x_i^*}, \quad \text{i.e.,} \quad x_i = x_i^* e^{y_i} \quad (i = 1, 2, \ldots, n), \tag{6.3.5}$$

we have $y : R_+^n := \{x \mid x > 0\} \to R_y^n$, i.e., the transformation (6.3.5) transforms R_+^n to $R_y^n = \{y \mid y \in R^n\}$, and thus system (6.3.4) becomes

$$\begin{aligned}\frac{dy_i}{dt} &= r_i + \sum_{j=1}^n a_{ij} x_j = -\sum_{j=1}^n a_{ij} x_j^* + \sum_{j=1}^n x_j^* a_{ij} e^{y_j} \\ &= \sum_{j=1}^n a_{ij} x_j^* \big(e^{y_j} - 1\big) := \sum_{j=1}^n a_{ij} f_j(y_j),\end{aligned} \tag{6.3.6}$$

where $f_j(y_j) = x_j^*(e^{y_j} - 1)$. Obviously, the equilibrium point $x = x^*$ of (6.3.4) is globally asymptotically stable if and only if the zero solution of (6.3.6) is globally asymptotically stable. However, (6.3.6) is a system with separable variables.

THEOREM 6.3.5. *If matrix A is Lyapunov–Volterra stable, then the zero solution of* (6.3.6) *is globally asymptotically stable. Hence, the equilibrium point $x = x^*$ is globally asymptotically stable in R_+^n.*

PROOF. The condition implies that there exists a positive definite matrix $P = \operatorname{diag}(p_1, p_2, \ldots, p_n)$ such that

$$PA + A^T P$$

is negative definite.

By choosing the Lyapunov function:

$$V(y) = \sum_{i=1}^n p_i \int_0^{y_i} f_i(y_i)\, dy_i = \sum_{i=1}^n p_i x_i^* \big(e^{y_i} - y_i - 1\big), \tag{6.3.7}$$

we have

$$\begin{aligned}\left.\frac{dV(y)}{dt}\right|_{(6.3.6)} &= \sum_{i=1}^n \sum_{j=1}^n p_i a_{ij} f_i(y_i) f_j(y_j) \\ &= \frac{1}{2}\big(f_1(y_1), \ldots, f_n(y_n)\big)\big(PA + A^T P\big)\big(f_1(y_1), \ldots, f_n(y_n)\big)^T \\ &< 0 \quad \text{for } f(y) \neq 0,\end{aligned}$$

implying that the conclusion holds. $\square$

THEOREM 6.3.6. *If system (6.3.4) has a positive equilibrium point $x = x^*$ and there exists an M matrix $-G(g_{ij})$ such that*

$$a_{ii} \leqslant g_{ii}, \quad |a_{ij}| \leqslant g_{ij} \quad (i \neq j,\ i, j = 1, 2, \dots, n),$$

then the equilibrium point $x = x^*$ of (6.3.4) is globally asymptotically stable in R^n_+.

PROOF. Since $-G(g_{ij})$ is an M matrix, $g_{ii} < 0$ and $-G^T$ is also an M matrix. Thus, $\forall (\xi_1, \dots, \xi_n)^T > 0$, the equation $-G^T P = \xi$ has a positive solution

$$P = \big(p_1, p_2, \dots, p_n\big)^T = \big(-G^T\big)^{-1}\xi > 0.$$

Choose the positive definite and radially unbounded Lyapunov function:

$$V(y) = \sum_{i=1}^{n} p_i |y_i|$$

to obtain

$$\begin{aligned} D^+V(y)|_{(6.3.6)} &\leqslant \sum_{i=1}^{n} p_i \operatorname{sign} y_i \sum_{j=1}^{n} a_{ij} f_j(y_j) \\ &\leqslant \sum_{j=1}^{n}\Bigg(p_j a_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} p_i |a_{ij}|\Bigg)\big|f_j(y_j)\big| \\ &\leqslant \sum_{j=1}^{n}\Bigg(p_j g_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} p_i g_{ij}\Bigg)\big|f_j(y_j)\big| \\ &\leqslant -\sum_{j=1}^{n} \xi_j \big|f_j(y_j)\big| \\ &< 0 \quad \text{for } f(y) \neq 0. \end{aligned}$$

So the conclusion is true. □

6.4. Partial variable stability for systems with separable variables

In this section, we preset partial variable stability of systems with separable variables. Let $y = (x_1, \dots, x_m)^T$, $z = (x_{m+1}, \dots, x_n)^T$. Then system (6.2.1) is reduced to

$$\begin{cases} \frac{dy}{dt} = \Big(\sum_{j=1}^{n} f_{1j}(x_j), \dots, \sum_{j=1}^{n} f_{mj}(x_j)\Big)^T, \\ \frac{dz}{dt} = \Big(\sum_{j=1}^{n} f_{m+1,j}(x_j), \dots, \sum_{j=1}^{n} f_{nj}(x_j)\Big)^T. \end{cases} \tag{6.4.1}$$

Similar to the discussion in the Sylvester's condition, we first establish a criterion for positive definiteness and negative definiteness of quadratic forms with respect to partial variables.

DEFINITION 6.4.1. The quadratic form

$$\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

is said to be positive (negative) definite with respect to y if there are constants $\varepsilon_i > 0$ $(i = 1, \ldots, m)$ such that

$$x^T A x \geqslant \sum_{i=1}^{m} \varepsilon_i x_i^2 \quad \left(x^T A x \leqslant -\sum_{i=1}^{m} \varepsilon_i x_i^2 \right),$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

LEMMA 6.4.2. *The quadratic form*

$$\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

is positive (*negative*) *definite with respect to variable* y *if and only if there exists a constant* $\varepsilon > 0$ *such that*

$$\begin{bmatrix} A_{11} - \varepsilon I_m & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ is positive semi-definite,}$$

$$\left(\begin{bmatrix} A_{11} + \varepsilon I_m & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ is positive semi-definite} \right),$$

where I_m *is an* $m \times m$ *unit matrix.*

PROOF. For an illustration, we prove the case of positive definite. The proof for other cases is similar and thus omitted.

Necessity. Since

$$\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

is positive definite with respect to variable y, there exist some constants $\varepsilon_i > 0$ $(i = 1, \ldots, m)$ such that

$$\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant \sum_{i=1}^{m} \varepsilon_i x_i^2. \tag{6.4.2}$$

Let $\varepsilon = \min_{1 \leqslant i \leqslant m} \varepsilon_i$. Then we can find

$$\begin{aligned}\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} &\geqslant \sum_{i=1}^{m} \varepsilon_i x_i^2 \geqslant \varepsilon \sum_{i=1}^{n} x_i^2 \\ &= \begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} \varepsilon I_m & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \end{aligned} \tag{6.4.3}$$

Thus,

$$\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} - \varepsilon I_m & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant 0 \quad \text{and} \quad \begin{bmatrix} A_{11} - \varepsilon I_m & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is positive semi-definite. In particular, A_{11} is positive definite.

Sufficiency. The assumptions can be reduced to

$$\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} - \varepsilon I_m & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant 0.$$

Thus, we have

$$\begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant \sum_{i=1}^{m} \varepsilon x_i^2.$$

This implies our claim. □

LEMMA 6.4.3. *If there exist functions $\phi_i(x_i)$ $(i = 1, \ldots, n)$ on $(-\infty, +\infty)$, which are continuous or have only finite discontinuous points of the first kind, such that*

(1) $\phi_i(x_i > 0)$ *for* $x_i \neq 0,\ i = 1, \ldots, m,\ \phi_i(x_i)x_i \geqslant 0,\ i = m+1, \ldots, n;$
(2) $\int_0^{\pm\infty} \phi_i(x_i)\, dx_i = +\infty,\ i = 1, \ldots, m;$
(3) *there is a positive definite function* $\psi(y)$ *satisfying*

$$G(x) = \sum_{i=1}^{n} \phi_i(x_i \pm 0) \sum_{j=1}^{n} f_{ij}(x_j) \leqslant -\psi(y);$$

then the zero solution of system (6.4.1) is globally stable with respect to variable y.

PROOF. Construct the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} \phi_i(x_i)\, dx_i. \tag{6.4.4}$$

Obviously, the conditions (1) and (2) imply that

$$V(x) \geqslant \sum_{i=1}^{m} \int_{0}^{x_i} \phi_i(x_i)\,dx_i = \varphi(y) \to +\infty \quad \text{as } \|y\| \to +\infty.$$

Hence, $V(x)$ is positive definite and radially unbounded with respect to y; and along the solution of (6.4.1), the Dini derivative of $V(x)$ is

$$D^+V(x)|_{(6.4.1)} = \begin{cases} \sum_{i=1}^n \phi_i(x_i) \sum_{j=1}^n f_{ij}(x_j) \\ \quad \text{at the continuous points of } \phi_i(x_i),\ i = 1, \dots, n, \\ \max\Big\{\sum_{i=1}^n \phi_i(x_i+0) \sum_{j=1}^n f_{ij}(x_j), \sum_{i=1}^n \phi_i(x_i-0) \sum_{j=1}^n f_{ij}(x_j)\Big\} \\ \quad \text{at the discontinuous points of } \phi_i(x_i),\ i = 1, \dots, n. \end{cases}$$

Hence, condition (3) implies that

$$D^+V(x)|_{(6.4.1)} \leqslant -\psi\big(\|y\|\big).$$

As a result, the zero solution of system (6.4.1) is globally asymptotically stable with respect to variable y. □

REMARK 6.4.4. In case $m = n$, conditions in Lemma 6.4.3 imply that the zero solution of system (6.4.1) is globally stable for all variables. In Theorem 6.4.5 below, when $m = n$, the statement follows from global stability of all variables.

THEOREM 6.4.5. *If system (6.4.2) satisfies*

(1) $f_{ii}(x_i)x_i < 0$ *for* $x_i \neq 0,\ i = 1, \dots, m,\ f_{ii}(x_i)x_i \leqslant 0,\ i = m+1, \dots, n,$ *and*

$$\int_0^{\pm\infty} f_{ii}(x_i)\,dx_i = -\infty, \quad i = 1, \dots, m;$$

(2) *there are constants* $c_i > 0$ $(i = 1, \dots, m)$, $c_j \geqslant 0$ $(j = m+1, \dots, n)$, $\varepsilon > 0$ *such that*

$$A\big(a_{ij}(x)\big)_{n\times n} + \begin{bmatrix} \varepsilon E_{m\times m} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$

is negative semi-definite, where

$$\big(a_{ij}(x)\big)_{n\times n} = \begin{cases} -\frac{1}{2}\Big(\frac{c_i f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{c_j f_{ji}(x_i)}{f_{ii}(x_i)}\Big), & x_i x_j \neq 0, \\ 0, & x_i x_j = 0, \end{cases} \quad i, j = 1, \dots, n;$$

then the zero solution of system (6.4.1) is globally stable with respect to y.

PROOF. Construct the Lyapunov function:

$$V(x) = -\sum_{i=1}^{n} \int_0^{x_i} c_i f_{ii}(x_i)\, dx_i. \tag{6.4.5}$$

Then, clearly $V(x)$ is positive definite and radially unbounded with respect to y, since

$$V(x) \geqslant -\sum_{i=1}^{m} \int_0^{x_i} c_i f_{ii}(x_i)\, dx_i := \varphi(y) \to +\infty \quad \text{as } \|y\| \to +\infty.$$

Now, we prove that

$$\left.\frac{dV}{dt}\right|_{(6.4.1)} = G(x) = -\sum_{i=1}^{n} c_i f_{ii}(x_i) \sum_{j=1}^{n} f_{ij}(x_j)$$

is negative definite with respect to y. For any $x = \xi \in R^n$, without loss of generality, we can assume that

$$\prod_{i=1}^{k} \xi_i \neq 0, \quad \sum_{i=k+1}^{n} \xi_i^2 = 0, \quad 1 \leqslant k \leqslant n.$$

Then, we obtain

$$\begin{aligned}
G(\xi) &= -\sum_{i=1}^{k} c_i f_{ii}(\xi_i) \sum_{j=1}^{k} f_{ij}(\xi_j) \\
&= -\frac{1}{2} \sum_{i,j=1}^{k} \left[c_i f_{ii}(\xi_i) f_{ij}(\xi_j) + c_j f_{jj}(\xi_j) f_{ji}(\xi_i)\right] \\
&= -\sum_{i=1}^{n} c_i f_{ii}^2(\xi_i) - \sum_{\substack{i,j=1\\ i\neq j}}^{k} \frac{1}{2}\left[\frac{c_i f_{ij}(\xi_j)}{f_{jj}(\xi_j)} + \frac{c_j f_{ji}(\xi_i)}{f_{ii}(\xi_i)}\right] f_{ii}(\xi_i) f_{jj}(\xi_j) \\
&= \sum_{i=1}^{k} a_{ii}(\xi) f_{ii}^2(\xi_i) + \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{k} a_{ij}(\xi) f_{ii}(\xi_i) f_{jj}(\xi_j) - \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) \\
&\leqslant -\sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) < 0.
\end{aligned} \tag{6.4.6}$$

Since ξ is arbitrary, we have shown that $G(x)$ is negative definite with respect to y. Then the zero solution of system (6.4.1) is globally stable with respect to variable y. $\square$

THEOREM 6.4.6. *Suppose that system (6.4.1) satisfies the following conditions*:

(1) *condition* (1) *of Theorem* 6.4.5 *holds*;
(2) *there exist n functions* $c_i(x_i)$ $(i = 1, \dots, n)$, *which are continuous or have only finite discontinuous points of the first or third kind, and satisfy*

$$\begin{aligned}&c_i(x_i)x_i > 0 \quad \text{for } x_i \neq 0 \quad \text{and}\\ &\int_0^{\pm\infty} c_i(x_i)\,dx_i = +\infty, \quad i = 1, \dots, m,\\ &c_i(x_i)x_i \geqslant 0, \quad \text{for } i = m+1, \dots, n;\end{aligned}$$

(3) *there exist functions* $\varepsilon_i(x_i) > 0$ $(i = 1, \dots, n)$ *such that*

$$\tilde{A}\big(\tilde{a}_{ij}(x)\big)_{n\times n} + \begin{pmatrix} \operatorname{diag}(\varepsilon_1(x_1), \dots, \varepsilon_m(x_m)) & 0\\ 0 & 0\end{pmatrix}_{n\times n}$$

is negative semi-definite, where

$$\tilde{a}_{ij}(x) = \begin{cases} \frac{1}{2}\Big[\frac{c_i(x_i)f_{ij}(x_j)}{\sqrt{|f_{ii}(x_i)f_{jj}(x_j)|}} + \frac{c_j(x_j)f_{ji}(x_i)}{\sqrt{|f_{jj}(x_j)f_{ii}(x_i)|}}\Big], & x_ix_j \neq 0,\\ 0, \quad x_ix_j = 0, \ i, j = 1, \dots, n. \end{cases}$$

Then the zero solution of system (6.4.1) is globally stable with respect to y.

PROOF. Choose

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} c_i(x_i)\,dx_i,$$

and proceed along the lines of Theorem 6.4.5 to complete the proof. $\square$

THEOREM 6.4.7. *If system (6.4.1) satisfies that*

(1) *condition* (1) *of Theorem* 6.4.5 *holds*;
(2) *there exist constants* $c_i > 0$ $(i = 1, \dots, m)$, $c_j \geqslant 0$ $(j = m+1, \dots, n)$ *such that*

$$-c_j\big|f_{jj}(x_j)\big| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i\big|f_{ij}(x_j)\big| < 0, \ \text{for } x_j \neq 0, \ j = 1, \dots, m,$$

$$-c_j\left|f_{jj}(x_j)\right| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i\left|f_{ij}(x_j)\right| \leqslant 0 \quad \textit{for } j = m+1, \ldots, n;$$

then the zero solution of system (6.4.1) is globally stable with respect to variable y.

PROOF. Construct the Lyapunov function

$$V(x) = \sum_{i=1}^{n} c_i |x_i|.$$

Clearly,

$$V(x) \geqslant \sum_{i=1}^{m} c_i |x_i| := \varphi(y) \to +\infty \quad \text{as } \|y\| \to +\infty,$$

and $\varphi(y)$ is positive definite. On the other hand, we have

$$\begin{aligned} D^+V(x)|_{(6.4.1)} &\leqslant \sum_{j=1}^{n}\left[-c_j\left|f_{jj}(x_j)\right| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i\left|f_{ij}(x_j)\right|\right] \\ &\leqslant \sum_{j=1}^{m}\left[-c_j\left|f_{jj}(x_j)\right| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i\left|f_{ij}(x_j)\right|\right] \\ &< 0 \quad \text{for } y \neq 0. \end{aligned}$$

Therefore, the zero solution of system (6.4.1) is globally stable with respect to y. □

THEOREM 6.4.8. *Suppose that system (6.4.1) satisfies the following conditions*:

(1) *condition* (1) *of Theorem* 6.4.5 *holds*;

(2) $\left|\frac{f_{ij}(x_j)}{f_{jj}(x_j)}\right| \leqslant b_{ij} = \textit{const.},\ i \neq j,\ i, j = 1, \ldots, n;$

(3)

$$\tilde{A} := \begin{bmatrix} 1 & -b_{21} & \cdots & -b_{n1} \\ -b_{12} & 1 & \cdots & -b_{n2} \\ \vdots & \vdots & & \vdots \\ -b_{1n} & -b_{2n} & \cdots & 1 \end{bmatrix} := \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

where $\tilde{A}_{11}$, $\tilde{A}_{22}$ and $I_m - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ are M matrices, in which $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$ and $\tilde{A}_{22}$ are $m \times m$, $m \times p$, $p \times m$ and $p \times p$ matrices, respectively.

Then the zero solution of system (6.4.1) is globally stable with respect to y.

PROOF. For any $\xi = (\xi_1, \dots, \xi_m)^T > 0,\ \eta = (\eta_1, \dots, \eta_p)^T \geqslant 0$ and $m+p = n$, we consider the linear algebraic equations with respect to $c = (c_1, \dots, c_m)^T$ and $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_p)^T$:

$$\begin{cases} \tilde{A}_{11}c + \tilde{A}_{12}\tilde{c} = \xi, \\ \tilde{A}_{21}c + \tilde{A}_{22}\tilde{c} = \eta, \end{cases} \tag{6.4.7}$$

or the equivalent ones:

$$\begin{cases} \tilde{c} = -\tilde{A}_{22}^{-1}\tilde{A}_{21}c + \tilde{A}_{22}^{-1}\eta, \\ c = \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}c - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\eta + \tilde{A}_{11}^{-1}\xi. \end{cases} \tag{6.4.8}$$

Since $\tilde{A}_{11}$ and $\tilde{A}_{22}$ are M matrices, we have

$$\tilde{A}_{11}^{-1} \geqslant 0, \quad \tilde{A}_{22}^{-1} \geqslant 0;$$

but $\tilde{A}_{12} \leqslant 0$ and $\xi > 0,\ \eta \geqslant 0$. Therefore, we obtain

$$-\tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\eta \geqslant 0, \quad \tilde{A}_{11}^{-1}\xi > 0,$$

since $(I_m - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})$ is an M matrix. The second equation in (6.4.8) has a positive solution with respect to c, and the first one in (6.4.8) has a nonnegative solution with respect to $\tilde{c}$. Thus, the conditions in Theorem 6.4.7 are satisfied.

Hence, we conclude that the zero solution of system (6.4.1) is globally stable with respect to variable y. □

In the following, we consider a more specific system:

$$\begin{aligned} \frac{dy}{dt} &= \left(\sum_{i=1}^{n} a_{1j} f_j(x_j), \dots, \sum_{j=1}^{n} a_{mj} f_j(x_j) \right)^T, \\ \frac{dz}{dt} &= \left(\sum_{j=1}^{n} a_{m+1,j} f_j(x_j), \dots, \sum_{j=1}^{n} a_{nj} f_j(x_j) \right)^T, \end{aligned} \tag{6.4.9}$$

where $f_j(x_j) \in C[R, R]$, $f_j(0) = 0$, $j = 1, \dots, n$. It is assumed that the solution of the initial value problem (6.4.9) is unique.

THEOREM 6.4.9. *Suppose system (6.4.9) satisfies the following conditions*:

(1) $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $\int_0^{\pm\infty} f_i(x_i)\,dx_i = +\infty$, $a_{ii} < 0$, $i = 1, \dots, m$, $a_{ii} \leqslant 0$, $f_i(x_i)x_i \geqslant 0$, $i = m+1, \dots, n$;
(2) *there exist constants* $c_i > 0$ $(i = 1, \dots, m)$, $c_j \geqslant 0$ $(j = m+1, \dots, n)$, $\varepsilon > 0$ *such that*

$$B(b_{ij})_{n\times n} + \begin{bmatrix} \varepsilon I_m & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$

is negative semi-definite, where

$$b_{ij} = \begin{cases} -c_i|a_{ii}|, & i = j = 1, \ldots, n, \\ -\frac{1}{2}(c_i a_{ij} + c_j a_{ji}), & i \neq j,\ i, j = 1, \ldots, n. \end{cases}$$

Then the zero solution of system (6.4.9) is globally stable with respect to variable y.

PROOF. Construct the positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n} c_i \int_0^{x_i} f(x_i)\, dx_i,$$

then the proof is analogous to that of Theorem 6.4.9, and thus omitted. □

THEOREM 6.4.10. *Suppose system (6.4.9) satisfy the following conditions*:

(1) $f_i(x_i)x_i < 0$ *for* $x_i \neq 0$, $a_{ii} > 0$, $i = 1, \ldots, m$, $f_i(x_i)x_i \leqslant 0$, $a_{ii} \geqslant 0$, $i = m+1, \ldots, n$;
(2) *there exist functions* $c_i(x_i)$ $(i = 1, \ldots, n)$, *which are continuous or have only finite discontinuous points of the first or third kind, such that*

$$c_i(x_i) > 0 \quad \textit{for } x_i \neq 0, \qquad \int_0^{\pm\infty} c_i(x_i)\, dx_i = +\infty, \qquad i = 1, \ldots, m,$$

$$c_i(x_i) \geqslant 0 \quad \textit{for } i = m+1, \ldots, n;$$

(3) *there exist functions* $\varepsilon_i(x_i) > 0$ $(i = 1, \ldots, m)$ *such that*

$$\tilde{B}\big(\tilde{b}_{ij}(x)\big)_{n\times n} + \begin{bmatrix} \mathrm{diag}(\varepsilon_1(x_1), \ldots, \varepsilon_m(x_m)) & 0 \\ 0 & 0 \end{bmatrix}$$

is negative semi-definite, where

$$\tilde{b}_{ij} = \begin{cases} \frac{1}{2}\Big[\frac{c_i(x_i)a_{ij}f_j(x_j)}{\sqrt{|f_i(x_i)f_j(x_j)|}} + \frac{c_j(x_j)a_{ji}f_i(x_i)}{\sqrt{|f_j(x_j)f_i(x_i)|}}\Big], & x_i x_j \neq 0, \\ 0, \quad x_i x_j = 0,\ i, j = 1, \ldots, n. \end{cases}$$

Then the zero solution of the system (6.4.9) is globally stable with respect to variable y.

PROOF. Choose the positive definite and radially unbounded Lyapunov function with respect to y:

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} c_i(x_i)\, dx_i.$$

Then the proof is similar to that of Theorem 6.4.5 and so omitted. □

THEOREM 6.4.11. *If system (6.4.9) satisfies the following conditions*:

(1) $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $a_{ii} < 0$, $i = 1, \ldots, m$, $f_i(x_i)x_i \geqslant 0$, $a_{ii} \leqslant 0$, $i = m+1, \ldots, n$;
(2) *there exist constants* $c_i > 0$ $(i = 1, \ldots, m)$, $c_j \geqslant 0$ $(j = m+1, \ldots, n)$ *such that*

$$-c_j|a_{jj}| + \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| < 0, \quad j = i, \ldots, m,$$

$$-c_j|a_{jj}| + \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| \leqslant 0, \quad j = m+1, \ldots, n;$$

then the zero solution of (6.4.9) *is globally stable with respect to* y.

PROOF. Construct the positive definite and radially unbounded Lyapunov function with respect to y:

$$V(x) = \sum_{i=1}^{n} c_i|x_i|,$$

and then follow the proof for Theorem 6.4.9. □

THEOREM 6.4.12. *Suppose system (6.4.9) satisfies the following conditions*:

(1) $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $i = 1, \ldots, m$, $f_i(x_i)x_i \geqslant 0$, $i = m+1, \ldots, n$, $a_{ii} < 0, i = 1, \ldots, n$;
(2)

$$\tilde{A} = \begin{bmatrix} 1 & -|\frac{a_{21}}{a_{11}}| & \cdots & -|\frac{a_{n1}}{a_{11}}| \\ -|\frac{a_{12}}{a_{22}}| & 1 & \cdots & -|\frac{a_{n2}}{a_{22}}| \\ \vdots & \vdots & & \vdots \\ = |\frac{a_{1n}}{a_{nn}}| & -|\frac{a_{2n}}{a_{nn}}| & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}.$$

where $\tilde{A}_{11}$, $\tilde{A}_{22}$ *and* $I_m - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ *are M matrices, in which* $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$ *and* $\tilde{A}_{22}$ *represent* $m \times m$, $m \times p$, $p \times m$ *and* $p \times p$ *matrices, respectively.*

Then the zero solution of system (6.4.9) *is globally asymptotically stable with respect to variable* y.

PROOF. We can follow the approach applied in the proof of Theorem 6.4.8. □

6.5. Autonomous systems with generalized separable variables

Consider the system with generalized separable variables:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} F_{ij}(x) \cdot f_{ij}(x_j), \quad i = 1, \ldots, n, \tag{6.5.1}$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $F_{ij} \in C[R^n, R]$, $f_{ij} \in C[R, R]$, $f_{ij}(0) = 0$, $i, j = 1, \ldots, n$. Suppose the solution of the initial value problem (6.5.1) is unique.

Let $y = (x_1, \ldots, x_m)^T$, $z = (x_{m+1}, \ldots, x_n)^T$. Rewrite system (6.5.1) as

$$\begin{cases} \frac{dy}{dt} = \Big(\sum_{j=1}^{n} F_{1j}(x) f_{1j}(x_j), \ldots, \sum_{j=1}^{n} F_{mj}(x) f_{mj}(x_j)\Big)^T, \\ \frac{dz}{dt} = \Big(\sum_{j=1}^{n} F_{m+1,j}(x) f_{m+1,j}(x_j), \ldots, \sum_{j=1}^{n} F_{nj}(x) f_{nj}(x_j)\Big)^T. \end{cases} \tag{6.5.2}$$

THEOREM 6.5.1. *If system (6.5.2) satisfies the following conditions*:

(1) $f_{ii}(x_i)x_i > 0$ *for* $x_i \neq 0$ *and* $\int_0^{\pm\infty} f_{ii}(x_i)\,dx_i = +\infty$, $i = 1, \ldots, m$, $f_{ii}(x_i)x_i \geqslant 0$, $i = m+1, \ldots, n$;

(2) *there exist constants* $c_i > 0$ $(i = 1, \ldots, m)$, $c_j \geqslant 0$ $(j = m+1, \ldots, n)$, $\varepsilon > 0$ *such that*

$$B\big(b_{ij}(x)\big)_{n\times n} + \begin{bmatrix} \varepsilon E_{m\times m} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$

is negative semi-definite, where

$$b_{ij}(x) = \begin{cases} \frac{1}{2}\Big(\frac{F_{ij}(x) f_{ij}(x_j)}{f_{jj}(x_j)} + \frac{F_{ji}(x) f_{ji}(x_i)}{f_{ii}(x_i)}\Big), & x_i x_j \neq 0, \\ 0, \quad x_i x_j = 0 \ (i, j = 1, \ldots, n); \end{cases}$$

then the zero solution of system (6.5.2) is globally asymptotically stable with respect to variable y.

PROOF. Choose the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} f_{ii}(x_i)\,dx_i.$$

Clearly,

$$V(x) \geqslant \sum_{j=1}^{n} \int_0^{x_i} f_{ii}(x_i)\,dx_i := \varphi\big(\|y\|\big) \to \infty \quad \text{as } \|y\| \to +\infty.$$

Hence, $V(x)$ is positive definite and radially unbounded with respect to y.

Now we proceed to prove that

$$\frac{dV}{dt}\bigg|_{(6.5.2)} = \sum_{i=1}^{n} f_{ii}(x_i) \sum_{j=1}^{n} F_{ij}(x) f_{ij}(x_j)$$

is negative definite with respect to y.

For any $x = \xi \in R^n$, without loss of generality, we assume that

$$\prod_{j=1}^{k} \xi_j \neq 0, \qquad \sum_{i=k+1}^{n} \xi_i^2 = 0, \qquad 1 \leqslant k \leqslant n.$$

Then it follows that

$$\begin{aligned}
G(\xi) &= \sum_{i=1}^{k} f_{ii}(\xi_i) \sum_{j=1}^{k} F_{ij}(\xi) f_{ij}(\xi_j) \\
&= \sum_{i=1}^{k} F_{ii}(\xi) f_{ii}^2(\xi_i) \\
&\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \big[f_{ii}(\xi_i) F_{ij}(\xi) f_{ij}(\xi_j) + f_{jj}(\xi_j) F_{ji}(\xi) f_{ji}(\xi_i) \big] \\
&= \sum_{i=1}^{k} b_{ii}(\xi) f_{ii}^2(\xi_i) + \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) \\
&\quad + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} b_{ij}(\xi) f_{ii}(\xi_i) f_{jj}(\xi_j) - \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) \\
&\leqslant - \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) \\
&< 0.
\end{aligned}$$

Since ξ is arbitrary, we have shown that $\frac{dV}{dt}|_{(6.5.2)}$ is negative definite with respect to y.

The proof is complete. □

THEOREM 6.5.2. *If system (6.5.2) satisfies the following conditions*:

(1) $F_{ii}(x) f_{ii}(x_i) x_i < 0$ *for* $x_i \neq 0$, $i = 1, \ldots, m$, $F_{ii}(x) f_{ii}(x_i) x_i \leqslant 0$, $i = m+1, \ldots, n$;

(2) *there exist constants* $c_i > 0$ $(i = 1, \dots, m)$, $c_j \geqslant 0$ $(j = m+1, \dots, n)$ *such that*

$$
\begin{aligned}
&-c_j \left|F_{jj}(x) f_{jj}(x_j)\right| + \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \left|F_{ij}(x) f_{ij}(x_j)\right| < 0 \\
&\quad \text{for } x_j \neq 0,\ \ j = 1, \dots, m, \\
&-c_j \left|F_{jj}(x) f_{jj}(x_j)\right| + \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \left|F_{ij}(x) f_{ij}(x_j)\right| \leqslant 0 \\
&\quad \text{for } j = m+1, \dots, m;
\end{aligned}
$$

then the zero solution of (6.5.2) *is globally stable with respect with* y.

PROOF. Choose the Lyapunov function:

$$
V(x) = \sum_{i=1}^{n} c_i |x_i|,
$$

which is positive definite and radially unbounded with respect to y, and then follow the proof of Theorem 6.4.7 to complete the proof. □

6.6. Nonautonomous systems with separable variables

Consider the nonautonomous system with separable variables:

$$
\begin{aligned}
\frac{dy}{dt} &= \left(\sum_{j=1}^{n} f_{1j}(t, x_j), \dots, \sum_{j=1}^{n} f_{mj}(t, x_j) \right)^T, \\
\frac{dz}{dt} &= \left(\sum_{j=1}^{n} f_{m+1,j}(t, x_j), \dots, \sum_{j=1}^{n} f_{nj}(t, x_j) \right)^T,
\end{aligned}
\tag{6.6.1}
$$

where $y = (x_1, \dots, x_m)^T$, $z = (x_{m+1}, \dots, x_n)^T$, $f_{ij}(t, x_j) \in C[I \times R^1, R^1]$, $f_{ij}(t, 0) \equiv 0$, $i, j = 1, \dots, n$. Suppose the solution of the initial value problem (6.6.1) is unique.

LEMMA 6.6.1. *If there exist functions* $\phi_i(x_i)$ $(i = 1, \dots, n)$ *on* $(-\infty, +\infty)$, *which are continuous or have only finite discontinuous points of the first or third kind, such that*

(1) $\phi_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $i = 1, \dots, m$, $\phi_i(x_i)x_i \geqslant 0$, $i = m+1, \dots, n$;
(2) $\int_0^{\pm\infty} \phi_i(x_i)\, dx_i = +\infty$, $i = 1, \dots, m$;

(3) *there is a positive definite function* ψ *satisfying*

$$G(x) = \sum_{i=1}^{n} \phi_i(x_i \pm 0) \sum_{j=1}^{n} f_{ij}(t, x_j) \leqslant -\psi(y);$$

then the zero solution of system (6.6.1) is globally stable with respect to y.

PROOF. The proof repeats that for Lemma 6.4.3 and is omitted. □

THEOREM 6.6.2. *Suppose system (6.6.1) satisfies the following conditions*:

(1) $f_{ii}(t, x_i)x_i < 0$ *for* $x_i \neq 0$, $i = 1, \ldots, m$, $f_{ii}(t, x_i)x_i \leqslant 0$, $i = m+1, \ldots, n$;
(2) *there exist functions* $F_{ii}(x_i) \in C[R^1, R^1]$ $(i = 1, \ldots, n)$, *which are continuous or have only finite discontinuous points of the first or third kind, such that*

$$\begin{cases} F_{ii}(x_i)x_i > 0 \quad \text{for } x_i \neq 0, & i = 1, \ldots, m, \\ F_{ii}(x_i)x_i \geqslant 0, & i = m+1, \ldots, n, \\ \int_0^{\pm\infty} F_{ii}(x_i)\, dx_i = +\infty, & i = 1, \ldots, m, \\ |F_{ii}(x_i)| \leqslant |f_{ii}(t, x_i)|, & i = 1, \ldots, n; \end{cases}$$

(3) *the matrix*

$$A\big(a_{ij}(t, x)\big)_{n \times n} + \begin{bmatrix} \varepsilon E_{m \times m} & 0 \\ 0 & 0 \end{bmatrix}_{n \times n}$$

is negative semi-definite, where $0 < \varepsilon \ll 1$, *and*

$$a_{ij}(t, x) = \begin{cases} -1, & i = j = 1, \ldots, n, \\ \frac{1}{2}\Big(\frac{f_{ij}(t, x_j)}{F_{jj}(x_j)} + \frac{f_{ji}(t, x_i)}{F_{ii}(x_i)}\Big), & i \neq j,\ x_i x_j \neq 0,\ i, j = 1, \ldots, n, \\ 0, & i \neq j,\ x_i x_j = 0,\ i, j = 1, \ldots, n. \end{cases}$$

Then the zero solution of system (6.6.1) is globally stable with respect to y.

PROOF. Consider the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} F_{ii}(x_i)\, dx_i.$$

Then

$$V(x) \geqslant \sum_{i=1}^{m} \int_0^{x_i} F_{ii}(x_i)\, dx_i = \varphi(y).$$

Clearly, we have

$$\varphi(y) \to +\infty \quad \text{as } \|y\| \to +\infty.$$

So $V(x)$ is positive definite and radially unbounded with respect to y.

We now prove that

$$\left.\frac{dV}{dt}\right|_{(6.6.1)} = G(t,x) = \sum_{i=1}^{n} F_{ii}(x_i) \sum_{j=1}^{n} f_{ij}(t,x_j)$$

is negative definite with respect to y.

For any $x = \xi \in R^n$, without loss of generality, we assume that

$$\prod_{i=1}^{k} \xi_i \neq 0, \qquad \sum_{i=k+1}^{n} \xi_i^2 = 0, \qquad 1 \leqslant k \leqslant n.$$

Then, it follows that

$$\begin{aligned}
G(t,\xi) &= \sum_{i=1}^{k} F_{ii}(\xi_i) \sum_{j=1}^{k} f_{ij}(t,\xi_j) \\
&\leqslant \sum_{i=1}^{k} a_{ii}(t,\xi) F_{ii}^2(\xi_i) + \sum_{i=1}^{m} \varepsilon F_{ii}^2(\xi_i) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{k} a_{ij}(t,\xi) F_{ii}(\xi_i) F_{jj}(\xi_j) - \sum_{i=1}^{m} \varepsilon F_{ii}^2(\xi_i) \\
&\leqslant -\sum_{i=1}^{m} \varepsilon F_{ii}^2(\xi_i) \\
&< 0.
\end{aligned}$$

Since ξ is arbitrary, we have shown that $\frac{dV}{dt}|_{(6.6.1)}$ is negative definite with respect to y. Hence, the zero solution of system (6.6.1) is globally stable with respect to y. □

THEOREM 6.6.3. *If system (6.6.1) satisfies the following conditions*:

(1) $f_{ii}(t,x_i)x_i < 0$ *for* $x_i \neq 0$, $i = 1,\ldots,m$, $f_{ii}(t,x_i)x_i \leqslant 0$, $i = m+1,\ldots,n$;
(2) *there exist constants* $c_i > 0$ $(i = i,\ldots,m)$, $c_j \geqslant 0$ $(j = m+1,\ldots,n)$ *such that*

$$-c_j\left|f_{jj}(t,x_j)\right| + \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i\left|f_{ij}(t,x_j)\right| < 0 \quad \textit{for } x_j \neq 0,\ j = 1,\ldots,m,$$

$$-c_j\left|f_{jj}(t,x_j)\right| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i\left|f_{ij}(t,x_j)\right| \leqslant 0 \quad \textit{for } j=m+1,\ldots,n;$$

then the zero solution of system (6.6.1) is globally stable with respect to variable y.

PROOF. Let us choose

$$V(x) = \sum_{i=1}^{n} c_i|x_i|,$$

then analogous to the proof of Theorem 6.4.7, we can verify the validity of this theorem. □

THEOREM 6.6.4. *Assume that system (6.6.1) satisfies the following assertion*:

(1) *there exist function $\phi_i(x_i) \in [R^1, R^1]$ ($i=1,\ldots,n$), which are continuous or have only finite discontinuous points of the first or third kind, such that $\phi_i(x_i)x_i > 0$ for $x_i \neq 0$ and $\int_0^{\pm\infty} \phi_i(x_i)\,dx_i = +\infty$, $i=1,\ldots,m$, $\phi_i(x_i)x_i \geqslant 0$, $i=m+1,\ldots,n$;*
(2) *there are functions $a_i(x_i)$ with $a_i(x_i) > 0$ for $x_i \neq 0$ ($i=1,\ldots,m$) such that*

$$\sum_{i=1}^{n} \phi_i(x_i) f_{ij}(x_j) \leqslant -a_j(x_j), \quad \textit{for } j=1,\ldots,m,$$

$$\sum_{i=1}^{n} \phi_i(x_i) f_{ij}(x_j) \leqslant 0, \quad \textit{for } j=m+1,\ldots,n.$$

Then the zero solution of system (6.6.1) is globally stable with respect to y.

PROOF. Construct the positive definite and radially unbounded Lyapunov function with respect to y:

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} \phi_i(x_i)\,dx_i.$$

Then, we have

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(6.6.1)} &= \sum_{i=1}^{n}\phi_i(x_i)\sum_{j=1}^{n} f_{ij}(t,x_j) \leqslant \sum_{j=1}^{n}\sum_{i=1}^{m}\phi_i(x_i)f_{ij}(t,x_j)\\ &\leqslant -\sum_{j=1}^{m} a_j(x_j),\end{aligned}$$

which indicates that $\frac{dV}{dt}|_{(6.6.1)}$ is negative definite with respect to y. Therefore, the zero solution of system (6.6.1) is globally stable with respect to variable y. $\square$

Chapter 7

Iteration Method for Stability

In the study of the stability of general nonlinear systems, the Lyapunov direct method is the main and powerful tool. However, generally great difficulties arise in applying this method to higher-dimensional systems with complicated structure, because there is no universal and systematic procedure available for constructing the required Lyapunov function. Therefore, it is necessary and important to develop explicit algebraic criteria for stability analysis.

In this chapter, we explore an approach for stability analysis, called iteration method. This method is based on the idea of Picard's iterative approach, which has been used in the proof of the existence and uniqueness theorem of differential equations. This method avoids the difficulties in constructing Lyapunov function and complex calculations. This method uses the estimation of the integral of the function on the right-hand side of the system, and yields algebraic criteria of stability. Because of the average property of an integral, this method does not require the slowly time-variant feature of a system; and can even be applied to systems with large time-variant amplitude of motions. It should be emphasized that though involved calculations appear for proving theorems, the procedure of calculation in each iterative step is not necessary. The convergent conditions of the iteration can yield various criteria of stability. Moreover, the internal relations between iterative convergence and stability are established. The advantage of this method is the independence of unknown functions or parameters of the systems. Further, this method can be used to consider the stability problems of other kind of dynamic systems, such as discrete systems, functional differential systems. In this chapter, we introduce this method only for ordinary differential equations. In the next chapter, this method will also be employed to study delay differential equations.

Part of materials presented in this chapter are chosen from [237,242,234,279, 244,245,361].

7.1. Picard iteration type method

We consider the following nonlinear time-dependent, large-scale dynamical system:

$$\frac{dx}{dt} = \operatorname{diag}\big(A_{11}(t), A_{22}(t), \dots, A_{rr}(t)\big)x + F(t,x), \tag{7.1.1}$$

where

$$\begin{aligned}
&F(t,x) \in C\big[I \times R^n, R^n\big], \quad F(t,0) = 0,\\
&F(t,x) = \big(F_1(t,x), \dots, F_r(t,x)\big)^T, \quad F_i(t,x) \in C\big[I \times R^n, R^{n_i}\big],\\
&x = (x_1, \dots, x_r)^T, \quad x_i \in R^{n_i}, \quad \sum_{i=1}^{r} n_i = n, \quad i = 1, 2, \dots, n,
\end{aligned}$$

and $F_i(t,x)$, $A_{ii}(t)$ are $n_i \times n_i$ matrices of continuous functions $F_i(t,x)$, satisfying the Lipschitz conditions:

$$\big\|F_i(t,x) - F_i(t,y)\big\| \leqslant \sum_{j=1}^{r} g_{ij}(t)\|x_j - y_j\|, \quad i = 1, 2, \dots, r.$$

At the same time, we consider the isolated subsystem:

$$\frac{dx}{dt} = \operatorname{diag}\big(A_{11}(t), \dots, A_{rr}(t)\big). \tag{7.1.2}$$

Assume that $K(t,t_0) = \operatorname{diag}(K_{11}(t,t_0), \dots, K_{rr}(t,t_0))$ is the standard fundamental solution matrix of (7.1.2), called Cauchy matrix solution (or simply Cauchy matrix), i.e.,

$$\begin{aligned}
&\frac{dK(t,t_0)}{dt} = \operatorname{diag}\big(A_{11}(t,t_0), \dots, A_{rr}(t,t_0)\big)K(t,t_0),\\
&K(t,t_0) = I_{n\times n}.
\end{aligned}$$

THEOREM 7.1.1. *Suppose the following conditions hold*:

(1) *there exist secular function* $\alpha_i(t) \in C[I, R^+]$, $\varepsilon(t) \in C[I, R^1]$ *and constants* M_i $(i = 1, 2, \dots, r)$ *such that*

$$\begin{aligned}
\big\|K_{ii}(t,t_0)\big\| &\leqslant M_i e^{-\int_{t_0}^t \alpha_i(\xi)\,d\xi}\\
&\leqslant M_i e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi}, \quad i = 1, \dots, r,\ t \geqslant t_0;
\end{aligned}$$

(2) *there exist constants* h_{ij} *such that*

$$\int_{t_0}^{t} M_i e^{-\int_{t_1}^t (\alpha_i(\xi)-\varepsilon(\xi))\,d\xi} g_{ij}(t_1)\,dt_1 \leqslant h_{ij}, \quad i, j = 1, 2, \dots, r;$$

(3) *the spectral radius of the matrix* $H(h_{ij})_{r\times r}$, $\rho(H)$, *is less than* 1, *i.e.*, $\rho(H) < 1$ (*particularly* $\|H\| < 1$).

Then the flowing conclusions hold:

(a) $\int_{t_0}^{t} \varepsilon(\xi)\,d\xi \geqslant l(t_0) = const.$ $(t \geqslant t_0 \geqslant 0)$;
(b) $\int_{t_0}^{t} \varepsilon(\xi)\,d\xi \geqslant l = const.$ $(t \geqslant t_0 \geqslant 0)$;
(c) $\int_{t_0}^{+\infty} \varepsilon(\xi)\,d\xi = +\infty$;
(d) $\int_{t_0}^{t} \varepsilon(\xi)\,d\xi \to +\infty$ *as* $t - t_0 \to +\infty$ *holds uniformly for* t_0;
(e) $\int_{t_0}^{t} \varepsilon(\xi)\,d\xi \geqslant l(t_0) \geqslant \alpha(t - t_0)$ $(\alpha = const. > 0)$;

which respectively imply that:

(a) *the zero solution of system* (7.1.1) *is stable*;
(b) *the zero solution of system* (7.1.1) *is uniformly stable*;
(c) *the zero solution of system* (7.1.1) *is asymptotically stable*;
(d) *the zero solution of system* (7.1.1) *is uniformly asymptotically stable*;
(e) *the zero solution of system* (7.1.1) *is exponentially stable.*

PROOF. By the method of constant variation, we can prove that the general solution $x(t, t_0, x_0)$ of system (7.1.1) is equivalent to the continuous solution of the following integral equations:

$$x_i(t) := x_i(t, t_0, x_0) = K_i(t, t_0)x_{0i} + \int_{t_0}^{t} K_i(t, t_1)F_i\big(t_1, x(t_1)\big)\,dt_1, \quad i = 1, 2, \ldots, r. \tag{7.1.3}$$

Applying the Picard iteration to equation (7.1.3), we have

$$x_i^{(0)}(t) = K_{ii}(t, t_0)x_{i0}, \quad i = 1, 2, \ldots, r, \tag{7.1.4}$$

$$x_i^{(m)}(t) = x_i^{(0)}(t) + \int_{t_0}^{t} K_i(t, t_1)F_i\big(t_1, x^{(m-1)}(t_1)\big)\,dt_1. \tag{7.1.5}$$

Hence, we obtain

$$\big(\big\|x_1^{(0)}(t)\big\|, \ldots, \big\|x_r^{(0)}(t)\big\|\big)^T \leqslant (M_1, \ldots, M_r)^T e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi}, \tag{7.1.6}$$

$$\begin{aligned}\big\|x_i^{(1)}(t)\big\| &\leqslant \big\|x_i^{(0)}(t)\big\| + \left(\int_{t_0}^{t} M_i e^{-\int_{t_1}^{t}(\alpha_i(\xi)-\varepsilon(\xi))\,d\xi} \sum_{j=1}^{n} g_{ij}(t_1), M_j\,dt_1\right) e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi} \\ &\leqslant M_i e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi} + \sum_{j=1}^{r} h_{ij} M_j e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi}, \quad i = 1, \ldots, n,\end{aligned} \tag{7.1.7}$$

$$
\begin{aligned}
&\left(\left\|x_1^{(1)}(t)\right\|, \ldots, \left\|x_r^{(1)}(t)\right\|\right)^T \\
&\quad \leqslant (I+H)(M_1, M_2, \ldots, M_r)^T e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (t \geqslant t_0).
\end{aligned} \tag{7.1.8}
$$

Then, one can easily prove that

$$
\begin{aligned}
&\left(\left\|x_1^{(1)}(t) - x_1^{(0)}(t)\right\|, \ldots, \left\|x_r^{(1)}(t) - x_r^{(0)}(t)\right\|\right)^T \\
&\quad \leqslant H(M_1, M_2, \ldots, M_r)^T e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (t \geqslant t_0).
\end{aligned} \tag{7.1.9}
$$

Suppose

$$
\begin{aligned}
&\left(\left\|x_1^{(m)}(t)\right\|, \ldots, \left\|x_r^{(m)}(t)\right\|\right)^T \\
&\quad \leqslant (I+H, \ldots, +H^m)(M_1, \ldots, M_r)^T e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (t \geqslant t_0),
\end{aligned} \tag{7.1.10}
$$

$$
\begin{aligned}
&\left(\left\|x_1^{(m)}(t) - x_1^{(m-1)}(t)\right\|, \ldots, \left\|x_r^{(m)}(t) - x_r^{(m-1)}(t)\right\|\right)^T \\
&\quad \leqslant H^m(M_1, M_2, \ldots, M_r)^T e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (t \geqslant_0).
\end{aligned} \tag{7.1.11}
$$

Let $H^m = (h_{ij}^{(m)})_{r\times r}$. Then, it follows that

$$
\begin{aligned}
\left\|x_i^{(m+1)}(t)\right\| &\leqslant \left\|x_i^{(0)}(t)\right\| + \left\{\int_{t_0}^t M_i e^{-\int_{t_1}^t (\alpha_i(\xi)-\varepsilon(\xi))\,d\xi} \sum_{j=1}^n g_{ij}(t_1)\right. \\
&\qquad \left.\times \left[\sum_{s=1}^r \left(\delta_{js} + h_{js} + \cdots + h_{js}^{(m)}\right) M_j\,dt_1\right]\right\} e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\leqslant M_i e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\qquad + \sum_{j=1}^r h_{ij}\left[\sum_{s=1}^r \delta_{js} + h_{js} + \cdots + h_{js}^{(m)}\right] M_s e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&= \sum_{s=1}^r \left[\delta_{js} + h_{js} + \cdots + h_{js}^{(m+1)}\right] M_s e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi}, \\
&\qquad i = 1, 2, \ldots, r,\ t \geqslant t_0,
\end{aligned} \tag{7.1.12}
$$

where

$$
\delta_{js} = \begin{cases} 1, & j = s, \\ 0, & j \neq s, \end{cases} \quad i = 1, 2, \ldots, r.
$$

Thus, we have

$$
\left\|x_i^{(m+1)}(t) - x_i^{(m)}(t)\right\|
$$

$$\leqslant \left\{ \int_{t_0}^{t} M_i e^{-\int_{t_1}^{t} (\alpha_i(\xi)-\varepsilon(\xi))\,d\xi} \sum_{j=1}^{r} g_{ij}(t_1) \left(\sum_{s=1}^{r} h_{js}^{(m)} M_s \right) dt_1 \right\} e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi}$$

$$\leqslant \sum_{j=1}^{r} h_{ij} \sum_{s=1}^{r} h_{js}^{(m)} M_s e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi}, \quad i = 1, 2, \ldots, n, \ t \geqslant t_0. \tag{7.1.13}$$

Hence, by the method of mathematical induction, the estimations given in (7.1.10) and (7.1.11) hold for arbitrary natural number m.

It is well known that $\sum_{m=1}^{\infty} H^m$ is convergent and

$$\sum_{m=0}^{+\infty} H^m = (I - Q)^{-1}.$$

So on any finite interval $[t_0, T]$, the convergence of

$$\sum_{m=0}^{+\infty} H^m (M_1, M_2, \ldots, M_r)^T e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi} \tag{7.1.14}$$

implies the convergence of

$$\sum_{m=0}^{+\infty} \left(\left\| x_1^{(m)}(t) - x_1^{(m-1)}(t) \right\|, \ldots, \left\| x_r^{(m)}(t) - x_r^{(m-1)}(t) \right\| \right)^T$$
$$+ \left(\left\| x_1^{(0)}(t) \right\|, \ldots, \left\| x_r^{(0)}(t) \right\| \right)^T. \tag{7.1.15}$$

But, the convergence of (7.1.15) implies the convergence of the following iteration

$$\left(x_1^{(m)}(t), \ldots, x_r^{(m)}(t) \right)^T. \tag{7.1.16}$$

Therefore, on an arbitrary finite interval $[t_0, T]$, we have

$$\left(x_1^{(m)}(t), \ldots, x_r^{(m)}(t) \right)^T \ \to \ \left(x_1(t), \ldots, x_r(t) \right)^T \quad \text{as } m \to +\infty.$$

Therefore,

$$\left(\left\| x_1(t) \right\|, \ldots, \left\| x_r(t) \right\| \right)^T$$
$$\leqslant (I - H)^{-1} (M_1, M_2, \ldots, M_r)^T e^{-\int_{t_0}^{t} \varepsilon(\xi)\,d\xi}, \tag{7.1.17}$$

from which we can conclude that the conclusions (a)–(e) are true.

The proof of Theorem 7.1.1 is complete. □

REMARK 7.1.2. In general, to find $K(t, t_0)$ of the isolated system (7.1.2) is difficult. But it is easy to find $K(t, t_0)$ for the following special cases.

(1) $A_{ii}(t) = a_{ii}(t)$, i.e., every isolated subsystems is one-dimensional, then

$$K_{ii}(t, t_0) = e^{-\int_{t_0}^t a_{ii}(t)\,dt}, \quad i = 1, 2, \ldots, n.$$

(2) $A_{ii}(t) \equiv A_{ii}$, i.e., A_{ii} is a constant matrix. Then $K_{ii}(t, t_0) = e^{A_{ii}(t-t_0)}$, $i = 1, 2, \ldots, r$.
(3) The variation of $A_{ii}(t)$ is sufficiently slow. Let $A_{ii}(t) = \mathring{A}_{ii}(t_0) + A_{ii}(t) - A_{ii}(t_0)$. In the isolated subsystem (7.1.2) with $\mathring{A}_{ii}(t_0)$ to replace $A_{ii}(t)$, we may combine $\mathring{A}_{ii}(t_0) - A_{ii}(t_0)$ with the associated terms.
(4) If $\lim_{t\to\infty} A_{ii}(t) = A_{ii}$, ($A_{ii}$ is a constant matrix). Let $A_{ii}(t) = A_{ii} + A_{ii}(t) - A_{ii}$. We may combine $A_{ii}(t) - A_{ii}$ with the associated terms as in (3).
(5) If $A_{ii}(t)\int_{t_0}^t A_{ii}(\xi)\,d\xi \equiv \int_{t_0}^t A_{ii}(\xi)\,d\xi\, A_{ii}(t)$, then $K_{ii}(t, t_0) = e^{\int_{t_0}^t A_{ii}(\xi)\,d\xi}$.
(6) We may use the expression of $K(t, t_0)$ for some particular linear isolated subsystem.

EXAMPLE 7.1.3. Consider the linear equations:

$$\frac{dx_i}{dt} = \sum_{j=1}^n a_{ij}x_j, \quad i = 1, 2, \ldots, n. \tag{7.1.18}$$

If

(1) $a_{ii} < 0$ ($i = 1, 2, \ldots, n$);
(2) $\rho(\tilde{A}) < 1$, where $\tilde{A} := I_n - \left(\frac{|a_{ij}|}{|a_{jj}|}\right)_{n\times n}$;

then system (7.1.18) is exponentially stable by Theorem 7.1.1.

7.2. Gauss–Seidel type iteration method

In this section, we still consider systems (7.1.1) and (7.1.2), but for (7.1.3) we apply the Gauss–Seidel type iteration:

$$x_i^{(0)}(t) = K_{ii}(t, t_0)x_{0i}, \quad i = 1, 2, \ldots, r, \tag{7.2.1}$$

$$\begin{aligned} x_i^{(m)}(t) &= K_{ii}(t, t_0)x_{0i} \\ &+ \int_{t_0}^t K_{ii}(t, t_1)F_i\big(t, x_1^{(m)}(t_1), \ldots, x_{i-1}^{(m)}(t_1), x_i^{(m-1)}(t_1), \ldots, x_r^{(m-1)}(t_1)\big)\,dt, \\ & i = 1, 2, \ldots, r. \end{aligned} \tag{7.2.2}$$

THEOREM 7.2.1. *If the conditions* (1) *and* (2) *in Theorem* 7.1.1 *are satisfied, and* (3) *the spectral radius* $\rho(\tilde{H})$ *of matrix* $\tilde{H} := (I - H_\triangle)^{-1}H^\triangle$ *is less than* 1 (*particularly* $\|\tilde{H}\| < 1$), *where*

$$H_{\triangle} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ h_{21} & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ h_{r1} & \cdots & h_{r,r-1} & 0 \end{bmatrix}, \qquad H^{\triangle} := H - H_{\triangle},$$

then the same conclusion of Theorem 7.1.1 *holds.*

PROOF. From condition (1) we have

$$\left\|x_i^{(0)}(t)\right\| \leqslant M_i e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} := \left\|y_i^{(0)}(t)\right\| \quad i = 1, 2, \ldots, r,\ t \geqslant t_0, \tag{7.2.3}$$

$$\begin{aligned}
&\left\|x_1^{(1)}(t) - x_1^{(0)}(t)\right\| \\
&\quad \leqslant \left[\int_{t_0}^{t} M_1 e^{-\int_{t_1}^t \alpha_1(\xi)d\xi} \sum_{j=1}^{n} g_{ij}(t_1) e^{\int_{t_1}^t \varepsilon(\xi)\,d\xi} M_j\,dt_1\right] e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\quad \leqslant \sum_{j=1}^{r} h_{ij} M_j e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\quad = \sum_{j=1}^{n} h_{ij} \left\|y_j^{(0)}(t)\right\| := \left\|y_1^{(1)}(t) - y_1^{(0)}(t)\right\| \quad (t \geqslant t_0),
\end{aligned} \tag{7.2.4$_1$}$$

$$\begin{aligned}
&\left\|x_2^{(0)}(t) - x_2^{(0)}(t)\right\| \\
&\quad \leqslant \left[\int_{t_0}^{t} M_2 e^{-\int_{t_1}^t \alpha_2(\xi)\,d\xi} g_{2i}(t_1) e^{\int_{t_1}^t \varepsilon(\xi)\,d\xi} \sum_{j=1}^{r} h_{1j} M_j\,dt_1\right] e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\qquad + \left[\int_{t_0}^{t} M_2 e^{-\int_{t_1}^t \alpha_2(\xi)\,d\xi} \sum_{j=2}^{r} g_{2j}(t_1) e^{\int_{t_1}^t \varepsilon(\xi)\,d\xi} M_j\,dt_1\right] e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\quad = h_{21} \left\|y_1^{(1)}(t) - y_1^{(0)}(t)\right\| + \sum_{j=2}^{r} h_{2j} M_j \left\|y_j^{(0)}(t)\right\| \\
&\quad := \left\|y_2^{(1)}(t) - y_2^{(0)}(t)\right\|
\end{aligned} \tag{7.2.4$_2$}$$

$$\vdots$$

$$\left\|x_r^{(1)}(t) - x_r^{(0)}(t)\right\|$$

$$
\begin{aligned}
&\leqslant \left[\int_{t_0}^{t} M_r e^{-\int_{t_1}^{t} \alpha_r(\xi)\,d\xi + \int_{t_1}^{t} \varepsilon(\xi)\,d\xi} \sum_{j=1}^{r-1} g_{rj}(t_1)\,dt_1 \right] \left\| y_j^{(1)}(t) - y_j^{(0)}(t) \right\| \\
&\quad + \left[\int_{t_0}^{t} M_r e^{-\int_{t_1}^{t} \alpha_r(\xi)\,d\xi} g_{rr}(t_1)\,dt_1 \right] \left\| g_r^{(0)}(t) \right\| \\
&= h_{r1} \left\| y^{(1)}(t) - y_1^{(0)}(t) \right\| + h_{r2} \left\| y_2^{(1)}(t) - y_2^{(0)}(t) \right\| + \cdots \\
&\quad + h_{r,r-1} \left\| y_{r-1}^{(1)}(t) - y_{r-1}^{(0)}(t) \right\| + h_{rr} \left\| y_r^{(0)}(t) \right\| \\
&:= \left\| y_r^{(1)}(t) - y_r^{(0)}(t) \right\|.
\end{aligned} \tag{7.2.4$_r$}
$$

We rewrite $(7.2.4)_1$–$(7.2.4)_r$ as

$$
\begin{aligned}
&I_n \left(\left\| y_1^{(1)}(t) - y_1^{(0)}(t) \right\|, \ldots, \left\| y_r^{(1)}(t) - y_r^{(0)}(t) \right\| \right)^T \\
&\quad = H_\triangle \left(\left\| y_1^{(1)}(t) - y_r^{(0)}(t) \right\|, \ldots, \left\| y_r^{(1)}(t) - y_r^{(0)}(t) \right\| \right)^T \\
&\qquad + H^\triangle \left(\left\| y_1^{(0)}(t) \right\|, \ldots, \left\| y_r^{(0)} \right\| \right)^T.
\end{aligned}
$$

Since the matrix $(I_n - H_\triangle)$ is nonsingular, we have

$$
\begin{aligned}
&\left(\left\| x_1^{(1)}(t) - x_r^{(0)}(t) \right\|, \ldots, \left\| x_r^{(1)}(t) - x_r^{(0)}(t) \right\| \right)^T \\
&\quad \leqslant \left(\left\| y_1^{(1)}(t) - y_1^{(0)}(t) \right\|, \ldots, \left\| y_r^{(0)}(t) - y_r^{(0)}(t) \right\| \right)^T \\
&\quad = (I_n - H_\triangle)^{-1} \left(\left\| y_1^{(0)}(t) \right\|, \ldots, \left\| y_r^{(0)}(t) \right\| \right)^T \\
&\quad := \tilde{H} \left(\left\| y_1^{(0)}(t) \right\|, \ldots, \left\| y_r^{(0)}(t) \right\| \right)^T.
\end{aligned} \tag{7.2.5}
$$

Let $\tilde{H}_m := (h_{ij}^{(m)})_{r\times r}$. Then we can express (7.2.5) as

$$
\left\| x_i^{(m)}(t) - x_i^{(m-1)}(t) \right\| \leqslant \sum_{j=1}^{n} h_{ij}^{(m)} \left\| y_j^{(0)}(t) \right\| \quad (i = 1, 2, \ldots, r). \tag{7.2.6}
$$

Similarly, following the derivation of $(7.2.4)_1$–$(7.2.4)_r$ we can prove that

$$
\begin{aligned}
\left\| x_1^{(m+1)}(t) - x_1^{(m)}(t) \right\| &\leqslant \sum_{j=1}^{r} \sum_{s=1}^{r} h_{ij} h_{js}^{(m)} \left\| y_j^{(0)}(t) \right\| \\
&:= \left\| y_1^{(m+1)}(t) - y_1^{(m)}(t) \right\|,
\end{aligned} \tag{7.2.7}
$$

$$
\begin{aligned}
\left\| x_2^{(m+1)}(t) - x_2^{(m)}(t) \right\| &\leqslant h_{21} \left\| y_1^{(m+1)}(t) - y_1^{(m)}(t) \right\| \\
&\quad + \sum_{j=2}^{r} h_{2j} \left\| y_j^{(m)}(t) - y_j^{(m-1)}(t) \right\| \\
&:= \left\| y_2^{(m+1)}(t) - y_2^{(m)}(t) \right\|,
\end{aligned} \tag{7.2.8}
$$

$$\begin{aligned}\left\|x_r^{(m+1)}(t) - x_r^{(m)}(t)\right\| \leqslant\ & h_{21}\left\|y_1^{(m+1)}(t) - y_1^{(m)}(t)\right\| \\ & + h_{r2}\left\|y_2^{(m+1)}(t) - y_2^{(m)}(t)\right\| \\ & + \cdots + h_{rr}\left\|y_r^{(m)}(t) - y_r^{(m-1)}(t)\right\| \\ := & \left\|y_r^{(m+1)}(t) - y_r^{(m)}(t)\right\|. \end{aligned} \tag{7.2.9}$$

So we obtain

$$\begin{aligned} & \left(\left\|x_1^{(m+1)}(t) - x_1^{(m)}(t)\right\|, \ldots, \left\|x_r^{(m+1)}(t) - x_r^{(m)}(t)\right\|\right)^T \\ & \quad \leqslant \tilde{H}\tilde{H}^m\left(\left\|y_1^{(0)}(t)\right\|, \ldots, \left\|y_r^{(0)}(t)\right\|\right)^T \\ & \quad = \tilde{H}^{m+1}(M_1, M_2, \ldots, M_r)^T e^{-\int_{t_0}^t \varepsilon(\xi)\, d\xi}. \end{aligned} \tag{7.2.10}$$

Hence, by the method of mathematical induction, we conclude that (7.2.10) holds for arbitrary natural numbers m, and

$$\begin{aligned} & \left(\left\|x_1^{(m+1)}(t)\right\|, \ldots, \left\|x_r^{(m+1)}(t)\right\|\right)^T \\ & \quad \leqslant \left(\left\|x_1^{(m+1)}(t) - x_1^{(m)}(t)\right\|, \ldots, \left\|x_r^{(m+1)}(t) - x_r^{(m)}(t)\right\|\right)^T \\ & \qquad + \cdots + \left(\left\|x_1^{(1)}(t) - x_1^{(0)}(t)\right\|, \ldots, \left\|x_r^{(1)}(t) - x_r^{(0)}(t)\right\|\right)^T \\ & \qquad + \left(\left\|x_1^{(0)}(t)\right\|, \ldots, \left\|x_r^{(0)}(t)\right\|\right)^T \\ & \quad \leqslant \left(I_n + \tilde{H}, \ldots, \tilde{H}^{m+1}\right)(M_1, \ldots, M_r)^T e^{-\int_{t_0}^t \varepsilon(\xi)\, d\xi} \\ & \quad \leqslant (I_n - \tilde{H})^{-1}(M_1, M_2, \ldots, M_r)^T e^{-\int_{t_0}^t \varepsilon(\xi)\, d\xi}. \end{aligned}$$

Finally, by proceeding as in the proof of Theorem 7.1.1, Theorem 7.2.1 is proved. □

THEOREM 7.2.2. *Suppose that*

(1) *condition* (1) *in Theorem* 7.1.1 *holds*;

(2) *let* $b_{ij}(t) := b_{ij}(t, t_0) := \int_{t_0}^t e^{-\int_{t_1}^t (\alpha_i(\xi) - \varepsilon(\xi))\, d\xi} g_{ij}(t_1)\, dt_1,\ i, j = 1, \ldots, r,$

$$\begin{aligned} \sum_{j=2}^n b_{ij}(t, t_0) &\leqslant \mu_1 < 1, \\ b_{21}(t, t_0)\mu_1 + \sum_{j=2}^n b_{ij}(t, t_0) &\leqslant \mu_2 < 1, \\ &\vdots \\ \sum_{j=1}^{r-1} b_{rj}(t, t_0)\mu_j + b_{rr}(t, t_0) &\leqslant \mu_r < 1. \end{aligned}$$

Then the conclusions (a)–(e) *in Theorem* 7.1.1 *hold, respectively, under the conditions* (a)–(e) *of condition* (3) *of Theorem* 7.1.1.

PROOF. Let $c_i = x_{i0}, x_i(t, t_0, c) := x_i(t)$. Take

$$
\begin{aligned}
c &:= \max_{1\leqslant i\leqslant r} |c_i|, \\
k &:= \sup_{\substack{1\leqslant i\leqslant r\\ t\geqslant t_0}} \left\{ \sum_{j=1}^{r} b_{ij}(t, t_0) + 1 \right\}, \\
\mu &:= \max_{1\leqslant i\leqslant r} \mu_i .
\end{aligned}
$$

We again consider systems (7.1.1) and (7.1.2), but for (7.1.3) we apply the following Gauss–Seidel type iteration:

$$
\begin{aligned}
x_i^{(1)}(t) = {}& K_{ii}(t, t_0)c_i \\
& + \int_{t_0}^{t} K_{ii}(t, t_1) F_i\big(t_1, x_i^{(0)}(t_1), \dots, x_{i-1}^{(0)}(t_1), 0, \dots, 0\big)\, dt_1, \\
& i = 1, \dots, r, \qquad (7.2.11)\\
x_i^{(m)}(t) = {}& K_{ii}(t, t_0)c_i \\
& + \int_{t_0}^{t} K_{ii}(t, t_1) F_i\big(t_1, x_1^{(m)}(t_1), \dots, x_{i-1}^{(m)}(t_1), x_i^{(m-1)}(t_1), \dots, \\
& x_r^{(m-1)}(t_1)\big)\, dt_1, \\
& m = 2, 3, \dots, \ i = 1, 2, \dots, n. \qquad (7.2.12)
\end{aligned}
$$

We obtain the following estimations:

$$
\begin{aligned}
\big\| x_1^{(1)}(t) \big\| &\leqslant cMe^{-\int_{t_0}^t \varepsilon(\xi)\, d\xi}, \\
\big\| x_2^{(1)}(t) \big\| &\leqslant cMe^{-\int_{t_0}^t \alpha_i(\xi)\, d\xi} \\
&\quad + \int_{t_0}^{t} e^{-\int_{t_1}^t [\alpha_2(\xi) - \varepsilon(\xi)]\, d\xi} g_{21}(t_1) e^{\int_{t_1}^t -\varepsilon(\xi)\, d\xi} cMe^{-\int_{t_0}^t -\varepsilon(\xi) d\xi}\, dt_1 \\
&\leqslant cMe^{\int_{t_0}^t -\varepsilon(\xi)\, d\xi} + cMb_{21}(t) e^{\int_{t_0}^t -\varepsilon(\xi)\, d\xi} \\
&\leqslant kcMe^{-\int_{t_0}^t \varepsilon(\xi)\, d\xi}, \\
&\quad \vdots
\end{aligned}
$$

$$
\begin{aligned}
\|x_r^{(1)}(t)\| &\leqslant cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} + \sum_{j=1}^{r-1} e^{-\int_{t_0}^t \alpha_j(\xi)\,d\xi} g_{rj}(t_1)\big|x_j^{(i)}(t_1)\big|\,dt_1 \\
&\leqslant cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} + b_{n1}(t)cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\quad + b_{r2}(t)kcMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\quad + \cdots + b_{r,r-1}(t)k^{r-1}cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\leqslant k^n cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi},
\end{aligned}
$$

i.e., it holds

$$
\|x_i^{(1)}(t)\| \leqslant k^r cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (i = 1, 2, \ldots, r). \tag{7.2.13}
$$

$$
\begin{aligned}
&\|x_1^{(2)}(t) - x_1^{(1)}(t)\| \\
&\quad \leqslant \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_{t_1}^t [\alpha_{11}(\xi)-\varepsilon(\xi)]\,d\xi} g_{1j}(t_1) e^{-\int_{t_1}^t \varepsilon(\xi)\,d\xi} k^r cMe^{-\int_{t_0}^t \varepsilon(\xi) d\xi}\,dt_1 \\
&\quad \leqslant \sum_{j=1}^{n} k^r cMb_{ij}(t)e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\quad \leqslant \mu_1 k^r cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi},
\end{aligned} \tag{7.2.14$_1$}
$$

$$
\begin{aligned}
&\|x_2^{(2)}(t) - x_2^{(1)}(t)\| \\
&\quad \leqslant \int_{t_0}^{t} e^{-\int_{t_1}^t \alpha_{22}(\xi)\,d\xi} g_{2i}(t_1)\|x_1^{(2)}(t_1) - x_1^{(1)}(t_1)\|\,dt_1 \\
&\qquad + \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_{t_1}^t \alpha_{22}(\xi)\,d\xi} g_{2j}(t_1)\|x_j^{(1)}(t_1)\|\,dt_1 \\
&\quad \leqslant \int_{t_0}^{t} e^{-\int_{t_1}^t [\alpha_{22}(\xi)-\varepsilon(\xi)]\,d\xi} g_{21}(t_1)e^{-\int_{t_1}^t \varepsilon(\xi)\,d\xi} \mu_1 k^r cMe^{-\int_{t_0}^{t_1} \varepsilon(\xi)\,d\xi}\,dt_1 \\
&\qquad + \sum_{j=2}^{n} \int_{t_0}^{t} e^{\int_{t_1}^t [\alpha_{22}(\xi)-\varepsilon(\xi)]\,d\xi} g_{2j}(t_1)k^r cMe^{-\int_{t_1}^t \varepsilon(\xi)\,d\xi} e^{-\int_{t_0}^{t_1} \varepsilon(\xi)\,d\xi}\,dt_1 \\
&\quad \leqslant \left[\mu_1 b_{21}(t) + \sum_{j=2}^{r} b_{2j}(t)\right] k^r cMe^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi}
\end{aligned}
$$

$$\leqslant \mu_2 k^r c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \tag{7.2.14$_2$}$$

$$\vdots$$

$$\begin{aligned}
&\left\|x_n^{(2)}(t) - x_n^{(1)}(t)\right\| \\
&\quad \leqslant \sum_{j=1}^{r-1} \int_{t_0}^{t} e^{-\int_{t_1}^t (\alpha_n(\xi)-\varepsilon(\xi))\,d\xi} g_{rj}(t_1) e^{-\int_{t_1}^t \varepsilon(\xi)\,d\xi} \left\|x_j^{(2)}(t_1) - x_j^{(1)}(t_1)\right\| dt_1 \\
&\qquad + \int_{t_0}^{t} e^{-\int_{t_1}^t (\alpha_n(\xi)-\varepsilon(\xi))\,d\xi} g_{rr}(t_1) e^{-\int_{t_1}^t \varepsilon(\xi)\,d\xi} \left\|x_j^{(1)}(t_1)\right\| dt_1 \\
&\quad \leqslant \sum_{j=1}^{n-1} \mu_j b_{nj}(t) k^n c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} + b_{rr}(t) k^n c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\quad \leqslant \mu_r k^r c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi}.
\end{aligned} \tag{7.2.14$_r$}$$

Now, by the method of mathematical induction, we can prove that

$$\left\|x_j^{(m)}(t) - x_j^{(m-1)}(t)\right\| \leqslant \mu^{m-1} k^r c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi}, \quad j = 1, \dots, r, \tag{7.2.15}$$

hold for arbitrary natural number m, which implies that

$$\begin{aligned}
\left\|x_i^{(m+1)}(t)\right\| &\leqslant \sum_{j=1}^{m} \left\|x_i^{(j+1)}(t) - x_i^{(j)}(t)\right\| + \left\|x_i^{(1)}(t)\right\| \\
&\leqslant \left(\sum_{j=0}^{m} \mu^j\right) k^{r-1} c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\leqslant \frac{k^r}{1-\mu} c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi},
\end{aligned} \tag{7.2.16}$$

and thus, for arbitrary natural number p, we have

$$\begin{aligned}
\left\|x_i^{(m+p)}(t) - x_i^{(m)}(t)\right\| &\leqslant \sum_{j=m+1}^{m+p} \left\|x_i^{(j)}(t) - x_i^{(j-1)}(t)\right\| \\
&\leqslant \left(\sum_{j=m+1}^{m+p} \mu^j\right) k^r c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \\
&\leqslant \frac{\mu^m}{1-\mu} k^r c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi}.
\end{aligned} \tag{7.2.17}$$

Therefore, on any finite interval $[t_0, T]$ $\{x_i^{(m)}(t)\}$ is a fundamental sequence (Cauchy sequence).

Next, we prove that the zero solution of (7.1.1) is stable. Otherwise, assume that $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists t_0$ and $\exists \tau > 0$, $\|x_0\| < \delta$ the system at east has the solution:

$$\left\| x_{0i}(t_0 + \tau, t_0, x_0) \right\| \geqslant \varepsilon. \tag{7.2.18}$$

However, since the continuous function space $C[t_0, t_0 + 2T]$ with uniformly convergent topology structure is a Banach space, we have

$$x_i^{(m)} \rightarrow x_i(t) \quad \text{as } m \to \infty,\ i = 1, \dots, n.$$

Thus, $x_i(t)$ admits the estimation:

$$\left\| x_i(t) \right\| \leqslant \frac{k^r cM}{1-\mu} e^{-\int_{t_0}^{t} \varepsilon(\xi)\, d\xi} \quad (i = 1, 2, \dots, n)$$

on $[t_0, t_0 + 2T]$.

Now we choose

$$0 < \delta < (1-\mu)\varepsilon / mk^r \max_{t_0 \leqslant t \leqslant t_0 + 2T} e^{-\int_{t_0}^{t} \varepsilon(\xi)\, d\xi},$$

and then for $0 < \|x_0\| < \delta$ we have

$$\begin{aligned} &\left\| x_i(t) \right\| \leqslant \frac{k^r cM}{1-\mu} e^{-\int_{t_0}^{t} \varepsilon(\xi)\, d\xi} < \varepsilon, \\ &\quad i = 1, 2, \dots, n,\ t \in [t_0, t_0 + 2T], \end{aligned} \tag{7.2.19}$$

which is a contradiction with (7.2.18). So the zero solution of system (7.1.1) is stable. By the estimation (7.2.16) one can obtain all conclusions in Theorem 7.2.2.

The proof of Theorem 7.2.2 is complete. □

COROLLARY 7.2.3. *If the following conditions hold*:

(1) $\|K_{ii}(t, t_0)\| \leqslant M_i e^{-\alpha_i (t - t_0)}$ $(t \geqslant t_0,\ i = 1, 2, \dots, r)$;

(2) $g_{ij}(t) = g_{ij} = const,\ i, j = 1, 2, \dots, r,$

$$\begin{aligned} \sum_{j=1}^{n} \frac{g_{1j}}{\alpha_1} M_1 &\leqslant \mu_1 < 1, \\ \frac{g_{21} M_2}{\alpha_2} \mu_1 + \sum_{j=2}^{n} \frac{M_2 g_{2j}}{\alpha_2} &\leqslant \mu_2 < 1, \\ &\vdots \\ \sum_{j=1}^{r-1} \frac{g_{rj} M_r}{\alpha_r} M_j + g_{rr} &\leqslant \mu_n < 1; \end{aligned} \tag{7.2.20}$$

then the zero solution of system (7.1.1) is exponentially stable.

PROOF. Choose $0 < \varepsilon \ll \alpha_i$ $(i = 1, 2, \dots, r)$. Then, we have

(1) $\|K_{ii}(t, t_0)\| \leqslant M_i e^{-\alpha_i(t-t_0)} \leqslant M e^{-\varepsilon}(t - t_0)$, i.e., the condition (1) in Theorem 7.2.2 holds.

(2) Since

$$b_{ij}(t, t_0) = \int_{t_0}^{t} M_i e^{-(\alpha_i-\varepsilon)(t-t_1)} g_{ij}\, dt_1 \leqslant e^{-(\alpha_i-\varepsilon)} M_i g_{ij} \cdot \frac{e^{(a_i-\varepsilon)t}}{a_i - \varepsilon}$$
$$= \frac{M_i g_{ij}}{a_i - \varepsilon},$$

one can take $0 < \varepsilon \ll 1$ such that

$$\sum_{j=1}^{r} \frac{g_{ij} M_1}{a_1 - \varepsilon} \leqslant \tilde{\mu}_1 < 1,$$
$$\frac{a_{21} M_2}{a_2 - \varepsilon} \mu_1 + \sum_{j=2}^{n} g_{2j} M_2 \leqslant \tilde{\mu}_2 < 1,$$
$$\vdots$$
$$\sum_{j=1}^{r-1} \frac{g_{rj} M_r}{a_n - \varepsilon} \tilde{\mu}_j + g_{rr} M_r \leqslant \tilde{\mu}_r < 1.$$

Hence, condition (2) of Theorem 7.2.2 hold.

The conclusion is true. □

THEOREM 7.2.4. *Assume that*

(1) *condition* (1) *of* Theorem 7.2.2 *holds*;

(2) $b_{ij}(t)$ *defined in condition* (2) *of* Theorem 7.2.2 *satisfies*

$$\max_{1 \leqslant j \leqslant n} b_{1j}(t, t_0) \leqslant \rho^{(1)},$$
$$b_{21}(t, t_0)\rho^{(1)} + \max_{2 \leqslant j \leqslant n} b_{2j}(t, t_0) \leqslant \rho^{(2)},$$
$$\vdots$$
$$\sum_{j=1}^{r-1} b_{rj}(t, t_0)\rho^{(j)} + b_{rr}(t, t_0) \leqslant \rho^{(r)},$$
$$\sum_{j=1}^{r} \rho^{(j)} := \rho < 1. \tag{7.2.21}$$

Then the conclusions of Theorem 7.2.2 hold.

PROOF. Along the derivation from (7.2.11) to (7.2.13) and let M, c, k be that defined in Theorem 7.2.2, we have

$$
\begin{aligned}
&\left\|x_1^{(2)}(t) - x_1^{(1)}(t)\right\| \\
&\quad \leqslant \sum_{j=1}^{n}\left(\int_{t_0}^{t} M_1 e^{-\int_{t_0}^{t}(\alpha_1(\xi)-\varepsilon(\xi))\,d\xi} g_{1j}(t_1)k^{j-1}cM\,dt_1\right)e^{-\int_{t_0}^{t}\varepsilon(\xi)\,d\xi} \\
&\quad \leqslant \rho^{(1)} r k^{r-1} cMe^{-\int_{t_0}^{t}\varepsilon(\xi)\,d\xi}, \\
&\left\|x_2^{(2)}(t) - x_2^{(1)}(t)\right\| \\
&\quad \leqslant \int_{t_0}^{t} M_2 e^{-\int_{t_1}^{t}(\alpha_2(\xi)-\varepsilon(\xi))\,d\xi} g_{21}(t_1)\left\|x_1^{(2)}(t_1) - x_1^{(1)}(t_1)\right\|dt_1 \\
&\qquad + \sum_{j=2}^{n}\int_{t_0}^{t} M_2 e^{-\int_{t_0}^{t}\alpha_2(\xi)} g_{2j}(t_1)\left\|x_j^{(1)}(t_1)\right\|dt_1 \\
&\quad \leqslant \Bigg(\int_{t_0}^{t} Me^{-\int_{t_0}^{t}(\alpha_2(\xi)-\varepsilon(\xi))} g_{2j}(t_1)\rho^{(1)} r k^{r-1} cM\,dt_1 \\
&\qquad + \max_{2\leqslant j\leqslant n}\int_{t_0}^{t} M_2 e^{-\int_{t_1}^{t}[\alpha_2(\xi)-\varepsilon(\xi)]\,d\xi} g_{2j}(t_1)\,dt_1 n k^{n-1} cM\Bigg)e^{-\int_{t_0}^{t}\varepsilon(\xi)\,d\xi} \\
&\quad \leqslant \rho^{(2)} r k^{r-1} cMe^{-\int_{t_0}^{t}\varepsilon(\xi)\,d\xi}, \\
&\quad \vdots \\
&\left\|x_r^{(2)}(t) - x_r^{(1)}(t)\right\| \\
&\quad \leqslant \sum_{j=1}^{r-1}\int_{t_0}^{t} M_r e^{-\int_{t_1}^{t}(\alpha_r(\xi)-\varepsilon(\xi))\,d\xi} g_{rj}(t_1)\left\|x_j^{(2)}(t_1) - x_j^{(1)}(t_1)\right\|dt_1 \\
&\quad \leqslant r k^{r-1} cM\rho^{(r)} e^{-\int_{t_0}^{t}\varepsilon(\xi)\,d\xi}.
\end{aligned}
$$

Thus, we obtain

$$
\sum_{j=1}^{r}\left\|x_j^{(2)}(t) - x_j^{(1)}(t)\right\| \leqslant r k^{r-1} cM\rho e^{-\int_{t_0}^{t}\varepsilon(\xi)\,d\xi}.
$$

Then, by the method of mathematical induction one can prove that

$$\sum_{j=1}^{r} \left\| x_j^{(m)}(t) - x_j^{(m-1)}(t) \right\| \leqslant \rho^{m-1} r k^{r-1} c M e^{-\int_{t_0}^{t} \varepsilon(\xi)\, d\xi}$$

holds for arbitrary natural number m, and

$$\begin{aligned}
\sum_{j=1}^{r} \left\| x_j^{(m)}(t) \right\| &\leqslant \sum_{j=1}^{n} \left\| x_j^{(m)}(t) - x_j^{(m-1)}(t) \right\| \\
&\quad + \sum_{j=1}^{n} \left\| x_j^{(m-1)}(t) - x_j^{(m-2)}(t) \right\| + \cdots + \sum_{j=1}^{n} \left\| x_j^{(1)}(t) \right\| \\
&\leqslant \left(\rho^{m-1} + \rho^{m-2} + \cdots + 1\right) r k^{r-1} c M e^{-\int_{t_0}^{t} \varepsilon(\xi)\, d\xi} \\
&\leqslant \frac{1}{1-\rho} r k^{r-1} c M e^{-\int_{t_0}^{t} \varepsilon(\xi)\, d\xi}.
\end{aligned}$$

Therefore, $\{x_j^{(m)}(t)\}$ is a fundamental sequence

Finally, it follows from the proof of Theorem 7.2.2 that the conclusion is true. □

COROLLARY 7.2.5. *If the following conditions are satisfied*:

(1) *condition* (1) *of Corollary* 7.2.3 *holds*;
(2) $g_{ij}(t) = g_{ij} = const.,\ i, j = 1, 2, \ldots, r,$

$$\begin{aligned}
\max_{1 \leqslant j \leqslant r} g_{ij} \frac{M_1}{\alpha_1} &\leqslant \rho^{(1)}, \\
\frac{g_{21} M_2}{\alpha_2} \rho^{(1)} + \max_{2 \leqslant j \leqslant r} \frac{g_{ij} M_2}{\alpha_2} &\leqslant \rho^{(2)}, \\
\sum_{j=1}^{r-1} g_{rj} \frac{M_r}{\alpha_r} \rho^{(j)} + \frac{M_r}{\alpha_r} \cdot g_{rr} &\leqslant \rho^{(r)}, \\
\sum_{j=1}^{r} \rho^{(j)} = \rho &< 1;
\end{aligned}$$

then the zero solution of (7.1.1) *is exponentially stable.*

PROOF. The proof is similar to that for Corollary 7.2.3 and is thus omitted. □

EXAMPLE 7.2.6. Consider the stability of the linear system:

$$\begin{cases} \frac{dx_1}{dt} = (-1 + 2\sin t)x_1 + a^2(\cos^m t)x_2, \\ \frac{dx_2}{dt} = b^2(\sin^n t)x_1 + (-1 + 2\cos t)x_2, \end{cases} \tag{7.2.22}$$

where $a^2 < \frac{0.9}{e^4}$, $b^2 < \frac{0.9^2}{e^4}$.

We check the conditions of Corollary 7.2.3 for system (7.2.22).

(1) Take $\varepsilon(t) = 0.1$, $M = e^4$. Then, we have

$$e^{\int_{t_0}^{t}[-1+2\sin\xi]\,d\xi} \leqslant e^4 e^{-\int_{t_0}^{t} d\xi} \leqslant e^4 e^{-0.1(t-t_0)} \to 0 \quad \text{as } t \to +\infty$$
$$e^{\int_{t_0}^{t}[-1+2\cos\xi]\,d\xi} \leqslant e^4 e^{-0.1(t-t_0)} \to 0 \quad \text{as } t \to +\infty;$$

(2)

$$\begin{aligned}
b_{12}(t,t_0) &:= \int_{t_0}^{t} e^{\int_{t_0}^{t}[-0.9+2\sin\xi]\,d\xi} \left|a^2 \cos^m t_1\right| dt_1 \\
&\leqslant a^2 e^4 \int_{t_0}^{t} e^{-0.9t} e^{+0.9t_1}\, dt_1 \\
&= \frac{a^2 e^4}{0.9} e^{-0.9t}\left(e^{0.9t} - e^{0.9t_0}\right) \\
&\leqslant \frac{a^2 e^4}{0.9} := \mu_1 < 1, \\
b_{21}(t,t_0)\mu_1 &= \frac{a^2 e^4}{0.9} \int_{t_0}^{t} e^{\int_{t_1}^{t}[-0.9+2\cos\xi]\,d\xi} \left|b^2 \sin^2 t_1\right| dt_1 \\
&\leqslant \frac{a^2 e^4}{0.9} b^2 e^4 e^{-0.9t}\left(e^{0.9t} - e^{0.9t_0}\right) \\
&\leqslant \frac{a^2 b^2}{0.9^2} e^8 \\
&= \mu_2 < 1.
\end{aligned}$$

Hence, the conditions are satisfied. So the zero solution of system (7.2.22) is exponentially stable.

For exponentially stable system, the estimation of exponential rate is very important, which is usually called decay rate, as appeared in automatic control systems.

EXAMPLE 7.2.7. Estimate the decay rate of the system:

$$\begin{cases} \frac{dx_1}{dt} = -3x_1 + (\sin t)x_2, \\ \frac{dx_2}{dt} = \frac{at}{1+t^2}x_1 + (-3+\cos t)x_2, \end{cases} \tag{7.2.23}$$

where it is assumed that $|a| < 4$.

Since $e^{\int_{t_0}^t a_{11}\,d\xi} = e^{-\int_{t_0}^t 3\,d\xi} = e^{-3(t-t_0)} \leqslant e^{-(t-t_0)}$ (taking $\varepsilon = 1$), we have

$$\int_{t_0}^{t} e^{-\int_{t_1}^t (-3+1)\,d\xi} |\sin t_1|\,dt_1 \leqslant \frac{1}{2} := \mu_1,$$

$$\int_{t_0}^{t} e^{\int_{t_1}^t (-3+\cos\xi+1)\,d\xi} \left|\frac{at_1}{1+t_1^2}\right| dt_1 \leqslant \int_{t_0}^{t} e^{-(t-t_1)} \frac{|a|}{2}\,dt_1 \frac{1}{2} < \frac{|a|}{4} = \mu_2 < 1.$$

Thus, the solution of (7.2.23) admits that estimation:

$$\|x(t)\| \leqslant M e^{-(t-t_0)} \quad \text{by taking } \alpha = 1,$$

i.e., the system (7.2.23) at least has the decay rate $\alpha = 1$.

7.3. Application of iteration method to extreme stability

In this section, we apply the iteration method to investigate the extreme stability for a class of nonlinear systems.

Consider the system:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} p_{ij}(t)x_j + f_i(t,x), \quad i = 1, 2, \ldots, n, \tag{7.3.1}$$

where $x = (x_1, \ldots, x_n)^T$, $f_i(t,x) \in C[S_H, R^n]$, $p_{ij}(t) \in C[I, R^1]$, $S_H = \{x \mid \|x\| \leqslant H\}$.

Suppose $f = (f_1, \ldots, f_n)^T$ satisfies the Lipschitz condition:

$$\begin{aligned} &\left|\sum_{\substack{j=1\\ j\neq i}}^{n} p_{ij}(t)(x_j - y_j) + f_i(t,x) - f_i(t,y)\right| \\ &\quad \leqslant \sum_{j=1}^{n} l_{ij}(t)|x_j - y_i| \quad \text{for } l_{ij}(t) \in C\big[I, R_+^1\big]. \end{aligned}$$

DEFINITION 7.3.1. System (7.3.1) is said to be extremely stable (uniformly extremely stable) with respect to H^* ($H^* < H$) if $\forall \varepsilon > 0$, $\forall t_0 \in I$, $\exists \delta(\varepsilon, t_0)$ ($\delta(\varepsilon)$), $\forall x_0 \in S_H^*$, $y_0 \in S_H^*$, $S_H^* = \{x \mid \|x\| \leqslant H^*\}$ such that when $\|x_0 - y_0\| < \delta(\varepsilon, t_0)$ it holds

$$\|x(t, t_0, x_0) - y(t, t_0, y_0)\| < \varepsilon, \quad t \geqslant t_0.$$

DEFINITION 7.3.2. System (7.3.1) is said to be extremely attractive (uniformly extremely attractive) with respect to (H^*, H) if $\forall \varepsilon > 0$, $\exists \delta(\varepsilon)$ and $T(\varepsilon, t_0) > 0$ ($T(\varepsilon) > 0$) such that $\forall x_0 \in S_H^*$, $y_0 \in S_H^*$, when $\|x_0 - y_0\| < \delta$ for all $t \geqslant t_0 + T(\varepsilon, t_0)$ ($t \geqslant t_0 + T(\varepsilon)$) it holds

$$\left\|x(t, t_0, x_0) - y(t, t_0, y_0)\right\| < \varepsilon.$$

DEFINITION 7.3.3. If system (7.3.1) is extremely stable and extremely attractive (extremely uniformly stable and extremely uniformly attractive), then system (7.3.1) is said to be extremely asymptotically stable (extremely uniformly asymptotically stable).

DEFINITION 7.3.4. System (7.3.1) is said to be extremely exponentially stable if $\forall x_0 \in S_H^*$, $y_0 \in S_H^*$, there exist $M(x_0, y_0) \geqslant 1$ and $\alpha > 0$ such that

$$\left\|x(t, t_0, x_0) - y(t, t_0, y_0)\right\| < M(x_0, y_0) e^{-\alpha(t-t_0)}.$$

THEOREM 7.3.5. *If system (7.3.1) satisfies the following conditions*:

(1) *there exist function $\varepsilon(t) \in C[I, R^1]$ and constant $M > 0$ such that*

$$e^{\int_{t_0}^t p_{ii}(\xi)\, d\xi} \leqslant M e^{-\int_{t_0}^t \varepsilon(\xi)\, d\xi};$$

(2) $a_{ij}(t) := \int_{t_0}^t e^{\int_{t_1}^t [p_{ii}(\xi)+\varepsilon(\xi)]\, d\xi} l_{ij}(t_1)\, dt_1$ ($i, j = 1, 2, \dots, n$), *and*

$$\sum_{j=1}^n a_{ij}(t) \leqslant \mu_1 < 1,$$

$$a_{21}(t)\mu_1 + \sum_{j=2}^n a_{2j}(t) \leqslant \mu_2 < 1,$$

$$\sum_{j=1}^{n-1} a_{nj}(t)\mu_j + a_{nn}(t) \leqslant \mu_n < 1,$$

where μ_i ($i = 1, 2, \dots, n$) are positive constants;

then,

(1) $\int_{t_0}^t \varepsilon(\xi)\, d\xi \geqslant k(t_0) = const.$ ($t \geqslant t_0 \geqslant 0$);
(2) $\int_{t_0}^t \varepsilon(\xi)\, d\xi \geqslant k = const.$ ($t \geqslant t_0 \geqslant 0$);
(3) $\int_{t_0}^{+\infty} \varepsilon(\xi)\, d\xi = +\infty$;
(4) $\int_{t_0}^t \varepsilon(\xi)\, d\xi \to +\infty$ *with respect to t_0, as $t - t_0 \to +\infty$*;
(5) $\int_{t_0}^t \varepsilon(\xi)\, d\xi \geqslant \alpha(t - t_0)$ ($\alpha = const. > 0$);

which imply, respectively, the solution of (7.3.1) *to be*

(1) *extremely stable*;
(2) *uniformly extremely stable*;
(3) *extremely asymptotically stable*;
(4) *uniformly extremely asymptotically stable*;
(5) *extremely exponentially stable*;

PROOF. Let the solution of (7.3.1) be

$$x_i(t, t_0, x_0) := x_i(t), \tag{7.3.2}$$

$$y_i(t, t_0, y_0) := y_i(t). \tag{7.3.3}$$

Applying the method of constant variation, we have

$$\begin{aligned} x_i(t) = x_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{\substack{j=1 \\ j\neq i}}^{n} p_{ij}(t_1) x_j(t_1) \\ + f_i\big(t_1, x_1(t_1), \dots, x_n(t_1)\big) \Bigg] dt_1, \quad i = 1, 2, \dots, n, \end{aligned} \tag{7.3.4}$$

$$\begin{aligned} y_i(t) = y_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{\substack{j=1 \\ j\neq i}}^{n} p_{ij}(t_1) y_j(t_1) \\ + f_i\big(t_1, y_1(t_1), \dots, y_n(t_1)\big) \Bigg] dt_1, \quad i = 1, 2, \dots, n. \end{aligned} \tag{7.3.5}$$

For equations (7.3.4) and (7.3.5), we respectively apply the iteration to obtain

$$\begin{aligned} x_i^{(m)}(t) &= x_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} \\ &\quad + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{j=1}^{j-1} p_{ij}(t_1) x_j^{(m)}(t_1) + \sum_{j=i+1}^{n} p_{ij}(t_1) x_j^{(m-1)}(t_1) \\ &\quad + f_i\big(t_1, x_1^{(m)}(t_1), \dots, x_{i-1}^{(m)}(t_1), x_i^{(m-1)}(t_1), \dots, x_n^{(m-1)}(t_1)\big) \Bigg] dt_1 \\ &\quad (m = 2, 3, \dots,\ i = 1, 2, \dots, n), \end{aligned} \tag{7.3.6}$$

$$x_i^{(1)}(t) = x_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{j=1}^{j-1} p_{ij}(t_1) x_j^{(1)}(t_1)$$

$$+ f_i\big(t_1, x_1^{(1)}(t_1), \dots, x_{i-1}^{(1)}(t_1), 0, \dots, 0\big)\Bigg] dt_1,$$

$$i = 1, 2, \dots, n, \tag{7.3.7}$$

$$\begin{aligned} y_i^{(m)}(t) = {} & y_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} \\ & + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{j=1}^{i-1} p_{ij}(t_1) y_j^{(m)}(t_1) + \sum_{j=i+1}^{n} p_{ij}(t_1) y_j^{(m-1)}(t_1) \\ & + f_i\big(t_1, y_1^{(m)}(t_1), \dots, y_{i-1}^{(m)}(t_1), y_i^{(m-1)}(t_1), \dots, y_n^{(m-1)}(t_1)\big)\Bigg] dt_1 \\ & (m = 2, 3, \dots,\ i = 1, 2, \dots, n), \end{aligned} \tag{7.3.8}$$

$$\begin{aligned} y_i^{(1)}(t) = {} & y_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{j=1}^{i-1} p_{ij}(t_1) y_j^{(1)}(t_1) \\ & + f_i\big(t_1, y_1^{(1)}(t_1), \dots, y_{i-1}^{(1)}(t_1), 0, \dots, 0\big)\Bigg] dt_1. \end{aligned} \tag{7.3.9}$$

Let $z_i^{(m)}(t) = x_i^{(m)}(t) - y_i^{(m)}(t)$, $z_i(t) = x_i(t) - y_i(t)$, $z_{0i} = x_{0i} - y_{i0}$, $i = 1, 2, \dots, n$, $m = 1, 2, \dots$. Then we have

$$\begin{aligned} z_i^{(m)}(t) = {} & z_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} \\ & + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{j=1}^{i-1} p_{ij}(t_1) z_j^{(m)}(t) + \sum_{j=i+1}^{n} p_{ij}(t_1) z_j^{(m-1)}(t_1) \\ & + f_i\big(t_1, x_1^{(m)}(t_1), \dots, x_{i-1}^{(m)}(t_1), x_i^{(m-1)}(t_1), \dots, x_n^{(m-1)}(t_1)\big) \\ & - f_i\big(t, y_1^{(m)}(t_1), \dots, y_{i-1}^{(m)}(t_1), y_i^{(m-1)}(t_1), \dots, y_n^{(m-1)}(t_1)\big)\Bigg] dt_1 \\ & (i = 1, 2, \dots, n,\ m = 2, 3, \dots), \end{aligned} \tag{7.3.10}$$

$$\begin{aligned} z_i^{(1)}(t) = {} & z_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} + \int_{t_0}^{t} e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{j=1}^{i-1} p_{ij}(t_1) z_j^{(1)}(t_1) \\ & + f_i\big(t_1, x_1^{(1)}(t_1), \dots, x_{i-1}^{(1)}(t_1), 0, \dots, 0\big) \\ & - f_i\big(t_1, y_1^{(1)}(t_1), \dots, y_{i-1}^{(1)}(t_1), 0, \dots, 0\big)\Bigg] dt_1. \end{aligned} \tag{7.3.11}$$

Following the proof of Theorem 7.2.2 and employing the method of mathematical induction, one can prove that for an arbitrary natural number m, it holds

$$\begin{aligned}
\left\| z_i^{(m+1)}(t) - z_i^{(m)}(t) \right\| &\leqslant \mu^m k^{n-1} c M e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} \\
&\leqslant \mu^m k^{n-1} c M e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (i = 1, 2, \ldots, n), \\
\left\| z_i^{(m+1)}(t) \right\| &\leqslant \frac{k^{n-1} c M}{1-\mu} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} \\
&\leqslant \frac{k^{n-1} c M}{1-\mu} e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (i = 1, 2, \ldots, n).
\end{aligned}$$

Hence, we have the estimation:

$$\begin{aligned}
\left\| z_i(t) \right\| &\leqslant \frac{k^{n-1} c M}{1-\mu} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} \\
&\leqslant \frac{k^{n-1} c M}{1-\mu} e^{-\int_{t_0}^t \varepsilon(\xi)\,d\xi} \quad (i = 1, 2, \ldots, n),
\end{aligned} \tag{7.3.12}$$

where $c = \max_{1 \leqslant i \leqslant n} |x_{0i} - y_{0i}|$. Equation (7.3.12) shows that all the conclusions of Theorem 7.3.5 are true. □

THEOREM 7.3.6. *If system (7.3.1) satisfies that*

(1) *condition* (1) *of Theorem* 7.3.5 *holds*;

(2)

$$\begin{aligned}
\max_{1 \leqslant j \leqslant n} a_{ij}(t) &\leqslant \rho^{(1)} = const., \\
a_{21}(t)\rho^{(1)} + \max_{2 \leqslant j \leqslant n} a_{ij}(t) &\leqslant \rho^{(2)} = const., \\
\sum_{j=1}^{n-1} a_{nj}(t)\rho^{(j)} + a_{nn}(t) &\leqslant \rho^{(n)} = const., \\
\sum_{j=1}^{n} \rho^{(j)} &= \rho < 1,
\end{aligned}$$

where

$$a_{ij}(t) = \int_{t_0}^{t} e^{\int_{t_1}^t [p_{ii}(\xi) + \varepsilon(\xi)]\,d\xi} l_{ij}(t_1)\,dt_1;$$

then all the conclusions of Theorem 7.2.1 *hold.*

PROOF. Following the procedures as in the proof of Theorems 7.3.5 and 7.2.4, one can complete the proof for this theorem. □

7.4. Application of iteration method to stationary oscillation

In this section, we consider a class of nonlinear time-varying periodic systems, described by

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} p_{ij}(t)x_j + f_i\big(t, x_i(t), \ldots, x_n(t)\big) + g_i(t), \\ i = 1, 2, \ldots, n, \tag{7.4.1}$$

where

$$\begin{aligned} &p_{ij}(t + w) \equiv p_{ij}(t), \quad f_i(t + w, x) \equiv f_i(t, x), \\ &g_i(t + w) \equiv g_i(t), \quad w = \text{const.} > 0, \\ &p_{ij}(t) \in C\big[I, R^1\big], \quad \xi_i(t) \in C\big[I, R^1\big], \\ &f_{ij}(t, x_1, \ldots, x_n) \in C\big[I \times R^n, R^1\big]. \end{aligned}$$

DEFINITION 7.4.1. If there exists a unique periodic solution $\eta(t, t_0, x_0)$ with period ω, which is globally asymptotically stable, then the system (7.4.1) is called a stationary oscillating system.

THEOREM 7.4.2. *If system (7.4.1) satisfies the following conditions*:

(1)

$$\left|\sum_{\substack{j=1\\ j\neq i}}^{n} p_{ij}(t)(x_j - y_j) + f_i(t, x) - f_i(t, y)\right| \leqslant \sum_{j=1}^{n} l_{ij}(t)|x_j - y_j|,$$

for $i = 1, 2, \ldots, n$ *and* $l_{ij}(t) \in C[I, R^1_+]$;

(2) *there exist function* $\varepsilon(t) \in C[I, R^1]$ *and constant* $M > 0$ *such that*

$$e^{\int_{t_0}^{t} p_{ii}(\xi)\, d\xi} \leqslant M e^{-\int_{t_0}^{t} \varepsilon(\xi)\, d\xi} \to 0 \quad \textit{as } t \to +\infty;$$

(3)

$$\begin{aligned} a_{ij}(t) := \int_{t}^{t} e^{\int_{t_1}^{t} (p_{ii}(\xi)+\varepsilon(\xi))\, d\xi} l_{ij}(t_1)\, dt_1 \\ \leqslant \tilde{a}_{ij} = \textit{const.} \quad (i, j = 1, 2, \ldots, n)\ (t \geqslant t_0), \end{aligned}$$

where the spectral radius $\rho(\tilde{A})$ of the matrix $\tilde{A}$ is less than 1*, i.e., $\rho(\tilde{A}) < 1$ (particularly $\|\tilde{A}\| < 1$);*

then the system (7.4.1) is a stationary oscillating system.

PROOF. First, we express the solution of (7.4.1) as

$$\begin{aligned} x_i(t) = e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} x_{0i} + \int_{t_0}^t \Bigg[\sum_{\substack{j=1\\ j\neq i}}^n e^{\int_{t_1}^t p_{ii}(\xi)} p_{ij}(t_1) x_j(t_1) \\ + e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} f_i\big(t_1, x_1(t_1)\big), \ldots, x_n(t_1) \Bigg] dt_1 \\ + \int_{t_0}^t e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} g_i(t_1)\,dt_1, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{7.4.2}$$

Since continuous periodic function is bounded, let $\|g(t)\| \leqslant k = \text{const}$. Then applying the Picard type iteration to (7.4.2) we obtain

$$\begin{aligned} x_i^{(1)}(t) &= x_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} + \int_{t_0}^t e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} g_i(t)\,dt_1 \quad (i = 1, 2, \ldots, n), \\ x_i^{(m)}(t) &= x_{0i} e^{\int_{t_0}^t p_{ii}(\xi)\,d\xi} + \int_{t_0}^t e^{\int_{t_1}^t p_{ii}(\xi)\,d\xi} \Bigg[\sum_{\substack{j=1\\ j\neq i}}^n p_{ij}(t_1) x_j^{(m-1)}(t_1) \\ &\quad + f_i\big(t_1, x^{(m-1)}(t_1), \ldots, x_n^{(m-1)}(t_1)\big) \Bigg] dt_1, \\ &\quad i = 1, 2, \ldots, n,\ m = 2, 3, \ldots. \end{aligned}$$

So, we have

$$\begin{aligned} \big|x_i^{(1)}(t)\big| &\leqslant |x_{0i}| e^{-\alpha_i(t-t_0)} + \int_{t_0}^t e^{-\alpha_i(t-t_1)} k\,dt_1 \\ &\leqslant |x_{0i}| + \frac{k}{\alpha_i} \\ &:= \tilde{M}_i \quad (i = 1, 2, \ldots, r), \end{aligned}$$

that is,

$$\big(\big|x_1^{(1)}(t)\big|, \ldots, \big|x_n^{(1)}(t_1)\big|\big)^T \leqslant I_n\big(\tilde{M}_1, \ldots, \tilde{M}_r\big)^T, \tag{7.4.3}$$

$$\begin{aligned}\left|x_i^{(1)}(t)\right| &\leqslant \tilde{M}_i + \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\alpha_i(t-t_1)} L_{ij}(t_1)\left|x_j^{(1)}(t_1)\,dt_1\right| \\ &= \tilde{M}_i + \sum_{j=1}^{n} \tilde{a}_{ij}\tilde{M}_j.\end{aligned}$$

Hence, $(|x_1^{(2)}(t)|, \ldots, |x_n^{(2)}(t)|)^T \leqslant (I_n + \tilde{A})(\tilde{M}_1, \ldots, \tilde{M}_r)^T$.

Finally, one can use the method of mathematical induction to prove $(|x_1^{(m)}(t)|, \ldots, |x_1^{(m)}(t)|)^T \leqslant (I_n + \tilde{A} + \cdots + \tilde{A}^m) \cdot (\tilde{M}_1, \tilde{M}_r)^T$. Since $\rho(\tilde{A}) < 1$, we have

$$\left(\left|x_1(t)\right|, \ldots, \left|x_n(t)\right|\right)^T \leqslant \left(I_n - \tilde{A}\right)^{-1}\left(\tilde{M}_1, \ldots, \tilde{M}_n\right)^T,$$

which shows that the solution is bounded. According to Theorem 7.3.5, we know that the solution of system (7.1.1) is extremely asymptotically stable, and thus the system (7.1.1) is a stationary oscillating system.

The proof of Theorem 7.4.2 is complete. □

7.5. Application of iteration method to improve frozen coefficient method

In this section, we employ the iteration method to improve frozen coefficient method. Consider the linear time-varying system:

$$\frac{dx}{dt} = A(t)x \quad A(t) \in C\left[I, R^{n\times n}\right],\ x \in R^n. \tag{7.5.1}$$

The classical frozen coefficient method can be stated as follows [344]. Assume that

(1) $\forall t_1, t_2 \in [t_0, +\infty]$, $\exists c = \text{constant} > 0$ such that

$$\sum_{i,j=1}^{n} \left|a_{ij}(t_1) - a_{ij}(t_2)\right| \leqslant c;$$

(2) $\operatorname{Re} \lambda(A(t)) < -r < 0$;

(3) the Cauchy matrix solution $K(t, t_0)$ of the system:

$$\frac{dx}{dt} = A(t_0)x \tag{7.5.2}$$

satisfies

$$\sum_{i,j=1}^{n} \left|k_{ij}(t, t_0)\right| \leqslant b e^{-\frac{r}{2}(t-t_0)}$$

and $bc < \frac{r}{4}$.

Then the solution $x(t)$ of system (7.5.1) admits the estimation:

$$\left\|x(t)\right\| \leqslant b\|x_0\|e^{-\frac{r}{4}(t-t_0)}.$$

The procedure of applying the iteration method to improve the classical frozen coefficient method is described below.

(1) The frozen is only needed at certain point t_0^*. The frozen coefficient matrix $A(t_0^*)$ is stable. Replace $\operatorname{Re}\lambda(A(t)) < -r$ by $\operatorname{Re}\lambda(A^*(t_0)) < -r$.
(2) Apply the integral average property to broad the demand of relaxation variety in classical frozen coefficient method.
(3) Make use of the different stable degree of the frozen coefficient matrix of the isolated subsystem to control the different coupling, and to estimate the solution's accessing property accurately.
(4) Convert the calculation of high-dimensional Cauchy matrix solution to the calculation of low-dimensional Cauchy matrix solution, which augments the possibility of calculation, and extends the application of frozen coefficient method adopted in solving engineering problems.

Rewrite system (7.5.1) as

$$\begin{aligned}\frac{dx}{dt} &= \operatorname{diag}\big(A_{11}(t_0), \dots, A_{rr}(t_0)\big)x \\ &\quad + \operatorname{diag}\big((A_{11}(t) - A_{11}(t_0)\big), \dots, \big(A_{11}(t) - A_{rr}(t_0)\big)x \\ &\quad + \big(A_{ij}(t)\sigma_{ij}\big)x, \end{aligned} \tag{7.5.3}$$

where $A_{ii}(t)$ and $A_{ij}(t)$ are $n_i \times n_i$ and $n_i \times n_j$ matrices, respectively. $A_{ii}(t_0)$ is the frozen matrix of $A_{ii}(t)$ at $t = t_0$, and

$$x_i = \big(x_1^{(1)}, \dots, x_{n_i}^{(i)}\big)^T, \quad x = \big(x_1^T, \dots, x_r^T\big)^T,$$
$$\sum_{i=1}^{r} n_i = n, \quad \sigma_{ij} = 1 - \delta_{ij}, \quad 1 \leqslant i, j \leqslant r.$$

For the isolated subsystem:

$$\frac{dx}{dt} = \operatorname{diag}\big(A_{11}(t_0), \dots, A_{rr}(t_0)\big)x, \tag{7.5.4}$$

we have the following theorem.

THEOREM 7.5.1. *If the following conditions are satisfied*:

(1) *the Cauchy matrix solution of* (7.5.4) *admits the estimation*:

$$\begin{aligned}P(t, t_0) &= \operatorname{diag}\big(P_{11}(t, t_0), \dots, P_{rr}(t - t_0)\big) \\ &= \operatorname{diag}\big(e^{A_{11}(t_0)(t-t_0)}, \dots, e^{A_{rr}(t_0)(t-t_0)}\big)\end{aligned}$$

with

$$\|P_{ii}(t,t_0)\| \leqslant m_i e^{-\alpha_i(t-t_1)}, \tag{7.5.5}$$

where $m_i \geqslant 1$ and α_i are positive constants;

(2)

$$\|A_{ii}(t) - A_{ii}(t_0)\| \leqslant l_{ii}(t), \quad \|A_{ij}(t)\| \leqslant l_{ij}(t), \tag{7.5.6}$$

where $l_{ij}(t) \in C[I, R^1_+]$;

(3) *there exists constant ε $(0 < \varepsilon < \min_{1\leqslant i\leqslant n} \alpha_i)$ such that*

$$b_{ij}(t) := \int_{t_0}^{t} m_i e^{-(\alpha_i-\varepsilon)(t-t_1)} l_{ij}(t_1)\,dt_1 \leqslant \tilde{b}_{ij} = const., \tag{7.5.7}$$

and the spectral radius $\rho(\tilde{B})$ of the matrix $\tilde{B}$ is $\rho(\tilde{B}) < 1$ (particularly $\|\tilde{B}\| < 1$);

then the zero solution of system (7.5.3) is exponentially stable.

PROOF. Any solution of system (7.5.3) $x(t) := x(t, t_0, x_0)$ satisfies

$$\begin{aligned} x_i(t) = {} & e^{A_{ii}(t_0)(t-t_0)} x_{0i} \\ & + \int_{t_0}^{t} e^{A_{ii}(t_0)(t-t_0)} \Bigg[\big(A_{ii}(t_1) - A_{ii}(t_0)\big) x_i(t_1) \\ & + \sum_{j=1}^{r} A_{ij}(t_1)\sigma_{ij} x_j(t_1)\,dt_1 \Bigg]. \end{aligned} \tag{7.5.8}$$

Applying the Picard iteration to (7.5.8) yields

$$\begin{cases} x_i^{(m)}(t) = e^{A_{ii}(t_0)(t-t_0)} x_{0i} + \int_{t_0}^{t} e^{A_{ii}(t_0)(t-t_1)} \Big[(A_{ii}(t_1) - A_{ii}(t_0)) x_i^{(m-1)}(t_1) \\ \qquad\qquad + \sum_{j=1}^{n} A_{ij}(t_1)\sigma_{ij} x_j^{(m-1)}(t_1) \Big]\,dt_1, \\ x_i^{(0)}(t) = e^{A_{ii}(t_0)(t-t_0)} x_{0i}. \end{cases} \tag{7.5.9}$$

Further, we can prove that

$$\begin{aligned} & \big(\|x_1^{(m)}(t)\|, \ldots, \|x_r^{(m)}(t)\|\big)^T \\ & \quad \leqslant \Big(\big(I_n + B + \cdots + B^m\big)\big(m_1\|x_{01}\|, \ldots, m_r\|x_{0r}\|\big) e^{-\varepsilon(t-t_0)} \Big)^T \\ & \quad \leqslant \left(\sum_{m=0}^{\infty} B^m \right) \big(m_1\|x_{01}\|, \ldots, m_r\|x_{0r}\|\big)^T e^{-\varepsilon(t-t_0)} \end{aligned}$$

$$= (I_n - B)^{-1}\big(m_1\|x_{01}\|, \dots, m_r\|x_{0r}\|\big)^T e^{-\varepsilon(t-t_0)} \tag{7.5.10}$$

holds for all natural numbers, which means that the zero solution of system (7.5.3) is exponentially stable. □

COROLLARY 7.5.2. *If*

(1) *condition* (1) *in Theorem* 7.5.1 *holds*;
(2) $l_{ij}(t) = l_{ij} = const.$ *in condition* (2) *of Theorem* 7.5.3;
(3) $\rho(B(\frac{m_i l_{ij}}{\alpha_i}))_{r\times r} < 1$ (*particularly* $\|B(\frac{m_i l_{ij}}{\alpha_i})\| < 1$);

then the zero solution of system (7.5.3) *is exponentially stable.*

PROOF. Choose $0 < \varepsilon \ll 1$ such that

$$b_{ij}(t) := \int_{t_0}^{t} m_i e^{-(\alpha_i-\varepsilon)(t-t_1)} l_{ij}\, dt_1 \leqslant \frac{m_i l_{ij}}{\alpha_i - \varepsilon} := \tilde{b}_{ij}.$$

Since

$$\rho\left(B\left(\frac{m_i l_{ij}}{\alpha_i}\right)\right) < 1 \quad \left(\left\|B\left(\frac{m_i l_{ij}}{\alpha_i}\right)\right\| < 1\right)$$

implies

$$\rho\left(B\left(\frac{m_i l_{ij}}{\alpha_i - \varepsilon}\right)\right) < 1 \quad \left(\left\|B\left(\frac{m_i l_{ij}}{\alpha_i - \varepsilon}\right)\right\| < 1\right)$$

when $0 < \varepsilon \ll 1$.

So the conclusion is true. □

THEOREM 7.5.3. *If the conditions* (1) *and* (2) *in Theorem* 7.5.1 *are satisfied, and*

(3) $b_{ij}(t)$ *defined by* (7.5.7) *satisfy*

$$\sum_{j=1}^{i-1} b_{ij}(t)\mu_j + \sum_{j=i}^{r} b_{ij}(t) \leqslant \mu_j = const. < 1$$
$$(i = 1, 2, \dots, r), \tag{7.5.11}$$

then the zero solution of system (7.5.3) *is exponentially stable.*

PROOF. Apply the following iterations to system (7.6.1):

$$x_i^{(m)}(t) = e^{A_{ii}(t_0)(t-t_0)} x_{0i} + \int_{t_0}^{t} e^{A_{ii}(t_0)(t-t_0)} \Bigg[\sum_{j=1}^{i-1} A_{ij}(t_1) x_j^{(m)}(t_1)$$

$$+ \big(A_{ii}(t_1) - A_{ii}(t_0)\big)x_i^{(m-1)}(t_1) + \sum_{j=i+1}^{r} A_{ij}(t_1)x_j^{(m-1)}(t_1)\Bigg] dt_1$$

$$(m = 2, 3, \ldots,\ i = 1, 2, \ldots, r),$$

$$x_i^{(1)}(t) = e^{A_{ii}(t_0)(t-t_0)}x_{0i} + \int_{t_0}^{t} e^{A_{ii}(t-t_1)}\left[\sum_{j=1}^{i-1} A_{ij}(t_1)x_j^{(1)}(t_1)\right] dt_1,$$

$$i = 1, 2, \ldots, r. \tag{7.5.12}$$

Let

$$M = \max_{1\leqslant i\leqslant r} m_i, \quad c = \max_{1\leqslant j\leqslant r} \|x_{0j}\|,$$

$$k = \sup_{\substack{1\leqslant i\leqslant r\\ t\geqslant t0}} \left\{\sum_{j=1}^{r} b_{ij}(t) + 1\right\}, \quad \mu = \max_{1\leqslant i\leqslant r} \mu_i,$$

$$\sigma = k^{r-1}cM.$$

Then, we can prove that for any natural numbers,

$$\big\|x_i^{(m)}(t)\big\| \leqslant \frac{\sigma}{1-\mu}e^{-\varepsilon(t-t_0)}, \quad x_i^{(m)}(t) \to x_i(t)$$

which implies that $\|x_i(t)\| \leqslant \frac{\sigma}{1-\mu}e^{-\varepsilon(t-t_0)}$. Hence, the zero solution of system (7.5.3) is exponentially stable. □

EXAMPLE 7.5.4. Consider a 4-dimensional time-varying linear system:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{bmatrix} -6 & 3 & \sin\frac{t}{8} & \frac{1}{8}\cos t\\ -5 & 2 & \frac{1}{7}\sin t & \frac{1}{7}\cos t\\ \frac{t}{1+t^2} & \frac{1}{2}\cos t & -4+\frac{1}{2}\sin t & 1\\ \frac{1}{2}\sin t & \frac{t}{1+t^2} & 1 & -4-\frac{1}{2}\sin t \end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

$$\xrightarrow[\text{frozen}]{\text{at } t_0=0} \begin{bmatrix} -6 & 3 & 0 & 0\\ -5 & 2 & 0 & 0\\ 0 & 0 & -4 & 1\\ 0 & 0 & 1 & -4 \end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \sin\frac{1}{8}t & \frac{1}{8}\cos t\\ 0 & 0 & \frac{1}{7}\sin t & \frac{1}{7}\cos t\\ \frac{t}{1+t^2} & \frac{1}{2}\cos t & \frac{1}{2}\sin t & 0\\ \frac{1}{2}\sin t & \frac{t}{1+t^2} & 0 & -\frac{1}{2}\sin t \end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}$$

or

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} A_{11}(0) & 0_2 \\ 0_2 & A_{22}(0) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$
$$+ \begin{bmatrix} A_{11}(t) - A_{11}(0) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) - A_{22}(0) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{7.5.13}$$

The eigenvalues of

$$A_{11}(0) = \begin{bmatrix} -6 & 3 \\ -5 & 2 \end{bmatrix}$$

are $\lambda_1 = -3$, $\lambda_2 = -1$, and that of

$$A_{22}(0) = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}$$

are $\lambda_1 = -5$, $\lambda_2 = -3$. Thus, the Cauchy matrix solution to the first isolated subsystem:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -6 & 3 \\ -5 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_{11}(0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7.5.14}$$

is

$$e^{A_{11}(0)t} = \begin{bmatrix} \frac{5}{2}e^{-3t} - \frac{3}{2}e^{-t} & -\frac{3}{2}e^{-3t} + \frac{3}{2}e^{-t} \\ \frac{3}{2}e^{-3t} - \frac{3}{2}e^{-t} & -\frac{3}{2}e^{-3t} + \frac{5}{2}e^{-t} \end{bmatrix}, \tag{7.5.15}$$

while the Cauchy matrix solution to the second isolated subsystem:

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = A_{22}(0) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{7.5.16}$$

is

$$e^{A_{22}(0)t} = \begin{bmatrix} \frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} \\ -\frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} \end{bmatrix}. \tag{7.5.17}$$

Hence, we have

$$\begin{aligned} \left\| e^{A_{11}(0)t} \right\| &\leqslant 3e^{-t}, \\ \left\| e^{A_{22}(0)t} \right\| &\leqslant \frac{3}{2}e^{-3t}, \\ \left\| A_{12}(t) \right\| &\leqslant \frac{2}{7}, \\ \left\| A_{21}(t) \right\| &\leqslant 1, \end{aligned}$$

$$\|A_{22}(t) - A_{22}(0)\| \leqslant \frac{1}{2},$$

and

$$\int_0^t \|e^{A_{11}(0)(t-\tau)}\| \|A_{12}(\tau)\| \, d\tau \leqslant 3 \int_0^t e^{-(t-\tau)} \frac{2}{7} \, d\tau \leqslant \frac{6}{7} < 1,$$

$$\int_0^t \|e^{A_{11}(0)(t-\tau)}\| \big[\|A_{21}(\tau)\| + \|A_{22}(\tau) - A_{22}(0)\|\big] \, d\tau$$

$$\leqslant \frac{3}{2} \int_0^t e^{-3(t-\tau)} \frac{3}{2} \, d\tau \leqslant \frac{3}{4} < 1.$$

Obviously, one can choose $0 < \varepsilon \ll 1$ such that any one of the conditions in Theorem 7.5.1 holds. Thus, system (7.5.13) is exponentially stable.

7.6. Application of iteration method to interval matrix

It is well known that an $n \times n$ internal matrix $N[P, Q]$ is a set of real matrices:

$$N[PQ] = \big[A = A(a_{ij}) \mid P(p_{ij}) \leqslant A(a_{ij}) \leqslant Q(q_{ij})\big],$$

i.e., $p_{ij} \leqslant a_{ij} \leqslant q_{ij}$, where p_{ij}, q_{ij} are known, while a_{ij} is unknown.

The set $N[P, Q]$ is said to be stable, if every $A \in N[P, Q]$ is stable.

The idea of frozen coefficient method can be applied to study the stability of interval matrix.

Let

$$\begin{aligned}
&A = \operatorname{diag}(A_{11}, A_{22}, \ldots, A_{rr}) + \big((1-\delta_{ij})A_{ij}\big),\\
&P = \operatorname{diag}(P_{11}, P_{22}, \ldots, P_{rr}) + \big((1-\delta_{ij})P_{ij}\big),\\
&Q = \operatorname{diag}(Q_{11}, Q_{22}, \ldots, Q_{rr}) + \big((1-\delta_{ij})Q_{ij}\big),\\
&A_{ij} \in N(P_{ij}, Q_{ij}),\\
&\mathring{A}_{ii} = \frac{1}{2}(P_{ii} + Q_{ii}), \quad B_{ii} = A_{ii} - \mathring{A}_{ii}, \quad i, j = 1, 2, \ldots, r,
\end{aligned}$$

where A_{ij}, P_{ij}, Q_{ij} are $n_i \times n_j$ matrices, and

$$\sum_{i=1}^{r} n_i = n.$$

δ_{ij} is the Kronecker delta function, $m_{ij} = \max \|A_{ij}\|$ and $A_{ij} \in N(P_{ij}, Q_{ij})$. Then $\|B_{ii}\| \leqslant \frac{m_{ii}}{2}$, $i = 1, 2, \ldots, r$.

We consider a linear dynamical system:

$$\begin{cases} \frac{dx}{dt} = \mathrm{diag}(A_{11}, \ldots, A_{rr})x + ((1-\delta_{ij})A_{ij})x \\ \quad = \mathrm{diag}(\mathring{A}_{11}, \ldots, \mathring{A}_{rr})x + \mathrm{diag}(B_{11}, \ldots, B_{rr})x + ((1-\delta_{ij})A_{ij})x \\ x(t_0) = x_n, \end{cases} \tag{7.6.1}$$

or

$$\begin{cases} \frac{dx_i}{dt} = \mathring{A}_{ii}x_i + B_{ii}x_i + \sum_{j=1}^{r} A_{ij}x_j, \\ x_i(t_0) = x_{0i}, \quad i = 1, 2, \ldots, r, \end{cases} \tag{7.6.2}$$

where

$$x_i = \left(x_1^{(i)}, \ldots, x_{n_i}^{(i)}\right)^T, \quad i = 1, 2, \ldots, r, \quad \sum_{i=1}^{r} n_i = n.$$

At the same time, we consider the isolated subsystem:

$$\begin{cases} \frac{dx_i}{dt} = \mathring{A}_{ii}x_i, \\ x = x_{0i}. \end{cases} \tag{7.6.3}$$

THEOREM 7.6.1. *Assume that*

(1) *there exist constants* $M_i > 1$, $\alpha_i > 0$ *such that*

$$e^{\mathring{A}_{ii}(t-t_0)} \leqslant M_i e^{-\alpha_i(t-t_0)}, \quad i = 1, 2, \ldots, r;$$

(2) *let* $c_{ij} := \frac{M_i}{\alpha_i}(1-\delta_{ij})m_{ij} + \frac{M_i m_{ii}}{2\alpha_i}$,

then $\rho(C) < 1$ (*in particular* $\|C\| < 1$) *implies that the interval matrix* $N[P, Q]$ *is stable. where* $\rho(C)$ *is the spectral radius of the matrix* C.

PROOF. From the method of constant variation, the solution of (7.6.2) can he written as

$$x_i(t) = e^{\mathring{A}_{ii}(t-t_0)}x_{0i} + \int_{t_0}^{t} e^{\mathring{A}_{ii}(t-t_1)}\left[B_{ii}(t_1)x_i(t_1) + \sum_{j=1}^{n}(1-\delta_{ij})A_{ij}x_j(t_1)\right]dt_1. \tag{7.6.4}$$

Let

$$\tilde{c}_{ij} = \frac{M_i}{\alpha_i - \varepsilon}\left(1 - \delta_{ij}m_{ij} + \frac{M_i m_{ij}}{2\alpha_i - \varepsilon}\right), \qquad \tilde{C} = (\tilde{c}_{ij}).$$

Since the eigenvalues of a matrix continuously depend on its elements, $\rho(C) < 1$ ($\|C\| < 1$) implies $\rho(\tilde{C}) < 1$ ($\|\tilde{c}\| < 1$) for $0 < \varepsilon \ll 1$. We now apply the iteration to (7.6.4) to obtain

$$\begin{cases} x_i^{(m)}(t) = e^{\mathring{A}_{ii}(t-t_0)}x_{0i} \\ \quad + \int_{t_0}^{t} e^{\mathring{A}_{ii}(t-t_1)}\Big[B_{ii}x_i(t_1) + \sum_{j=1}^{r}(1-\delta_{ij})A_{ij}x_j^{(m-1)}\Big]\,dt_1 \\ x_i^{(0)}(t) = e^{\mathring{A}_{ii}(t-t_0)}x_{0i}, \quad i = 1, 2, \ldots, r,\ m = 1, 2, \ldots, \end{cases} \tag{7.6.5}$$

where $t \geqslant t_0$, $0 < \varepsilon < \min_{1 \leqslant i \leqslant r} \alpha_i$. We obtain

$$\begin{aligned} &\big\|x_i^{(0)}(t)\big\| \leqslant M_i\|x_{0i}\|e^{-\alpha_i(t-t_0)} \leqslant M_i\|x_{0i}\|e^{-\varepsilon(t-t_0)}, \\ &\quad \big(\big\|x_1^{(0)}(t)\big\|, \ldots, \big\|x_r^{(0)}(t)\big\|\big)^T \leqslant I_r\big(M_1\|x_{0i}\|, \ldots, M_r\|x_{0r}\|\big)^T e^{-\varepsilon(t-t_0)}, \end{aligned}$$

$$\begin{aligned} \big\|x_i^{(1)}(t)\big\| &\leqslant M_i\|x_{0i}\|e^{-\varepsilon(t-t_0)} + \int_{t_0}^{t} M_i r^{-(\alpha_i-\varepsilon)(t-t_1)} \\ &\qquad \times \left[\|B_{ii}\|\|x_{0i}\| + \sum_{j=1}^{r}(1-\delta_{ij})\|A_{ij}\|M_j\|x_{0j}\|\,dt_1\right]e^{-\varepsilon(t-t_0)} \\ &\leqslant M_i\|x_{0i}\|e^{-\varepsilon(t-t_0)} + \frac{M_i m_{ii}}{2(\alpha_i-\varepsilon)}\|x_{0i}\| \\ &\qquad + \sum_{j=1}^{r}\frac{(1-\delta_{ij})m_{ij}M_j\|x_{0j}\|}{\alpha_i-\varepsilon}e^{-\varepsilon(t-t_0)}, \end{aligned}$$

$$\begin{aligned} &\big\|x_i^{(1)}(t) - x_i^{(0)}(t)\big\| \leqslant \left(\sum_{j=1}^{r}\left(\frac{1-\delta_{ij}}{\alpha_i-\varepsilon}\right)m_{ij}M_j + \frac{M_i m_{ij}}{2(\alpha_i-\varepsilon)}\right)e^{-\varepsilon(t-t_0)}, \\ &\quad i = 1, 2, \ldots, r, \end{aligned} \tag{7.6.6}$$

that is,

$$\begin{aligned} &\big(\big\|x_1^{(1)}(t)\big\|, \ldots, \big\|x_r^{(1)}(t)\big\|\big)^T \\ &\quad \leqslant (E + \tilde{B})\big(M_1\|x_{01}\|, \ldots, M_r\|x_{0r}\|\big)^T e^{-\varepsilon(t-t_0)}, \end{aligned} \tag{7.6.7}$$

$$\begin{aligned} &\big(\big\|x_1^{(1)}(t) - x_0^{(1)}(t)\big\|, \ldots, \big\|x_r^{(1)}(t) - x_r^{(0)}(t)\big\|\big)^T \\ &\quad \leqslant \tilde{B}\big(M_1\|x_{01}\|, \ldots, M_r\|x_{0r}\|\big)^T e^{-\varepsilon(t-t_0)}. \end{aligned} \tag{7.6.8}$$

Finally, by the method of mathematical induction we obtain

$$\begin{aligned} &\big\|x_1^{(m)}(t)\big\|, \ldots, \big\|x_r^{(m)}(t)\big\| \\ &\quad \leqslant \big(I_r + \tilde{C} + \cdots + \tilde{C}^m\big)\big(M_1\|x_{01}\|, \ldots, M_r\|x_{0r}\|\big)^T \\ &\quad \leqslant (I_r - \tilde{C})^{-1}\big(M_1\|x_{01}\|, \ldots, M_r\|x_{0r}\|\big)^T e^{-\varepsilon(t-t_0)}, \end{aligned} \tag{7.6.9}$$

and

$$\begin{aligned}&\left(\|x_1^{(m)}(t)\|, \ldots, \|x_r^{(m)}(t)\|\right)^T \\ &\quad \leqslant (I_r - \tilde{C})^{-1}\left(M_1\|x_{0i}\|, \ldots, M_r\|x_{0r}\|\right)^T e^{-\varepsilon(t-{}_0)} \end{aligned} \tag{7.6.10}$$

which shows that the conclusion of Theorem 7.6.1 is true. □

EXAMPLE 7.6.2. Consider the stability of a mechanical system with angular speed ω [361]:

$$\begin{aligned}\dot{X}_1 &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 \\ \omega^2 & 0 \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 \\ 0 & 2\omega \end{bmatrix} X_2, \\ \dot{X}_2 &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} X_2 + \begin{bmatrix} 0 & 0 \\ 0 & -2\omega \end{bmatrix} X_1 + \begin{bmatrix} 0 & 0 \\ \omega^2 & 0 \end{bmatrix} X_2,\end{aligned}$$

where

$$X_1 = (x_1, x_2)^T, \quad X_2 = (x_3, x_4)^T.$$

We are interested in estimating the stability region for the coupling parameter $\omega \geqslant 0$. Siljak [361] applied the decomposition-aggregation method with Lyapunov vector function to determine the stability interval for ω as $0 \leqslant \omega \leqslant 0.05$, i.e., the interval matrix

$$\begin{aligned}&N(P, Q) \\ &\quad = \begin{bmatrix} 0 & 1 & 0 & 0 \\ [-2, -1 + (0.05)^2] & -1 & 0 & [0, 0.1] \\ 0 & 0 & 0 & 1 \\ 0 & [-0.1, 0] & [-1, -1 + (0.05)^2] & -1 \end{bmatrix}\end{aligned}$$

is stable. Here, we have applied Theorem 7.6.1 to obtain a greater interval of ω for the stability as $0 \leqslant \omega \leqslant 0.08768$.

For the isolated subsystem:

$$\dot{X}_i = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} X_i, \quad i = 1, 2,$$

we have

$$\begin{aligned}&\exp\left(\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} t\right) \\ &\quad = e^{-\frac{t}{2}} \begin{bmatrix} \cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t & \frac{2\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t \\ \frac{-2\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t & \cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t \end{bmatrix}.\end{aligned}$$

Therefore,

$$\left\|\exp\left(\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} t\right)\right\| \leqslant (1+\sqrt{3})e^{-\frac{t}{2}},$$

$$\int_{t_0}^{t} (1+\sqrt{3})e^{-\frac{t-\tau}{2}}\omega^2\,d\tau \leqslant 2(1+\sqrt{3})\omega^2,$$

$$\int_{0}^{t} (1+\sqrt{3})e^{-\frac{t-\tau}{2}} \leqslant 4\omega(1+\sqrt{3}).$$

Let $2(1+\sqrt{3})\omega^2 + 4(1+\sqrt{3})\omega - 1 < 0$. Then $0 \leqslant \omega \leqslant 0.08768$, i.e., the interval matrix, given by

$$N(P,Q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ [-1, -1+(0.08768)^2] & -1 & 0 & [0, 0.17436] \\ 0 & 0 & 0 & 1 \\ 0 & [-0.17436, 0] & [-1, -1+(0.08768)^2] & -1 \end{bmatrix},$$

is stable.

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Chapter 8

Dynamical Systems with Time Delay

As we know, ordinary differential equation is an important mathematical tool, not only in solving problems in natural science and social science, but also playing a significant role in the development of dynamical system theory and methodologies. As a matter of fact, general theory of Lyapunov stability theory was developed over a century ago by using ordinary differential equations. Later, the theory was extended to different types of dynamic systems, described by difference equations, differential difference equations, functional equations, partial differential equations, stochastic differential equations and so on. The research on the direct Lyapunov method for constructing stability criteria is still a main topic in dynamical systems.

Whether from development of theory or from applications, the most important part is still the stability of dynamical systems described by differential difference equations with time delays. In this chapter, we systematically introduce the stability theory of the differential difference equations.

The materials presented in this chapter are chosen from various sources, in particular, [231] for Section 8.1, [344,231] for Section 8.2, [280] for Section 8.7, [139] for Section 8.8, [243] for Section 8.9, Section 8.10 and [155,280] for Section 8.11.

8.1. Basic concepts

In natural or social phenomena, the trend or the future status of many systems are not only determined by their current situation but also determined by their history. Such phenomena are called delay or genetic effect. Many mathematical models arising from engineering, physics, mechanics, control theory, chemical reaction, or biomedicine always involve delays. For example, the delitescence of a contagion in biomedicine, the hysteresis in elastic mechanics, and especially, any automatic control systems with feedback in general always contain time delay. This is because these systems have only limited time to receive information and react accordingly. Such a system cannot be described by purely differential equations, but has to be treated with differential difference equations or the so called differential equations with difference variables.

For illustration, let us consider the motion of a swing ship. Let $\theta(t)$ denote the angle to the vertical position. Then θ satisfies the following differential equation:

$$m\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = f(t). \tag{8.1.1}$$

To reduce the swing, besides increasing the damping c, some ships are equipped with water pumps on both sides, which can transfer water from one cabin to another in order to increase the damping $q\dot{\theta}(t)$. Since there always exists delay in the servo of control system, system (8.1.1) should be more precisely described by

$$m\ddot{\theta}(t) + C\dot{\theta}(t) + q\dot{\theta}(t-\tau) + k\theta(t) = f(t), \tag{8.1.2}$$

where τ is the delay. This is a typical time-delay differential equation.

Although some differential equation with difference variables appeared as early as in 1750 in Euler's geometrical problem, systematic studies started only in the 20th century. The development of this research and its applications were first related to automatic control theory. Recently, researchers paid particular attention to the stability of neural network with time delay.

When different values are introduced to the independent variable t of an unknown function $x(t)$ in a differential equation, the equation is called differential difference equation. For example,

$$\dot{x}(t) = g\big(t, x(t), x\big(t-\tau(t)\big)\big) \tag{8.1.3}$$

$$\dot{x}(t) = g\big(t, x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_n)\big) \tag{8.1.4}$$

$$\ddot{x}(t) = g\big(t, x(t), \dot{x}(t), x\big(t-\tau(t)\big), \dot{x}\big(t-\tau(t)\big)\big) \tag{8.1.5}$$

$$\ddot{x}(t) = g\left(t, x\left(\frac{t}{2}\right), \dot{x}\left(\frac{t}{2}\right), x(t), \dot{x}(t)\right). \tag{8.1.6}$$

If the independent variable in the highest-order derivatives is not less than any other order derivatives as well as the independent variable t, it is called delayed differential difference equation. For example, if $\tau_1 > 0$ in (8.1.3) (8.1.5), $\tau_1 > 0, \tau_2 > 0, \ldots, \tau_n > 0$ in (8.1.4), and $\frac{t}{2} = t - \frac{t}{2} \geqslant 0$ in (8.1.6), then all these equations are called differential difference equations.

Another type of delayed differential difference equations is called neutral differential difference equation. For example,

$$\dot{x}(t) = g\big(t, x(t), x(t-\tau), \dot{x}(t-\tau)\big), \quad \tau > 0, \tag{8.1.7}$$

$$\ddot{x}(t) = g\big(t, x(t), \dot{x}\big(t-\tau(t)\big), \dot{x}\big(t-\tau(t)\big)\ddot{x}\big(t-\tau(t)\big)\big). \tag{8.1.8}$$

The independent variable in the highest-order derivatives can take different values. So, general time-delay differential difference equations can be described by

$$\frac{dy}{dt} = g\big(t, y(t), y\big(t - \tau_1(t)\big), \ldots, y\big(t - \tau_m(t)\big)\big), \quad \tau_i(t) \geqslant 0,$$
$$y(t) = \tilde{\psi}(t) \in E_{t_0} = \Big\{t_0 - \sup_{t \geqslant t_0} \tau_i(t),\ i = 1, \ldots, m\Big\}. \tag{8.1.9}$$

Suppose the function on the right-hand side of (8.1.9) is smooth enough, such that the existence and uniqueness of Cauchy problem is guaranteed.

To study the stability of any solution of (8.1.9), $\hat{y}(t)$, we only need a transformation $x(t) = y(t) - \hat{y}(t)$ so that it is equivalent to study the stability of the zero solution of the following equation

$$\frac{dx}{dt} = f\big(t, x(t), x\big(t - \tau_1(t)\big), \ldots, x\big(t - \tau_m(t)\big)\big). \tag{8.1.10}$$

Let $x \in R^n$, $f \in [I \times \overbrace{R^n \times R^n \times \cdots \times R^n}^{m}, R^n]$. In the following, we first give the definitions of different stabilities for differential difference equations. For the neural type equations, it is not difficult for readers to obtain their definitions.

DEFINITION 8.1.1. The zero solution of (8.1.10) is said to be stable, if $\forall t_0 \in I$, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon, t_0) > 0$, $\forall \xi(t) \in E_{t_0}$, when $\|\xi(t)\| < \delta$, it holds

$$\big\|x(t, t_0, \xi)\big\| < \varepsilon \quad \forall t \geqslant t_0.$$

If $\delta(\varepsilon, t_0)$ in the above definition is independent of t_0, then the zero solution of (8.1.10) is said to be uniformly stable.

If the zero solution of (8.1.10) is stable, and $\exists \sigma(t_0) > 0$, $\forall \xi(t) \in E_{t_0}$, when $\|\xi(t)\| < \sigma(t_0)$, it holds

$$\lim_{t \to +\infty} \big\|x(t, t_0, \xi)\big\| = 0,$$

then the zero solution of (8.1.10) is said to be asymptotically stable.

DEFINITION 8.1.2. The zero solution of system (8.1.10) is said to be uniformly asymptotically stable, if it is uniformly stable, and there exists $\sigma > 0$ (independent of t_1), $\forall \varepsilon > 0$, $\exists T(\varepsilon) > 0$ (also independent of t_1), when $t > t_1 + T(\varepsilon)$, $\|\xi(t)\| < \sigma$, $\|x(t, t_0, \xi)\| < \varepsilon$ holds. Here, $\xi(t)$ is an arbitrary function defined on E_{t_1} for $t_1 \geqslant t_0$.

DEFINITION 8.1.3. The zero solution of system (8.1.10) is said to be exponentially stable, if there exist $\delta > 0$, $\alpha > 0$, $B(\delta) \geqslant 1$ such that when $\|\xi\| < \delta$ and

$t > T$,

$$\left\|x(t, t_0, \xi)\right\| \leqslant B(\delta)\Big(\max_{t \in E_{t_0}} \|\xi\|\Big)e^{-\alpha(t-t_0)}. \tag{8.1.11}$$

DEFINITION 8.1.4. The zero solution of system (8.1.10) is said to be globally asymptotically stable, if it is stable and for any initial value function $\xi(t)$ (i.e., the δ in Definition 8.1.1 can be arbitrarily large),

$$\lim_{t\to+\infty} \left\|x(t, t_0, \xi)\right\| = 0$$

holds.

For the neutral equation:

$$\begin{cases} \frac{dx(t)}{dt} = f(x, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_m(t)), \\ \qquad\qquad \dot{x}(t-\tau_1(t)), \ldots, \dot{x}(t-\tau_m(t))), \\ x(t) = \xi(t), \quad t \in E_{t_0}, \\ \dot{x}(t) = \dot{\xi}(t), \quad t \in E_{t_0}, \end{cases} \tag{8.1.12}$$

the definitions of different stabilities for its solution are similar to Definitions 8.1.1–8.1.4, one only needs to change the initial condition $\|\phi(t)\| < \delta$ in Definitions 8.1.1–8.1.4 to $\|\xi(t)\| < \delta$, $\|\dot{\xi}(t) < \delta\|$.

8.2. Lyapunov function method for stability

It is natural to extend the basic ideas and approaches of Lyapunov function method developed for the stability of ordinary differential equations to that of the differential difference equations.

To make easy understand and readable, we, instead of (8.1.10), use a simpler system:

$$\frac{dx}{dt} = f\big(t, x(t), x\big(t-\tau(t)\big)\big) \tag{8.2.1}$$

where $x \in R^n$, $f \in C[I \times R^n \times R^n, R^n]$, $f(t, 0, 0) \equiv 0$, $0 \leqslant \tau(t) < t < +\infty$, and

$$x\big(t-\tau(t)\big) = \big(x_1\big(t-\tau_1(t)\big), \ldots, x_n\big(t-\tau_n(t)\big)\big)^T,$$
$$0 \leqslant \tau_i(t) \leqslant \tau_i = \text{constant}.$$

Similar to the results of stability obtained in Chapter 4, it is easy to obtain the following theorem.

For convenience, let $x(t) := x(t, t_0, \xi)$. $x(\theta) = \xi(\theta)$ when $\theta \in [t_0 - \tau, t_0]$.

THEOREM 8.2.1. *If there exists a positive definite function $V(t, x)$ in some region $G_H := \{(t, x), t \geqslant t_0, \|x\| \leqslant H\}$ such that*

$$D^+V(t, x)|_{(8.2.1)} \leqslant 0. \tag{8.2.2}$$

(*Specifically,* $\frac{dV(t,x)}{dt}|_{(8.2.1)} = \frac{\partial V}{\partial t} + \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(t, x(t), x(t-\tau(t))) \leqslant 0$.) *Then the zero solution of* (8.2.1) *is stable.*

PROOF. Since $V(t, x)$ is positive definite, there exists

$$\varphi(\|x\|) \leqslant V(t, x), \quad \forall \varepsilon > 0. \tag{8.2.3}$$

Because $V(t, 0) = 0$, there exists $\delta(t_0)$ such that when $\|x_0\| \leqslant \delta(t_0)$, $V(t_0, x_0) < \varphi(\varepsilon)$. It then follows from (8.2.2) and (8.2.3) that

$$\varphi(\|x(t)\|) \leqslant V(t_0, x(t)) \leqslant V(t_0, x_0) \leqslant \varphi(\varepsilon),$$

implying that $\|x(t)\| \leqslant \varepsilon,\ t \geqslant t_0$. Thus, the zero solution of (8.2.1) is stable. □

THEOREM 8.2.2. *If there exists a positive definite function $V(t, x)$ with infinitesimal upper bound in some region G_H such that*

$$D^+V(t, x)|_{(8.2.1)} \leqslant 0.$$

(*In particular,* $\frac{dV(t,x)}{dt}|_{(8.2.1)} = \frac{\partial V}{\partial t} + \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(t, x(t), x(t-\tau(t))) \leqslant 0$.) *Then, the zero solution of* (8.2.1) *is uniformly stable.*

PROOF. Because $V(t, x)$ is positive definite with infinitesimal upper bound, there exist $\varphi_1, \varphi_2 \in K$ such that

$$\varphi_1(\|x\|) \leqslant V(t, x) \leqslant \varphi_2(\|x\|) \quad \forall \varepsilon > 0.$$

Choose $\delta(\varepsilon) = \varphi_2^{-1}(\varphi_1(\varepsilon))$. When $\|x_0\| < \delta(\varepsilon)$, from the assumption we have

$$\varphi_1(\|x(t)\|) \leqslant V(t, x(t)) \leqslant V(t_0, x_0) \leqslant \varphi_2(\|x_0\|) \leqslant \varphi_2(\|\delta(\varepsilon)\|).$$

So $\|x(t)\| \leqslant \varphi_1^{-1}\varphi_2(\|\delta(\varepsilon)\|) \leqslant \varepsilon\|$, implying that the zero solution of (8.2.1) is uniformly stable. □

EXAMPLE 8.2.3. Consider the stability of the zero solution of the following two dimensional delayed system:

$$\begin{cases} \frac{dx_1}{dt} = (-3 + \sin t)x_1(t)(1 + \sin(x_1^2(t - \tau_2(t)))) \\ \qquad\quad + x_2(t)\sin(x_1)(t - \tau_1(t)), \\ \frac{dx_2}{dt} = -2x_1 \sin(x_1(t - \tau_1(t))) - 2x(1 + \frac{1}{2}\cos(x_2(t - \tau_2(t))))x_2, \end{cases} \tag{8.2.4}$$

where $0 \leqslant \tau_i(t) \leqslant \tau_i = \text{constant}$, $i = 1, 2$.

PROOF. Consider the positive definite Lyapunov function with infinitesimal upper bound:

$$V = x_1^2 + \frac{1}{2}x_2^2.$$

We have

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(8.2.4)} &= 2(-3+\sin t)x_1^2(t)\big(1+\sin x_1^2\big(t-\tau_1(t)\big)\big)\\ &\quad + 2x_1(t)x_2(t)\sin x_1\big(t-\tau_1(t)\big)\\ &\quad - 2x_1(t)x_2(t)\sin x_1\big(t-\tau_1(t)\big)\\ &\quad - 2\bigg(1+\frac{1}{2}\cos\big(x_2\big(t-\tau_1(t)\big)\big)\bigg)x_2^2(t)\\ &\leqslant -4x_1^2(t) - x_2^2(t) < 0 \quad \text{when } x_1^2+x_2^2 \neq 0.\end{aligned}$$

Thus the zero solution of (8.2.4) is uniformly stable. □

THEOREM 8.2.4. *If there exists a positive definite function $V(t,x)$ with infinitesimal upper bound in some region G_H such that*

$$D^+V(t,x)|_{(8.2.4)}$$

is negative definite, in particular, if

$$\frac{dV(t,x)}{dt}\bigg|_{(8.2.4)} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n}\frac{\partial V}{\partial x_i}f_i\big(t,x(t),x\big(t-\tau(t)\big)\big)$$

is negative definite, then the zero solution of (8.2.4) is uniformly asymptotically stable.

When $G_H = R^n$, the conclusion of Theorem 8.2.4 becomes globally asymptotically stable.

PROOF. Let $V(t) := V(t,x(t))$. From the condition we know that there exist $\varphi_1, \varphi_2 \in K$ such that

$$\varphi_1\big(\|x\|\big) \leqslant V(t,x) \leqslant \varphi_2\big(\|x\|\big).$$

Because $D^+V(t,x)|_{(8.2.4)}$ is negative definite, $\frac{dV}{dt}|_{(8.2.4)}$ is negative definite. This implies that $D^+V(t,x)|_{(8.2.4)}$ ($\frac{dV}{dt}|_{(8.2.4)}$) is negative definite about $x(t)$. Therefore, $\exists\varphi_3(\|x\|) \in K$ such that

$$D^+V\big(t,x(t)\big)\big|_{(8.2.4)} \leqslant -\varphi_3\big(\|x(t)\|\big) \leqslant -\varphi_3\big(\|\varphi_2^{-1}\big(V(t)\big)\|\big) \leqslant 0,$$

i.e.,

$$\int_{V(t_0)}^{V(t)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} \leqslant -(t-t_0),$$

or

$$\int_{V(t)}^{V(t_0)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} \geqslant t-t_0.$$

$\forall \varepsilon < H$, using

$$\varphi_1\big(\|x(t)\|\big) \leqslant V(t) := V\big(t, x(t)\big)$$

and

$$V(t_0) \leqslant \varphi_2\big(\|x_0\|\big) \leqslant \varphi_2(H),$$

we have

$$\begin{aligned}\int_{\varphi_1(\|x(t)\|)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} &= \int_{\varphi_1(\|x(t)\|)}^{\varphi_1(\varepsilon)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} + \int_{\varphi_1(\varepsilon)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} \\ &\geqslant \int_{V(t)}^{V(t_0)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} \geqslant t-t_0.\end{aligned}$$

Thus,

$$T = T(\varepsilon, H) > \int_{\varphi_1(\varepsilon)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))}.$$

Obviously, when $t \geqslant t_0 + T$, it is easy to obtain from the above equation:

$$\begin{aligned}\int_{\varphi_1(\|x(t)\|)}^{\varphi_1(\varepsilon)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} &\geqslant t-t_0-\int_{\varphi_1(\varepsilon)}^{\varphi_2(H)} \frac{dV}{\varphi_3(\varphi_2^{-1}(V(t)))} \\ &\geqslant t-t_0-T \geqslant 0 \end{aligned} \tag{8.2.5}$$

from which we can deduce that

$$\varphi_1\big(\|x(t)\|\big) < \varphi_1(\varepsilon), \quad \text{when } t \geqslant t_0 + T(\varepsilon, H).$$

Since $T = T(\varepsilon, H)$ is independent of t_0 and x_0, $x = 0$ is uniformly attractive.

Further, noticing that the conditions in Theorem 8.2.4 imply the conditions of Theorem 8.2.2, we know that $x = 0$ is uniformly stable. Therefore, $x = 0$ is uniformly asymptotically stable. □

Again we consider Example 8.2.3. Obviously, for this example, the conditions in Theorem 8.2.4 are satisfied. Hence, $x = 0$ of Example 8.2.3 is uniformly asymptotically stable.

THEOREM 8.2.5. *If there exists function $V(t,x) \in [I \times G_H, R]$ in G_H satisfying the following conditions*:

(1) $\|x\| \leqslant V(t,x) \leqslant k(H)\|x\|$, $x \in G_H$;
(2) $\frac{dV}{dt}\big|_{(8.2.4)} \leqslant -cV(t,x)$, *where $c > 0$ is a constant*;

then the zero solution of (8.2.4) is exponentially stable.

If $G_H = R^n$, the conclusion of Theorem 8.2.5 is globally exponentially stable.

PROOF. From condition (2), we know that

$$V\big(t, x(t)\big) \leqslant V\big(t_0, x(t_0)\big) \leqslant V(t_0, x_0)e^{-c(t-t_0)}.$$

Thus, the conclusion is true. □

EXAMPLE 8.2.6. Consider the stability of the zero solution for the following nonlinear system with variable time delays:

$$\begin{cases} \frac{dx_1}{dt} = -x_1(t)(1 + x_2^2(t - \tau_2(t))) \\ \quad + 2x_2(t)x_1(t - \tau_1(t))x_2(t - \tau_2(t)), \\ \frac{dx_2}{dt} = -3x_1(t)x_1(t - \tau_1(t))x_2(t - \tau_2(t)) \\ \quad - x_2(t)(2 + \sin x_1(t - \tau_1(t))), \end{cases} \tag{8.2.6}$$

where $0 \leqslant \tau_i(t) \leqslant \tau_i = \text{constant},\ i = 1, 2, \ldots, n.$

Construct the positive definite and radially unbounded Lyapunov function:

$$V = \frac{(3x_1^2 + 2x_2^2)}{2}.$$

Then, we have

$$\begin{aligned} \frac{dV}{dt}\bigg|_{(8.2.6)} &= -3x_1^2(t)\big(1 + x_2^2\big(t - \tau_1(t)\big)\big) \\ &\quad + 6x_1(t)x_2(t)x_1\big(t - \tau_1(t)\big)x_2\big(t - \tau_2(t)\big) \\ &\quad - 6x_1(t)x_2(t)x_1\big(t - \tau_1(t)\big)x_2\big(t - \tau_2(t)\big) \\ &\quad - 2x_2^2\big(2 + \sin x_1\big(t - \tau_1(t)\big)\big) \end{aligned}$$

$$\begin{aligned}&\leqslant -3x_1^2(t) - 2x_2^2(t)\\&= -2V\big(t, x(t)\big),\end{aligned}$$

which satisfies the conditions in Theorem 8.2.5. Thus, the zero solution of (8.2.6) is globally exponentially stable.

THEOREM 8.2.7. *If there exists function $V(t,x) \in C[G_H, R]$ such that $V(t,0) = 0$ and*

(1) *for $t \geqslant t_0$, in any neighborhood of the origin, there exists region such that $V > 0$*;
(2) *in the region where $V > 0$, $V(t,x)$ is bounded*;
(3) *in the region where $V > 0$, $\frac{dV}{dt}|_{(8.2.6)}$ is positive definite*;

then the zero solution of (8.2.6) is unstable.

PROOF. In the region $V > 0$, the meaning of $\frac{dV}{dt}|_{(8.2.6)}$ being positive definite is: $\forall \varepsilon > 0$, $\exists l > 0$ in the region $V \geqslant \varepsilon > 0$, $\forall t \geqslant t_0$, it holds

$$\left.\frac{dV}{dt}\right|_{(8.2.6)} \geqslant l > 0.$$

Now choose $\varepsilon > 0$ such that $0 < \varepsilon < H$. We want to show that there exists x_0, no matter how small $\|x_0\|$ is, the solution $x(t, t_0, x_0)$ moves out the region defined by $\|x\| < \varepsilon$.

To achieve this, from condition (1), in the region of $V > 0$, we can choose any x_0 such that $|x_0|$ is arbitrarily small and satisfies $V(t_0, x_0) > 0$.

Suppose the solution orbit $x(t, t_0, x_0)$ does not move out the region $\|x\| < \varepsilon$. Since

$$\left.\frac{dV}{dt}\right|_{(8.2.1)} \geqslant 0 \quad (t > t_0), \quad \text{and} \quad V\big(t, x(t, t_0, x_0)\big) \geqslant V(t_0, x_0) > 0,$$

so the solution $x(t, t_0, x_0)$ would stay in the region $V(t,x) > 0$ for $t \geqslant t_0$. Then from condition (3) we know that there exists $l > 0$ such that

$$\frac{dV(t, x(t))}{dt} \geqslant l > 0.$$

Hence,

$$V\big(t, x(t)\big) = V(t_0, x_0) + \int_{t_0}^{t} \frac{dV}{dt}\,dt \geqslant V(t_0, x_0) + l(t - t_0), \tag{8.2.7}$$

which implies that when $t \gg 1$, $V(t, x(t))$ can be arbitrary large. This contradicts condition (2). So the conclusion of Theorem 8.2.7 is true. □

Theorems 8.2.1–8.2.7 are almost parallel extensions of the Lyapunov stabilities of ordinary differential equations.

Generally, the expression of the derivative of $V(t,x)$, $\frac{dV}{dt}$, involves both instant state variables, $x_1(t), \dots, x_n(t)$, and past (delayed) state variables, $x_1(t-\tau_1(t)), \dots, x_n(t-\tau_n(t))$. Sometimes it even contains the derivatives of the past state variables, $\dot{x}_1(t-\tau_1(t)), \dots, \dot{x}_n(t-\tau_n(t))$. So it is very difficult to determine whether $\frac{dV}{dt}$ is negative definite or not.

The above simple examples are proved by enlarging the inequality to eliminate the delayed state variables. Certainly, such cases are very rare.

There are approaches to overcome this difficulty. The first is to consider $\frac{dV}{dt}$ as a function of $2n$ variables, $x_1(t), \dots, x_n(t)$, $x_1(t-\tau_1(t)), \dots, x_n(t-\tau_n(t))$ or even $3n$ variables, $x_1(t), \dots, x_n(t)$, $x_1(t-\tau_1(t)), \dots, x_n(t-\tau_n(t))$, $\dot{x}_1(t-\tau_1(t)), \dots, \dot{x}_n(t-\tau_n(t))$. Then if $\frac{dV}{dt}$ is negative definite for these variables, one can conclude that $\frac{dV}{dt}$ is negative definite about $x_1(t-\tau_1(t)), \dots, x_n(t-\tau_n(t))$. This brings difficulty due to the dimension increase of the system.

Another effective method is the Razumikhin method, which will be discussed in the next section.

8.3. Lyapunov function method with Razumikhin technique

First, we use a very simple example to illustrate the original idea of the Lyapunov function method with Razumikhin technique, proposed in the 1960s by the famous former Soviet scholar, Razumikhin [228].

Consider the stability of a 1-dimensional ordinary differential equation with a constant delay:

$$\frac{dx}{dt} = ax(t) + bx\bigl(t-\tau(t)\bigr), \tag{8.3.1}$$

where $0 \leqslant \tau(t) \leqslant \tau = \text{constant}$.

If we construct the positive definite and radially unbounded Lyapunov function

$$V = \frac{1}{2}x^2,$$

then,

$$\frac{dV}{dt} = ax^2(t) + bx(t)x\bigl(t-\tau(t)\bigr). \tag{8.3.2}$$

In this case, even if assume $a < 0$, $|b| \ll 1$, we cannot deduce the negative definiteness of $\frac{dV}{dt}$ about $x(t)$ even if $\frac{dV}{dt}$ is negative definite about $x(t)$ and $x(t-\tau)$.

For more complicated systems, it is more difficult to determine whether $\frac{dV}{dt}$ is negative definite or not about $x(t)$ by increasing the dimension of the system with additional variables $x(t-\tau)$.

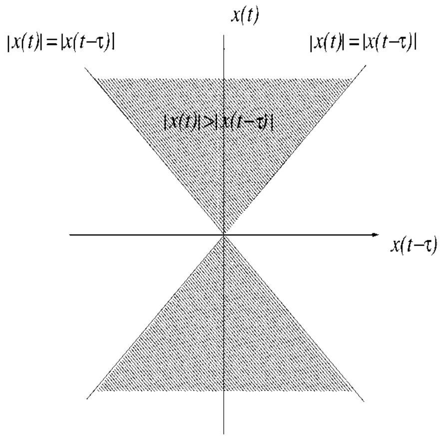

Figure 8.3.1. Geometric expression of the Lyapunov direct method.

The right-hand side of (8.3.2) is a function of $x(t)$ and $x(t-\tau)$. Roughly speaking, the stability of a system is to study the relation between the future state and the current state. When $|x(t-\tau)| > |x(t)|$, the trend of the solution becomes decreasing (or stable), which does not add any constrains on $\frac{dV}{dt}$. Thus, one only needs to apply restriction to the increasing trend. In other words, when $|x(t)| \geqslant |x(t-\tau)|$, we use the condition $\frac{dV}{dt} < 0$, which is exactly used to overcome the instability of the system. However, there does not necessarily exist relation between the condition $|x(t)| \geqslant |x(t-\tau)|$ and the sign of $\frac{dV}{dt}$.

Consequently, we only need to show $\frac{dV}{dt} \leqslant 0$ when $|x(t)| \geqslant |x(t-\tau)|$ (i.e., $|V(x(t))| \geqslant |V(x(t-\tau))|$). In other words, it is not necessary to show $\frac{dV}{dt} \leqslant 0$ for the whole $x(t)$-$x(t-\tau)$ plane, but just for the shaded sector, as shown in Figure 8.3.1.

Back to system (8.3.1), when $|x(t-\tau)| \leqslant |x(t)|$,

$$\frac{dV}{dt} \leqslant ax^2(t) + \left|bx(t)\right|\left|x(t-\tau)\right| \leqslant ax^2(t) + |b|x^2(t).$$

When $a + |b| < 0$, $\frac{dV}{dt}$ is negative definite about $x(t)$. Thus, if $a + |b| < 0$, we can conclude that the zero solution of (8.3.2) is globally asymptotically stable.

Now, consider a general delayed system:

$$\frac{dx}{dt} = f\big(t, x(t), x\big(t-\tau(t)\big)\big), \tag{8.3.3}$$

where $x \in R^n$, $f \in C[I \times R^n \times R^n, R^n]$ and $f(t, 0, 0) = 0$, $0 \leqslant \tau(t) \leqslant \tau < \infty$.

THEOREM 8.3.1. *(See [228].)*

(1) *If there exists function* $V(t,x) \in C[G_H, R^1]$ *and* $\varphi_1, \varphi_2 \in K$ *such that*

$$\varphi_1\big(\|x\|\big) \leqslant V(t,x) \leqslant \varphi_2\big(\|x\|\big)$$

in G_H;

(2) *the Dini derivative of* $V(t,x)$ *along the solution of* (8.3.3) *satisfies*

$$\begin{aligned} D^+V(t,x)|_{(8.3.3)} &:= \overline{\lim_{h\to 0^+}} \frac{1}{h}\big[V\big(t+h, x(t+h,t,x_0)\big) - V(t,x_0)\big] \\ &\leqslant g(t)F\big(V\big(t,x(t)\big)\big) \end{aligned}$$

when

$$V\big(t-\tau(t)\big), x\big(t-\tau(t)\big) \leqslant V\big(t,x(t)\big), \tag{8.3.4}$$

where $F(V) > 0$ *and when* $V > 0$, $F(0) = 0$;

(3)

$$\lim_{h\to 0^+} \int_a^b \frac{dr}{F(r)} = +\infty. \tag{8.3.5}$$

Then the zero solution of (8.3.3) *is uniformly stable. Here,* $a > b, g(t) \geqslant 0$, $\int_0^{+\infty} g(t)\,dt = M > 0$.

PROOF. Construct the following equation:

$$\frac{dy}{dt} = \bar{g}(t)F(y), \quad \text{where } \bar{g}(t) = g(t) + \frac{1}{t^2+1}. \tag{8.3.6}$$

Then from condition (3) we know that there exists y_0 such that

$$\int_{y(0)}^{\varepsilon_1} \frac{dy}{F(y)} = \overline{M} = \int_0^{+\infty} \bar{g}(t)\,dt. \tag{8.3.7}$$

Assuming that $y(t)$ is the solution of (8.3.6) and $y(t_0) = y_0 > 0$, we have

$$\int_{y_0}^{y(t)} \frac{dy}{F(y)} = \int_{t_0}^{t} \bar{g}(s)\,ds < \int_0^{+\infty} \bar{g}(s)\,ds := \overline{M} = \int_{y(0)}^{\varepsilon} \frac{dy}{F(y)}. \tag{8.3.8}$$

So, $0 < y(t) < \varepsilon_1, t \geqslant t_0$

Let

$$0 < \delta < y_0, \qquad \varphi_2(\delta) < y_0, \tag{8.3.9}$$

where the initial value function $\xi(t)$ satisfies

$$\|\xi(t)\| < \delta.$$

Now, we prove that for the solution of (8.3.3) $x(t) := x(t, t_0, \xi)$ satisfying $\|\xi\| \leqslant \delta$, we have

$$V(t, x(t)) := V(t) \leqslant y(t), \quad t \geqslant t_0. \tag{8.3.10}$$

Otherwise, there exists $t_1 > t_0$ such that

$$V(t) \leqslant y(t), \quad t \leqslant t_1, \quad V(t_1) = y(t_1),$$

and there exists $0 < r \ll 1$ for which there is an infinite number of $\bar{t}_i$ ($i = 1, 2, \ldots$) such that $\bar{t}_i \to +\infty$, $\bar{t}_i \in [t_i, x_1 + r]$ and $V(\bar{t}_i) > y(\bar{t}_i)$. It follows from the definition of upper limit that

$$D^+V(t_1) \geqslant \left.\frac{dy}{dt}\right|_{t=t_1} = \bar{g}(t_1)F(y(t_1)) = \bar{g}(t_1)F(V(t_1)). \tag{8.3.11}$$

But

$$\begin{aligned} &V(\xi) \leqslant V(t_1), \quad t_1 - r \leqslant \xi \leqslant t_1, \\ &D^+V(t_1) \leqslant g(t_1)F(V(t_1)) < \bar{g}(t_1)F(V(t_1)). \end{aligned} \tag{8.3.12}$$

This contradicts (8.3.11), so (8.3.10) holds. Thus,

$$\varphi_1(\|x(t)\|) \leqslant V(t) \leqslant g(t) < \varepsilon_1 \leqslant \varphi_1(\varepsilon),$$

implying that $\|x(t)\| < \varepsilon$, $t \geqslant t_0$. Since $\delta < y_0$ is independent of t_0, the zero solution of (8.3.3) is uniformly stable. □

COROLLARY 8.3.2. *If*

(1) *condition* (1) *in Theorem* 8.3.1 *holds*; *and*
(2) *condition* (2) *in Theorem* 8.3.1 *is changed to*

$$D^+V(t, x)|_{(8.3.1)} \leqslant 0;$$

then the zero solution of (8.3.3) *is uniformly stable.*

This is the improved Razumikhin theorem which can be used to find the stability of the zero solution of differential difference equation (8.3.3).

THEOREM 8.3.3. *Assume the following conditions hold*:

(1) *condition* (1) *in Theorem* 8.3.1 *holds*;
(2) *there exist nonnegative continuous functions* $F(t, x)$, $\psi(t, x)$ *such that* $\forall \sigma > 0$, *when* $|x| \geqslant \delta$, $t \geqslant t_0$ *it holds*

$$F\big(t, x(t)\big) \geqslant \psi(t, \sigma) \geqslant 0, \tag{8.3.13}$$

and

$$\int_{t_0}^{t} \psi(t, \sigma)\, dt \Rightarrow +\infty \quad (\textit{uniformly about } t_0) \textit{ as } t \to +\infty; \tag{8.3.14}$$

(3) *there exists continuous nondecreasing function $P(s) > s$, when $s > 0$ and*

$$V\big(t - \tau(t), x\big(t - \tau(t)\big)\big) \leqslant PV\big(t, x(t)\big), \tag{8.3.15}$$

it holds

$$D^+V\big(t, x(t)\big)\big|_{(8.3.3)} \leqslant -F\big(t, x(t)\big); \tag{8.3.16}$$

then the zero solution of (8.3.3) is uniformly stable.

PROOF. Let $x(t) := x(t, t_0, \varphi)$, $V(t) := V(t, x(t))$. Obviously,

$$-F\big(t, x(t)\big) \leqslant \frac{1}{t^2 + 1} V(t).$$

So the conditions in Theorem 8.3.1 are satisfied, and thus the zero solution of (8.3.1) is uniformly stable.

Next, we prove that the zero solution is uniformly asymptotically stable. $\forall \sigma < \varepsilon_1 < H$, $\exists \delta > 0$, when the initial value function $|\varphi| < \delta$, there exist $|x(t_0)| < H$ and

$$V(t, x_0) < \varphi_2(\varepsilon_1), \quad t \geqslant t_0,$$
$$\forall \eta > 0, \quad 0 < \eta < \min(\varepsilon_1, \delta),$$

such that $0 < \varphi_1(\eta) < \varphi_2(\varepsilon)$.

We want to prove that $\exists t_N$ such that $|x(t)| \leqslant \eta$ when $t \geqslant t_N$. To achieve this, let

$$a = \inf\big[P(s) - s\big] > 0 \quad \text{when } \varphi_1(\eta) \leqslant s \leqslant \varphi_2(\varepsilon_1).$$

Further, there exists $\sigma > 0$ such that $\varphi_1(\eta) \geqslant \varphi_2(\sigma) > 0$. So, when $\varphi_2(\|x(t_0)\|) \geqslant \varphi_1(\eta) \geqslant \varphi_2(\sigma) > 0$, we have $\|x(t_0)\| \geqslant \sigma > 0$ and

$$F\big(t, \big\|x(t_0)\big\|\big) \geqslant \psi(t, \sigma). \tag{8.3.17}$$

Assume that N is a positive integer such that

$$\varphi_1(\eta) + Na \geqslant \varphi_2(\varepsilon_1) > \varphi_1(\eta_1) + (N - 1)a. \tag{8.3.18}$$

Then, there must exists $T_1 \geqslant t_0 + r$ such that

$$V(T_1) < \varphi_1(\eta_1) + (N - 1)a. \tag{8.3.19}$$

Otherwise, we have

$$V(t) \geqslant \varphi_1(\eta_1) + (N-1)a, \quad t > t_0 + r.$$

On the other hand,

$$\begin{aligned} P\big(V(t)\big) &\geqslant V(t) + a \geqslant \varphi_1(\eta) + Na \geqslant \varphi_2(\varepsilon_1) \\ &> V(\xi) \quad \text{when } t - r \leqslant \xi \leqslant t. \end{aligned}$$

Since

$$\varphi_2\big(\|x(t)\|\big) \geqslant V(t) \geqslant \varphi_1(\eta) > \varphi_2(\sigma)$$

and $\|x(t)\| \geqslant \sigma$, from condition (2) and (3) we can get

$$D^+V(t) \leqslant -F\big(t, \|x(t)\|\big) \leqslant -\psi(t,\sigma), \quad t \geqslant t_0 + r.$$

Therefore,

$$V(t) \leqslant V(t_0 + r) - \int_{t_0 - r}^{t} \psi(s,\sigma)\,ds \to -\infty \quad \text{when } t \to \infty.$$

This is impossible. Thus, (8.3.19) is true.

Since $D^+V(t) \leqslant 0$, $\forall t \geqslant T_1$,

$$V(t) \leqslant \varphi_2(\eta) + (N-1)a$$

holds. Repeating the above process, we can show that there must exist $T_2, \ldots, T_N$ such that

$$V(t) \leqslant \varphi_2(\eta) + (N-l)a, \quad t \geqslant T_l,\ l = 1, \ldots, N,$$

from which we obtain

$$\varphi_1\big(\|x(t)\|\big) \leqslant V\big(t, x(t)\big) \leqslant \varphi_1(\eta), \quad t \geqslant T_N.$$

Thus,

$$\big\|x(t)\big\| \leqslant \eta, \quad t \geqslant T_N.$$

Due to the arbitrary of η, we know that the zero solution of (8.3.12) is uniformly asymptotically stable. □

COROLLARY 8.3.4. *Assume that the conditions in Theorem 8.3.3 are satisfied, while $F(t, \|x(t_0)\|) = \psi(t)W(\|x(t_0)\|)$, $\psi(t) > 0$, $t \geqslant t_0$, and for arbitrarily given $\beta > 0$, there exists $\alpha(\beta) > 0$ such that $\int_{t_1}^{t} \psi(s)\,ds \geqslant \beta$ holds uniformly for $t \geqslant t_1 + \alpha(\beta)$ (irrelevant to $t_1 \geqslant t_0 \geqslant 0$). Then the zero solution of (8.3.3) is uniformly asymptotically stable.*

Particularly, if take $\psi(t) \equiv 1$, then all conditions in Corollary 8.3.4 hold, and thus the zero solution of (8.3.3) is uniformly asymptotically stable.

EXAMPLE 8.3.5. Consider the stability of the following 2-dimensional time-delay prototype system:

$$\begin{cases} \frac{dx_1}{dt} = -3x_1(t) + a_{12}x_2(t) + \frac{1}{3}x_1(t-\tau(t)) + \frac{1}{2}x_2(t-\tau(t)), \\ \frac{dx_2}{dt} = a_{21}x_1(t) - 4x_2(t) + \frac{2}{3}x_1(t-\tau(t)) + \frac{1}{2}x_2(t-\tau(t)), \end{cases} \tag{8.3.20}$$

where a_{12} and a_{21} are real constants.

Construct the Lyapunov function:

$$V(x_1, x_2) = |x_1| + |x_2|.$$

Obviously, $V(x_1, x_2)$ is positive definite and radially unbounded. Then, we have that when $|x_1(t-\tau(t))| + |x_2(t-\tau(t))| \leqslant |x_1(t)| + |x_2(t)|$,

$$\begin{aligned} D^+V(x_1, x_2)|_{(8.3.20)} &\leqslant \big(-3 + |a_{21}|\big)\big|x_1(t)\big| + \big(-4 + |a_{12}|\big)\big|x_2(t)\big| \\ &\quad + \big|x_1\big(t-\tau(t)\big)\big| + \big|x_2\big(t-\tau(t)\big)\big| \\ &\leqslant \big(-3 + |a_{21}|\big)\big|x_1(t)\big| + \big(-4 + |a_{12}|\big)\big|x_2(t)\big| \\ &\quad + \big|x_1(t)\big| + \big|x_2(t)\big| \\ &= \big(-2 + |a_{21}|\big)\big|x_1(t)\big| + \big(-3 + |a_{12}|\big)\big|x_2(t)\big| \\ &\leqslant 0 \quad (x \neq 0). \end{aligned} \tag{8.3.21}$$

Based on Corollary 8.3.2, when $|a_{21}| \leqslant 2$ and $|a_{12}| \leqslant 3$, the zero solution of (8.3.20) is stable.

When $|a_{21}| < 2$ and $|a_{12}| < 3$, according to Corollary 8.3.4, the zero solution of (8.3.20) is asymptotically stable. In fact, let $2 - |a_{21}| = \varepsilon_1$, $3 - |a_{12}| = \varepsilon_2$. Choose $P(S) = (1 + \frac{1}{2}\min(\varepsilon_1, \varepsilon_2))S$. Then, when $V(x_1(t-\tau(t)), x_2(t-\tau(t))) \leqslant PV(x_1(t), x_2(t))$, we have

$$\begin{aligned} D^+V(x_1, x_2)|_{(8.3.19)} &\leqslant \big(-3 + |a_{21}|\big)\big|x_1(t)\big| + \big(-4 + |a_{12}|\big)\big|x_2(t)\big| \\ &\quad + \big|x_1\big(t-\tau(t)\big)\big| + \big|x_2\big(t-\tau(t)\big)\big| \\ &\leqslant \big(-3 + |a_{21}|\big)\big|x_1(t)\big| + \big(-4 + |a_{12}|\big)\big|x_2(t)\big| \\ &\quad + \left(1 + \frac{1}{2}\min(\varepsilon_1, \varepsilon_2)\right)\big(\big|x_1(t)\big| + \big|x_2(t)\big|\big) \\ &= \left(-2 + \frac{1}{2}\varepsilon_1 + |a_{21}|\right)\big|x_1(t)\big| \\ &\quad + \left(-3 + \frac{1}{2}\varepsilon_1 + |a_{21}|\right)\big|x_2(t)\big| \\ &\leqslant -\varepsilon_1\big|x_1(t)\big| - \varepsilon_2\big|x_2(t)\big| \end{aligned}$$

$$< 0 \quad \text{when } x_1^2 + x_1^2 \neq 0.$$

In addition, we have

$$V\big(x_1(t-\tau_t), x_2(t-\tau_t)\big) \leqslant P\big(V\big(x_1(t), x_2(t)\big)\big).$$

Thus, the zero solution of (8.3.20) is asymptotically stable.

EXAMPLE 8.3.6. Consider the time-delay system:

$$\begin{cases} \frac{dx_1}{dt} = -a_{11}^2 x_1(t) - a_{12}x_2(t) + b_{11}x_1(t-\tau(t)) \\ \quad + b_{12}x_2(t-\tau(t)), \\ \frac{dx_2}{dt} = 2a_{12}x_1(t) - a_{22}^2 x_2(t) + b_{21}x_1(t-\tau(t)) \\ \quad + b_{22}x_2(t-\tau(t)), \end{cases} \tag{8.3.22}$$

and determine the range of the values of the coefficients such that the zero solution of the system is stable.

Construct the Lyapunov function:

$$V(x_1, x_2) = x_1^2 + \frac{1}{2}x_2^2.$$

Then, we have

$$\begin{aligned} \frac{dV}{dt}\bigg|_{(8.3.22)} &= -2a_{11}^2 x_1^2(t) - a_{22}^2 x_2^2(t) + 2b_{11}x_1(t)x_1\big(t-\tau(t)\big) \\ &\quad + 2b_{12}x_1(t)x_2\big(t-\tau(t)\big) + b_{21}x_2(t)x_1\big(t-\tau(t)\big) \\ &\quad + b_{22}x_2(t)x_2\big(t-\tau(t)\big) \\ &\leqslant -2a_{11}^2 x_1^2(t) - a_{22}^2 x_2^2(t) + b_{11}^2 x_1^2(t) + x_1^2(t)\big(t-\tau(t)\big) \\ &\quad + 2b_{12}^2 x_1^2(t) + \frac{x_2^2(t-\tau(t))}{2} + \frac{b_{21}^2}{4}x_2^2(t) \\ &\quad + x_1^2\big(t-\tau(t)\big) + \frac{b_{22}^2}{2}x_2^2(t) + \frac{x_2^2(t-\tau(t))}{2} \\ &= -\big[2a_{11}^2 - b_{11}^2 - 2b_{12}^2 - 1\big]x_1^2(t) - \left[a_{22}^2 - \frac{b_{21}^2}{4} - \frac{b_{22}^2}{2}\right]x_2^2(t) \\ &\quad + 2V\big(x_1\big(t-\tau(t)\big), x_2\big(t-\tau(t)\big)\big) \\ &\leqslant -\big[2a_{11}^2 - b_{11}^2 - 2b_{12}^2 - 1\big]x_1^2(t) - \left[a_{22}^2 - \frac{b_{21}^2}{4} - \frac{b_{22}^2}{2}\right]x_2^2(t). \end{aligned}$$

Thus, when

$$\begin{cases} 2a_{11}^2 \geqslant b_{11}^2 + 2b_{12}^2 + 1, \\ a_{22}^2 \geqslant \frac{b_{21}^2}{4} + \frac{b_{22}^2}{2}, \end{cases}$$

the zero solution of system (8.3.22) is stable.

When the above two inequalities take strict inequalities, similarly following the proof of Example 8.3.5 we can show that the zero solution of system (8.3.22) is globally asymptotically stable.

8.4. Lyapunov functional method for stability analysis

Razumikhin method is still an important research topic that has been continuously studied, promoted and improved by scientists. One of its advantages is that it does not need to verify whether the derivative of function V, $\frac{dV}{dt}$, is negative definite or not about the state variables $x_1(t-\tau_1(t)), \dots, x_n(t-\tau_n(t))$, but only needs to show that $\frac{dV}{dt}$, satisfying $V(x(t-\tau(t))) < P(V(x(t)))$ (where $P(s) > s$ when $s \neq 0$ is a function to be determined), is negative definite about the variables $x_1, \dots, x_n$. However, in general, there are various types of $\frac{dV}{dt}$, and it is hard to obtain a $V(x(t))$ satisfying the form of $P(V(x(t-\tau(t)))) > V(x(t-\tau(t)))$, except for some very special V and $\frac{dV}{dt}$.

In this section, we discuss the method introduced by Krasovskii, which uses Lyapunov functional to replace Lyapunov function to study stability. This is one of the most important methods in the study of stability of differential difference equations.

First, we introduce several norms of functionals defined on $C[-\tau, 0]$ as follows:

$$\|x(s)\|_\tau = \sup_{\substack{-\tau \leqslant s \leqslant 0 \\ 1 \leqslant i \leqslant n}} |x_i|,$$

$$\|x(s)\|_\tau^2 = \left(\int_{-\tau}^{0} \sum_{i=1}^{n} x_i^2(s)\, ds \right)^{1/2},$$

$$\|x\|_1 = \sup_{1 \leqslant i \leqslant n} |x_i|,$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_1^2 \right)^{1/2}.$$

DEFINITION 8.4.1. Functional $V(t, x(s))$, $t \geqslant t_0$, $-\tau \leqslant s \leqslant 0$, is said to be positive definite, if there exists $\varphi_1 \in K$ such that $\varphi_1(\|x(s)\|_\tau) \leqslant V(t, x(s))$. Functional $V(t, x(s))$, $t \geqslant t_0$, $-\tau \leqslant s \leqslant 0$, is said to have infinitesimal upper bound, if there exists $\varphi \in K$ such that $V(t, x(s)) \leqslant \varphi_2(\|x(s)\|_\tau)$.

Similarly, we can define the negative definite and radially unboundedness of functional $V(t, x(s))$.

Let $x_t(\xi)$ denote the solution of (8.3.3) satisfying the initial value condition $x(t) = \xi(t)$, $-\tau \leqslant t \leqslant 0$.

THEOREM 8.4.2. *If there exists positive definite functional $V(t, x(s))$ with infinitesimal upper bound in some region $G_H = \{(t, x), \|x(s)\|_\tau < H, -\tau \leqslant s \leqslant 0,\ t \geqslant t_0\}$ such that the Dini derivative of $V(t, x(s))$ along the solution (8.3.3), defined by*

$$\begin{aligned} &D^+V\big(t, x(x)\big)\big|_{(8.3.3)} \\ &\quad := \lim_{h\to 0^+} \frac{1}{h}\Big(V\big(t+h, x_{t+h}(\xi)\big)\Big) - V\big(t, x_t(\xi)\big) \leqslant 0, \end{aligned} \tag{8.4.1}$$

is positive definite, then the zero solution of (8.3.3) is uniformly stable.

PROOF. Assume that there exist $\varphi_1, \varphi_2 \in K$ satisfying

$$\varphi_1\big(\big\|x(s)\big\|_\tau\big) \leqslant V\big(t, x(s)\big) \leqslant \varphi_2\big(\big\|x(s)\big\|_\tau\big).$$

$\forall \varepsilon > 0$ $(0 < \varepsilon < h)$, choose $\delta(\varepsilon) > 0$ such that $\varphi_2(t) < \varphi_1(t)$. Then, when $\|\xi(s)\| < \delta$, we have

$$\varphi_1\big(\big\|x_t(\xi)\big\|\big) \leqslant V\big(t, x_t(\xi)\big) \leqslant V\big(t_0, \xi(s)\big) \leqslant \varphi_2\big(\|\xi\|_\tau\big) \leqslant \varphi_2(\delta) < \varphi_1(\varepsilon).$$

Thus,

$$\big\|x_t(\xi)\big\|_T < \varepsilon,$$

which indicates that the zero solution of (8.3.3) is uniformly stable. □

THEOREM 8.4.3. *If there exists positive definite functional $V(t, x(s))$ in region G_H, with infinitesimal upper bound, such that its Dini derivative is negative definite along the solution of (8.3.3), then the zero solution of (8.3.3) is uniformly asymptotically stable.*

PROOF. Assume that there exist φ_1, $\varphi_2 \in K$ and positive definite function W such that

$$\begin{aligned} &\varphi_1\big(\big\|x(s)\big\|_\tau\big) \leqslant V\big(t, x(s)\big) \leqslant \varphi_2\big(\big\|x(s)\big\|_\tau\big), \\ &D^+V\big(t, x(s)\big)\big|_{(8.3.3)} \leqslant -W\big(x(0)\big). \end{aligned} \tag{8.4.2}$$

Since the conditions in Theorem 8.4.3 imply that the conditions in Theorem 8.4.2 are satisfied, the zero solution of (8.3.3) is uniformly stable.

Now we prove that the zero solution of (8.3.3) is uniformly attractive. $\forall \eta > 0$, choose $\sigma(\eta) > 0$ such that

$$\sup_{\|x(s)\|_\tau \leqslant \sigma_1(\eta)} V\big(t, x(s)\big) < \inf_{\|x(s)\|_\tau = \eta} V\big(t, x(s)\big), \quad t \geqslant t_0.$$

Next, we show that there exists $T > 0$, when $t^* = t_0 + T$, it holds

$$\left\|x_t^*(\xi)\right\|_\tau < \sigma(\eta).$$

Otherwise, suppose

$$\sigma(\eta) \leqslant \left\|x_t(\xi)\right\|_\tau \leqslant H.$$

Let $\alpha = \inf_{\alpha \leqslant \|x\|\| \leqslant H} W(x)$. Then, there exists

$$D^+V\big(t, x(s)\big) \leqslant -\alpha \quad \text{on } \alpha \leqslant \|x\| \leqslant H.$$

Thus, $\forall t \geqslant t_0$, we have

$$V\big(t, x_t(\xi)\big) \leqslant V\big(t_0, x_{t_0}(\xi)\big) - \alpha(t - t_0) \to -\infty \quad \text{as } t \to +\infty.$$

This implies that there exists $T > \frac{1}{2}V[t_0, x_0(\xi)]+\alpha t_0$ such that $\|x_T(\xi)\|_\tau < \sigma(\eta)$. Thus, when $t > t_0 + T$, we have $\|x_t(\xi)\| < \eta$, while $\sigma(\eta)$ is independent on t_0. So, the zero solution is uniformly asymptotically stable. □

THEOREM 8.4.4.

(1) *If there exist functional $V(t, x(s))$ and $\varphi_1, \varphi_2, W_1, W_2 \in K$ and positive definite function $V(t, x(s))$ satisfying*

$$V\big(t, x(s)\big) \leqslant \varphi_1\big(\left\|x(0)\right\|\big) + \varphi_2\big(\left\|x(\xi)\right\|_\tau^2\big),$$
$$V\big(t, x(s)\big) \geqslant W_1\big(\left\|x(0)\right\|\big);$$

(2) $D^+V(t, x(s))|_{(8.3.3)} \leqslant -W_2(\|x(0)\|)$;

then the zero solution of (8.3.3) is uniformly asymptotically stable.

PROOF. One can follow the proofs of Theorems 8.4.2 and 8.4.3 to finish the proof of Theorem 8.4.4. Thus, the details are omitted. □

In the above we only discussed general principle and method of how to use the Lyapunov functional to study the stability of time-delay systems. The key problem is how to construct the Lyapunov functional, which will be discussed further later with more examples given in Section 8.6, Chapter 9 (Absolute stability), and Chapter 10 (Stability of neural networks).

EXAMPLE 8.4.5. Consider an n-dimensional time-delay system with constant coefficients

$$\frac{dx}{dt} = Ax(t) + Bx(t - \tau), \tag{8.4.3}$$

where $R \in R^n$, A, B are all $n \times n$ constant matrices, and $\tau = \text{constant} > 0$.

The stability of the zero solution of system (8.4.3) can be determined by using eigenvalue. Here, we use the Lyapunov functional method to solve the problem.

Assuming A is stable, for an arbitrary given $n \times n$ symmetric positive definite matrix D, the Lyapunov matrix equation

$$A^T C + CA = -D$$

has symmetric positive definite matrix solution $C_{n\times n}$.

Choose Lyapunov functional

$$V(x) = x^T Cx + \int_{t-\tau}^{t} x^T(s)Gx(s)\,ds. \tag{8.4.4}$$

Then we have

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(8.4.3)} &= -x^T(t)Dx(t) + 2x^T(t)CBx(t-\tau)\\ &\quad + x^T(t)Gx(t) - x^T(t-r)Gx(t-r).\end{aligned} \tag{8.4.5}$$

Consider the right-hand side of the above equation as a quadratic form about $x(t)$ and $x(t-\tau)$, with some constrains on matrices A and B so that C and G can be solved, under which the quadratic form (8.4.4) is negative definite. Thus, the zero solution of (8.4.3) is asymptotically stable. To achieve this, we only need

$$\begin{bmatrix} D-G & -CB \\ -B^T C & G \end{bmatrix}$$

being negative definite.

The above condition can be even further weakened by requiring the above quadratic form being negative definite only about $x(t)$, not necessarily for both $x(t)$ and $x(t-\tau)$.

8.5. Nonlinear autonomous systems with various time delays

Nonlinear systems with separate various time delays are often encountered in autonomic control, biomathematics, and specially, in neural networks. In this section, we discuss general methods in determining the stability of nonlinear autonomous systems with separate various time delays.

Consider the following nonlinear autonomous system with separate various time delays:

$$\begin{aligned}\frac{dx_i}{dt} &= \sum_{j=1}^{n} a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(x\big(t-\tau_j(t)\big)\big),\\ &\quad i = 1, 2, \ldots, n,\end{aligned} \tag{8.5.1}$$

where $f_j(\cdot) \in C[(-\infty+\infty), R]$, $g_j(\cdot) \in C[(-\infty+\infty), R]$, but $|g_i(\cdot)| \leqslant |f_i(\cdot)|$. Suppose that these conditions guarantee the uniqueness of the Cauchy problem, and

$$f_j(0) = 0, \quad j = 1, 2, \ldots, n, \quad 0 \leqslant \tau_j(t) \leqslant \tau_j = \text{constant},$$

and $A = (a_{ij})_{n\times n}$, $B = (B_{ij})_{n\times n}$ are given real matrices.

THEOREM 8.5.1. *For system (8.5.1), if the following conditions are satisfied*:

(1) $f_j(x_j)x_j > 0$ *when* $x_j \neq 0$, $\int_0^{+\infty} f_j(x_j)\,dx_j = +\infty$;
(2) $1 - \dot\tau_j(t) \geqslant \delta_j > 0$, $j = 1, 2, \ldots, n$; *and*
(3) *there exist positive definite diagonal matrices* $P = \operatorname{diag}(p_1, \ldots, p_n)$ *and* $\xi = \operatorname{diag}(\xi_1, \ldots, \xi_n)$ *such that the matrix*

$$\begin{bmatrix} PA + A^TP + \xi\delta^{-1} & PB \\ B^TP & -\xi \end{bmatrix}$$

is negative definite, where $\delta^{-1} = \operatorname{diag}(\frac{1}{\delta_1}, \ldots, \frac{1}{\delta_n})$,

then the zero solution of (8.5.1) *is globally asymptotically stable.*

PROOF. We construct a positive definite and radially unbounded Lyapunov functional as follows:

$$V\big(x(t), t\big) = \sum_{i=1}^n 2p_i \int_0^{x_i} f_i(x_i)\,dx_i + \sum_{j=1}^n \frac{\xi_i}{\delta_i} \int_{t-\tau_i(t)}^t g_i\big(x(s)\big)\,ds. \tag{8.5.2}$$

Differentiating $V(x(t), t)$ with respect to time t along the solution of (8.5.1) yields

$$\begin{aligned}
\frac{dV(x(t), t)}{dt} &= \sum_{i=1}^n \sum_{j=1}^n 2p_i a_{ij} f_i\big(x_i(t)\big) f_j\big(x_j(t)\big) \\
&\quad + 2\sum_{i=1}^n \sum_{j=1}^n p_i b_{ij} f_i\big(x_i(t)\big) g_j\big(x_j\big(t - \tau_j(t)\big)\big) \\
&\quad + \sum_{i=1}^n \frac{\xi_i}{\delta_i} g_i^2\big(x_i(t)\big) - \sum_{i=1}^n \frac{\xi_i}{\delta_i} g_i^2\big(x_i\big(t - \tau_i(t)\big)\big)\big(1 - \dot\tau_i(t)\big) \\
&\leqslant \sum_{i=1}^n \sum_{j=1}^n 2p_i a_{ij} f_i\big(x_i(t)\big) f_j\big(x_j(t)\big) \\
&\quad + \sum_{i=1}^n \sum_{j=1}^n 2p_i b_{ij} f_i\big(x_i(t)\big) f_j\big(x_j\big(t - \tau_j(t)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{n} \frac{\xi_i}{\delta_i} f_i^2\big(x_i(t)\big) - \sum_{i=1}^{n} \xi_i g_i^2\big(x_i\big(t - \tau_i(t)\big)\big) \\
&\leqslant \begin{pmatrix} f(x(t)) \\ g(x(t - \tau(t))) \end{pmatrix}^T \begin{bmatrix} PA + +A^T P + \xi\delta^{-1} & PB \\ B^T P & -\xi \end{bmatrix} \\
&\quad \times \begin{pmatrix} f(x(t)) \\ g(x(t - \tau(t))) \end{pmatrix} \\
&< 0 \quad \text{when } x \neq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
f\big(x(t)\big) &= \big(f_1\big(x_1(t)\big), \ldots, f_n\big(x_n(t)\big)\big)^T \\
g\big(x\big(t - \tau(t)\big)\big) &= \big(g_1\big(x_1\big(t - \tau_1(t)\big)\big), \ldots, g_n\big(x_n\big(t - \tau_n(t)\big)\big)\big)^T.
\end{aligned}
$$

Thus, the conclusion is true. □

Particularly, choosing $P = I_n$, we have

COROLLARY 8.5.2. *If the conditions* (1) *and* (2) *in Theorem* 8.5.1 *hold, and* (3)

$$
\begin{bmatrix} A + A^T + \xi\delta^{-1} & B \\ B^T & -\xi \end{bmatrix}
$$

is negative definite, then the conclusion in Theorem 8.5.1 *holds.*

REMARK 8.5.3. It is generally assumed that $\dot{\tau}_j(t) \leqslant 0$ in the literature. The identity matrix I_n is often used to replace ξ, i.e., using the negative definite matrix

$$
\begin{bmatrix} PA + A^T P + I_n & PB \\ B^T P & -I_n \end{bmatrix}
$$

to replace condition (3) in Theorem 8.5.1. Here, we use ξ to replace I_n so that ξ or $\xi_1, \ldots, \xi_n$ can be appropriately chosen depending upon the diagonal elements of $PA + A^T P$. For example, when the diagonal elements of $PA + A^T P$, $2p_i a_{ii} > -1$, choosing I_n would not satisfy the conditions. Therefore, sometimes one can adjust $\xi_1, \ldots, \xi_n$ to satisfy the conditions in Theorem 8.5.1 or Corollary 8.5.2.

EXAMPLE 8.5.4. Consider a 2-dimensional nonlinear system with constant time delays:

$$
\begin{cases}
\frac{dx_1}{dt} = -\frac{1}{2} f_1(x_1(t)) + 8 f_2(x_2(t)) \\
\quad + \frac{1}{4} f_1(x_1(t - \tau_1)) + \frac{1}{5} f_2(x_2(t - \tau_2)), \\
\frac{dx_2}{dt} = -8 f_1(x_1(t)) - f_2(x_2(t)) \\
\quad - \frac{1}{5} f_1(x_1(t - \tau_1)) + \frac{1}{4} f_2(x_2(t - \tau_2)),
\end{cases}
\tag{8.5.3}
$$

where τ_i = constant, $i = 1, 2$, $f_i(0) = 0$, $i = 1, 2$, satisfying the basic assumptions in Theorem 8.5.1.

Obviously,

$$\begin{bmatrix} A + A^T + I_2 & B \\ B^T & -I_2 \end{bmatrix} = \begin{bmatrix} -1+1 & 0 & \frac{1}{4} & \frac{1}{5} \\ 0 & -2+1 & -\frac{1}{5} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{5} & -1 & 0 \\ \frac{1}{5} & \frac{1}{4} & 0 & -1 \end{bmatrix}$$

is not negative definite, but negative semi-definite.

If we choose $\xi_1 = \frac{1}{2}$ and $\xi_2 = 1$, then

$$\begin{bmatrix} A + A^T + \xi & B \\ B^T & -\xi \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{5} \\ 0 & -1 & -\frac{1}{5} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{5} & -\frac{1}{2} & 0 \\ \frac{1}{5} & \frac{1}{4} & 0 & -1 \end{bmatrix}$$

is negative definite. So the conditions in Corollary 8.5.2 are satisfied. Thus, the zero solution of Example 8.5.4 is globally asymptotically stable.

In the following, we discuss another more general class of nonlinear time-delay systems with separable variables, described by

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(x_j\big(t - \tau_{ij}(t)\big)\big),$$
$$i = 1, 2, \ldots, n, \tag{8.5.4}$$

where $\tau_{ij}(t) \not\equiv \tau_j(t)$, and $f_j(\cdot)$ satisfies the basic assumptions given for system (8.5.1).

THEOREM 8.5.5. *If the following conditions are satisfied for system* (8.5.4)

(1) $f_j(x_j)x_j > 0$ *when* $x_j \neq 0$;
(2) $\tau_{ij}(t)$ *is differentiable and* $1 - \dot{\tau}_{ij}(t) \geqslant \delta_{ij} =$ *constant* > 0;
(3) $|g_i(\cdot)| \leqslant f_i(\cdot)$;
(4) $\overline{A} = (\bar{a}_{ij})_{m\times n}$ *is an M matrix*;

then the zero solution of (8.5.4) *is globally asymptotically stable. Here,*

$$\bar{a}_{ij} = \begin{cases} -a_{ij}, & i = j = 1, 2, \ldots, n \\ -|a_{ij}| - \frac{|b_{ij}|}{\delta_{ij}}, & i = j = 1, 2, \ldots, n,\ i \neq j. \end{cases}$$

PROOF. Since $\overline{A} = (\bar{a}_{ij})_{m\times n}$ is an M matrix, there exists positive numbers $p_i > 0$, $i = 1, 2, \ldots, n$, such that

$$p_j a_{jj} + \sum_{\substack{i=1\\i\neq j}}^{n} p_i|a_{ij}| + \sum_{i=1}^{n} p_i \frac{|b_{ij}|}{\delta_{ij}} < 0, \quad j = 1, 2, \ldots, n.$$

Construct a positive definite and radially unbounded Lyapunov functional:

$$V\big(x(t), t\big) = \sum_{i=1}^{n} p_i\big|x_i(t)\big| + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \frac{|b_{ij}|}{\delta_{ij}} \int_{t-\tau_{ij}(t)}^{t} \big|g_j\big(x_j(s)\big)\big|\,ds. \tag{8.5.5}$$

Taking the Dini derivative of $V(x(t), t)$ along the solution of (8.5.4) yields

$$\begin{aligned}
D^+V\big(x(t), t\big) &\leqslant \sum_{i=1}^{n} p_i D^+\big|x_i(t)\big| + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \frac{|b_{ij}|}{\delta_{ij}}\big|g_j\big(x_j(t)\big)\big| \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} p_i \frac{|b_{ij}|}{\delta_{ij}}\big|g_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\big|\big(1-\dot{\tau}_{ij}(t)\big) \\
&\leqslant \sum_{j=1}^{n} p_j a_{jj}\big|f_j\big(x_j(t)\big)\big| + \sum_{\substack{i=1\\i\neq j}}^{n} p_i|a_{ij}|\big|f_j\big(x_j(t)\big)\big| \\
&\quad + \sum_{i=1}^{n} p_i|b_{ij}|\big|g_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\big| \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{n} p_i \frac{|b_{ij}|}{\delta_{ij}}\big|g_j\big(x_j(t)\big)\big| \\
&\quad - \sum_{j=1}^{n}\sum_{i=1}^{n} p_i|b_{ij}|\big|g_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\big| \\
&\leqslant \sum_{j=1}^{n}\left(p_j a_{jj} + \sum_{\substack{i=1\\i\neq j}}^{n} p_i|a_{ij}| + \sum_{i=1}^{n} p_i\frac{|b_{ij}|}{\delta_{ij}}\right)\big|f_j\big(x_j(t)\big)\big| \\
&< 0 \quad \text{when } x \neq 0.
\end{aligned}$$

So, the conclusion of Theorem 8.5.5 is true. $\square$

COROLLARY 8.5.6. *If the conditions* (1), (2) *and* (3) *in Theorem* 8.5.5 *hold, and*

$$a_{jj} + \sum_{\substack{i=1\\i\neq j}}^{n} |a_{jj}| + \sum_{i=1}^{n} \frac{|b_{jj}|}{\delta_{ij}} < 0,$$

or

$$a_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{jj}| + \sum_{j=1}^{n} \frac{|b_{jj}|}{\delta_{ij}} < 0,$$

then the conclusion of Theorem 8.5.5 holds.

Since the conditions in Corollary 8.5.6 imply that $\overline{A}_{ij}(\bar{a}_{ij})_{n\times n}$ is an M matrix, the conclusion of Corollary 8.5.6 is true.

REMARK 8.5.7. In Theorems 8.5.1, 8.5.5 and in Corollary 8.5.6, when the time delays are constants ($\tau_{ij}(t) = \tau_{ij} =$ constant), or $\dot{\tau}_{ij}(t) \leqslant 0$, the conclusions still hold.

EXAMPLE 8.5.8. Consider the following nonlinear system with variable time delays:

$$\begin{cases} \frac{dx_1(t)}{dt} = a_{11} f_1(x_1(t)) + a_{12} f_2(x_2(t)) + b_{11} g_1(x_1(t-\tau_{11}(t))) \\ \qquad + b_{12} g_2(x_2(t-\tau_{12}(t))) \\ \qquad = -3x_1^3(t) + \frac{1}{2}x_2^5(t) + \frac{1}{2}\frac{2x_1(t-\frac{1}{2}\sin(t))}{1+x_1^2(t-\frac{1}{2}\sin(t))} x_1^3(t-\frac{1}{2}\sin(t)) \\ \qquad + \frac{1}{2}e^{-x_2^2(t-\frac{1}{3}\cos(t))} x_2^5(t-\frac{1}{3}\cos(t)), \\ \frac{dx_2(t)}{dt} = a_{21} f_1(x_1(t)) + a_{22} f_2(x_2(t)) + b_{21} g_1(x_1(t-\tau_{11}(t))) \\ \qquad + b_{22} g_2(x_2(t-\tau_{22}(t))) \\ \qquad = \frac{1}{2}x_1^3(t) - 3x_2^5(t) + \frac{2}{3}\frac{2x_1(t-\frac{1}{3}\sin(t))}{1+x_1^2(t-\frac{1}{2}\sin(t))} x_1^3(t-\frac{1}{3}\sin(t)) \\ \qquad + \frac{2}{3}e^{-x_2^2(t-\frac{1}{3}\cos(t))} x_2^5(t-\frac{1}{3}\cos(t)), \end{cases} \tag{8.5.6}$$

where $1 - \dot{\tau}_{i1}(t) \geqslant \frac{1}{2} = \delta_{i1},\ i = 1, 2$, and $1 - \dot{\tau}_{i2}(t) \geqslant \frac{2}{3} = \delta_{i2}, i = 1, 2$.

Obviously,

$$\left|g_1(x_1)\right| = \left|\frac{2x_1}{1+x_1^2} x_1^3\right| \leqslant x_1^3 = \left|f_1(x_1)\right|,$$

$$\left|g_2(x_2)\right| = \left|e^{-x_2^2} x_2^5\right| \leqslant x_2^5 = \left|f_2(x_2)\right|,$$

$$a_{11} + |a_{21}| + \frac{|b_{11}|}{\delta_{11}} + \frac{|b_{21}|}{\delta_{21}} = -3 + \frac{1}{2} + \frac{1}{2}\frac{1}{\frac{1}{2}} + \frac{2}{3}\frac{1}{\frac{1}{2}}$$

$$= -3 + \frac{1}{2} + 1 + \frac{4}{3} = -\frac{1}{6} < 0,$$

$$a_{22} + |a_{12}| + \frac{|b_{12}|}{\delta_{12}} + \frac{|b_{22}|}{\delta_{22}} = -3 + \frac{1}{2} + \frac{2}{3}\frac{3}{2} + \frac{2}{3}\frac{3}{2}$$
$$= -\frac{1}{2} < 0.$$

Hence, this Example 8.5.8 satisfies all the conditions in Corollary 8.5.6, and so the zero solution of (8.5.6) is globally asymptotically stable.

Next we consider nonlinear time-varying system with varying time delays:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) f_j x_j(t) + \sum_{j=1}^{n} b_{ij}(t) g_j x_j\big(t - \tau_j(t)\big), \tag{8.5.7}$$

where $f(\cdot)$, $g(\cdot)$ and $\tau_j(t)$ satisfy the conditions given in (8.5.1) and (8.5.4). $A(t) = (a_{ij}(t))_{n\times n}$ and $B(t) = (b_{ij}(t))_{n\times n}$ are continuous matrix functions. The condition (1) and (2) in Theorem 8.5.1 still hold, but only to replace Condition (3) by the following: there exist positive definite diagonal matrices $P = \operatorname{diag}(p_1, \ldots, p_n)$ and $\xi = \operatorname{diag}(\xi_1, \ldots, \xi_n)$ such that

$$\begin{bmatrix} PA(t) + A^T(t)P + \xi\delta^{-1} & PB(t) \\ B^T(t)P & -\xi \end{bmatrix}$$

is negative definite, then the zero solution of (8.5.7) is globally asymptotically stable. However, it is very difficult to verify whether a time varying matrix is negative definite or not.

Similarly, if the conditions (1), (2) and (3) in Theorem 8.5.5 still hold, but condition (4) is changed to that $\overline{A}(\bar{a}_{ij}(t))_{n\times n}$ is an M matrix, where

$$\bar{a}_{ij}(t) = \begin{cases} -a_{ij}(t) & \text{for } i = j = 1, \ldots, n, \\ -\big|a_{ij}(t)\big| - \frac{|b_{ij}(t)|}{\delta_{ij}} & \text{for } i, j = 1, \ldots, n,\ i \neq j, \end{cases} \tag{8.5.8}$$

then the zero solution of (8.5.7) is globally asymptotically stable. Again, it is very difficult to verify whether a time varying matrix is an M matrix or not.

If the conditions in Corollary 8.5.6 are changed to

$$a_{jj}(t) + \sum_{\substack{i=1\\ i\neq j}}^{n} \big|a_{ij}(t)\big| + \sum_{i=1}^{n} \frac{|b_{ij}(t)|}{\delta_{ij}} < 0$$

or

$$a_{ii}(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} \big|a_{ij}(t)\big| + \sum_{j=1}^{n} \frac{|b_{ij}(t)|}{\delta_{ij}} < 0,$$

(then the zero solution of (8.8.7) is globally asymptotically stable), then these conditions are relatively easy to verify.

For convenience in verifying the conditions, in the following we present some results with stronger conditions.

THEOREM 8.5.9. *If the following conditions are satisfied for system* (8.5.7):

(1) $f_i(x_i)x_i > 0$ *when* $x_i \neq 0$, $f_i(0) = 0$, $\int_0^{\pm\infty} f_i(x_i)\,dx_i = +\infty$;
(2) $\tau_j(t)$ *is differentiable and* $1 - \dot{\tau}_j(t) \geqslant \delta_j = constant > 0$;
(3) $|g_i(\cdot)| \leqslant |f_i(\cdot)|$;
(4) *there exists positive definite diagonal matrices* $P = \mathrm{diag}(p_1, \dots, p_n) > 0$ *and* $\xi = \mathrm{diag}(\xi_1, \dots, \xi_n) > 0$ *such that*

$$\begin{bmatrix} P\overline{A}^* + \overline{A}^{*T}P + \xi\delta^{-1} & P\overline{B}^* \\ \overline{B}^{*T}P & -\xi \end{bmatrix}$$

is negative definite;

then the zero solution of (8.5.7) *is globally asymptotically stable. Here,*

$$\bar{a}_{ij}^* = \begin{cases} a_{ii}^* = \sup_{t\in[t_0,+\infty)}\{a_{ii}(t)\}, & i = j = 1, \dots, n, \\ a_{ij}^* = \sup_{t\in[t_0,+\infty)}|a_{ij}(t)|, & i \neq j, i, j = 1, \dots, n, \end{cases}$$

$$b_{ij}^* = \sup_{t\in[t_0,+\infty)} \bigl|b_{ij}(t)\bigr|, \quad j = 1, \dots, n.$$

PROOF. Construct the positive definite and radially unbounded Lyapunov functional:

$$V\bigl(x(t), t\bigr) = \sum_{i=1}^n \int_0^{x_i} f_i(x_i)\,dx_i + \sum_{i=1}^n \xi_i\delta_i^{-1} \int_{t-\tau_i(t)}^{t} g_i^2\bigl(x_i(s)\bigr)\,ds.$$

Then, we have

$$\begin{aligned}
\frac{dV}{dt}\bigg|_{(8.5.7)} &\leqslant \sum_{i=1}^n p_i f_i(x_i)\frac{dx_i}{dt} + \sum_{i=1}^n \xi_i\delta_i^{-1} g_i^2\bigl(x_i(t)\bigr) \\
&\quad - \sum_{i=1}^n \xi_i g_i^2\bigl(x_i\bigl(t - \tau_i(t)\bigr)\bigr)\bigl(1 - \dot{\tau}_i(t)\bigr) \\
&\leqslant \sum_{i=1}^n p_i a_{ij}^* \bigl|f_i\bigl(x_i(t)\bigr)\bigr|\bigl|f_j\bigl(x_i(t)\bigr)\bigr| \\
&\quad + \sum_{i=1}^n p_i b_{ij}^* \bigl|f_i\bigl(x_i(t)\bigr)\bigr|\bigl|g_j\bigl(x_i\bigl(t - \tau_j(t)\bigr)\bigr)\bigr| \\
&\quad + \sum_{i=1}^n \xi_i\delta_i^{-1} f_j^2\bigl(x_i(t)\bigr) - \sum_{i=1}^n \xi_i g_i^2\bigl(x_i\bigl(t - \tau_i(t)\bigr)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \begin{pmatrix} |f(x(t))| \\ |g(x(t-\tau(t)))| \end{pmatrix}^T \begin{bmatrix} P\overline{A}^* + \overline{A}^* P + \xi\delta^{-1} & P\overline{B}^* \\ \overline{B}^{*T} P & -\xi \end{bmatrix} \\
&\quad \times \begin{pmatrix} |f(x(t))| \\ |g(x(t-\tau(t)))| \end{pmatrix} \\
&< 0 \quad \text{when } x \neq 0,
\end{aligned}
$$

where,

$$
\left|f\big(x(t)\big)\right| = \left(\left|f_1\big(x_1(t)\big)\right|, \ldots, \left|f_n\big(x_n(t)\big)\right|\right)^T
$$

and

$$
\left|g\big(x\big(t-\tau(t)\big)\big)\right| = \left(\left|g_1\big(x_1\big(t-\tau(t)\big)\big)\right|, \ldots, \left|g_n\big(x_n\big(t-\tau_n(t)\big)\big)\right|\right)^T.
$$

Thus, the conclusion is true. □

COROLLARY 8.5.10. *If the conditions* (1) *and* (2) *in Theorem* 8.5.9 *hold, and condition* (3) *is changed to*

$$
\begin{bmatrix} \overline{A}^* + \overline{A}^{*T} + \xi\delta^{-1} & \overline{B}^* \\ \overline{B}^{*T} & -\xi \end{bmatrix}
$$

being negative definite, then the zero solution of (8.5.7) *is globally asymptotically stable.*

Now we allow $\tau_{ij}(t) = \tau_j(t)$ in (8.5.7), yielding

THEOREM 8.5.11. *If condition* (1) *in Theorem* 8.5.9 *holds, and* $|g_i(\cdot)| \leqslant |f_i(\cdot)|$,

(2) $\tau_{ij}(t)$ *is differentiable and* $1 - \dot{\tau}_{ij}(t) \geqslant \delta_{ij} = constant > 0$, $j = 1, 2, \ldots, n$;
(3) *there exist* $p_i > 0$, $i = 1, 2, \ldots, n$, *such that*

$$
p_j \bar{a}^*_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^n p_i \bar{a}^*_{ij} + \sum_{i=1}^n p_i \frac{\bar{b}^*_{ij}}{\delta_j} < 0;
$$

then the zero solution of (8.5.7) *is globally asymptotically stable.*

PROOF. Construct the positive definite and radially unbounded Lyapunov functional:

$$
V\big(x(t), t\big) = \sum_{i=1}^n p_i \left|x_i(t)\right| + \sum_{i=1}^n \sum_{j=1}^n p_i \frac{\bar{b}^*_{ij}}{\delta_{ij}} \int\limits_{t-\tau_j(t)}^{t} \left|g_i\big(x_i(s)\big)\right| ds,
$$

from which we obtain

$$\begin{aligned}
D^+V\big(x(t),t\big)\big|_{(8.5.7)} &\leqslant \sum_{j=1}^{n} p_i a_{jj}(t)\big|f_j\big(x_j(t)\big)\big| + \sum_{\substack{i=1\\ i\neq j}}^{n} p_i\big|a_{ij}(t)\big|\big|f_j\big(x_j(t)\big)\big| \\
&\quad + \sum_{i=1}^{n} p_i\big|b_{ij}(t)\big|\big|g_j\big(x_j\big(t-\tau_j(t)\big)\big)\big| \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i\frac{\bar{b}_{ij}}{\delta_{ij}}\big|g_j\big(x_j(t)\big)\big| \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} p_i\frac{\bar{b}^*_{ij}}{\delta_{ij}}\big|g_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\big|\big(1-\dot{\tau}_{ij}\big) \\
&\leqslant \sum_{j=1}^{n}\left(p_i\bar{a}^*_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} p_i a^*_{ij} + \sum_{i=1}^{n} P_i\frac{b^*_{ij}}{\sigma}\right)\big|f_j\big(x_j(t)\big)\big| \\
&< 0 \quad \text{when } x\neq 0.
\end{aligned}$$

So the zero solution of (8.5.7) is globally asymptotically stable. □

COROLLARY 8.5.12. *If the conditions* (1) *and* (2) *in Theorem* 8.5.9 *hold, and* (3)

$$a^*_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} a^*_{ij} + \sum_{i=1}^{n}\frac{b^*_{ij}}{\delta_{ij}} < 0,$$

or

$$a^*_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} a^*_{ij} + \sum_{j=1}^{n}\frac{b^*_{ij}}{\delta} < 0,$$

then the zero solution of (8.5.7) *is globally asymptotically stable.*

8.6. Application of inequality with time delay and comparison principle

Differential inequality and comparison principle play a great role in the study of stability in ordinary differential equation. It is Halanay who first extended differential inequality to differential difference inequality. Here, we first extend the Halanay one-dimensional time-delay differential inequality from standard derivative to Dini derivative inequality.

LEMMA 8.6.1 *(Extended Halanay one-dimensional time-delay differential inequality). Assume that constants $a > b > 0$, and that function $x(t)$ is a nonnegative unitary continuous function on $[t_0 - \tau, t]$ with the following inequality held*

$$D^+ x(t) \leqslant -ax(t) + b\bar{x}(t), \tag{8.6.1}$$

where

$$\bar{x}(t) := \sup_{t-\tau \leqslant s \leqslant t} \{x(s)\}, \quad \tau \geqslant 0,$$

is a constant. Then, there exists

$$x(t) \leqslant \bar{x}(t_0) e^{-\lambda(t-t_0)}, \quad \forall t \geqslant t_0, \tag{8.6.2}$$

where λ is the unique positive root of the following transcendent equation:

$$\lambda = a - be^{\lambda\tau}. \tag{8.6.3}$$

PROOF. First, we prove that the transcendent equation (8.6.3) has only one positive solution. To achieve this, consider

$$\begin{aligned}
&\Delta(\mu) = \mu - a + be^{\mu t}, \quad \mu \in [0, a], \\
&\Delta(0) = -a + b < 0, \quad \Delta(a) = be^{at} > 0, \\
&\Delta'(\mu) = 1 + b\tau e^{\mu t} > 0,
\end{aligned}$$

where $\Delta(\mu)$ is a strictly monotone increasing function. Thus, there exists a unique $\lambda \in (0, \alpha)$ on $[0, a]$ satisfying (8.6.3).

Let

$$y(t) = \bar{x}(t_0) e^{-\lambda(t-t_0)}, \quad t_0 - \tau \leqslant t < \beta. \tag{8.6.4}$$

Assume $k > 1$ is an arbitrary constant. Then $x(t) < ky(t)$ holds on $t_0 - \tau \leqslant t \leqslant t_0$.

Now, suppose that for some $t \in (t_0, \beta)$, where $\beta > t_0$ is arbitrary, we have $x(t) = ky(t)$. Since $x(t)$ and $y(t)$ are continuous functions, there must exist $t_1 \in (t_0, \beta)$ such that

$$x(t) < ky(t), \quad t_0 - \tau \leqslant t < t_1, \quad x(t_1) < ky(t_1). \tag{8.6.5}$$

Thus, $D^+ x(t_1) \geqslant ky'(t_1)$.

On the other hand, due to (8.6.1), we have

$$D^+ x(t_1) \leqslant -ax(t_1) + b\bar{x}(t_1) < -aky(t_1) + bky(t_1 - \tau) = ky'(t_1),$$

i.e.,

$$D^+ x(t_1) < ky'(t_1). \tag{8.6.6}$$

This shows that (8.6.5) and (8.6.6) are contradictory, thus, for any $\beta > t$ we have

$$x(t) < ky(t) \quad \forall t \in [t_0, \beta].$$

Finally, letting $k \to 1$ results in

$$x(t) \leqslant y(t) = x(t_0)e^{-\lambda(t-t_0)}. \tag{8.6.7}$$

The proof is complete. □

Next, we extend the case of one-dimensional inequality to higher dimensional cases.

Assume that C^n is a set consisting of all functions $f \in C[I \times R^n \times C^n, R^n]$.

DEFINITION 8.6.2. $G(t, x, y)$ is said to be an H_n class function, if the following conditions are satisfied:

(1) $\forall t \in I$, $\forall x \in R^n$, $\forall y^{(1)}, y^{(2)} \in C^n$, when $y^{(1)} \leqslant y^{(2)}$ (i.e., $y_i^{(1)} \leqslant y_i^{(2)}$, $i = 1, 2, \dots, n$) it holds

$$G\big(t, x, y^{(1)}\big) \leqslant G\big(t, x, y^{(2)}\big); \tag{8.6.8}$$

(2) $\forall t \in I, \forall y \in C^n, \forall x^{(1)}, x^{(2)} \in R^n$, when

$$x^{(1)} \leqslant x^{(2)} \quad \text{(but } x_i^{(1)} = x_i^{(2)} \text{ for some } i) \tag{8.6.9}$$

it holds

$$g_i\big(t, x^{(1)}, y\big) \leqslant g_i\big(t, x^{(2)}, y\big) \quad \text{for these } i\text{'s}. \tag{8.6.10}$$

LEMMA 8.6.3. *If a continuous n-dimensional vector function*

$$x(t),\ y(t), \quad \bar{x}(t) := \sup_{\xi \in [t-\tau, t]} x(\xi), \quad \bar{y}(t) := \sup_{\xi \in [t-\tau, t]} y(\xi)$$

satisfies the following conditions:

(1) $x(\theta) < y(\theta)$, $\theta \in [-\tau, 0]$;
(2) $D^+ y_i(t) > g_i(t, y(t), \bar{y}(t))$, $i = 1, 2, \dots, n$, $t \geqslant 0$; $D^+ x_i(t) \leqslant g_i(t, x(t), \bar{x}(t))$, $i = 1, 2, \dots, n$, $t \geqslant 0$;

then $x(t) < y(t)$ for $t > 0$. Here,

$$G\big(t, x(t), \bar{x}(t)\big) = \big(g_1\big(t, x(t), \bar{x}(t)\big), \dots, g_n\big(t, x(t), \bar{x}(t)\big)\big)^T.$$

PROOF. By contradiction. Suppose there exists a constant $\eta > 0$ and some i such that

$$x_i(\eta) = y_i(\eta).$$

Let $Z = \{\eta \mid x_i(\eta) = y_i(\eta)$ for some $i\}$. Obviously, Z is not empty. Thus, from condition (1), $\exists \eta_0 = \inf_{\eta \in z} \eta$ such that

$$x(\theta) < y(\theta), \quad \theta \in [-\tau, 0],$$

and thus $\eta_0 > 0$, $x(\theta) \leqslant y(\eta_0)$ and $\bar{x}(\theta) \leqslant \bar{y}(\eta)$. This implies that there exists j, $1 \leqslant j \leqslant n$, such that

$$x_j(\eta_0) = y_j(\eta_0).$$

From $G \in H_n$ and condition (2) we have

$$\begin{aligned} D^+ x_j(t_0) &\leqslant g_j\big(\eta_0, x(\eta_0), \bar{x}(\eta_0)\big) \\ &\leqslant g_j\big(\eta_0, y(\eta_0), \bar{y}(\eta_0)\big) < D^+ y_j(\eta_0). \end{aligned} \tag{8.6.11}$$

However, when $0 < t < \eta_0$, we have

$$x(t) < y(t), \quad \text{i.e.,} \quad x_j(t) < y_j(t), \quad \text{but} \quad x_j(\eta_0) < y_j(\eta_0).$$

Hence, we obtain

$$D^+ x_j(\eta_0) \geqslant D^+ y_j(\eta_0). \tag{8.6.12}$$

$\square$

Consider the following n-dimensional varying time delay linear system:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \tilde{a}_{ij} x_j(t) + \sum_{j=1}^{n} \tilde{b}_{ij} x\big(t - \tau_j(t)\big), \quad i = 1, 2, \ldots, n, \tag{8.6.13}$$

where a_{ij}'s are real coefficients and $0 \leqslant \tau_j(t) \leqslant \tau_j = \text{constant}$, $j = 1, 2, \ldots, n$.

THEOREM 8.6.4. *Assume that the following conditions are satisfied*:

(1)

$$D^+|x_i| \leqslant \sum_{j=1}^{n} a_{ij} \big|x_j(t)\big| + \sum_{j=1}^{n} b_{ij} \bar{x}\big(t - \tau_j(t)\big), \quad i = 1, 2, \ldots, n,$$

where

$$a_{ij} = \begin{cases} \tilde{a}_{ii}, & i = j = 1, \ldots, n, \\ |\tilde{a}_{ij}|, & i \neq j,\ i, j = 1, \ldots, n, \end{cases}$$

$$b_{ij} = \big|\tilde{b}_{ij}\big|, \quad i, j = 1, \ldots, n,$$

$$a_{ij} \geqslant 0, \quad i \neq j,\ b_{ij} \geqslant 0,\ i, j = 1, \ldots, n,$$

and

$$\sum_{i=1}^{n} \bar{x}_i(t_0) > 0, \qquad \bar{x}_j(t_0) = \sup_{t-\tau \leqslant s \leqslant t} \{x_j(s)\};$$

(2) $M := -(a_{ij} + b_{ij})_{n\times n}$ *is an* M *matrix.*

Then, there exist constants $\gamma_i > 0$, $a_i > 0$ *such that the solution of differential inequality* (8.6.3) *has the following estimation*

$$|x_i(t)| \leqslant \gamma_i \left[\sum_{j=1}^{n} |\bar{x}_j(t_0)| \right] e^{-a(t-t_0)}. \tag{8.6.14}$$

This implies that the zero solution of (8.6.13) *is globally exponentially stable.*

PROOF. Let

$$g_i(t, |x(t)|, |\bar{x}(t)|) = \left(\sum_{j=1}^{n} a_{ij} |x_j(t)| + \sum_{j=1}^{n} b_{ij} |\bar{x}_j(t)| \right),$$

$$G(t, |x(t)|, |\bar{x}(t)|)$$
$$= \left(g_1(t, |x(t)|, |\bar{x}(t)|), \ldots, g_n(t, |x(t)|, |\bar{x}(t)|)\right)^T \in H_n.$$

Then from condition (2) we know that there exist $\delta > 0$ and $d_j > 0$ ($j = 1, 2, \ldots, n$) such that

$$\sum_{j=1}^{n} (a_{ij} + b_{ij}) d_j < -\delta, \quad i = 1, 2, \ldots, n. \tag{8.6.15}$$

Choose $0 < a \ll 1$ such that

$$a d_i + \sum_{j=1}^{n} a_{ij} d_i + \sum_{j=1}^{n} b_{ij} d_j e^{a\tau} < 0. \tag{8.6.16}$$

When $t \in [t_0 - \tau, t_0]$, $j \neq i$, choose $R \gg 1$ such that

$$R d_i e^{a\tau} > 1. \tag{8.6.17}$$

$\forall \varepsilon > 0$, let

$$q_i(t) = R d_i \left[\sum_{j=1}^{n} |\bar{x}_j(t_0)| + \varepsilon \right] e^{-a(t-t_0)}. \tag{8.6.18}$$

Then, from (8.6.16) we have

$$D^+ q_i(t) = -a R d_i \left[\sum_{j=1}^{n} |\bar{x}_j(t_0)| + \varepsilon \right] e^{-a(t-t_0)}$$

$$
\begin{aligned}
&\geqslant \sum_{j=1}^{n}\left[a_{ij}d_j + b_{ij}d_j e^{a\tau}\right]R\left[\sum_{j=1}^{n}\left|\bar{x}_j(t_0)\right| + \varepsilon\right]e^{-a(t-t_0)}\\
&= \sum_{j=1}^{n} a_{ij}d_j R\left[\sum_{j=1}^{n}\left|\bar{x}_j(t_0)\right| + \varepsilon\right]e^{-a(t-t_0)}\\
&\quad + \sum_{j=1}^{n} b_{ij}d_j R\left[\sum_{j=1}^{n}\left|\bar{x}_j(t_0)\right| + \varepsilon\right]e^{-a(t-t_0)}e^{a\tau}\\
&\geqslant \sum_{j=1}^{n} a_{ij}q_j(t) + \sum_{j=1}^{n} b_{ij}\bar{q}_j(t)\\
&= g_i\big(t, q(t), \bar{q}(t)\big),
\end{aligned}
$$

which indicates that $D^+q_i(t) > g_i(t, q(t), \bar{q}(t))$. Thus, when $t \in [t_0 - \tau, t_0]$, it follows from (8.6.17) that

$$
q_i(t) = Rd_i\left[\sum_{j=1}^{n}\left|\bar{x}_j(t_0)\right| + \varepsilon\right]e^{-a(t-t_0)} > \sum_{j=1}^{n}\left|\bar{x}_j(t_0)\right| + \varepsilon.
$$

Now, let

$$
\left|x_i(t)\right| \leqslant \sum_{j=1}^{n}\left|\bar{x}_j(t_0)\right| + \varepsilon.
$$

Then, when $t \in [t_0 - \tau, t_0]$, from Lemma 8.6.3 we know that

$$
\left|x_i(t)\right| < q_i(t) = Rd_i\left[\sum_{j=1}^{n}\left|\bar{x}_j(t_0)\right| + \varepsilon\right]e^{-a(t-t_0)}.
$$

Finally, let $\varepsilon \to 0^+$, $Rd_i = \gamma_i$. Then, by using Theorem 8.6.4, we obtain

$$
\left|x_i(t)\right| \leqslant \gamma_i\left[\sum_{j=1}^{n}\left|x_j(t_0)\right|\right]e^{-a(t-t_0)} \quad \text{when } t \geqslant t_0,\ i = 1, \ldots, n,
$$

and thus the zero solution of (8.6.13) is globally exponentially stable. □

REMARK 8.6.5. The greatest advantage of using differential inequality method is that it avoids the strict constrains that the delay $I(t)$ is differentiable when applying the Lyapunov functional method. In this section, we only discussed linear differential systems with constant coefficients. More applications will be given in Chapter 10 for neural networks. Readers can try, instead of Lyapunov functional method, to employ Lyapunov function method with Razumikhin principle to get similar results.

8.7. Algebraic method for LDS with constant coefficients and time delay

In this section, we introduce algebraic methods for studying the stability of linear delay systems (LDS) with constant coefficients and constant time delays.

Consider a general linear system with constant coefficients and constant time delay [153]

$$\frac{dx}{dt} = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - \gamma_k \cdot r), \tag{8.7.1}$$

where $x \in R^n$, every A_k is an $n \times n$ real constant matrix, $r = (r_1, \ldots, r_m)$ and $\gamma_k = (\gamma_{k_1}, \ldots, \gamma_{k_m})$, $\gamma_{k_j} > 0$ are given nonnegative integers, and

$$\gamma_k r = \sum_{j=1}^{m} \gamma_{k_j} r_j \quad (k = 1, 2, \ldots, m).$$

The characteristic equation of (8.7.1) is

$$f(\lambda, r, A) = \det\left[\lambda I - A_0 - \sum_{k=1}^{n} A_k e^{-\lambda \gamma_k \cdot r}\right] = 0, \tag{8.7.2}$$

where $A_0, A_2, \ldots, A_n \in R^{n^2}$.

Similar to linear systems described by ordinary differential equations with constant coefficients, the distribution of the roots of characteristic equation (8.7.2) plays the key role in determining the stability of the zero solution of the system.

LEMMA 8.7.1. *The necessary and sufficient conditions for the zero solution of* (8.7.1) *being time delay independent, asymptotically stable are*

(1) *when the time delay is zero, all roots of*

$$f(\lambda, 0, A) = \det\left[\lambda I - \sum_{k=0}^{N} A_k\right] = 0$$

are located on the open left-half of the complex plane, i.e., $\operatorname{Re}\lambda < 0$;

(2)

$$f(iy, r, A) = \det\left[iyI - A_0 - \sum_{k=1}^{N} A_k e^{-\lambda r_k \cdot r}\right] \neq 0 \quad \forall y \in R.$$

The basic idea of this lemma is simple. Condition (1) implies that when $r = 0$, the roots of (8.7.2) are all located in the open left-half of the complex plane.

Because of the continuous dependence of the roots on the coefficients, if some roots move to right-half of the complex plane or reach the imaginary axis, they must reach the imaginary axis at some $\tau_0 > 0$. However, condition (2) excludes this possibility, and thus the conclusion is true.

If (8.7.2) has eigenvalues with positive real part, then the zero solution of (8.7.2) is unstable. But it is very difficult to verify the conditions (1) and (2) for the characteristic equation (8.7.2). In this section, we will give some simple sufficient conditions in determining stability.

DEFINITION 8.7.2. System (8.7.1) is said to be asymptotically stable about (r, A), if its characteristic equation has only eigenvalues with negative real parts, i.e., $\operatorname{Re}\lambda < 0$.

For given $r \in (R^+)^m$, define the radial line Γ_r passing through r as

$$\Gamma_r := \left\{r \mid dr \in \left(R^+\right)^m,\ d \geqslant 0\right\}. \tag{8.7.3}$$

DEFINITION 8.7.3. For given $r^0 \in (R^+)^M$, the asymptotically stable cone, S_{r^0}, about r is defined as

$$\begin{aligned} S_{r^0} := \big\{A \in \left(R^{n^2(N+1)}\right) \mid \text{(8.7.1) is asymptotically stable about } (r, A) \\ \text{for every } r \in \Gamma_{r^0}\big\}; \end{aligned} \tag{8.7.4}$$

and the asymptotically stable cone S is defined as [153]

$$S = \bigcap_r \left(S_r\colon\ r \in \left(R^+\right)^m\right). \tag{8.7.5}$$

Let $\operatorname{Re}\lambda(A)$ denote the real part of the eigenvalues of matrix A. To obtain some practically useful criteria, we need the following lemma.

LEMMA 8.7.4. *(See [280].) Assume $G(g_{ij})_{n\times n}$ is a complex matrix, and $H(h_{ij})_{n\times n}$ is an M matrix, where*

$$h_{ij} = \begin{cases} |g_{ij}|, & i = j = 1, 2, \dots, n, \\ -|g_{ij}|, & i \neq j,\ i, j = 1, 2, \dots, n. \end{cases}$$

Then $\det G(g_{ij}) \neq 0$.

PROOF. Since M is an M matrix, there exist constants $\beta_i > 0$ $(i = 1, 2, \dots, n)$ such that

$$|g_{ii}|\beta_i - \sum_{\substack{j=1\\ j\neq i}}^{n} \beta_i |g_{ij}| > 0 \quad (i = 1, 2, \dots, n). \tag{8.7.6}$$

From the Gershgorin disk theorem we know that zero is not an eigenvalue of matrix $G\,\mathrm{diag}(\beta_1, \dots, \beta_n)$. Thus,

$$\det(G)\,\mathrm{diag}(\beta_1, \dots, \beta_n) = \det(G)\det\bigl(\mathrm{diag}(\beta_1, \dots, \beta_n)\bigr) \neq 0.$$

But

$$\det\bigl(\mathrm{diag}(\beta_1, \dots, \beta_n)\bigr) = \prod_{i=1}^{n} \beta_i \neq 0.$$

Therefore, $\det G \neq 0$. The lemma is proved. □

LEMMA 8.7.5. *(See [280].) For the two given matrices $H(h_{ij})$ and $\hat{H}(\tilde{h}_{ij})$, if*

(1) *H is an M matrix*;
(2) $h_{ij} \leqslant \tilde{h}_{ij}$, $i, j = 1, 2, \dots, n$, $i, j = 1, 2, \dots, n$, $\tilde{h}_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, 2, \dots, n$;

then $\hat{H}$ is also an M matrix.

PROOF. From the properties of M matrix we know that H is an M matrix, if and only if there exist n constants $\beta_i > 0$ $(i = 1, 2, \dots, n)$ such that $\sum_{j=1}^{n} h_{ij}\beta_j > 0$. However, $\tilde{h}_{ii} \geqslant h_{ii} > 0$, $i = 1, 2, \dots, n$, $h_{ii} \leqslant \tilde{h}_{ii} \leqslant 0$, $i \neq j$, $i, j = 1, 2, \dots, n$. Thus,

$$\sum_{j=1}^{n} \tilde{h}_{ij}\beta_j \geqslant \sum_{j=1}^{n} h_{ij}\beta_j > 0,$$

which means that $\tilde{H}$ is an M matrix. □

Let the elements of A_k be $a_{ij}^{(k)}$, i.e., $A_k = (a_{ij}^{(k)})_{n\times n}$.

THEOREM 8.7.6. *(See [280].) Assume*

$$b_{ij} := \begin{cases} |a_{ii}^{0}| - \sum_{k=1}^{N} |a_{ii}^{(k)}|, & i = j = 1, 2, \dots, n, \\ -\sum_{k=1}^{N} |a_{ij}^{(k)}|, & i \neq j,\ i, j = 1, 2, \dots, n. \end{cases}$$

If $B(b_{ij})_{n\times n}$ is an M matrix, and $\mathrm{Re}\,\lambda(\sum_{k=0}^{N} A_k) < 0$ (particularly, $a_{ii}^{(0)} < 0$, $i = 1, 2, \dots, n$), then $A \in S$, i.e., the zero solution of (8.7.1) is globally asymptotically stable.

PROOF. Define

$$\hat{B}(\hat{b}_{ij}) := \begin{bmatrix} |a_{11}^{(0)} - iy| - \sum_{k=1}^{N}|a_{11}^{(k)}| & -\sum_{k=0}^{N}|a_{12}^{(k)}| & \cdots & -\sum_{k=0}^{N}|a_{1n}^{(k)}| \\ -\sum_{k=0}^{N}|a_{21}^{(k)}| & |a_{22}^{0} - iy| - \sum_{k=1}^{N}|a_{22}^{(k)}| & \cdots & -\sum_{k=0}^{N}|a_{2n}^{(k)}| \\ \vdots & \vdots & \vdots & \vdots \\ -\sum_{k=0}^{N}|a_{n1}^{(k)}| & -\sum_{k=0}^{n}|a_{n2}^{(k)}| & \cdots & |a_{nn}^{(0)} - iy| - \sum_{k=1}^{N}|a_{nn}^{(k)}| \end{bmatrix}.$$

Since $B(b_{ij})_{n\times n}$ is an M matrix and satisfies

$$\hat{b}_{ij} = b_{ij} \leqslant 0, \quad i \neq j,$$
$$\hat{b}_{ii} = |a_{ii}^{(0)} - iy| - \sum_{k=1}^{N}|a_{ii}^{(k)}| \geqslant |a_{ii}^{(0)}| - \sum_{k=1}^{N}|a_{ii}^{(k)}| = b_{ii} > 0. \tag{8.7.7}$$

By Lemma 8.7.4, we know that $\hat{B}(\hat{b}_{ij})_{n\times n}$ is an M matrix. Further, also from Lemma 8.7.4, we have

$$\det_{\substack{s_j\in C\\|s_j|=1}} = \left[iyI - A_0 - \sum_{k=1}^{N} A_k s_{r_{k_1}}, \ldots, s_{r_{k_m}}\right] \neq 0,$$

which, together with the condition $\mathrm{Re}(\sum_{k=1}^{N} A_k) < 0$, we know that the zero solution of (8.7.1) is asymptotically stable by Lemma 8.7.1. □

COROLLARY 8.7.7. *If* $b_{ii} > 0$ $(i = 1, 2, \ldots, n)$, $\mathrm{Re}\,\lambda(\sum_{k=1}^{n} A_k) < 0$ (*particularly,* $a_{ii}^{(0)} < 0$, $i = 1, 2, \ldots, n$), *and any one of the following conditions is satisfied*:

(1) $\displaystyle b_{jj} > \sum_{\substack{i=1\\i\neq j}}^{N} |b_{ij}|, \quad j = 1, 2, \ldots, n;$

(2) $\displaystyle b_{ii} > \sum_{\substack{j=1\\j\neq i}}^{n} |b_{ij}|, \quad i = 1, 2, \ldots, n;$

(3) $\displaystyle b_{ii} > \frac{1}{2}\sum_{\substack{j=1\\j\neq i}}^{n} \big(|b_{ij}| + |b_{ji}|\big), \quad i = 1, 2, \ldots, n;$

(4) $\displaystyle \sum_{\substack{i=1\\i\neq j}}^{n} \left(\frac{|b_{ij}|}{|b_{ii}|}\right)^2 < 1;$

(5) $b_{ii}^2 > \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{2}\big(|b_{ii}b_{ij}| + |b_{jj}b_{ji}|\big)$;

then $A \in S$, the zero solution of system (8.7.1) is asymptotically stable.

PROOF. According to the property of M matrix, we know that any of the above conditions implies that $B(b_{ij})_{n\times n}$ is an M matrix. Thus, the conditions of Theorem 8.7.6 hold. □

EXAMPLE 8.7.8. Consider the following system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}, \tag{8.7.8}$$

where $\tau = \text{constant} > 0$, and

$$A_0 = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} \end{bmatrix}, \qquad A_1 = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix}.$$

Assume that

(1) $a_{11}^{(0)} + a_{22}^{(0)} + a_{11}^{(1)} + a_{22}^{(1)} < 0$ and

$$\det\begin{bmatrix} a_{11}^{(0)} + a_{11}^{(1)} & a_{12}^{(0)} + a_{12}^{(1)} \\ a_{21}^{(0)} + a_{21}^{(1)} & a_{22}^{(0)} + a_{22}^{(1)} \end{bmatrix} > 0;$$

(2) $|a_{ii}^{(0)}| - |a_{ii}^{(1)}| > 0$, $i = 1, 2$, and

$$\begin{aligned} &\big(|a_{11}^{(0)}| - |a_{11}^{(1)}|\big)\big(|a_{22}^{(0)}| - |a_{22}^{(1)}|\big) \\ &\quad - \big(|a_{12}^{(0)}| + |a_{12}^{(1)}|\big)\big(|a_{21}^{(0)}| + |a_{21}^{(1)}|\big) > 0; \end{aligned}$$

then $(A_0, A_1) \in S$.

Since $(A_0, A_1) \in S$, if and only if

(1°) $\operatorname{Re}\lambda(A_0 + A_1) < 0$;
(2°) $\forall \tau > 0$, $\forall y \in R$,

$$\det \Delta := \begin{vmatrix} a_{11}^{(0)} + a_{11}^{(1)} e^{-iy\tau} - iy & a_{12}^{(0)} + a_{12}^{(1)} e^{-iy\tau} \\ a_{21}^{(0)} + a_{21}^{(1)} e^{-iy\tau} & a_{22}^{(0)} + a_{22}^{(1)} e^{-iy\tau} - iy \end{vmatrix} \neq 0.$$

Obviously (1°) and (1) are equivalent. Now we prove that (2) implies (2°).

In fact,

$$\begin{aligned} |\Delta| \geqslant{} &\big(a_{11}^{(0)} + a_{11}^{(1)} e^{-iy\tau} - iy\big)\big(a_{22}^{(0)} + a_{22}^{(1)} e^{-iy\tau} - iy\big) \\ &- \big(a_{12}^{(0)} + a_{12}^{(1)} e^{-iy\tau}\big)\big(a_{21}^{(0)} + a_{21}^{(1)} e^{-iy\tau}\big) \end{aligned}$$

$$\geqslant \left|a_{11}^{(0)} + a_{11}^{(1)} \cos \tau y\right|\left|a_{22}^{(0)} + a_{22}^{(1)} \cos \tau y\right| - \left|a_{12}^{(0)}\right| + \left|a_{12}^{(1)}\right|\left|a_{21}^{(0)}\right| + \left|a_{21}^{(1)}\right|$$
$$\geqslant \left(\left|a_{11}^{(0)}\right| - \left|a_{11}^{(1)}\right|\right)\left(\left|a_{22}^{(0)} - a_{22}^{(1)}\right|\right) - \left(\left|a_{12}^{(0)}\right| + \left|a_{12}^{(1)}\right|\right)\left(\left|a_{21}^{(0)}\right| + \left|a_{21}^{(1)}\right|\right)$$
$$> 0.$$

Thus, $\Delta \neq 0$ and so $(A_1, A_2) \in S$. This implies that the zero solution of (8.7.8) is asymptotically stable.

EXAMPLE 8.7.9. Consider the following scalar equation:

$$\frac{dx}{dt} = a_0 x(t) + \sum_{k=1}^{N} a_k x(t - \tau_k). \tag{8.7.9}$$

From Theorem 8.7.6 we know that if

$$|a_0| > \sum_{k=1}^{N} |a_k| \quad \text{and} \quad a < 0,$$

then $a \in S$, which indicates that the zero solution of (8.7.9) is asymptotically stable.

J. Hale [155] once raised the following question: Does an asymptotically stable cone have the property of convexity? That is, given $A \in S$ and $\hat{A} \in S$, does that $\alpha A + (1 - \alpha)\tilde{A}$ also belong to S for $0 \leqslant \alpha \leqslant 1$? The significance of this question is that if the answer is positive, then any combinations of two convex asymptotically stable cones yields a new asymptotically stable cone, and thus it can generate infinite number of asymptotically stable cones.

However, generally, the answer to this question is no, unless the system is a scalar system or the system has symmetric coefficients.

In the following, we give sufficient conditions for a system to be asymptotically stable cone.

THEOREM 8.7.10. *(See [280].) Assume system* (8.7.1) *satisfies the following conditions*:

(1) $\quad a_{ii}^{(0)} < 0, \quad i = 1, 2, \ldots, n;$

$(2)_1 \quad b_{ii} - \sum_{j=1, j\neq i}^{n} \frac{1}{2}|b_{ij} + b_{ji}| > 0, \quad i = 1, 2, \ldots, n,$ *or*

$(2)_2 \quad b_{ii} - \sqrt{\sum_{j=1, j\neq i}^{n} \frac{(b_{ii}b_{ij} + b_{jj}b_{ji})}{2}} > 0, \quad i = 1, 2, \ldots, n$ *or*

$(2)_3 \quad B(b_{ij})_{n\times n}$ *is a symmetric M matrix*;

where b_{ij}'s are defined in Theorem 8.7.6. Denote the system satisfying the conditions (1) and $(2)_q$ (($q = 1, 2, 3$) as S_q).

Then $S_q \subset S$ and S_q is convex.

PROOF. $\forall A, \hat{A} \in S_1$, $\forall \alpha \in [01]$, we have

$$\alpha A + (1-\alpha)\hat{A} = \left(\alpha \sum_{k=0}^{N} a_{ij}^{(k)} + (1-\alpha)\sum_{k=0}^{N} \hat{a}_{ij}^{(k)}\right).$$

Since $a_{ii}^{(0)} < 0$ and $\tilde{a}_{ii}^{(0)} < 0$, it holds

$$\alpha a_{ii}^{(0)} + (1-\alpha)\tilde{a}_{ii}^{(0)} < 0, \quad i = 1, 2, \dots, n,$$

$$\begin{aligned}\alpha b_{ii} + (1-\alpha)\tilde{b}_{ii} &> \alpha \frac{|b_{ij} + b_{ji}|}{2} + (1-\alpha)\frac{|\tilde{b}_{ij} + \tilde{b}_{ji}|}{2} \\ &\geqslant \frac{1}{2}\left|\alpha b_{ij} + (1-\alpha)\tilde{b}_{ij} + \alpha b_{ji} + (1-\alpha)\tilde{b}_{ji}\right|,\end{aligned}$$

i.e.,

$$\alpha b_{ii} + (1-\alpha)\tilde{b}_{ii} - \frac{1}{2}\left|\alpha b_{ij} + (1-\alpha)\tilde{b}_{ij} + \alpha b_{ji} + (1-\alpha)\tilde{b}_{ji}\right| > 0.$$

Moreover, $\alpha A + (1-\alpha)\tilde{A} \in S_1$, S_1 is convex. $S_1 \in S$ has been proved in Theorem 8.7.6 and Corollary 8.7.7.

Similarly, we can prove that $S_2 \in S$ is convex.

$S_3 \in S$ has been proved in Theorem 8.7.6 and Corollary 8.7.7. Now we prove that S_3 is convex. $\forall A, \hat{A} \in S_3$, $\forall \alpha \in [01]$, $a_{ii}^{(0)} < 0$, $\hat{a}_{ii}^{(0)} < 0$ we have

$$\alpha a_{ii}^{(0)} + (1-\alpha)a_{ii}^{(0)} < 0, \quad \alpha \in [01].$$

Because B and $\hat{B}$ are symmetric, so is $\alpha B + (1-\alpha)\hat{B}$. Moreover, it is an M matrix. So, $\alpha A + (1-\alpha)\hat{A} \in S_3$. Thus, S_3 is convex. □

8.8. A class of time delay neutral differential difference systems

Consider a general neutral differential difference system [139]

$$\frac{dx(t)}{dt} = f\big(t, x(t), x\big(t - \tau(t)\big), \dot{x}\big(t - \tau(t)\big)\big), \tag{8.8.1}$$

where $x \in R^n$, $f \in C[I \times R^n \times R^n \times R^n, R^n]$, $f(t, 0, 0, 0) = 0$, $0 \leqslant \tau(t) < +\infty$,

$$x\big(t - \tau(t)\big) = \big(x_1\big(t - \tau_1(t)\big), \dots, x_n\big(t - \tau_n(t)\big)\big),$$

and

$$\dot{x}\big(t-\tau(t)\big)=\big(\dot{x}_1\big(t-\tau_1(t)\big),\ldots,\dot{x}_n\big(t-\tau_n(t)\big)\big).$$

All results in Theorems 8.2.1 to 8.2.7, about delayed differential difference equation (8.2.1), can be extended to neutral systems. If we have method to determine the negative definiteness of $\frac{dV}{dt}$ about $x_1,\ldots,x_n$, then similar conclusions can be obtained. However, generally $\frac{dV}{dt}$ is a function of $3n$ variables, it is difficult to determine whether $\frac{dV}{dt}$ is negative definite or not.

First, we consider a special class of time-delay linear neutral differential difference equations. This class of systems was first developed from electrical networks.

In the following, we extend the results from the constant time-delay system (8.2.1) to varying time-delay systems.

Consider the following more general time-varying linear neutral system [139]:

$$\begin{aligned}\frac{dx_i(t)}{dt}&=\sum_{j=1}^{n}a_{ij}(t)x_j(t)+\sum_{j=1}^{n}b_{ij}(t)x_j\big(t-\tau_{ij}^{(1)}(t)\big)\\&\quad+\sum_{j=1}^{n}c_{ij}(t)\dot{x}_j\big(t-\tau_{ij}^{(2)}(t)\big),\end{aligned}\tag{8.8.2}$$

where

$$\dot{x}_j\big(t-\tau_{ij}^{(2)}(t)\big)=\frac{dx_j(t-\tau_{ij}^{(2)}(t))}{dt},\quad 0\leqslant\tau_{ij}^{(i)}(t)\leqslant\tau_{ij}^{(i)}=\text{constant},\tag{8.8.3}$$

$$x_i(s)=\varphi_i(s),\quad s\in[-\tau,0],\tag{8.8.4}$$

$$\frac{dx_i(s)}{dt}=\psi_i(s),\quad s\in[-\tau,0],\ \tau=\max_{1\leqslant i,j\leqslant n}\big(\tau_{ij}^{(1)},\tau_{ij}^{(2)}\big),\ i=1,2,\ldots,n.$$

Here, ψ_i $(i=1,2,\ldots,n)$ is almost piecewise continuous on $[-\tau,0]$, φ_i $(i=1,2,\ldots,n)$ is bounded and integrable on $[-\tau,0]$, satisfying

$$\frac{d\varphi_i(s)}{ds}=\psi_i(s),\quad i=1,2,\ldots,n.$$

THEOREM 8.8.1. *Assume that the varying coefficients in* (8.8.1) *satisfy*

(1) $a_{ij}(t),b_{ij}(t),c_{ij}(t)$ *are all continuous bounded functions on* $[-\tau,+\infty]$;
(2) *there exists a nonnegative constant c such that*

$$\begin{aligned}&\max_{i\leqslant j\leqslant n}c_j^*:=c<1,\quad \textit{where } c_j^*=\sup_{t\geqslant-\tau}\sum_{i=1}^{n}\big|c_{ij}(t)\big|,\\&\quad j=1,2,\ldots,n;\end{aligned}\tag{8.8.5}$$

(3) *there exist positive constants* α *and* β *such that*

$$a_{jj}(t) \leqslant -\alpha < 0$$

and

$$\left|a_{jj}(t)\right| - \sum_{\substack{i=1\\ i\neq j}}^{n}\left|a_{ij}(t)\right| - \sum_{i=1}^{n}\left|b_{ij}\left(t+\tau_{ij}^{(2)}(t)\right)\right| \geqslant \beta > 0,$$

$$t \geqslant 0; \tag{8.8.6}$$

(4) $\dot{\tau}_{ij}^{(i)} \leqslant 0,\ i, j = 1, 2, \ldots, n;$

then the zero solution of (8.8.2) *is asymptotically stable.*

PROOF. If $\dot{x}_i(t)$ has an infinite number of isolated zero points on $[0, +\infty]$, then for any $t > 0$, choose t_i^* such that $\dot{x}_i$ does not change its sign on (t_i^*, t); if t is a zero of $\dot{x}_i$, let $t_i^* = t$; if $\dot{x}_i$ has a finite number of isolated zero points on $(0, +\infty)$, let t_i^* be the last zero of $\dot{x}_i$; if $\dot{x}_i$ does not have zero points on $(0, +\infty)$, let $t_i^* = 0$.

Now, we construct the positive definite and radially unbounded Lyapunov functional:

$$\begin{aligned}
V(t) = {} & \sum_{i=1}^{n}\left[\left|x_i(t)\right| + \left|x_i\left(t_i^*\right)\right| - c_i^* \int_{t_i^*}^{t}\left|\frac{dx_i(t)}{ds}\right| ds\right] \\
& + \sum_{j=1}^{n}\int_{t-\tau_{ij}^{(1)}(t)}^{t}\left|b_{ij}\left(s+\tau_{ij}^{(1)}(s)\right)\right|\left|x_j(s)\right| ds \\
& + \sum_{j=1}^{n}\left[\int_{t-\tau_{ij}^{(2)}(t)}^{t}\left|c_{ij}\left(s+\tau_{ij}^{(2)}(s)\right)\right|\left|\dot{x}_j(s)\right| ds\right].
\end{aligned} \tag{8.8.7}$$

From the initial condition and the assumption of the coefficients we know that $V(0)$ is finite. Thus,

$$V(t) \geqslant \sum_{i=1}^{n}\left[\left|x_i(t)\right| - c_i^*\left|x_i(t)\right|\right] \geqslant (1-c)\sum_{i=1}^{n}\left|x_i(t)\right|, \quad t \geqslant 0. \tag{8.8.8}$$

Using $\dot{\tau}_{ij}^{(1)}(t) \leqslant 0$, we have

$$\begin{aligned}
D^+V|_{(8.8.2)} \leqslant {} & \sum_{i=1}^{n}\left[\left(\frac{dx_i}{dt}\right)\operatorname{sign} x_i - c_i^*\left|\dot{x}_i(t)\right|\right] \\
& + \sum_{j=1}^{n}\left\{\left|b_{ij}\left(t+\tau_{ij}^{(1)}(t)\right)\right|\left|x_j(t)\right| - \left|b_{ij}(t)\right|\left|x_j\left(t-\tau_{ij}^{(1)}(t)\right)\right|\right\}\left(1-\dot{\tau}_{ij}^{(1)}(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\left\{\left|c_{ij}\left(t+\tau_{ij}^{(2)}(t)\right)\right|\left|x_j(t)\right|-\left|c_{ij}(t)\right|\left|x_j\left(t-\tau_{ij}^{(2)}(t)\right)\right|\right\}\left(1-\dot{\tau}_{ij}^{(2)}(t)\right)\\
&\quad\leqslant\sum_{i=1}^{n}\Bigg[-\left|a_{ii}(t)\right|\left|x_i(t)\right|+\sum_{\substack{j=1\\ j\neq 1}}^{n}\left|a_{ij}(t)\right|\left|x_j(t)\right|-c_i^*\left|\dot{x}_i(t)\right|\\
&+\sum_{j=1}^{n}\left|b_{ij}\left(t+\tau_{ij}^{(1)}(t)\right)\right|\left|x_j(t)\right|+\sum_{j=1}^{n}\left|c_{ij}\left(t+\tau_{ij}^{(2)}(t)\right)\right|\left|\dot{x}_j(t)\right|\Bigg]\\
&\quad\leqslant\sum_{j=1}^{n}\Bigg[-\left|a_{jj}(t)\right|+\sum_{\substack{i=1\\ i\neq j}}^{n}\left|a_{ij}(t)\right|+\sum_{i=1}^{n}\left|b_{ij}\left(t+\tau_{ij}^{(1)}(t)\right)\right|\Bigg]\left|x_j(t)\right|\\
&+\sum_{j=1}^{n}\Bigg[\sum_{i=1}^{n}\left|c_{ij}\left(t+\tau_{ij}^{(2)}(t)\right)\right|-c_j^*\Bigg]\left|\dot{x}_j(t)\right|\\
&\quad\leqslant-\beta\sum_{j=1}^{n}\left|x_j(t)\right|+\sum_{j=1}^{n}\Bigg[\sum_{i=1}^{n}\left|c_{ij}\left(t+\tau_{ij}^{(2)}(t)\right)\right|-c_j^*\Bigg]\left|\dot{x}_j(t)\right|\\
&\quad\leqslant-\beta\sum_{j=1}^{n}\left|x_j(t)\right|.
\end{aligned}
\tag{8.8.9}
$$

It then follows from (8.8.9) that

$$
V(t)\leqslant V(0)-\beta\int_0^t\left(\sum_{i=1}^{n}\left|x_i(s)\right|\right)ds.
$$

Further, by using (8.8.8) we obtain

$$
(1-c)\sum_{i=1}^{n}\left|x_i(t)\right|+\beta\int_0^t\left(\sum_{i=1}^{n}\left|x_i(s)\right|\right)ds\leqslant V(0)<\infty,\quad t\geqslant 0,\tag{8.8.10}
$$

which means that $x_i(t)$ $(i=1,\ldots,n)$ is bounded.

From (8.8.2), we have

$$
\begin{aligned}
\left|\frac{dx}{dt}\right|&=\left|\sum_{i=1}^{n}\frac{dx_i}{dt}\right|\\
&\leqslant\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}(t)\right|\left|x_j(t)\right|+\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}(t)\right|\left|x_j\left(t_j\left(t-\tau_{ij}^{(1)}(t)\right)\right)\right|
\end{aligned}
$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\left|c_{ij}(t)\right|\left|\dot{x}_j\left(t_j\left(t-\tau_{ij}^{(2)}(t)\right)\right)\right|. \tag{8.8.11}$$

Define

$$m(t) := \sum_{i=1}^{n}\Big(\sup_{s\in[t-\tau,t]}\left|\dot{x}_i(s)\right|\Big). \tag{8.8.12}$$

By the assumption, we have

$$\sup_{t\geqslant-\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left|a_{ij}(t)\right| + \left|\sup_{t\geqslant-\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}(t)\right|\right] < \infty. \tag{8.8.13}$$

Then it follows from (8.8.10)–(8.8.12) that

$$\begin{aligned} m(t) &\leqslant \left\{\sup_{t\geqslant-\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left|a_{ij}(t)\right| + \left|\sup_{t\geqslant-\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}(t)\right|\right]\right\}\Big(\sup_{t\geqslant-\tau}\left|x_j(t)\right|\Big) \\ &\quad + \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{t\geqslant-\tau}\left|c_{ij}(t)\right|\right]\Big(\sup_{s\in[t-\tau,t]}\left|\dot{x}_j(s)\right|\Big) \\ &\leqslant \sigma + cm(t), \end{aligned} \tag{8.8.14}$$

where

$$\begin{aligned} \sigma = &\sup_{t\geqslant-\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left|a_{ij}(t)\right| + \left|\sup_{t\geqslant-\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}(t)\right|\right] \\ &\times\left(\sum_{j=1}^{n}\sup_{t\geqslant-\tau}\left|x_j(t)\right|\right) < \infty. \end{aligned} \tag{8.8.15}$$

Since $c < 1$, from (8.8.13) we know that $m(t) \leqslant \frac{\sigma}{1-c}$ for $t \geqslant 0$.

Therefore, the derivative of $x_i(t)$ is bounded on $[0, \infty]$, so $\sum_{i=1}^{n}|x_i(t)|$ is uniformly continuous on $[0, \infty]$, and $\sum_{i=1}^{n}|x_i(t)| \in L_1(0, \infty)$. Thus, from (8.8.15) we know that $\lim_{t\to\infty}\sum_{i=1}^{n}|x_i(t)| = 0$, implying that the zero solution of (8.8.2) is asymptotically stable. □

8.9. The method of iteration by parts for large-scale neural systems

The basic idea for analyzing large-scale systems is to decompose the large-scale system into a number of lower order isolated subsystems, and then determine the stability of the whole large system on the basis of stability degree of the subsystems as well as connection strength between the subsystems. The current mostly used methods are still the Lyapunov vector function (functional) method or the weight and scalar Lyapunov function (functional) method. The main difficulty in using these methods is that there do not exist general rules or skills in constructing such Lyapunov functions (functionals). So the results obtained are mainly for existence.

In this section, we introduce a comparison method for stability study of large-scale neutral systems. The main idea of this method is as follows: Based on the integral estimation of the subsystem's Cauchy matrix solution and couping matrix solution, for the solution of the whole system, we find estimations for each subsystem, and then construct explicit comparison system from which we determine the stability of the whole large system.

First we give a lemma as follows.

LEMMA 8.9.1. *Assume* $H(h_{ij}(t))_{r\times r} \in C[I, R^{r\times r}]$, $h_{ij}(t) \geqslant 0$, $i \neq j$, $i, j = 1, \dots, r$, $f(t) \in C[I, R^r]$. *Then the solution of the differential inequality*:

$$\begin{cases} \frac{dx(t)}{dt} \leqslant H(t)x(t) + f(t), \\ x(t_0) = x_0, \end{cases} \tag{8.9.1}$$

is right-up limited by the solution of the differential equation:

$$\begin{cases} \frac{dy(t)}{dt} = H(t)y(t) + f(t), \\ y(t_0) = y_0 \geqslant x_0, \end{cases} \tag{8.9.2}$$

i.e., $x(t) \leqslant y(t)$, $t \geqslant t_0$ *or* $x_i(t) \leqslant y_i(t)$, $i = 1, \dots, r$.

PROOF. Let

$$z(t) := y(t) - x(t) := y(t, t_0, y_0) - x(t, t_0, x_0) = z(t, t_0, z_0)$$

and

$$g(t) := \frac{dz(t)}{dt} - H(t)z \geqslant 0, \qquad D_H(t) := \operatorname{diag}\bigl(h_{11}(t), \dots, h_{\tau\tau}(t)\bigr),$$

then $z(t)$ satisfies

$$\begin{cases} \frac{dz(t)}{dt} = \bigl[D_H(t) + H(t) - D_H(t)\bigr]z(t) + g(t), \\ z(t_0) = z_0, \end{cases} \tag{8.9.3}$$

and

$$z(t) = z_0 e^{\int_{t_0}^t D_H(t)\,dt} + \int_{t_0}^t \Bigl[e^{\int_\tau^t D_H(\xi)\,d\xi}\bigl(H(\tau) - D_H(\tau)\bigr)z(\tau) + g(\tau)\Bigr]\,d\tau. \tag{8.9.4}$$

Choosing $z^{(0)}(t) = z^{(0)} e^{\int_{t_0}^{t} D_H(t)\,dt} \geqslant 0$, one may apply the Picard iteration method to obtain

$$z^{(m)}(t) = z^{(0)} e^{\int_{t_0}^{t} D_H(t)\,dt} + \int_{t_0}^{t} \Big[e^{\int_{\tau}^{t} D_H(\xi)\,d\xi} \big(H(\tau) - D_H(\tau) \big) z^{(m-1)}(\tau) + g(\tau) \Big]\,d\tau.$$

Since for any natural number m, $z^{(m)}(t) \geqslant 0$, and thus

$$\lim_{m\to\infty} z^{(m)}(t) = z(t) \geqslant 0, \quad \text{i.e.,} \quad x(t, t_0, x_0) \leqslant y(t, t_0, y_0). \qquad \square$$

Consider the following large-scale linear system described by the neutral differential difference equation [243]:

$$\frac{dx(t)}{dt} = \operatorname{diag}\big(A_{11}(t), \ldots, A_{11}(t)\big)x(t) + \big((1-\delta_{ij})A_{ij}(t)\big)x(t) + \big(B_{ij}(t)\big)x(t-\tau) + \big(C_{ij}(t)\big)\dot{x}(t-\tau), \tag{8.9.5}$$

where $C_{ij}(t)$ is $n_i \times n_i$ continuously differentiable matrix function on $[t_0, +\infty]$, $A_{ij}(t)$ and $B_{ij}(t)$ are $n_i \times n_i$, continuous matrix functions, and τ is a positive constant.

Assume that $\varphi(t)$ is a continuously differentiable matrix function on $[t_0-\tau, t_0]$, and $P_{ii}(t, t_0)$ is the Cauchy matrix solution of the subsystems:

$$\frac{dx_i}{dt} = A_{ii}(t)x_i, \quad x_i = \big(x_1^i, \ldots, x_{n_i}^i\big)^T, \quad i = 1, 2, \ldots, r. \tag{8.9.6}$$

Let

$$\begin{aligned} f_i(t) := {} & P_{ii}(t, t_0)\varphi_i(t_0) + \int_{t_0-\tau}^{t_0} P_{ii}(t, t_1+\tau) \sum_{j=1}^{r} B_{ij}(t_1+\tau)\varphi_j(t_1)\,dt_1 \\ & - P_{ii}(t, t_0) \sum_{j=1}^{r} C_{ij}(t_0)\varphi_j(t_0-\tau) \\ & + \int_{t_0-\tau}^{t_0} A_{ii}(t_1+\tau)P_{ii}(t, t_1+\tau) \sum_{j=1}^{r} C_{ij}(t_1+\tau)\varphi_j(t_1)\,dt_1 \\ & - \int_{t_0-\tau}^{t_0} P_{ii}(t, t_1+\tau) \sum_{j=1}^{r} \dot{C}_{ij}(t_1+\tau)\varphi_j(t_1)\,dt_1, \quad i = 1, 2, \ldots, r. \end{aligned}$$

THEOREM 8.9.2. *Suppose the following conditions are satisfied*:

(1) *there exist scalar function* $\alpha(t) \in C[t_0, +\infty]$ *and constants* $\overline{M}_i \geqslant 1$, $M_i \geqslant 1$ *such that*

$$\|f_i(t)\| \leqslant \overline{M}_i e^{-\int_{t_0}^t \alpha(\xi)\,d\xi}, \quad t \geqslant t_0,$$
$$\|P_{ii}(t, t_0)\| \leqslant M_i e^{-\int_{t_0}^t \alpha(\xi)\,d\xi}, \quad t \geqslant t_0;$$

(2) $\|C_{ij}(t)\|$ *is a monotone nonincreasing function of* t;

(3) *there exists scalar function* $\beta(t) \in C[t_0, +\infty]$ *such that the Cauchy matrix solution* $R(t, t_0)$ *of the following ordinary differential equations*:

$$\begin{aligned}\frac{d\xi_i}{dt} = \sum_{j=1}^{r} \Big[& M_i(1-\delta_{ij})\|A_{ij}(t)\| + M_i K \|B_{ij}(t+\tau)\| \\ & + M_i K \|A_{ii}(t+\tau)\| \|C_{ij}(t+\tau)\| \\ & + M_i K \|\dot{C}_{iJ}(t+\tau)\| + \frac{K}{\tau}\|C_{ij}(t+\tau)\|\Big]\xi_j, \quad i = 1, 2, \ldots, r,\end{aligned}$$

has the estimation

$$\|R(t, t_0)\| \leqslant N \exp\left(\int_{t_0}^{t} \beta(\xi)\,d\xi\right),$$

where

$$K := \sup_{t \geqslant t_0} \exp\left(\int_{t-\tau}^{t} \alpha(\xi)\,d\xi\right) < +\infty,$$

$N \geqslant 1$ *is a constant.*

Then the following five conditions:

(1) $\int_{t_0}^t [\alpha(\xi) - \beta(\xi)]\,d\xi \geqslant c(t_0) > -\infty$ $(t \geqslant t_0)$,

(2) $\int_{t_0}^t [\alpha(\xi) - \beta(\xi)]\,d\xi \geqslant c > -\infty$ $(t \geqslant t_0)$,

(3) $\int_{t_0}^{+\infty} [\alpha(\xi) - \beta(\xi)]\,d\xi = +\infty$,

(4) $\int_{t_0}^t [\alpha(\xi) - \beta(\xi)]\,d\xi = +\infty$ *when* $(t - t_0) \to +\infty$ (*uniformly about* t_0),

(5) $\int_{t_0}^t [\alpha(\xi) - \beta(\xi)]\,d\xi \geqslant \gamma(t - t_0)$ $(t \geqslant t_0, \gamma > 0$ *being a constant*)

imply that the trivial zero solution of (8.9.5) *is stable, uniformly stable, asymptotically stable, uniformly asymptotically stable, and exponentially stable, respectively.*

PROOF. Denote the solution of (8.9.5) as $x(t)$ which satisfies the initial condition $x(t) = \varphi(t)$, $\dot{x}(t) = \psi(t)$ and $t_0 - \tau \leqslant t \leqslant t_0$. By using the variation of constants formula for integration substitution and integrate by parts, we can show that

$$
\begin{aligned}
x_i(t) = P_{ii}(t,t_0)\varphi_i(t_0) &+ \int_{t_0-\tau}^{t_0} P_{ii}(t,t_1+\tau)\sum_{j=1}^{r} B_{ij}(t_1+\tau)\varphi_j(t_1)\,dt_1 \\
&- P_{ii}(t,t_0)\sum_{j=1}^{r} C_{ij}(t_0)\varphi_j(t_0-\tau) \\
&+ \int_{t_0-\tau}^{t_0} A_{ii}(t_1+\tau)P_{ii}(t,t_1+\tau)\sum_{j=1}^{r} C_{ij}(t_1+\tau)\varphi_j(t_1)\,dt_1 \\
&- \int_{t_0-\tau}^{t_0} P_{ii}(t,t_1+\tau)\sum_{j=1}^{r} \dot{C}_{ij}(t_1+\tau)\varphi_j(t_1)\,dt_1 \\
&+ \int_{t_0}^{t} P_{ii}(t,t_1)\sum_{j=1}^{r}(1-\delta_{ij})A_{ij}(t_1)x_j(t_1)\,dt_1 \\
&+ \int_{t_0}^{t-\tau} P_{ii}(t,t_1+\tau)\sum_{j=1}^{r} B_{ij}(t_1+\tau)x_j(t_1)\,dt_1 \\
&+ \int_{t_0}^{t-\tau} A_{ii}(t_1+\tau)P_{ii}(t,t_1+\tau)\sum_{j=1}^{r} C_{ij}(t_1+\tau)x_j(t_1)\,dt_1 \\
&- \int_{t_0}^{t-\tau} P_{ii}(t,t_1+\tau)\sum_{j=1}^{r} \dot{C}_{ij}(t_1+\tau)x_j(t_1)\,dt_1 \\
&+ \sum_{j=1}^{r} C_{ij}(t)x_j(t-\tau).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|x_i(t)\| &= \sum_{j=1}^{r} M_i \int_{t_0}^{t} (1-\delta_{ij})\|A_{ij}(t_1)\| e^{-\int_{t_1}^{t}\alpha(\xi)\,d\xi}\|x_j(t_1)\|\,dt_1 \\
&+ \sum_{j=1}^{r}\int_{t_0}^{t} M_i\|B_{ij}(t_1+\tau)\| e^{-\int_{t_1+\tau}^{t}\alpha(\xi)\,d\xi}\|x_j(t_1)\|\,dt_1
\end{aligned}
$$

$$+\sum_{j=1}^{r}\int_{t_0}^{t} M_i\|A_{ij}(t_1+\tau)\|e^{-\int_{t_1+\tau}^{t}\alpha(\xi)\,d\xi}\|C_{ij}(t_1+\tau)\|\|x_j(t_1)\|\,dt_1$$

$$+\sum_{j=1}^{r}\int_{t_0}^{t} M_i e^{-\int_{t_1+\tau}^{t}\alpha(\xi)\,d\xi}\|\dot{C}_{ij}(t_1+\tau)\|\|x_j(t_1)\|\,dt_1$$

$$+\sum_{j=1}^{r}\|C_{ij}(t)\|\|x_1(t-\tau)\|+\overline{M}_i e^{-\int_{t_0}^{t}\alpha(\xi)\,d\xi},$$

which, in turn, gives

$$\|x_i(t)\|e^{\int_{t_0}^{t}\alpha(\xi)\,d\xi}\leqslant\sum_{j=1}^{r}\int_{t_0}^{t}(1-\delta_{ij})M_i\|A_{ij}(t_1)\|e^{\int_{t_0}^{t_1}\alpha(\xi)\,d\xi}\|x_j(t_1)\|\,dt_1$$

$$+\sum_{j=1}^{r}\int_{t_0}^{t} M_i\|B_{ij}(t_1+\tau)\|e^{\int_{t_1}^{t_1+\tau}\alpha(\xi)\,d\xi}\|x_j(t_1)\|e^{\int_{t_0}^{t_1}\alpha(\xi)\,d\xi}\,dt_1$$

$$+\sum_{j=1}^{r}\int_{t_0}^{t} M_i\|A_{ij}(t_1+\tau)\|e^{\int_{t_1}^{t_1+\tau}\alpha(\xi)\,d\xi}$$

$$\times\|C_{ij}(t_1+\tau)\|\|x_j(t_1)\|e^{\int_{t_0}^{t_1}\alpha(\xi)\,d\xi}\,dt_1$$

$$+\sum_{j=1}^{r}\int_{t_0}^{t} M_i e^{\int_{t_1}^{t_1+\tau}\alpha(\xi)\,d\xi}\|\dot{C}_{ij}(t_1+\tau)\|\|x_j(t_1)\|e^{\int_{t_0}^{t_1}\alpha(\xi)\,d\xi}\,dt_1$$

$$+\sum_{j=1}^{r}\|C_{ij}(t)\|e^{\int_{t-\tau}^{t}\alpha(\xi)\,d\xi}\|x_i(t-\tau)\|e^{\int_{t_0}^{t-\tau}\alpha(\xi)\,d\xi}+\overline{M}_i.$$

Let

$$y_i(t)=\sum_{t_0-\tau\leqslant t_1\leqslant t}\|x_i(t_1)\|\exp\int_{t_0}^{t_1}\alpha(\xi)\,d\xi.$$

Then, using the properties of monotone nondecreasing of $y_i(t)$ and monotone nonincreasing of $\|C_{ij}(t)\|$, we obtain

$$\|x_i(t)\|e^{\int_{t_0}^{t}\alpha(\xi)\,d\xi}$$

$$\leqslant\overline{M}_i+\sum_{j=1}^{r}\int_{t_0}^{t}\big[M_i(1-\delta_{ij})\|A_{ij}(t_1)\|+M_iK\|B_{jj}(t_1+\tau)\|$$

$$
\begin{aligned}
&+ M_i K \big\| A_{ii}(t_1+\tau) \big\| \big\| C_{ij}(t_1+\tau) \big\| \\
&+ M_i K \big\| \dot{C}_{ij}(t_1+\tau) \big\| \big] y_i(t_1)\, dt_1 \\
&+ \frac{K}{\tau} \int\limits_{t-\tau}^{t} \sum_{j=1}^{r} \big\| \dot{C}_{ij}(t_1) \big\| y_1(t_1)\, dt_1 \\
\leqslant\ & M_i^* + \sum_{j=1}^{r} \int\limits_{t_0}^{t} \Big[(1-\delta_{ij}) \big\| A_{ij}(t_1) \big\| \\
&+ M_i K \big\| B_{ij}(t_1+\tau) \big\| + M_i K \big\| A_{ii}(t_1+\tau) \big\| \big\| C_{ij}(t_1+\tau) \big\| \\
&+ M_i K \big\| \dot{C}_{ij}(t_1+\tau) \big\| + \frac{K}{\tau} \big\| C_{ij}(t_1) \big\| \Big] y_i(t_1)\, dt_1,
\end{aligned}
\tag{8.9.7}
$$

where

$$
\begin{aligned}
M_i^* &:= \overline{M}_i + \frac{K}{\tau} \int\limits_{t_0-\tau}^{t_0} \sum_{j=1}^{r} \big\| C_{ij}(t_1) \big\| y_j^*(t_1)\, dt_1, \\
y_i^* &:= \max_{t_0-\tau<t_1<t<t_0} \big\| \varphi_j(t_1) \big\| e^{\int_{t_0}^{t_1} \alpha(\xi)\, d\xi}.
\end{aligned}
$$

Denote the right-hand side of (8.9.7) as $\eta_i(t)$. Since $\eta_i(t)$ is monotone nondecreasing, $y_i(t) \leqslant \eta_i(t)$ and $\eta_i(t)$ satisfies the following set of differential inequalities:

$$
\begin{cases}
\frac{d\eta_i}{dt} \leqslant \sum_{j=1}^{r} \big[M_i(1-\delta_{ij}) \|A_{ij}(t)\| + M_i K \|B_{ij}(t+\tau)\| \\
\qquad + M_i K \|A_{ii}(t+\tau)\| \|C_{ij}(t+\tau)\| \\
\qquad + M_i K \|C_{ij}(t+\tau)\| + \frac{K}{\tau} \|C_{ij}(t)\| \big] \eta_i, \\
\eta_i(t) = M_i^*, \quad i = 1, 2, \ldots, r.
\end{cases}
\tag{8.9.8}
$$

Assume that $\xi_i(t)$ satisfies the following set of differential equalities:

$$
\begin{cases}
\frac{d\xi_i}{dt} = \sum_{j=1}^{r} \big[M_i(1-\delta_{ij}) \|A_{ij}(t)\| + M_i K \|B_{ij}(t+\tau)\| \\
\qquad + M_i K \|A_{ii}(t+\tau)\| \|C_{ij}(t+\tau)\| \\
\qquad + M_i K \|\dot{C}_{ij}(t+\tau)\| + \frac{K}{\tau} \|C_{ij}(t)\| \big] \xi_i, \\
\xi_i(t_0) = M_i^*, \quad i = 1, 2, \ldots, r.
\end{cases}
\tag{8.9.9}
$$

Then, from Lemma 8.9.1 we know that $\eta_i(t) \leqslant \xi_i(t),\ i = 1, 2, \ldots, r$. Further, by condition (3) we have

$$
\big\| \big(\|x_1(t)\|, \ldots, \|x_r(t)\| \big)^T \big\| e^{\int_{t_0}^{t} \alpha(\xi)\, d\xi} \leqslant N \big(M_1^*, \ldots, M_r^* \big)^T e^{\int_{t_0}^{t} \beta(\xi)\, d\xi},
$$

and finally we obtain

$$\begin{aligned}&\left\|\left(\|x_1(t)\|,\ldots,\|x_r(t)\|\right)^T\right\|\\&\quad\leqslant N\left(M_1^*,\ldots,M_r^*\right)^T eY-\int_{t_0}^{t}\left[\alpha(\xi)-\beta(\xi)\right]d\xi,\end{aligned}\tag{8.9.10}$$

which implies that the conclusion of the theorem is true.

The proof is complete. □

8.10. Stability of large-scale neutral systems on C^1 space

Consider a class of large-scale nonlinear neutral system with varying time delays:

$$\begin{cases}\frac{dx}{dt}=\operatorname{diag}(A_{11},\ldots,A_{rr})x(t)\\ \qquad\quad+F\big(\big(t,x\big(t-\Delta(t)\big)\big),\dot{x}\big(t-\Delta(t)\big)\big),\\ x(t)=\Phi,\quad \dot{x}(t)=\dot{\Phi}(t),\quad \text{for } t_0-\Delta\leqslant t\leqslant t_0,\end{cases}\tag{8.10.1}$$

and the isolated subsystems:

$$\frac{dx}{dt}=\operatorname{diag}(A_{11},\ldots,A_{rr})x(t),\tag{8.10.2}$$

where

$$F\in C\left[I\times R^n\times R^n,R^n\right],\quad F(t,0,0)\equiv 0,\quad x\in R^n.$$

Let $P(t,t_0)=\operatorname{diag}(P_{11}(t,t_0),\ldots,P_{rr}(t,t_0))x(t)$ be a Cauchy matrix solution of (8.10.2).

If the trivial solution of (8.10.1) is asymptotically stable about x and $\dot{x}$, then it is said to be asymptotically stable on C^1.

THEOREM 8.10.1. *If*

(1) *in an open neighborhood of D enclosing the origin, $F(t,x,y)$ satisfies*

$$\left\|F_i(t,x,y)\right\|\leqslant\sum_{j=1}^{r}g_{ij}(t)\left[\|x_j\|+\|y_j\|\right],\tag{8.10.3}$$

where $g_{ij}(t)\in C[t_0,+\infty]$ is a nonnegative bounded function, and

$$x_i\in R^n,\quad y_i\in R^n,\quad \sum_{j=1}^{r}n_i=n;$$

(2) *the Cauchy matrix solution of* (8.10.2), $P(t, t_0)$, *has the estimation*:

$$\|P_{11}(t, t_0)\| \leqslant M_i e^{-\int_{t_0}^{t} \alpha_i(\xi)\, d\xi} \to 0 \quad \text{as } t \to +\infty,$$

where $M_i > 0$ *is a known constant, and* $\alpha_i(t) \in C[t_0, +\infty]$ $(i = 1, 2, \ldots, r)$ *are given functions*;

(3) *the spectral radius of matrix* $\Omega(\sigma_{ij})$, $\rho(\Omega) < 1$ (*particularly* $\|\Omega\| < 1$), *where*

$$\begin{aligned}
\sigma_{ii} &= \left(1 - K\tilde{g}_{ii}\right)^{-1} \int_0^{+\infty} m_i K_i g_{ii}(t_1)\, dt_1 + \delta_{ij}\left(1 - k\tilde{g}_{ii}\right)^{-1} K_i \tilde{g}_{ij}, \\
K_i &= \sup_{t \geqslant t_0} e^{\int_{t-\Delta(t)}^{t} \alpha_i(\xi)\, d\xi}, \\
m_i &= \left(1 + \tilde{A}_{ii}\right) M_i, \qquad \tilde{A}_{ii} = \sup_{t \geqslant t_0} \|A_{ii}(t)\|, \\
\tilde{g}_{ij} &= \sup_{t \geqslant t_0} g_{ij}(t), \qquad K_i \tilde{g}_{ij} < 1;
\end{aligned}$$

then the trivial solution of (8.10.1) *is asymptotically stable on* C^1.

PROOF. Let the solution of (8.10.1), $x(t)$, be expressed as

$$\begin{aligned}
x_i(t) &= P_{ii}(t, t_0)\Phi_i(t_0) \\
&\quad + \int_{t_0}^{t} P_{ii}(t, t_1)\Big(F_i\big(t_1, x\big(t_1 - \Delta(t_1)\big)\big), \dot{x}\big(t_1 - \Delta(t_1)\big)\Big)\, dt_1, \qquad (8.10.4) \\
\dot{x}_i(t) &= A_{ii} P_{ii}(t, t_0)\Phi_i(t_0) \\
&\quad + \int_{t_0}^{t} A_{ii} P_{ii}(t, t_1)\Big(F_i\big(t_1, x\big(t_1 - \Delta(t_1)\big)\big), \dot{x}\big(t_1 - \Delta(t_1)\big)\Big)\, dt_1 \\
&\quad + \Big(F_i\big(t, x\big(t - \Delta(t)\big)\big), \dot{x}\big(t - \Delta(t)\big)\Big), \qquad (8.10.5) \\
\|x_i(t)\| &\leqslant M_i \|\Phi_i(t_0)\| e^{-\int_{t_0}^{t} \alpha_i(\xi)\, d\xi} \\
&\quad + \int_{t_0}^{t} M_i e^{-\int_{t_1}^{t} \alpha_i(\xi)\, d\xi} \sum_{j=1}^{r} g_{ij}(t_1)\big[\|x_j\big(t_1 - \Delta(t_1)\big)\| \\
&\quad + \|\dot{x}_j\big(t_1 - \Delta(t_1)\big)\|\big]\, dt_1, \qquad (8.10.6) \\
\|x_i(t)\| &\leqslant \|A_{ii}(t)\| M_i \|\Phi_i(t_0)\| e^{-\int_{t_0}^{t} \alpha_i(\xi)\, d\xi}
\end{aligned}$$

$$+\int_{t_0}^{t}\|A_{ii}(t)\|M_i e^{-\int_{t_1}^{t}\alpha_i(\xi)\,d\xi}\sum_{j=1}^{r}g_{ij}(t_1)\big[\big\|x_j\big(t_1-\Delta(t_1)\big)\big\|$$

$$+\big\|\dot{x}_i\big(t_1-\Delta(t_1)\big)\big\|\big]\,dt_1$$

$$+\sum_{j=1}^{r}g_{ij}(t)\big[\big\|x_j\big(t-\Delta(t)\big)\big\|+\big\|\dot{x}_j\big(t-\Delta(t)\big)\big\|\big]. \tag{8.10.7}$$

Let

$$\tilde{A}_{ii}=\sup_{t\geqslant t_0}\|A_{ii}\|,\qquad K_i=\sup_{t\geqslant t_0}e^{\int_{t-\Delta(t)}^{t}\alpha_i(\xi)\,d\xi},$$

$$m_i=\big(1+\tilde{A}_{ii}\big)M_i\quad\text{and}\quad\tilde{g}_{ij}=\sup_{t\geqslant t_0}g_{ij}(t).$$

Then, it follows from (8.10.6) and (8.10.7) that

$$\big(\|x_i(t)\|+\|\dot{x}_i(t)\|\big)e^{\int_{t_0}^{t}\alpha_i(\xi)\,d\xi}$$

$$\leqslant m_i\|\Phi_i(t_0)\|+\int_{t_0}^{t}M_i e^{\int_{t_1-\Delta(t_1)}\alpha_i(\xi)\,d\xi}\sum_{j=1}^{r}g_{ij}(t_1)\big[\big\|x_j\big(t_1-\Delta(t_1)\big)\big\|$$

$$+\big\|\dot{x}_j\big(t_1-\Delta(t_1)\big)\big\|\big]e^{\int_{t_0}^{t-\Delta(t)}\alpha_i(\xi)\,d\xi}+e^{\int_{t_0-\Delta(t)}^{t}\alpha_i(\xi)\,d\xi}$$

$$\times\sum_{j=1}^{n}g_{ij}(t)\big[\big\|x_j\big(t-\Delta(t)\big)\big\|\big\|\dot{x}_j\big(t-\Delta(t)\big)\big\|\big]e^{\int_{t_0}^{t-\Delta(t)}\alpha_i(\xi)\,d\xi}$$

$$\leqslant m_i\|\Phi_i(t_0)\|+\int_{t_0}^{t}K_i m_i\Bigg[\sum_{j=1}^{r}g_{ij}(t_1)\big\|x_j\big(t_1-\Delta(t_1)\big)\big\|$$

$$+K_i\sum_{j=1}^{r}g_{ij}(t)\big[\big\|x_j\big(t-\Delta(t)\big)\big\|$$

$$+\big\|\dot{x}_i\big(t-\Delta(t)\big)\big\|\big]e^{\int_{t_0}^{t-\Delta t}\alpha_i(\xi)\,d\xi}\Bigg]\,dt. \tag{8.10.8}$$

Let

$$u_i(t)=\sup_{t_0-\Delta\leqslant t_1\leqslant t}\big\{\|x_i(t_1)\|+\|\dot{x}_i(t_1)\|\big\}e^{\int_{t_0}^{t}\alpha_i(\xi)\,d\xi},$$

where $x_i(t)=\Phi_i(t)$, $\dot{x}_i(t)=\dot{\Phi}_i(t)$, $t_0-\Delta\leqslant t_1\leqslant t_0$. Then, we have

$$
\begin{aligned}
&\big(\|x_i(t)\| + \|\dot{x}_i(t)\|\big)e^{\int_{t_0}^t \alpha_i(\xi)\,d\xi} \\
&\quad \leqslant m_i \|\Phi_i(t_0)\| + \int_{t_0}^{t} m_i K_i \sum_{j=1}^{r} g_{ij}(t_1)u_j(t_1)\,dt_1 \\
&\qquad + K \sum_{j=1}^{r} \tilde{g}_{ij} u_j(t).
\end{aligned} \tag{8.10.9}
$$

Since the right-hand side of (8.10.9) is a monotone nondecreasing function of t, we obtain

$$
\begin{aligned}
u_i(t) \leqslant{}& m_i \|\Phi_i(t_0)\| + \int_{t_0}^{t} m_i k_i \sum_{j=1}^{r} g_{ij}(t_1)u_j(t_1)\,dt_1 \\
&+ K \sum_{j=1}^{r} \tilde{g}_{ij} u_j(t).
\end{aligned} \tag{8.10.10}
$$

Further, because of the right-hand side of (8.10.10) being nonnegative, we have

$$
\begin{aligned}
u_i(t) \leqslant{}& m_i \|\Phi_i(t_0)\| \big(1 - K_i \tilde{g}_{ii}\big)^{-1} \\
&+ \big(1 - K_i \tilde{g}_{ii}\big)^{-1} \int_{t_0}^{t} m_i K_i \sum_{j=1}^{r} g_{ij}(t_1)u_j(t_1)\,dt_1 \\
&+ \big(1 - K_i \tilde{g}_{ii}\big)^{-1} K_i \sum_{\substack{j=1 \\ j\neq i}}^{r} \tilde{g}_{ij} u_j(t).
\end{aligned} \tag{8.10.11}
$$

Consider the following integral equation:

$$
\begin{aligned}
\tilde{u}_i(t) ={}& m_i \|\Phi_i(t_0)\| \big(1 - K_i \tilde{g}_{ii}\big)^{-1} \\
&+ \big(1 - K_i \tilde{g}_{ii}\big)^{-1} \int_{t_0}^{t} m_i K_i \sum_{j=1}^{r} g_{ij}(t_1)\tilde{u}_j(t_1)\,dt_1 \\
&+ \big(1 - K_i \tilde{g}_{ii}\big)^{-1} K_i \sum_{\substack{j=1 \\ j\neq i}}^{r} \tilde{g}_{ij} \tilde{u}_j(t) \\
:={}& L(\tilde{u}).
\end{aligned} \tag{8.10.12}
$$

We prove that on any finite interval $[t_0, T]$, the solutions of the integral inequality system (8.10.11) and the integral equation (8.10.12), with the same initial value, satisfy

$$u_i(t) \leqslant \tilde{u}_i(t).$$

Due to the condition in the theorem, $\rho(\Omega) < 1$, integral operator $L(u)$ is essentially a contraction operator, and thus the following iteration

$$\begin{aligned}
\tilde{u}_i^{(0)}(t) &= m_i \left\|\Phi_i(t_0)\right\| \left(1 - K_i \tilde{g}_{ii}\right)^{-1}, \\
\tilde{u}_i^{(m)}(t) &= m_i \left\|\Phi_i(t_0)\right\| \left(1 - K_i \tilde{g}_{ii}\right)^{-1} \\
&\quad + (1 - K_i \tilde{g}_{ii})^{-1} \int_{t_0}^{t} m_i K_i \sum_{j=1}^{r} g_{ij}(t_1) \tilde{u}_j^{(m-1)}(t_1)\, dt_1 \\
&\quad + \left(1 - K_i \tilde{g}_{ii}\right)^{-1} K_i \sum_{\substack{j=1 \\ j\neq i}}^{r} \tilde{g}_{ij} \tilde{u}_j^{(m-1)}(t), \quad m = 1, 2, \ldots, \qquad (8.10.13)
\end{aligned}$$

converges to some limit function $u_i^*(t)$ $(i = 1, 2, \ldots, r)$ in any finite interval $[t_0, T]$. This leads to the following estimation:

$$\begin{aligned}
\left(\tilde{u}_1^{(m)}(t), \ldots, \tilde{u}_r^{(m)}(t)\right)^T &\leqslant \sum_{m=0}^{\infty} \Omega^m(\sigma_{ij}) \left(\tilde{u}_1^{(0)}(t), \ldots, \tilde{u}_r^{(0)}(t)\right)^T \\
&= (E - \Omega)^{-1} \left(\tilde{u}_1^{(0)}(t), \ldots, \tilde{u}_r^{(0)}(t)\right)^T,
\end{aligned}$$

implying that

$$\left(u_1^*(t), \ldots, u_r^*(t)\right)^T \leqslant (E - \Omega)^{-1} \left(\tilde{u}_1^{(0)}(t), \ldots, \tilde{u}_r^{(0)}(t)\right)^T. \qquad (8.10.14)$$

On the interval $[t_0, T]$, in principle, one can repeatedly employ varying step sizes to transform the neutral system (8.10.11) into several corresponding ordinary differential systems and then to solve the IVP (initial value problem) of these differential systems. The ordinary differential equations deduced from the conditions of the theorem will take some linear ordinary differential equations as their control equations. Therefore, they have lower limit on $[t_0, T]$, and thus we may take

$$\sup_{t_o \leqslant t \leqslant T} \left\{u_i(t) - u_i^{(0)}(t)\right\} := N_i < +\infty.$$

It follows from (8.10.11) and (8.10.13) that

$$u_i(t) - \tilde{u}_i^{(1)} \leqslant \left(1 - K_i \tilde{g}_{ii}\right)^{-1} \int_{t_0}^{t} m_i K_i \sum_{j=1}^{r} g_{ij}(t_1) u_j\left(t_1 - u_j^{(0)}(t_1)\right) dt_1$$

$$
\begin{aligned}
&+ \left(1 - K_i \tilde{g}_{ii}\right)^{-1} K_i \sum_{\substack{j=1 \\ j \neq i}}^{r} \tilde{g}_{ij} \left[u_j(t) - u_j^{(0)}(t)\right] \\
&\leqslant \left(1 - K_i \tilde{g}_{ii}\right)^{-1} \int_{t_0}^{t} m_i K_i \sum_{j=1}^{r} g_{ij}(t_1) N_j \, dt_1 \\
&\quad + \left(1 - K_i \tilde{g}_{ii}\right)^{-1} K_i \sum_{\substack{j=1 \\ j \neq i}}^{r} \tilde{g}_{ij} N_i \\
&\leqslant \sum_{j=1}^{r} \sigma_{ij} N_j. \qquad (8.10.15)
\end{aligned}
$$

So, $(u_1(t) - \tilde{u}_1^{(1)}(t), \dots, u_r(t) - \tilde{u}_r^{(1)}(t))^T \leqslant \Omega(N_1, N_2, \dots, N_r)^T$. Thus, we can apply the method of mathematical induction to show that

$$\left(u_1(t) - \tilde{u}_1^{(m)}(t), \dots, u_r(t) - \tilde{u}_r^{(m)}(t)\right)^T \leqslant \Omega^{(m)}(N_1, N_2, \dots, N_r)^T.$$

Since $\rho(\Omega) < 1$, we have $\lim_{n \to \infty} \Omega^n = 0$. When $[t_0, T]$ is determined, N_i $(i = 1, 2, \dots, r)$ are fixed. Thus,

$$\left(u_1(t) - \tilde{u}_1^{*}(t), \dots, u_r(t) - \tilde{u}_r^{*}(t)\right)^T = 0. \qquad (8.10.16)$$

Since $[t_0, T]$ is an arbitrarily finite interval, (8.10.16) holds on $[t_0, \infty]$. This indicates that

$$
\begin{aligned}
\left(u_1(t), u_2(t), \dots, u_r(t)\right)^T &\leqslant \left(u_1^*(t), u_2^*(t), \dots, u_r^*(t)\right)^T \\
&\leqslant (E - \Omega)^{-1} \left(u_1^{(0)}(t), u_2^{(0)}(t), \dots, u_r^{(0)}(t)\right)^T,
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
&\left(\|x_1(t)\| + \|\dot{x}_1(t)\|, \dots, \|x_r(t)\| + \|\dot{x}_r(t)\|\right)^T \\
&\quad \leqslant (E - \Omega)^{-1} \operatorname{col}\left(u_1^{(0)}(t) e^{\int_{t_0}^{t} \alpha_1(\xi)\, d\xi}, \dots, u_r^{(0)}(t) e^{-\int_{t_0}^{t} \alpha_r(\xi)\, d\xi}\right). \qquad (8.10.17)
\end{aligned}
$$

Obviously, the last inequality (8.10.17) implies that the trivial solution of (8.10.11) is asymptotically stable on C^1. □

8.11. Algebraic methods for GLNS with constant coefficients

Consider the following general linear neutral system (GLNS) with constant coefficients:

$$\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} B_k x(t - \gamma_k, r)\right] = A_0 x_t + \sum_{k=1}^{N} A_k x(t - \gamma_k r), \qquad (8.11.1)$$

where B_k is an $n \times n$ real matrix, while all the other notations are the same as that defined in (8.7.1).

To study the stability of system (8.11.1), we first consider difference system

$$x(t) - \sum_{k=1}^{N} B_k x(t - \gamma_k r) = 0. \tag{8.11.2}$$

The characteristic equation of (8.11.2) is

$$e(\lambda, r, B) := \det\left[I - \sum_{k=1}^{N} B_k e^{-\lambda \gamma_k r} \right] = 0.$$

DEFINITION 8.11.1. (See [155].) System (8.11.2) is said to be uniformly asymptotically stable about (r, B) if

$$\{\operatorname{Re}\lambda\colon\ e(\lambda, r, B) = 0\} \cap [-\delta, +\infty] = \emptyset, \tag{8.11.3}$$

where $\delta > 0$ is a constant. Equation (8.11.3) means that the eigenvalue of (8.11.2) $\operatorname{Re}\lambda \leqslant -\delta < 0$.

DEFINITION 8.11.2. (See [155].) System (8.11.2) is said to be locally asymptotically stable about (r^0, B), if there exists a neighborhood of r^0, $U(r^0)$, such that $\forall r \in U(r^0)$, system (8.11.2) is asymptotically stable about B. If for every $r \in (R^+)^M$, (8.11.2) is asymptotically stable about (r, B), then system (8.11.2) is said to be globally asymptotically stable about B.

It should be noted that the above definitions of asymptotic stability and global asymptotic stability are different from those of Lyapunov stabilities.

Now we discuss how to determine these stabilities. Let

$$\eta_{ij} := \begin{cases} -\sum_{k=1}^{N} |b_{ii}^{(k)}|, & i = j,\ i, j = 1, 2, \ldots, n, \\ -\sum_{k=1}^{N} |b_{ij}^{(k)}|, & i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases} \tag{8.11.4}$$

THEOREM 8.11.3. *If $(\eta_{ij})_{n\times n}$ is an M matrix, then (8.11.2) is globally asymptotically stable.*

PROOF. $\forall \mu(\theta)$, $|\mu(\theta)| \geqslant 1$, we obtain

$$\left| \mu(\theta) - \sum_{k=1}^{N} b_{ii} e^{i\gamma_k \theta} \right| \geqslant |\mu(\theta)| - \sum_{k=1}^{N} |b_{ii}^{(k)}| \geqslant 1 - \sum_{k=1}^{N} |b_{ii}^{(k)}| > 0.$$

According to Lemmas 8.7.4 and 8.7.5 we have

$$\det\left[\mu(\theta)I - \sum_{k=1}^{N} B_k e^{i\gamma_k\theta}\right] \neq 0,$$

which shows that the difference system (8.11.2) is globally asymptotically stable about B. □

COROLLARY 8.11.4. *If*

$$\sum_{j=1}^{n}\sum_{k=1}^{N}\left|b_{ij}^{(k)}\right| < 1, \quad i = 1, 2, \ldots, n,$$

or

$$\sum_{i=1}^{n}\sum_{k=1}^{N}\left|b_{ij}^{(k)}\right| < 1, \quad j = 1, 2, \ldots, n,$$

then system (8.11.2) *is globally asymptotically stable about* B.

In the following, when we discuss the stability of the neutral system (8.11.1), we always assume that (8.11.2) is uniformly asymptotically stable.

Now, we study the stability of the neutral system (8.11.1). The characteristic equation of (8.11.1) is

$$\begin{aligned} g(\lambda, r, A, B) := \det\Bigg[\lambda\left(I - \sum_{k=1}^{N} B_k e^{-\lambda\gamma_k r}\right) - A_0 \\ - \sum_{k=1}^{N} A_k e^{-\lambda\gamma_k r}\Bigg] = 0, \end{aligned} \tag{8.11.5}$$

where $A = (A_0, A_1, \ldots, A_N)$ and $B = (B_1, \ldots, B_N)$.

DEFINITION 8.11.5. (See [155].) System (8.11.1) is said to be uniformly asymptotically stable about (r, A, B), if

$$\left\{\operatorname{Re}\lambda\colon\ g(\lambda, r, A, B) = 0\right\} \cap [-\delta, +\infty] = \emptyset. \tag{8.11.6}$$

DEFINITION 8.11.6. (See [155].) For a given $r^0 \in (R^+)^M$,

$$\begin{aligned} S_{r^0} = \Big\{(A, B) \in R^{n^2(N+1)} \times R^{n^2(N+1)},\ \text{system (8.11.1) is} \\ \text{asymptotically stable about } (r, A, B) \text{ for every } r = \alpha r^0, \alpha \geqslant 0\Big\} \end{aligned}$$

is called asymptotically stable cone about r^0.

Let

$$S = \bigcap_r S_r \tag{8.11.7}$$

be an asymptotically stable cone.

LEMMA 8.11.7. *(See [155].) For system (8.11.1), the necessary and sufficient conditions for $(A, B) \in S_r$ are*

(1)

$$\operatorname{Re}\left[\left(I - \sum_{k=1}^{N} B_k\right)^{-1} \sum_{k=0}^{N} A_k\right] < 0; \tag{8.11.8}$$

(2)

$$g(iy, \alpha, r, A, B) \neq 0 \quad \textit{for all } y \in R,\ y \neq 0,\ \alpha \geqslant 0. \tag{8.11.9}$$

Similar to Lemma 8.7.1, the idea of Lemma 8.11.7 is simple and the proof is omitted.

Let

$$\sigma_{ij} := \begin{cases} \left|a_{ij}^0 + iy\right| - \sum_{k=1}^{N} \left|b_{ii}^{(k)} iy\right| - \sum_{k=1}^{N} \left|a_{ii}^{(k)}\right|, \\ \quad i = j = 1, \ldots, n, \\ -\sum_{k=1}^{N} \left|iy b_{ij}^{(k)}\right| - \sum_{k=1}^{N} \left|a_{ij}^{(k)}\right|, \\ \quad i \neq j,\ i, j = 1, \ldots, n, \end{cases} \tag{8.11.10}$$

$$\eta(\eta_{ij}) := \left[\left(I - \sum_{k=1}^{N} B_k\right)^{-1} \sum_{k=1}^{N} A_k\right], \tag{8.11.11}$$

$$\tilde{\eta}_{ii} := \begin{cases} -\delta_{ij}, & i = j = 1, 2, \ldots, n, \\ -|\delta_{ij}|, & i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases} \tag{8.11.12}$$

THEOREM 8.11.8. *(See [280].) If for all $y \in R$, $\alpha \geqslant 0$, and $\sigma(\sigma_{ij})_{n\times n}$ is an M matrix, in addition, $(\tilde{\eta}_{ij})_{n\times n}$ is also an M matrix, then $(A, B) \in S_r$, i.e., system (8.11.1) is globally asymptotically stable.*

PROOF. According to the condition of the theorem, and Lemmas 8.7.4 and 8.11.7, it is easy to prove that

$$g(iy, \alpha\gamma, A, B) := \det\left[iy\left(I - \sum_{k=1}^{N} B_k e^{-iy_k\alpha\gamma}\right) - A_0\right.$$

$$- \sum_{k=1}^{N} A_k e^{-iY_{r_k}\alpha\gamma} \Bigg] \neq 0$$

for all $\alpha \geqslant 0$ and $y \in R$. Thus condition (2) in Lemma 8.11.1 is satisfied.

Since $\tilde{\eta}(\tilde{\eta}_{ij})_{n\times n}$ is an M matrix, $\eta(\eta_{ij})_{n\times n}$ is stable. Therefore,

$$\operatorname{Re}\lambda\left(\left(I - \sum_{k=1}^{N} B_k\right)^{-1} \sum_{k=1}^{N} A_k\right) < 0, \tag{8.11.13}$$

which indicates that Condition (1) in Lemma 8.11.7 holds, so $(A, B) \in S_r$. □

COROLLARY 8.11.9. *If any of the following conditions is satisfied*:

(1)

$$\left|-a_{ii}^{(0)} - iy\right| > \sum_{j=1}^{n}\sum_{k=1}^{N}\left|b_{ij}^{(k)} iy\right| + \sum_{k=1}^{n}\sum_{k=1}^{N}\left|a_{ij}^{(k)}\right| + \sum_{j=1, j\neq 1}^{n}\left|a_{ij}^{(0)}\right|;$$

(2)

$$-\eta_{ii} > \sum_{\substack{j=1\\ j\neq i}}^{n}\left|\eta_{ij}^{(0)}\right|, \quad i = 1, 2, \ldots, n;$$

then $(A, B) \in S_r$, *so system* (8.11.1) *is globally asymptotically stable.*

PROOF. It is easy to verify that the conditions in Lemma 8.11.7 are satisfied. □

COROLLARY 8.11.10. *If any of the following conditions is satisfied*:

(1) *both*

$$\frac{1}{\sqrt{2}}\left|a_{ii}^{(0)}\right| > \sum_{j=1}^{n}\sum_{k=1}^{N}\left|a_{ij}^{(k)}\right| + \sum_{\substack{j=1\\ j\neq i}}^{n}\left|a_{ij}^{(0)}\right| \quad \text{and} \quad \frac{1}{\sqrt{2}} > \sum_{j=1}^{n}\sum_{k=1}^{N}\left|a_{ij}^{(k)}\right|$$

hold;

(2) *condition* (2) *in Corollary* 8.11.9 *is satisfied*;

then $(A, B) \in S_r$, *so system* (8.11.1) *is globally stable.*

PROOF. Since

$$\left|-a_{ii}^{(0)} + iy\right| = \sqrt{\left(a_{ii}^{(0)}\right)^2 + y^2} \geqslant \frac{|a_{ii}^{(0)}|}{\sqrt{2}} + \frac{|iy|}{\sqrt{2}}$$

$$> \sum_{j=1}^{n} \sum_{k=1}^{N} \left| b_{ij}^{(k)} iy \right| + \sum_{j=1}^{n} \sum_{k=1}^{N} \left| a_{ij}^{(k)} \right| + \sum_{j=1, j \neq i}^{n} \left| a_{ij}^{(0)} \right|, \tag{8.11.14}$$

all the conditions in Corollary 8.11.9 are satisfied. Thus, the conclusion of Corollary 8.11.10 is true. □

Let

$$\begin{aligned}
(\xi_{ij})_{n \times n} &:= -\left(I - \sum_{k=1}^{N} B_k s_1^{\gamma k_1}, \ldots, s_m^{\gamma k_m} \right)^{-1} \\
&\quad \times \left(A_0 + \sum_{k=1}^{N} A_k s_1^{\gamma k_1}, \ldots, s_m^{\gamma k_m} \right), \\
\tilde{\xi}_{ij} &= \begin{cases} |\xi_{ij}|, & i = j = 1, 2, \ldots, n, \\ -|\xi_{ij}|, & i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases}
\end{aligned}$$

Similarly, we can prove the following

THEOREM 8.11.11. *(See [280].) If system (8.11.11) satisfies that*

(1) *all the conditions in Theorem 8.11.3 hold*;
(2) $(\tilde{\xi}_{ij})_{n \times n}$ *is an M matrix*;
(3) $(\tilde{\eta}_{ij})_{n \times n}$ *is an M matrix*;

then $(A, B) \in S$*, and so system (8.11.1) is globally stable.*

J. Hale [155] once raised a question: Is the asymptotically stable cone of the neutral system (8.11.1) convex? In general, the answer is no. Here is a counter example.

EXAMPLE 8.11.12. Consider the following scalar function:

$$\frac{d}{dt}\big(x(t) + Bx(t - \tau)\big) = A_0 x(t) + A_1 x(t - \tau),$$
$$N = 1,\ n = 2,\ x \in R^2. \tag{8.11.15}$$

Take

$$B^{(1)} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad A_0^{(1)} = \begin{bmatrix} -2 & 0 \\ 4 & -2 \end{bmatrix}, \qquad A_1^{(1)} = \begin{bmatrix} -1 & 0 \\ 6 & -1 \end{bmatrix},$$
$$B^{(2)} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}, \qquad A_0^{(2)} = \begin{bmatrix} -2 & 4 \\ 0 & -2 \end{bmatrix}, \qquad A_1^{(2)} = \begin{bmatrix} -1 & 6 \\ 0 & -1 \end{bmatrix}.$$

It is easy to verify that $(A_0^{(1)}, A_1^{(1)}, B^{(1)}) \in S$ and $(A_0^{(2)}, A_1^{(2)}, B^{(2)}) \in S$.

Further, choose $r = \frac{1}{2}$. Then, we have

$$
\begin{aligned}
&B = rB^{(1)} + (1-r)B^{(2)} = \frac{5}{12}I,\\
&A_0 = rA_0^{(1)} + (1-r)A_0^{(2)} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix},\\
&A_1 = rA_1^{(1)} + (1-r)A_1^{(2)} = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix},\\
&\operatorname{Re}\lambda\big[(I-B)^{-1}(A_0 + A_1)\big] = \operatorname{Re}\lambda \begin{bmatrix} -\frac{60}{7} & \frac{75}{7} \\ \frac{75}{7} & -\frac{60}{7} \end{bmatrix} \not< 0.
\end{aligned}
$$

This shows that $(AB) \notin S$, and so S is not a convex set.

To prove that the asymptotically stable cone of the neutral system (8.11.1) is convex, more conditions are needed.

THEOREM 8.11.13. *(See [280].) For the scalar neutral function*:

$$
\frac{d}{dt}\left[x(t) - \sum_{k=1}^{N} b_k x(t-\gamma_k)\right] = a_0 x(t) + \sum_{k=1}^{N} a_k x(t-\gamma_k), \tag{8.11.16}
$$

S is a convex set.

PROOF. It is known [154] that $(a, b) \in S$, if and only if the following conditions are satisfied:

(1) $\sum_{k=1}^{N} |b_k| < 1$,
(2) $\sum_{k=1}^{N} a_k < 0$ and $\sum_{k=0}^{N} |a_k| \leqslant |a_0|$.

Choose $a_i^{(j)}$ and $b_i^{(j)}$ ($i = 1, 2, \ldots, N$, $j = 1, 2$) to satisfy conditions (1) and (2). Then, $a_0^{(j)} < 0$, and for $0 \leqslant r \leqslant 1$ we have

$$
\begin{aligned}
\sum_{k=1}^{N} \big|rb_k^{(1)} + (1-r)b_k^{(2)}\big| &\leqslant r\sum_{k=1}^{N}\big|b_k^{(1)}\big| + (1-r)\sum_{k=1}^{N}\big|b_k^{(2)}\big| \\
&\leqslant \max\left[\sum_{k=1}^{N}\big|b_k^{(1)}\big|, \sum_{k=1}^{N}\big|b_k^{(2)}\big|\right] \\
&< 1,\\
\sum_{k=0}^{N} ra_k^{(1)} + \sum_{k=1}^{N}(1-r)a_k^{(2)} &= r\sum_{k=1}^{N} a_k^{(1)} + (1-r)\sum_{k=1}^{N} a_k^{(2)} \\
&< 0,
\end{aligned}
$$

$$\begin{aligned}\sum_{k=0}^{N}\left|ra_k^{(1)}+(1-r)a_k^{(2)}\right| &\leqslant r\sum_{k=1}^{N}\left|a_k^{(1)}\right|+(1-r)\sum_{k=1}^{N}\left|a_k^{(1)}\right|\\ &\leqslant \left|ra_0^{(1)}\right|+(1-r)\left|a_0^{(2)}\right|\\ &=\left|ra_0^{(1)}+(1-r)a_0^{(2)}\right|.\end{aligned}$$

These conditions imply that S is convex. □

THEOREM 8.11.14. *(See [280].) Assume in system (8.11.1) that* $B_k := \mathrm{diag}(b_{11}^{(k)}, \ldots, b_{nn}^{(k)})$ *which satisfy*

(1) $\sum_{k=1}^{N}|b_{ii}^{(k)}| < 1,\ i = 1, 2, \ldots, n;$

(2) $a_{ii}^{(0)} < 0,\ i = 1, 2, \ldots, n$*, and*

$$\left|a_{ii}^{(0)}\right| > \sum_{k=1}^{N}\sum_{j=1}^{n}\left|a_{ii}^{(k)}\right| + \sum_{\substack{j=1\\ j\neq i}}^{n}\left|a_{ij}^{(0)}\right|, \quad i = 1, 2, \ldots, n.$$

These conditions mean that S_1 *includes all* (A, B) *that satisfies conditions* (1) *and* (2)*. Thus,* $(A, B) \in S_1 \subset S$ *and* S_1 *is convex.*

PROOF. Let

$$\mu_{ij} := \begin{cases} \frac{1}{1-\sum_{k=1}^{N}|b_{ii}^{(k)}|}\Big(\left|a_{ii}^{(0)}\right| - \sum_{k=1}^{N}\left|a_{ii}^{(k)}\right|\Big), \\ \quad i = j = 1, 2, \ldots, n, \\ -\frac{1}{1-\sum_{k=1}^{N}|b_{ii}^{(k)}|}\Big(\sum_{k=0}^{N}|a_{ij}^{(k)}|\Big), \\ \quad i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases} \tag{8.11.17}$$

From conditions (1), (2), and Lemmas 8.7.1 and 8.11.7, it is easy to prove that $S_1 \subset S$. So, we only prove that S_1 is convex.

For $(A^{(1)}, B^{(1)}) \in S_1$ and $(A^{(2)}, B^{(2)}) \in S_1$, let

$$(A, B) := \big(\gamma A^{(1)} + (1-\lambda)A^{(2)}, \gamma B^{(1)} + (1-\gamma)B^{(2)}\big), \quad 0 \leqslant \gamma \leqslant 1,$$

$$\tilde{\mu}_{ij} := \begin{cases} \frac{|\gamma a_{ii}^{(1.0)}+(1-\gamma)a_{ii}^{(2.0)}|-(\sum_{k=1}^{N}|\gamma a_{ii}^{(1.k)}+(1-\gamma)a_{ii}^{(2.k)}|)}{1-\sum_{k=1}^{N}|\gamma b_{ii}^{(1.k)}+(1-\gamma)b_{ii}^{(2.k)}|}, \\ \quad i, j = 1, \ldots, n, \\ \frac{-(\sum_{k=1}^{N}|\gamma a_{ij}^{(1.k)}+(1-\gamma)a_{ij}^{(2.k)}|)}{1-\sum_{k=1}^{N}|\gamma b_{ii}^{(2.k)}+(1-\gamma)b_{ii}^{(2.k)}|}, \\ \quad i \neq j,\ i, j = 1, \ldots, n, \end{cases} \tag{8.11.18}$$

$$\sum_{k=1}^{N}\left|\gamma b_{ii}^{(1.k)}+(1-\gamma)b_{ii}^{(2.k)}\right| \leqslant \gamma\sum_{k=1}^{N}\left|b_{ii}^{(1.k)}\right|+(1-\gamma),$$

$$\sum_{k=1}^{N}\left|b_{ii}^{(2.k)}\right| \leqslant \max\left(\sum_{k=1}^{N}\left|b_{ii}^{(1.k)}\right|,\sum_{k=1}^{N}\left|b_{ii}^{(2.k)}\right|\right)<1. \tag{8.11.19}$$

Since $a_{ii}^{(1.0)}<0$, $j=1,2,\ldots,n$, we have

$$\gamma a_{ii}^{(1.0)}+(1-\gamma)a_{ii}^{(2.0)}<0, \quad i=1,2,\ldots,n,$$

and

$$\begin{aligned}
\left|\gamma a_{ii}^{(1.0)}+(1-\gamma)a_{ii}^{(2.0)}\right| &= \gamma\left|a_{ii}^{(1.0)}\right|+(1-\gamma)\left|a_{ii}^{(2.0)}\right| \\
&> \gamma\left[\sum_{k=1}^{N}\left|a_{ii}^{(1.k)}\right|+\sum_{j=1}^{n}\sum_{k=1}^{N}\left|a_{ij}^{(1.k)}\right|+\sum_{\substack{j=1\\j\neq i}}^{n}\left|a_{ij}^{(1.0)}\right|\right] \\
&\geqslant \sum_{k=1}^{N}\left|\gamma a_{ii}^{(1.k)}\right|+(1-\gamma)\left|a_{ii}^{(2.k)}\right| \\
&\quad+\sum_{\substack{j=1\\j\neq i}}^{n}\left|\gamma a_{ij}^{(1.0)}\right|+(1-\gamma)\left|a_{ij}^{(2.0)}\right|.
\end{aligned}$$

Thus, the conditions in Theorem 8.11.11 are satisfied, and so S_1 is convex. □

THEOREM 8.11.15. *(See [280].) Assume that* $B_k := \operatorname{diag}(b_{11}^{(k)},\ldots,b_{nn}^{(k)})$, $k=1,2,\ldots,N$, *and*

(1) $\sum_{k=1}^{N}|b_{ii}^{(k)}|<1$, $i=1,2,\ldots,n$;
(2) $a_{ii}^{(0)}<0$, $i=1,2,\ldots,n$, *and* $(\mu_{ij})_{n\times n}$ *is a symmetric positive definite matrix.*

Then $(A,B)\subset S_2\subset S$ *and* S_2 *is convex.*

PROOF. By employing a similar method as that used in Theorem 8.11.14, it is easy to prove that $S_2\subset S$. We thus only prove that S_2 is convex.

$\forall(A^{(1)},B^{(1)})\in S_2$, $(A^{(2)},B^{(2)})\in S_2$, consider

$$(A,B)=\left(\gamma A^{(1)}+(1-\lambda)A^{(2)},\gamma B^{(1)}+(1-\gamma)B^{(2)}\right).$$

Choose (8.11.18), so (8.11.19) holds and condition (1) in Theorem 8.11.15 is satisfied.

Since $a_{ii}^{(j.0)} < 0$, we have

$$\begin{aligned}
&\big|\gamma a_{ii}^{(1.0)} + (1-\gamma)a_{ii}^{(2.0)}\big| - \big|\gamma a_{ii}^{(1.0)} + (1-\gamma)a_{ii}^{(2.0)}\big| \\
&\quad - \sum_{k=1}^{N}\big|\gamma a_{ii}^{(1.k)} + (1-\gamma)a_{ii}^{(2.k)}\big| \\
&\qquad \geqslant \gamma\big|a_{ii}^{(1.0)}\big| + (1-\gamma)\big|a_{ii}^{(2.0)}\big| - \gamma\sum_{k=1}^{N}\big|a_{ii}^{(1.k)}\big| - (1-\gamma)\sum_{k=1}^{N}\big|a_{ii}^{(2.k)}\big|,
\end{aligned}$$

$$\gamma\mu_{ij}^{(1)} := \begin{cases} \dfrac{\gamma|a_{ii}^{(1.0)}| - \gamma\sum_{k=1}^{N}|a_{ii}^{(1.k)}|}{1-\sum_{k=1}^{N}|\gamma b_{i}^{(1.k)} + (1-\gamma)b_{ii}^{(2.k)}|}, & i = j = 1, 2, \ldots, n, \\ \dfrac{-\gamma\sum_{k=0}^{N}|a_{ij}^{(1.k)}|}{1-\sum_{k=1}^{N}|\gamma b_{ii}^{(1.k)} + (1-\gamma)b_{ii}^{(2.k)}|}, & i \neq j,\ i, j = 1, 2, \ldots, n, \end{cases}$$

and

$$(1-\gamma)\mu_{ij}^{(2)} := \begin{cases} \dfrac{(1-\gamma)|a_{ii}^{(2.0)}| - (1-\gamma)\sum_{k=1}^{N}|a_{ii}^{(2.k)}|}{1-\sum_{k=1}^{N}|\gamma b_{ii}^{(1.k)} + (1-\gamma)b_{ii}^{(2.k)}|}, & i = j = 1, 2, \ldots, n, \\ \dfrac{(1-\gamma)\sum_{k=0}^{N}|a_{ij}^{(2.k)}|}{1-\sum_{k=1}^{N}|\gamma b_{ii}^{(1.k)} + (1-\gamma)b_{ii}^{(2.k)}|}, & i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases}$$

Since $(\mu_{ij}^{(1)})$ and $(\mu_{ij}^{(2)})$ are symmetric and positive definite for $\gamma \in [0, 1]$,

$$\big(\gamma\mu_{ij}^{(1)}\big), \quad \big((1-\gamma)\mu_{ij}^{(2)}\big) \quad \text{and} \quad \big(\gamma\mu_{ij}^{(1)} + (1-\gamma)\big(\mu_{ij}^{(2)}\big)\big)$$

are all symmetric and positive definite. Thus S_2 is convex. $\square$

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Chapter 9

Absolute Stability of Nonlinear Control Systems

The automatic control theory was originally motivated by the stability analysis of Watt centrifugal governor by Maxwell. In early 1940s, the former Soviet Union scholars Lurie, Postnikov and others developed method to deal with a class of nonlinear systems, now called nonlinear isolate method. Lurie and his co-workers studied many real control systems, including the Bulgakov problem of aircraft automatic control. They first isolated the nonlinear part from the system and considered it as a feedback control of the system so that the system has a closed-loop form. Thus, the well-known Lurie problem was proposed, which initiated the research on the robust control and robust stability for nondeterministic systems or multivalued differential equations. It promoted the application and development of stability theory.

This chapter introduces the background of Lurie problem, the methodology for solving Lurie problem, and in particular present three classical methods for studying absolute stability—the Lyapunov–Lurie type V-function method (i.e., the V function containing integrals and quadric form), the S-program method, and Popov frequency-domain criteria. Main attention is given to sufficient and necessary conditions for absolute stability and some simple algebraic sufficient conditions.

The materials presented in this chapter include both classical results and new results obtained recently. Details can be found from [368] for Section 9.1, [403, 346] for Section 9.2, [456,458] for Section 9.3, [337,350,451,452] for Section 9.4, [239] for Section 9.5, [246] for Section 9.6, [247,242] for Section 9.7, [255,282] for Section 9.8, [271] for Section 9.9, [261] for Section 9.10 and [285] for Section 9.11.

9.1. The principal of centrifugal governor and general Lurie systems

First we introduce the centrifugal governor, as shown in Figure 9.1.1, which is the earliest example of Lurie control system. The work principal of the centrifugal

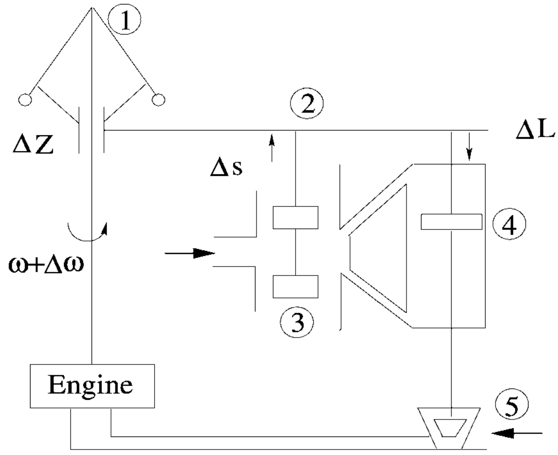

Figure 9.1.1. Centrifugal governor model.

governor is described as follows. The angle velocity of the generator, ω, is measured by the centrifugal governor. The centrifugal governor is connected to the server 4 through the level 2 and the sliding valve 3. The server 4 makes the regulator 5 move so that the generator is rotating with a constant speed. When the load of the generator is reducing and so the speed of the generator is increasing, the governor sleeve is moving up to raise the sliding valve via the level. Thus, high pressure gasoline enters the upper part of the server cylinder, and the gasoline left in the lower part of the cylinder is drained off through the lower narrow passage. Therefore, the piston descends to move the regulator to reduce the amount of gasoline and so the angle velocity of the generator is reduced. On the other hand, when the speed of the generator is below the normal speed, the server moves up to adjust the regulator to increase the amount of gasoline. Thus, the speed of the generator increases. Due to the negative feedback of the centrifugal governor, the generator's angle velocity can be kept in a constant value. As we know, a generator is usually working in the environment that the end-users' loads are frequently varying. If the generator cannot be kept in a constant angle velocity, the users will not have a constant currency, which could cause damage or even disaster.

Now, we consider the mathematical model of the centrifugal governor. The differential equation of motion of the generator is

$$J\frac{d\Delta\omega}{dt} = k_1\Delta\omega + k_2\Delta L,$$

where J is the angular inertia, $\Delta\omega$ is the increment of the angular velocity, ΔL is the position increment of the regulator, and k_1 and k_2 are the rates of the changes of the moments per unit.

The dynamic equation of motion for the centrifugal activator is given by

$$M\frac{d^2\Delta Z}{dt^2}+C\frac{d\Delta z}{dt}=F_1\Delta\omega+F_2\Delta z,$$

where M is the generalized mass, C is the damping coefficient, ΔZ is the measurement of the position change of the governor sleeve, F_1 and F_2 denote the generalized forces per unit, respectively, for ω and Z.

The equation of motion for the server is

$$A\frac{d\Delta L}{dt}=f(\Delta s),$$

where A is the cross-area of the governor cylinder, Δs is the amount of change of the sliding valve, $f(\Delta s)$ is the quantity of gasoline entering into the cylinder per unit time. The nonlinear function f is determined by the shape of the sliding valve. In general, it is difficulty to know the exact form of function f.

The kinematics of the feedback lever is

$$\Delta s=a\Delta Z-b\Delta L,$$

where a and b are constants.

The above four equations can be transformed to the following dimensionless equations:

$$\begin{cases} a_1\varphi+a_2\varphi=-\mu,\\ b_1\eta+b_2\eta+b_3\eta=\varphi,\\ \dot{\mu}=f(\sigma),\\ \sigma=c_1\eta-c_2\mu, \end{cases}$$

where φ, η, μ, σ are, respectively, the variables proportional to $\Delta\omega$, ΔZ, ΔL, Δs, while σ is the control signal which determines the amount of the gasoline entering into the cylinder.

Let $\varphi=x_1, \eta=x_2, \frac{d\eta}{dt}=x_3, \mu=x_4$. Then, the above equations can be rewritten as

$$\begin{cases} \frac{dx_1}{dt}=-\frac{a_2}{a_1}x_1-\frac{1}{a_1}x_4,\\ \frac{dx_2}{dt}=x_3,\\ \frac{dx_3}{dt}=\frac{1}{b_1}x_1-\frac{b_3}{b_1}x_2-\frac{b_2}{b_1}x_3,\\ \frac{dx_4}{dt}=f(\sigma),\\ \sigma=c_1x_1-c_2x_2. \end{cases}$$

When the generator is working in normal condition, we have $x_1=x_2=x_3=x_4=0$ which is the equilibrium position of the system, which required to be kept globally stable.

Next, we introduce the Lurie type nonlinear control system [295–297], i.e., the so-called Lurie problem and its mathematical description. Around 1944, the former Soviet Union mathematical control scholar, A.I. Lurie, based on the study of aircraft automatic control system, proposed a control model, described by the following general differential equations:

$$\begin{cases} \frac{dx}{dt} = Ax + bf(\sigma), \\ \sigma = c^T x = \sum_{i=1}^n c_i x_i, \end{cases} \tag{9.1.1}$$

where $x \in R^n$ is the state variable, $b, c \in R^n$ are known vectors, σ is the feedback control variable, $f(\sigma)$ is a nonlinear function. The form of f is not specified, but it is known that it belongs to some type of functions $F_{[0,k]}$, $F_{[0,k)}$, or F_∞. Here,

$$F_{[0,k]} := \{f \mid f(0) = 0,\ 0 < \sigma f(\sigma) \leqslant k\sigma^2,\ \sigma \neq 0,\ f \text{ continuous}\};$$
$$F_{[0,k)} := \{f \mid f(0) = 0,\ 0 < \sigma f(\sigma) < k\sigma^2,\ \sigma \neq 0,\ f \text{ continuous}\};$$
$$F_\infty := \{f \mid f(0) = 0,\ \sigma f(\sigma) > 0,\ \sigma \neq 0,\ f \text{ continuous}\}.$$

Many practical nonlinear feedback control problems can be described by system (9.1.1), but the form of f is usually not known. Partial information about f may be obtained from experiments. However, experiments can only be carried out under specific loads, and thus f depends upon the leads. Usually one only knows that f belongs to $F_{[0,k)}$, $F_{[0,k]}$, or F_∞. No any other information is available in practice. So strictly speaking, system (9.1.1) is indefinite or is called a multiple valued system.

The classification of system (9.1.1) is given follows:

(1) If A is a Hurwitz matrix, (9.1.1) is called direct control system.
(2) If A has only one zero eigenvalue, and others have negative real parts, (9.1.1) is called indirect control system.
(3) If $\operatorname{Re}\lambda(A) \leqslant 0$ and it does not belong to indirect control, then (9.1.1) is called critical control system.

DEFINITION 9.1.1. The zero solution of system (9.1.1) is said to be absolutely stable, if for any $f \in F_\infty$, the zero solution of (9.1.1) is globally asymptotically stable. The zero solution of (9.1.1) is said to be absolutely stable in the Hurwitz angle $[0, k]$, ($[0, k)$), if $\forall f \in F_{[0,k]}$ ($f \in F_{[0,k)}$) the zero solution of (9.1.1) is globally asymptotically stable.

In the following, we introduce some frequently used necessary conditions for the zero solution of system (9.1.1) to be absolutely stable.

THEOREM 9.1.2. *If the zero solution of system (9.1.1) is absolutely stable, then one of the following conditions must hold*:

(1) $\forall \varepsilon > 0$ $(0 < \varepsilon < k)$, $A + \varepsilon bc^T$ *is a Hurwitz matrix*;
(2) $\operatorname{Re} \lambda(A) \leqslant 0$, *i.e., A has no eigenvalues with positive real parts*;
(3) $c^T b \leqslant 0$;
(4) *if A is stable, then* $c^T A^{-1} b \geqslant 0$.

PROOF. (1) Let $f(\sigma) = \varepsilon\sigma = \varepsilon c^T x$. Then the first equation of (9.1.1) becomes

$$\frac{dx}{dt} = \big(A + \varepsilon bc^T\big)x, \tag{9.1.2}$$

whose zero solution is asymptotically stable. So $A + \varepsilon bc^T$ is a Hurwitz matrix.

(2) Suppose there exists an eigenvalue λ_0 with $\operatorname{Re} \lambda_0(A) > 0$. By taking $f(\sigma) = \varepsilon\sigma$ $(0 < \varepsilon \ll 1)$, the eigenvalues of $A + \varepsilon bc^T$ depend continuously on the coefficients. For $0 < \varepsilon \ll 1$, the coefficient matrix (9.1.2) must have an eigenvalue $\tilde{\lambda}_0$ with $\operatorname{Re} \bar{\lambda}_0 > 0$. This is a contradiction with the fact that the zero solution of (9.1.1) is absolutely stable. So $\operatorname{Re} \lambda(A) \leqslant 0$.

(3) Let $f(\sigma) = h\sigma$. Then system (9.1.1) becomes $\frac{dx}{dt} = (A + hbc^T)x$. So the trace of the matrix $A + hbc^T$ satisfies

$$tr\big(A + hbc^T\big) = tr\, A + h\, tr\, bc^T = \sum_{i=1}^{n} a_{ii} + h \sum_{i=1}^{n} c_i b_i < 0.$$

But when $c^T b > 0$, for $h \gg 1$, it follows

$$\sum_{i=1}^{n} a_{ii} + h \sum_{i=1}^{n} c_i b_i > 0,$$

which is impossible, showing that $c^T b \leqslant 0$.

(4) We first verify the identity of the linear algebraic equation:

$$\det(I + GH) = \det(I + HG),$$

where G and H are arbitrary matrices for which GH and HG exist, The unit matrices on both sides of the equation can be in different order. Then,

$$\begin{aligned}
\det(I + GH) &= \det \begin{bmatrix} I + GH & 0 \\ H & I \end{bmatrix} \\
&= \det \left(\begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -G \\ H & I \end{bmatrix} \right) \\
&= \det \left(\begin{bmatrix} I & -G \\ H & I \end{bmatrix} \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \right) \\
&= \det \begin{bmatrix} I & 0 \\ H & HG + I \end{bmatrix} \\
&= \det(I + HG).
\end{aligned}$$

Next, we prove $c^T A^{-1} b \geqslant 0$. Since A and $A + \varepsilon bc^T$ $(\varepsilon > 0)$ are stable, for $\lambda = 0$, we have

$$\begin{aligned} \det\big(I\lambda - \big(A + \varepsilon bc^T\big)\big) &= \det(I\lambda - A)\det\big[I - \varepsilon(I\lambda - A)^{-1}bc^T\big] \\ &= \det(I\lambda - A)\det\big[I - \varepsilon c^T(I\lambda - A)^{-1}b\big] \neq 0 \end{aligned}$$

if $c^T A^{-1} b < 0$. Then, there must exist $\varepsilon_0 > 0$ such that

$$1 - \varepsilon_0 c^T (-A)^{-1} b = 1 + \varepsilon_0 c^T A^{-1} b = 0, \tag{9.1.3}$$

leading to a contradiction. As a result, $c^T A^{-1} b \geqslant 0$.

The proof of Theorem 9.1.2 is complete. □

9.2. Lyapunov–Lurie type V function method

In this section, we present the sufficient conditions for absolute stability by using the Lyapunov–Lurie type V function.

Let (9.1.1) be a direct control system. That is, A is a Hurwitz matrix, Lurie first constructed the following Lyapunov–Lure type function:

$$V(t) = x^T P x + \beta \int_0^{\sigma} f(\sigma)\, d\sigma, \tag{9.2.1}$$

for which, for any given symmetric positive matrix B, the Lyapunov matrix equation:

$$A^T P + PA = -B \tag{9.2.2}$$

has a unique symmetric positive matrix solution P.

THEOREM 9.2.1. *(See [296].) Assume that there exist constant $\beta \geqslant 0$ and an $n \times n$ symmetric positive definite matrix such that*

$$V(x) = x^T P x + \beta \int_0^{\sigma} f(\sigma)\, d\sigma, \tag{9.2.3}$$

and the derivative of V along the solution of (9.1.1), given by

$$\frac{dV}{dt}\bigg|_{(9.1.1)} = -x^T B x + 2\left(Pb + \frac{1}{2}\beta A^T c\right)^T x f(\sigma) + \beta c^T b f^2(\sigma), \tag{9.2.4}$$

is negative definite. Then the zero solution of system (9.1.1) is absolutely stable.

PROOF. For any $f \in F_\infty$ (or $f \in F_{[0,k]}$ or $f \in F_{[0,k)}$), the Lyapunov function (9.2.3) is positive definite and radially unbounded. In addition to that $\frac{dV}{dt}|_{(9.1.1)}$ is negative, by the Lyapunov globally asymptotic stability theorem, the conclusion is true. □

Determining the negative definite propriety of (9.2.4) is a difficult problem. Equation (9.2.4) looks like a quadric form for x, $f(\sigma)$, but the Sylvester condition is not satisfied here. Lurie developed a method called S-program method, which can be used as a sufficient condition to determine the negative definite propriety of (9.2.4).

First, we give a lemma which is needed in the following analysis.

LEMMA 9.2.2. *Given a real symmetric positive matrix*:

$$G = \begin{bmatrix} K & d \\ d^T & r \end{bmatrix},$$

where $K \in R^{n\times n}$, $K^T = K$, $d \in R^n$, $r \in R^1$. *Then, G is positive definite if and only if* (1) *K is a positive definite and* $r - d^T K^{-1} d > 0$; *or* (2) $r > 0$, $K - \frac{1}{r} dd^T$ *is positive definite.*

PROOF. By a direct calculation, we obtain

$$\begin{bmatrix} I & -\frac{1}{r}d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} K & d \\ d^T & r \end{bmatrix}\begin{bmatrix} I & 0 \\ -\frac{1}{r}d^T & 1 \end{bmatrix} = \begin{bmatrix} K - \frac{1}{r}dd^T & 0 \\ 0 & r \end{bmatrix},$$

$$\begin{bmatrix} I & 0 \\ -d^T K^{-1} & 1 \end{bmatrix}\begin{bmatrix} K & d \\ d^T & r \end{bmatrix}\begin{bmatrix} I & -K^{-1}d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & r - d^T K^{-1} d \end{bmatrix}.$$

So the conclusion is true. □

Therefore, we can observe that the Lyapunov–Lurie function:

$$V(x) = x^T P x + \beta \int_0^t f(\sigma)\,d\sigma, \tag{9.2.5}$$

where $\beta > 0$ is constant and P is a symmetric positive matrix, is equivalent to

$$W(x) = x^T Q x + \int_0^t f(\sigma)\,d\sigma, \tag{9.2.6}$$

where Q is a symmetric positive matrix.

It is easy to see from (9.2.5) and (9.2.6) that letting $\beta = 1$, $Q = \frac{P}{\beta}$ and so $W(x) = \frac{V(x)}{\beta}$, then (9.2.5) becomes (9.2.6). Therefore, in the following analysis we will use (9.2.6), instead of (9.2.5) since it has one less parameter.

Now, we present Lurie's S-program method, which is a sufficient condition to assure that (9.2.4) is negative definite. $\forall f(\sigma) \in F_\infty$, we can rewrite (9.2.4) as

$$-\frac{dV}{dt}\bigg|_{(9.1.1)} = x^T Bx - 2\left(\frac{1}{2}c^T A + b^T P + \tau c^T\right)xf(\sigma) - c^T bf^2(\sigma) \\ + 2\tau\sigma f(\sigma) := S(x, f) + 2\tau\sigma f(\sigma), \tag{9.2.7}$$

where $\tau > 0$ is a certain constant. Obviously, if

$$S(x, f) := x^T Bx - 2\left(\frac{1}{2}c^T A + b^T P + \tau c^T\right)xf(\sigma) - c^T bf^2(\sigma), \tag{9.2.8}$$

is positive definite with respect to x, f, then (9.2.4) is negative definite.

However, to determine whether $S(x, f)$ is positive definite or not, one may use the famous Sylvester condition. The Sylvester condition holds if and only if a one-variable quadratic equation has positive solution. To be more specific, since B is positive definite, $\forall x \neq 0$, let $y = Bx$, then we have

$$y^T B^{-1} y = x^T B^T B^{-1} Bx = x^T BB^{-1} Bx = x^T Bx > 0 \quad \text{for } x \neq 0.$$

So B^{-1} is also positive definite. Further, since B is positive definite, the conditions for $S(x, f)$ to be positive definite with respect to x, $f(\sigma)$ are given by

$$\det\begin{bmatrix} B & -\left(\frac{1}{2}A^T c + Pb + \tau c\right) \\ -\left(\frac{1}{2}A^T c + Pb + \tau c\right)^T & -c^T b \end{bmatrix} > 0, \tag{9.2.9}$$

which is equivalent to

$$\begin{aligned} &\det\begin{bmatrix} B^{-1} & 0 \\ 0 & 1 \end{bmatrix} \det\begin{bmatrix} B & -\left(\frac{1}{2}A^T c + Pb + \tau c\right) \\ -\left(\frac{1}{2}A^T c + Pb + \tau c\right)^T & -c^T b \end{bmatrix} \\ &\quad = \det\begin{bmatrix} I & -B^{-1}\left(\frac{1}{2}A^T c + Pb + \tau c\right) \\ -\left(\frac{1}{2}A^T c + Pb + \tau c\right)^T & -c^T b \end{bmatrix} \\ &\quad = -c^T b - \left(\frac{1}{2}A^T c + Pb + \tau c\right)^T B^{-1}\left(\frac{1}{2}A^T c + Pb + \tau c\right) \\ &\quad > 0. \end{aligned} \tag{9.2.10}$$

Therefore, we have the following theorem.

THEOREM 9.2.3. *(See [345].) If there exist symmetric positive matrix B and a positive number τ such that*

$$-c^T b > \left(\frac{1}{2}A^T c + Pb + \tau c\right)^T B^{-1}\left(\frac{1}{2}A^T c + Pb + \tau c\right), \tag{9.2.11}$$

then (9.2.4) is negative definite. Thus, the zero solution of system (9.1.1) is absolutely stable.

COROLLARY 9.2.4. *If there exists a symmetric positive matrix B such that*

$$-c^T b > \left(\frac{1}{2}A^T c + Pb + c\right)^T B^{-1}\left(\frac{1}{2}A^T c + Pb + c\right), \tag{9.2.12}$$

then (9.2.4) is negative definite, and hence the zero solution of system (9.1.1) is absolutely stable.

If $f(\sigma) \in F_{[0,k)}$, then in (9.2.4) we can add and subtract a same term: $2\tau f(\sigma)(\sigma - \frac{1}{k}f(\sigma))$ ($\tau > 0$ is a constant) to obtain

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(9.1.1)} &= -x^T Bx + 2\left(\frac{1}{2}A^T c + Pb + \tau c\right)^T x f(\sigma) \\
&\quad + \left(c^T b - \frac{2\tau}{k}\right) f^2(\sigma) - 2\tau f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right) \\
&:= -S_1\big(x, f(\sigma)\big) - 2\tau f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right).
\end{aligned} \tag{9.2.13}$$

Since $f(\sigma)(\sigma - \frac{1}{k}f(\sigma)) > 0$, $\forall \sigma \neq 0$, if

$$\begin{aligned}
S_1\big(xf(\sigma)\big) = x^T Bx &- 2\left(\frac{1}{2}A^T c + Pb + \tau c\right)^T x f(\sigma) \\
&- \left(c^T b - \frac{2\tau}{k}\right) f^2(\sigma)
\end{aligned}$$

is positive definite for $(x, f(\sigma))$, then (9.2.13) is negative definite for x. Hence, the following result is obtained.

THEOREM 9.2.5. *(See [345].) If there exist a symmetric positive matrix B and a constant $\tau > 0$ such that*

$$-\left(c^T b - \frac{2\tau}{k}\right) > \left(\frac{1}{2}A^T c + Pb + \tau c\right)^T B^{-1}\left(\frac{1}{2}A^T c + Pb + \tau c\right),$$

then the zero solution of system (9.1.1) is absolutely stable within the Hurwitz angle $[0, k)$.

If $\forall f(\sigma) \in F_{[0,k]}$, similarly we can add and subtract a same term

$$2\tau\left[f(\sigma)\left(\sigma - \frac{1}{k+\varepsilon}f(\sigma)\right)\right]$$

into (9.2.4) to obtain

$$\left.\frac{dV}{dt}\right|_{(9.1.1)} = -x^T Bx + 2\left(\frac{1}{2}A^T c + Pb + \tau c\right)^T x f(\sigma)$$

$$
\begin{aligned}
&\quad + \left(c^T b - \frac{2\tau}{k+\varepsilon}\right) f^2(\sigma) - 2\tau f(\sigma)\left(\sigma - \frac{f(\sigma)}{k+\varepsilon}\right) \\
&:= -S_2\left(x, f(\sigma)\left(\sigma - \frac{1}{k+\varepsilon} f(\sigma)\right)\right) \\
&= -S_2\big(x, f(\sigma)\big) - 2\tau f(\sigma)\left(\sigma - \frac{f(\sigma)}{k+\varepsilon}\right).
\end{aligned}
\tag{9.2.14}
$$

So if

$$
\begin{aligned}
S_2(x, f) = x^T B x - 2\left(\frac{1}{2} A^T c + Pb + \tau c\right)^T x f(\sigma) \\
- \left(c^T b - \frac{2\tau}{k}\right) f^2(\sigma)
\end{aligned}
$$

is positive definite for $(x, f(\sigma))$, then (9.2.14) is negative definite, leading to the following theorem.

THEOREM 9.2.6. *If there exist a symmetric positive matrix B and a positive number $\tau > 0$ such that*

$$
-\left(c^T b - \frac{2\tau}{k+\varepsilon}\right) > \left(\frac{1}{2} A^T c + Pb + \tau c\right)^T B^{-1} \left(\frac{1}{2} A^T c + Pb + \tau c\right),
$$

then the zero solution of system (9.1.1) is absolutely stable within the Hurwitz angle $[0, k]$.

The method described above is called S-method, which is simple and practically useful. However, note that the positive definite property is only a sufficient condition for the negative definite of $\frac{dV}{dt}\big|_{(9.1.1)}$, which is not necessary.

EXAMPLE 9.2.7. (See [458].) Consider the following system:

$$
\begin{cases}
\frac{dx_1}{dt} = -2x_1 + f(x_2), \\
\frac{dx_2}{dt} = -x_2 - \frac{1}{2} f(x_2).
\end{cases}
\tag{9.2.15}
$$

We can show that the Lyapunov–Lurie V function exists such that $\frac{dV}{dt}\big|_{(9.2.15)}$ is negative definite, but there does not exist $\tau > 0$ such that $S(x, f)$ is positive definite.

9.3. NASCs of negative definite for derivative of Lyapunov–Lurie type function

In this section, we present the necessary and sufficient conditions (NASCs) for the derivative of the following Lyapunov–Lurie type V function:

$$V(x) = x^T P x + \int_0^t f(\sigma)\, d\sigma \quad \text{(in which } \beta \text{ has been set 1)} \tag{9.3.1}$$

to be negative definite along the solution of system (9.1.1).

THEOREM 9.3.1. *(See [456].) Let system (9.1.1) be a direct control systems. Then the derivative of (9.3.1) with respect to time t along the solution of system (9.1.1), given by*

$$\left.\frac{dV}{dt}\right|_{(9.1.1)} = -x^T Bx - \left(c^T A + 2b^T P\right)x + f(\sigma) + c^T bf^2(\sigma), \tag{9.3.2}$$

is negative definite with respect x if and only if

(1) $W(x) := x^T Bx - (c^T A + 2b^T P)x - c^T b \geqslant 0$ *when* $c^T x = \sigma = 0$*, i.e.,* $W(x)$ *is positive semi-definite on hyperplane* $c^T x = \sigma = 0$;
(2) $c^T HH(A^T c + 2P^T b) \leqslant 0$*, where* $H^T BH = I_n$.

The proof of the theorem is quite long and is omitted here. Interested readers can find the detailed proof in [456]. Later, Zhu [463] showed that the condition (1) in Theorem 9.3.1 is still difficult to verify. His improved result is given in the following theorem.

THEOREM 9.3.2. *(See [463].) Let A be stable. Then $\forall f \in F_{[0,k]}$, the derivative of V given in (9.3.1),*

$$\left.\frac{dV}{dt}\right|_{(9.1.1)} = -x^T Bx + \left(c^T A + 2b^T P\right)xf(\sigma) + c^T bf^2(\sigma),$$

is negative definite with respect to x if and only if

$$\frac{1}{k} - c^T B^{-1} d > 0, \tag{9.3.3}$$

and

$$\frac{1}{c^T B^{-1} c}\left(\frac{1}{k} - c^T B^{-1} d\right)^2 - d^T B^{-1} d - c^T b > 0, \tag{9.3.4}$$

where $d = Pb + \frac{1}{2} A^T c$. Thus, when (9.3.3) and (9.3.4) hold, the zero solution of system (9.1.1) is absolutely stable.

PROOF. B is symmetric positive definite and so is B^{-1}. In addition, there exists a matrix H such that $B = H^T H$ and $B^{-1} = H^{-1}(H^{-1})^T$. Clearly, $V(x)$ is positive definite and radially unbounded. Equation (9.3.2) can be reduced to

$$
\begin{aligned}
-\frac{dV}{dt}\bigg|_{(9.1.1)} &= x^T Bx - 2x^T\left(Pb + \frac{1}{2}Ac^T\right)f(\sigma) - c^T bf^2(\sigma) \\
&= x^T H^T Hx - 2x^T df(\sigma) - c^T bf^2(\sigma) \\
&= (Hx)^T(Hx) - 2(Hx)^T\left(H^{-1}\right)^T df(\sigma) - c^T bf^2(\sigma) \\
&= \left[Hx - \left(H^{-1}\right)^T df(\sigma)\right]^T\left[Hx - \left(H^{-1}\right)^T df(\sigma)\right] \\
&\quad - \left(d^T B^{-1} dc^T b\right)f^2(\sigma) \\
&= \begin{cases} 0, & \text{when } x = 0, \\ (Hx)^T(Hx) = x^T Bx > 0, & \text{when } f(\sigma) = 0,\ x \neq 0, \\ Uf^2(\sigma)f, \end{cases}
\end{aligned} \tag{9.3.5}
$$

where

$$
\begin{aligned}
U :=& \left[H\frac{x}{f(\sigma)} - \left(H^{-1}\right)d\right]^T\left[H\frac{c}{f(\sigma)} - \left(H^{-1}\right)^T d\right] \\
&- \left(d^T B^{-1}\right)d + c^T b.
\end{aligned} \tag{9.3.6}
$$

Therefore, it is suffice to show that $U > 0$ for any $c^T x \geqslant \frac{1}{k}$. To complete the proof, we use the topological transformation:

$$y = Hx - \left(H^{-1}\right)^T d$$

to reduce U to $U = y^T y - \rho$, where $\rho = d^T B^{-1} d + c^T b$, and the condition $c^T x \geqslant \frac{1}{k}$ is equivalent to

$$c^T H^{-1}\left[y + \left(H^{-1}\right)^T d\right] \geqslant \frac{1}{k}. \tag{9.3.7}$$

As a result, we only need to show that in the half space of y satisfying (9.3.7) the expression $U = y^T y - \rho$ satisfies $U > 0$. It is easy to prove that for $\rho \geqslant 0$, when $y = 0$ we have $U \leqslant 0$. Therefore, it only needs to guarantee that $y = 0$ is not on the half space of (9.3.7), i.e,.

$$c^T H^{-1}\left[0 + \left(H^{-1}\right)^T d\right] = c^T H^{-1}\left(H^{-1}\right)^T d = c^T B^{-1} d < \frac{1}{k},$$

which is just exactly the condition (9.3.3).

Clearly, on the half space of (9.3.7) V reaches its minimum at $y = y^*$, i.e., at the intersection point of the hyperplane

$$c^T H^{-1}\left[y + \left(H^{-1}\right)^T d\right] = \frac{1}{k},$$

and the normal line passing through the point $y = 0$.

Let $y^* = \lambda(c^T H^{-1})^T$, where λ is to be determined. Then the expression

$$c^T H^{-1}\big[\lambda\big(c^T H^{-1}\big)^T d\big] = \frac{1}{k}$$

leads to

$$\lambda = \frac{\frac{1}{k} - c^T B^{-1} d}{c^T B^{-1} c}.$$

As a result, we have

$$\begin{aligned} V(y^*) &= \big[\lambda\big(c^T H^{-1}\big)^T\big]^T \big[\lambda\big(c^T H^{-1}\big)^T\big] - P \\ &= \lambda^2 c^T B^{-1} c - \big(d^T B^{-1} d + c^T b\big) \\ &= \frac{1}{c^T B^{-1} c}\left(\frac{1}{k} - c^T B^{-1} d\right)^2 - d^T\big(B^{-1}\big)d - c^T d > 0. \end{aligned}$$

This is just the condition (9.3.4), and the proof of theorem is complete. □

Through the proof for Theorem 9.3.2, we may find the following results.

COROLLARY 9.3.3. *(See [463].) Let A be a Hurwitz matrix. Then $\forall f(\sigma) \in F_{[0,k)}$, the derivative of V of (9.3.1),*

$$\frac{dV}{dt}\bigg|_{(9.1.1)} = -x^T B x + \big(c^T A + 2b^T P\big) x f(\sigma) + c^T b f^2(\sigma),$$

is negative definite with respect to x if and only if

$$\frac{1}{k} - c^T B^{-1} d \geqslant 0, \tag{9.3.8}$$

$$\frac{1}{c^T B^{-1} c}\left(\frac{1}{k} - c^T B^{-1} d\right)^2 - d^T B^{-1} d - c^T b \geqslant 0. \tag{9.3.9}$$

Taking $k \to +\infty$, then $\frac{1}{k} = 0$ and we have

COROLLARY 9.3.4. *(See [463].) Let A be stable. Then $\forall f(\sigma) \in F_\infty$, the derivative of V of (9.3.1),*

$$\frac{dV}{dt}\bigg|_{(9.1.1)} = -x^T B x + \big(c^T A + 2b^T P\big) x f(\sigma) + c^T b f^2(\sigma),$$

is negative definite with respect to x if and only if

$$c^T B^{-1} d \leqslant 0, \tag{9.3.10}$$

$$\frac{1}{c^T B^{-1} c}\big(c^T B^{-1} d\big)^2 - d^T B^{-1} d - c^T b \geqslant 0. \tag{9.3.11}$$

9.4. Popov's criterion and improved criterion

THEOREM 9.4.1. *(See [337].) Let A be stable. If there exists a real constant q such that*

$$\mathrm{Re}\big\{(1+iq\omega)W(i\omega)\big\}+\frac{1}{k}>0, \quad \omega\in[0,+\infty), \tag{9.4.1}$$

then the zero solution of system (9.1.1) is absolutely stable within the Hurwitz angle $[0,k]$, *where*

$$W(i\omega):=\frac{K(i\omega)}{D(i\omega)}=\frac{\det\begin{bmatrix} i\omega I_n-A & b\\ c^T & 0\end{bmatrix}}{\det(i\omega I_n-A)}. \tag{9.4.2}$$

The above theorem is a well-known result. Its proof is very long and omitted here. Readers can find the details in [337,350].

THEOREM 9.4.2. *Assume* $\mathrm{Re}\,\lambda(A)\leqslant 0$. *If there exists a real constant q such that (9.4.1) holds, then for* $\forall f(\sigma)\in F_{[\varepsilon,k]}:=[f\mid 0<\varepsilon\leqslant\frac{f(\sigma)}{\sigma}\leqslant k,\sigma\neq 0\}$, *the zero solution of system (9.1.1) is globally stable. In this case, the zero solution of (9.1.1) is said to be absolutely stable within the Hurwitz angle* $[\varepsilon,k]$.

By setting $k\to\infty$, we easily obtain the conditions of absolute stability in Popov's absolutely stable condition within the Hurwitz angle $[0,+\infty)$, $([\varepsilon,+\infty])$ as

$$\mathrm{Re}\big\{(1+iq\omega)W(iw)\big\}>0. \tag{9.4.3}$$

Since the frequency ω is varied on the infinite interval $[0,+\infty)$, verifying these conditions is still difficult. Now, we introduce an improved criterion due to Zhang [451], which changes the infinite $[0,+\infty)$ into a finite interval $[0,\rho]$. We still consider the direct control system (9.1.1).

Let

$$W(i\omega):=-c^T(i\omega I_n-A)^{-1}b:=\frac{K(i\omega)}{D(i\omega)}, \tag{9.4.4}$$

where

$$D(i\omega):=\det(i\omega I-A). \tag{9.4.5}$$

Denote

$$X(\omega):=\mathrm{Re}\,W(i\omega)=\frac{\mathrm{Re}\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2} \tag{9.4.6}$$

$$Y(\omega):=I_m W(i\omega)=\frac{I_m\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2}:=\frac{H(\omega)}{|D(i\omega)|^2}, \tag{9.4.7}$$

where

$$H(\omega) = h_{2n} + h_{2n-1}\omega + \cdots + h_1\omega^{2n-1}. \tag{9.4.8}$$

According to the estimation of the boundedness of zeros of polynomials, all the zero points are located within the circle:

$$\|z\| < \rho := 1 + \max_{1\leqslant i\leqslant 2n} |h_i|.$$

LEMMA 9.4.3. *Let $f(\sigma) = \varepsilon\sigma$ $(0 \leqslant \varepsilon \leqslant k)$ in (9.1.1). Then the necessary and sufficient condition for the corresponding linearized system of (9.1.1) to be asymptotically stable is that the frequency characteristic curve $W(i\omega)$ $(0 \leqslant \omega \leqslant \rho)$ and the line $(-\infty, -\frac{1}{k})$ on the real axis has no intersection points.*

PROOF. *Necessity*. By Theorem 9.4.1, the necessary and sufficient condition for the asymptotic stability of the linearized system of (9.1.1) is that the frequency characteristic curve $W(i\omega)$ $(-\infty \leqslant \omega \leqslant \infty)$ and the line $(-\infty, -\frac{1}{k})$ on the real axis has no intersection points. So, the necessity is obvious.

Sufficiency. Since $W(i\omega) = X(\omega) + iY(\omega)$, all the zero points ω_j satisfying $|w_j| < \rho,\ 1 \leqslant j \leqslant 2n - 1$. When $|W| > \rho,\ H(\omega) \neq 0$, i.e., $y(\omega) \neq 0$. By

$$x(\omega) = \frac{K(i\omega)D(-i\omega) + K(-i\omega)D(i\omega)}{2D(i\omega)D(-i\omega)},$$

we get $X(-\omega) = X(\omega)$. So if the curve $W(i\omega)$ $(0 \leqslant \omega \leqslant \rho)$ and the line $(-\infty, -\frac{1}{k})$ have no intersection points, then the curve $W(i\omega)$ $(-\rho \leqslant \omega \leqslant 0)$ and the line $(-\infty, -\frac{1}{k})$ have no intersection points. Hence $W(i\omega)$ $(-\infty \leqslant \omega \leqslant +\infty)$ and the line $(-\infty, -\frac{1}{k})$ do not intersect. The sufficiency is proved. □

In the following, for convenience, we use the notations:

$$X^*(\omega) = \operatorname{Re} W(i\omega) = \frac{\omega I_m\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2} := \frac{A(\omega)}{I_n(\omega)},$$
$$Y^*(\omega) = \omega \operatorname{Im} W(i\omega) = \frac{\omega I_m\{K(i\omega)\bar{D}(i\omega)\}}{|D(i\omega)|^2} := \frac{B(\omega)}{I_n(\omega)}.$$

THEOREM 9.4.4. *If there exists a real number q such that*

$$\operatorname{Re}\bigl\{1 + iq\omega W(i\omega)\bigr\} + \frac{1}{k} > 0, \quad \omega \in [0, \rho], \tag{9.4.9}$$

then the zero solution of system (9.1.1) is absolutely stable within the Hurwitz angle $[0, k]$*, where*

$$\rho = 1 + \max_{2\leqslant i\leqslant 2n+1} \rho_i \quad (i = 2, 3, \ldots, 2n + 1).$$

Here, ρ_i's are the coefficients of the polynomial function:

$$P(\omega) := A(\omega) - qB(\omega) + \frac{1}{k} I_n(\omega). \tag{9.4.10}$$

PROOF. The condition (9.4.9) is equivalent to

$$X^*(\omega) - qy^*(\omega) + \frac{1}{k} > 0, \quad \omega \in [0, \rho], \tag{9.4.11}$$

i.e.,

$$\frac{A(\omega)}{I_n(\omega)} - q\frac{B(\omega)}{I_n(\omega)} + \frac{1}{k} > 0, \quad \omega \in [0, \rho],$$

or

$$\frac{A(\omega) - qB(\omega) + \frac{1}{k} I_n(\omega)}{I_n(\omega)} = \frac{P(\omega)}{I_n(\omega)}, \quad \omega \in [0, \rho]. \tag{9.4.12}$$

By the definition of ρ and the zero points ω_i of the polynomial function $P(\omega)$ with $|\omega_i| < \rho$, we have $P(\omega) \neq 0$ when $\omega \geqslant \rho$ and $P(\rho) > 0$. Hence, $P(\omega) > 0$ when $\omega > \rho$, i.e.,

$$X^*(\omega) - qy^*(\omega) + \frac{1}{k} = \frac{P(\omega)}{E(\omega)} > 0 \quad \text{when } \omega > \rho. \tag{9.4.13}$$

Combining (9.4.8) and (9.4.13) yields

$$\operatorname{Re}(Hiq\omega)W(i\omega) + \frac{1}{k} > 0 \quad \text{for all } \omega \geqslant 0.$$

Hence, by Popov theorem, we know that the zero solution of system (9.1.1) is absolutely stable within the Hurwitz angle $[0, k]$.

The proof of Theorem 9.4.4 is complete. □

However, it is noted in Theorem 9.4.4 that although ρ is independent of q, k, q is a parameter for existence. Determining ρ is still a difficult task. Therefore, we need to further specify the parameters.

For a given $k > 0$, let

$$\begin{aligned}
G(\omega) &:= A(\omega) + \frac{1}{k} I_n(\omega) = x_{2n} + c_{2n-1}\omega + \cdots + c_0\omega^{2n}, \\
B(\omega) &:= b_{2n} + b_{2n-1}\omega + \cdots + b_1\omega^{2n-1} + b_0\omega^{2n}, \\
\rho_1 &= 1 + \max_{1 \leqslant i \leqslant 2n} \left| \frac{c_i}{c_0} \right|, \\
\rho_2 &= 1 + \max_{1 \leqslant i \leqslant 2n} \left| \frac{b_i}{b_0} \right|.
\end{aligned}$$

THEOREM 9.4.5. *(See [417].) If one of the following two conditions holds*:

(1) $b_0 < 0$ (*or* $b_0 = 0,\ b_1 < 1$) *and there exists* $q \geqslant 0$ *such that*

$$\mathrm{Re}\big\{(1 + iq\omega)W(i\omega)\big\} + \frac{1}{k} > 0, \quad \omega \in [0, \rho]; \tag{9.4.14}$$

(2) $b_0 > 0$ (*or* $b_0 = 0,\ b_1 > 1$) *and there exists* $q \leqslant 0$ *such that*

$$\mathrm{Re}\big\{(1 + iq\omega)W(i\omega)\big\} + \frac{1}{k} > 0, \quad \omega \in [0, \rho]; \tag{9.4.15}$$

then the zero solution of system (9.1.1) *is absolutely stable within the Hurwitz angle* $[0, k]$.

PROOF. The condition (9.4.14) is equivalent to

$$\frac{A(\omega) - qB(\omega) + \frac{1}{k}I_n(\omega)}{I_n(\omega)} > 0, \quad \omega \in [0, \rho]. \tag{9.4.16}$$

Thus one only needs to prove that under the condition:

$$A(\omega) - qB(\omega) + \frac{1}{k}I_n(\omega) = G(\omega) - qB(\omega) > 0 \quad \text{for } \omega > \rho, \tag{9.4.17}$$

the conclusion is true.

If condition (1) holds, then by the definition of ρ it must hold $G(\omega) \neq 0$, $B(\omega) \neq 0$ when $\omega > \rho$, and by $b_0 < c$ (or $b_0 = 0,\ b_1 < 0$), when $\omega \gg 1$, $B(\omega) < 0$. However, the degree of $A(\omega)$ is lower than the degree of $E(\omega)$ at least by one. So the coefficient of the highest degree of $G(\omega)$ is the same as that of $\frac{1}{k}$. Thus, when $\omega \gg 1$, $G(\omega) > 0$. Further, we have $G(\omega) > 0$ and $B(\omega) < 0$ when $\omega > \rho$, i.e., (9.4.17) holds. By a similar method, one can show that the condition implies (9.4.17) to be held.

If the condition holds, one can similarly prove (9.4.17).

Theorem 9.4.5 is proved. □

Next, we present a criterion in which ρ is independent of q, k.

Let

$$A(\omega) = a_{2n-1} + a_{2n-2}\omega + \cdots + a_1\omega^{2n-2} + a_o\omega^{2n-1},$$

where

$$a_0, \quad a_1, \quad a_0^2 + a_1^2 \neq 0, \quad \rho_3 = 1 + \max_{1 \leqslant i \leqslant 2n-1}\left|\frac{a_i}{a_0}\right|.$$

THEOREM 9.4.6. *(See [451].) If one of the following two conditions holds*:

(1) $a_0 > 0$ (*or* $a_0 = 0$, $a_1 < 0$), $b_0 < 0$ (*or* $b_0 = 0$, $b_1 > 0$) *and there exists* $q \geqslant 0$ *such that*

$$\mathrm{Re}\big\{(1 + iq\omega)W(i\omega)\big\} + \frac{1}{k} > 0, \quad \forall \omega \in [0, \rho]; \tag{9.4.18}$$

(2) $a_0 < 0$ (*or* $a_0 = 0$, $a_1 < 0$), $b_0 > 0$ (*or* $b_0 = 0$, $b_1 > 0$) *and there exists* $q \leqslant 0$ *such that*

$$\mathrm{Re}\big\{(1 + iq\omega)W(i\omega)\big\} + \frac{1}{k} > 0, \quad \forall \omega \in [0, \rho]; \tag{9.4.19}$$

then the zero solution of system (9.1.1) *is absolutely stable within the Hurwitz angle* $[0, k]$, *where* $\rho = \max\{\rho_1, \rho_2\}$.

PROOF. If condition (1) holds, then (9.4.18) is equivalent to

$$\begin{aligned} &X^*(\omega) - q\bar{y}^*(\omega) + \frac{1}{k} \\ &\quad = \frac{A(\omega)}{I_n(\omega)} - q\frac{B(\omega)}{I_n(\omega)} + \frac{1}{k} > 0, \quad \forall \omega \in [0, \rho]. \end{aligned} \tag{9.4.20}$$

By the definition of ρ, we have $A(\omega) \neq 0$, $B(\omega) < 0$ when $\omega > \rho$. But, on the other hand, condition (1) implies $A(\omega) > 0$, $B(\omega) < 0$ when $\omega > \rho$, so when $\omega > \rho$, $A(\omega) - qB(\omega) > 0$ is true. Thus, when $\omega > \rho$ it holds

$$\frac{A(\omega)}{I_n(\omega)} - q\frac{B(\omega)}{I_n(\omega)} + \frac{1}{k} > \frac{1}{k} > 0. \tag{9.4.21}$$

combining (9.4.18) and (9.4.21) yields

$$\mathrm{Re}\big\{(1 + iq\omega)W(i\omega)\big\} + \frac{1}{k} > 0 \quad \text{when } \omega \geqslant 0,$$

implying that the zero solution of system (9.1.1) is absolutely stable.

If condition (2) holds, by a same argument one can prove that the conclusion is true. □

The improved Popov's criterion described above only requests that the independent variable is varied in the range $[0, \rho]$ for the frequency characteristic curve. Therefore, the computation of the characteristic curve can be executed on a computer system. Although the conditions in Theorems 9.4.5 and 9.4.6 are a little bit stronger than those in the Popov frequency criterion, the former is more convenient in application.

So far, we have introduced the main methods and results of the Lurie problem, namely, the quadratic form V function method with integrals, the S-method based on the V function method, and the Popov frequency domain method. There methods were developed during the same period of time, and motivated the development of new mathematical theory and methodology, such as complex function, mathematical analysis and matrix theory, positive real function theory, etc.

9.5. Simple algebraic criterion

In the previous sections, we introduced several classical methods for determining the derivative of the Lyapunov–Lurie function. However, a common point is observed: whether the V function method, or the Lyapunov Lurie type method combined with the S-method, or the Popov frequency domain method, all the conditions are based on the existence description and are merely sufficient conditions. In particular, when the dimension of a given system is large, the computation demanding increases significantly even with computers. Moreover, note that we must repeat a process so that p is obtained from solving the Lyapunov matrix equation. From the view point of application, practical engineers or designers prefer algebraic expressions since they are easy to verify. Especially, it is convenient for control designs since it enables one to easily consider stability requirement. In this section, we introduce some constructive, explicit algebraic conditions, which are independent of the existence matrices or parameters. For most of the results presented here, we only briefly state the results without giving detailed proof, but provide the necessary references.

We still consider system (9.1.1). Let

$$
\begin{aligned}
g(\sigma) &= \begin{cases} \frac{f(\sigma)}{\sigma}, & \text{when } \sigma \neq 0, \\ 0, & \text{when } \sigma = 0, \end{cases} \\
F_{ij}(\sigma) &:= \big(a_{ij} + b_i c_j g(\sigma)\big).
\end{aligned}
$$

Then (9.1.1) can be rewritten as

$$
\frac{dx_i}{dt} = \sum_{j=1}^{n} F_{ij}(\sigma) x_j. \tag{9.5.1}
$$

Let

$$
\begin{aligned}
&0 \leqslant g(\sigma) \leqslant k < \infty, \\
&\tilde{b}_{ii} = \begin{cases} a_{ii}, & \text{when } b_i c_i \leqslant 0 \\ a_{ii} + b_i c_i k, & \text{when } b_i c_i > 0 \end{cases} \quad (i = 1, 2, \dots, n), \\
&\tilde{b}_{ij} = \tilde{b}_{ji} = \begin{cases} \frac{1}{2} \max\big[|a_{ij} + a_{ji}|, k|b_i c_j + b_j c_i|\big] \\ \quad \text{when } (a_{ij} + a_{ji})(b_i c_j + c_j b_i) \leqslant 0, \quad i \neq j, \\ \frac{1}{2}|a_{ij} + a_{ji} + k(b_i c_j + b_j c_i)| \\ \quad \text{when } (a_{ij} + a_{ji})(b_i c_j + b_j c_i) > 0 \end{cases} \\
&\quad (i = 1, 2, \dots, n).
\end{aligned}
$$

PROPOSITION 9.5.1. *If $\tilde{B}(\tilde{b}_{ij})_{n\times n}$ is a Hurwitz matrix, then the zero solution of system (9.5.1) is absolutely stable within the Hurwitz angle* $[0, k]$.

REMARK 9.5.2. One can use the simple Lyapunov function:

$$V(t) = \sum_{i=1}^{n} x_i^2$$

to prove the proposition.

PROPOSITION 9.5.3. *If the matrix $-\tilde{A}(\tilde{a}_{ij})_{n\times n}$ is an M matrix, then the zero solution of system (9.5.1) is absolutely stable within the Hurwitz angle* $[0, k]$, *where*

$$\tilde{a}_{ii} = \begin{cases} a_{ii}, & \text{for } b_i c_i \leqslant 0, \\ a_{ii} + k b_i c_i, & \text{for } b_i c_i > 0 \end{cases}$$
$$(i = 1, 2, \ldots, n),$$
$$\tilde{a}_{ij} := \begin{cases} \max\{|a_{ij}| k b_i c_j|\}, & \text{for } a_{ij} b_i c_j \leqslant 0,\ i \neq j, \\ |a_{ij} + k|b_i c_j|, & \text{for } a_{ij} b_i c_j > 0 \end{cases}$$
$$(i \neq j,\ i, j = 1, 2, \ldots, n).$$

REMARK 9.5.4. One can employ the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \eta_i |x_i|$$

to complete the proof, where η_i satisfy

$$\eta_j \tilde{a}_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} \tilde{a}_{ij} \eta_i < 0, \quad j = 1, 2, \ldots, n.$$

In the following, we consider a more general case. Without loss of generality, let

$$b_i c_i < 0 \quad (i = 1, 2, \ldots, i_1), \qquad b_i c_i > 0 \quad (i = i_1 + 1, \ldots, i_2),$$
$$b_i = c_i = 0 \quad (i = i_2 + 1, \ldots, i_3), \qquad b_i = 0,$$
$$c_i \neq 0 \quad (i = i_3 + 1, \ldots, i_4),$$
$$b_i \neq 0, \quad c_i = 0 \quad (i = i_4 + 1, \ldots, n,\ 1 \leqslant i_1 \leqslant i_2 \leqslant i_3 \leqslant i_4 \leqslant n).$$

PROPOSITION 9.5.5. *If the matrix $[R(r_{ij})_{n\times n} + S(s_{ij})_{n\times n}]$ is negative definite, then the zero solution of system (9.5.1) is absolutely stable within the Hurwitz* *angle* $[0, k]$, *where*

$$r_{ij} = r_{ji} = \begin{cases} -\frac{c_i}{b_i} a_{ii} & (i = j = 1, 2, \ldots, i_1), \\ \frac{c_i}{b_i} a_{ii} & (i = j = i_1 + 1, \ldots, i_2), \\ a_{ii} & (i = j = i_2 + 1, \ldots, n), \\ \frac{1}{\alpha}\Big[\frac{r_{ii} a_{ij}}{a_{ii}} + \frac{r_{jj} a_{ji}}{a_{jj}}\Big] & (i \neq j,\ i = j = 1, 2, \ldots, n), \end{cases}$$

$$S_{ij} = S_{ji} = \begin{cases} 0 & (i = j = 1, 2, \ldots, i_1, i_1 + 1, \ldots, i_2), \\ 0 & (i \neq j,\ i, j = 1, 2, \ldots, n), \\ 3(i_2 - i_1) k c_i^2 & (i = j = i_2 + 1, \ldots, i_3), \\ \frac{3}{4}(i_4 - i_3) k c_i^2 & (i = j = i_3 + 1, \ldots, i_4), \\ \frac{3}{4}(n - i_4) k b_i^2 & (i = j = i_4 + 1, \ldots, n). \end{cases}$$

REMARK 9.5.6. One may choose the following Lyapunov function:

$$V(x) = -\sum_{i=1}^{i_1} \frac{c_i}{b_i} x_i^2 + \sum_{i=i_1+1}^{i_2} \frac{c_i}{b_i} x_i^2 + \sum_{i=i_2+1}^{n} x_2^2$$

to prove the proposition.

PROPOSITION 9.5.7. *If* $a_{ii} < 0$ $(i = 1, 2, \ldots, n)$, *the matrix* $G(g_{ij})_{n \times n}$ *is negative definite, then the zero solution of system* (9.5.1) *is absolutely stable within the Hurwitz angle* $[0, k]$, *where*

$$g_{ij} = g_{ji} = \begin{cases} -a_{ii}^2 \quad \text{when } b_i c_i \leqslant 0,\ i = j = 1, 2, \ldots, n, \\ -a_{ii}^2 - a_{ii}^2 b_i c_i k \quad \text{when } b_i c_i > 0,\ i = j = 1, 2, \ldots, n, \\ \max\Big[\frac{1}{2}|a_{ii} a_{ij} + a_{jj} a_{ji}|,\ \frac{1}{2} k |b_i c_j a_{ii} + b_j c_i a_{ij}|\Big] \\ \quad \text{when } (a_{ii} a_{ij} + a_{jj} a_{ji})(b_i c_j a_{ij} + b_j c_i a_{ji}) \leqslant 0, \\ \quad i \neq j,\ i, j = 1, 2, \ldots, n, \\ \frac{1}{2}\big|a_{ii} a_{ij} + a_{jj} a_{ji} + k(b_i c_j a_{ij} + b_j a_i a_{ji})\big| \\ \quad \text{when } (a_{ii} a_{ij} + a_{jj} a_{ji})(b_i c_j a_{ii} + b_j c_i a_{ji}) > 0, \\ \quad i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases}$$

REMARK 9.5.8. We take the Lyapunov function:

$$V(t) = -\sum_{i=1}^{n} \frac{a_{ii}}{2} x_i^2$$

to prove that $\frac{dV}{dt}\big|_{(9.5.1)}$ is negative definite. So the conclusion is true.

PROPOSITION 9.5.9. *If there exist constants* $r_i > 0$ $(i = 1, 2, \ldots, n)$ *and* $\alpha > 0$ *such that* $r_i b_i = -\alpha c_i$, $i = 1, 2, \ldots, n$, *and the matrix* $U(u_{ij})_{n \times n}$ *is negative* *definite, then the zero solution of system (9.5.1) is absolutely stable, where*

$$u_{ij} = u_{ji} = \frac{1}{2}(r_i a_{ij} + r_j a_{ji}), \quad i, j = 1, 2, \ldots, n.$$

REMARK 9.5.10. Choosing

$$V(x) = \sum_{i=1}^{n} r_i x_i^2,$$

we can show that $\frac{dV}{dt}|_{(9.5.1)}$ is negative definite, and thus the zero solution of system (9.5.1) is absolutely stable.

In the following, we consider a class of simplified systems, called first standard form:

$$\begin{aligned}
&\frac{dx_i}{dt} = -\rho_i x_i + f(\sigma), \quad \rho_i > 0,\ i = 1, 2, \ldots, n,\\
&\sigma = \sum_{i=1}^{n} c_i x_i, \quad f(0) = 0,\\
&0 < \sigma f(\sigma) \leqslant k\sigma^2, \quad \sigma \neq 0,\ 0 < k \leqslant \infty.
\end{aligned} \tag{9.5.2}$$

Let

$$c_i \begin{cases} > 0 & \text{when } i = 1, \ldots, i_1,\\ = 0 & \text{when } i = i_1 + 1, \ldots, i_2,\\ < 0 & \text{when } i = i_2 + 1, \ldots, n. \end{cases}$$

PROPOSITION 9.5.11. *If $\rho_i > 2k(n - i_2)c_i$, $i = i_2 + 1, \ldots, n$, then the zero solution of system (9.5.2) is absolutely stable within the Hurwitz angle* $[0, k]$.

REMARK 9.5.12. By using the Lyapunov function

$$V(x) = -\sum_{i=1}^{i_1} c_i x_i^2 + \sum_{i=i_1+1}^{i_2} \varepsilon_i x_i^2 + \sum_{i=i_2+1}^{n} c_i x_i^2,$$

where $0 < \varepsilon_i < \frac{2\rho_i}{k(i_2 - i_1)}$, one can easily prove that $\frac{dV}{dt}|_{(9.5.2)}$ is negative definite. So the conclusion is true.

COROLLARY 9.5.13. *If $c_i \leqslant 0$ $(i = 1, 2, \ldots, n)$, then the zero solution of system (9.5.2) is absolutely stable within the Hurwitz angle* $[0, k]$.

COROLLARY 9.5.14. *If $c_i < 0$ $(i = 1, 2, \ldots, n)$, then the zero solution of system (9.5.2) is absolutely stable.*

PROPOSITION 9.5.15. *Suppose the system (9.5.2) is a critical control system. Let* $\rho_i > 0$ $(i = 1, 2, \ldots, n-1)$, $\rho_n = 0$, $c_n < 0$, $c_i < \frac{\rho_i}{2k(n-1)}$, $i = 1, 2, \ldots, n-1$. *Then the zero solution of system (9.5.2) is absolutely stable within the Hurwitz angle* $[0, k]$.

PROOF. Consider a system of quadric inequalities for y_i:

$$k(n-1)y_i^2 + 2\big[k(n-1)c_i - \rho_i\big]y_i + k(n-1)c_i^2 < 0 \quad (i = 1, 2, \ldots, n-1) \tag{9.5.3}$$

which has positive real solutions if and only if

$$c_i < \frac{\rho_i}{2k(n-1)}, \quad i = 1, 2, \ldots, n.$$

Let $y_i = r_i$ $(i = 1, 2, \ldots, n-1)$ be some positive solutions for (9.5.3).

Construct the Lyapunov function:

$$V(x) = \sum_{i=1}^{n-1} r_i x_i^2 - c_n x_n^2.$$

Then, we have

$$\begin{aligned}
G(x) := \frac{dV}{dt}\bigg|_{(9.5.2)} &= -2\sum_{i=1}^{n-1} r_i \rho_i x_i^2 + 2\sum_{i=1}^{n-1} r_i x_i f(\sigma) - 2c_n x_n f(\sigma) \\
&= -2\sum_{i=1}^{n-1} r_i \rho_i x_i^2 + 2\sum_{i=1}^{n-1} (r_i + c_i) x_i f(\sigma) - \sum_{i=1}^{n} 2c_i x_i f(\sigma) \\
&\leqslant -2\sum_{i=1}^{n-1} r_i \rho_i x_i^2 + 2\left|\sqrt{k}\sum_{i=1}^{n-1} (r_i + c_i) x_i\right| x \left|\frac{f(\sigma)}{\sqrt{k}}\right| - 2\sigma f(\sigma) \\
&\leqslant -2\sum_{i=1}^{n-1} r_i \rho_i x_i^2 + k(n-1)\sum_{i=1}^{n-1} (r_i + c_i)^2 x_i^2 + \sigma f(\sigma) - 2\sigma(\sigma) \\
&\leqslant -\sum_{i=1}^{n-1} \big[k(n-1)(r_i + c_i)^2 - 2r_i\rho_i\big] x_i^2 - \sigma f(\sigma) \\
&\leqslant -\sum_{i=1}^{n-1} \big[k(n-1)r_i^2 + 2\big[k(n-1)c_i - \rho_i\big] r_i + K(n-1)c_i^2\big] x_i^2 \\
&\quad - \sigma f(\sigma) := W(x) \\
&\leqslant 0.
\end{aligned}$$

Since $G(0) = W(0) = 0,\ G(x) \leqslant W(x) \leqslant 0$, if there exists $\tilde{x}$ such that $W(\tilde{x}) = 0$, then

$$\sum_{i=1}^{n-1} \tilde{x}_i^2 = 0,$$

and $\sigma f(\sigma) = 0$, i.e., $c_n \tilde{x}_n f(\tilde{x}_n) = 0$. Thus, $c_n \tilde{x}_n = 0$, $W(x)$ is negative definite. Further, we know that $G(x)$ is negative definite, implying that the zero solution of system (9.5.2) is absolutely stable within the Hurwitz angle $[0, k]$. □

COROLLARY 9.5.16. *If $\rho_i > 0$ $(i = 1, 2, \ldots, n-1)$, $\rho_n = 0,\ c_n < 0,\ c_i \leqslant 0$, $i = 1, 2, \ldots, n-1$, then the zero solution of system* (9.5.2) *is absolutely stable within the Hurwitz angle* $[0, k]$.

COROLLARY 9.5.17. *If $\rho_i > 0$ $(i = 1, 2, \ldots, n-1)$, $\rho_n = 0, c_i < 0,\ i = 1, 2, \ldots, n-1$, then the null solution of* (9.5.2) *is absolutely stable.*

Now we turn to consider the algebraic criterion for the Popov method [253,254, 417].

PROPOSITION 9.5.18. *Consider a class of specific control systems in the form* (9.1.1):

$$\begin{cases} \frac{dx}{dt} = Ax + bf(\sigma), \\ \sigma = c^T x, \end{cases} \tag{9.5.4}$$

where

$$A = \begin{bmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 0 & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda \end{bmatrix}, \quad \lambda > 0.$$

Then the zero solution of system (9.5.4) *is absolutely stable if and only if*

$$c^T b \leqslant 0 \quad \textit{and} \quad c^T A^{-1} b \geqslant 0. \tag{9.5.5}$$

PROOF. *Necessity* has been proved in Theorem 9.1.2. We only need to prove *sufficiency*. To show this, suppose there exists constants $q \geqslant 0$ such that

$$\operatorname{Re}\big\{(1 + i\omega q) W(i\omega)\big\} \geqslant 0 \quad \text{for } \omega \geqslant 0, \tag{9.5.6}$$

where

$$W(z) = -c^T (I_n z - A)^{-1} b,$$

then the zero solution of system (9.5.4) is absolutely stable. Rewrite (9.5.6) to an equivalent form:

$$\mathrm{Re}\big\{(1+i\omega q)c^T A_{i\omega}^{-1} b\big\} \leqslant 0 \quad \text{for } \omega \geqslant 0,$$

where $A_{i\omega} = i\omega I_n - A$. Then, we have

$$A_{i\omega} = \begin{bmatrix} i\omega+\lambda & -1 & 0 & \cdots & 0 \\ 0 & i\omega+\lambda & & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & i\omega+\lambda \end{bmatrix},$$

$$A_{i\omega}^{-1} = \begin{bmatrix} \frac{1}{i\omega+\lambda} & \frac{1}{(i\omega+\lambda)^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{i\omega+\lambda} & & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & & & \frac{1}{i\omega+\lambda} \end{bmatrix},$$

$$\begin{aligned} c^T A_{i\omega}^{-1} b &= (c_1, c_2, \ldots, c_n) A_{i\omega}^{-1} (b_1, \ldots, b_n)^T \\ &= \sum_{j=1}^{n} \frac{b_j c_j}{i\omega+\lambda} + \frac{c_1 b_2}{(i\omega+\lambda)^2} \\ &= \frac{c^T b(\lambda - i\omega)}{\lambda^2+\omega^2} + \frac{c_1 b_2(\lambda^2-\omega^2) - 2c_1 b_2 W_i \lambda_i}{(\lambda^2-\omega^2)^2 + 4\omega^2\lambda^2}. \end{aligned}$$

It follows that

$$\begin{aligned} &\mathrm{Re}\big\{(1+i\omega q)c^T A_{i\omega}^{-1} b\big\} \\ &\quad = \frac{c^T b\lambda}{\lambda^2+\omega^2} + \frac{c_1 b_2(\lambda^2-\omega^2)}{(\lambda^2-\omega^2)^2+4\omega^2\lambda^2} + \frac{q\omega^2(c^T b)}{\lambda^2+\omega^2} + \frac{2q\omega^2 c_1 b_2 \lambda}{(\lambda^2-\omega^2)+4\omega^2\lambda^2} \\ &\quad := \frac{F(\omega^3)}{(\lambda^2+\omega^2)[(\lambda^2-\omega^2)^2+4\lambda^2\omega^2]}, \end{aligned}$$

where

$$\begin{aligned} F(\omega^2) &= \big[(\lambda^2-\omega^2)+4\omega^2\lambda^2\big]\big[(c^t b)\lambda + q\omega^2(c^T b)\big] \\ &\quad + \big[\lambda^2+\omega^2\big]\big[c_1 b_2(\lambda^2-\omega^2) + 2q\omega^2 c_1 b_2 \lambda\big] \\ &= (c^T b)\big[(\lambda^4+2\omega^2\lambda^2+\omega^4)\big]\big[\lambda+q\omega^2\big] + c_1 b_2(\lambda^4-\omega^4) \\ &\quad + 2q\lambda^3\omega^2 c_1 b_2 + 2q\omega^4 c_1 b_2 \lambda \\ &= q(c^T b)\omega^6 + \big[(c^T b)\lambda + 2(c^T b)\lambda^2 q - c_1 b_2 + 2q c_1 b_2 \lambda\big]\omega^4 \\ &\quad + \big[(c^T b)\lambda^4 q + 2(c^T b)\lambda^3 + 2q\lambda^3 c_1 b_2\big]\omega^2 + \big[(c^T b)\lambda^5 + c_1 b_2 \lambda^4\big]. \end{aligned}$$

The conditions $c^T b \leqslant 0$ and $-c^T A^{-1} b = \frac{1}{\lambda^2}[(c^b)\lambda + c_1 b_1] \leqslant 0$ imply that the first term and the constant term in $F(\omega^2)$ are not positive. Now, consider the coefficients of the terms ω^4 and ω^2.

(1) If $c_1 b_2 \leqslant 0$, then for any $q \geqslant 0$ it holds $(c^T b)x^4 q + 2(c^T b)\lambda^3 + 2q\lambda^3 a b_2 \leqslant 0$. Choose $q > \frac{1}{2\lambda}$ to get

$$\begin{aligned}&(c^T b)\lambda + 2(c^T b)\lambda^2 q - c_1 b_2 + 2q c_1 b_2 \lambda\\ &\quad = (c^T b)\lambda + 2(c^T b)\lambda^2 q + c_1 b_2 (2q\lambda - 1)\\ &\quad < 0.\end{aligned}$$

Therefore, the coefficients of ω^4 and ω^2 are not positive.

(2) If $a_1 b_2 > 0$, then choose $q = 0$ so that the coefficient of ω^4 term is $(c^T b)\lambda - c_1 b_2 \leqslant 0$, and the coefficient of ω^2 term is $2(c^T b)\lambda^3 \leqslant 0$.

Thus, in any case, one can find $q \geqslant 0$ such that $F(\omega^2) \leqslant 0$. So the zero solution of system (9.5.4) is absolutely stable. □

COROLLARY 9.5.19. *If there exists a similar transformation $x = By$, $B \in R^{n\times n}$ such that system (9.1.1) is transformed into system (9.5.4), then the zero solution of system (9.1.1) is absolutely stable.*

PROOF. We only need to prove that $c^T b$ and $c^T A^{-1} b$ are invariant under the similar transformation. Since system (9.1.1) becomes

$$\frac{dy}{dt} = B^{-1}ABy + B^{-1}bf(c^T By) = \tilde{A}y + \tilde{b}f(\tilde{c}^T y),$$

where $\tilde{A} = B^{-1}AB$, $\tilde{b} = B^{-1}b$, $\tilde{c} = B^T c$, we have

$$\begin{aligned}\tilde{c}^T \tilde{b} &= c^T B B^{-1} b = c^T b,\\ \tilde{c}^T \tilde{A}^{-1} \tilde{b} &= c^T B B^{-1} A^{-1} B B^{-1} b = c^T A^{-1} b.\end{aligned}$$

Therefore, the conclusion is true. □

PROPOSITION 9.5.20. *In system (9.1.1) let*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} -\lambda I_1 & 0 \\ 0 & -\rho I_2 \end{bmatrix}$$

where $\lambda > 0$, $\rho > 0$, $I_1 \in R^{n_1\times n_1}$, $I_2 \in R^{n_2\times n^2}$, $n_1 + n_2 = n$. Then the zero solution of system (9.1.1) is absolutely stable if and only if $c^T b \leqslant 0$ and $c^T A^{-1} b \geqslant 0$.

PROOF. The *necessity* has been proved. We only need to prove the *sufficiency*. Since

$$A_{i\omega}\begin{bmatrix}(i\omega+\lambda)I_1 & 0\\ 0 & (i\omega+\rho)I_2\end{bmatrix}, \qquad A_{i\omega}^{-1}\begin{bmatrix}\frac{1}{i\omega+\lambda}I_1 & 0\\ 0 & \frac{1}{i\omega+\rho}I_2\end{bmatrix},$$

we take

$$\hat{c}_1=\begin{bmatrix}I_1 & 0\\ 0 & 0\end{bmatrix}_{n\times n}c, \qquad \hat{c}_2=\begin{bmatrix}0 & 0\\ 0 & I_n\end{bmatrix}_{n\times n}c,$$

$$\hat{b}_1=\begin{bmatrix}I_1 & 0\\ 0 & 0\end{bmatrix}b, \qquad \hat{b}_2=\begin{bmatrix}0 & 0\\ 0 & I_2\end{bmatrix}b.$$

Then,

$$\begin{aligned}
c^T A_{i\omega}^{-1}b &= \frac{\hat{c}_1^T\hat{b}_1}{i\omega+\lambda}+\frac{\hat{c}_2^T\hat{b}_2}{i\omega+\rho}\\
&= \frac{\hat{c}_1^T\hat{b}_1\lambda-i\hat{c}_1^T\hat{b}_1\omega}{\omega^2+\lambda^2}+\frac{\hat{c}_2^T\hat{b}_2\rho-i\hat{c}_2^T\hat{b}_2\omega}{\omega^2+\rho^2},\\
\mathrm{Re}\big\{(1+i\omega q)c^T A_{i\omega}^{-1}b\big\} &= \frac{\hat{c}_1^T\hat{b}_1\lambda+q\omega^2\hat{c}_1^T\hat{b}_1}{\omega^2+\lambda^2}+\frac{\hat{c}_2^T\hat{b}_2\rho+q\omega^2\tilde{c}_2^T\tilde{b}_2}{\omega^2+\rho^2}\\
&:= \frac{F(\omega^2)}{(\omega^2+\lambda^2)(\omega^2+\rho^2)},
\end{aligned}$$

where

$$\begin{aligned}
F\big(\omega^2\big) &= q\big(\bar{c}_1^T\bar{b}_1+\bar{c}_2^T\bar{b}_2\big)\omega^4+\big[\big(\hat{c}_1^T\bar{b}_1\lambda+\hat{c}_1^T\bar{b}_2\rho\big)\\
&\quad +q\big(\bar{c}_1^T\bar{b}_1\rho^2+\bar{c}_2^T\bar{b}_2\lambda^2\big)\big]\omega^2+\left(\frac{\bar{c}_1^T\bar{b}_1}{\lambda}+\frac{\bar{c}_2^T\bar{b}_2}{\rho}\right)\lambda^2\rho^2.
\end{aligned}$$

By the conditions we have

$$\begin{aligned}
c^T b &= \tilde{c}_1^T\tilde{b}_1+\tilde{c}_2^T\hat{b}_2\leqslant 0,\\
-c^T A^{-1}b &= \frac{\hat{c}_1^T\hat{b}_1}{\lambda}+\frac{\hat{c}_2^T\hat{b}_2}{\rho}\leqslant 0.
\end{aligned}$$

Then it is easy to prove that there exists $q\geqslant 0$ such that

$$\big(\tilde{c}_1^T\tilde{b}_1\lambda+\tilde{c}_2^T\hat{b}_2\rho\big)+q\big(\tilde{c}_1^T\tilde{b}_1\rho^2+\tilde{c}_2^T\hat{b}_2\lambda^2\big)\leqslant 0,$$

i.e., there exists $q\geqslant 0$ satisfying $F(\omega^2)\leqslant 0$. So the zero solution of system (9.1.1) is absolutely stable under the conditions of Proposition 9.5.20. □

PROPOSITION 9.5.21. *Let b a right eigenvector of A and c be the corresponding left eigenvector of A. Then, the zero solution of system* (9.1.1) *is absolutely stable if and only if $c^T b\geqslant 0$.*

PROOF. *Necessity* is obvious, so we only prove *sufficiency*. Let $Ab = -\lambda b$ ($\lambda > 0$). Then

$$(i\omega A)b = [\lambda + i\omega]b, \quad \omega \geqslant 0,$$
$$(i\omega A)^{-1} = \frac{1}{\lambda + i\omega} b,$$

and then $c^T(i\omega I - A)^{-1}b = \frac{c^T b}{i\omega+\lambda}$ holds. By the condition $c^T b \leqslant 0$ we have

$$\mathrm{Re}\{c^T(i\omega I - A)^{-1}b\} = \frac{\lambda c^T b}{\lambda^2 + \omega^2} \leqslant 0.$$

Hence, the zero solution of system (9.1.1) is absolutely stable.

Next, let $A^T c = -\lambda c$ ($\lambda > 0$). Then, we have

$$\left(i\omega I - A^T\right)^{-1} c = \frac{1}{i\omega + \lambda} c,$$
$$\left[\left(i\omega I - A^T\right)^{-1} c\right]^T b = \frac{1}{i\omega + \lambda} c^T b,$$

i.e.,

$$c^T(i\omega I - A)^{-1}b = \frac{c^T b}{i\omega + \lambda}.$$

So the zero solution of system (9.1.1) is absolutely stable. □

EXAMPLE 9.5.22. Consider a 3-dimensional control system:

$$\begin{cases} \frac{dx_1}{dt} = -x_1 + x_2 - x_3 + f(x_1 - x_2 - x_3), \\ \frac{dx_2}{dt} = x_1 - x_2 - x_3 + f(x_1 - x_2 - x_3), \\ \frac{dx_3}{dt} = x_1 + x_2 - 3x_3 + f(x_1 - x_2 - x_3), \end{cases} \tag{9.5.7}$$

with

$$A := \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -3 \end{pmatrix}, \quad b := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad f \in F_\infty.$$

It is easy to cheek that A is stable, b is an eigenvector of A associated with the eigenvalue $\lambda = -1$. Since $c^T b = -1$, the zero solution of system (9.5.5) is absolutely stable.

PROPOSITION 9.5.23. *In system (9.1.1), let*

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_m),$$
$$b = \left(\tilde{b}_1^T, \tilde{b}_2^T, \ldots, \tilde{b}_m^T\right)^T,$$

$$c = \big(\tilde{c}_1^T, \tilde{c}_2^T, \dots, \tilde{c}_m^T\big)^T,$$

where A_r $(r = 1, 2, \dots, m)$ is an $n_r \times n_r$ matrix, and

$$\sum_{r=1}^{m} n_r = m.$$

If $\tilde{b}_r$ is a right eigenvector of A_r and $\tilde{c}_r$ is the corresponding left eigenvector of A_R^T $(r = 1, 2, \dots, m)$, then $\tilde{c}_r^T \tilde{b}_r \leqslant 0$ $(r = 1, 2, \dots, m)$ implies that the zero solution of system (9.1.1) is absolutely stable.

PROOF. Since $A_r\tilde{b}_r = -\lambda\tilde{b}_r$, we have

$$\tilde{c}_r^T (i\omega I_r - A_r)^{-1}\tilde{b}_r = \frac{\lambda\tilde{c}_r^T\tilde{b}_r}{\lambda + i\omega}.$$

Then,

$$\mathrm{Re}\big\{\tilde{c}_r^T (i\omega I_r - A_r)^{-1}\tilde{b}_r\big\} = \frac{\lambda\tilde{c}_r^T\tilde{b}_r}{\lambda^2 + \omega^2}, \quad i = 1, 2, \dots, m.$$

Further, since $A_r^T\tilde{c}_r = -\lambda\tilde{c}$, it holds

$$\begin{aligned}
\tilde{c}_r^T (i\omega I_r - A_r)^{-1}\tilde{b}_r &= \frac{\lambda\tilde{c}_r^T\tilde{b}_r}{i\omega + \lambda}, \\
\mathrm{Re}\big\{\tilde{c}_r^T (i\omega I_r - A_r)^{-1}\tilde{b}_r\big\} &\leqslant 0, \quad r = 1, 2, \dots, m.
\end{aligned}$$

From

$$\begin{aligned}
A &= \mathrm{diag}(A_1, \dots, A_n), \\
A_{i\infty} &= \mathrm{diag}(i\omega I_1 - A_1, i\omega I_2 - A_2, \dots, i\omega I_m - A_m), \\
c^T A_{i\omega}^{-1} b &= \big(\tilde{c}_1^T, \dots, \tilde{c}_m^T\big) \cdot \mathrm{diag}\big((i\omega I_1 - \lambda_1)^{-1}, \dots, (I_m - \lambda_m)^{-1}\big) \\
&\quad \cdot \big(\tilde{b}_1^T, \dots, \tilde{b}_m^T\big)^T \\
&= \sum_{r=1}^{m} \tilde{c}_i^T (i\omega I_r - A_r)^{-1}\tilde{b}_r,
\end{aligned}$$

it follows that

$$\mathrm{Re}\big(c^T A_{i\omega b}^{-1}\big) = \sum_{r=1}^{m} \mathrm{Re}\,\tilde{c}_r^T (i\omega I_r, A_r)^{-1}\tilde{b}_r \leqslant 0.$$

Hence, the zero solution of system (9.1.1) is absolutely stable. $\square$

Now, we consider the first standard form (9.5.2) for real number $q \geqslant 0$. Let

$$a_j = \begin{cases} c_j q & \text{if } c_j(qp_j - 1) > 0, \\ c_j/\rho_j & \text{if } c_j(q\rho_j - 1) < 0, \\ c_j/\rho_j & \text{if } c_j(q\rho_j - 1) = 0,\ c_j \neq 0, \\ 0 & \text{if } c_j(q\rho_j - 1) = 0,\ c_j = 0. \end{cases}$$

PROPOSITION 9.5.24. *If there exists $q \geqslant 0$ such that*

$$\sum_{j=1}^{n} a_j < \frac{1}{k},$$

then the zero solution of system (9.5.2) is absolutely stable within the Hurwitz angle $[0, k]$.

PROOF. Since

$$A = \begin{bmatrix} -\rho_1 & 0 & \cdots & 0 \\ 0 & -\rho_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & -\rho_n \end{bmatrix}, \qquad A_{i\omega}^{-1} = \begin{bmatrix} \frac{1}{i\omega+\rho_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{i\omega+\rho_2} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{i\omega+\rho_n} \end{bmatrix},$$

we have

$$c^T A_{i\omega}^{-1} b = \sum_{j=1}^{n} \frac{c_j}{i\omega + \rho_j},$$

$$\operatorname{Re}\bigl\{(1 + i\omega q)c^T A_{i\omega}^{-1} b\bigr\} = \sum_{j=1}^{n} \frac{c_j(\rho_j + q\omega^2)}{\omega^2 + \rho_j^2}.$$

Let

$$f_j(\omega) = \frac{c_j(\rho_j + q\omega^2)}{\omega^2 + \rho_j^2}.$$

Then, it holds

$$\frac{d}{d\omega} f_j(\omega) = \alpha c_j q\omega\bigl(\omega^2 + \rho_j^2\bigr) - \bigl(c_j\rho_j + c_j q\omega^2\bigr)2\omega = \frac{\alpha c_j\rho_j(q\rho_j - 1)\omega}{(\omega^2 + \rho_j^2)^2}.$$

It can been seen that $\frac{df_j(\omega)}{d\infty} = 0$ only when $\omega = 0$, and $f_j(\omega)$ is monotone increasing in $[0, +\infty)$ when $c_j(q\rho_j - 1) > 0$, so $f_j(\omega) \leqslant f_j(\infty) = \rho_i q$; and $f_i(\omega)$ is monotone decreasing in $[0, +\infty)$ when $c_j(qp_j - 1) < 0$, so $f_j(\omega) \leqslant f_j(0) = c_i/\rho_j$.

Since $\frac{df_j(\omega)}{d\omega} = 0$, when $c_j(q\rho_j - 1) = 0$, $c_j \neq 0$, i.e., $q = 1/\rho_j$, we obtain

$$f_j(\omega) = \frac{c_j(\rho_j + \omega^2/\rho_j)}{\omega^2 + \rho_j^2} = \frac{c_j}{\rho_j}.$$

Thus, when $c_j(q\rho_j - 1) = 0,\ c_j = 0,\ f_i(\omega) = 0$.

Choose α_j such that $f_j(\omega) \leqslant \alpha_j$. Then, from the conditions of the theorem we have

$$\operatorname{Re}\{(1 + i\omega q)c^T A_{i\omega}^{-1} b\} = \sum_{j=1}^{n} \frac{c_j(\rho_j + q\omega^2)}{\omega^2 + \rho_j^2} = \sum_{j=1}^{n} f_j(\omega) \leqslant \sum_{j=1}^{n} a_j < \frac{1}{k},$$

i.e.,

$$\operatorname{Re}\{(1 + i\omega q)c^T A_{i\omega}^{-1} b\} - \frac{1}{k} < 0.$$

Therefore, the zero solution of system (9.5.2) is absolutely stable. □

COROLLARY 9.5.25. *Let*

$$c_j = \begin{cases} < 0 & \textit{for } j = 1, 2, \ldots, i_1, \\ = 0 & \textit{for } j = i_1 + 1, \ldots, i_2, \\ > 0 & \textit{for } j = i_2 + 1, 2, \ldots, n. \end{cases}$$

If

$$\sum_{j=j_2+1}^{n} \frac{c_j}{\rho_j} < \frac{1}{k},$$

then the zero solution of system (9.5.2) *is absolutely stable within the Hurwitz angle* $[0, k]$.

This corollary can be proved by using the approach in proving Proposition 9.5.23 with $q = 0$.

COROLLARY 9.5.26. *If there exists $q \geqslant 0$ such that*

$$\sum_{j=1}^{n} d_j \leqslant 0,$$

then the zero solution of system (9.5.2) *is absolutely stable, where α_j is defined in* (9.5.6).

PROOF. Following the proof of Proposition 9.5.23 one can show that

$$f_j(\omega) \leqslant \alpha_j.$$

Thus, we have

$$\operatorname{Re}\left\{(1+i\omega I)c^T A_{i\omega}^{-1} b\right\} = \sum_{j=1}^{n} f_j(\omega) \leqslant \sum_{j=1}^{n} \alpha_j \leqslant 0. \qquad \square$$

9.6. NASCs of absolute stability for indirect control systems

The main objective in the study of the Lurie problem is to find the necessary and sufficient condition (NASC) for absolute stability. This problem has been studied by many authors. As Burton [46] has pointed out that the major advantage is based on finding NASCs of certain Lyapunov functions which are positive definite with negative definite derivative. However, the approaches introduced in the preceding sections do not necessarily imply NASCs for the Lurie problem, since the Lyapunov function may be poorly chosen. We have studied the Lurie problem and found that it is possible to obtain the NASCs for all solutions of Lurie systems which tend to zero under any nonlinear restriction on $f(\sigma)$.

In this section, we present a method which transforms the Lurie indirect control systems into a nonlinear system with separate variables by a topological transformation. We then introduce a definition of absolute stability with respect to partial variables. The NASCs for absolute stability of Lurie indirect control systems are obtained, and some sufficient conditions are also given.

Consider the mth-order indirect control systems of Lurie type:

$$\begin{cases} \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + h_i\xi, \quad i = 1, 2, \dots, n, \\ \frac{d\xi}{dt} = f(\sigma), \end{cases} \tag{9.6.1}$$

$$\sigma = \sum_{i=1}^{n} c_i x_i - \rho\xi,$$

where a_{ij}, h_i, ρ $(i, j = 1, 2, \dots, n)$ are real constants, $\rho \neq 0$, $y = (y_1, \dots, y_n, \sigma)^T$, $x = (x_1, \dots, x_n, \xi)^T$. By the topological transformation

$$\begin{cases} y_i = x_i, \quad i = 1, 2, \dots, n, \\ y_{n+1} = \sigma = \sum_{j=1}^{n} c_i x_i - \rho\xi. \end{cases} \tag{9.6.2}$$

We can transform system (9.6.1) into a system with separate variables:

$$\frac{dy_i}{dt} = \sum_{j=1}^{n+1} \tilde{a}_{ij} y_j, \quad i = 1, 2, \dots, n,$$

$$\frac{dy_{n+1}}{dt} = \sum_{j=1}^{n+1} \tilde{a}_{n+1,j} y_j - \rho f(y_{n+1}), \tag{9.6.3}$$

where

$$
\begin{aligned}
\tilde{a}_{ij} &= a_{ij} + \frac{h_i}{\rho} c_j \quad (i, j = 1, 2, \dots, n), \\
\tilde{a}_{i,n+1} &= -\frac{h_i}{\rho} \quad (i, j = 1, 2, \dots, n), \\
\tilde{a}_{n+1,j} &= \sum_{i=1}^{n} c_i \tilde{a}_{ij} = \sum_{i=1}^{n} c_i \left(a_{ij} + \frac{h_i c_j}{\rho} \right) \quad (j = 1, 2, \dots, n), \\
\tilde{a}_{n+1,n+1} &= \sum_{i=1}^{n} c_i \tilde{a}_{i,n+1} = \sum_{i=1}^{n} c_i \left(\frac{-h_i}{\rho} \right).
\end{aligned}
\tag{9.6.4}
$$

Obviously, the absolute stabilities of the zero solutions of systems (9.6.1) and (9.6.3) are equivalent.

DEFINITION 9.6.1. The zero solution of system (9.6.3) is absolutely stable with respect to partial variable y_{n+1}, if $\forall f(y_{n+1}) \in F_\infty$, and $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that the solution $y(t, t_0, y_0)$ of system (9.6.3) satisfies $\|y_{n+1}(t, t_0, y_0)\| < \varepsilon$ for $t \geqslant t_0$, when the initial value $\|y_0\| < \delta(\varepsilon)$, $\forall y_0 \in R^{n+1}$, and in addition,

$$
\lim_{t \to +\infty} y_{n+1}(t, t_0, y_0) = 0.
$$

DEFINITION 9.6.2. The zero solution of system (9.6.3) is absolutely stable withe respect to partial variables $y_j, \dots, y_{n+1}$ $(1 < j \leqslant n+1)$, if $\forall f(\sigma) \in F_\infty$, $\forall \varepsilon > 0$, $\forall \delta > 0$ and $\|y_0\| < \delta(\varepsilon)$, the following condition

$$
\sum_{i=j}^{n+1} y_i^2(t, t_0, y_0) < \varepsilon, \quad t \geqslant t_0;
$$

is satisfied, and $\forall y(t_0) \in R^{n+1}$ it holds

$$
\lim_{t \to +\infty} \sum_{i=j}^{n+1} y_i^2(t, t_0, y_0) = 0 \quad \text{with } y_0 = y(t_0). \tag{9.6.5}
$$

THEOREM 9.6.3. *The NASC for the zero solution of system* (9.6.3) *to be absolutely stable is subject to*

(1) *The zero solution of system* (9.6.3) *is absolutely stable with respect to partial variable* y_{n+1};
(2) *The matrix* $B = (b_{ij})_{(n+1)\times(n+1)}$ *is Hurwitz stable, where*

$$
b_{ij} = \begin{cases} \tilde{a}_{n+1,n+1} - \rho & \text{for } i = j = n+1, \\ \tilde{a}_{ij} & \text{otherwise}. \end{cases}
$$

PROOF. *Necessity*. The zero solution is absolutely stable, and so is y_{n+1}. If $f(y_{n+1}) = y_{n+1}$, then system (9.6.3) can be transformed into

$$\frac{dy_1}{dt} = \sum_{j=1}^{n+1} b_{ij} y_j \quad (i = 1, 2, \dots, n+1).$$

Hence, $B(b_{ij})$ is Hurwitz stable, and thus the necessity is proved.

Sufficiency. According to the method of constant variation, the solution of system (9.6.3) is given by

$$y(t) = y(t, t_0, y_0),$$

satisfying

$$y(t) = e^{B(t-t_0)} y(t_0) + \int_{t_0}^{t} e^{B(t-\tau)} \tilde{h}\big[f\big(y_{n+1}(\tau)\big) - y_{n+1}(\tau)\big]\, d\tau, \tag{9.6.6}$$

where $\tilde{h} = (\overbrace{0, \dots, 0}, -\rho)^T$. Since B is stable, there exist constants $\alpha > 0$ and $k \geqslant 1$ such that

$$\big\| e^{B(t-t_0)} \big\| \leqslant k e^{-\alpha(t-t_0)} \quad \text{for } t \geqslant t_0.$$

Because $y_{n+1}(t, t_0, y_0) \to 0$ as $t \to +\infty$, $y_{n+1}(t)$ continuously depends on the initial value y_0, and $f(y_{n+1}(t, t_0, y_0))$ is a compound continuous function of y_0 and $f(y_{n+1}(t, t_0, y_0)) \to 0$ as $t \to +\infty$, so $\forall \varepsilon > 0$, $\exists \varepsilon > 0$, $\exists \delta_1(\varepsilon) > 0$, and $t_1 > t_0$, when $\|y_0\| < \delta_1(\varepsilon)$, we have

$$\int_{t_0}^{t_1} k e^{-\alpha(t-\tau)} \big[\big\|\tilde{h} f\big(y_{n+1}(\tau)\big)\big\| + \big\|\tilde{h} y_{n+1}(\tau)\big\|\big]\, d\tau < \frac{\varepsilon}{3}, \tag{9.6.7}$$

$$\int_{t_1}^{t} k e^{-\alpha(t-\tau)} \big[\big\|\tilde{h} f\big(y_{n+1}(\tau)\big)\big\| + \big\|\tilde{h} y_{n+1}(\tau)\big\|\big]\, d\tau < \frac{\varepsilon}{3}. \tag{9.6.8}$$

Take $\delta_2(\varepsilon) = \frac{\varepsilon}{3k}$, $\delta(\varepsilon) = \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$. For $\|y(t_0)\| < \delta(\varepsilon)$, it follows from (9.6.6)–(9.6.8) that

$$\begin{aligned}\big\|y(t)\big\| \leqslant{}& \big\|e^{B(t-t_0)}\big\| \|y_0\| + \int_{t_0}^{t} \big\|e^{B(t-\tau)} \tilde{h} f\big(y_{n+1}(\tau)\big)\big\| \\ &+ \int_{t_0}^{t} \big\|e^{B(t-\tau)} \tilde{h} y_{n+1}(\tau)\big\|\, d\tau\end{aligned}$$

$$
\begin{aligned}
&\leqslant ke^{-\alpha(t-t_0)}\|y_0\| + \int_{t_0}^{t_1} ke^{-\alpha(t-\tau)}\big\|\tilde{h}f\big(y_{n+1}(\tau)\big)\big\|\,d\tau \\
&\quad + \int_{t_1}^{t} ke^{-\alpha(t-\tau)}\big\|\tilde{h}f\big(y_{n+1}(\tau)\big)\big\|\,d\tau + \int_{t_0}^{t_1} ke^{-\alpha(t-\tau)}\big\|\tilde{h}y_{n+1}(\tau)\big\|\,d\tau \\
&\quad + \int_{t_1}^{t} ke^{-\alpha(t-\tau)}\big\|\tilde{h}y_{n+1}(\tau)\big\|\,d\tau \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
\tag{9.6.9}
$$

Therefore, the zero solution of system (9.6.3) is stable. Further, $\forall y_0 \in R^{n+1}$, by using the L'Hospital rule we have

$$
\begin{aligned}
0 &\leqslant \lim_{t\to+\infty}\big\|y(t)\big\| \leqslant \lim_{t\to+\infty} ke^{-\alpha(t-t_0)} \\
&\quad + \lim_{t\to+\infty}\int_{t_0}^{t} ke^{-\alpha(t-\tau)}\big[\big\|\tilde{h}f\big(y_{n+1}(\tau)\big)\big\| + \big\|\tilde{h}y_{n+1}(\tau)\big\|\big]\,d\tau \\
&= 0 + \lim_{t\to+\infty}\frac{1}{e^{\alpha t}}\int_{t_0}^{t} ke^{\alpha\tau}\big[\big\|\tilde{h}f\big(y_{n+1}(\tau)\big)\big\| + \big\|\tilde{h}y_{n+1}(\tau)\big\|\big]\,d\tau \\
&= 0.
\end{aligned}
\tag{9.6.10}
$$

Hence, the zero solution of system (9.6.3) is absolutely stable.

The proof of Theorem 9.6.3 is complete. □

THEOREM 9.6.4. *The zero solution of system (9.6.3) is absolutely stable if and only if*

(1) *condition* (2) *in Theorem* 9.6.3 *is satisfied*;
(2) *the zero solution of system (9.6.3) is absolutely stable with respect to partial variables* $y_j, \dots, y_{n+1}$ $(1 < j \leqslant n+1)$.

PROOF. Since when condition (1) holds, the absolute stabilities of the zero solution of system (9.6.3) with respect to $y_j, \dots, y_{n+1}$ and y_{n+1} are equivalent. On account of that the absolute stability with respect to partial variable y_{n+1} implies the absolute stability with respect to partial variables $y_1, \dots, y_{n+1}$ (by Theorem 9.6.3), and in particular implies the absolute stability with respect to partial variables $y_j, \dots, y_{n+1}$; while the absolute stability with respect to partial variables $y_j, \ldots, y_{n+1}$ obviously implies the absolute stability with respect to the variable y_{n+1}. Thus, Theorem 9.6.4 is proved. □

Since the matrix $B = \begin{bmatrix} A & h \\ c^T & -\rho \end{bmatrix}$ is Hurwitz stable, B is a nonsingular matrix. By the following nonsingular linear transformation:

$$\begin{aligned} &z = Ax + h\xi, \quad z = (z_1, \ldots, z_n)^T, \\ &z_n = \sum_{i=1}^{n} c_i x_i - \rho\xi, \quad x = (x_1, \ldots, x_n)^T, \end{aligned} \tag{9.6.11}$$

we can transform system (9.6.1) to

$$\begin{aligned} \frac{dz_i}{dt} &= \sum_{j=1}^{n} \tilde{a}_{ij} z_j + \tilde{h}_i f(z_{n+1}), \quad i = 1, 2, \ldots, n, \\ \frac{dz_{n+1}}{dt} &= \sum_{j=1}^{n} c_j z_j - \rho f(z_{n+1}). \end{aligned} \tag{9.6.12}$$

Following the proof of Theorems 9.6.3 and 9.6.4 we have the following results.

THEOREM 9.6.5. *The zero solution of system (9.6.12) is absolutely stable if and only if*

(1) *the zero solution of system (9.6.12) is absolutely stable with respect to partial variable z_{n+1};*
(2) *the matrix $B = \begin{bmatrix} \tilde{A} & \tilde{h} \\ c^T & -\rho \end{bmatrix}$ is Hurwitz stable, where $\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_n)^T$, $c^T = (c_1, \ldots, c_n)$.*

THEOREM 9.6.6. *The zero solution of system (9.6.12) is absolutely stable if and only if*

(1) *the zero solution of system (9.6.12) is absolutely stable with respect to partial variables $z_j, \ldots, z_{n+1}$ $(1 < j \leqslant n+1)$;*
(2) *condition (2) in Theorem 9.6.5 is satisfied.*

Theorems 9.6.5 and 9.6.6 can be proved analogously.

From the above theorems, we can see that the key step is to determine the stability with respect to partial variables. In the following, we present some sufficient conditions for the stability with respect to partial variables. Since $\rho \geqslant 0$ is the necessary condition for the absolute stability, in the following we suppose $\rho > 0$.

THEOREM 9.6.7. *If there exist some constants $\tilde{c}_i$ $(i = 1, 2, \ldots, n+1)$ such that*

$$-\tilde{c}_j\tilde{a}_{jj} \geqslant \sum_{\substack{i=1\\ i\neq 1}}^{n+1} \tilde{c}_i|\tilde{a}_{ij}|, \quad j = 1, 2, \ldots, n+1, \tag{9.6.13}$$

then the zero solution of system (9.6.3) is absolutely stable with respect to partial variable y_{n+1}.

PROOF. Construct the Lyapunov function:

$$V(y) = \sum_{i=1}^{n+1} \tilde{c}_i|y_i|,$$

for $y_{n+1} \neq 0$. Here, $V(y) > 0$ and $V(y)$ is positive definite and radially unbounded for y_{n+1}. Next, we show that

$$\begin{aligned} D^+V|_{(9.6.3)} &\leqslant \sum_{j=1}^{n+1}\Bigg[\tilde{c}_j\tilde{a}_{jj} + \sum_{\substack{i=1\\ i\neq 1}}^{n+1}\tilde{c}_i|\tilde{a}_{ij}|\Bigg]\big|y_j(t)\big| - \rho\tilde{c}_{n+1}\big|f(y_{n+1})\big| \\ &\leqslant -\rho\tilde{c}_{n+1}|f(y_{n+1})| < 0 \quad \text{when } y_{n+1} \neq 0. \end{aligned} \tag{9.6.14}$$

Therefore, $D^+V|_{(9.6.3)}$ is negatively definite for y_{n+1}, indicating that the zero solution of system (9.5.3) is absolutely stable with respect to partial variable y_{n+1}. □

THEOREM 9.6.8. *If there exist some constants $r_i > 0$ $(i = 1, 2, \ldots, n+1)$ such that*

$$\begin{aligned} -r_j\tilde{a}_{jj} &\geqslant \sum_{i=1,\, i\neq j}^{n} r_i|\tilde{a}_{ij}| + r_{n+1}|c_j|, \quad j = 1, 2, \ldots, n, \\ r_{n+1}\rho &> \sum_{i=1}^{n} r_i|\tilde{h}_i|, \end{aligned} \tag{9.6.15}$$

then the zero solution of system (9.6.12) is absolutely stable with respect to partial variable z_{n+1}.

PROOF. We choose the Lyapunov function:

$$V = \sum_{i=1}^{n+1} r_i|z_i|.$$

and then obtain

$$\begin{aligned}
D^+V|_{(9.6.12)} &\leqslant \sum_{j=1}^{n}\Bigg[r_j\tilde{a}_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n}|\tilde{a}_{ij}|r_i + r_{n+1}|c_j|\Bigg]|z_j| \\
&\quad + \Bigg[-\rho r_{n+1} + \sum_{i=1}^{n} r_i|\tilde{h}_i|\Bigg]\big|f(z_{n+1})\big| \\
&\leqslant \Bigg[-\rho r_{n+1} + \sum_{i=1}^{n} r_i|\tilde{h}_i|\Bigg]\big|f(z_{n+1})\big| < 0 \\
&\quad \text{for } z_{n+1} \neq 0. \qquad (9.6.16)
\end{aligned}$$

Consequently, the zero solution of system (9.6.12) is absolutely stable with respect to partial variable z_{n+1}. □

THEOREM 9.6.9. *If there exist some constants $r_i > 0$ $(i = 1, 2, \ldots, n)$ such that*

$$\begin{aligned}
-r_j\tilde{a}_{jj} &\geqslant \sum_{\substack{i=1\\ i\neq j}}^{n}|\tilde{a}_{ij}|r_i + r_{n+1}|c_j|, \quad j = 1, 2, \ldots, n,\ j \neq j_0, \\
r_{n+1}\rho &\geqslant \sum_{i=1}^{n} r_i|\tilde{h}_i|, \\
-r_{j_0}\tilde{a}_{j_0j_0} &> \sum_{\substack{i=1\\ i\neq j_0}}^{n}|\tilde{a}_{ij}|r_i + r_{n+1}|c_{j_0}|,
\end{aligned}$$

then the zero solution of system (9.6.12) is absolutely stable with respect to partial variable z_{i_0}.

PROOF. Construct the Lyapunov function:

$$V = \sum_{i=1}^{n+1} r_i|z_i|,$$

and we then obtain

$$\begin{aligned}
D^+V|_{(9.6.12)} \leqslant{}& \sum_{\substack{j=1\\ j\neq j_0}}^{n}\Bigg[r_j\tilde{a}_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}|r_i + r_{n+1}|c_j|\Bigg]|z_j| \\
&+ \Bigg[-\rho r_{n+1} + \sum_{i=1}^{n} r_i|\tilde{h}_i|\Bigg]\big|f(z_{n+1})\big|
\end{aligned}$$

$$\leqslant \Bigg[r_{j_0}\tilde{a}_{j_0 j_0} + \sum_{\substack{i=1 \\ i \neq j_0}}^{n} |a_{ij_0}| r_i + r_{n+1}|c_{j_0}| \Bigg] |z_{j_0}| < 0$$

$$\text{for } |z_{j_0}| \neq 0. \tag{9.6.17}$$

This shows that the zero solution system (9.6.12) is absolutely stable with respect to partial variable z_{j_0}. The conclusion is true. □

THEOREM 9.6.10. *If there exists a symmetric and positive semi-definite matrix in the form*

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ b_{n1} & \cdots & b_{nn} & 0 \\ 0 & \cdots & 0 & b_{n+1,n+1} \end{bmatrix}, \quad \text{where } b_{n+1,n+1} > 0 \tag{9.6.18}$$

such that $\tilde{A}^T B + B\tilde{A}$ is negative semi-definite, then the zero solution of system (9.6.3) is absolutely stable with respect to partial variable y_{n+1}.

PROOF. Choose the Lyapunov function $V(y) = y^T By$. Then,

$$V(y) \geqslant b_{n+1,n+1} y_{n+1}^2 \to +\infty \quad \text{as } |y_{n+1}| \to \infty,$$

and $V(y) > 0$ for $y_{n+1} \neq 0$. Let

$$\tilde{h} = \big(\overbrace{0, \ldots, 0}^{n}, -\rho\big)^T.$$

Then,

$$\begin{aligned} \frac{dV}{dt}\bigg|_{(9.6.3)} &= y^T By + y^T By \\ &= y^T \tilde{A}^T By + y^T BAy + \big(\tilde{h}^T By + y^T B\tilde{h}\big) f(y_{n+1}) \\ &= y^T \big(\tilde{A}^T B + B\tilde{A}\big) y - \rho b_{n+1,n+1} f(y_{n+1}) y_{n+1} \\ &\leqslant -\rho b_{n+1,n+1} f(y_{n+1}) \cdot y_{n+1} < 0, \quad \text{for } y_{n+1} \neq 0. \end{aligned} \tag{9.6.19}$$

This completes the proof of Theorem 9.6.10. □

THEOREM 9.6.11. *If there exist a constant $\varepsilon > 0$ and the symmetric positive definite matrix B of $(n+1)$-order such that*

$$\begin{bmatrix} \tilde{A}^T B + B\tilde{A} & B\tilde{h} + \frac{1}{2}\tilde{A}_{n+1} + \varepsilon e_n \\ (B\tilde{h} + \frac{1}{2}\tilde{A}_{n+1} + \varepsilon e_n)^T & -\rho \end{bmatrix}$$

is negative semi-definite, where

$$\tilde{A}_{n+1} = (\tilde{a}_{n+1,1}, \ldots, \tilde{a}_{n+1,n+1})^T,$$

$$\tilde{h} = \big(\overbrace{0, \ldots, 0}^{n}, -\rho\big)^T, \qquad e_n = \big(\overbrace{0, \ldots, 0}^{n-1}, 1\big)^T,$$

$$\int\limits_0^{\pm\infty} f(y_{n+1})\, dy_{n+1} = +\infty,$$

then the zero solution system (9.6.3) is absolutely stable with respect to partial variable y_{n+1}.

PROOF. Take a Lyapunov function as follows:

$$V = y^T By + \int\limits_0^{y_{n+1}} f(y_{n+1})\, dy_{n+1}. \tag{9.6.20}$$

Obviously,

$$V(y) \geqslant \int\limits_0^{y_{n+1}} f(y_{n+1})\, dy_{n+1} > 0 \quad \text{for } y_{n+1} \neq 0,$$

and $V(y) \to +\infty$ as $|y_{n+1}| \to +\infty$. Further, we obtain

$$\begin{aligned}
\frac{dV}{dt}\bigg|_{(9.6.3)} &= y^T By + y^T By + \big[\tilde{A}_{n+1} y - \rho f(y_{n+1})\big] f(y_{n+1}) \\
&= \big[\tilde{A} y + \tilde{h} f(y_{n+1})\big]^T By + y^T B\big[\tilde{A} y + \tilde{h} f(y_{n+1})\big] \\
&\quad + \big[\tilde{A}_{n+1} y - \rho f(y_{n+1})\big] f(y_{n+1}) \\
&= y^T \tilde{A} By + y^T B\tilde{A} y + \big[\tilde{h}^T By + y^T B\tilde{h} + \tilde{A}_{n+1} y\big] f(y_{n+1}) \\
&\quad - \rho f^2(y_{n+1}) \\
&= \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \\ f(y_{nn}) \end{pmatrix}^T \begin{bmatrix} \tilde{A}^T B + B\tilde{A} & B\tilde{h} + \frac{1}{2}\tilde{A}_{n+1} + \varepsilon e_n \\ (B\tilde{h} + \frac{1}{2}\tilde{A}_{n+1} + \varepsilon e_n)^T & -\rho \end{bmatrix} \\
&\quad \times \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \\ f(y_{n+1}) \end{pmatrix}
\end{aligned}$$

$$
-\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \\ f(y_{n+1}) \end{pmatrix}^T \begin{bmatrix} 0 & \varepsilon e_n \\ (\varepsilon e_n)^T & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \\ f(y_{n+1}) \end{pmatrix}
$$

$$
\leqslant -2\varepsilon y_{n+1} f(y_{n+1}) < 0 \quad \text{for } y_{n+1} \neq 0. \tag{9.6.21}
$$

Theorem 9.6.11 is proved. □

THEOREM 9.6.12. *Suppose that there exists an $n \times n$ symmetric positive matrix B such that $\tilde{A}^T B + B\tilde{A} = -P$ is negative semi-definite, and there exists a constant $\varepsilon > 0$ such that*

$$
\det \begin{bmatrix} P & -(Bh^* + \frac{1}{2}c) \\ -(Bh^* + \frac{1}{2}c)^T & \rho - \varepsilon \end{bmatrix} \geqslant 0,
$$

$$
\int_0^{\pm\infty} f(z_{n+1})\, dz_{n+1} = +\infty. \tag{9.6.22}
$$

Then the zero solution of system (9.6.12) is absolutely stable with respect to partial variable z_{n+1}, where

$$
h^* = \left(\tilde{h}_1, \dots, \tilde{h}_n\right)^T.
$$

PROOF. Choose the Lyapunov function:

$$
V(x) = z^T B z + \int_0^{x_{n+1}} f(z_{n+1})\, dz_{n+1},
$$

where $z = (z_1, \dots, z_n)^T$. Obviously, $V(z)$ is positively definite for z_{n+1} and $V(z) \to \infty$ as $z_{n+1} \to \infty$. Further, we obtain

$$
\begin{aligned}
\left.\frac{dV(z)}{dt}\right|_{(9.6.12)} &= z^T B z + z^T B z + z_n f(_{n+1}) \\
&= z^T \left(\tilde{A}^T B + B\tilde{A}\right) z + \left(h^{*T} B z + z^T B h^* + c^T z\right) f(z_{n+1}) \\
&\quad - \rho f^2(z_{n+1}) \\
&= -z^T P I + 2 f(z_{n+1}) \left(B h^* + \frac{1}{2} c\right)^T z - \rho f^2(z_{n+1}) \\
&= \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ f(z_{n+1}) \end{pmatrix}^T \begin{bmatrix} -P & (Bh^* + \frac{1}{2}c) \\ (Bh^* + \frac{1}{2}c)^T & -\rho + \varepsilon \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\times \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ f(z_{n+1}) \end{pmatrix} - \varepsilon f^2(z_{n+1}) \\
&\leqslant -\varepsilon f^2(z_{n+1}) < 0.
\end{aligned} \tag{9.6.23}
$$

Therefore, the zero solution of system (9.6.12) is absolutely stable with respect to partial variable z_{n+1}. The proof of Theorem 9.6.12 is finished. □

EXAMPLE 9.6.13. Consider the third-order system:

$$
\begin{cases}
\frac{dx_1}{dt} = 0x_1 - x_2 + f(\sigma), \\
\frac{dx_2}{dt} = x_1 + 0x_2 - f(\sigma), \\
\frac{d\sigma}{dt} = -x_1 + x_2 - \rho f(\sigma),
\end{cases} \tag{9.6.24}
$$

where $\rho > 0$, $f(\sigma) \in F_\infty$) and $\int_0^{\pm\infty} f(\sigma)\, d\sigma = +\infty$, and $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

Given an arbitrary positive definite matrix P, the Lyapunov matrix $A^T B + BA = -P$ does not have positive definite matrix solution, and thus the conventional method fails. Using our method described above, we can prove the stability. To achieve this, we choose the Lyapunov function:

$$
V(x, \sigma) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + \int_0^\sigma f(\sigma)\, d\sigma,
$$

and then obtain

$$
\begin{aligned}
\left.\frac{dV}{dt}\right|_{(9.6.24)} &= -x_1x_2 + x_1 f(\sigma) + x_1x_2 - x_2 f(\sigma) - x_1 f(\sigma) \\
&\quad + x_2 f(\sigma) - \rho f^2(\sigma) \\
&= -\rho f^2(\sigma) < 0 \quad \text{for } \sigma \neq 0.
\end{aligned}
$$

Then, the zero solution of system (9.6.24) is absolutely stable with respect to σ. Let $f(\sigma) \equiv \sigma$. Then, we have the following system:

$$
\begin{cases}
\frac{dx_1}{dt} = 0x_1 - x_2 + \sigma, \\
\frac{dx_2}{dt} = x_1 + 0x_2 - \sigma, \\
\frac{dx_3}{dt} = -x_1 + x_2 - \rho\sigma.
\end{cases} \tag{9.6.25}
$$

The characteristic polynomial of the above system is

$$
\det \begin{bmatrix} \lambda & 1 & -1 \\ -1 & \lambda & 1 \\ 1 & -1 & \lambda + \rho \end{bmatrix} = \lambda^2(\rho + \lambda) + \lambda + \lambda + \lambda + \rho
$$

$$= \lambda^3 + \rho\lambda^2 + 3\lambda + \rho = 0$$

which is a Hurwitz polynomial if and only if

$$\rho > 0, \quad \Delta_1 = 3 > 0, \quad \Delta_2 = \begin{vmatrix} 3 & \rho \\ 1 & \rho \end{vmatrix} = 2\rho > 0.$$

One can verify that when $\rho > 0$, the conditions of Theorem 9.6.3 are satisfied, and therefore the zero solution of system (9.6.24) is absolutely stable.

In the following, we give an application to the second canonical form [226]. Consider the second canonical form of control system:

$$\begin{aligned} \frac{dx_i}{dt} &= -\rho_i x_i + \sigma \quad (i = 1, 2, \dots, n), \\ \frac{d\sigma}{dt} &= \sum_{j=1}^{n} \beta_j x_j - p\sigma - rf(\sigma), \end{aligned} \tag{9.6.26}$$

with constants $p > 0$, $r > 0$, $\rho_i > 0$ $(i = 1, 2, \dots, n)$.

THEOREM 9.6.14. *(See [246].) If*

$$p \geqslant \sum_{i=1}^{n} \left(\frac{1 + \operatorname{sign}(\beta_i)}{2} \right) \frac{\beta_i}{\rho_i}, \tag{9.6.27}$$

then the zero solution of system (9.6.26) is absolutely stable.

PROOF. Choose the Lyapunov function:

$$V(x, \sigma) = \sum_{j=1}^{n} c_i x_i^2 + \sigma^2,$$

where

$$c_i = \begin{cases} -\beta_i & \text{if } \beta_i < 0, \\ \varepsilon_i, \ 0 < \varepsilon_i \ll 1, & \text{if } \beta_i = 0, \\ \beta_i & \text{if } \beta_i > 0. \end{cases}$$

One can show that

$$\left. \frac{dV}{dt} \right|_{(9.6.26)} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}^T \begin{bmatrix} -2c_1\rho_1 & 0 & \cdots & (c_1 + \beta_1) \\ 0 & -2c_1\rho_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & (c_n + \beta_n) \\ (c_1 + \beta_1) & (c_2 + \beta_2) & & -2p \end{bmatrix}$$

$$\times \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \sigma \end{pmatrix} - 2r\sigma f(\sigma)$$
$$\leqslant -2r\sigma f(\sigma) < 0 \quad \text{for } \sigma \neq 0. \tag{9.6.28}$$

Then the zero solution of (9.6.26) is absolutely stable about σ.

On the other hand, let $f(\sigma) = \sigma$ in (9.6.26). Then the following equalities are valid:

$$\begin{aligned} \frac{dx_i}{dt} &= -\rho_i x_i + \sigma, \\ \frac{d\sigma}{dt} &= \sum_{j=1}^{n} \beta_j x_j - (p+r)\sigma. \end{aligned} \tag{9.6.29}$$

Applying the Lyapunov function $V(x, \sigma) = \sum_{i=1}^{n} c_i x_i^2 + \sigma^2$, we have

$$\left.\frac{dV}{dt}\right|_{(9.6.26)} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}^T \begin{bmatrix} -2c_1\rho_1 & 0 & \cdots & (c_1+\beta_1) \\ 0 & -2c_1\rho_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & (c_n+\beta_n) \\ (c_1+\beta_1) & \cdots & -(c_n+\beta_n) & -2p-2r \end{bmatrix}$$
$$\times \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}. \tag{9.6.30}$$

Since

$$D = \frac{1}{2^{n-2}} \begin{vmatrix} 2c_1\rho_1 & 0 & \cdots & -(c_1+\beta_1) \\ 0 & 2c_2\rho_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 2c_n\beta_n & -(c_n\beta_n) \\ -(c_1+\beta_1) & \cdots & -(c_n+\beta_n) & 2p+2r \end{vmatrix}$$
$$= 4\prod_{i=1}^{n} c_i\rho_i(p+r) - \sum_{j=1}^{n}\prod_{j=1}^{n} c_i\rho_i(c_j+\alpha_j)^2$$
$$\geqslant 4\prod_{i=1}^{n} c_i\rho_i r > 0,$$

so $\frac{dV}{dt}|_{(9.6.29)}$ is negative definite. Hence, the coefficient matrix of system (9.6.29) is stable. By Theorem 9.6.3 we know that the zero solution of system (9.6.26) is absolutely stable. The proof is complete. □

As a particular example of system (9.6.26), we consider the equation of the longitudinal motion of an air plane:

$$\begin{aligned} \frac{dx_i}{dt} &= -\rho_i x_i + \sigma, \quad i = 1, 2, 3, 4, \dots, \\ \frac{d\sigma}{dt} &= \sum_{i=}^{n} \beta_i x_i - rp_2\sigma - f(\sigma), \end{aligned} \tag{9.6.31}$$

where $r\rho_2 > 0$, $\rho_i > 0$, $f(\sigma) \in F_\infty$. It is known that some stability parametric region are given as [316,346]

(1)

$$\min \rho_i^2 r^2 p_2^2 - 16\left(\sum_{i=1}^{4} \beta_i^2\right) > 0, \tag{9.6.32}$$

(2)

$$\min_{1\leqslant i\leqslant 4} \rho_i r p_2 - 4 \max_{1\leqslant i\leqslant 4} |\beta_i| > 0, \tag{9.6.33}$$

(3)

$$\min \rho_i^2 r^2 p_2^2 - 4\left(\sum_{i=1}^{4} \beta_i^2\right) > 0. \tag{9.6.34}$$

As a corollary, we have the following result.

COROLLARY 9.6.15. *If*

$$rp_2 \geqslant \sum_{j=1}^{4} \frac{1}{2}\big(1 + \operatorname{sign}(\beta_i)\big)\frac{\beta_i}{\rho_i}, \tag{9.6.35}$$

then the zero solution of system (9.6.27) is absolutely stable.

Let

$$R = \min_{1\leqslant i\leqslant 4} \rho_i r p_i.$$

Then the stability parameter regions defined in (9.6.32)–(9.6.34) have the following geometric interpretations, as shown in Figure 9.6.1.

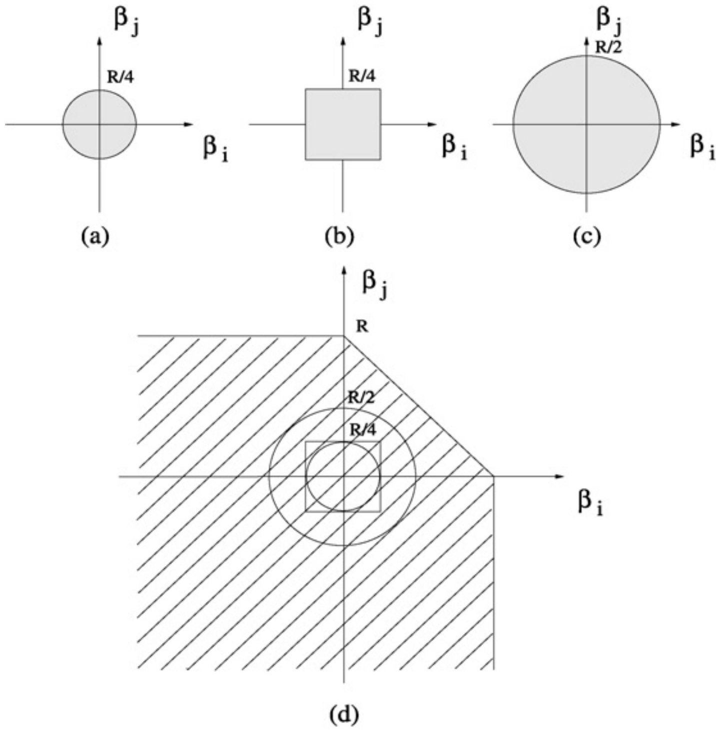

Figure 9.6.1. Stable parameter regimes for (a) $\sum_{i=1}^{4}\beta_i^2 < \frac{R}{4}$; (b) $\max_{1\leqslant i\leqslant 4}|\beta_i| < \frac{R}{4}$; (c) $\max_{1\leqslant i\leqslant 4}|\beta_i| < \frac{R}{2}$; and (d) $rp_2 \geqslant \sum_{i=1}^{4}\frac{1+\operatorname{sign}\beta_i}{2}\frac{\beta_i}{\rho_i}$.

9.7. NASCs of absolute stability for direct and critical control system

In this section, we discuss the necessary and sufficient condition (NASC) of absolute stability for direct and critical Lurie control systems. For most of results, we only present the conclusions, but omit the detailed proofs because the proofs are similar to those given in the previous section. Without loss generality, let $c_{11} \neq 0$. By the topological transformation:

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ c_1 & c_2 & \cdots & \cdots & c_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix},$$

which is simply denoted as $y = Gx$.

We can transform system (9.1.1) to a system with separate variables:

$$
\begin{aligned}
\frac{dy_i}{dt} &= \sum_{j=1}^{n} \tilde{a}_{ij} y_j + \tilde{b}_i f(y_n), \quad i = 1, 2, \ldots, n-1, \\
\frac{dy_n}{dt} &= \sum_{j=1}^{n} \tilde{a}_{nj} y_j + \tilde{b}_n f(y_n),
\end{aligned} \tag{9.7.1}
$$

where $y_n = \sigma$ is an independent variable. The absolute stabilities of the zero solution of (9.1.1) and (9.7.1) are equivalent. Here,

$$
\begin{aligned}
\tilde{a}_{ij} &= \left(a_{ij} - \frac{a_{in}}{c_n} c_j\right) \quad (i, j = 1, 2, \ldots, n-1), \\
\tilde{a}_{in} &= \frac{a_{in}}{c_n} \quad (i = 1, 2, \ldots, n-1), \\
\tilde{a}_{nj} &= \sum_{j=1}^{n} c_i a_{ij} - \sum_{i=1}^{n} c_i \frac{a_{in}}{c_n} c_j \quad (j = 1, 2, \ldots, n-1), \\
\tilde{a}_{nn} &= \frac{\sum_{i=1}^{n} c_i a_{in}}{a_{in}}, \\
\tilde{b}_i &= b_i, \quad i = 1, 2, \ldots, n-1, \quad \tilde{b}_n = \sum_{i=1}^{n} c_i b_i.
\end{aligned} \tag{9.7.2}
$$

THEOREM 9.7.1. *When system (9.1.1) is a direct control system, i.e., A is Hurwitz stable, then the zero solution of system (9.1.1) is absolutely stable if and only if the zero solution of system (9.7.1) is absolutely stable with respect to partial variable y_n.*

PROOF. One can follow the proof of Theorem 9.6.3 to prove this theorem. □

THEOREM 9.7.2. *When A is Hurwitz stable, then the zero solution of system (9.1.1) is absolutely stable it and only if the zero solution of system (9.7.1) is absolutely stable with respect to partial variables $y_j, \ldots, y_n$.*

PROOF. If the zero solution of system (9.7.1) is absolutely stable with respect to partial variables $y_j, \ldots, y_n$ $(1 \leqslant j \leqslant n)$. Particularly, it is absolutely stable with respect to variable y_n. So the conditions of Theorem 9.7.1 are satisfied. On the other hand, if the zero solution of system (9.1.1) is absolutely stable, and in particular, it is absolutely stable with respect to partial variables $x_j, x_{j+1}, \ldots, x_n$, $\sum_{i=1}^{n} c_i x_i$ is absolutely stable. However, since $y_i = x_i$, $i = j, j+1, \ldots, n-1$,

and

$$y_n = \sum_{i=1}^{n} c_i x_i,$$

the zero solution of system (9.7.1) is absolutely stable with respect to partial variables $y_j, \ldots, y_n$. □

THEOREM 9.7.3. *When* $\operatorname{Re}\lambda(A) \leqslant 0$*, the zero solution of system* (9.1.1) *is absolutely stable if and only if*

(1) *the matrix* $A + bc^T$ *is stable*;
(2) *the zero solution of system* (9.7.1) *is absolutely stable with respect to partial variable* y_n.

PROOF. *Necessity*. Take $f(\sigma) = \sigma$. Then system (9.1.1) becomes

$$\frac{dx}{dt} = \left(A + bc^T\right)x. \tag{9.7.3}$$

So $A + bc^T$ must be stable, i.e., condition (1) holds. The absolute stability of the zero solution of system (9.1.1) implies the absolute stability of the zero solution with respect to y_n, i.e., condition (2) holds.

Sufficiency. We rewrite system (9.1.1) as

$$\frac{dx}{dt} = \left(A + bc^T\right)x + bf(\sigma) - b\sigma. \tag{9.7.4}$$

Since the zero solution of system (9.7.1) is absolutely stable with respect to partial variable y_n, $bf(\sigma(t)) - b\sigma(t) \to 0$ as $t \to +\infty$. Since $A + bc^T$ is stable, similar to the proof of Theorem 9.6.3 one can finish the proof of this theorem. □

EXAMPLE 9.7.4. Suppose $A(a_{ij})_{n\times n}$ of (9.1.1) is Hurwitz stable, and the coefficients of (9.7.1) satisfy

$$\sum_{j=1}^{n-1} \tilde{a}_{nj}^2 = 0 \quad \text{and} \quad \tilde{a}_{nn} \leqslant 0; \quad \tilde{b}_n = \sum_{i=1}^{n} c_i b_i < 0,$$

or

$$\sum_{j=1}^{n-1} \tilde{a}_{nj}^2 = 0 \quad \text{and} \quad \tilde{a}_{nn} < 0; \quad \tilde{b}_n = \sum_{i=1}^{n} c_i b_i \leqslant 0.$$

Then the zero solution of system (9.1.1) is absolutely stable.

We can choose the Lyapunov function $V(y) = y_n^2$ to show that

$$\left.\frac{dV}{dt}\right|_{(9.7.1)} = 2\tilde{a}_{nn} y_n^2 + 2\sum_{i=1}^{n} c_i b_i y_n f(y_n) < 0 \quad \text{when } y_n \neq 0.$$

Thus, by Theorem 9.7.2 we know that the conclusion is true.

THEOREM 9.7.5. *When* $\operatorname{Re}\lambda(A) \leqslant 0$, *the zero solution of system* (9.1.1) *is absolutely stable if and only if*

(1) *the zero solution of system* (9.1.1) *is absolutely stable with respect to partial variables* $y_j, y_{j+1}, \ldots, y_n$;
(2) $A + bc^T$ *is stable.*

THEOREM 9.7.6. *When* $\operatorname{Re}\lambda(A) \leqslant 0$, *the zero solution of system* (9.1.1) *is absolutely stable if and only if*

(1) *the zero solution of system* (9.1.1) *is absolutely stable with respect to partial variable* y_n;
(2) *there exists a constant vector* $\eta = (\eta_1, \ldots, \eta_n)^T$ *such that the matrix* $\tilde{B}(\tilde{b}_{ij})_{n\times n}$ *is stable, where*

$$\tilde{b}_{ij} = \begin{cases} \tilde{a}_{ij}, & 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n-1, \\ \tilde{a}_{in} + \eta_i, & 1 \leqslant i \leqslant n,\ j = n. \end{cases}$$

PROOF. *Necessity*. The existence of η is obvious. For example, take $\eta = b$.

Sufficiency. We rewrite system (9.7.1) as

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \tilde{b}_{ij} y_j + \tilde{b}_i f(y_n) - \eta_i y_n, \quad i = 1, 2, \ldots, n.$$

By the method of constant variation formula, the solution $y(t) = y(t, t_0, y_0)$ can be expressed as

$$y(t, t_0, y_0) e^{\tilde{B}(t-t_0)} y_0 + \int_{t_0}^{t} e^{\tilde{B}(t-\tau)} \tilde{b} f\bigl(y_n(\tau)\bigr)\, d\tau - \int_{t_0}^{t} e^{\tilde{B}(t-\tau)} \eta y_n(\tau)\, d\tau.$$

Similar to the proof of Theorem 9.6.3, one can show that the conclusion is true. □

EXAMPLE 9.7.7. If the zero solution of system (9.7.1) is absolutely stable with respect to y_n, and

$$\tilde{A}_{(n-1)} := \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n-1} \\ \vdots & & \vdots \\ \tilde{a}_{n-11} & \cdots & \tilde{a}_{n-1,n-1} \end{bmatrix}_{(n-1)\times(n-1)}$$

is Hurwitz stable, then the zero solution of system (9.7.1) is absolutely stable.

In fact, as an application of Theorem 9.7.6, we take $\eta_i = -\tilde{a}_{in}$ $(1 \leqslant i \leqslant n-1)$, $\eta_n = -\tilde{a}_{nn} - 1$. Thus,

$$\tilde{B}(\tilde{b}_{ij})_{n\times n} = \begin{bmatrix} & \tilde{A}_{n-1} & & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{n,n-1} & -1 \end{bmatrix}_{n\times n}$$

where the matrix $\tilde{A}_{n-1}(\tilde{a}_{ij})_{(n-1)\times(n-1)}$ is stable, implying that $\tilde{B}(\tilde{b}_{ij})_{n\times n}$ is stable. So the conditions of Theorem 9.7.6 are satisfied.

THEOREM 9.7.8. *If the zero solution of system (9.7.1) is absolutely stable with respect to partial variables $y_{i+1}, \dots, y_n$ $(1 \leqslant j \leqslant n)$, and the matrix*

$$\tilde{A}^{(j)} = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1j} \\ \vdots & & \vdots \\ \tilde{a}_{j1} & \cdots & \tilde{a}_{jj} \end{bmatrix}$$

it Hurwitz stable, then the zero solution of system (9.7.1) is absolutely stable.

PROOF. Let

$$\begin{aligned}
y^{(j)}(t) &:= y^{(j)}(t, t_0, y_0) := \bigl(y_1(t_1, t_0, y_0), \dots, y_j(t, t_0, y_0)\bigr)^T, \\
y^{(n-j)}(t) &= y^{(n-j)}(t, t_0, y_0) = \bigl(y_{j+1}(t_1, t_0, y_0), \dots, y_n(t_1, t_0, y_0)\bigr)^T, \\
b^{(j)} &:= \bigl(\tilde{h}_1, \dots, \tilde{b}_j\bigr)^T, \\
b^{(n-j)} &:= \bigl(\tilde{b}_{j+1}, \dots, \tilde{b}_n\bigr)^T, \\
\tilde{A}^{(n-j)} &= \begin{bmatrix} \tilde{a}_{1j+1} & \cdots & \tilde{a}_{1n} \\ \vdots & & \vdots \\ a_{jj+1} & \cdots & \tilde{a}_{jn} \end{bmatrix}.
\end{aligned}$$

The first j components of the solution of system (9.7.1) satisfy

$$\begin{aligned}
y^{(j)}(t) = e^{A_{(j)}(t-t_0)} y^{(j)}(t_0) + \int_{t_0}^{t} e^{A^{(j)}(t-\tau)} \tilde{A}_r^{(n-j)} y^{(n-j)}(\tau)\, d\tau \\
+ \int_{t_0}^{t} e^{A^{(j)}(t-\tau)} A^{(n-j)} f\bigl(y_n(\tau)\bigr)\, d\tau. \qquad (9.7.5)
\end{aligned}$$

Then following the proof for the sufficiency of Theorem 9.7.1, one can prove that the zero solution of system (9.7.1) is absolutely stable. □

In the following, we give some sufficient conditions for absolute stability with respect to partial variables.

THEOREM 9.7.9. *Let $\tilde{A}_n = (\tilde{a}_{n1}, \tilde{a}_{n2}, \ldots, \tilde{a}_{nn})^T$. If there exists a function*

$$V = y^T B y + \int_0^{y_n} f(y_n)\, dy_n$$

which is positive definite and radially unbounded with respect to partial variable y_n, and moreover, there exists a constant $\varepsilon > 0$ such that the matrix

$$\begin{bmatrix} A^T B + B\tilde{A} & B\tilde{b} + \frac{1}{2}\tilde{A}_n + \varepsilon e_n \\ (B\tilde{b} + \frac{1}{2}\tilde{A}_n + \varepsilon e_n)^T & b_n \end{bmatrix}$$

is negative semi-definite, then the zero solution of system (9.7.1) is absolutely stable with respect to partial variable y_n.

PROOF. By employing the Lyapunov function:

$$V(y) = y^T B y + \int_0^{y_n} f(y_n)\, dy_n,$$

we obtain

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(9.7.1)} &= \dot{y}^T B y + y^T B \dot{y} + \big[\tilde{A}_n y + \tilde{b}_n f(y_n)\big] f(y_n) \\
&= \big[\tilde{A} y + \tilde{b}_n f(y_n)\big]^T B y + y^T B\big[\tilde{A} y + \tilde{b} f(y_n)\big] \\
&\quad + \big[\tilde{A}_n y + \tilde{b}_n f(y_n)\big] f(y_n) \\
&= \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ f(y_n) \end{pmatrix}^T \begin{bmatrix} \tilde{A}^T B + B\tilde{A} & B\tilde{b} + \frac{1}{2}A_n + \varepsilon e_n \\ (B\tilde{b} + \frac{1}{2}A_n + \varepsilon e_n)^T & \tilde{b}_n \end{bmatrix} \\
&\quad \times \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ f(y_n) \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ f(y_n) \end{pmatrix}^T \begin{bmatrix} 0 & \varepsilon e_n \\ (\varepsilon e_n)^T & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ f(y_n) \end{pmatrix} \\
&\leqslant -2\varepsilon y_n f(y_n) < 0 \quad \text{when } y_n \neq 0.
\end{aligned}$$

Thus, the zero solution of system (9.7.1) is absolutely stable with respect to y_n. □

Note that $\tilde{b}_n \leqslant 0$ is the necessary condition for absolute stability. Let $b_n < 0$. By a nonsingular linear transformation $z = Hy$, where

$$H := \begin{bmatrix} 1 & 0 & \cdots & \frac{-b_1}{b_n} \\ 0 & 1 & \cdots & \frac{-b_2}{b_n} \\ \vdots & & \ddots & \vdots \\ \vdots & & \cdots & -\frac{b_{n-1}}{b_n} \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

we can transform system (9.7.1) to

$$\begin{aligned} \frac{dz_i}{dt} &= \sum_{j=1}^{n} d_{ij} z_i, \quad i = 1, 2, \ldots, n-1, \\ \frac{dz_n}{dt} &= \sum_{j=1}^{n} d_{nj} z_i + \tilde{b}_n f(z_n), \end{aligned} \tag{9.7.6}$$

where $D(d_{ij}) = H\tilde{A}H^{-1}$. Let $\tilde{b}^* = (0, \ldots, 0, \tilde{b}_n)^T$. We have the following result.

THEOREM 9.7.10. *Assume that there exists a matrix, given in the form*:

$$S = \begin{bmatrix} s_{11} & \cdots & s_{1,n-1} & 0 \\ \vdots & & & \vdots \\ s_{n-1,1} & \cdots & s_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & s_{nn} \end{bmatrix} = S^T,$$

which is positive definite, such that $SD + D^T S$ is negative semi-definite. Then the zero solution of system (9.7.6) is absolutely stable with respect to partial variable z_n.

PROOF. Choose the Lyapunov function $V(t) = z^T S z$, which is positive definite and radially unbounded with respect to partial variable z_n. Further, we have

$$\begin{aligned} \frac{dV}{dt}\bigg|_{(9.7.9)} &= \dot{z}^T s z + z^T S \dot{z} \\ &= z^T D^T S z + z^T S D z + \left(z^T S \tilde{b}^* + \tilde{n}^{*T} S z\right) f(z_n) \\ &= z^T \left(SD + D^T S\right) z + (2\tilde{b}_n s_{nn}) f(z_n) \cdot z_n \\ &\leqslant 2\tilde{b}_n s_{nn} f(z_n) z_n < 0 \quad \text{when } z_n \neq 0. \end{aligned}$$

Thus, the zero solution of system (9.7.6) is absolutely stable with respect to partial variable z_n. $\square$

COROLLARY 9.7.11. *If there exist constants $p_i \geqslant 0$, $i = 1, \ldots, n-1$, $p_n > 0$ such that*

$$PD + P^T P$$

is negative semi-definite, then the zero solution of system (9.7.6) is absolutely stable with respect to partial variable z_n, where

$$P = \operatorname{diag}(p_1, p_2, \ldots, p_n).$$

PROOF. Take the Lyapunov function:

$$V(z) = \sum_{i=1}^{n} p_i z_i^2$$

which is positive definite and radially unbounded with respect to partial variable z_n. Then we have

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(9.7.1)} &= z^T\big(PD + D^T P\big)z + 2p_n\tilde{b}_n f(z_n)z_n \\ &\leqslant 2p_n\tilde{b}_n z_n f(z_n) < 0 \quad \text{when } z_n \neq 0.\end{aligned}$$

Thus, the zero solution of system (9.7.1) is absolutely stable with respect to partial variable z_n. □

THEOREM 9.7.12. *Assume that there exist positive constant ξ_i $(i = 1, 2, \ldots, n)$ such that*

$$-\xi_j\tilde{a}_{jj} \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i|\tilde{a}_{ij}|, \quad j = 1, 2, \ldots, n-1, \tag{9.7.7}$$

$$-\xi_n\tilde{a}_{nn} \geqslant \sum_{i=1}^{n-1} \xi_i|\tilde{a}_{in}|, \tag{9.7.8}$$

$$-\xi_n\tilde{b}_{nn} \geqslant \sum_{i=1}^{n-1} \xi_i|\tilde{b}_i|. \tag{9.7.9}$$

(1) *If the strict inequalities hold in equations (9.7.8) or (9.7.9), then the zero solution of system (9.7.1) is absolutely stable with respect to partial variable y_n.*
(2) *If the $(n - j_0 + 1)$ strict inequalities hold in (9.7.7), let*

$$-\xi_j\tilde{a}_{jj} > \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i|\tilde{a}_{ij}|, \quad j = j_0, j_0 + 1, \ldots, n-1,$$

then the zero solution of system (9.7.1) is absolutely stable with respect to partial variables $y_{j_0}, y_{j_0+1}, \ldots, y_{n-1}$.

PROOF. Choose the Lyapunov function:

$$V(y) = \sum_{i=1}^{n} \xi_i |y_i|$$

which is positive definite and radially unbounded. Then, we have

$$\begin{aligned}
&D^+ V(y)|_{(9.7.1)} \\
&\quad \leqslant \sum_{j=1}^{n} \left[\xi_j \tilde{a}_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \xi_i |\tilde{a}_{ij}| \right] |y_j| + \left[\xi_n \tilde{b}_n + \sum_{i=1}^{n-1} \xi_i |\tilde{b}_i| \right] \big| f(y_n) \big| \\
&\quad \leqslant \sum_{j=1}^{n} \left[\xi_n \tilde{a}_{nn} + \sum_{i=1}^{n-1} |\tilde{a}_{in}| \right] |y_n| + \left[\xi_n \tilde{b}_n + \sum_{i=1}^{n-1} \xi_i |\tilde{b}_i| \right] \big| f(y_n) \big| \\
&\quad < 0 \quad \text{for } y_n \neq 0 \text{ (when condition (1) holds)}, \\
&D^+ V(y)|_{(9.7.1)} \\
&\quad \leqslant \sum_{j=j_0}^{n} \left[\xi_j \tilde{a}_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \xi_i |\tilde{a}_{ij}| \right] |y_j| + \left[\xi_n \tilde{b}_n + \sum_{i=1}^{n-1} \xi_i |\tilde{b}_i| \right] \big| f(y_n) \big| \\
&\quad < 0 \quad \text{for } y_n \neq 0 \text{ (when condition (2) holds)}
\end{aligned}$$

So, the conclusion is true. □

9.8. NASCs of absolute stability for control systems with multiple nonlinear controls

Consider the control system with m nonlinear control terms

$$\begin{cases} \frac{dx}{dt} = Ax + \sum_{j=1}^{m} b_j f_j(\sigma_j), \\ \sigma_j = c_j^T x = \sum_{i=1}^{n} c_{ij} x_i, \quad j = 1, \ldots, m, \end{cases} \tag{9.8.1}$$

where

$$\begin{aligned}
&A \in R^{n \times n}, \quad x = (x_1, \ldots, x_n)^T, \\
&b_j = (b_{1j}, \ldots, b_{nj})^T, \quad c_j = (c_{1j}, \ldots, c_{nj})^T, \\
&f_i \in F = \big\{ f\colon\ f(0) = 0,\ f(\sigma)\sigma > 0,\ \sigma \neq 0,\ f(\sigma) \in \big[(-\infty, +\infty), R^1\big]\big\}, \\
&\quad j = 1, \ldots, m,
\end{aligned}$$

with $\operatorname{Re}\lambda(A) \leqslant 0$. Let

$$\Omega_i = \{x\colon \sigma_i = c_i^T x = 0\}, \quad i = 1, \ldots, m,$$
$$\Omega = \left\{x\colon \|\sigma\| = \sum_{j=1}^{n} |\sigma_j| = \sum_{j=1}^{n} \left|c_j^T x\right| = 0\right\}.$$

DEFINITION 9.8.1. The zero solution of system (9.8.1) is said to be absolutely stable for the set Ω (Ω_j) if for any $f_j(\sigma_j) \in F$ $(j = 1, \ldots, m)$ and any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\|x_0\| < \delta(\varepsilon)$, then the distance from the solution $x(t) := x(t, t_0, x_0)$ to the set Ω (Ω_j) satisfies

$$\rho(x, \Omega) := \sum_{j=1}^{m} \left|c_j^T x(t)\right| < \varepsilon \quad \left(\rho(x, \Omega_j) := \left|c_j^T x(t)\right| < \varepsilon\right)$$

and such that

$$\lim_{t\to+\infty} \sum_{j=1}^{m} \left|c_j^T x(t)\right| = 0 \quad \left(\lim_{t\to+\infty} \left|c_j^T x(t)\right| = 0\right)$$

for every $x_0 \in R^n$.

DEFINITION 9.8.2. The function $V(x) \in C[R^n, R]$ is said to be positive definite with respect to the set Ω (Ω_j) if

$$V(x) \begin{cases} = 0 & \text{for } x \in \Omega, \\ > 0 & \text{for } x \notin \Omega. \end{cases} \quad \left(V(x) = \begin{cases} = 0 & \text{for } x \in \Omega_j, \\ > 0 & \text{for } x \notin \Omega_j. \end{cases}\right)$$

The function $V(x) \in C[R^n, R]$ is said to be negative definite with respect to the set Ω (Ω_j) if $-V(x)$ is positive definite for Ω (Ω_j).

DEFINITION 9.8.3. The function $V(x) \in C[R^n, R]$ is said to be positive definite and radially unbounded for Ω (Ω_j) if $V(x)$ is positive definite for Ω (Ω_j) and $V(x) \to +\infty$ as $\sum_{j=1}^{m} |\sigma_j| \to +\infty$ $(|\sigma_j| \to +\infty)$.

THEOREM 9.8.4. *The necessary and sufficient conditions of absolute stability for the zero solution of system* (9.8.1) *are*

(1) $B = A + \sum_{j=1}^{n} \theta_j b_j c_j^T$ *is Hurwitz stable with* $\theta_j = 1$ *or* $\theta_j = 0$, $j = 1, \ldots, m$;
(2) *the zero solution of system* (9.8.1) *is absolutely stable with respect to* Ω.

PROOF. *Necessity*. (1) In the case $\operatorname{Re}\lambda(A) < 0$, we take $\theta_j = 0$, $j = 1, \ldots, m$, and $B = A$. Then B is obviously Hurwitz stable. In the case $\operatorname{Re}\lambda(A) \leqslant 0$, we take some $\theta_j = 1$. Let $f_j(\sigma_j) = \sigma_j = c_j^T x$ $(j = 1, \dots, m)$. Then system (9.8.1) can be transformed into

$$\frac{dx}{dt} = \left[A + \sum_{j=1}^{m} \theta_j b_j c_j^T\right] x.$$

Therefore,

$$B = A + \sum_{j=1}^{m} \theta_j b_j c_j^T$$

is Hurwitz stable.

(2) For any $\varepsilon > 0$, choose

$$\bar{\varepsilon} = \frac{\varepsilon}{\sum_{j=1}^{m} \|c_j^T\|}.$$

There exists $\delta(\bar{\varepsilon}) > 0$ such that if $\|x_0\| < \delta(\bar{\varepsilon})$, then

$$\|x(t)\| := \|x(t, t_0, x_0)\| := \sum_{j=1}^{m} |x_j(t)| < \bar{\varepsilon} \quad \text{for all } t \geqslant t_0.$$

This implies that

$$\sum_{j=1}^{m} \|c_j^T x(t)\| \leqslant \sum_{j=1}^{m} \|c_j^T\| \|x(t)\| < \sum_{j=1}^{m} \|c_j^T\| \bar{\varepsilon} = \varepsilon \quad \text{for all } t \geqslant t_0.$$

Furthermore, we have $\lim_{t\to+\infty} \|x(t)\| = 0$ for every $x_0 \in R^n$, and thus

$$0 \leqslant \lim_{t\to+\infty} \sum_{j=1}^{m} \|c_j^T x(t)\| \leqslant \sum_{j=1}^{m} \|c_j^T\| \lim_{t\to+\infty} \|x(t)\| = 0.$$

Consequently, the zero solution of system (9.8.1) is absolutely stable for Ω. The necessity is proved.

Sufficiency. In accordance with the method of constant variation, the solution $x(t) := x(t, t_0, x_0)$ of system (9.8.1) satisfies

$$x(t) = e^{B(t-t_0)} x_0 + \int_{t_0}^{t} e^{B(t-\tau)} \left[\sum_{j=1}^{m} b_j f_j\big(\sigma_j(\tau)\big) - \sum_{j=1}^{m} \theta_j b_j \sigma_j(\tau)\right] d\tau.$$

Since B is stable, there exist constants $M \geqslant 1$ and $\alpha > 0$ such that

$$\left\|e^{B(t-t_0)}\right\| \leqslant M e^{-\alpha(t-t_0)} \quad \text{for } t \geqslant t_0.$$

Define $\sigma_j(t) = \sigma_j(t, t_0, x_0)$. Since $\sigma = \sum_{j=1}^{n} |\sigma_j(t)| \to 0$ as $t \to +\infty$, we have $\lim_{t\to+\infty} \sigma_j(t) = 0$. Because $\sigma_j(t)$ continuously depends on the initial value x_0

and $f_j(\sigma_j(t))$ is a composite continuous function of x_0 and $f_j(\sigma_j(t)) \to +\infty$ as $t \to +\infty$, thus for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $t_1 > t_0$ such that $\|x_0\| < \delta(\varepsilon)$ implies that

$$\left\|e^{B(t-t_0)}x_0\right\| \leqslant \left\|e^{B(t-t_0)}\right\| \|x_0\| < \frac{\varepsilon}{3},$$

$$\int_{t_0}^{t_1} Me^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\| + \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau < \frac{\varepsilon}{3},$$

and

$$\int_{t_1}^{t} Me^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\| + \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau < \frac{\varepsilon}{3}.$$

Thus, we have

$$\begin{aligned}
\left\|x(t)\right\| \leqslant{}& \left\|e^{B(t-\tau)}x_0\right\| \\
&+ \int_{t_0}^{t_1} Me^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\| + \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau \\
&+ \int_{t_1}^{t} Me^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\| + \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau \\
<{}& \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

For any $x_0 \in R^n$, by the L'Hospital rule, we get

$$\begin{aligned}
0 &\leqslant \lim_{t\to+\infty}\left\|x(t)\right\| \\
&\leqslant \lim_{t\to+\infty} Me^{-\alpha(t-t_0)} \\
&\quad + \lim_{t\to+\infty}\int_{t_0}^{t} Me^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\| + \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau \\
&= 0.
\end{aligned}$$

Thus, the zero solution of system (9.8.1) is absolutely stable. The proof of the theorem is complete. □

THEOREM 9.8.5. *The zero solution of system* (9.8.1) *is absolutely stable if and only if*

(1) *$A + \sum_{j=1}^{m} \theta_j b_j c_j^T := B$ is Hurwitz stable, where $\theta_j = 0$ or $\theta_j = 1$, $j = 1, \dots, m$;*
(2) *there exists a differential function $V_f \in [R^n, R]$, where $V_f(x)$ is positive definite and radially unbounded for Ω, i.e., there exists $\varphi_f \in KR$ and $\psi_f \in K$ such that*

$$V_f(x) \geqslant \varphi_f(\sigma), \quad \frac{dV}{dt}\bigg|_{(9.8.1)} \leqslant -\psi_f(\sigma). \tag{9.8.2}$$

PROOF. *Sufficiency*. It is suffice to prove that condition (2) implies that the zero solution of system (9.8.1) is absolutely stable about Ω.

Since $V_f(0) = 0, 0 \in \Omega$ and $V_f(x)$ is a continuous function of x, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$V_f(x_0) < \varphi_f(\varepsilon) \quad \text{for } \|x_0\| < \delta(\varepsilon).$$

It follows from (9.8.2) that

$$\varphi_f\big(|\sigma(t)|\big) \leqslant V_f\big(x(t)\big) \leqslant V_f(x_0) \leqslant \varphi_f(\varepsilon),$$

and therefore $|\sigma(t)| < \varepsilon$. Thus, the zero solution of (9.8.1) is stable about Ω.

Now, we prove that $\lim_{t\to+\infty} \sigma(t, t_0, x_0) = 0$ for any $x_0 \in R^n$. Since $V_f(x(t)) := V_f(t)$ is a monotone decreasing and bounded function, we have

$$\inf_{t \geqslant t_0} V_f\big(x(t)\big) := \lim_{t\to+\infty} V_f\big(x(t)\big) := \alpha \geqslant 0.$$

Next, we show that α can be reached only in Ω. If otherwise, suppose it can be reached outside Ω, then there must exists a constant $\beta > 0$ such that $|\sigma(t)| \geqslant \beta > 0$ for $t_0 \leqslant t < +\infty$, or there must exist a sequence $\{t_k\}$ with $t_k \to +\infty$ as $k \to +\infty$ such that $\lim_{t_k\to+\infty} \sigma(t_k) = 0$. As a result,

$$\alpha = \lim_{t_k\to+\infty} V_f(t_k) = \lim_{\substack{t_k\to+\infty \\ s(t_k)=0}} V_f(t_k).$$

In other words, α can be reached in Ω, leading to the contradiction with the presumption that α can reached outside Ω.

For any $x_0 \in R^n$, the expression (9.8.2) gives

$$\big|\sigma(t)\big| \leqslant \big|\sigma(t_0)\big| := h < H < +\infty.$$

We now prove that $\lim_{t\to+\infty} \sigma(t) = 0$. Assume that $\lim_{t\to+\infty} \sigma(t) \neq 0$. Since $\sigma(t)$ is uniformly continuous, there exist constants $\beta > 0$, $\eta > 0$ and point sequence $\{t_j\}$ such that $|\sigma(t)| \geqslant \beta$ for $t \in [t_j - \eta, t_j + \eta]$, $j = 1, 2, \dots$.

Setting $\eta_f = \inf_{\beta \leqslant \sigma} \psi_f(\sigma)$, we deduce

$$0 \leqslant V_f(t) \leqslant V_f(t_0) + \int_{t_0}^{t} \frac{dV_f}{dt}\, dt$$

$$\leqslant V_f(t_0) - \int_{t_0}^{t} \psi_f\big(|\sigma(\tau)|\big)\,d\tau$$

$$\leqslant V_f(t_0) - \sum_{j=1}^{n} \int_{t_j-\psi}^{t_j+\psi} \psi_f\big(|\sigma(\tau)|\big)\,d\tau$$

$$\leqslant V_f(t_0) - 2n\eta\eta_f \to -\infty \quad \text{as } n \to +\infty.$$

This yields a contradiction, and thus $\lim_{t\to+\infty} \sigma(t) = 0$. This proves that the zero solution of (9.8.1) is absolutely stable for Ω. The sufficiency is proved.

Necessity. Suppose the zero solution of system (9.8.1) is absolutely stable. R^n is an attractive space. For any $f_j \in F$ $(j = 1, \ldots, m)$ and any $x \in R^n$, let

$$W_f(x) := \sup\big\{\|x_f(t, 0, x)\|^2,\ t \geqslant 0\big\},$$

where $x_f(t)$ denotes a solution of (9.8.1). From Theorem 9.8.4 of Bhatia and Szegö [29], we know that $W_f(x)$ has the following properties:

(1) $W_f(x) \geqslant 0$, and $W_f(x) = 0$ if and only if $x = 0$, $W_f(x)$ is positive definite and radially unbounded;
(2) $W_f(x)$ is a monotone decreasing function;
(3) $W_f(x)$ is continuous in R^n.

Furthermore, we define

$$V_f(x) := \int_0^{+\infty} W_f\big(x_f(\eta, 0, x)\big)e^{-\eta}\,d\eta. \tag{9.8.3}$$

Obviously, $V_f(x)$ is positive definite and radially unbounded. Thus, there exists $\bar{\varphi}_f \in KR$ such that

$$V_f(x) \geqslant \bar{\varphi}_f\big(\|x\|\big).$$

Let

$$\Phi = \int_0^{t+\eta} W_f\big(x_f(\xi)\big)\,d\xi.$$

It follows that

$$\Phi'_\eta = \Phi'_t = W_f\big(x_f(t+\eta)\big).$$

Integrating (9.8.3) by parts yields

$$V_f\big(x_f(t)\big) = \int_0^{+\infty} e^{-\eta}\,d\Phi$$

$$= e^{-\eta} \int_0^{t+\eta} W_f\big(x_f(\xi)\big)\, d\xi \bigg|_0^{+\infty} + \int_0^{+\infty} \Phi(t+\eta) e^{-\eta}\, d\eta$$

$$= -\int_0^{t} W_f\big(x_f(\xi)\big)\, d\xi + \int_0^{+\infty} \Phi(t+\eta) e^{-\eta}\, d\eta.$$

Since $W_f(x_f(t))$ is a monotone nonincreasing function, $W_f(x_f(t))$ is bounded. Furthermore, we note that

$$\lim_{\eta\to+\infty} e^{-\eta} \int_0^{t+\eta} W_f\big(x_f(\xi)\big)\, d\xi = 0,$$

$$\begin{aligned} \frac{dV_f}{dt}\bigg|_{(9.8.1)} &= -W_f\big(x_f(t)\big) + \int_0^{+\infty} \Phi_t' e^{-\eta}\, d\eta \\ &= -W_f\big(x_f(t)\big) + \int_0^{+\infty} W_f\big(x_f(t+\eta)\big) e^{-\eta}\, d\eta \\ &= \int_0^{+\infty} \big[W_f\big(x_f(t+\eta)\big) - W_f\big(x_f(t)\big)\big]\, d\eta. \end{aligned} \tag{9.8.4}$$

Since $W_f(x_f(t))$ is a monotone nonincreasing function, we obtain

$$W_f\big(x_f(t)\big) \geqslant W_f\big(x_f(t+\eta)\big) \quad \text{for } \eta \geqslant 0.$$

In particular, if $x(t)$ is a nonzero solution of system (9.8.1), we have

$$W_f\big(x_f(t)\big) \neq W_f\big(x_f(t+\eta)\big),$$

or

$$W_f\big(x_f(t)\big) \equiv W_f\big(x_f(t+\eta)\big) \to 0 \quad \text{as } \eta \to +\infty.$$

Thus, $W_f(x_f(t)) \equiv 0$, which contradicts the fact that

$$W_f(x) = \sup\big\{\big\|x_f(t,0,x)\big\|^2,\ t \geqslant 0\big\} \neq 0.$$

Therefore, if $x_f(t) \not\equiv 0$, we have

$$\int_0^{+\infty} \big[W_f\big(x_f(t+\eta)\big) - W_f\big(x_f(t)\big)\big] e^{-\eta}\, d\eta < 0,$$

i.e.,

$$\left.\frac{dV_f}{dt}\right|_{(9.8.1)} < 0 \quad \text{for } x \neq 0.$$

Consequently, we obtain

$$\left.\frac{dV_f}{dt}\right|_{(9.8.1)} \leqslant -u_f(x),$$

with $u_f(x)$ being a positive definite function. Thus, we have

$$\begin{aligned}
u_f(x) &\geqslant \bar{\varphi}_f\big(\|x\|\big) := \tilde{\varphi}_f\left(\sum_{i=1}^n |x_i|\right) \\
&\geqslant \bar{\varphi}_f\left(\frac{1}{m}\sum_{j=1}^n \frac{1}{\max_{1\leqslant i,j\leqslant n}|c_{ij}|}\sum_{i,j=1}^n |c_{ij}x_j|\right) \\
&\geqslant \bar{\varphi}_f\left(\frac{1}{m}\frac{1}{\max_{1\leqslant i,j\leqslant n}|c_{ij}|}\sum_{i,j=1}^n |c_{ij}x_j|\right) \\
&:= \varphi_f(\sigma) \in KR. \qquad (9.8.5)
\end{aligned}$$

Hence, $u_f(x)$ is positive definite and radially unbounded for Ω. Further, we can show that

$$\left.\frac{dV_f}{dt}\right|_{(9.8.1)} \leqslant -u_f(x) \leqslant -\bar{\varphi}_f\big(\|x\|\big) \leqslant -\varphi_f(\sigma) \quad \text{for } \varphi_f \in K,$$

indicating that condition (2) of Theorem 9.8.4 is satisfied. Satisfactory of condition (1) of this theorem is trivial. The necessity is proved. □

THEOREM 9.8.6. *The zero solution of system (9.8.1) is absolutely stable if and only if*

(1) *condition* (1) *in Theorem* 9.8.4 *is satisfied*;
(2) *for any* $f_j \in F$ $(j = 1, \dots, m)$, *there exist* m *Lyapunov functions* $V_j(x) \in [R^n, R]$ $(j = 1, \dots, m)$ *such that*

$$V_j(x) \geqslant \varphi_j\big(|\sigma_j|\big), \quad \varphi_j \in KR,\ j = 1, \dots, m,$$
$$\left.\frac{dV_j}{dt}\right|_{(9.8.1)} \leqslant -\psi_j\big(|\sigma_j|\big), \quad \psi_j \in K,\ j = 1, \dots, m.$$

PROOF. *Necessity*. Theorem 9.8.5 guarantees that the condition (1) is satisfied, and that there exists $V_f(x) \geqslant \varphi_f(\sigma) \in KR$ such that

$$\left.\frac{dV_f(x)}{dt}\right|_{(9.8.1)} \leqslant -\psi_f(\sigma) \quad \text{for } \varphi_f \in K.$$

Take $V_j = V_f(x)$, $j = 1, \dots, m$. Due to that

$$V_j = V_f(x) \geqslant \varphi_f(\sigma) = \varphi_f\left(\sum_{j=1}^{m} \left|c_j^T x\right|\right) \geqslant \varphi_f\left(\left|c_j^T x\right|\right)$$

$$= \varphi_f\left(|\sigma_j|\right) \in KR, \quad j = 1, \dots, m,$$

we have

$$\left.\frac{dV_j}{dt}\right|_{(9.8.1)} = \left.\frac{dV_f}{dt}\right|_{(9.8.1)} \leqslant -\psi_f(\sigma) = -\psi_f\left(\sum_{j=1}^{m} \left|c_j^T x\right|\right)$$

$$\leqslant -\psi_f\left(\left|c_j^T x\right|\right) = -\psi_j(\sigma_j), \quad \sigma_j \in K, j = 1, \dots, m.$$

This verifies the necessity.

Sufficiency. Similar to the proof for the sufficiency of Theorem 9.8.5, condition (2) implies that the zero solution of system (9.8.1) is absolutely stable about Ω_j, $j = 1, \dots, m$. Then, as in the proof for Theorem 9.8.4 one can show that the zero solution of system (9.8.1) is absolutely stable. This verifies the sufficiency. □

THEOREM 9.8.7. *If the following conditions are satisfied*:

(1) *condition* (1) *in Theorem* 9.8.4 *holds*;
(2) *there exist an* $n \times n$ *real symmetric matrix* B *and constants* $\beta_i \geqslant 0$ ($i = 1, \dots, m$), $\alpha > 0$ *such that*

$$V(x) = x^T B x + \sum_{j=1}^{m} \beta_j \int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j \tag{9.8.6}$$

with

$$x^T B x \geqslant \alpha \sum_{i=1}^{n} x_i^2$$

or

$$V(x) \geqslant \sum_{j=1}^{m} \beta_i \int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j, \quad \beta_j > 0,$$

$$\int_0^{+\infty} f(\sigma_j)\, d\sigma_j = +\infty, \quad j = 1, \dots, m;$$

(3)

$$\left.\frac{dV}{dt}\right|_{(9.8.1)} \leqslant -\varepsilon\tau, \quad \tau \in \left\{\sigma^2, \sum_{j=1}^{m} \sigma_j f_j(\sigma_j), \sum_{j=1}^{m} f_j^2(\sigma_j)\right\};$$

then the zero solution of system (9.8.1) is absolutely stable.

PROOF. It is suffice to prove that the conditions (2) and (3) of this theorem imply the condition of Theorem 9.8.4.

In fact, for the Lyapunov function:

$$V(x) = x^T Bx + \sum_{j=1}^{m} \beta_j \int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j, \tag{9.8.7}$$

condition (2) implies that there exists $0 < \xi \ll 1$ such that

$$V(x) \geqslant \xi \sum_{i=1}^{m} \sigma_j^2 + \sum_{j=1}^{m} \beta_j \int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j = \varphi(\sigma) \in KR$$

or

$$V(x) \geqslant \sum_{j=1}^{m} \beta_j \int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j := \varphi_1(\sigma) \in KR$$

with

$$\left.\frac{dV}{dt}\right|_{(9.8.1)} = -\varepsilon\tau := -\psi(\sigma) \quad \text{for } \psi \in K.$$

Therefore, the conditions of Theorem 9.8.4 are satisfied, and Theorem 9.8.7 is thus proved. □

COROLLARY 9.8.8. *Suppose that there exist constants $\beta_j \geqslant 0$ $(j = 1, \ldots, m)$ and a symmetric positive definite matrix P such that the function*

$$V(x) = x^T Px + \sum_{j=1}^{m} \beta_j \int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j \tag{9.8.8}$$

satisfies $\frac{dV}{dt}|_{(9.8.1)} < 0$, $x \neq 0$. Then the zero solution of system (9.8.1) is absolutely stable.

PROOF. It is suffice to prove that $V(x)$ is positive definite and radially unbounded for Ω, while $\frac{dV}{dt}|_{(9.8.1)}$ is negative definite for Ω.

Let

$$\bar{c} = \max_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} |c_{ij}|, \qquad \underline{\lambda} = \min_{1\leqslant i\leqslant n} \lambda_i(P),$$

λ_i being the eigenvalues of P. Then

$$V(x) \geqslant x^T P x \geqslant \underline{\lambda} x^T x \geqslant \underline{\lambda} \frac{\sum_{j=1}^m \sigma_j^2}{nm\bar{c}} := \varphi\big(|\sigma|\big) \in KR.$$

Therefore, $V(x)$ is positive definite and radially unbounded for Ω, and

$$\frac{dV}{dt}\bigg|_{(9.8.1)} \leqslant -\psi\big(\|x\|\big) \leqslant -\psi\left(\frac{1}{m\bar{c}} \sum_{j=1}^m |\sigma_j|\right) := -\psi_1(\sigma) \quad \text{for } \psi_1 \in K.$$

Thus, $\frac{dV}{dt}|_{(9.8.1)}$ is negative definite for Ω. The conditions of Theorem 9.8.6 are satisfied. The proof of Corollary 9.8.8 is complete. □

To end this section, we present some simple sufficient conditions for absolute stability. Without loss of generality, we assume that $c_i = (c_{i1}, \ldots, c_{in})^T$ $(i = 1, \ldots, m)$ are linearly independent. By an n-dimensional nonsingular linear transformation, system (9.8.1) can be transformed into the following form:

$$\frac{dx}{dt} = Ax + \sum_{j=n-m+1}^{n} b_j f_j(x_j), \tag{9.8.9}$$

or in the vector component form:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=n-m+1}^{n} b_{ij} f_j(x_j), \quad j = 1, \ldots, m. \tag{9.8.10}$$

THEOREM 9.8.9. *Suppose that*

(1) $A = (a_{ij})_{n\times n}$ *is a Hurwitz matrix*;
(2) *there exist constants* $r_i \geqslant 0$ $(i = 1, \ldots, n-m)$, $r_i > 0$ $(i = n-m+1, \ldots, n)$ *such that*

$$\begin{cases} -r_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n r_i |a_{ij}| \leqslant 0, & j = 1, \ldots, n-m, \\ -r_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n r_i |a_{ij}| < 0, & j = n-m+1, \ldots, n, \\ -r_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n r_i |b_{ij}| \leqslant 0, & j = n-m+1, \ldots, n, \end{cases}$$

or

$$\begin{cases} -c_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n c_i|a_{ij}| \leqslant 0, & j = 1,\dots,n-m, \\ -c_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n c_i|a_{ij}| \leqslant 0, & j = n-m+1,\dots,n, \\ -c_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n c_i|b_{ij}| < 0, & j = n-m+1,\dots,n. \end{cases}$$

Then the zero solution of system (9.8.9) is absolutely stable.

PROOF. Construct the Lyapunov function:

$$V = \sum_{i=1}^n c_i|x_i|.$$

Obviously, we have

$$V = \sum_{i=1}^n c_i|x_i| \geqslant \sum_{i=n-m+1}^n c_i|x_i| \to +\infty \quad \text{as} \quad \sum_{i=n-m+1}^n |x_i| \to +\infty.$$

Thus, V is positive definite and radially unbounded for $x_{n-m+1},\dots,x_n$. Since

$$\begin{aligned} D^+V|_{(9.8.9)} &\leqslant \sum_{j=1}^n \Bigg[c_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^n c_i|a_{ij}| \Bigg] |x_i| \\ &\quad + \sum_{i=n-m+1}^n \Bigg[c_i b_{ii} + \sum_{\substack{i=1\\ i\neq j}}^n |c_i b_j| \Bigg] \big|f_i(x_i)\big| \\ &< 0 \quad \text{for} \sum_{j=n-m+1}^n |x_j| \neq 0. \end{aligned}$$

Thus, the zero solution of system (9.8.9) is absolutely stable with respect to partial variables $x_{n-m+1},\dots,x_n$. Since matrix A is stable, there exist $M \geqslant 1$ and $\alpha > 0$ such that

$$\big\| e^{A(t-t_0)} \big\| \leqslant M e^{\alpha(t-t_0)}.$$

The solution of system (9.8.9) can be expressed in the following form:

$$x(t,t_0,x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)} \sum_{j=1}^m b_j f_j\big(x_{n-m+k}(\tau)\big)\,d\tau.$$

Following the proof of Theorem 9.8.4, we can prove that the zero solution of system (9.8.9) is absolutely stable. $\square$

EXAMPLE 9.8.10. Consider the problem of absolute stability of the system:

$$\begin{cases} \frac{dx_1}{dt} = -x_1 - 2f_1(x_1) + 2f_2(x_2), \\ \frac{dx_2}{dt} = -x_2 + 2f_1(x_1) - 2f_2(x_2), \end{cases} \quad f_1, f_2 \in F. \tag{9.8.11}$$

We choose the Lyapunov function:

$$V = |x_1| + |x_2|,$$

and then obtain

$$\begin{aligned} D^+V(x)|_{(9.8.11)} &\leqslant -|x_1| - |x_2| + [-2+2]|f_1(x_1)| + [-2+2]\big|f_2(x_2)\big| \\ &\leqslant -|x_1| - |x_2| < 0 \quad \text{for } x \neq 0. \end{aligned}$$

Therefore, the zero solution of system (9.8.11) is absolutely stable.

9.9. NASCs of absolute stability for systems with feedback loops

Consider the general control system with multi-nonlinear and loop feedback terms:

$$\begin{cases} \frac{dx}{dt} = Ax + \sum_{j=1}^m b_j f_j(\sigma_j), \\ \sigma_j = c_j^T x - d_j f_j(\sigma_j), \end{cases} \tag{9.9.1}$$

where $A \in R^{n\times n}$, $\operatorname{Re}\lambda(A) \leqslant 0$, $x, b_j, c_j \in R^n$.

Let F, Ω and Ω_i be defined as that in Section 9.8. The definitions for the absolute stability of the zero solution of system with respect to Ω, Ω_i, and the definition for $V(x)$ being positive definite and radially unbounded with respect to set Ω, Ω_i are given below.

DEFINITION 9.9.1. The zero solution of system (9.9.1) is said to be absolutely stable about the variable σ (σ_j), if $\forall f_j \in F$ and $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$\|\sigma\| = \sum_{j=1}^m |\sigma_j| < \varepsilon \quad \big(|\sigma_j| < \varepsilon\big)$$

when $\|x_0\| < \delta(\varepsilon)$, and

$$\lim_{t\to+\infty} \sum_{j=1}^m |\sigma_j| = 0 \quad \big(\lim_{t\to+\infty} |\sigma_j| = 0\big),$$

$$\forall x_0 \in R^n,\ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)^T.$$

DEFINITION 9.9.2. The function $V \in C[R^n, R]$ is said to be positive definite with respect to variable σ (σ_j) if

$$V(\sigma)\begin{cases} = 0 & \text{for } \sigma = 0 \\ > 0 & \text{for } \sigma \neq 0 \end{cases} \quad \left(V(\sigma_j)\begin{cases} = 0 & \text{for } \sigma_j = 0 \\ > 0 & \text{for } \sigma_j \neq 0 \end{cases}\right).$$

DEFINITION 9.9.3. The function $V \in C[R^n, R]$ is said to be positive definite and radially unbounded with respect to the variable σ (σ_j), if $V(\sigma)$ is positive definite for σ (σ_j), and $V(x, \sigma) \to \infty$ as $\sum_{j=1}^m |\sigma_j| \to \infty$ $(|\sigma_j| \to \infty)$, whereas $V(\sigma)$ is negative definite with respect to σ (σ_j) if $-V(\sigma)$ is positive definite for σ (σ_j).

REMARK 9.9.4. Verifying the conditions of Definitions 9.9.2 and 9.9.3 are very difficulty, because $\sigma_j + d_j f_j(\sigma_j) = 0$ is an implicit formula.

Assume that $|\sigma_j - d_j f_j(\sigma_j)|$ is positive definite for σ. Obviously. If any one of the following conditions is satisfied:

(1) $d_j \geqslant 0$ (which was considered by Gan and Ge [128]);

(2) $d_j < 0$ but $|f_j(\sigma_j)| < \frac{1}{|d_j|}|\sigma_j|$, $\sigma_j \neq 0$;

(3) $d_j < 0$ but $|f_j(\sigma_j)| > \frac{1}{|d_j|}|\sigma_j|$, $\sigma_j \neq 0$;

then $|\sigma_j - d_j f_j(\sigma_j)|$ is positive definite for σ.

THEOREM 9.9.5. *The zero solution of system (9.9.1) is absolutely stable if and only if it is absolutely stable for $G(G_i)$.*

PROOF. Let the zero solution of (9.9.1) be absolutely stable for σ. $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, when $|\sigma_0| < \delta$, we have

$$\left|\sigma_j(t, t_0, \sigma_0)\right| < \frac{\varepsilon}{2m} \quad \text{and} \quad \left|d_f f_j\big(\sigma_j(t, t_0, \sigma_0)\big)\right| < \frac{\varepsilon}{2m}.$$

This is possible because $f_j \in F$, $f_j(0) = 0$, and $\forall x_0 \in R^n$,

$$\begin{aligned}\lim_{t\to+\infty}\sum_{i=1}^m\left|c_j^T x_j(t, t_0, x_0)\right| &\leqslant \lim_{t\to+\infty}\sum_{j=1}^m\left|\sigma_j(t, t_0, \sigma_0)\right| \\ &\quad + \lim_{t\to+\infty}\sum_{j=1}^m\left|d_j f_j\big(\sigma_j(t, t_0, \sigma_0)\big)\right| \\ &= 0.\end{aligned}$$

So the zero solution of system (9.9.1) is absolutely stable for $G(G_i)$.

Otherwise, let the zero solution of system (9.9.1) be absolutely stable for Ω $(\Omega_j) = 0$. Since $|\sigma_j - d_j f_j(\sigma_j)|$ is positive definite, according to the equivalence relation of positive definite function and K-class function, we know that there exists $\varphi(|\sigma_j|) \in K$ such that

$$\varphi_j(|\sigma_j|) \leqslant \left|\sigma_j - d_j f_j(\sigma_j)\right| = \left|c_j^T x(t, t_0, x_0)\right|,$$

and so

$$\left|\sigma_j(t, t_0, \sigma_0)\right| < \varphi_j^{-1}\left(\left|c_j^T x(t, t_0, x_0)\right|\right).$$

Then $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, when $|x_0| < \delta(\varepsilon)$, it holds

$$\varphi_j^{-1}\left|c_j^T x(t, t_0, x_0)\right| < \frac{\varepsilon}{m}.$$

Hence,

$$\left|\sigma(t, t_0, \sigma_0)\right| \leqslant \sum_{j=1}^{m}\left|\sigma_j(t, t_0, \sigma_0)\right| < \sum_{j=1}^{m} \varphi_j^{-1}\left|c_j^T x(t, t_0, x_0)\right| < \varepsilon.$$

Further, $\forall x_0 \in R^n$ we have

$$\begin{aligned}\left|\sigma(t, t_0, \sigma_0)\right| &< \sum_{j=1}^{m}\left|\sigma_j(t, t_0, \sigma_0)\right| \\ &< \sum_{j=1}^{m} \varphi_j^{-1}\left(\left|c_j^T x(t, t_0, x_0)\right|\right) \to 0 \quad \text{as } t \to +\infty.\end{aligned}$$

Therefore, the zero solution of system (9.9.1) is absolutely stable with respect to σ (σ_j). □

REMARK 9.9.6. Theorem 9.9.5 was improved and generalized by Gan and Ge [128].

THEOREM 9.9.7. *The NASCs of the zero solution of system* (9.9.1) *are*:

(1) $B := A + \sum_{i=1}^{m} \theta_i b_i c_i^T$ *is a Hurwitz matrix, where*

$$\begin{cases} \theta_i = \frac{1}{2} & \text{when } d_i = 0, \\ \theta_i = \frac{1}{3d_i} & \text{when } d_i \neq 0, \end{cases} \qquad i = 1, 2, \dots, n;$$

(2) *the zero solution of system* (9.9.1) *is absolutely stable with respect to* Ω.

PROOF. *Sufficiency*. When $d_j \neq 0$, we take $f_j(\sigma_j) = \frac{1}{2d_j}\delta_j$; when $d_j = 0$ take $f_j(\sigma_j) = \frac{1}{2}C_j^T x$. Obviously, $|\sigma_j + d_j f_j(\sigma_j)|$ is positive definite with respect to σ_j.

By the formula of constant variation, the solution $x(t) = x(t, t_0, x_0)$ of system (9.9.1) can be expressed as

$$x(t) = e^{B(t-t_0)}x_0 + \int_{t_0}^{t} e^{B(t-\tau)}\left[\sum_{j=1}^{m} b_j f_j\big(\sigma_j(\tau)\big) - \sum_{j=1}^{m}\theta_j b_j \sigma_j(\tau)\right]d\tau. \tag{9.9.2}$$

Since B is a Hurwitz matrix, then it holds:

$$\left\|e^{B(t-t_0)}\right\| \leqslant Me^{-\alpha(t-t_0)}, \quad t \geqslant t_0,$$

where $M \geqslant 1$, $\alpha > 0$.

Let $\sigma_j = \sigma_j(t, t_0, x_0)$. Then $\sigma_j(t, t_0, x_0) \to 0$ as $t \to +\infty$, $j = 1, 2, \ldots, n$, is equivalent to

$$\sigma(t, t_0, x_0) = \sum_{j=1}^{n}\left|\sigma_j(t, t_0, x_0)\right| \to 0 \quad \text{as } t \to +\infty.$$

So we have $\lim_{t\to\infty}\sigma_j(t) = 0$. Obviously, $\sigma_j(t)$ continuously depends on the initial value x_0, and $f_j(\sigma_j(t))$ is a composite continuous function of x_0, then, $\lim_{t\to\infty} f_j(\sigma_j(t)) = 0$. Thus $\forall\varepsilon > 0$, $\exists\delta(\varepsilon) > 0$, and $t_1 > t_0$ such that $\|x_0\| < \delta(\varepsilon)$ and for every fixed $f_j(\sigma_j)$, it is implied that

$$\left\|e^{B(t-t_0)}x_0\right\| \leqslant \left\|e^{B(t-t_0)}\right\|\|x_0\| < \frac{\varepsilon}{3},$$

$$\int_{t_0}^{t_1} Me^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\| + \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau < \frac{\varepsilon}{3},$$

$$\int_{t_1}^{t} Me^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\| + \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau < \frac{\varepsilon}{3}$$

for $t \geqslant t_1$.

Thus, $\|x(t)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, i.e., $x = 0$ is stable.

Next, $\forall x_0 \in R^n$, by the L'Hospital rule, we deduce that

$$\begin{aligned} 0 \leqslant \lim_{t\to+\infty}\left\|x(t)\right\| &\leqslant \lim_{t\to+\infty} Me^{-\alpha(t-t_0)} \\ &\quad + \lim_{t\to+\infty}\int_{t_0}^{t} Me^{-\alpha(t-t_0)}\left[\sum_{j=1}^{m}\left\|b_j f_j\big(\sigma_j(\tau)\big)\right\|\right. \\ &\quad \left.+ \sum_{j=1}^{m}\theta_j\left\|b_j\sigma_j(\tau)\right\|\right]d\tau = 0, \end{aligned}$$

indicating that the zero solution of system (9.9.1) is absolutely stable.

Necessity. When $d_j \neq 0$, take $f_j(\sigma_j) = \frac{\sigma_j}{2d_j}$; when $d_j = 0$, take $f_j(\sigma_j) = \frac{1}{2}c_j^T x$. Then system (9.9.1) can be transformed into

$$\frac{dx}{dt} = \left[A + \sum_{j=1}^{n} \theta_j b_j c_j^T\right] x, \tag{9.9.3}$$

so $B = A + \sum_{j=1}^{n} \theta_j b_j c_j^T$ is a Hurwitz matrix. $\forall \varepsilon > 0$, take

$$\hat{\varepsilon} = \frac{\varepsilon}{\sum_{j=1}^{m} \|c_j^T\|},$$

there exists $\delta(\varepsilon)$ such that when $\|x_0\| < \delta(\varepsilon)$ it holds

$$\|x(t)\| = \sum_{j=1}^{n} \|x_j(t)\| < \tilde{\varepsilon} \quad \text{for } t \geqslant t_0.$$

Therefore,

$$\sum_{j=1}^{m} \|c_j^T x(t)\| \leqslant \sum_{j=1}^{m} \|c_j^T\| \|x(t)\| \leqslant \sum_{j=1}^{m} \|c_j^T\| \tilde{\varepsilon} = \varepsilon,$$

and $\lim_{t\to\infty} \|x(t)\| = 0$ for $\forall x_0 \in R^n$.

Consequently, the zero solution of (9.9.1) is absolutely stable for G, and the necessity is proved. The proof of Theorem 9.9.7 is complete. □

THEOREM 9.9.8. *The zero solution of system* (9.9.1) *is absolutely stable if and only if*

(1) *condition* (1) *in Theorem* 9.9.7 *holds*;
(2) *there exists* $V(x) \in C[R^n, R]$, $\varphi \in KR$ *and* $\psi \in K$ *such that*

$$V \geqslant \varphi(\|c^T x\|), \qquad \left.\frac{dV}{dt}\right|_{(9.9.1)} \leqslant -\psi(\|c^T x\|).$$

The proof of Theorem 9.9.8 is similar to that for Theorem 9.8.5, and thus omitted.

EXAMPLE 9.9.9. Consider the following system:

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} f_1(\sigma_1) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} f_2(\sigma_2), \\ \sigma_1 = (1\ 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 0.5 f_1(\sigma_1), \\ \sigma_2 = (0\ 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 0.3 f_2(\sigma_2). \end{cases} \tag{9.9.4}$$

(1) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is stable.
(2) Take the Lyapunov function as

$$V(x) = \sum_{j=1}^{m} \int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j - \frac{1}{2} \sum_{j=1}^{m} d_j f_j^2(\sigma_j)$$

with

$$\int_0^{\sigma_j} f_j(\sigma_j)\, d\sigma_j = +\infty, \quad \text{or} \quad \lim_{t \to +\infty} f_j^2(\sigma_j) = +\infty,$$

where $d_1 = -0.5$, $d_2 = -0.3$.
(3)

$$\left.\frac{dV}{dt}\right|_{(9.9.4)} = -\sigma_1 f_1(\sigma_1) - \sigma_2 f_2(\sigma_2) - f_1^2(\sigma_1) - f_2^2(\sigma_2) < 0,$$

while $\sigma_j \neq 0$, $j = 1, 2$.

Hence, the zero solution of system (9.9.4) is absolutely stable.

9.10. Chaos synchronization as a stabilization problem of Lurie system

Chaos synchronization has been a focusing topic of intensive researching the past decade (see, for example, Chen and Dong [67]. One unified approach to chaos synchronization is to reformulate it as a (generalized) Lurie system and then discuss the absolute stability of its error dynamics [92].

Consider a uni-directional feedback-controlled chaos synchronization system in the following form of a classical Lurie system:

$$\begin{cases} \frac{dx}{dt} = A_1 x + B_1 f(c^T x, t) \quad \text{(drive)}, \\ \frac{dy}{dt} = A_2 x + B_2 f(c^T y, t) - K(x - y) - g(y, t) \quad \text{(response)}, \end{cases} \tag{9.10.1}$$

where $x(t),\ y(t),\ c \in R^n$, $A_i,\ B_i,\ K \in R^{n \times n}$, and $f \in C[R^1 \times (t_0, \infty), R^1]$. Let the synchronization error be

$$e = x - y. \tag{9.10.2}$$

Then, system (9.10.1) can be reformulated to

$$\frac{de}{dt} = A_1 x - A_2 y + B_1\big(f\big(c^T x, t\big) - B_2 f\big(c^T y, t\big)\big) + Ke + g(y, t). \tag{9.10.3}$$

The objective is to choose the constant feedback gain K and a simplest possible control signal $g(y,t)$ such that $e(t) \to 0$ as $t \to \infty$, thereby achieving synchronization $y(t) \to x(t)$ as $t \to \infty$.

Note that in system (9.10.1) for synchronization purpose the control term $g(y,t)$ is only a function of the y signal but not the x signal. Therefore, this setting can be easily implemented as a component of the response (receiver), which usually has the same structure as the drive (transmitter).

LEMMA 9.10.1. *For the chaos synchronization system* (9.10.1) *there always exist a constant feedback gain K and a control signal $g(y,t)$ such that the error dynamics* (9.10.2) *can be rewritten in the classical Lurie system form as follows*:

$$\frac{de}{dt} = (A_1 + K)e + B_1 F\big(c^T e, t\big), \tag{9.10.4}$$

where

$$F(\sigma, t) := f\big(c^T x, t\big) - f\big(c^T y, t\big).$$

PROOF.

$$\begin{aligned}
\frac{de}{dt} &= \frac{dx}{dt} - \frac{dy}{dt} \\
&= A_1 x - A_2 y + B_1 f\big(c^T x, t\big) - B_2 f\big(c^T y, t\big) + K(x-y) + g(y,t) \\
&= A_1 e + (A_1 - A_2)y + B_1\big(f\big(c^T x, t\big) - f\big(c^T y, t\big)\big) \\
&\quad + (B_1 - B_2) f\big(x^T y, t\big) + Ke + g(y,t) \\
&= (A_1 + K)e + B_1 F\big(c^T e, t\big)
\end{aligned}$$

as claimed, where

$$g(y,t) = (A_2 - A_1)y + (B_2 - B_1) f\big(c^T y, t\big).$$

□

Note that if $K \neq 0$ in (9.10.4) can be arbitrarily chosen, then we can determine a K such that $(A_1 + K)$ is Hurwitz stable, if, moreover, the function $F(\cdot, t)$ satisfies the sector condition:

$$0 \leqslant \frac{F(\sigma,t)}{\sigma} = \frac{f(c^T e + c^T y, t) - f(c^T y, t)}{c^T e} \leqslant \beta < \infty, \tag{9.10.5}$$

then we arrive at a typical Lurie system. This Lurie system may satisfy some existing sufficient conditions for synchronization. However, if K is in a restricted form so that $K + A_1$ cannot be stabilized, then even if the function $F(\cdot, t)$ satisfies the sector condition, it dose not result in a classical Lurie system. In this case, we have to develop a new criterion for the intended synchronization, which will be carried out in the next section.

First, consider the case where $B_1 = (b_1, \ldots, b_n)^T := b$. Without loss of generality, suppose that $c_n \neq 0$ in $c = (c_1, \ldots, c_n)^T$. Let

$$\Omega = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_n \end{bmatrix},$$

and consider the transform

$$\eta = \Omega e, \tag{9.10.6}$$

where $\eta = (\eta_1, \ldots, \eta_n)^T$ and $e = (e_1, \ldots, e_n)^T$. Since Ω is nonsingular, it is clear that $e(t) \to 0$ if and only if $\eta(t) \to 0$ as $t \to +\infty$.

Under transform (9.10.6), system (9.10.4) becomes

$$\begin{aligned} \frac{d\eta}{dt} &= \Omega(A_1 + K)\Omega^{-1}\eta + \Omega b F\big(\eta_n(t), t\big) \\ &:= \big(\bar{A} + \bar{K}\big)\eta + \tilde{b} F\big(\eta_n(t), t\big), \end{aligned} \tag{9.10.7}$$

where

$$\begin{aligned} \bar{a}_{ij} + \tilde{k}_{ij} &= \left(a_{ij} + k_{ij} - \frac{a_{in} + k_{in}}{c_n} a_{ij}\right), \quad i, j = 1, 2, \ldots, n-1, \\ \bar{a}_{in} + \tilde{k}_{in} &= \frac{a_{in} + k_{in}}{c_n}, \quad i = 1, 2, \ldots, n-1, \\ \bar{a}_{nj} + \tilde{k}_{nj} &= \sum_{i=1}^{n} c_i(a_{ij} + k_{ij}) - \sum_{i=1}^{n} c_i \frac{a_{in} + k_{in}}{c_n} c_j, \quad j = 1, 2, \ldots, n-1, \\ \bar{a}_{nn} + \tilde{k}_{nn} &= \frac{1}{c_n} \sum_{i=1}^{m} c_i(a_{in} + k_{in}), \\ \bar{b}_i &= b_i, \quad i = 1, 2, \ldots, n-1, \\ \bar{b}_n &= \sum_{i=1}^{n} c_i b_i. \end{aligned}$$

THEOREM 9.10.2. *If the zero solution of system (9.10.7) is absolutely stable about its partial variable η_n, with $F(\cdot, t)$ satisfying (9.10.5) and if the matrix $(\tilde{A} + \tilde{K})$ is Hurwitz stable, then the zero solution of system (9.10.7) is absolutely stable. Consequently, the zero solution of system (9.10.4) is globally stable, and so system (9.10.1) synchronizes.*

PROOF. Since the general solution of equation (9.10.7) can be expressed as

$$\eta(t, t_0, \eta_0) = e^{(t-t_0)(\tilde{A}+\bar{K})}\eta_0 + \int_{t_0}^{t} e^{(t-\tau)(\bar{A}+\bar{K})}\bar{b} f\big(\eta_n(\tau), \tau\big)\, d\tau,$$

and since $\bar{A} + \bar{K}$ is Hurwitz, there exist constants $\alpha > 0$ and $M \geqslant 1$ such that

$$\big\| e^{(t-t_0)(\bar{A}+\bar{K})} \big\| \leqslant M e^{-\alpha(t-t_0)}. \tag{9.10.8}$$

Since (9.10.7) is absolutely stable about its partial variables η_n, we have $\eta_n(t, t_0, \eta_0) \to 0$ as $t \to \infty$. Therefore, for any $\varepsilon > 0$, there is a $t^* \geqslant t_0$ such that

$$\int_{t^*}^{t} \big\| e^{(t-\tau)(\tilde{A}+\bar{K})}\bar{b} F(\eta_n(\tau), \tau) \big\|\, d\tau \leqslant \int_{t^*}^{t} M e^{-\alpha(t-\tau)} \|\tilde{b}\| \big\| F\big(\eta_n(\tau), \tau\big) \big\|\, d\tau$$
$$\leqslant \int_{t^*}^{t} M e^{-\alpha(t-\tau)} \|\bar{b}\| \big\| \beta \eta_n(\tau) \big\|\, d\tau < \frac{\varepsilon}{3}. \tag{9.10.9}$$

Note that solution $\eta(t)$ is continuously dependent on the initial condition η_0, so is the continuous function $F(\cdot, t)$. Hence, for the $\varepsilon > 0$ above, there is a constant $\delta_1(\varepsilon) > 0$ such that when $\|\eta_0\| < \delta_1(\varepsilon)$,

$$\int_{t_0}^{t^*} M e^{-\alpha(t-\tau)} \big\| \bar{b} F\big(\eta_n(\tau), \tau\big) \big\|\, d\tau < \frac{\varepsilon}{3}. \tag{9.10.10}$$

Now, take $\delta_2(\varepsilon) = \varepsilon/(3M)$ and $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$. It then follows from (9.10.8), (9.10.10) that, when $\|\eta_0\| < \delta(\varepsilon)$,

$$\begin{aligned}
\big\|\eta(t)\big\| &\leqslant \big\| e^{(t-t_0)(\bar{A}+\tilde{K})}\eta_0 \big\| + \int_{t_0}^{t} \big\| e^{(t-\tau)(\bar{A}+\bar{K})}\bar{b} F\big(\eta_n(\tau), \tau\big) \big\|\, d\tau \\
&\leqslant M e^{-\alpha(t-t_0)} \|\eta_0\| + \int_{t_0}^{t^*} M e^{-\alpha(t-\tau)} \|\bar{b}\| \beta \big\| \eta_n(\tau) \big\|\, d\tau \\
&\quad + \int_{t^*}^{t} M e^{-\alpha(t-\tau)} \|\bar{b}\| \beta \big\| \eta_n(\tau) \big\|\, d\tau \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Therefore, the zero solution of system (9.10.7) is stable.

On the other hand, for any $\eta_0 \in R^n$, we have

$$\begin{aligned}\|\eta(t,t_0,\eta_0)\| &\leqslant Me^{-\alpha(t-t_0)}\|\eta_0\| + \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\|\bar{b}\|\|F(\eta_n(\tau),\tau)\|\,d\tau \\ &= Me^{-\alpha(t-t_0)}\|\eta_0\| + \frac{M\|\tilde{b}\|\int_{t_0}^{t} e^{\alpha\tau}\|\beta\eta_n(\tau)\|\,d\tau}{e^{\alpha t}}.\end{aligned}$$

By using L'Hospital's rule, it follows that

$$\begin{aligned}0 &\leqslant \lim_{t\to+\infty}\|\eta(t,t_0,\eta_0)\| \\ &\leqslant \lim_{t\to+\infty} Me^{-\alpha(t-t_0)}\|\eta_0\| + \lim_{t\to+\infty}\frac{M\|\tilde{b}\|\int_{t_0}^{t} e^{\alpha\tau}\|\beta\eta_n(\tau)\|\,d\tau}{e^{\alpha t}} \\ &= 0 + \lim_{t\to+\infty}\frac{M\|\tilde{b}\|e^{\alpha\tau\|\beta\eta_n(\tau)\|}}{\alpha e^{\alpha t}} = 0.\end{aligned}$$

In conclusion, the zero solution of system (9.10.7) is absolutely stable, so that the zero solution of system (9.10.4) is also absolutely stable, implying that system (9.10.1) synchronizes. This completes the proof of the theorem. □

It is well known that for an autonomous Lurie system, since $F(\sigma(t),t) = F(\sigma(t))$, a necessary condition for absolute stability is $\tilde{b}_n \leqslant 0$. For this reason, we will assume this condition below.

THEOREM 9.10.3. *Assume that $\bar{b}_n \leqslant 0$. Then, it is always possible to select appropriate elements $\tilde{K}_{ij}$ in K such that $\tilde{a}_{nj} + \tilde{k}_{nj} = 0$, $j = 1, 2, \ldots, n-1$, and $\tilde{a}_{nn} + k_{nn} < 0$, and, moreover, $(\bar{A} + \tilde{K})$ is a Hurwitz stable matrix. As a result, the zero solution of system* (9.10.7) *is absolutely stable, and so is the zero solution of system* (9.10.4) *implying that system* (9.10.1) *synchronizes.*

PROOF. For system (9.10.7), use the Lyapunov function $V = \eta_n^2$, which gives

$$\frac{d\eta_n^2}{dt} = 2(\bar{a}_{nn} + \bar{k}_{nn})\eta_n^2 + 2\bar{b}_n\eta_n F(\eta_n(t),t) \leqslant 2(\bar{a}_{nn} + \bar{k}_{nn})\eta_n^2 < 0$$

for $\eta_n \neq 0$. Therefore, the zero solution of system (9.10.7) is absolutely stable about its partial variable $\eta_n(t)$. The rest of the proof is similar to that in the proof of Theorem 9.10.2. □

THEOREM 9.10.4. *Assume that*

$$0 \leqslant \frac{F(\sigma(t),t)}{\sigma(t)} \leqslant \infty, \quad \textit{with } F(\sigma(t),t) \to 0$$

uniformly as $\sigma(t) \to 0$ for all $t \geqslant t_0$. Also, assume that $\tilde{b}_n \leqslant 0$. Then, it is always possible to select appropriate elements $\tilde{k}_{ij}$ in K such that $\tilde{a}_{nj} + \tilde{k}_{nj} = 0$, $j = 1, 2, \ldots, n-1$, and $\tilde{a}_{nn} + \tilde{k}_{nn} < 0$, and moreover, $(\tilde{A} + \bar{K})$ is a Hurwitz stable matrix. As a result, the zero solution of system (9.10.7) is absolutely stable, and so is the zero solution of system (9.10.4) implying that system (9.10.1) synchronizes.

PROOF. For system (9.10.7), use the Lyapunov function $V = \eta_n^2$, which yields

$$\frac{dV}{dt} = 2(\tilde{a}_{nn} + \tilde{k}_{nn})\eta_n^2 + 2\bar{b}_n \eta_n F(\eta_n(t), t) \leqslant 2(\tilde{a}_{nn} + \tilde{k}_{nn})\eta_n^2 < 0$$

for all $\eta_n \neq 0$. Therefore, the zero solution of system (9.10.7) is absolutely stable about its partial variable $\eta_n(t)$. Since $F(\sigma(t), t) \to 0$ uniformly as $\sigma(t) \to 0$ for all $t \geqslant t_0$. Similarly to the proof of Theorem 1, we can prove that

$$\begin{aligned}\eta(t, t_0, \eta_0) &= e^{(t-t_0)(\tilde{A}+\tilde{K})}\eta_0 \\ &\quad + \int_{t_0}^{t} e^{(t-\tau)(\tilde{A}+\bar{K})}\bar{b}F(\eta_n(\tau), \tau)\, d\tau \to 0 \quad \text{as } t \to +\infty.\end{aligned}$$

This completes the proof of the theorem. □

THEOREM 9.10.5. *Assume that $\tilde{b}_n < 0$ but condition* (5) *holds. Then, it is still always possible to select appropriate elements $\tilde{k}_{ij}$ in $\tilde{K}$ such that $\bar{a}_{nj} + \tilde{k}_{nj} = 0$, $j = 1, 2, \ldots, n-1$, and $\bar{a}_{nn} + \tilde{k}_{nn} + \bar{b}_n\beta < 0$, and moreover, $(\tilde{A} + \tilde{K})$ is a Hurwitz stable matrix. As a result, the zero solution of system (9.10.4) is absolutely stable, and so is the zero solution of system (9.10.4) implying that system* (9.10.1) *synchronizes.*

PROOF. For system (9.10.7) we still use the Lyapunov function $V = \eta_n^2$, which yields

$$\begin{aligned}\frac{dV}{dt} &= 2(\bar{a}_{nn} + \tilde{k}_{nn})\eta_n^2 + 2\tilde{b}_n \eta_n F(\eta_n(t), t) \\ &\leqslant 2(\bar{a}_{nn} + \tilde{k}_{nn})\eta_n^2 + 2\tilde{b}_n\beta\eta_n^2 \\ &= 2(\bar{a}_{nn} + \tilde{k}_{nn} + \tilde{b}_n\beta)\eta_n^2 < 0\end{aligned}$$

for $\eta_n \neq 0$. Therefore, the zero solution of system (9.10.7) is absolutely stable about its partial variable $\eta_n(t)$. The rest of the proof is similar to that in the proof of Theorem 9.10.3. □

THEOREM 9.10.6. *Assume that in system (9.10.7) $F(\cdot, t)$ satisfies (9.10.5). If it is possible to select an appropriate $\tilde{K}$ such that the zero solution of (9.10.7) is absolutely stable about its partial variables $\eta_{j+1}, \eta_{j+2}, \ldots, \eta_n$, where $j =$*

$1, 2, \ldots, n-1$, and such that

$$B^{(j)} := \begin{bmatrix} \bar{a}_{11} + \tilde{k}_{11} & \ldots & \bar{a}_{1j} + \tilde{k}_{1j} \\ \vdots & & \vdots \\ \bar{a}_{j1} + \tilde{k}_{j1} & \ldots & \bar{a}_{jj} + \tilde{k}_{jj} \end{bmatrix}, \quad j = 1, 2, \ldots, n-1,$$

is Hurwitz stable, then the zero solution of system (9.10.7) is absolutely stable, so is the zero solution of system (9.10.4), implying that system (9.10.7) synchronizes.

PROOF. Let

$$\begin{aligned} \eta^{(j)}(t) &:= \big(\eta_1(t, t_0, \eta_0), \ldots, \eta_j(t, t_0, \eta_0)\big)^T, \\ \eta^{(n-j)}(t) &:= \big(\eta_{j+1}(t, t_0, \eta_0), \ldots, \eta_n(t, t_0, \eta_0)\big)^T, \\ \bar{b}^{(j)} &:= (\bar{b}_1, \ldots, \tilde{b}_j)^T, \\ \bar{b}^{(n-j)} &:= (\bar{b}_{j+1}, \ldots, \tilde{b}_n)^T, \\ B^{(n-j)} &:= \begin{bmatrix} \tilde{a}_{1,j+1} + \bar{k}_{1,j+1} & \cdots & \tilde{a}_{1j} + \bar{k}_{1j} \\ \vdots & & \vdots \\ \tilde{a}_{j,j+1} + \bar{k}_{j,j+1} & \cdots & \tilde{a}_{jn} + \bar{k}_{jn} \end{bmatrix}, \quad j = 1, 2, \ldots, n-1. \end{aligned}$$

Then, the first j components of (9.10.7) are given by

$$\eta^{(j)}(t) = e^{(t-t_0)B(j)}\eta^{(j)}(t_0) + \int_{t_0}^{t} e^{(t-\tau)B(j)} B^{(n-j)}\eta^{(n-j)}(\tau)\, d\tau$$
$$+ \int_{t_0}^{t} e^{(t-\tau)B(j)} \tilde{b}^{(n-j)} F\big(\eta_n(\tau), \tau\big)\, d\tau. \tag{9.10.11}$$

Since $B^{(j)}$ is Hurwitz stable, there are constants $M \geqslant 1$ and $\alpha > 0$ such that $\|e^{t-t_0}\eta^{(j)}\| \leqslant Me^{-\alpha(t-t_0)}$. Similar to the proof of Theorem 9.10.2 we have

$$\begin{aligned} \big\|\eta^{(j)}(t)\big\| &\leqslant \big\|e^{(t-t_0)B(j)}\eta^{(j)}(t_0)\big\| + \int_{t_0}^{t} \big\|e^{(t-\tau)B^{(j)}}\big\| \big\|B^{(n-j)}\big\| \big\|\eta^{(n-j)}(\tau)\big\|\, d\tau \\ &\quad + \int_{t_0}^{t} \big\|e^{(t-\tau)B^{(j)}}\big\| \big\|\bar{b}^{(n-j)}\big\| \big\|F(\eta_n(\tau), \tau)\big\|\, d\tau \\ &\leqslant Me^{-\alpha(t-t_0)}\big\|\eta^{(j)}(t_0)\big\| + \int_{t_0}^{t} Me^{-\alpha(t-\tau)}\big\|B^{(n-j)}\big\| \big\|\tilde{b}^{(n-j)}\big\|\, d\tau \end{aligned}$$

$$+\int_{t_0}^{t} M e^{-\alpha(t-\tau)} \|\tilde{b}^{(n-j)}\| \|F(\eta_n(\tau),\tau)\| \, d\tau \to 0 \quad \text{as } t \to \infty.$$

Therefore, the zero solution of system (9.10.7) is also absolutely stable about its partial variables $\eta_j, \ldots, \eta_n$. Consequently, the zero solution of system (9.10.4) is absolutely stable, so that system (9.10.1) synchronizes. □

Consider the following Lurie type system:

$$\begin{cases} \frac{dx}{dt} = Ax + f(t, q_1^T x, \ldots, q_n^T x)(bc^T - cb^T)x, \\ \frac{dy}{dt} = Ay + f(t, q_1^T x, \ldots, q_n^T x)(bc^T - cb^T)y - K(x-y), \end{cases} \tag{9.10.12}$$

where $f \in [t_0, \infty) \times R^r, R]$ is continuous, $1 \leqslant r \leqslant n$, and $K \in R^{n\times n}$ is the gain matrix to be determined.

Let $e = x - y$. Then, the error dynamics system is

$$\frac{de}{dt} = Ae + f\big(t, q_1^T x, \ldots, q_n^T x\big)\big(bc^T - cb^T\big)e + Ke. \tag{9.10.13}$$

The following is the main result of Curran and Chua [92].

THEOREM 9.10.7. *Suppose that f is bounded for any bounded variables, and that there exists a positive definite and symmetric matrix P such that*

$$(A+P)^T P + P(A+K) < 0 \quad \text{and} \quad B^T P + PB = 0. \tag{9.10.14}$$

where $B = bc^T - cb^T$. Then, the zero solution of system (9.10.3) is globally asymptotically stable, so that system (9.10.12) synchronizes.

Note that $B = bc^T - cb^T$ is a very special asymmetric matrix, therefore the second equation in (9.10.14) always has a solution (e.g., the identity matrix or a quasi-diagonal matrix). As a result, by selecting an appropriate gain matrix K, equations in (9.10.4) are always solvable.

Note also that although system (9.10.12) includes the Lorenz system as a special case, it does not include many other chaotic systems such as the simple yet similar Chen and Rössler systems.

To be even more general, we now consider the following generalized Lurie system:

$$\begin{cases} \frac{dx}{dt} = Ax + f(t, q_1^T x, \ldots, q_n^T x)Bx, \\ \frac{dy}{dt} = Ay + f(t, q_1^T x, \ldots, q_n^T x)By - K(x-y), \end{cases} \tag{9.10.15}$$

where $B \in R^{n\times n}$ is an arbitrary constant matrix and $K \in R^{n\times n}$ is the gain matrix to be determined. Again, let $e = x - y$, so as to obtain the error dynamics system

$$\frac{de}{dt} = Ae + f\big(t, q_1^T x, \ldots, q_n^r x\big)Be + Ke. \tag{9.10.16}$$

It should be pointed out that under the conditions given in Theorem 9.10.8, the zero solution of system (9.10.16) can be globally asymptotically stable, if the equations in (9.10.14) have a solution. However, for a general matrix $B \neq bc^T - cb^T$, these matrix equations in (9.10.14) usually do not have a solution. Thus, Theorem 9.10.8 can not be applied. We hence have to resort to a different approach, which is further discussed below.

Notice that most of people believe that chaotic orbits are ultimately bounded. Recently, we have proved that the general Lorenz system, the Chen system and Lü system [69] as well the smooth Chua system [287] all have globally exponentially attractive sets. Moreover, we gave explicit estimations for these attractive sets. Therefore, for such well-known typical chaotic systems, the answer for the existence of globally exponentially attractive set is positive.

Since f is continuous and the system is ultimately bounded, there exists a constant $M > 0$ such that

$$\left|f\left(t, q_1^T x, \ldots, q_n^T x\right)\right| \leqslant M < \infty$$

in the chaos synchronization problem studied here. Denote the matrix $D = (M|b_{ij}|)_{n\times n}$. We have the following results, which provide some very simple algebraic conditions and are very easy to use for synchronization verification and design.

THEOREM 9.10.8. *If the gain matrix K is selected such that*

$$(A + K + D) + (A + K + D)^T < 0,$$

then the zero solution of system (9.10.16) is absolutely stable, implying that system (9.10.15) synchronizes.

Note that obviously it is very easy to choose a K to satisfy the above negative definiteness condition, which does not require solving any matrix equation or inequality. The following two corollaries will make this point even clearer.

PROOF. Choose the Lyapunov function $V = e^T e$. Then

$$\begin{aligned}
\frac{dV}{dt} &= e^T\big((A+K) + (A+K)^T + f\big(t, q_1^T x, \ldots, q_n^T x\big)B \\
&\quad + f\big(t, q_1^T x, \ldots, q_r^T x\big)B^T\big)e \\
&\leqslant e^T\big((A+K) + (A+K)^T\big)e + e^T De \\
&= e^T((A+K+D) + (A+K+D)^T)e < 0
\end{aligned} \tag{9.10.17}$$

for all $e \neq 0$. This yields the result, and completes the proof of the theorem. □

COROLLARY 9.10.9. *Let $\lambda_{\max}$ be the largest eigenvalue of matrix $(A+A^T+D+D^T)$. If $\lambda_{\max} < 0$, then one may simply choose $K = 0$; if $\lambda_{\max} \geqslant 0$, then one may choose $K = \mu I$ with $2\mu < -\lambda_{\max}$. In both cases, the zero solution of the error dynamites system (9.10.16) is absolutely stable, implying that system (9.10.15) synchronizes.*

PROOF. When $\lambda_{\max} < 0$, we select $K = 0$, so

$$(A + K + D) + (A + K + D)^T = (A + D) + (A + D)^T < 0,$$

implying the result of (9.10.17). When $\lambda_{\max} \geqslant 0$, since

$$\begin{aligned}
&e^T\big((A + K + D) + (A + K + D)^T\big)e \\
&\quad = e^T\big(A + D + (A + D)\big)^T e + e^T\big(K + K^T\big)e \\
&\quad \leqslant \lambda_{\max} e^T e + 2\mu e^T e < 0
\end{aligned}$$

for all $e \neq 0$, we also have the result of (9.10.17) completing the proof of the corollary. □

COROLLARY 9.10.10. *Let*

$$\begin{aligned}
&H = \big(A + A^T + E + D^T\big) := (h_{ij})_{n\times n}, \\
&l = \max_{1\leqslant i\leqslant n} \sum_{j=1,\, j\neq i}^{n} |h_{ij}|.
\end{aligned}$$

If $K = \mu I$ and $2\mu + h_{ii} < -l$, then the zero solution of the error dynamics system (9.10.15) is absolutely stable, implying that system (9.10.15) synchronizes.

PROOF. Since

$$(A + K + D) + (A + K + D)^T = \big(A + A^T + D + D^T\big) + 2\mu I$$

and

$$2\mu + h_{ii} < -l = -\max_{1\leqslant i\leqslant n} \sum_{j=1,\, j\neq i}^{n} |h_{ij}|,$$

it follows from the well-known Gershgorin theorem that all the eigenvalues of the matrix $(A + K + D) + (A + K + D)^T$ are located on the left-hand side of the complex plane, which means that this matrix is negative definite. As a result, the zero solution of the error dynamics system (9.10.16) is absolutely stable. □

The Chen system [69] is given by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xy + cy, \\ \dot{z} = xy - bz. \end{cases} \tag{9.10.18}$$

With constants $a = 35$, $b = 3$, and $c = 8$, it generates chaos.

Let (x_d, y_d, z_d) and x_r, y_r, z_r be the variables of the drive and the response, respectively, and let $e_1 = x_d - x_r$, $e_2 = y_d - y_r$, $e_3 = z_d - z_r$, and $y := y_d = y_r$. Then the error dynamics system is

$$\begin{cases} \dot{e}_1 = -ae_1, \\ \dot{e}_2 = (c - a)e_1 - ye_1 + ce_2, \\ \dot{e}_3 = e_1 y - be_3. \end{cases} \tag{9.10.19}$$

It can be verified that the analytic method developed in [261,262] can only prove that $e_1 \to 0$, $\dot{e}_2 \to 0$, and $e_3 \to 0$ exponentially, but cannot prove $e_2 \to 0$, as $t \to \infty$. Here, it can be easily verified that by using a simple feedback on the second equation of (9.10.19) $-ke_2$ with $k > c_1$ conditions of Theorem 9.10.8 are satisfied, so two Chen systems synchronize. In fact, a direct calculation yields

$$\begin{aligned} e_1(t) &= e_1(t_0)e^{-a(t-t_0)} \to 0, \quad \text{as } t \to \infty, \\ e_2(t) &= e_2(t_0)e^{-(k-c)(t-t_0)} + \int_{t_0}^{t} e^{-(k-c)(t-\tau)}\big[(c - a)e_1(t_0)e^{-a(\tau-t_0)} \\ &\qquad - y(\tau)e_1(t_0)e^{-a(\tau-t_0)}\big]\, d\tau \to 0 \quad \text{as } t \to \infty \\ e_3(t) &= e_3(t_0)e^{-b(t-t_0)} + \int_{t_0}^{t} e^{-b(t-\tau)} e_1(t_0)e^{-a(\tau-t_0)} y(\tau)\, d\tau \to 0, \\ &\qquad \text{as } t \to \infty \end{aligned}$$

all exponentially. Therefore, two Chen systems synchronize, as claimed.

9.11. NASCs for absolute stability of time-delayed Lurie control systems

Consider the following Lurie system with constant time delays:

$$\begin{cases} \frac{dz}{dt} = \tilde{A}z(t) + \tilde{B}z(t - \tau_1) + \tilde{h} f(\sigma(t - \tau_2)), \\ \quad \tilde{h} = (\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)^T, \\ \sigma = c^T z = \sum_{i=1}^{n} c_i z_i, \end{cases} \tag{9.11.1}$$

where $c, \tilde{h} \in R^n$ are constant vectors, $z \in R^n$ is the state vector, and $\tilde{A}, \tilde{B} \in R^{n\times n}$ are real matrices; the time delays $\tau_1 > 0$ and $\tau_2 > 0$ are constants; the function f

is defined as

$$f(\cdot) \in F_\infty := \{ f \mid f(0) = 0,\ \sigma f(\sigma) > 0,\ \sigma \neq 0,\ f \in (-\infty, +\infty) \}$$

or

$$f(\cdot) \in F_{[0,k]} := \{ f \mid f(0) = 0,\ 0 < \sigma f(\sigma) \leqslant k\sigma^2,\ \sigma \neq 0,\\ f \in (-\infty, +\infty) \},$$

where k is a positive real number. Let $\tau = \max\{\tau_1, \tau_2\}$, then $C[[-\tau, 0], R^n]$ represents a Banach space with uniform, continuous topological structure.

DEFINITION 9.11.1. If $\forall f(\cdot) \in F_\infty$ (or $\forall f(\cdot) \in F_{[0,k]}$), the zero solution of system (9.11.1) is globally asymptotically stable for any values of $\tau_1, \tau_2 \geqslant 0$, then the zero solution of system (9.11.1) is said to be time-delay independent absolutely stable (or time-delay independent absolutely stable in the Hurwitz sector $[0, k]$).

It is easy to show that the necessary condition for system (9.11.1) to be time-delay independent absolutely stable is $c^T \tilde{h} \leqslant 0$; and the necessary condition for system (9.11.1) to be time-delay independent absolutely stable in the Hurwitz sector $[0, k]$ is that $\forall \mu \in [0, k]$, the matrix $\tilde{A} + \tilde{B} + \mu \tilde{h} c^T$ is a Hurwitz matrix.

In fact, let

$$f(\sigma(t - \tau_2)) = \mu\sigma(t - \tau_2) = \sum_{i=1}^{n} \mu c_i z_i (t - \tau_2).$$

Then system (9.11.1) becomes

$$\frac{dz}{dt} = \tilde{A} z(t) + \tilde{B} z(t - \tau_1) + \mu \tilde{h} c^T z(\sigma(t - \tau_2)). \tag{9.11.2}$$

In particular, when $\tau_1 = \tau_2 = 0$, for an arbitrary $\mu \in [0, +\infty)$, the matrix $\tilde{A} + \tilde{B} + \mu \tilde{h} c^T$ is a Hurwitz matrix. Thus, we have

$$tr(\tilde{A} + \tilde{B} + \mu \tilde{h} c^T) = tr\,\tilde{A} + tr\,\tilde{B} + tr(\tilde{h} c^T) = tr\,\tilde{A} + tr\,\tilde{B} + \mu c^T \tilde{h} < 0,$$

which holds for $\mu \gg 1$, implying that $c^T \tilde{h} \leqslant 0$.

Next, take $\mu \in [0, k]$, $f(\sigma(t - \tau_2)) = \mu\sigma(t - \tau_2)$, $\tau_1 = \tau_2 = 0$. Then it is easy to see that $\tilde{A} + \tilde{B} + \mu \tilde{h} c^T$ must be a Hurwitz matrix.

In the following, we use two nonsingular linear transforms to change system (9.11.1) into a separable nonlinear system. There are two cases.

(1) When $c^T \tilde{h} \leqslant 0$. Without loss of generality, suppose $c_n \neq 0$. Let

$$x = \Omega z, \tag{9.11.3}$$

where

$$\Omega = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}.$$

Then system (9.11.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= \Omega \tilde{A} \Omega^{-1} x(t) + \Omega \tilde{B} \Omega^{-1} x(t-\tau_1) + \Omega \tilde{h} f\big(x_n(t-\tau_2)\big) \\ &:= Ax(t) + BX(t-\tau_1) + hf\big(x_n(t-\tau_2)\big), \end{aligned} \tag{9.11.4}$$

where $A = \Omega \tilde{A} \Omega^{-1}$, $B = \Omega \tilde{B} \Omega^{-1}$, and $h = \Omega \tilde{h}$. Since (9.11.3) is a nonsingular linear transformation, the time-delay independent absolute stabilities of the zero solutions of systems (9.11.1) and (9.11.4) are equivalent.

(2) When $c^T \tilde{h} < 0$. Without loss of generality, assume $\tilde{h}_n c_n \neq 0$. Let $y = Gz$, where

$$G = \begin{bmatrix} \tilde{h}_n & 0 & \cdots & 0 & -\tilde{h}_1 \\ 0 & \tilde{h}_n & \cdots & 0 & -\tilde{h}_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \tilde{h}_n & -\tilde{h}_{n-1} \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}.$$

Then system (9.11.4) can be rewritten as

$$\begin{aligned} \frac{dy}{dt} &= G\tilde{A}G^{-1} y(t) + G\tilde{B}G^{-1} y(t-\tau_1) + G\tilde{h} f\big(y_n(t-\tau_2)\big) \\ &:= Py(t) + Qy(t-\tau_1) + bf\big(y_n(t-\tau_2)\big), \end{aligned} \tag{9.11.5}$$

where $P = G\tilde{A}G^{-1}$, $Q = G\tilde{B}G^{-1}$, and $b = G\tilde{h} = (\overbrace{0, 0, \ldots, 0}^{n-1},\ c^T\tilde{h})^T$. Similarly, due to the nonsingularity of G, the time-delay independent absolute stabilities of the zero solutions of systems (9.11.1) and (9.11.5) are equivalent.

DEFINITION 9.11.2. The zero solution of system (9.11.4) is said to be time-delay independent absolutely stable (or time-delay independent absolute stable in the Hurwitz sector $[0, k]$) with respect to (w.r.t.) partial variable x_n of the system, if $\forall f(\cdot) \in F_\infty$ (or $\forall f(\cdot) \in F_{[0,k]}$), the zero solution of system (9.11.4) is globally asymptotically time-delay independently stable (or globally asymptotically time-delay independently stable in the Hurwitz sector $[0, k]$) w.r.t. partial variable x_n of the system.

Similarly, we can define for the zero solution of system (9.11.5) to be time-delay independently stable w.r.t. partial variable y_n.

THEOREM 9.11.3. *The sufficient and necessary conditions for the zero solution of system (9.11.4) to be time-delay independently stable are*:

(1) *The matrix* $A + B + (O_{n\times(n-1)}, h\theta)$ *is a Hurwitz matrix, where* $\theta = 0$ *or* $\theta = 1$, *and*

$$(O_{n\times(n-1)}, h\theta) = \begin{bmatrix} 0 & \cdots & 0 & h_1\theta \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & h_n\theta \end{bmatrix}_{n\times n}.$$

(2) $\det(i\sigma - A - Be^{-i\sigma\tau_1} - (O_{n\times(n-1)}, h\theta)e^{-i\sigma\tau_2}) \neq 0\ \forall\sigma \in R,\ \forall\tau_1, \tau_2 \geqslant 0.$

(3) *The zero solution of system (9.11.4) is time-delay independent absolutely stable w.r.t. partial variable* x_n.

PROOF. *Necessity*. Suppose that the zero solution of system (9.11.4) is time-delay independent absolutely stable. When $A + B$ is a Hurwitz matrix, we can choose $\theta = 0$ and thus $A+B+(O_{n\times(n-1)}, h\theta) = A+B$ is a Hurwitz matrix. When $A+B$ is not a Hurwitz matrix, we take $f(x_n) = x_n$. Then system (9.11.4) becomes a linear time-delayed system:

$$\frac{dx}{dt} = Ax(t) + Bx(t-\tau) + hx_n(t-\tau_2). \tag{9.11.6}$$

From the sufficient and necessary conditions of global time-delay independent stability for constant time-delayed systems with constant coefficients, we know that all the eigenvalues of the characteristic equation of system (9.11.5), given by

$$\det\bigl(\lambda I - A - Be^{-i\lambda\tau_1} - (O_{n\times(n-1)}, h\theta)e^{-i\lambda\tau_2}\bigr) = 0, \tag{9.11.7}$$

must have negative real parts. This is equivalent to the conditions (1) and (2) in Theorem 9.11.3 ($\theta = 1$). The condition (3) of Theorem 9.11.3 is obvious. The necessity is proved.

Sufficiency. Rewrite system (9.11.4) as

$$\begin{aligned} \frac{dx}{dt} &= Ax(t) + Bx(t-\tau) + h\theta x_n(t-\tau_2) \\ &\quad + hf\bigl(x_n(t-\tau_2) - \theta hx_n(t-\tau_2)\bigr). \end{aligned} \tag{9.11.8}$$

Let $x^*(t) = x(t_0, \phi)(t)$ be the solution of the following system:

$$\begin{aligned} &\frac{dx}{dt} = Ax(t) + Bx(t-\tau) + h\theta x_n(t-\tau_2), \\ &x(t) = \phi(t), \quad t_0 - \tau \leqslant t \leqslant t_0. \end{aligned} \tag{9.11.9}$$

Then from the method of constant variation, we know that the solution of (9.11.8) passing through the initial point (t_0, ϕ) can be expressed as

$$x(t) = x^*(t) + \int_{t_0}^{t} U(t,s)\big[hf\big(x_n(s-\tau_2) - \theta h x_n(t-\tau_2)\big)\big]\,ds, \qquad (9.11.10)$$

where $U(t,s)$ is the fundamental matrix solution, satisfying

$$\frac{\partial U(t,s)}{\partial t} = AU(t,s) + BU(t-\tau_1, s) + (O_{n\times(n-1)}, h\theta)U(t-\tau_2, s),$$
$$U(t,s) = \begin{cases} 0 & \text{when } \tau - s \leqslant t \leqslant s, \\ I & \text{when } t = s. \end{cases}$$

From the conditions given in Theorem 9.11.3, it is known that there exist constants $M \geqslant 1$, $N \geqslant 1$ and $\alpha > 0$ such that

$$\big\|x^*(t)(t_0,\phi)(t)\big\| \leqslant M\|\phi\|e^{-\alpha(t-t_0)} \quad \text{when } t \geqslant t_0, \qquad (9.11.11)$$
$$\big\|U(t,s)\big\| \leqslant Ne^{-\alpha(t-s)} \quad \text{when } t \geqslant s. \qquad (9.11.12)$$

Therefore, we have

$$\begin{aligned}\big\|x(t)\big\| \leqslant{}& M\|\phi\|e^{-\alpha(t-t_0)} + N\int_{t_0}^{t} e^{-\alpha(t-s)}\big[\big\|hf\big(x_n(t-\tau_1)\big)\big\| \\ &+ \big\|\theta h x_n(s-\tau_2)\big\|\big]\,ds \quad (t \geqslant t_0).\end{aligned} \qquad (9.11.13)$$

$\forall\epsilon > 0$, since $x_n(t) \to 0$ as $t \to +\infty$ and $f(\cdot)$ is a continuous function of x_0, there exists $\delta_1(\epsilon) > 0$ such that when $\|\phi\| < \delta_1(\epsilon)$, the following inequalities hold:

$$N\int_{t_0}^{t_1} e^{-\alpha(t-s)}\big[\big\|hf\big(x_n(t-\tau_1)\big)\big\| + \big\|\theta h x_n(s-\tau_2)\big\|\big]\,ds < \frac{\epsilon}{3}$$
$$\text{when } t_1 > t_0,$$
$$N\int_{t_1}^{t} e^{-\alpha(t-s)}\big[\big\|hf\big(x_n(t-\tau_1)\big)\big\| + \big\|\theta h x_n(s-\tau_2)\big\|\big]\,ds < \frac{\epsilon}{3}$$
$$\text{when } t > t_1. \qquad (9.11.14)$$

Further, let $\delta_2 = \frac{\epsilon}{3M}$, and $M\|\phi\|e^{-\alpha(t-t_0)} \leqslant \frac{\epsilon}{3}$ when $\|\phi\| \leqslant \delta_2$. Then define

$$\delta(\epsilon) = \min\big(\delta_1(\epsilon), \delta_2(\epsilon)\big). \qquad (9.11.15)$$

Now combining equations (9.11.13), (9.11.14) and (9.11.15) yields $\|x(t)\| < \epsilon$ when $t \geqslant t_0$ and $\|\phi\| < \delta(\epsilon)$. Hence, the zero solution of system (9.11.8) is stable in the sense of Lyapunov.

Further, it can be shown by applying L'Hospital rule to equation (9.11.13) that $\forall x_0 \in R^n$,

$$\begin{aligned}\lim_{t\to+\infty}\|x(t)\| &\leqslant \lim_{t\to+\infty} M\|\phi\|e^{-\alpha(t-t_0)}\\ &\quad+\lim_{t\to+\infty}\frac{1}{e^{\alpha t}}\int_{t_0}^{t} e^{\alpha s}\big[\big\|hf\big(x_n(s-\tau_1)\big)\big\|+\big\|\theta h x_n(s-\tau_2)\big\|\big]\,d\tau\\ &= 0+\frac{1}{\alpha}\lim_{t\to+\infty}\big[\big\|hf\big(x_n(t-\tau_1)\big)\big\|+\big\|\theta h x_n(t-\tau_2)\big\|\big]\\ &= 0,\end{aligned}$$

which implies that the zero solution of system (9.11.8) is globally asymptotically stable. Due to the arbitrary of $f(\cdot) \in F$, the zero solution of system (9.11.4) is time-delay independent absolutely stable. The sufficiency is also proved. □

THEOREM 9.11.4. *The sufficient and necessary conditions for system (9.11.4) to be time-delay independent absolutely stable are*:

(1) *There exists nonnegative vector $\eta = (\eta_1, \ldots, \eta_n)^T$ such that the matrix $A + B + (O_{n\times(n-1)}, \eta)$ is a Hurwitz matrix.*
(2) $\det(i\sigma - A - Be^{-i\sigma\tau_1} - (O_{n\times(n-1)}, \eta)e^{-i\sigma\tau_2}) \neq 0\ \forall\sigma \in R$.
(3) *The zero solution of system (9.11.4) is time-delay independent absolutely stable w.r.t. partial variable x_n.*

The proof of Theorem 9.11.4 is similar to that for Theorem 9.11.3, and thus omitted for brevity.

REMARK 9.11.5. The existence of the vector $\eta = (\eta_1, \ldots, \eta_n)^T$ is obvious. For example, $\eta = \theta$. [θ is defined in condition (1) of Theorem 9.11.3, which is a constructive condition, while condition (1) in Theorem 9.11.4 is an existence condition, which is certainly not as good as condition (1) of Theorem 9.11.3.] The condition (1) of Theorem 9.11.3 is easy to be verified. However, if an appropriate η is chosen, it may simplify the validation of other conditions in the theorem.

Similar to Theorems 9.11.3 and 9.11.4, we have the following theorems.

THEOREM 9.11.6. *The sufficient and necessary conditions for system (9.11.5) to be time-delay independent absolutely stable are*:

(1) *The matrix $P + Q + (O_{n\times(n-1)}, \theta b)$ is a Hurwitz matrix, where*

$$(O_{n\times(n-1)}, \theta b) = \begin{bmatrix} 0 & \cdots & 0 & \theta b_1\\ \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & \theta b_n\end{bmatrix}_{n\times n} := (\theta_{ij})_{n\times n}.$$

(2) $\det(i\sigma - P - Qe^{-i\sigma\tau_1} - (O_{n\times(n-1)}, \theta b)e^{-i\sigma\tau_2}) \neq 0 \; \forall \sigma \in R$.
(3) *The zero solution of system (9.11.5) is time-delay independent absolutely stable w.r.t. partial variable y_n.*

THEOREM 9.11.7. *The sufficient and necessary conditions for system (9.11.5) to be time-delay independent absolutely stable are*:

(1) *There exists nonnegative vector $\eta = (\eta_1, \dots, \eta_n)^T$ such that the matrix $P + Q + (O_{n\times(n-1)}, \eta)$ is a Hurwitz matrix.*
(2) $\det(i\sigma - P - Qe^{-i\sigma\tau_1} - (O_{n\times(n-1)}, \eta)e^{-i\sigma\tau_2}) \neq 0 \; \forall \sigma \in R$.
(3) *The zero solution of system (9.11.5) is time-delay independent absolutely stable w.r.t. partial variable y_n.*

Now we derive some simple and easy-applicable algebraic criteria. These criteria are easy to be verified in practice and can be applied in designing absolutely stable systems or stabilizing an existing control system.

In system (9.11.4), assume that $f(\cdot) \in F_{[0,k]} := \{f \mid f(0) = 0,\ 0 \leqslant x_n f(x_n) \leqslant kx_n^2\}$, and f is continuous. Then we have the following theorem.

THEOREM 9.11.8. *If system (9.11.4) satisfies the following conditions*:

(1) $a_{ii} < 0,\ i = 1, 2, \dots, n$.
(2) $G = (-(-1)^{\delta_{ij}}|a_{ij}| - |b_{ij}| - \theta_{ij}|h_i|k)_{n\times n}$ *is an M matrix,*

where

$$\theta_{ij} = \begin{cases} 1 & \text{when } i = 1, \dots, n,\ j = n, \\ 0 & \text{when } i = 1, \dots, n,\ j = 1, \dots, n-1, \end{cases}$$

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Then the zero solution of system (9.11.4) is time-delay independent absolutely stable in the Hurwitz sector $[0, k]$.

PROOF. Since G is an M matrix, it is known from the property of M matrix that $\forall \beta = (\beta_1, \dots, \beta_n)^T > 0$ (i.e., $\beta_i > 0,\ i = 1, 2, \dots, n$), there exist constants $c_i > 0,\ i = 1, 2, \dots, n$, such that $c = (G^T)^{-1}\beta,\ c = (c_1, \dots, c_n)^T$, i.e.,

$$-c_j a_{jj} - \left(\sum_{i=1, i\neq j}^{n} |a_{ij}|c_i + \sum_{i=1}^{n} |b_{ij}|c_i + \sum_{i=1}^{n} c_i\theta_{ij}\|h_i\|k \right) = \beta_j,$$
$$j = 1, 2, \dots, n.$$

Consider a positive definite and radially unbounded Lyapunov functional:

$$\begin{aligned} V(t) = \sum_{i=1}^{n} c_i \Bigg[\big|x_i(t)\big| + \sum_{j=1}^{n} |b_{ij}| \int\limits_{t-\tau_1}^{t} \big|x_j(s)\big|\, ds \\ + \sum_{i=1}^{n} \theta_{ij} |h_i| k \int\limits_{t-\tau_2}^{t} \big|x_i(s)\big|\, ds \Bigg]. \end{aligned}$$

Suppose that the initial condition for the solution of system (9.11.4) is given by $x(t) = \phi(t)$, $-\tau \leqslant t \leqslant 0$. Then we have

$$\begin{aligned} V(t_0) &\leqslant \sum_{i=1}^{n} c_i \Bigg[\big|x_i(t_0)\big| + \sum_{j=1}^{n} |b_{ij}| \|\phi\| \tau + \sum_{i=1}^{n} \theta_{ij} |h_i| k \|\phi\| \tau_2 \Bigg] \\ &:= M < +\infty \end{aligned}$$

and $V(t) \geqslant \sum_{i=1}^{n} c_i |x_i(t)|$. Thus along the trajectory of system (9.11.4) differentiating V with respect to time yields

$$\begin{aligned} D^+ V(t)|_{(9.11.4)} \leqslant{} & \sum_{i=1}^{n} c_i \Bigg[\frac{dx_i}{dt} \operatorname{sign}(x_i) + \sum_{j=1}^{n} |b_{ij}| \big|x_j(t)\big| \\ & - \sum_{j=1}^{n} |b_{ij}| \big|x_j(t-\tau_1)\big| + \sum_{i=1}^{n} \theta_{ij} |h_j| \big|x_i(t)\big| - \sum_{i=1}^{n} \theta_{ij} |h_j| \big|x_j(t-\tau_2)\big| \Bigg] \\ \leqslant{} & \sum_{i=1}^{n} c_i \Bigg\{ \Bigg[\sum_{i=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t-\tau_1) \\ & + \sum_{i=1}^{n} \theta_{ij} h_i f\big(x_n(t-\tau_2)\big) \Bigg] \operatorname{sign}(x_i) + \sum_{j=1}^{n} |b_{ij}| \big|x_j(t)\big| \\ & - \sum_{j=1}^{n} |b_{ij}| \big|x_j(t-\tau_1)\big| + \sum_{i=1}^{n} \theta_{ij} |h_j| \big|x_i(t)\big| - \sum_{i=1}^{n} \theta_{ij} |h_j| \big|x_i(t-\tau_2)\big| \Bigg\} \\ \leqslant{} & \sum_{j=1}^{n} \Bigg[c_j a_{jj} + \sum_{\substack{j=1 \\ j \neq i}}^{n} c_i |a_{ij}| + \sum_{j=1}^{n} c_i |b_{ij}| + \sum_{i=1}^{n} \theta_{ij} |h_i| k \Bigg] \big|x_j(t)\big| \\ \leqslant{} & -\sum_{j=1}^{n} \beta_j \big|x_j(t)\big|. \end{aligned} \tag{9.11.16}$$

Hence,

$$0 \leqslant V(t) \leqslant V(t_0) - \int_{t_0}^{t} \sum_{j=1}^{n} \beta_j \left|x_j(\tau)\right| d\tau \leqslant V(t_0). \tag{9.11.17}$$

Equation (9.11.17) clearly indicates that the zero solution of system (9.11.4) is time-delay independently stable in the Hurwitz sector $[0, k]$.

Next, we show that the zero solution of system (9.11.4) is time-delay independently attractive in the Hurwitz sector $[0, k]$.

Because

$$0 \leqslant \min_{1 \leqslant i \leqslant n} c_i \sum_{i=1}^{n} \left|x_i(t)\right| \leqslant V(t) \leqslant V(t_0) < +\infty,$$

$\sum_{i=1}^{n} |x_i(t)|$ is bounded, and thus $\sum_{i=1}^{n} |\frac{dx_i}{dt}|$ is bounded in $[t_0, +\infty)$. This implies that $\sum_{i=1}^{n} |x_i(t)|$ is uniformly continuous in $[t_0, +\infty)$. On the other hand, it follows from equation (9.11.17) that

$$\int_{t_0}^{t} \sum_{j=1}^{n} \beta_j \left|x_j(t)\right| dt \leqslant V(t_0),$$

which, in turn, results in $\sum_{i=1}^{n} |x_i(t)| \in L_1[0, +\infty)$. Therefore, it follows from calculus that $\forall \phi \in C[[-\tau, 0], R^n]$,

$$\lim_{t \to +\infty} \sum_{i=1}^{n} \left|x_i(t)\right| = 0,$$

which implies that the zero solution of system (9.11.4) is time-delay independently attractive in the Hurwitz sector $[0, k]$. The proof of Theorem 9.11.8 is complete. □

COROLLARY 9.11.9. *If one of the following conditions are satisfied*:

$$(1) \quad -a_{jj} > \sum_{i=1, i \neq j}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_{ij}| + \sum_{i=1}^{n} \theta_{ij} |h_i| k, \quad j = 1, 2, \ldots, n.$$

$$(2) \quad -a_{ii} > \sum_{j=1, j \neq i}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}| + \sum_{j=1}^{n} \theta_{ij} |h_i| k, \quad i = 1, 2, \ldots, n.$$

$$(3) \quad -a_{ii} > \frac{1}{2} \sum_{j=1, j \neq i}^{n} \left(|a_{ij}| + |a_{ji}|\right) + \frac{1}{2} \sum_{j=1}^{n} \left(|b_{ij}| + |b_{ji}|\right) + \frac{1}{2} \sum_{j=1}^{n} \left(\theta_{ij} |h_i| k + \theta_{ji} |h_j| k\right).$$

Then the zero solution of system (9.11.4) is time-delay independently attractive in the Hurwitz sector $[0, k]$.

This is simply because that any of the above conditions implies that $a_{ii} < 0$ $(i = 1, 2, \dots, n)$ and G is an M matrix.

Similar to Theorem 9.11.8 and Corollary 9.11.9, we have the following theorem and corollary.

THEOREM 9.11.10. *If system (9.11.5) satisfies the following conditions*:

(1) $p_{ii} < 0$, $i = 1, 2, \dots, n$.
(2) $\tilde{G} = (-(-1)^{\delta_{ij}}|p_{ij}| - |q_{ij}| - \theta_{ij}|b_i|k)_{n\times n}$ *is an M matrix, where* θ_{ij} *and* δ_{ij} *are defined in Theorem 9.11.3.*

Then the zero solution of system (9.11.4) is time-delay independent absolutely stable in the Hurwitz sector $[0, k]$.

COROLLARY 9.11.11. *If one of the following conditions is satisfied*:

$$(1)\quad -p_{jj} > \sum_{i=1, i\neq j}^{n} |p_{ij}| + \sum_{i=1}^{n} |q_{ij}| + \sum_{i=1}^{n} \theta_{ij}|h_i|k, \quad j = 1, 2, \dots, n.$$

$$(2)\quad -p_{ii} > \sum_{j=1, j\neq i}^{n} |p_{ij}| + \sum_{j=1}^{n} |q_{ij}| + \sum_{j=1}^{n} \theta_{ij}|h_i|k, \quad i = 1, 2, \dots, n.$$

$$(3)\quad -p_{ii} > \frac{1}{2}\sum_{j=1, j\neq i}^{n} \big(|p_{ij}| + |p_{ji}|\big) + \frac{1}{2}\sum_{j=1}^{n}\big(|q_{ij}| + |q_{ji}|\big) + \frac{1}{2}\sum_{j=1}^{n}\big(\theta_{ij}|b_i|k + \theta_{ji}|b_j|k\big).$$

Then the zero solution of system (9.11.5) is time-delay independently attractive in the Hurwitz sector $[0, k]$.

Further, we have the following result.

THEOREM 9.11.12. *If system (9.11.4) satisfies the following conditions*:

(1) *There exist constants* c_i, $i = 1, 2, \dots, n$, *such that*

$$c_j a_{jj} + \sum_{i=1, i\neq j}^{n} c_i|p_{ij}| + \sum_{i=1}^{n} c_i|q_{ij}| \leqslant 0, \quad j = 1, 2, \dots, n-1;$$

and

$$c_n a_{nn} + \sum_{i=1}^{n-1} c_i |a_{in}| + \sum_{i=1}^{n} c_i |b_{in}| + \sum_{i=1}^{n} c_i \theta_{in} |h_i| k \leqslant -\delta < 0.$$

(2) *All eigenvalues of* $\det(\lambda I_n - A - Be^{-i\lambda\tau_1}) = 0$ *have negative real part.*

Then the zero solution of system (9.11.4) *is time-delay independent absolutely stable in the Hurwitz sector* $[0, k]$.

PROOF. Construct a positive definite and radially unbounded Lyapunov functional as follows:

$$V(t) = \sum_{i=1}^{n} c_i \Bigg[\big|x_i(t)\big| + \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_1}^{t} \big|x_j(s)\big|\, ds + \sum_{j=1}^{n} \theta_{ij} |h_i| k \int_{t-\tau_2}^{t} \big|x_j(s)\big|\, ds \Bigg]. \tag{9.11.18}$$

Following the proof of Theorem 1 we obtain

$$\begin{aligned} D^+ V(t)|_{(9.11.4)} &\leqslant \sum_{j=1}^{n} \Bigg[c_j a_{jj} + \sum_{j=1, j\neq i}^{n} c_i |a_{ij}| + \sum_{j=1}^{n} c_i |b_{ij}| \Bigg] \big|x_j(t)\big| \\ &\leqslant \Bigg[c_n a_{nn} + \sum_{i=1}^{n-1} c_i |a_{in}| + \sum_{i=1}^{n} c_i |b_{in}| + \sum_{j=1}^{n} c_i \theta_{ij} |h_i| k \Bigg] \big|x_n(t)\big| \\ &\leqslant -\delta \big|x_n(t)\big|. \end{aligned} \tag{9.11.19}$$

Hence, we have

$$0 \leqslant V(t) \leqslant V(t_0) - \delta \int_{t_0}^{t} \big|x_n(s)\big|\, ds \leqslant V(t_0), \tag{9.11.20}$$

which indicates that the zero solution of system (9.11.4) is time-delay independently stable.

Again, similar to Theorem 9.11.3, we can prove that $\int_{t_0}^{t} \delta |x_n(s)|\, ds \leqslant V(t_0)$ and $|x_n(t)| \in L_1[0, +\infty)$. Therefore, $\lim_{t\to+\infty} |x_n(t)| = 0$, which implies that the zero solution of system (9.11.4) is time-delay independently attractive w.r.t. partial variable x_n in the Hurwitz sector $[0, k]$.

Following the proof of Theorem 9.11.3, we can express the solution of system (9.11.4) as

$$x(t) = x^*(t) + \int_{t_0}^{t} U(t,s)\big[hf\big(x_n(s-\tau)\big)\big]\,ds,$$

where $U(t,s)$ satisfies the following system:

$$\begin{aligned} \frac{\partial U(t,s)}{\partial t} &= AU(t,s) + BU(t-\tau,s) + (O_{n\times(n-1)}, h\theta)U(t-\tau_2,s), \\ U(t,s) &= \begin{cases} 0 & \text{when } \tau - s_0 \leqslant t \leqslant s_0, \\ I & \text{when } t = s_0, \end{cases} \end{aligned} \tag{9.11.21}$$

and $x^*(t) = x(t_0,\phi)$ is the solution of the following equation:

$$\begin{aligned} &\frac{dx}{dt} = Ax(t) + Bx(t-\tau_1) + h\theta x(t-\tau_2), \\ &x(t) = \phi(t), \quad -\tau \leqslant t \leqslant t_0. \end{aligned} \tag{9.11.22}$$

The remaining part of the proof can follow the derivation given in the proof of Theorem 9.11.3 from equations (9.11.11) to (9.11.15). The details are omitted here. This finishes the proof of Theorem 9.11.12. □

Similar to Theorem 9.11.12, we have

THEOREM 9.11.13. *If system (9.11.5) satisfies the following conditions*:

(1) *There exist constants* c_i, $i = 1, 2, \ldots, n$, *such that*

$$c_j p_{jj} + \sum_{i=1,\, i\neq j}^{n} c_i|p_{ij}| + \sum_{i=1,\, i\neq j}^{n} c_i|q_{ij}| \leqslant 0, \quad j = 1, 2, \ldots, n-1;$$

and

$$c_n p_{nn} + \sum_{i=1}^{n-1} c_i|p_{in}| + \sum_{i=1}^{n} c_i|q_{in}| + \sum_{i=1}^{n} c_i\theta_{in}|h_i|k \leqslant -\delta < 0.$$

(2) *All eigenvalues of* $\det(\lambda I_n - P - Qe^{-i\lambda\tau_1}) = 0$ *have negative real part.*

Then the zero solution of system (9.11.5) is time-delay independent absolutely stable in the Hurwitz sector $[0, k]$.

To end this section, we give an example to demonstrate the applicability of the theoretical results obtained in this section.

EXAMPLE 9.11.14. Consider a 3-dimensional Lurie control system in the form of (9.11.4), given by

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{bmatrix} -4 & 0 & \frac{3}{4} \\ \frac{3}{2} & -4 & \frac{5}{4} \\ \frac{1}{2} & 1 & -6 \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \\ x_3(t-\tau_1) \end{pmatrix}$$
$$+ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} f\big(x_3(t-\tau_2)\big),$$

where $f(\cdot) \in F_{[0,1/2]}$. It is seen that $a_{11} = -4 < 0$, $a_{22} = -4 < 0$ and $a_{33} = -6 < 0$, and easy to verify that

$$G = \begin{bmatrix} 4-1 & -\frac{1}{2} & -\frac{3}{4}-\frac{1}{2}-1 \\ -\frac{3}{2}-\frac{1}{2} & 4-\frac{2}{3} & -\frac{5}{4}-\frac{1}{4}-\frac{3}{2} \\ -\frac{1}{2}-\frac{1}{2} & -1-\frac{1}{3} & 6-\frac{1}{4}-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{1}{2} & -\frac{7}{4} \\ -2 & \frac{10}{3} & -3 \\ -1 & -\frac{4}{3} & \frac{21}{4} \end{bmatrix}$$

is an M matrix. Thus, the conditions given in Theorem 9.11.8 are satisfied, and the zero solution of this example is time-delay absolutely stable in the Hurwitz sector $[0, \frac{1}{2}]$.

In the following, we consider using a decomposition method to obtain the absolute stability of the whole system's states, based on lower dimensional linear system which has negative real part for its eigenvalues and that partial variables of the system are stable.

Let

$$A := \begin{bmatrix} A_{m\times m} & A_{m\times(n-m)} \\ A_{(n-m)\times m} & A_{(n-m)\times(n-m)} \end{bmatrix},$$
$$B := \begin{bmatrix} B_{m\times m} & B_{m\times(n-m)} \\ B_{(n-m)\times m} & B_{(n-m)\times(n-m)} \end{bmatrix},$$
$$(O_{n\times(n-1)}\ h\theta) := D = \begin{bmatrix} D_{m\times m} & D_{m\times(n-m)} \\ D_{(n-m)\times m} & D_{(n-m)\times(n-m)} \end{bmatrix},$$
$$h_{(m)} := (h_1, h_2, \dots, h_m)^T, \quad h_{(n-m)} := (h_{m+1}, h_{m+2}, \dots, h_n)^T,$$
$$x_{(m)} := (x_1, x_2, \dots, x_m)^T, \quad x_{(n-m)} := (x_{m+1}, x_{m+2}, \dots, x_n)^T.$$

Then system (9.11.4) can be rewritten as

$$\begin{aligned} \frac{dx_{(m)}}{dt} &= A_{m\times m}x_{(m)}(t) + B_{m\times m}x_{(m)}(t-\tau_1) + D_{m\times m}x_{(m)}(t) \\ &\quad + A_{m\times(n-m)}x_{(n-m)}(t) + B_{m\times(n-m)}x_{(n-m)}(t-\tau_1) \\ &\quad + D_{m\times(n-m)}x_{(n-m)}(t-\tau_1) \\ &\quad + k_{(m)}\big[f\big(x_n(t-\tau_2)\big) - \theta x_n(t-\tau_2)\big], \end{aligned} \tag{9.11.23}$$

$$\begin{aligned}\frac{dx_{(n-m)}}{dt} &= A_{(n-m)\times m}x_{(m)}(t) + B_{(n-m)\times m}x_{(n-m)}(t-\tau_1)\\ &\quad + D_{(n-m)\times m}x_{(n-m)}(t)\\ &\quad + A_{(n-m)\times(n-m)}x_{(n-m)}(t) + B_{(n-m)\times(n-m)}x_{(n-m)}(t-\tau_1)\\ &\quad + D_{(n-m)\times(n-m)}x_{(n-m)}(t-\tau_1)\\ &\quad + k_{(n-m)}\big[f\big(x_n(t-\tau_2)\big) - \theta x_n(t-\tau_2)\big]. \end{aligned} \tag{9.11.24}$$

It is obvious that the solutions of system (9.11.4) are equivalent to that of equations (9.11.23) and (9.11.24).

THEOREM 9.11.15. *If the following conditions are satisfied*:

(1) *All the eigenvalues of* $\det(\lambda I_m - A_{m\times m} - B_{m\times m}e^{-i\lambda\tau_1} - D_{m\times m}e^{-i\lambda\tau_2}) = 0$ *have negative real part.*
(2) *There exist constants* $c_i \geqslant 0,\ i = 1, 2, \dots, m$, *and* $c_i > 0,\ j = m+1, m+2, \dots, n$, *such that*

$$c_j a_{jj} + \left(\sum_{i=1,\, i\neq j}^{n} c_i|a_{ij}| + \sum_{i=1}^{n} c_i|b_{ij}|\right) \leqslant 0, \quad j = 1, 2, \dots, m.$$

$$c_j a_{jj} + \left(\sum_{i=1,\, i\neq j}^{n} c_i|a_{ij}| + \sum_{i=1}^{n} c_i|b_{ij}|\right) < 0,$$

$$j = m+1, m+2, \dots, n.$$

$$c_n a_{nn} + \left(\sum_{i=1,\, i\neq j}^{n} c_i|a_{in}| + \sum_{i=1}^{n} c_i|b_{in}| + \sum_{i=1}^{n} c_i|h_i|k\right) < 0.$$

Then the zero solution of system (9.11.23)–(9.11.24) *is time-delay independent absolutely stable in the Hurwitz sector* $[0, k]$.

PROOF. For the partial variables of the system: $x_{m+1}, x_{m+2}, \dots, x_n$, construct the positive definite and radially unbounded Lyapunov functional:

$$\begin{aligned} V(x,t) &= \sum_{i=1}^{n} c_i|x_i| + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_1}^{t} c_i|b_{ij}|\big|x_j(s)\big|\,ds\\ &\quad + \sum_{i=1}^{n} c_i|h_i| \int_{t-\tau_2}^{t} k\big|x_n(s)\big|\,ds. \end{aligned}$$

Then

$$D^+V(x,t)|_{(9.11.4)} \leqslant \sum_{i=1}^{n} c_i \frac{dx_i}{dt}\,\mathrm{sign}(x_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i|b_{ij}|\big|x_j(t)\big|$$

$$
\begin{aligned}
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} c_i|b_{ij}|\big|x_j(t-\tau_1)\big| \\
&\quad + \sum_{i=1}^{n} c_i|h_i|k\big|x_n(t)\big| - \sum_{i=1}^{n} c_i|h_i|k\big|x_n(t-\tau_2)\big| \\
&\leqslant \sum_{j=1}^{n-1}\left[c_j a_{jj} + \left(\sum_{i=1,i\neq j}^{n} c_i|a_{ij}| + \sum_{i=1}^{n} c_i|b_{ij}|\right)\right]\big|x_j(t)\big| \\
&\quad + \left[c_n a_{nn} + \left(\sum_{i=1}^{n-1} c_i|a_{in}| + \sum_{i=1}^{n} c_i|b_{in}| + \sum_{i=1}^{n} c_i|h_i|k\right)\right]\big|x_n(t)\big| \\
&\leqslant \sum_{j=m+1}^{n-1}\left[c_j a_{jj} + \left(\sum_{i=1,i\neq j}^{n} c_i|a_{ij}| + \sum_{j=1}^{n} c_i|b_{ij}|\right)\right]\big|x_j(t)\big| \\
&\quad + \left[c_n a_{nn} + \left(\sum_{i=1}^{n-1} c_i|a_{in}| + \sum_{i=1}^{n} c_i|b_{in}| + \sum_{i=1}^{n} c_i|h_i|k\right)\right]\big|x_n(t)\big| \\
&< 0 \quad \text{when } \|x_{(n-m)}\| \neq 0.
\end{aligned}
$$

Therefore, the complete solution of system (9.11.23)–(9.11.24), namely the complete solution of system (9.11.4), is absolutely stable w.r.t. partial variables $x_{m+1}, x_{m+2}, \ldots, x_n$.

Now let $x^*_{(m)}(t) = x_m(t_0, \phi_m)(t)$ be the solution of the homogeneous part of equation (9.11.23):

$$
\begin{aligned}
&\frac{dx_{(m)}}{dt} = A_{m\times m}x_{(m)}(t) + B_{m\times m}x_{(m)}(t-\tau_1) + D_{m\times m}x_{(m)}(t), \\
&x_m(t) = \phi_m(t), \quad -\tau \leqslant t \leqslant 0.
\end{aligned}
$$

Then we may follow the proof of Theorem 9.11.3 to write the solution of equation (9.11.23) as

$$
\begin{aligned}
x_m(t) = x^*_{(m)}(t) + \int_{t_0}^{t} U_{(m)}(t,s)\big\{&A_{m\times(n-m)}x_{(n-m)}(s) \\
&+ B_{m\times(n-m)}x_{(n-m)}(s-\tau_1) + D_{m\times(n-m)}x_{(n-m)}(s-\tau_1) \\
&+ k_{(n-m)}\big[f\big(x_n(s-\tau_2) - \theta x_n(s-\tau_2)\big)\big]\big\}\,ds,
\end{aligned}
$$

where $U_{(m)}(t,s)$ is the fundamental matrix solution of the system:

$$
\begin{aligned}
\frac{\partial U_{(m)}(t,s)}{\partial t} = {}& A_{m\times m}U_{(m)}(t,s) + B_{m\times m}U_{(m)}(t-\tau_1,s) \\
&+ D_{m\times m}U_{(m)}(t-\tau_2,s),
\end{aligned}
$$

$$U_{(m)}(t,s) = \begin{cases} 0 & \text{when } \tau - s \leqslant t \leqslant s_0, \\ I_m & \text{when } t = s_0. \end{cases}$$

Finally, we can follow the last part of the proof for Theorem 9.11.3 to show that the zero solution of system (9.11.4) w.r.t. $x_{(m)}$ is also time-delay independent absolutely stable. This completes the proof of Theorem 9.11.15. □

Similarly, let

$$P := \begin{bmatrix} P_{m\times m} & P_{m\times(n-m)} \\ P_{(n-m)\times m} & P_{(n-m)\times(n-m)} \end{bmatrix},$$

$$Q := \begin{bmatrix} Q_{m\times m} & Q_{m\times(n-m)} \\ Q_{(n-m)\times m} & Q_{(n-m)\times(n-m)} \end{bmatrix},$$

$$(O_{n\times(n-1)}\ b\theta) := \overline{D} = \begin{bmatrix} \overline{D}_{m\times m} & \overline{D}_{m\times(n-m)} \\ \overline{D}_{(n-m)\times m} & \overline{D}_{(n-m)\times(n-m)} \end{bmatrix},$$

$$b_{(m)} := (b_1, b_2, \dots, b_m)^T, \qquad b_{(n-m)} := (b_{m+1}, b_{m+2}, \dots, b_n)^T,$$

$$y_{(m)} := (y_1, y_2, \dots, y_m)^T, \qquad y_{(n-m)} := (y_{m+1}, y_{m+2}, \dots, y_n)^T.$$

Then system (9.11.5) can be equivalently written as

$$\begin{aligned} \frac{dy_{(m)}}{dt} = {} & P_{m\times m} y_{(m)}(t) + Q_{m\times m} y_{(m)}(t-\tau_1) + \overline{D}_{m\times m} y_{(m)}(t) \\ & + P_{m\times(n-m)} y_{(n-m)}(t) + Q_{m\times(n-m)} y_{(n-m)}(t-\tau_1) \\ & + \overline{D}_{m\times(n-m)} y_{(n-m)}(t-\tau_1) \\ & + b_{(m)}\big[f\big(y_n(t-\tau_2) - \theta y_n(t-\tau_2)\big)\big], \end{aligned} \tag{9.11.25}$$

$$\begin{aligned} \frac{dy_{(n-m)}}{dt} = {} & P_{(n-m)\times m} y_{(m)}(t) + Q_{(n-m)\times m} y_{(n-m)}(t) + \overline{D}_{(n-m)\times m} y_{(n-m)}(t) \\ & + P_{(n-m)\times(n-m)} y_{(n-m)}(t) + Q_{(n-m)\times(n-m)} y_{(n-m)}(t) \\ & + \overline{D}_{(n-m)\times(n-m)} y_{(n-m)}(t) \\ & + b_{(n-m)}\big[f\big(y_n(t-\tau_2) - \theta y_n(t-\tau_2)\big)\big]. \end{aligned} \tag{9.11.26}$$

Thus we have a similar theorem, as given below.

THEOREM 9.11.16. *If the following conditions are satisfied*:

(1) *All the eigenvalues of* $\det(\lambda I_m - P_{m\times m} - Q_{m\times m}e^{-i\lambda\tau_1} - \overline{D}_{m\times m}e^{-i\lambda\tau_2}) = 0$ *have negative real part.*
(2) *There exist constants* $c_i \geqslant 0,\ i = 1, 2, \dots, m$, *and* $c_i > 0,\ j = m+1, m+2, \dots, n$, *such that*

$$c_j p_{jj} + \left(\sum_{i=1,\, i\neq j}^{n} c_i |p_{ij}| + \sum_{i=1}^{n} c_i |q_{ij}| \right) \leqslant 0, \quad j = 1, 2, \dots, m.$$

$$c_j p_{jj} + \left(\sum_{i=1,\, i\neq j}^{n} c_i |p_{ij}| + \sum_{i=1}^{n} c_i |q_{ij}| \right) < 0,$$
$$j = m+1, m+2, \ldots, n.$$
$$c_n p_{nn} + \left(\sum_{i=1,\, i\neq j}^{n} c_i |p_{in}| + \sum_{i=1}^{n} c_i |q_{in}| + \sum_{j=1}^{n} c_i |h_i| k \right) < 0.$$

Then the zero solution of system (9.11.25)–(9.11.26), namely the zero solution of system (9.11.5), is time-delay independent absolutely stable in the Hurwitz sector $[0, k]$.

The proof of Theorem 9.11.16 is similar to Theorem 9.11.15 and thus omitted.

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Chapter 10

Stability of Neural Networks

In the past decade, the research in the area of neural networks has received considerable attention from scientists and engineers. So far, about 20 well-known Journals are related to this research. It is unprecedented that neural network has covered a wide range of subjects, not to mention a wide range of applications and an incredible diversity of theoretical results. Hopfield and his co-workers [172–177] introduced an energy function to investigate the stability of neural networks, which has been applied to study associative memory and optimal computing for symmetric feedback neural networks. This method has subsequently been modelled by many other scholars.

In this chapter, we first present an exact definition of stability in the sense of Hopfield for neural networks to show the difference between this stability and Lyapunov stability. We also present general methods of constructing energy function. Then we introduce some Lyapunov stability results, related to cell neural networks, bidirectional associative memory neural networks, and general neural networks.

Some materials presented in this chapter are chosen from [172–174] for Section 10.1, [276] for Section 10.2, [234] for Section 10.3, [265] for Section 10.4, [72,278] for Section 10.5, [269] for Section 10.6, [270] for Section 10.7, [372, 373] for Section 10.8, [87] for Section 10.9, [82,83] for Section 10.10,

10.1. Hopfield energy function method

The neural network proposed by Hopfield and his co-workers [173,176,177] can be described by a system of ordinary differential equations as follows:

$$\begin{cases} C_i \frac{du_i}{dt} = \sum_{j=1}^{n} T_{ij} V_j - \frac{u_i}{R_i} + I_i^* := f_i, \quad i = 1, 2, \dots, n, \\ V_i = g_i(u_i), \end{cases} \tag{10.1.1}$$

where the variable $u_i(t)$ represents the voltage on the input of the ith neuron. Each neuron is characterized by an input capacitance, C_i, and a transfer function, $g_i(u_i)$, called sigmoid function. The parallel resistance at the input of the

ith neuron is expressed by R_i, I_i is the current on the input of the ith neuron. The nonlinear transfer function $g_i(u)$ is a sigmoid function, saturating at ± 1 with maximum slope at $u_i = 0$. In term of mathematics, $g_i(u_i)$ is strictly monotone increasing function.

Let $T_{ij} = T_{ji}$. Hopfield [172] first constructed the following energy function:

$$E = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} V_i V_j - \sum_{i=1}^{n} V_i I_i + \sum_{i=1}^{n} \frac{1}{R_i}\int_0^{V_i} g_i^{-1}(\xi)\, d\xi. \tag{10.1.2}$$

Differentiating E with respect to time t along the solution of system (10.1.1) yields

$$\begin{aligned}
\frac{dE}{dt}\bigg|_{(10.1.1)} &= \sum_{i=1}^{n} \frac{\partial E}{\partial V_i}\frac{dV_i}{dt} \\
&= \sum_{i=1}^{n}\left[-\frac{1}{2}\sum_{j=1}^{n} T_{ij}V_j - \frac{1}{2}\sum_{j=1}^{n} T_{ji}V_j + \frac{u_i}{R_i} - I_i\right]\frac{dV_i}{dt} \\
&= \sum_{i=1}^{n}\left[-\frac{1}{2}\sum_{j=1}^{n}(T_{ij} - T_{ij})V_j - \left(\sum_{j=1}^{n} T_{ij}V_j - \frac{u_i}{R_i} + I_i\right)\right]\frac{dV_i}{dt} \\
&= \sum_{i=1}^{n}\left(-C_i \frac{du_i}{dt}\frac{dV_i}{dt}\right) \\
&= -\sum_{i=1}^{n} c_i \dot{g}_i^{-1}(V_i)\left(\frac{dV_i}{dt}\right)^2 \leqslant 0.
\end{aligned} \tag{10.1.3}$$

Since $\dot{g}_i > 0$,

$$\frac{dE}{dt}\bigg|_{(10.1.1)} = 0 \quad \Leftrightarrow \quad \frac{dV_i}{dt} = 0 \quad (i = 1, 2, \ldots, n),$$

implying that

$$-\frac{u_i}{R_i} + \sum_{j=1}^{n} T_{ij}V_j + I_i = 0 \quad (i = 1, 2, \ldots, n). \tag{10.1.4}$$

Based on (10.1.3) and (10.1.4), one obtains the following conclusions:

(1) the neural network (10.1.1) is stable;
(2) the network tends to some static points which are the minimum of the energy function E.

However, we would like to point out that this class stability of neural networks is essentially different from Lyapunov stability. For Lyapunov stability, the equilibria of dynamical systems are known, and the constructed Lyapunov function V has a definite sign in the neighborhood of a known equilibrium, while the time derivative of V on trajectories of the system has the opposite sign of V. For neural networks, the equilibria are unknown, even if the existence and uniqueness of the equilibria are known. The constructed Hopfield type energy functions may have varying signs. In fact, the stability of a neural network means that the network tends to transfer from dynamic state to static state. Thus, it is necessary to establish an exact definition for the stability of neural networks.

DEFINITION 10.1.1. The set Ω_E of equilibria u of system (10.1.1) is said to be attractive, if for any $K > 0$ and any $u_0 \in S_K := \{u, \|u\| < K\}$, the solution of (10.1.1), $u(t, t_0, u_0)$, with the initial point $u = u_0$, satisfies

$$\rho\big\{u(t, t_0, u_0), \Omega_E\big\} \to 0 \quad \text{as } t \to \infty,$$

where $\rho(u, \Omega_E)$ denotes the distance from the point u to Ω_E. System (10.1.1) is said to be Hopfield stable, if the set Ω_E of equilibria of (10.1.1) is attractive.

Generally, there are many equilibria in a neural network. Attraction of the set of equilibria is not equivalent to the attraction of every equilibrium. The Hopfield energy function method can merely guarantee that a solution $u(t, t_0, u_0)$ tends to a certain equilibrium $u^*(u_0)$ of system (10.1.1), where $u^*(u_0)$ depends on the initial point u_0, u^* may be unstable in the sense of Lyapunov. Even for a given equilibrium $u = u^*$, Hopfield energy function method is not able to answer whether it is stable or attractive. Hence, in the following, we first present a series of results concerning the Lyapunov asymptotic stability of a given equilibrium point of certain neural networks.

For the energy function described above, we try to explore the principles of constructing the energy function (10.1.2) of neural network (10.1.1). The principles of other energy functions may be analyzed analogously.

Based upon the gradient method for constructing Lyapunov functions, the time derivative of $E(V)$ on the trajectory of (10.1.1) can be expressed as

$$\left.\frac{dE}{dt}\right|_{(10.1.1)} = (\operatorname{grad} E)^T \bullet f = \sum_{i=1}^{n} \frac{\partial E}{\partial V_i} f_i, \tag{10.1.5}$$

where

$$\operatorname{grad} E = \left(\frac{\partial E}{\partial V_i}, \ldots, \frac{\partial E}{\partial V_n}\right)^T$$

denotes the gradient of $E(u)$, $f = (f_1, \ldots, f_n)^T$. Set $\operatorname{grad} E = -f$. Then

$$(\operatorname{grad} E)^T \cdot f = -\sum_{i=1}^{n} f_i^2 \leqslant 0, \tag{10.1.6}$$

which is equal to zero if and only if $f_i = 0$, $i = 1, 2, \ldots, n$.

The hypotheses $T_{ij} = T_{ji}$ is merely the result of the general rotation:

$$\frac{\partial f_i}{\partial V_j} = T_{ij} = T_{ji} = \frac{\partial f_j}{\partial V_i}. \tag{10.1.7}$$

Integrating $\operatorname{grad} E = -f$ and noting that the integration is not related to the integral path, we have

$$\begin{aligned} E(V) = &-\int_0^{V_1} f_1(\xi_1, 0, \ldots, 0)\, d\xi_1 - \int_0^{V_2} f_2(V_1, \xi_2, 0, \ldots, 0)\, d\xi_1 + \cdots \\ &- \int_0^{V_n} f_n(V_1, V_2, \ldots, V_{n-1}, \xi_n)\, d\xi_n \\ &= -\frac{1}{2}\sum_{j=1}^{n}\sum_{j=1}^{n} T_{ij} V_i V_j - \sum_{i=1}^{n} I_i V_i + \sum_{i=1}^{n} \frac{1}{R_i} \int_0^{V_i} g_i^{-1}(\xi)\, d\xi, \end{aligned} \tag{10.1.8}$$

which, in fact, is the energy function (10.1.2). This is the reason behind development of Hopfield type energy function.

EXAMPLE 10.1.2. Consider a Hopfield network ($n = 2$):

$$\begin{cases} C_1 \frac{du_1}{dt} = -\frac{u_1}{R_1} + T_{11} V_1 + T_{12} V_2 + I_1 := f_1, \\ C_2 \frac{du_2}{dt} = -\frac{u_2}{R_2} + T_{21} V_1 + T_{22} V_2 + I_2 := f_2. \end{cases}$$

Let $T_{12} = T_{21}$. Then we have

$$\begin{aligned} E(V) &= -\int_0^{V_1} f_1(\xi_1, 0)\, d\xi_1 - \int_0^{V_2} f_2(V_1, \xi_2)\, d\xi_2 \\ &= \int_0^{V_1} \left(\frac{u_1}{R_1} - T_{11} V_1 - I_1\right) dV_1 + \int_0^{V_2} \left(\frac{u_2}{R_2} - T_{21} V_1 - T_{22} V_2 - I_2\right) dV_2 \\ &= \int_0^{V_1} \frac{g_1^{-1}(V_1)}{R_1}\, dV_1 + \int_0^{V_2} \frac{g_2^{-1}(V_2)}{R_2}\, dV_2 - \frac{1}{2} T_{11} V_1^2 - I_1 V_1 \end{aligned}$$

$$
\begin{aligned}
& - T_{21} V_1 V_2 - \frac{1}{2} T_{22} V_2^2 - I_2 V_2 \\
= & -\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} T_{ij} V_i V_j + \sum_{i=1}^{2} \frac{1}{R_i} \int_0^{V_i} g_i^{-1}(\xi)\, d\xi - \sum_{i=1}^{2} I_i V_i . \qquad (10.1.9)
\end{aligned}
$$

10.2. Lagrange globally exponential stability of general neural network

In this section, we consider the general recursive neural network model with multiple time delays

$$
\begin{aligned}
C_i \frac{du_i}{dt} = & -d_i u_i(t) + \sum_{j=1}^{n} T_{ij} g_j\big(u_j(t)\big) \\
& + \sum_{j=1}^{n} r_{ij} g_j\big(u_j(t-\tau_{ij})\big) + I_i, \quad i = 1, 2, \ldots, n, \qquad (10.2.1)
\end{aligned}
$$

where $T = (T_{ij})_{n \times n} \in R^{n \times n}$, $r = (r_{ij})_{n \times n} \in R^{n \times n}$, $\sigma < \tau_{ij} \leqslant \tau =$ constant.

Assume that the $g_i(u_i)$ is a sigmoidal function, defined as

$$
S_i = \big\{ g_i(u) \mid g(0) = 0,\ \big|g_i(u)\big| \leqslant k_i,\ i = 1, 2, \ldots, n,\ D^+ g_i(u_i) \geqslant 0 \big\}.
$$

We first consider the Lagrange globally exponential stability. We have found that all neural networks with bounded activation functions, such as Hopfield neural network, bidirectional associative memory neural network, cellular neural network (CNN), are Lagrange globally exponentially stable. This general characteristics, independent of time delay, plays the key role in the study of neural networks. More specifically, its importance has several major aspects.

Firstly, one of the important applications of neural networks is computational optimization in finding the equilibrium points of neural networks. One of the novelty of neural networks is that it can be used to solve nonlinear algebraic or transcendent equations by using electronic circuits to realize differential equations. The globally exponentially attractive set can provide prior knowledge for optimization. That is, having found such an attractive set, the bound of the optimization solution can be roughly determined.

Secondly, it has been recently found the neural network described by (10.2.1) can exhibit chaotic motions. To identify chaos, one usually needs to assure that the system is ultimately bounded and the system has at least one positive Lyapunov exponent. Therefore, it is important to study stability in the sense of Lagrange since it is directly related to the ultimate boundedness of the system.

Thirdly, the global stability in the sense of Lyapunov on unique equilibrium point and the stability in the sense of Hopfield on equilibrium point set can be treated as special cases of stability in the sense of Lagrange.

Let

$$\left|g_i(u_i)\right| \leqslant k_i, \qquad 2M_i := \sum_{j=1}^{n}\bigl(|T_{ij}| + |r_{ij}|\bigr)k_j + |I_i|,$$

and

$$\Omega = \left\{u \;\middle|\; \sum_{i=1}^{n} \frac{1}{2}C_i u_i^2 \leqslant \frac{\sum_{i=1}^{n} M_i^2/\varepsilon_i}{2\min_{1\leqslant j\leqslant n}(d_i - \varepsilon_i)/C_i}\right\}, \quad \text{where } \sigma < \varepsilon_i < d_i.$$

THEOREM 10.2.1. *If $g(u) \in S$, then the neural network (10.2.1) is Lagrange globally exponentially stable, and the set Ω is a globally exponentially attractive set.*

PROOF. We employ the positive definite and radially unbounded Lyapunov function:

$$V(u) = \frac{1}{2}\sum_{i=1}^{n} C_i u_i^2,$$

and choose ε_i such that $0 < \varepsilon_i < d_i$. Then,

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(10.2.1)} &= \sum_{i=1}^{n} C_i u_i \frac{du_i}{dt} \\
&\leqslant \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\bigl(|T_{ij}| + |r_{ij}|\bigr)k_j|u_i| - d_i u_i^2 + |I_i||u_i|\right] \\
&= -\sum_{i=1}^{n}\left\{-\left[d_i u_i^2 - \left(\sum_{j=1}^{n}\bigl(|T_{ij}| + |r_{ij}|\bigr)k_j + |I_i|\right)|u_i|\right]\right\} \\
&= -\sum_{i=1}^{n} d_i u_i^2 + \sum_{i=1}^{n} 2M_i|u_i| \\
&\leqslant -\sum_{i=1}^{n} d_i u_i^2 + \sum_{i=1}^{n} \varepsilon_i u_i^2 + \sum_{i=1}^{n} \frac{M_i^2}{\varepsilon_i} \\
&\leqslant -\min_{1\leqslant i\leqslant n}\frac{d_i - \varepsilon_i}{C_i}\sum_{i=1}^{n} C_i u_i^2 + \sum_{i=1}^{n}\frac{M_i^2}{\varepsilon_i}
\end{aligned}$$

$$
\begin{aligned}
&= -2 \min_{1\leqslant i\leqslant n} \frac{d_i - \varepsilon_i}{C_i} V\big(u(t)\big) + \sum_{i=1}^{n} \frac{M_i^2}{\varepsilon_i} \\
&\leqslant -2 \min_{1\leqslant i\leqslant n} \frac{d_i - \varepsilon_i}{C_i} \\
&\quad \times \left[V\big(u(t)\big) - \frac{\sum_{i=1}^{n} M_i^2/\varepsilon_i}{2\min_{1\leqslant j\leqslant n}(d_i - \varepsilon_i)/C_i} \right].
\end{aligned} \tag{10.2.2}
$$

Hence, when

$$
V\big(u(t)\big) > \frac{\sum_{i=1}^{n} M_i^2/\varepsilon_i}{2\min_{1\leqslant j\leqslant n}(d_i - \varepsilon_i)/c_i} \quad \text{for } t \geqslant t_0,
$$

we have

$$
\begin{aligned}
&V\big(u(t)\big) - \frac{\sum_{i=1}^{n} M_i^2/\varepsilon_i}{2\min_{1\leqslant j\leqslant n}(d_i - \varepsilon_i)/C_i} \\
&\quad \leqslant \left[V\big(u(0)\big) - \frac{\sum_{i=1}^{n} M_i^2/\varepsilon_i}{2\min_{1\leqslant j\leqslant n}(d_i - \varepsilon_i)/C_i} \right] \\
&\quad \times e^{-2\min_{1\leqslant i\leqslant n} \frac{d_i-\varepsilon_i}{C_i}(t-t_0)}.
\end{aligned} \tag{10.2.3}
$$

This indicates that the ellipse Ω is a globally exponentially attractive set of (10.2.1), i.e., system (10.2.1) is Lagrange globally exponentially stable. □

10.3. Extension of Hopfield energy function method

In this section, by using LaSalle invariant principle, we prove more general stability theorems for Hopfield neural network.

THEOREM 10.3.1. *If there exist some positive constants β_i $(i = 1, 2, \ldots, n)$ such that $(\beta_i T_{ij})_{n\times n}$ is a symmetric matrix, then the network* (10.1.1) *is stable in the sense of Hopfield.*

PROOF. We employ the general Hopfield type energy function:

$$
\begin{aligned}
W(u) = {} & -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_i T_{ij}) V_i V_j - \sum_{i=1}^{n} \beta_i I_i V_i \\
& + \sum_{i=1}^{n} \beta_i \frac{1}{R_i} \int_0^{V_i} g_i^{-1}(\xi)\, d\xi,
\end{aligned} \tag{10.3.1}
$$

and obtain

$$
\begin{aligned}
\left.\frac{dW(u)}{dt}\right|_{(10.1.1)} &= \sum_{i=1}^{n} \frac{\partial E}{\partial V_i} \frac{dV_i}{dt} \\
&= \sum_{i=1}^{n} \left[-\frac{1}{2} \sum_{j=1}^{n} (\beta_j T_{ji}) V_j - \frac{1}{2} \sum_{j=1}^{n} (\beta_i T_{ij}) V_j + \beta_i \frac{u_i}{R_i} - \beta_i I_i \right] \frac{dV_i}{dt} \\
&= \sum_{i=1}^{n} \left[-\frac{1}{2} \big(\beta_j T_{ji} - \beta_i T_{ij}\big) V_j - \sum_{i=1}^{n} (\beta_i T_{ij}) V_j + \beta_i \frac{u_i}{R_i} - \beta_i I_i \right] \frac{dV_i}{dt} \\
&= -\sum_{i=1}^{n} C_i \beta_i \frac{du_i}{dt} \frac{dV_i}{dt} \\
&= -\sum_{i=1}^{n} C_i \beta_i \dot{g}_i(u_i) \left(\frac{du_i}{dt} \right)^2 \leqslant 0.
\end{aligned} \tag{10.3.2}
$$

Since $\dot{g}_i(u_i) > 0$,

$$
\frac{dW}{dt} = 0 \quad \Longleftrightarrow \quad \frac{du_i}{dt} = 0 \quad (i = 1, 2, \dots, n),
$$

implying that

$$
-\frac{u_i}{R_i} + \sum_{j=1}^{n} T_{ij} V_j + I_i = 0, \quad i = 1, 2, \dots, n.
$$

Since $W(u(t))$ is monotone decreasing, $W(u(t))$ is bounded in D. Thus, $\lim_{t\to\infty} W(u(t)) = W_0$. $\forall u_0 \in D$, let $\Omega(u_0)$ be ω-limit set of $u(t, t_0, u_0)$, E the set of equilibria of (10.1.1), M the largest invariant set of (10.1.1) in D. Then, by LaSalle invariance principle [222], we have

$$
u(t, t_0, u_0) \to M \quad \text{as } (t \to +\infty),
$$

but $\Omega(u_0) \subseteq M \subseteq E$.

Hence, E is attractive, i.e., system (10.1.1) is stable in the sense of Hopfield. □

Especially, for $\beta_i \equiv 1$, $i = 1, 2, \dots, n$, $W(u)$ is the original energy function proposed by Hopfield [175].

EXAMPLE 10.3.2. (See [234].) For the following neural network:

$$
\begin{cases}
\frac{du_1}{dt} = -u_1 + 2V_1 + V_2 + I_1, \\
\frac{du_2}{dt} = -u_1 + 3V_1 + 4V_2 + I_2,
\end{cases} \tag{10.3.3}
$$

the weight matrix $T = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ is not symmetric. However, if we take $\beta_1 = 3$, $\beta_2 = 1$, then

$$(\beta_i T_{ij}) = \begin{bmatrix} 6 & 3 \\ 3 & 4 \end{bmatrix}$$

is a symmetric matrix.

Obviously, we cannot determine the asymptotic behavior of this system by the original Hopfield energy function. But we can do so using our extensive energy function.

EXAMPLE 10.3.3. (See [234].) Consider the following Hopfield network (n =3):

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ 1/2 & 1 & 1 \end{bmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}. \tag{10.3.4}$$

Take

$$\beta_1 = 2, \quad \beta_2 = 1, \quad \beta_3 = 4,$$

and thus

$$(\beta_i T_{ij}) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

is a symmetric matrix.

So system (10.3.4) is stable in the sense of Hopfield.

In the following, by the method and results of interval matrix stability, we analyze the Hopfield type stability of (10.1.1).

Let $y_i = \frac{du_i}{dt}$ ($i = 1, 2, \ldots, n$). Then system (10.1.1) can be rewritten as

$$\frac{dy_i}{dt} = -\frac{y_i}{C_i R_i} + \sum_{j=1}^{n} \frac{T_{ij}}{C_i} g_j'(u_j) y_j. \tag{10.3.5}$$

Let

$$\underline{T}_{ij} = \inf_{u_j \in R^1} \left\{ \frac{T_{ij}}{C_i} g_j'(u_j) \right\}, \qquad \overline{T}_{ij} = \sup_{u_j \in R^1} \left\{ \frac{T_{ij}}{C_i} g_j'(u_j) \right\}.$$

We consider an interval dynamical system:

$$\frac{dy_i}{dt} = -\frac{y_i}{R_i C_i} + \sum_{j=1}^{n} a_{ij} y_j \quad (i = 1, 2, \ldots, n), \tag{10.3.6}$$

where $a_{ij} \in [\underline{T}_{ij}, \overline{T}_{ij}]$.

THEOREM 10.3.4. *Assume that*

(1) $-\frac{1}{C_i R_i} + \overline{T}_{ii} < 0 \quad (i = 1, 2, \ldots, n)$;

(2)

$$B := \left\{\left(-\frac{1}{C_i R_i} - \overline{T}_{ii}\right) - (1 - \delta_{ij}) \max\left\{\left|\underline{T}_{ij}\right|, \left|\overline{T}_{ij}\right|\right\}\right\}_{n\times n}$$
$$:= (b_{ij})_{n\times n}$$

is an M matrix.

Then the zero solution of system (10.3.6) is asymptotically stable. Hence, the system (10.1.1) is stable in the sense of Hopfield.

PROOF. Since M is an M matrix, $(B^T)^{-1} \geqslant 0$. Then, $\forall \xi = (\xi_1, \ldots, \xi_n)^T > 0$, $\eta := (B^T)^{-1}\xi > 0$, i.e., $\xi_i > 0$, $\eta_i > 0$ $(i = 1, 2, \ldots, n)$. $\forall a_{ij} \in [\underline{T}_{ij}, \overline{T}_{ij}]$, by employing the positive definite and radially unbounded Lyapunov function:

$$W(y) = \sum_{i=1}^{n} \eta_i |y_i|,$$

we have

$$\begin{aligned}
D^+W(y)|_{(10.3.6)} &= \sum_{i=1}^{n} \operatorname{sgn}(y_i)\left(-\frac{1}{C_i R_i} y_i + \sum_{j=1}^{n} a_{ij} y_j\right)\eta_i \\
&\leqslant -|y|^T B^T \eta \\
&= -\sum_{i=1}^{n} \xi_i |y_i| < 0 \quad \text{when } y \neq 0.
\end{aligned}$$

So, the conclusion is true. □

THEOREM 10.3.5. *If the matrix $B^* = (b^*_{ij})_{n\times n}$ is positive definite, then the zero solution of system (10.3.6) is asymptotically stable and thus the system (10.1.1) is stable in the sense of Hopfield, where*

$$\begin{aligned}
b^*_{ii} &= \frac{1}{C_i R_i} - \overline{T}_{ii}, \quad i = 1, 2, \ldots, n, \\
b^*_{ij} &= \max_{i\neq j}\left[\frac{|\underline{T}_{ij} + \underline{T}_{ji}|}{2}, \frac{|\overline{T}_{ij} + \overline{T_{ji}}|}{2}\right], \quad i \neq j,\ i, j = 1, 2, \ldots, n.
\end{aligned}$$

PROOF. Choose the positive definite and radially unbounded Lyapunov function:

$$W(y) = \sum_{i=1}^{n} \frac{1}{2} y_i^2,$$

which results in

$$\begin{aligned}
\left.\frac{dW(y)}{dt}\right|_{(10.3.6)} &= \sum_{i=1}^{n} y_i \frac{dy_i}{dt} = \sum_{i=1}^{n} \frac{y_i^2}{C_i R_i} + \sum_{j=1}^{n}\sum_{j=1}^{n} a_{ij} y_i y_j \\
&\leqslant -\sum_{i=1}^{n}\left(\frac{1}{C_i R_i} - \overline{T}_{ii}\right) y_i^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(\frac{a_{ij}+a_{ji}}{2}\right) y_i y_j \\
&\leqslant -\sum_{i=1}^{n}\left(\frac{1}{C_i R_i} - \overline{T}_{ii}\right)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n} b_{ij}^{*} |y_i||y_j| \\
&\leqslant \big(|y_1| \cdots |y_n|\big)\big(B^{*}\big)\big(|y_1|, \ldots, |y_n|\big)^{T} \\
&< 0 \quad \text{when } y \neq 0.
\end{aligned}$$

Thus, the conclusion is true. □

EXAMPLE 10.3.6. (See [234].) Consider a 2-dimension Hopfield neural network:

$$\begin{cases} C_1 \frac{du_1}{dt} = -\frac{1}{R_1} + T_{11} g_1(u_1) + T_{12} g_2(u_2) + I_1, \\ C_2 \frac{du_2}{dt} = -\frac{1}{R_2} + T_{21} g_1(u_1) + T_{22} g_2(u_2) + I_2. \end{cases} \tag{10.3.7}$$

Assume that

$$\begin{aligned}
&C_i = \frac{1}{2}, \quad R_i = \frac{1}{3} \quad (i = 1, 2), \quad \overline{T}_{ii} = 2 \quad (i = 1, 2), \\
&\underline{T}_{ii} = -2 \quad (i = 1, 2), \\
&\underline{T}_{12} = -3.5, \quad \overline{T}_{12} = 3, \quad \underline{T}_{21} = -3, \quad \overline{T}_{21} = 3.5. \\
&\underline{T}_{ii} = \inf_{u_j \in R^1} \left\{ \frac{T_{ij}}{C_i} \dot{g}_i(u_j) \right\}, \qquad \overline{T}_{ij} = \sup_{u_j \in R^1} \left\{ \frac{T_{ij}}{C_i} \dot{g}_j(u_j) \right\}.
\end{aligned} \tag{10.3.8}$$

Thus,

$$-\frac{1}{C_i R_i} + \overline{T}_{ii} = -6 + 2 = -4 < 0 \quad (i = 1, 2),$$

$$\begin{aligned}
B &:= \begin{bmatrix} \frac{1}{C_1 R_1} - \overline{T}_{11} & -\max(|\underline{T}_{12}|, |\overline{T}_{12}|) \\ -\max(|\underline{T}_{21}|, |\overline{T}_{21}|) & \frac{1}{C_2 R_2} - \overline{T}_{22} \end{bmatrix} \\
&= \begin{bmatrix} 4 & -3.5 \\ -3.5 & 4 \end{bmatrix}.
\end{aligned}$$

It is clear that B is positive definite, and so system (10.3.7) is stable in the sense of Hopfield by Theorem 10.3.4.

EXAMPLE 10.3.7. (See [234].) Consider a 3-dimensional Hopfield neural network:

$$C_i \frac{du_i}{dt} = -\frac{1}{R_i} + \sum_{j=1}^{3} T_{ij} g_j(u_j) + I_i \quad (i = 1, 2, 3). \tag{10.3.9}$$

Let

$$\begin{aligned}
&C_i = \frac{1}{2}, \quad R_i = \frac{1}{2} \quad (i = 1, 2, 3), \quad \overline{T}_{ii} = 1 \quad (i = 1, 2, 3), \quad \underline{T}_{12} = -1,\\
&\underline{T}_{21} = -2, \quad \overline{T}_{12} = 2, \quad \overline{T}_{21} = 1, \quad \underline{T}_{13} = \underline{T}_{31} = -1,\\
&\overline{T}_{13} = \overline{T}_{31} = 1.4, \quad \underline{T}_{23} = \underline{T}_{32} = -1.2, \quad \overline{T}_{23} = \overline{T}_{32} = 1.3,
\end{aligned}$$

where $\underline{T}_{ij}, \overline{T}_{ij}$ are obtained by using (10.3.8). By Theorem 10.3.5, construct the matrix B^* as

$$B^* = \begin{bmatrix} 3 & 1.5 & 1.4 \\ 1.5 & 3 & 1.3 \\ 1.4 & 1.3 & 3 \end{bmatrix}$$

which is positive definite. So, system (10.3.9) is stable in the sense of Hopfield.

In the following, we use Karsovskii's theorem [193] to study the Hopfield type stability. Rewrite (10.1.1) as

$$\frac{du_i}{dt} = -\frac{u_i}{C_i R_i} + \sum_{j=1}^{n} \frac{T_{ij}}{C_i} V_j + \frac{I_i}{C_i} := f_i(u_i), \quad j = 1, 2, \ldots, n. \tag{10.3.10}$$

THEOREM 10.3.8. *If there exists a symmetric positive matrix $P = (p_{ij})_{n\times n}$ such that the matrix $Q := (PJ + J^T P)$ is negative definite, then system (10.3.10) is stable in the sense of Hopfield, where $J = (\frac{\partial f_i}{\partial t_{ij}})_{n\times n}$ is the Jacobian matrix of f.*

PROOF. Take the Lyapunov function:

$$W(x) = f^T(u) P f(u),$$

where $f = (f_1(u), \ldots, f_n(u))^T$. Obviously, $W(u) \geqslant 0$, $W(u) = 0$ if and only if u is an equilibrium point. Further, we have

$$\begin{aligned}
\left.\frac{dW}{dt}\right|_{(10.3.10)} &= f^T P \dot{f} + \dot{f}^T P f = f^T P J f + (Jf)^T P f \\
&= f^T \big(PJ + J^T P\big) f = f^T Q f \leqslant 0,
\end{aligned} \tag{10.3.11}$$

and $\frac{dW}{dt} = 0$ if and only if u is an equilibrium point of (10.3.10). So, the conclusion is true. □

The Hopfield energy function method can only guarantee that a solution $u(t, t_0, u_0)$ tends to a certain equilibrium $u^*(u)$ of (10.1.1) but cannot answer whether $u = u^*$ is stable or attractive in the sense of Lyapunov. Now we employ another Lyapunov function:

$$E^*(V) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} V_i V_j + \sum_{i=1}^{n}\int_{V_i^*}^{V_i} \frac{g_i^{-1}(V_i)}{R_i}\, dV_i$$
$$- \sum_{i=1}^{n} I_i\big(V_i - V_i^*\big) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} V_i^* V_j^* \tag{10.3.12}$$

where $V_i^* = g_i(u_i^*)$, $u = u^*$ is an equilibrium point of system (10.3.10).

THEOREM 10.3.9. *If the matrix*

$$\left(\frac{\partial^2 E^*}{\partial V^2}\right)_{V=V^*} := \left(\frac{\partial^2 E^*}{\partial V_i \partial V_j}\right)_{n\times n}\bigg|_{V=V^*} \tag{10.3.13}$$

is positive definite, then $u = u^*$ of (10.3.10) *is asymptotically stable in the sense of Lyapunov.*

PROOF. We have $E^*(V^*) = 0$, and

$$\begin{aligned}
\frac{\partial E^*(V)}{\partial V_i}\bigg|_{V=V^*} &= -\sum_{j=1}^{n} T_{ij} V_j^* + \frac{1}{R_i} g_i^{-1}\big(V_i^*\big) - I_i = 0 \quad (i = 1, \dots, n),\\
\frac{\partial^2 E^*(V)}{\partial V_i^2}\bigg|_{V=V^*} &= -T_{ii} + \frac{1}{R_i}\big(g_i^{-1}\big(V_i^*\big)\big) \quad (i = 1, \dots, n),\\
\frac{\partial^2 E^*(V)}{\partial V_i \partial V_j}\bigg|_{V=V^*} &= \frac{\partial^2 E^*}{\partial V_j \partial V_i}\bigg|_{V=V^*}\\
&= -T_{ij} = -T_{ji} \quad (i \neq j,\ i, j = 1, \dots, n).
\end{aligned}$$

Since $(\frac{\partial^2 E}{\partial V^2})_{V=V^*}$ is positive definite, so is $E^*(V)$ in certain neighborhood of $u = u^*$. Moreover,

$$\begin{aligned}
\frac{dE^*(V)}{dt}\bigg|_{(10.3.10)} &= \sum_{i=1}^{n}\left(-\sum_{j=1}^{n} T_{ij} V_j + \frac{g_i^{-1}(V_i)}{R_i} - I_i\right)\frac{dV_i}{dt}\\
&\quad - \sum_{i=1}^{n} C_i \dot{g}_i^{-1}(V_i)\left(\frac{dV_i}{dt}\right)^2 \begin{cases} < 0\ (V \neq V^*)\\ = 0\ (V = V^*)\end{cases}\\
&\quad \text{for } \big\|V - V^*\big\| \ll 1.
\end{aligned} \tag{10.3.14}$$

Therefore, $u = u^*$ of (10.3.10) is asymptotically stable in the sense of Lyapunov. □

COROLLARY 10.3.10. *If $(g_i^{-1}(V_i))' \geqslant 0$ and the matrix $(T_{ij})_{n\times n}$ is symmetric negative definite, then any equilibrium point of (10.3.10) is asymptotically stable in the sense of Lyapunov.*

PROOF. Since $\frac{1}{R_i}(g_i^{-1}(V_i))' \geqslant 0$, we have that

$$\operatorname{diag}\left(\frac{1}{R_1}\dot{g}_1^{-1}\left(V_1^*\right), \dots, \frac{1}{R_n}g_n'^{-1}\left(V_n^*\right)\right)$$

is positive semi-definite. Hence, the quadratic form:

$$\begin{aligned}
&\left(V - V^*\right)^T \left(\frac{\partial^2 E}{\partial V_i \partial V_j}\right)_{n\times n}\bigg|_{V=V^*}\left(V - V^*\right) \\
&\quad = -\left(V - V^*\right)^T (T_{ij})_{n\times n}\left(V - V^*\right) + \left(V - V^*\right)^T \\
&\qquad \times \operatorname{diag}\left[\left(\frac{1}{R_1}g_1'^{-1}\left(V_1^*\right), \dots, \frac{1}{R_n}g_n'^{-1}\left(V_n^*\right)\right)\right]\left(V - V^*\right) \\
&\quad \geqslant \left(V - V^*\right)^T (T_{ij})_{n\times n}\left(V - V^*\right) \\
&\quad > 0 \quad \text{for } V \neq V^*,\ \left\|V - V^*\right\| \ll 1,
\end{aligned} \tag{10.3.15}$$

indicating that the conclusion is true. □

COROLLARY 10.3.11. *If $\dot{g}_i(u_i) > 0$ $(i = 1, 2, \dots, n)$ and the matrix $(T_{ij})_{n\times n}$ is symmetric negative semi-definite, then any equilibrium point of (10.3.10) is asymptotically stable.*

PROOF. By (10.3.15) we have

$$\begin{aligned}
&\left(V - V^*\right)^T \left(\frac{\partial^2 E}{\partial V_i \partial V_j}\right)\left(V - V^*\right) \\
&\quad = -\left(V - V^*\right)^T (T_{ij})_{n\times n}\left(V - V^*\right) + \left(V - V^*\right)^T \\
&\qquad \times \operatorname{diag}\left[\left(\frac{1}{R_1}\dot{g}_1^{-1}(u_1), \dots, \frac{1}{R_n}\dot{g}_n^{-1}(u_n)\right)\right]\left(V - V^*\right) \\
&\quad \geqslant \left(V - V^*\right)^T \operatorname{diag}\left[\frac{1}{R_1}\dot{g}^{-1}\left(V_1^*\right), \dots, -\frac{1}{R_n}\dot{g}^{-1}\left(V_n^*\right)\right]\left(V - V^*\right) \\
&\quad > 0 \quad \text{for } V \neq V^*,
\end{aligned}$$

which implies that the conditions in Theorem 10.3.9 are satisfied, and thus the conclusion holds. □

THEOREM 10.3.12. *If the matrix* $(\frac{\partial^2 E}{\partial V_i \partial V_j})_{n\times n}|_{V=V^*}$ *is negative definite, then any equilibrium point of* (10.1.1) *is unstable.*

PROOF. The condition of Theorem 10.3.12 implies that $E^*(V)$ is negative definite in certain neighborhood of $u = u^*$ and so is $\frac{dE^*(V)}{dt}$. By Lyapunov unstable theorem, the conclusion is true. □

In the following, we use the first approximation theory to study the stability. Let

$$B_{ij} = \begin{cases} \frac{T_{ii}}{C_i} \frac{\partial g_i(u_i)}{\partial u_i}\Big|_{u_i=u_i^*} - \frac{1}{C_i}, & i = j = 1, 2, \ldots, n, \\ \frac{1}{C_j} T_{ij} \frac{\partial g_i(u_i)}{\partial u_i}\Big|_{u_i=u_i^*}, & i \neq j,\ i = j = 1, 2, \ldots, n. \end{cases} \tag{10.3.16}$$

THEOREM 10.3.13. *If* $((-1)^{\delta_{ij}}|b_{ij}|)_{n\times n}$ *is a Hurwitz matrix, then* (1) $b_{ii} < 0$ *implies that* $u = u^*$ *of* (10.3.10) *is asymptotically stable*; *and* (2) $u = u^*$ *of* (10.3.10) *is unstable if there exists at least one* $b_{i_0 i_0} > 0$.

PROOF. The first approximation of system (10.3.10) is

$$\frac{du}{dt} = B(u - u^*), \quad B = (b_{ij})_{n\times n}.$$

(1) $((-1)^{\delta_{ij}}|b_{ij}|)$ is a Hurwitz matrix $\Longleftrightarrow$ $\exists$ constants $\xi_i > 0$ $(i = 1, \ldots, n)$ such that

$$\xi_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i |b_{ij}| < 0, \quad j = 1, 2, \ldots, n,$$

for (10.3.10). Then it holds

$$\begin{aligned} E^+ W(u)|_{(10.3.10)} &= \sum_{i=1}^{n} \xi_i \frac{du_i}{dt} \operatorname{sgn}\big|u_i - u_i^*\big| \\ &\leqslant \sum_{j=1}^{n} \Bigg[\xi_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i b_{ij} \Bigg] \big|u_i - u_i^*\big| < 0 \end{aligned} \tag{10.3.17}$$

for $u \neq u^*$. So B is stable. According to the first approximation theory, we know that $u = u^*$ of (10.3.10) is asymptotically stable.

(2) $((-1)^{\delta_{ij}}|b_{ij}|)$ is a Hurwitz matrix and at least one $b_{i_0 i_0} > 0$. Without loss of generality, we assume that

$$b_{ii} < 0, \quad i = 1, 2, \ldots, m,$$

$$b_{ii} > 0, \quad i = m+1, \dots, n, \ 1 \leqslant m < n.$$

Therefore, there exist constants $\xi_i > 0$ such that

$$\xi_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i |b_{ij}| < 0, \quad \text{for } j = 1, \dots, m,$$

$$-\xi_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i |b_{ij}| < 0, \quad \text{for } j = m+1, \dots, m.$$

Choose the function

$$W(u) = \sum_{i=1}^{m} \xi_i \left|u_i - u_i^*\right| - \sum_{i=m+1}^{n} \xi_i \left|u_i - u_i^*\right|. \tag{10.3.18}$$

Then, we obtain

$$\begin{aligned} D^+ W(u)|_{(10.3.10)} \leqslant{} & \sum_{j=1}^{m} \left[\xi_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i |b_{ij}|\right] \left|u_j - u_j^*\right| \\ & - \sum_{j=m+1}^{n} \left[\xi_j b_{jj} - \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i |b_{ij}|\right] \left|u_j - u_j^*\right| < 0 \end{aligned}$$

for $u = u^*$. Furthermore, we can prove that $D^+ W(u)|_{(10.3.10)}$ is negative definite in the neighborhood of $u = u^*$. So $u = u^*$ is unstable. □

10.4. Globally exponential stability of Hopfield neural network

In recent years, research on global stability of Hopfield neural network has received considerable attention. However in all existing results, the activation functions of neural networks are restricted to sigmoid functions, or piecewise linear monotone nondecreasing functions, or Lipschitz type functions, i.e., g_i or $\dot{g}_i$ is bounded.

In this section, we first consider a strapping nonlinear activation function, which only requires $\dot{g}_i(u_i) \geqslant 0$, without assuming that: (1) g_i and $\dot{g}_i$ are bounded; and (2) the weight matrices are symmetry.

Two global exponential stability theorems are given below. Let $u = (u_1, \dots, u_n)^T$, and $u^* = (u_1^*, \dots, u_n^*)^T$ be an equilibrium of system (10.1.1).

THEOREM 10.4.1. *If there exist constants $\xi_i > 0$, $\eta_i > 0$, $i = 1, \dots, n$, such that the following matrix A is negative definite, then*

(1) *the equilibrium $u = u^*$ of system (10.1.1) is globally exponentially stable*;

(2) $\varepsilon = \frac{\lambda}{\mu}$ *is estimation of the Lyapunov exponent, where*

$$
\begin{aligned}
&A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}_{2n\times 2n}, \qquad A_{11} = \operatorname{diag}\left(-\frac{\xi_1}{R_1}, \dots, -\frac{\xi_n}{R_n}\right), \\
&A_{12} = \operatorname{diag}\left(-\frac{\eta_1}{2R_1}, \dots, -\frac{\eta_n}{2R_n}\right)_{n\times n} + \left(\frac{\xi_i T_{ij} + \xi_j T_{ji}}{2}\right)_{n\times n}, \\
&A_{22} = \left(\frac{\eta_i T_{ij} + \eta_j T_{ij}}{2}\right)_{n\times n}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}_{2n\times 2n}, \\
&B_{11} = \operatorname{diag}\left(\frac{C_1\xi_1}{2}, \dots, \frac{C_n\xi_n}{2}\right)_{n\times n}, \\
&B_{12} = B_{12}^T = \operatorname{diag}\left(\frac{\eta_1 C_1}{2}, \dots, \frac{\eta_n C_n}{2}\right)_{n\times n}, \\
&B_{22} = O_{n\times n}, \\
&-\lambda = \lambda_{\max}(A), \qquad \mu = \lambda_{\max}(B)
\end{aligned}
$$

in which $\lambda_{\max}(A)$ *and* $\lambda_{\max}(B)$ *denote the maximum eigenvalues for* A *and* B*, respectively.*

PROOF. (1) Let

$$
\begin{aligned}
x = (x_1, \dots, x_n)^T &= \left(u_1 - u_1^*, \dots, u_n - u_n^*\right)^T, \\
f_i(x_i) &= g_i\left(x_i + u_i^*\right) - g_i\left(u_i^*\right).
\end{aligned}
$$

We rewrite (10.1.1) as

$$
C_i \frac{dx_i}{dt} = \sum_{j=i}^{n} T_{ij} f_j(x_j) - \frac{x_i}{R_i} \quad (i = 1, \dots, n). \tag{10.4.1}
$$

Then the stability of u^* of (10.1.1) is equivalent to the stability of the zero solution of (10.4.1). We employ the Lyapunov function:

$$
V(x) = \sum_{i=1}^{n} \frac{C_i \xi_i}{2} x_i^2 + \sum_{i=1} \eta_i C_i \int_0^{x_i} f_i(x_i)\, dx_i, \tag{10.4.2}
$$

with

$$
\begin{aligned}
&V(0) = 0, \quad V(x) > 0 \text{ for } x \neq 0, \\
&V(x) \geqslant \sum_{i=1}^{n} \frac{C_i \xi_i}{2} x_i^2 \to \infty \quad \text{as } x \to \infty.
\end{aligned}
$$

Thus, $V(x)$ is a positive definite and radially unbounded Lyapunov function. Differentiating V with respect to time t along the solution of (10.4.1) yields

$$\begin{aligned}\frac{dV}{dt}\bigg|_{(10.4.1)} &= -\sum_{i=1}^{n}\frac{\xi_i}{R_i}x_i^2+\sum_{i=1}^{n}\xi_i x_i\sum_{j=1}^{n}T_{ij}f_j(x_j)\\ &\quad -\sum_{i=1}^{n}\frac{\eta_i x_i}{R_i}f_i(x_i)+\sum_{i=1}^{n}\eta_i\sum_{j=1}^{n}T_{ij}f_j(x_j)f_i(x_i)\\ &= \begin{pmatrix}x\\ f(x)\end{pmatrix}^T\begin{bmatrix}A_{11} & A_{12}\\ A_{12}^T & A_{22}\end{bmatrix}\begin{pmatrix}x\\ f(x)\end{pmatrix}. \end{aligned}\tag{10.4.3}$$

Let $W = e^{\varepsilon t}V$, where $0 < \varepsilon \ll 1$. Due to $D^+ f_i(x_i) \geqslant 0$, we have

$$\begin{aligned}&\frac{dW}{dt}\bigg|_{(10.4.1)}\\ &= \varepsilon e^{\varepsilon t}V + e^{\varepsilon t}\frac{dV}{dt}\bigg|_{(10.4.1)}\\ &= e^{\varepsilon t}\left\{\varepsilon V+\begin{pmatrix}x\\ f(x)\end{pmatrix}^T\begin{bmatrix}A_{11} & A_{12}\\ A_{12}^T & A_{22}\end{bmatrix}\begin{pmatrix}x\\ f(x)\end{pmatrix}\right\}\\ &= e^{\varepsilon t}\left\{\varepsilon\left(\sum_{i=1}^{n}\frac{C_i\xi_i}{2}x_i^2+\sum_{i=1}^{n}\eta_i C_i\int_0^{x_i}f_i(x_i)\,dx_i\right)\right.\\ &\qquad\left.+\begin{pmatrix}x\\ f(x)\end{pmatrix}^T\begin{bmatrix}A_{11} & A_{12}\\ A_{12}^T & A_{22}\end{bmatrix}\begin{pmatrix}x\\ f(x)\end{pmatrix}\right\}\\ &\leqslant e^{\varepsilon t}\left\{\varepsilon\left(\sum_{i=1}^{n}\frac{C_i\xi_i}{2}x_i^2+\sum_{i=1}^{n}\eta_i C_i x_i f_i(x_i)\right)\right.\\ &\qquad\left.+\begin{pmatrix}x\\ f(x)\end{pmatrix}^T\begin{bmatrix}A_{11} & A_{12}\\ A_{12}^T & A_{22}\end{bmatrix}\begin{pmatrix}x\\ f(x)\end{pmatrix}\right\}\\ &= e^{\varepsilon t}\begin{pmatrix}x\\ f(x)\end{pmatrix}^T\begin{bmatrix}A_{11}+B_{11}\varepsilon & A_{12}+B_{12}\varepsilon\\ A_{12}^T+B_{12}^T\varepsilon & A_{22}+B_{22}\varepsilon\end{bmatrix}\begin{pmatrix}x\\ f(x)\end{pmatrix}\end{aligned}\tag{10.4.4}$$

when $0 < \varepsilon \ll 1$. The negative definite property of

$$\begin{bmatrix}A_{11} & A_{12}\\ A_{12}^T & A_{22}\end{bmatrix}$$

implies that

$$\begin{bmatrix}A_{11}+B_{11}\varepsilon & A_{12}+B_{12}\varepsilon\\ A_{12}^T+B_{12}^T\varepsilon & A_{22}+B_{22}\varepsilon\end{bmatrix}$$

is negative definite. So, when $0 < \varepsilon \ll 1$

$$\frac{dW}{dt}\bigg|_{(10.4.1)} \leqslant 0. \tag{10.4.5}$$

Integrating both sides of (10.4.5) from 0 to arbitrary $t^* > 0$, we obtain

$$e^{\varepsilon t} V\big(x(t)\big) = W\big(x(t)\big) \leqslant W\big(x(0)\big) := W_0 < \infty.$$

Hence, it holds

$$\sum_{i=1}^{n} \frac{C_i \xi_i}{2} x_i^2 \leqslant V\big(x(t)\big) \leqslant e^{-\varepsilon t} W_0, \tag{10.4.6}$$

$$\sum_{i=1}^{n} x_i^2 \leqslant \frac{W_0}{\min_{1 \leqslant i \leqslant n} \frac{C_i \xi_i}{2}} e^{-\alpha t} \quad (\varepsilon > 0). \tag{10.4.7}$$

The inequality (10.4.7) shows that the zero solution of (10.4.1) is globally exponentially stable, i.e., the $u = u^*$ of (10.1.1) is globally exponentially stable.

(2) Let $-\lambda = \lambda_{\max}(A)$, $\mu = \lambda_{\max}(B)$. Then, we have

$$\begin{aligned}
&\frac{dW}{dt}\bigg|_{(10.4.1)} \\
&\quad \leqslant e^{\varepsilon t} \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} \right\} \\
&\qquad + e^{\varepsilon t} \left[\varepsilon \left(\sum_{i=1}^{n} \frac{C_i \xi_i}{2} x_i^2 + \sum_{i=1}^{n} \eta_i C_i x_i f_i(x_i) \right) \right] \\
&\quad \leqslant e^{\varepsilon t}(-\lambda) \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} f_i^2(x_i) \right) + e^{\varepsilon t} \varepsilon \begin{pmatrix} x \\ f(x) \end{pmatrix}^2 \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_0 \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} \\
&\quad \leqslant e^{\varepsilon t}(-\lambda + \varepsilon \mu) \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} f_i^2(x_i) \right) \leqslant 0.
\end{aligned} \tag{10.4.8}$$

Therefore, $\varepsilon = \frac{\lambda}{\mu}$ can be taken as the Lyapunov exponent.

The proof of Theorem 10.4.1 is complete. □

Let

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12}^T & \tilde{A}_{22} \end{bmatrix}, \quad \text{where } \tilde{A}_{11} = -\operatorname{diag}\left(\frac{1}{R_1}, \ldots, \frac{1}{R_n}\right)_{n \times n},$$

$$\tilde{A}_{12} = -\operatorname{diag}\left(\frac{1}{2R_1}, \ldots, \frac{1}{2R_n}\right)_{n \times n} + \left[\frac{T_{ij} + T_{ji}}{2}\right]_{n \times n},$$

$$\tilde{A}_{22} = \left[\frac{T_{ij} + T_{ji}}{2}\right]_{n\times n},$$
$$\tilde{B} = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12}, \\ \tilde{B}_{12}^T & \tilde{B}_{22} \end{bmatrix}_{2n\times 2n},$$
$$\tilde{B}_{11} = \operatorname{diag}\left(\frac{C_1}{2}, \dots, \frac{C_n}{2}\right)_{n\times n},$$
$$\tilde{B}_{12} = \tilde{B}_{12}^T = \operatorname{diag}\left(\frac{C_1}{2}, \dots, \frac{C_n}{2}\right)_{n\times n},$$
$$\tilde{B}_{22} = O_{n\times n}.$$

COROLLARY 10.4.2. *If $\tilde{A}$ is negative definite, then the equilibrium $u = u^*$ is globally exponentially stable, and $\tilde{\varepsilon} = \frac{\tilde{\lambda}}{\tilde{\mu}}$ can be taken as the Lyapunov exponent, where $\tilde{\lambda} = \lambda_{\max}(\tilde{A})$, $\tilde{\mu} = \lambda_{\max}(\tilde{B})$.*

PROOF. One can take $\xi_i = \eta_i = 1,\ i = 1, \dots, n$, in Theorem 10.4.1 to directly prove this corollary. □

THEOREM 10.4.3. *If there exist constants $\xi_i > 0$, $\eta_i > 0$, $i = 1, 2, \dots, n$, such that the matrix G is negative definite, and moreover it holds*:

$$\varepsilon = \min\left[\min_{1\leqslant i\leqslant n} \frac{\eta_i R_i - \xi_i T_{ji}}{\eta_i C_i}, \frac{\lambda^*}{\mu^*}\right] > 0, \tag{10.4.9}$$

then the conclusion of Theorem 10.4.1 is true. Here,

$$-\lambda^* = \lambda_{\max}(G),$$
$$\mu^* = \max_{1\leqslant i\leqslant n} \frac{C_i\xi_i}{2},$$
$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}_{2n\times 2n},$$
$$G_{11} = -\operatorname{diag}\left(\frac{\xi_1}{R_1}, \dots, \frac{\xi_n}{R_n}\right),$$
$$G_{12} = G_{12}^T = \left[(1 - \delta_{ij})\frac{\xi_i T_{ij} + \xi_j T_{ji}}{2}\right]_{n\times n},$$
$$G_{22} = \left[\frac{\eta_i T_{ij} + \eta_j T_{ji}}{2}\right]_{n\times n}.$$

PROOF. The negative definite condition of G implies $T_{ii} < 0$ $(i = 1, \dots, n)$. Again using the Lyapunov function (10.4.2) and similar to the derivation leading to (10.4.3), we obtain

$$\frac{dV}{dt} = \begin{pmatrix} x \\ f(x) \end{pmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} + \sum_{i=1}^n \xi_i T_i x_i f_i(x_i) - \sum_{R_i}^n \frac{\eta_i}{R_i} x_i f_i(x_i). \tag{10.4.10}$$

Let $W = e^{\varepsilon t} V$. Then,

$$\begin{aligned}
\frac{dW}{dt}\bigg|_{(10.4.1)} &= e^{\varepsilon t}\left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} \begin{pmatrix} x \\ f(x) \end{pmatrix} + \varepsilon \sum_{i=1}^n \frac{\xi_i C_i}{2} x_i^2 \right. \\
&\quad \left. + \sum_{i=1}^n \left(\varepsilon \eta_i C_i + \xi_i T_{ij} - \frac{\eta_i}{R_i} \right) x_i f_i(x_i) \right\} \\
&\leqslant e^{\varepsilon t}\left\{ -\lambda^* \left(\sum_{i=1}^n x_i^2 + \sum_{i=1}^n f_i^2(x_i) \right) + \mu^* \varepsilon \sum_{i=1}^n x_i^2 \right. \\
&\quad \left. - \sum_{i=1}^n \left(\varepsilon \eta_i C_i + \xi_i T_{ii} - \frac{\eta_i}{R_i} \right) x_i f_i(x_i) \right\} \\
&\leqslant 0.
\end{aligned}$$

The remaining proof is similar to the last part of the proof of Theorem 10.4.1, and thus is omitted. □

In the following, we consider a weaker nonlinear activation function. Assume that

$$0 < \sup_{u_i \in R^1} D^+ g_i(u_i) \leqslant M_i, \quad i = 1, \dots, n. \tag{10.4.11}$$

Let

$$\varOmega_1 = \mathrm{diag}\left(\frac{1}{R_1} - T_{11} M_1, \dots, \frac{1}{R_n} - T_{nn} M_n \right)_{n \times n} - \big(\sigma_{ij} |T_{ij}| M_j \big)_{n \times n},$$

$$\varOmega_2 = \mathrm{diag}\left(-\frac{1}{R_1 M_1}, \dots, -\frac{1}{R_n M_n} \right) + (T_{ij})_{n \times n},$$

where

$$\sigma_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases} \quad i, j = 1, \dots, n.$$

THEOREM 10.4.4. *If $\varOmega_1$ is an M matrix, then $u = u^*$ of (10.1.1) is globally exponentially stable.*

PROOF. Since Ω_1 is an M matrix, there exist constants $\xi_i > 0$ such that

$$\xi_j\left(-\frac{1}{R_j} + T_{jj}M_j\right) + \sum_{i=1}^{n} \xi_i \sigma_{ij}|T_{ij}|M_j < 0, \quad j = 1, \ldots, n. \tag{10.4.12}$$

Without loss of generality, we have

$$\xi_j T_{jj} + \sum_{i=1}^{n} \xi_i \sigma_{ij}|T_{ij}| \leqslant 0, \quad j = 1, \ldots, n_0,\ 1 \leqslant n_0 \leqslant n,$$
$$\xi_j T_{jj} + \sum_{i=1}^{n} \xi_i \sigma_{ij}|T_{ij}| > 0, \quad j = n_0 + 1, \ldots, n.$$

Let

$$\lambda = \min_{\substack{1\leqslant k\leqslant n_0\\ n_0+1\leqslant j\leqslant n}} \left[\frac{1}{R_k C_k}, \frac{1}{\xi_j C_j}\left(\frac{\xi_j}{R_j} - \xi_j T_{jj} M_j - \sum_{i=1}^{n} \sigma_{ij}\xi_i |T_{ij}| M_j\right)\right].$$

Choose the positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \xi_i C_i |x_i|. \tag{10.4.13}$$

Then, we have

$$\begin{aligned}
D^+ V(x)|_{(10.4.1)} &= \sum_{i=1}^{n} \xi_i C_i x_i \operatorname{sgn}(x_i) \\
&\leqslant \sum_{i=1}^{n} \xi_i \left[-\frac{1}{R_i}|x_i| + T_{ii}\big|f_i(x_i)\big| + \sum_{j=1}^{n} \sigma_{ij}|T_{ij}|\big|f_j(x_j)\big|\right] \\
&\leqslant \sum_{j=1}^{n_0} \xi_i \left(-\frac{1}{R_j}\right)|x_j| + \sum_{j=1}^{n_0}\left(\xi_i T_{jj} + \sum_{i=1}^{n} \sigma_{ij}|T_{ij}|\right)\big|f_j(x_j)\big| \\
&\quad + \sum_{j=n_0+1}^{n}\left(-\frac{\xi_j}{R_j}\right)|x_j| + \sum_{j=n_0+1}^{n}\left(\xi_i T_{jj} + \sum_{i=1}^{n} \sigma_{ij}|T_{ij}|\right)M_j\big|(x_j)\big| \\
&\leqslant \sum_{j=1}^{n_0}\left(-\frac{\xi_j}{R_j}\right)|x_j| \\
&\quad + \sum_{j=n_0+1}^{n}\left[-\frac{\xi_j}{R_j} + \left(\xi_j T_{jj} M_j + \sum_{i=1}^{n} \sigma_{ij}\xi_i |T_{ij}| M_j\right)\right]|x_j| \\
&\leqslant -\lambda V(x).
\end{aligned} \tag{10.4.14}$$

So

$$0 < V\big(x(t)\big) \leqslant V\big(x(0)\big)e^{-\lambda t} \tag{10.4.15}$$

$$\big|x_i(t)\big| \leqslant \frac{1}{\min(\xi_i C_i)} V\big(x(0)\big)e^{-\lambda t}. \tag{10.4.16}$$

The inequality (10.4.16) shows that the $u = u^*$ of (10.1.1) is globally exponentially stable. □

REMARK 10.4.5. Assume that

$$R_i = M_i = 1, \quad i = 1, 2, \ldots, n.$$

The sufficient conditions for globally asymptotic stability in the literature can be summarized as follows [147,169,196,309,315,412]:

$$\|T\|_\infty := \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |T_{ij}| < 1,$$

$$\mu_\infty(T) := \max_{1 \leqslant i \leqslant n} \left(T_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |T_{ij}| \right) < 1,$$

$$\mu_1(T) := \max_{1 \leqslant j \leqslant n} \left(T_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |T_{ij}| \right) < 1,$$

$$\|T\|_m := \left(\sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij}^2 \right)^{1/2} < 1,$$

$$\max_{1 \leqslant i \leqslant n} \left[T_{ii} + \frac{1}{2} \sum_{\substack{j \neq 1 \\ j = i}}^{n} \big(|T_{ij}| + |T_{ji}|\big) \right] < 1,$$

$$\|T\|_2 = \sup_x \big[|TX|_2 / |x|_2\big] = \lambda_{\max}\big(T^T T\big)^{1/2} < 1.$$

By the propriety of M matrix, any of the above conditions is a sufficient condition for Ω_1 to be M matrix.

PROOF. If the matrix Ω_2 is Lyapunov diagonally stable, then there exists a positive definite matrix $H = \operatorname{diag}(h_1, \ldots, h_n)$ such that $Q = \frac{1}{2}(H\Omega_2 + H_2 H)$ is negative definite.

Let $\lambda \in (0, \min_{1 \leqslant i \leqslant n} \frac{1}{R_i C_i})$ be the maximal positive number such that

$$Q + \lambda \operatorname{diag}\left(\frac{C_1 h_1}{M_1}, \ldots, \frac{C_n h_n}{M_n} \right)$$

is negative semi-definite. We employ the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} C_i h_i \int_0^{x_i} f_i(x_i)\,dx_i,$$

which is obviously positive definite.

Let $\underline{m_i} = \min_{|x|\leqslant 1} |f_i'(x_i)|$, $\overline{m_i} = \min[f_i(1), |f_i(-1)|]$. $\forall x \in R^n$, without loss of generality let $|x_1| \leqslant 1$, $i = 1, 2, \ldots, l_0$, and $|x_1| > 1$, $i = l_0 + 1, \ldots, n$. Then, by the monotone nondecreasing propriety of $f_i(x_i)$, we have

$$\begin{aligned} V(x) &= \sum_{i=1}^{n} C_i h_i \int_0^{x_i} f_i(x_i)\,dx_i \\ &\geqslant \sum_{i=1}^{l_0} C_i h_i \underline{m_i} x_i^2 + \sum_{i=l_0+1}^{n} C_i h_i \overline{m_i} |x_i| \to \infty \quad \text{as } |x| \to \infty. \end{aligned}$$

Thus, $V(x)$ is positive definite and radially unbounded. Further, similarly we obtain

$$\begin{aligned} \frac{de^{\lambda t} V}{dt} &= \lambda e^{\lambda t} \sum_{i=1}^{n} C_i h_i \int_0^{x_i} f_i(x_i)\,dx_i + e^{\lambda t} \sum_{i=1}^{n} C_i h_i f_i(x_i) \frac{dx_i}{dt} \\ &\leqslant e^{\lambda t} \left[\lambda \sum_{i=1}^{n} C_i h_i x_i f_i(x_i) - \sum_{i=1}^{n} \frac{h_i}{R_i} x_i f_i(x_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} h_i f_i(x_i) T_{ij} f_j(x_j) \right] \\ &= e^{\lambda t} \left[-\sum_{i=1}^{n} \left(\frac{h_i}{R_i} - C_i h_i \lambda \right) x_i f_i(x_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} h_i T_{ij} f_i(x_i) f_j(x_j) \right] \\ &= e^{\lambda t} \left[-\sum_{i=1}^{n} \left(\frac{h_i}{R_i M_i} - \frac{h_i C_i \lambda}{M_i} \right) f_i^2(x_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} h_i T_{ij} f_i(x_j) f_j(x_j) \right] \\ &= e^{\lambda t} \Big\{ \big(f_1(x_1), \ldots, f_n(x_n)\big) \Big[\frac{1}{2} \big(H \Omega_2 + \Omega_2^T H \big) \\ &\quad + \lambda \operatorname{diag}\left(\frac{C_1 h_1}{M_1}, \ldots, \frac{C_n h_n}{M_n} \right) \big(f_1(x_1), \ldots, f_n(x_n)\big)^T \Big] \Big\} \\ &= e^{\lambda t} \left\{ \begin{pmatrix} f_1(x_1) \\ \cdots \\ f_n(x_n) \end{pmatrix}^T \left[Q + \lambda \operatorname{diag}\left(\frac{C_1 h_1}{M_1}, \ldots, \frac{C_n h_n}{M_n} \right) \right.\right. \\ &\quad \left.\left. \times \begin{pmatrix} f_1(x_1) \\ \cdots \\ f_n(x_n) \end{pmatrix} \right] \right\}. \end{aligned} \tag{10.4.17}$$

Integrating both sides of (10.4.17) from 0 to arbitrary t yields

$$V\big(x(t)\big) \leqslant e^{-\lambda t} V\big(x(0)\big) := e^{-\lambda t} V_0,$$

i.e.,

$$\sum_{i=1}^{l_0} \frac{1}{2} C_i h_i \underline{m_i} x_i^2 + \sum_{i=l_0+1}^{n} C_i h_i \overline{m_i} |x_i| \leqslant V_0 e^{-\lambda t} \tag{10.4.18}$$

from which we get

$$\big|x_i(t)\big| \leqslant \sqrt{\frac{2}{C_i h_i \underline{m}_i}} \sqrt{V_0 e}^{-\frac{\lambda}{2}t} \quad \text{for } 1 \leqslant i \leqslant l_0,$$

$$\big|x_i(t)\big| \leqslant \frac{1}{C_i h_i \overline{m_i}} V_0 e^{-\lambda t} \leqslant \frac{1}{C_i h_i \overline{m_i}} V_0 e^{-\frac{\lambda}{2}t} \quad \text{for } l_0 + 1 \leqslant i \leqslant n.$$

Let

$$k = \max_{\substack{1 \leqslant i \leqslant l_0 \\ l_0+1 \leqslant i \leqslant n}} \left[\sqrt{\frac{2}{C_i h_i \underline{m_i}} V_0},\ \frac{1}{C_i h_i \overline{m_i}} V_0 \right].$$

Then, we have

$$\big|x_i(t)\big| \leqslant k e^{-\frac{\lambda}{2}t}, \quad i = 1, \dots, n, \tag{10.4.19}$$

which shows that the conclusion is true. □

THEOREM 10.4.6. *If there exists a positive definite matrix $P = \mathrm{diag}(p_1, p_2, \dots, p_n) > 0$ such that $T^T P + PT \leqslant 0$ is negative semi-definite, then the equilibrium point $u = u^*$ of (10.1.1) is globally exponentially stable, where T^T is the transpose of T.*

PROOF. Rewrite (10.1.1) as the following equivalent form:

$$C_i \frac{d(u_i - u_i^*)}{dt} = \sum_{j=1}^{n} T_{ij} \big(g_j(u_j) - g_j\big(u_i^*\big)\big) - \frac{u_i - u_i^*}{R_i}. \tag{10.4.20}$$

Since there exists $P = \mathrm{diag}(p_1, p_2, \dots, p_n) > 0$ such that

$$PT + T^T P \leqslant 0,$$

we choose the Lyapunov function:

$$W(u) = \sum_{i=1}^{n} p_i C_i \int_{u_i^*}^{u_i} \big(g_i(u_i) - g_i\big(u_i^*\big)\big)\, du_i \tag{10.4.21}$$

which obviously satisfies $W(u^*) = 0$ and $W(u) > 0$ for $u \neq u^*$. So $W(u)$ is a positive definite and radially unbounded Lyapunov function.

Computing the derivative of $W(u)$ along the solution of (10.1.1) and simplifying the result, we obtain

$$\begin{aligned}
\frac{dW(u)}{dt}\bigg|_{(10.4.20)} &= \big(\big(g(u) - g(u^*)\big)^T (PT + T^T P)\big(g(u) - g(u^*)\big) \\
&\quad - \sum_{i=1}^{n} p_i \left(\frac{u_i - u_i^*}{R_i}\right)\big(g_i(u_i) - g_i(u_i^*)\big) \\
&\leqslant - \sum_{i=1}^{n} \frac{p_i}{R_i}\big(g_i(u_i) - g_i(u_i^*)\big)\big(u_i - u_i^*\big) \\
&\leqslant - \min_{1\leqslant i\leqslant n} \frac{1}{R_i C_i} \sum_{i=1}^{n} C_i p_i \int_0^{u_i} \big(g_i(u_i) - g_i(u_i^*)\big)\, du_i \\
&\leqslant - \min_{1\leqslant i\leqslant n} \frac{1}{R_i C_i} W\big(u(t)\big).
\end{aligned}$$

Thus,

$$W\big(u(t)\big) \leqslant W\big(u(t_0)\big) e^{-\min_{1\leqslant i\leqslant n}\{\frac{1}{R_i C_i}\}(t-t_0)},$$

indicating that the equilibrium point $u = u^*$ of (10.1.1) is globally asymptotically stable. □

COROLLARY 10.4.7. *If one of the following conditions is satisfied*:

(1) $T = T^T \leqslant 0$;
(2) $T + T^T \leqslant 0$;
(3) T *is antisymmetric*;

then $u = u^*$ *of* (10.1.1) *is globally exponentially stable.*

The proof for this corollary can follow the proof for Theorem 10.4.4, and the estimations (10.3.7) and (10.3.8).

THEOREM 10.4.8. *If one of following conditions is satisfied*:

(1) *there exist constants* $\xi_i > 0$ $(i = 1, \ldots, n)$ *and* η_i $(i = 1, \ldots, n)$ *such that*

$$\xi_j T_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} \xi_i |T_{ij}| \leqslant 0;$$

(2)

$$\eta_{ii} T_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \eta_j |T_{ij}| \leqslant 0;$$

then $u = u^*$ of (10.4.20) is globally exponentially stable.

PROOF. If condition (1) holds, we choose the positive definite and radially unbounded Lyapunov function

$$W(u) = \sum_{i=1}^{n} \xi_i C_i \left| u_i - u_i^* \right|,$$

and then have

$$\begin{aligned} D^+ W(u)|_{(10.4.20)} &\leqslant \sum_{j=1}^{n} \left[\xi_j T_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \xi_i |T_{ij}| \right] \left[g_j(u_j) - g_j\left(u_j^*\right) \right] \\ &\quad - \sum_{i=1}^{n} \frac{\xi_i}{R_i} \left| u_i - u_i^* \right| \\ &\leqslant - \sum_{j=1}^{n} \frac{\xi_j}{R_j} \left| u_j - u_j^* \right| \\ &\leqslant - \min_{1 \leqslant i \leqslant n} \frac{1}{R_i C_i} \sum_{j=1}^{n} \xi_j C_j \left| u_j - u_j^* \right| \\ &\leqslant - \min_{1 \leqslant i \leqslant n} \frac{1}{R_i C_i} W(u). \end{aligned}$$

So,

$$W\big(u(t)\big) \leqslant W\big(u(t_0)\big) e^{-\min_{1 \leqslant i \leqslant n} \frac{1}{R_i C_i} (t - t_0)}$$

in the neighborhood of $u = u^*$.

If condition (2) is satisfied, then we can employ another positive definite and radially unbounded Lyapunov function:

$$W(u) = \eta_l \left| u_l - u_l^* \right| := \max_{1 \leqslant i \leqslant n} \eta_i \left| u_i - u_i^* \right|. \tag{10.4.22}$$

Then it holds

$$D^+ W(u)|_{(10.4.20)} \leqslant \frac{1}{C_l} \left[\eta_l T_{ll} + \sum_{\substack{j=1 \\ j \neq l}}^{n} \eta_j |T_{lj}| \right] \left[g_l(u_l) - g_l\left(u_l^*\right) \right]$$

$$
\begin{aligned}
&- \frac{\eta_l}{C_l}\left|u_l - u_l^*\right| \\
&\leqslant -\frac{1}{C_l} W(u), \qquad (10.4.23)
\end{aligned}
$$

which implies that

$$W\big(u(t)\big) \leqslant W\big(u(t_0)\big) e^{-\frac{1}{C_l}(t-t_0)}$$

in the neighborhood of $u \neq u^*$. Therefore, the conclusion of Theorem 10.4.8 is true. □

COROLLARY 10.4.9. *If $T_{ii} < 0$ $(i = 1, 2, \ldots, n)$ and $(-1)^{\delta_{ij}}|T_{ij}|$ is a Hurwitz matrix, then $u = u^*$ of (10.1.1) is globally exponentially stable. Here,*

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise}, \end{cases} \qquad i, j = 1, 2, \ldots, n.$$

PROOF. Since $(-1)^{\delta_{ij}}|T_{ij}|$ being a Hurwitz matrix is equivalent to $-(-1)^{\delta_{ij}}|T_{ij}|$ being an M matrix. According to the property of M matrix, there exist constants $\xi_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$\xi_j T_{jj} + \sum_{i=1}^{n} \xi_i |T_{ij}| < 0, \qquad j = 1, 2, \ldots, n,$$

indicating that the conditions in Theorem 10.4.8 are satisfied, and thus the conclusion of the corollary is true. □

COROLLARY 10.4.10. *Assume that $T_{ii} < 0$, $i = 1, 2, \ldots, n$, and one of the following conditions holds*:

(1) $|T_{jj}| > \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} |T_{ij}|, \quad j = 1, 2, \ldots, n;$

(2) $|T_{ii}| > \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} |T_{ij}|, \quad i = 1, 2, \ldots, n;$

(3) $\sum\limits_{i,j=1}^{n} \left(\frac{(1 - \delta_{ij})T_{ij}}{T_{ii}}\right)^2 < 1;$

(4) $|T_{jj}| > \frac{1}{2} \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \big(|T_{ij}| + |T_{ji}|\big), \quad j = 1, 2, \ldots, n;$

then $u = u^*$ of (10.1.1) is globally exponentially stable.

10.5. Globally asymptotic stability of a class of Hopfield neural networks

So far, most of the results on globally asymptotic stability are actually globally exponential stability. Very rare cases have been found that a system is globally asymptotically stable, but not globally exponentially stable. However, such cases do exist and need further investigation. In this section, we further discuss stability problems of Hopfield network, but particular attention is given to the neural networks with a special class of functions and consider the problem that such systems may be globally asymptotically stable, but not globally exponentially stable. More specifically, we consider system (10.1.1) but with the following activation functions:

$$g_i(u_i) = \tanh(\beta_i u_i), \quad i = 1, 2, \ldots, n.$$

Some new results presented here with weaker conditions and thus less restrictive, improving and generalizing those results obtained in [72].

According to Theorem 10.2.1, we only need to consider the stability in the global attractive sets. Before presenting the main theorem, we introduce a lemma.

LEMMA 10.5.1. *If $a_{ii} \geqslant 0$, $a_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$, there exist constants $\xi_i > 0$, $i = 1, 2, \ldots, n$, such that*

$$\sum_{j=1}^{n} \xi_i a_{ij} \geqslant 0, \tag{10.5.1}$$

then

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} \end{bmatrix} \geqslant 0, \quad i = 1, 2, \ldots, n.$$

PROOF. $\forall \varepsilon > 0$, by the condition of lemma we have

$$\xi_i(a_{ii} + \varepsilon) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \xi_j a_{ij} \geqslant \xi_i \varepsilon > 0.$$

According to the property of M matrix, we know that the matrix

$$\begin{bmatrix} a_{11} + \varepsilon & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} + \varepsilon \end{bmatrix}$$

is an M matrix. Thus,

$$\det\begin{bmatrix} a_{11}+\varepsilon & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii}+\varepsilon \end{bmatrix} > 0, \quad i = 1, 2, \ldots, n.$$

Letting $\varepsilon \to 0$ results in

$$\det\begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} \end{bmatrix} \geqslant 0, \quad i = 1, 2, \ldots, n,$$

and the proof is complete. □

Let $g_i(u_i) = \tanh(\beta_i u_i)$ in (10.1.1) and (10.4.1), and

$$B = \left[-\frac{1}{R_i\beta_i}\delta_{ij} + T_{ij}\right]_{n\times n}, \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{for } i = j = 1, \ldots, n, \\ 0 & \text{for } i \neq j. \end{cases}$$

THEOREM 10.5.2. *If there exists a positive definite matrix* $\xi = \operatorname{diag}(\xi_i, \ldots, \xi_n)$ *such that* $\xi B + B^T\xi$ *is negative semi-definite, then the equilibrium* $u = u^*$ *of* (10.1.1) *is globally asymptotically stable.*

PROOF. We employ the positive definite and unbounded Lyapunov function:

$$V = \sum_{i=1}^{n} C_i\xi_i \int_{u_i^*}^{u_i} \left(g_i(u_i) - g_i\left(u_i^*\right)\right) du_i. \tag{10.5.2}$$

Then, we have

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(10.4.1)} &= -\sum_{j=1}^{n} \frac{\xi_i}{R_i}\left(u_i - u_i^*\right)\left[g_i(u_i) - g_i\left(u_i^*\right)\right] \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i T_{ij}\left[g_i(u_i) - g_i\left(u_i^*\right)\right]\left[g_j(u_j) - g_j\left(u_j^*\right)\right] \\ &= -\sum_{i=1}^{n} \frac{\xi_i}{R_i\beta_i}\left(g_i(u_i) - g_i\left(u_i^*\right)\right)^2 \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i T_{ij}\left[g_i(u_i) - g_i\left(u_i^*\right)\right]\left[g_j(u_j) - g_j\left(u_j^*\right)\right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\left\{\frac{\xi_i}{R_i\beta_i}\big[g_i(u_i)-g_i\big(u_i^*\big)\big]^2-\frac{(u_i-u_i^*)\xi_i}{R_i}\right\}\big[g_i(u_i)-g_i\big(u_i^*\big)\big]\\
=&\ \frac{1}{2}\begin{pmatrix} g_1(u_1)-g_1(u_1^*)\\ \vdots\\ g_n(u_n)-g_n\big(u_n^*\big)\end{pmatrix}^T\big[\xi B+B^T\xi\big]\begin{pmatrix} g_1(u_1)-g_1(u_1^*)\\ \vdots\\ g_n(u_n)-g_n(u_n^*)\end{pmatrix}\\
&+\sum_{i=1}^{n}\xi_i\left[\frac{g_i(u_i)-g_i(u_i^*)}{R_i\beta_i}-\frac{(u_i-u_i^*)}{R_i}\right]\big[g_i(u_i)-g_i\big(u_i^*\big)\big]\\
\leqslant&\ \sum_{i=1}^{n}\xi_i\left[-\frac{(u_i-u_i^*)}{R_i}+\frac{g_i(u_i)-g_i(u_i^*)}{R_i\beta_i}\right]\big|g_i(u_i)-g_i\big(u_i^*\big)\big|\\
=&\ \sum_{i=1}^{n}\xi_i\left[-\frac{|u_i-u_i^*|}{R_i}+\frac{|g_i(u_i)-g_i(u_i^*)|}{R_i\beta_i}\right]\big|g_i(u_i)-g_i\big(u_i^*\big)\big|\leqslant 0.
\end{aligned}
$$

Therefore, by Theorem 4.7.9 or Theorem 5.1.3 we know that the conclusion is true and Theorem 10.5.2 is proved. □

THEOREM 10.5.3. *If there exist constants $\xi_i>0$, $i=1,\ldots,n$, such that*

$$
-\frac{\xi_i}{R_i}+\xi_i T_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n}\xi_j|T_{ij}|\leqslant 0, \tag{10.5.3}
$$

then the equilibrium $u=u^*$ of (10.1.1) is globally asymptotically stable.

PROOF. Choose the positive definite and radially unbounded Lyapunov function:

$$
V=\max_{1\leqslant j\leqslant n}\big|\xi_i^{-1}\big(u_i-u_i^*\big)\big|:=\big|\xi_l^{-1}\big(u_l-u_l^*\big)\big|. \tag{10.5.4}
$$

By the property of the sigmoid function,

$$
g_i(u_i)=\tanh(\beta_i u_i),
$$

we have $|g_i(u_i)-g_i(u_i^*)|<\beta_i|u_i-u_i^*|$ for $u_i\neq u_i^*$, which yields

$$
\begin{aligned}
&D^+V|_{(10.4.1)}\\
&\quad\leqslant C_l^{-1}\xi_l^{-1}\left\{-\xi_l\left|\xi_l^{-1}\frac{(u_l-u_l^*)}{R_l}\right|+\xi_l T_{ll}\xi_l^{-1}\big|g_l(u_l)-g_l\big(u_l^*\big)\big|\right.\\
&\qquad\left.+\sum_{\substack{j=1\\ j\neq l}}^{n}\xi_j|T_{lj}|\big|\xi_j^{-1}\big(g_j(u_j)-g_j\big(u_j^*\big)\big)\big|\right\}
\end{aligned}
$$

$$\leqslant C_l^{-1}\xi_l^{-1}\Bigg\{-\frac{\xi_l}{R_l}\big|\xi_l^{-1}\big(u_l-u_l^*\big)\big| + \Bigg[\Bigg|\xi_l T_{ll}+\sum_{\substack{j=1\\ j\neq l}}^{n}\xi_j|T_{lj}|\Bigg|\Bigg]\big[\xi_l^{-1}\big(g_l(u_l)-g_l\big(u_l^*\big)\big)\big]\Bigg\}$$

$$< C_l^{-1}\xi_l^{-1}\Bigg\{-\frac{\xi_l}{R_l}\big|\xi_l^{-1}\big(u_l-u_l^*\big)\big| + \Bigg[\xi_l T_{ll}+\sum_{\substack{j=1\\ j\neq l}}^{n}\xi_j|T_{lj}|\Bigg]\beta_l\big|\xi_l^{-1}\big(u_l-u_l^*\big)\big|\Bigg\}$$

$$\leqslant 0 \quad \text{when } u_l \neq u_l^*.$$

So by Theorem 4.7.9 or Theorem 5.1.3, the conclusion is true. □

THEOREM 10.5.4. *If there exist constants $\xi_i > 0$, $i = 1, \dots, n$, such that*

$$\xi_i\left[\beta_i^* T_{ii} - \frac{1}{R_i}\right] + \sum_{i=1}^{n}\frac{1}{2}\big[\xi\beta_i|T_{ij}| + \xi_j\beta_j|T_{ij}|\big] \leqslant 0, \tag{10.5.5}$$

then the equilibrium $u = u^*$ of (10.1.1) is globally asymptotically stable, where

$$\beta_i^* = \begin{cases}\beta_i & \text{for } T_{ii}\geqslant 0,\\ \underline{\beta}_i & \text{for } T_{ii} < 0,\end{cases} \qquad \underline{\beta}_i = \inf_{u_i\in\Omega} D_+g_i(u_i) = D_+g_i(l_i),$$

where Ω is the globally exponentially attractive set defined in Theorem 10.2.1.

PROOF. Choose the Lyapunov function:

$$V(u) = \frac{1}{2}\sum_{i=1}^{n}\xi_i\big(u_i-u_i^*\big)^2. \tag{10.5.6}$$

Then for $u = u^*$, we have

$$\frac{dV}{dt}\bigg|_{(10.3.1)}$$

$$= -\sum_{i=1}^{n}\frac{\xi_i}{R_i}\big(u_i-u_i^*\big)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i T_{ij}\big(u_i-u_i^*\big)\big[g_j(u_j)-g_j\big(u_j^*\big)\big]$$

$$< \begin{pmatrix} u_1-u_1^*\\ \vdots\\ u_n-u_n^*\end{pmatrix}^T \left[\xi_i\left(\beta_i^* T_{ii} - \frac{1}{R_i}\right)\delta_{ij}\right.$$

$$+ \frac{1}{2}\big(\xi_i\beta_j|T_{ij}| + \xi_j\beta_i|T_{ji}|\big)(1-\delta_{ij})\bigg]_{n\times n} \begin{pmatrix} u_1 - u_1^* \\ \vdots \\ u_n - u_n^* \end{pmatrix}$$
$$\leqslant 0.$$

Hence, by Theorem 4.7.9 or Theorem 5.1.3, the conclusion is true. □

THEOREM 10.5.5. *(See [72].) Let $y^+ = \max\{0, y\}$, where y is an arbitrary real number. If there exist constants $\xi_i > 0$, $i = 1, \dots, n$, such that*

$$\xi_i\big[\beta_i T_{ii}^+ - R_i^{-1}\big] + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{2}\big[\xi_i\beta_i|T_{ij}| + \xi_j\beta_j|T_{ji}|\big] \leqslant 0,$$
$$i = 1, \dots, n, \tag{10.5.7}$$

then the equilibrium $u = u^*$ of (10.4.1) is globally asymptotically stable.

THEOREM 10.5.6. *If there exist constants $\xi > 0$, $i = 1, 2, \dots, n$, such that*

$$-\frac{\xi_i}{R_i} + \xi_i\beta_i^* T_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} \xi_j\beta_j|T_{ij}| \leqslant 0, \quad i = 1, 2, \dots, n,$$

then the equilibrium $u = u^*$ of (10.4.1) is globally asymptotically stable, where

$$\beta_i^* = \begin{cases} \beta_i & \text{for } T_{ii} \geqslant 0, \\ \underline{\beta}_i & \text{for } T_{ii} < 0, \end{cases} \qquad \underline{\beta}_i = \inf_{u_i\in\Omega} D_+ g_i(u_i).$$

PROOF. We again use the Lyapunov function (10.5.4). By the property of the sigmoid function $g_i(u_i) = \tanh(\beta_i u_i)$, we have

$$\underline{\beta}_i\big|u_i - u_i^*\big| < \big|g_i(u_i) - g_i\big(u_i^*\big)\big| < \beta_i\big|u_i - u_i^*\big| \quad \text{for } u_i \in \Omega,$$

and

$$\begin{aligned} D^+V|_{(10.4.1)} &\leqslant C_l^{-1}\xi_l^{-1} D^+\big|C_l\big(u_l - u_l^*\big)\big| \\ &= C_l^{-1}\xi_l^{-1} C_l \frac{d(u_i - u_i^*)}{dt} \operatorname{sgn}\big(u_l - u_l^*\big) \\ &= C_l^{-1}\xi_l^{-1}\Bigg\{-\xi_l\bigg|\xi_l^{-1}\frac{u_l - u_l^*}{R_l}\bigg| + T_{ll}\big|g_l(u_l) - g_l\big(u_l^*\big)\big| \\ &\qquad + \sum_{\substack{j=1\\ j\neq l}}^{n} \xi_j|T_{lj}|\big|\xi_j^{-1}\big(g_j(u_j) - g_j\big(u_j^*\big)\big)\big|\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= C_l^{-1}\xi_l^{-1}\Bigg\{-\frac{\xi_l}{R_l}\big|\xi_l^{-1}\big(u_l-u_l^*\big)\big| + \xi_l T_{ll}\xi_l^{-1}\beta_l^*\big|u_l-u_l^*\big| \\
&\quad + T_{ll}\big|g_l(u_l)-g_l\big(u_l^*\big)\big| - \xi_l T_{ll}\xi_l^{-1}\beta_l^*\big|u_l-u_l^*\big| \\
&\quad + \sum_{\substack{j=1\\ j\neq l}}^{n} \xi_j|T_{lj}|\xi_j^{-1}\beta_j\big|u_j-u_j^*\big| \\
&\quad + \sum_{\substack{j=1\\ j\neq l}}^{n} |T_{lj}|\big|\xi_j^{-1}\big(g_j(u_j)-g_j\big(u_j^*\big)\big)\big| \\
&\quad - \sum_{\substack{j=1\\ j\neq i}}^{n} \xi_j|T_{lj}|\xi_j^{-1}\beta_j\big|u_j-u_j^*\big|\Bigg\} \\
&< C_l^{-1}\xi_l^{-1}\Bigg\{-\frac{\xi_l}{R_l}\big|u_l-u_l^*\big| + \xi_l T_{ll}\beta_l^*\xi_l^{-1}\big|u_l-u_l^*\big| \\
&\quad + \sum_{\substack{j=1\\ j\neq l}}^{n} \xi_j|T_{lj}|\beta_j\big|\xi_j^{-1}\big(u_j-u_j^*\big)\big|\Bigg\} \\
&\leqslant \xi_l^{-1}C_l^{-1}\Bigg\{-\frac{\xi_l}{R_l} + \xi_l\beta_l^*T_{ll} + \sum_{\substack{j=1\\ j\neq l}}^{n} \xi_j\beta_j|T_{lj}|\Bigg\}\big|\xi_l^{-1}\big(u_l-u_l^*\big)\big| \\
&\leqslant 0,
\end{aligned}
$$

i.e., $D^+V|_{(10.4.1)}$ is negative definite. So the conclusion is true, and the proof is complete. □

REMARK 10.5.7. When $T_{ii} \geqslant 0$, Theorem 10.5.4 is equivalent to Theorem 10.5.5, but when some $T_{ii} < 0$, the conditions in Theorem 10.5.4 are weaker than that of Theorem 10.5.5. When $T_{ii} \geqslant 0$ $(i = 1, 2, \ldots, n)$, Theorem 10.5.6 is equivalent to the following theorem [72].

THEOREM 10.5.8. *If for some constants $\xi_i > 0$, $i = 1, \ldots, n$, there exist*

$$
\xi_i\big[\beta_i T_{ii}^+ - R_i^{-1}\big] + \sum_{\substack{j=1\\ j\neq l}}^{n} \xi_j\beta_j|T_{ij}| \leqslant 0, \quad i = 1, 2, \ldots, n, \tag{10.5.8}
$$

then the equilibrium $u = u^*$ of (10.1.1) is globally asymptotically stable.

When some $T_{ii} < 0$, Theorem 10.5.6 is better than Theorem 10.5.8.

THEOREM 10.5.9. *If there exist constants $\xi_i > 0$, $i = 1, \ldots, n$, such that*

$$-\frac{\xi_j}{R_j} + \beta_j^* \xi_j T_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \xi_i |T_{ij}| \leqslant 0, \quad j = 1, 2, \ldots, n, \tag{10.5.9}$$

then the equilibrium $u = u^*$ of (10.1.1) is globally asymptotically stable.

PROOF. Choosing the positive definite and radially unbounded Lyapunov function:

$$V = \sum_{i=1}^{n} c_i \xi_i \left| u_i - u_i^* \right|, \tag{10.5.10}$$

we have

$$\begin{aligned}
D^+ V|_{(10.4.1)} \leqslant \sum_{j=1}^{n} \xi_i \Biggl\{ & -\frac{1}{R_i \beta_i} \bigl[g_i(u_i) - g_i\bigl(u_i^*\bigr) \bigr] \\
& + \sum_{j=1}^{n} T_{ij} \bigl[g_j(u_j) - g_j\bigl(u_j^*\bigr) \bigr] \\
& - \left[\frac{u_i - u_i^*}{R_i} - \frac{1}{R_i \beta_i} \bigl(g_i(u_i) - g_i\bigl(u_i^*\bigr) \bigr) \right] \Biggr\} \operatorname{sgn}\bigl(u_i - u_i^*\bigr) \\
= \sum_{j=1}^{n} & \left(-\frac{\xi_j}{R_j \beta_j} + \xi_j \beta_j^* T_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \xi_i |T_{ij}| \right) \bigl| g_j(u_j) - g_j\bigl(u_j^*\bigr) \bigr| \\
& - \sum_{i=1}^{n} \left(\frac{|u_i - u_i^*|}{R_i} - \frac{|g_i(u_i) - g_i(u_i^*)|}{R_i \beta_i} \right) \xi_i \\
\leqslant - \sum_{j=1}^{n} & \left(\frac{|u_i - u_i^*|}{R_i} - \frac{|g_i(u_i) - g_i(u_i^*)|}{R_i \beta_i} \right) \xi_i < 0 \quad \text{for } u \neq u^*,
\end{aligned}$$

so the conclusion is true. □

REMARK 10.5.10. Theorem 10.5.9 is equivalent to the following theorem.

THEOREM 10.5.11. *If for some constants* $\xi_i > 0$, $i = 1, \ldots, n$, *the following inequalities*:

$$-\frac{\xi_j}{R_j} + \beta_j \left\{ \xi_j T_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} \xi_i |T_{ij}| \right\}^+ \leqslant 0, \quad j = 1, \ldots, n, \tag{10.5.11}$$

are satisfied, then the equilibrium $u = u^*$ *of* (10.1.1) *is globally asymptotically stable.*

However, by Theorem 10.5.9 one can obtain better estimation for the convergent rate.

EXAMPLE 10.5.12. Consider a two-state neural network:

$$\begin{pmatrix} \dot u_1(t) \\ \dot u_2(t) \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{bmatrix} -2 & 8 \\ -16 & 1 \end{bmatrix} \begin{pmatrix} g_1(u_1(t)) \\ g_2(u_2(t)) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \tag{10.5.12}$$

where $g_i(u_i) = \tanh(u_i)$ and $R_i = C_i = \beta_i = 1$, $i = 1, 2$. According to Theorem 10.5.2, we have

$$B = \begin{bmatrix} -1-2 & 8 \\ -16 & -1+1 \end{bmatrix} = \begin{bmatrix} -3 & 8 \\ -16 & 0 \end{bmatrix}.$$

Take $\xi_1 = 2$, $\xi_1 = 1$. Then $\xi B + B^T \xi = \left[\begin{smallmatrix} -12 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is negative semi-definite. The conditions in Theorem 10.5.2 are satisfied. So the equilibrium $u = u^*$ of (10.1.1) is globally asymptotically stable.

EXAMPLE 10.5.13. Consider a Hopfield neural networks, described by

$$\begin{pmatrix} \dot u_1(t) \\ \dot u_2(t) \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{bmatrix} -2 & 3/2 \\ 3/2 & -2 \end{bmatrix} \begin{pmatrix} g_1(u_1(t)) \\ g_2(u_2(t)) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \tag{10.5.13}$$

where $R_i = \beta_i = C_i = 1$, $i = 1, 2$, $T_{ii} = -2$, $T_{12} = T_{21} = \frac{3}{2}$. Suppose $\beta_i^* = \frac{1}{3}$, $i = 1, 2$. Then,

$$\begin{bmatrix} -\frac{1}{R_1} & 0 \\ 0 & -\frac{1}{R_2} \end{bmatrix} + \begin{bmatrix} \beta_1^* T_{11} & |T_{12}| \\ |T_{21}| & \beta_2^* T_{12} \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{3} \end{bmatrix},$$

and it is easy to verify that the conditions in Theorems 10.5.2, 10.5.3, 10.5.4 and Theorem 10.5.6 are satisfied, while the conditions of Theorems 10.5.8 and 10.5.11 are not satisfied by Lemma 10.5.1.

EXAMPLE 10.5.14. Consider a 2-dimensional neural networks, given below:

$$\begin{pmatrix} \dot u_1(t) \\ \dot u_2(t) \end{pmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix} \begin{pmatrix} g_1(u_1(t)) \\ g_2(u_2(t)) \end{pmatrix}, \quad (10.5.14)$$

where $R_i = \frac{1}{2}$, $i = 1, 2$, $T_{ii} = -5$, $i = 1, 2$, $T_{12} = T_{21} = 1$, $\beta_i = 2$, $i = 1, 2$. Further, suppose $\beta_u^* = 1$.

Choose the Lyapunov function:

$$V = \sum_{i=1}^{2} |y_i|.$$

By Theorem 10.5.11, one can only obtain

$$D^+ V \leqslant -2V, \quad \text{and} \quad \sum_{i=1}^{2} \bigl|y_i(t)\bigr| = V(t) \leqslant V_0 e^{-2(t-t_0)}.$$

However, applying Theorem 10.5.9 yields

$$D^+ V \leqslant -6V,$$

leading to

$$\sum_{i=1}^{2} \bigl|y_i(t)\bigr| = V(t) \leqslant V_0 e^{-6(t-t_0)},$$

which is better than the above estimation.

In the following, we study globally asymptotical stability of a class of neural networks with variable time delay. Consider the following system

$$\begin{aligned} &C\frac{dy(t)}{dt} = -R^{-1} y(t) + Af\bigl(y(t)\bigr) + Bf\bigl(y\bigl(t-\tau(t)\bigr)\bigr) + I, \\ &\quad y(t) = \eta(t) \quad \text{for } t \in [t-\tau, t_0], \end{aligned} \qquad (10.5.15)$$

where

$$\begin{aligned} &C = \operatorname{diag}(C_1, \dots, C_n)^T > 0, \quad R^{-1} = \operatorname{diag}\bigl(R_1^{-1}, \dots, R_n^{-1}\bigr)^T > 0, \\ &A, B \in R^{n\times n}, \quad I = (I_1, \dots, I_n)^T, \\ &y = (y_1, \dots, y_n)^T, \quad f\bigl(y(t)\bigr) = \bigl(f_1\bigl(y_1(t)\bigr), \dots, f_1\bigl(y_n(t)\bigr)\bigr)^T, \\ &f\bigl(y\bigl(t-\tau(t)\bigr)\bigr) = \bigl(f_1\bigl(y_1\bigl(t-\tau_1(t)\bigr)\bigr), \dots, f_1\bigl(y_n\bigl(t-\tau_n(t)\bigr)\bigr)\bigr)^T, \\ &\tau(t) = \bigl(\tau_1(t), \dots, \tau_n(t)\bigr)^T, \quad 0 \leqslant \tau_i(t) \leqslant \tau_i = \text{constant}, \quad i = 1, 2, \dots, n. \end{aligned}$$

Assume $\dot\tau_i(t) \leqslant 0$.

Let y^* be a unique equilibrium of (10.5.15), and $x = y - y^*$, $f(x) = f(x + y^*) - f(y^*)$, $g(x(t - \tau(t)) = f(x(t - \tau(t)) + y^*) - f(y^*)$. Then (10.5.15) can be rewritten as

$$C\frac{dx(t)}{dt} = -R^{-1}x(t) + Ag\bigl(x(t)\bigr) + Bg\bigl(x\bigl(t - \tau(t)\bigr)\bigr). \tag{10.5.16}$$

Further, define

$$\begin{aligned}
&G := \bigl\{f_i(y_i) \in C\bigl[R^1, R^1\bigr],\ D^+ f_i(y_i) \geqslant 0,\ i = 1, 2, \dots, n\bigr\},\\
&L = \Biggl\{f_i(y_i) \in C\bigl[R^1, R^1\bigr],\\
&\qquad \frac{f_i(y_i^{(1)}) - f_i(y_i^{(i)})}{y_i^{(1)} - y_i^{(i)}} \leqslant L_i < +\infty,\ i = 1, 2, \dots, n\Biggr\}.
\end{aligned}$$

THEOREM 10.5.15. *Let $f_i(\cdot) \in G$, $g_i(\cdot) \in G$ and*

$$\begin{aligned}
L_i &:= \sup_{x_i \in R} D^+ g_i(x_i) = \sup_{y_i \in R} D^+ f_i(y_i)\\
&= D^+ f_i(0) \neq D^+ f_i(y_i), \quad y_i \neq 0,\ i = 1, \dots, n.
\end{aligned}$$

If there exist positive definite diagonal matrices $P = \operatorname{diag}(p_1, \dots, p_n) > 0$ and $\xi = \operatorname{diag}(\xi_1, \dots, \xi_n) > 0$ such that the matrix

$$Q := PA + A^T P - 2PL^{-1}R^{-1} + \xi + PB\xi^{-1}B^T P \leqslant 0$$

(*i.e., negative semi-definite*), *then the equilibrium $y = y^*$ of* (10.5.15) (*i.e., $x = 0$ of* (10.5.16)) *is globally asymptotically stable.*

PROOF. Construct the positive definite and radially unbounded Lyapunov function:

$$V(t, x) = 2\sum_{i=1}^{n} C_i p_i \int_0^{x_i} g_i(x_i)\,dx_i + \sum_{j=1}^{n} \xi_j \int_{t-\tau_j(t)}^{t} g_j^2\bigl(x_i(s)\bigr)\,ds. \tag{10.5.17}$$

Calculating the time derivative of $V(t, x)$ along the positive half trajectory of (10.5.16) yields

$$\begin{aligned}
&\frac{dV(t, x)}{dt}\bigg|_{(10.5.16)}\\
&\quad = \bigl(g(t)\bigr)^T \bigl(PA + A^T P\bigr)g\bigl(x(t)\bigr) - \bigl(x(t)\bigr)^T PR^{-1}g\bigl(x(t)\bigr)\\
&\qquad + \sum_{j=1}^{n} \xi_j g_j^2\bigl(x_j(t)\bigr) - \sum_{j=1}^{n} \xi_j g_j^2\bigl(x_j\bigl(t - \tau_j(t)\bigr)\bigr)\bigl(1 - \dot{\tau}_j(t)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{n} 2p_i g_i\big(x_i(t)\big) \sum_{j=1}^{n} b_{ij} g_j\big(x_j\big(t-\tau_j(t)\big)\big) \\
&\leqslant \big(g(t)\big)^T \big[PA + A^T P - 2PL^{-1}R^{-1} + \xi\big] g\big(x(t)\big) \\
&\quad - \sum_{j=1}^{n} \xi_j \left[g_j\big(x_j\big(t-\tau_j(t)\big)\big) - \frac{\sum_{i=1}^{n} p_i g_i(x_i(t)) b_{ij}}{\xi_j} \right]^2 \\
&\quad + \sum_{j=1}^{n} \frac{(\sum_{i=1}^{n} p_i g_i(x_i(t)) b_{ij})^2}{\xi_j} \\
&\leqslant g\big((t)\big)^T \big[PA + A^T P - 2PL^{-1}R^{-1} + \xi + PB\xi^{-1}B^T P\big] g\big(x(t)\big) \\
&\leqslant 0,
\end{aligned} \tag{10.5.18}
$$

where $\xi^{-1} = \mathrm{diag}(\xi_1^{-1}, \ldots, \xi_n^{-1})$, $L^{-1} = \mathrm{diag}(L_1^{-1}, \ldots, L_n^{-1})$, and $\frac{dV}{dt} = 0$ is reached at $x = 0$ or $x = -y^*$, but $x = -y^*$ is not the equilibrium of (10.5.16) since $x = 0$ is the unique equilibrium of (10.5.16).

Next, we prove that the set $E := \{x \mid \frac{dV(t,x)}{dt}|_{(10.5.16)} = 0\}$ does not include the positive half trajectory except $x = 0$, i.e., E does not include any other invariant set of (10.5.16), except $x = 0$.

Since

$$
\begin{aligned}
L_j &= \sup_{x\in R} D^+ g_j(x_j) = \sup_{y\in R} D^+ f_j(y_j) = D^+ f_j(0) \\
&\quad \neq D^+ g_j\big(-y_j^*\big) > D^+ g_j(y_j) \neq 0, \quad y_i \neq y_i^*.
\end{aligned}
$$

Without loss of generality, let

$$
\begin{aligned}
&x = (x_1, \ldots, x_{j-1}, x_j, \ldots, x_n) \neq 0 \quad \text{and} \\
&x_i \neq -y_i^*, \quad i = j, j+1, \ldots, n,
\end{aligned}
$$

be any nonequilibrium state in positive half trajectory $x_i \neq 0$ for $j \leqslant i \leqslant n$. Then,

$$
\begin{aligned}
&\delta_i := D^+ g_i(x_i) < \sup_{x\in R} D^+ g_i(x_i) = D^+ g_i\big(-y_i^*\big) = L_j, \\
&\quad i = j, j+1, \ldots, n,
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
\frac{dV}{dt}\bigg|_{(10.5.16)} &= g^T\big(x(t)\big)\bigg[PB\xi^{-1}B^T P + \xi \\
&\quad + P\bigg(A - \mathrm{diag}\bigg(\frac{1}{L_1 R_1}, \ldots, \frac{1}{L_{j-1}R_{j-1}}, \frac{1}{\delta_j R_j}, \ldots, \frac{1}{\delta_n R_n}\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + A^T - \operatorname{diag}\left(\frac{1}{L_1R_1},\ldots,\frac{1}{L_{j-1}R_{j-1}},\frac{1}{\delta_jR_j},\ldots,\frac{1}{\delta_nR_n}\right)\Bigg)P\Bigg]g\big(x(t)\big)\\
&= g^T\big(x(t)\big)\Big[PB\xi^{-1}B^TP + \xi\\
&+ P\left(A - \operatorname{diag}\left(\frac{1}{L_1R_1},\ldots,\frac{1}{L_{j-1}R_{j-1}},\frac{1}{\delta_jR_j},\ldots,\frac{1}{\delta_nR_n}\right)\right.\\
&+ A^T - \operatorname{diag}\left(\frac{1}{L_1R_1},\ldots,\frac{1}{L_{j-1}R_{j-1}},\frac{1}{\delta_jR_j},\ldots,\frac{1}{\delta_nR_n}\right)\Bigg)P\Bigg]g\big(x(t)\big)\\
&- g^T\big(x(t)\big)\Big[PB\xi^{-1}B^TP + \xi\\
&+ P\left(A - \operatorname{diag}\left(\frac{1}{L_1R_1},\ldots,\frac{1}{L_{j-1}R_{j-1}},\frac{1}{L_jR_j},\ldots,\frac{1}{L_nR_n}\right)\right)\\
&+ \left(A^T - \operatorname{diag}\left(\frac{1}{L_1R_1},\ldots,\frac{1}{L_{j-1}R_{j-1}},\frac{1}{L_jR_j},\ldots,\frac{1}{L_nR_n}\right)\right)P\Bigg]g\big(x(t)\big)\\
&+ g^T\big(x(t)\big)\big[PB\xi^{-1}B^TP + \xi\\
&+ P\big(A - PL^{-1}R^{-1}\big) + \big(A - PL^{-1}R^{-1}\big)^TP\big]g\big(x(t)\big)\\
&\leqslant g^T\big(x(t)\big)\Bigg[P\left(\operatorname{diag}\left(0,\ldots,0,\frac{1}{L_jR_j}-\frac{1}{\delta_jR_j},\ldots,\frac{1}{L_nR_n}-\frac{1}{\delta_nR_n}\right)\right.\\
&\quad + \operatorname{diag}\left(0,\ldots,0,\frac{1}{L_jR_j}-\frac{1}{\delta_jR_j},\ldots,\frac{1}{L_nR_n}-\frac{1}{\delta_nR_n}\right)\Bigg)P\Bigg]g\big(x(t)\big)\\
&= \sum_{i=j}^{n} p_i\left(\frac{1}{L_iR_i}-\frac{1}{\delta_iR_i}\right)g_i^2\big(x_i(t)\big) < 0 \quad \forall x \neq 0.
\end{aligned}
\tag{10.5.19}
$$

So $\frac{dV}{dt}|_{(10.5.16)} \leqslant 0$ and $E = \{x \mid \frac{dV}{dt}|_{(10.4.23)} = 0\}$. Thus, except $x = 0$, any other positive half trajectory (i.e., invariant set) is not included in E.

According to the LaSalle invariant principle [220], we know that the equilibrium point $x = 0$ of (10.5.16) (i.e., $y = y^*$ of (10.5.15)) is globally asymptotically stable. □

COROLLARY 10.5.16. *If $Q_1^* := A + A^T - 2L^{-1}R^{-1} + I_n + BB^T$ is negative semi-definite, then $x = 0$ of (10.5.16) (i.e., $y = y^*$ of (10.5.15)) is globally asymptotically stable.*

COROLLARY 10.5.17. *If $B = 0$, then the conditions in Theorem 10.5.15 become that*

$$Q_1 := PA + A^TP - 2L^{-1}R^{-1} \quad \textit{is negative semi-definite.}$$

EXAMPLE 10.5.18. For the following two-state neural network (no delay terms):

$$\begin{cases} \frac{dy_1}{dt} = -y_1 - 2f_1(y_1) + 9f_2(y_2) + I_1, \\ \frac{dy_2}{dt} = -y_2 + f_1(y_1) - 2f_2(y_2) + I_2, \end{cases} \tag{10.5.20}$$

suppose $f_i(y_i) \in L$ and $C_i = R_i = L_i = 1,\ i = 1, 2$.

For system (10.5.16), we have

$$\bar{A} := A - L^{-1}R^{-1} = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} := \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix}.$$

$\bar{A}$ is Lyapunov diagonally stable if and only if $\bar{a}_{11} < 0$, $\bar{a}_{22} < 0$, and $\bar{a}_{11}\bar{a}_{22} > \bar{a}_{12}\bar{a}_{21}$. However, here $\bar{a}_{11}\bar{a}_{22} = \bar{a}_{12}\bar{a}_{21} = 9$, hence $\bar{A}$ is not Lyapunov diagonally stable. Now, we take $P = \operatorname{diag}(p_1, p_2) = \operatorname{diag}(1/3, 3)$. Then,

$$\begin{aligned} PA + A^T P &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 6 & -18 \end{bmatrix} < 0, \end{aligned}$$

indicating that the conditions in Corollary 10.5.17 are satisfied. Therefore, the equilibrium $y = y^*$ of (10.5.16) is globally asymptotically stable.

THEOREM 10.5.19. *Let $f(\cdot) \in L$ and*

$$L_i = \sup_{y \in R} D^+ f_i(y_i) = D^+ f_i(0) > D^+ f_i(y_i), \quad y_i \neq 0.$$

If there exist n positive constants $p_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$p_j\left(a_{jj} - \frac{1}{L_j R_j}\right) + \sum_{\substack{i=1 \\ i \neq j}}^{n} p_i |a_{ij}| + \sum_{i=1}^{n} p_i |b_{ij}| \leqslant 0,$$

then the equilibrium point $x = 0$ of (10.5.16) (i.e., $y = y^*$ of (10.5.15)) is globally asymptotically stable.

PROOF. Let the positive definite and radially unbounded Lyapunov function be

$$V(t, x) = \sum_{i=1}^{n} p_i C_i |x_i| + \sum_{i,j=1}^{n} p_i \int_{t-\tau_j(t)}^{t} |b_{ij}| \left|g_j(s)\right| ds.$$

Evaluating the right-upper Dini derivative of V, we obtain

$$
\begin{aligned}
&D^+V(t,x)|_{(10.5.16)}\\
&\quad\leqslant \sum_{i=1}^n p_i C_i \frac{dx_i}{dt}\,\mathrm{sign}(x_i) + \sum_{i=1}^n p_i \sum_{j=1}^n |b_{ij}|\big|g\big(x_j(t)\big)\big|\\
&\qquad - \sum_{i=1}^n \sum_{j=1}^n |b_{ij}|\big|g_j\big(x_j\big(t-\tau_j(t)\big)\big)\big|\big(1-\dot{\tau}_j(t)\big)\\
&\quad\leqslant \sum_{j=1}^n \Bigg[p_j\bigg(a_{jj}-\frac{1}{R_jL_j}\bigg)+\sum_{\substack{i=1\\ i\neq j}}^n p_i|a_{jj}|\Bigg]\big|g_j\big(x_j(t)\big)\big|\\
&\qquad + \sum_{j=1}^n \ p_i \sum_{j=1}^n |b_{ij}|\big|g_j\big(x_j\big(t-\tau_j(t)\big)\big)\big|\\
&\qquad + \sum_{i=1}^n p_i \sum_{j=1}^n |b_{ij}|\big|g_j\big(x_j(t)\big)\big|\\
&\qquad - \sum_{j=1}^n \ p_i \sum_{j=1}^n |b_{ij}|\big|g_j\big(x_j\big(t-\tau_j(t)\big)\big)\big|\\
&\quad\leqslant \sum_{j=1}^n \Bigg[p_j\bigg(a_{jj}-\frac{1}{R_jL_j}\bigg)+\sum_{\substack{i=1\\ i\neq j}}^n p_i|a_{ij}|+\sum_{i=1}^n p_i|b_{ij}|\Bigg]\big|g_j\big(x_j(t)\big)\big|\\
&\quad\leqslant 0.
\end{aligned}
\tag{10.5.21}
$$

Since $x=-y^*$ is not an equilibrium of (10.5.16), we only need to prove that the set $E := \{x \mid D^+V|_{(10.5.16)}\}$ does not include the positive half trajectory except $x=0$.

Without loss of generality, let any nonequilibrium state in the positive half trajectory of (10.5.16) be $x=(0,\dots,0,x_i,\dots,x_n)\neq 0$, and $\delta_j = D^+y_j(x_j) > D^+y_j(-y^*) = L_j$, $j=i,i+1,\dots,n$. Then, $x_i \neq -y_i^*$, $x_i \neq 0$, $i=j$, $j+1,\dots,n$. We have

$$
\begin{aligned}
&D^+V\big(t,x(t)\big)\big|_{(10.5.16)}\\
&\quad\leqslant \sum_{j=1}^{i-1} \Bigg[p_j\bigg(a_{jj}-\frac{1}{L_jR_j}\bigg)+\sum_{\substack{i=1\\ i\neq j}}^n p_i|a_{ij}|+\sum_{i=1}^n p_i|b_{ij}|\Bigg]\big|g_j\big(x_j(t)\big)\big|\\
&\qquad + \sum_{j=i}^n \Bigg[p_j\bigg(a_{jj}-\frac{1}{\delta_jR_j}\bigg)+\sum_{\substack{i=1\\ i\neq j}}^n p_i|a_{ij}|+\sum_{i=1}^n p_i|b_{ij}|\Bigg]\big|g_j\big(x_j(t)\big)\big|
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=1}^{n} \Bigg[p_j \bigg(a_{jj} - \frac{1}{L_j R_j} \bigg) + \sum_{\substack{i=1\\ i\neq j}}^{n} p_i |a_{ij}| + \sum_{i=1}^{n} p_i |b_{ij}| \Bigg] \big| g_j\big(x_j(t)\big)\big| \\
&+ \sum_{j=1}^{n} \Bigg[p_j \bigg(a_{jj} - \frac{1}{L_j R_j} \bigg) + \sum_{\substack{i=1\\ i\neq j}}^{n} p_i |a_{ij}| + \sum_{i=1}^{n} p_i |b_{ij}| \Bigg] \big| g_j\big(x_j(t)\big)\big| \\
\leqslant{} & \sum_{j=i}^{n} \bigg[-p_j \bigg(\frac{1}{\delta_j R_j} - \frac{1}{L_j R_j} \bigg) \bigg] \big| g_j\big(x_j(t)\big)\big| < 0 \quad \text{when } x \neq 0. \qquad (10.5.22)
\end{aligned}
$$

Thus, the equilibrium $x = 0$ of (10.5.16) (i.e., $y = y^*$ of (10.5.15)) is globally asymptotically stable. □

COROLLARY 10.5.20. *Let $g(\cdot) \in L$. If*

$$
\bigg(a_{jj} - \frac{1}{L_j R_j} \bigg) + \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_{ij}| \leqslant 0,
$$

then the equilibrium $x = 0$ of (10.5.16) (*i.e., $y = y^*$ of* (10.5.15)) *is globally asymptotically stable.*

EXAMPLE 10.5.21. Consider the following 3-dimensional neural network without time delay (i.e., $B = 0$):

$$
\begin{cases}
\frac{dy_1}{dt} = -y_1 - 3f_1(y_1) - 2f_2(y_2) - 2f_3(y_3) + I_1, \\
\frac{dy_2}{dt} = -y_2 + 3f_1(y_1) - 3f_2(y_2) + f_3(y_3) + I_2, \\
\frac{dy_3}{dt} = -y_3 + f_1(y_1) + 2f_2(y_2) - 2f_3(y_3) + I_3,
\end{cases}
\qquad (10.5.23)
$$

where $f_i(y_i) \in L$, $C_i = R_i = L_i = 1$, $i = 1, 2, 3$.

Because

$$
\tilde{A} = \begin{bmatrix} a_{11} - \frac{1}{L_1 R_1} & |a_{12}| & |a_{13}| \\ |a_{21}| & a_{22} - \frac{1}{L_2 R_2} & |a_{23}| \\ |a_{31}| & |a_{32}| & a_{33} - \frac{1}{L_3 R_3} \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ -3 & 4 & -1 \\ -1 & -2 & 3 \end{bmatrix},
$$

$\det(\tilde{A}) = 0$, indicating that $\tilde{A}$ is not an M matrix. So, for this example, one cannot use M matrix theory to determine the stability of the equilibrium of (10.5.23).

Now, let us take $p_1 = p_2 = p_3 = 1$, which satisfies the condition in Corollary 10.5.20. Thus, the equilibrium point of (10.5.23) is globally asymptotically stable.

10.6. Stability of bidirectional associative memory neural network

The stability and encoding properties of two-layer nonlinear feedback neural network were studied by Kösko [208] based on the following systems:

$$\dot{x}_i = -x_i - \sum_{j=1}^{m} S(y_j)m_{ij} + I_i, \quad i = 1, 2, \ldots, n, \tag{10.6.1}$$

$$\dot{y}_j = -y_j - \sum_{i=1}^{n} S(x_i)m_{ij} + I_j, \quad j = 1, 2, \ldots, m. \tag{10.6.2}$$

This dynamical model is a direct generalization of Hopfield continuous circuit model, where the meaning of the constants I_i, I_j, m_{ij} and the variables x_i, y_j can be found in [208]. Assume that $0 < \frac{dS(\xi)}{d\xi} \leqslant 1$.

Now, we present the results about the energy function obtained by Roska [352], as well as some methods and results concerning with globally exponential stability of a specific equilibrium position for more general bidirectional associative memory systems.

With the following energy function:

$$\begin{aligned} E(x, y) = {} & \sum_{i=1}^{n} \int_0^{x_i} \dot{S}(x_i)x_i \, dx_i - \sum_{i=1}^{n} \sum_{j=1}^{m} S(x_i)S(y_i)m_{ij} \\ & - \sum_{i=1}^{n} S(x_i)I_i + \sum_{j=1}^{m} \int_0^{y_i} \dot{S}(y_j)y_i \, dy_j - \sum_{j=1}^{m} S(y_j)I_j. \end{aligned} \tag{10.6.3}$$

Kosko [206] analyzed the stability of equations (10.6.1) and (10.6.2). However, for a specific equilibrium position $x = x^*$, $y = y^*$, one cannot determine its Lyapunov stability by using this method. In the following, we carry out a further study and develop a new result.

Let $x = (x_1, \ldots, x_n)^T, \ y = (y_1, \ldots, y_m)^T$.

THEOREM 10.6.1. *Assume that*

(1) $(x, y) = (x^*, y^*)$ *is an equilibrium position of equations* (10.6.1) and (10.6.2);
(2) $S(\xi) \in C^2, \xi \in R$;
(3) *The matrix* $H(h_{ij})_{(n+m)(n+m)}$ *is positive definite*;

then the equilibrium position (x^*, y^*) *of equations* (10.6.1) and (10.6.2) *is asymptotically stable, where*

$$h_{ij} = -\dot{S}(x_i^*)S(y_j^*)m_{ij} = -\dot{S}(y_j^*)S(x_i^*)m_{ji}$$

$$
\begin{aligned}
&= h_{ji}, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ i \neq j,\\
h_{ii} &= \dot{S}\big(x_i^*\big), \quad i = 1, 2, \ldots, n,\\
h_{jj} &= \dot{S}\big(y_j^*\big), \quad j = 1, 2, \ldots, m.
\end{aligned}
$$

PROOF. Here, we construct a new positive definite and radially unbounded Lyapunov function:

$$
W(x, y) = E(x, y) - E\big(x^*, y^*\big).
$$

Obviously, $W(x^*, y^*) = 0$ and

$$
\begin{aligned}
\frac{\partial W}{\partial x_i}\bigg|_{\substack{x=x^*\\ y=y^*}} &= \left(x_i^* - \sum_{j=1}^{m} S\big(y_j^*\big)m_{ij} - I_i\right)\dot{S}\big(x_i^*\big) = 0, \quad i = 1, \ldots, n,\\
\frac{\partial W}{\partial y_j}\bigg|_{\substack{x=x^*\\ y=y^*}} &= \left(y_j^* - \sum_{i=1}^{n} S\big(y_i^*\big)m_{ij} - J_j\right)\dot{S}\big(y_j^*\big) = 0, \quad j = 1, \ldots, m,
\end{aligned}
$$

$$
\begin{aligned}
h_{ij} &= \frac{\partial^2 W}{\partial x_i \partial y_j}\bigg|_{\substack{(x=x^*)\\ (y=y^*)}} = -\dot{S}(x_i^*)S\big(y_j^*\big)m_{ij} = -\dot{S}\big(y_j^*\big)\dot{S}\big(x_i^*\big)m_{ji}\\
&= \frac{\partial^2 W}{\partial y_j \partial x_i}\bigg|_{\substack{(x=x^*)\\ (y=y^*)}} = h_{ji},\\
&\qquad i \neq j,\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\\
h_{ii} &= \frac{\partial^2 W}{\partial x_i^2}\bigg|_{\substack{(x=x^*)\\ (y=y^*)}} = \dot{S}_i\big(x_i^*\big), \quad i = 1, 2, \ldots, n,\\
h_{jj} &= \frac{\partial^2 W}{\partial y_j^2}\bigg|_{\substack{(x=x^*)\\ (y=y^*)}} = \dot{S}_j\big(y_j^*\big), \quad j = 1, 2, \ldots, m.
\end{aligned}
$$

According to the minimax theorem of multivariate function, we know that the positive (negative) definiteness of matrix $H(h_{ij})$ implies that the equilibrium position (x^*, y^*) is the minimal (maximal) value point of $W(x, y)$, i.e., $W(x, y)$ is positive (negative) definite in some neighborhood of $(x, y) = (x^*, y^*)$. In addition,

$$
\frac{dW}{dt}\bigg|_{(10.6.1)\cup(10.6.2)} = \sum_{i=1}^{n} \dot{S}(x_i)x_i^2 - \sum_{j=1}^{m} \dot{S}(y_j)y_j^2 \leqslant 0.
$$

Therefore,

$$
\frac{dW}{dt}\bigg|_{(10.6.1),\,(10.6.2)} < 0 \quad \text{if and only if } x \neq x^* \text{ or } y \neq y^*,
$$

$$\left.\frac{dW}{dt}\right|_{(10.6.1),\,(10.6.2)} = 0 \quad \text{if and only if } x = x^* \text{ and } y = y^*.$$

By Lyapunov asymptotic stability theorem, the equilibrium point $(x, y) = (x^*, y^*)$ is asymptotically stable.

The proof is complete. □

Now we consider more general bidirectional associative memory models, described by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -E_n & 0 \\ 0 & -E_m \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix} \begin{pmatrix} S(x) \\ S(y) \end{pmatrix} + \begin{pmatrix} I \\ J \end{pmatrix}, \tag{10.6.4}$$

where

$$\begin{aligned} &x = (x_1, \ldots, x_n)^T, \quad y = (y_1, \ldots, y_m)^T, \quad T = (T_{ij})_{n\times m}, \\ &R = (R_{jk})_{n\times m}, \quad I = (I_1, \ldots, I_n)^T, \quad J = (J_1, \ldots, J_m)^T, \\ &S(x) = \big(S(x_1), \ldots, S(x_n)\big)^T, \quad S(y) = \big(S(y_1), \ldots, S(y_m)\big)^T. \end{aligned}$$

E_n and E_m are $n \times n$ and $m \times m$ unit matrices, respectively.

We admit $R^T \neq T$. Obviously, when $R^T = T$, equation (10.6.4) becomes equations (10.5.1) and (10.5.2). Here, R^T is a transpose of R.

Let $(x, y) = (x^*, y^*)$ be an equilibrium position of (10.6.4). Then equation (10.6.4) can be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -E_n & 0 \\ 0 & -E_m \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} + \begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix} \begin{pmatrix} S(x) - S(x^*) \\ S(y) - S(y^*) \end{pmatrix}. \tag{10.6.5}$$

THEOREM 10.6.2. *If there exist two positive definite matrices $\xi = \operatorname{diag}(\xi, \ldots, \xi_n)$ and $\eta = \operatorname{diag}(\eta_1, \ldots, \eta_m)$ such that the matrix*

$$\Omega = \begin{bmatrix} -\xi E_n & 0 & 0 & \xi T \\ 0 & -\eta E_m & \eta R & 0 \\ 0 & R^T \eta & -\xi E_m & 0 \\ T^T \xi & 0 & 0 & -\eta E_m \end{bmatrix}$$

is negative definite, then the equilibrium position (x^*, y^*) of equation (10.6.5) is globally exponentially stable and λ/μ can be defined as the Lyapunov exponent, where $-\lambda$ is the biggest eigenvalue of Ω, and $\mu = \max_{1\leqslant i\leqslant n}\{\xi_i, \eta_i\}$.

PROOF. We employ the positive definite and radially unbounded Lyapunov function:

$$V(a, b) = \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}. \tag{10.6.6}$$

Using the facts:

$$\left|S(x_i) - S(x_i^*)\right| \leqslant \left|x_i - x_i^*\right|, \quad i = 1, 2, \ldots, n,$$
$$\left|S(y_j) - S(y_j^*)\right| \leqslant \left|y_j - y_j^*\right|, \quad j = 1, 2, \ldots, m,$$

along the solution of equation (10.6.5), we compute the derivatives of $V(x, y)$ and $e^{\lambda t/\mu} V(x, y)$, respectively, and obtain the following results:

$$\begin{aligned}
&\frac{dV}{dt}\bigg|_{(10.6.5)} \\
&\quad = \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \left\{ \begin{bmatrix} -E_n & 0 \\ 0 & -E_m \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \right. \\
&\qquad \left. + \begin{pmatrix} 0 & T \\ R & 0 \end{pmatrix} \begin{pmatrix} S(x) - S(x^*) \\ S(y) - S(y^*) \end{pmatrix} \right\} \\
&\qquad + \left\{ \begin{bmatrix} -E_n & 0 \\ 0 & -E_m \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} + \begin{bmatrix} 0 & T \\ R & 0 \end{bmatrix} \begin{pmatrix} S(x) - S(x^*) \\ S(y) - S(y^*) \end{pmatrix} \right\}^T \\
&\qquad \times \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&\quad \leqslant -\left(x - x^*\right)^T (\xi E_n) \left(x - x^*\right) \\
&\qquad - \left(S(x) - S(x^*)\right)^T (\xi E_n) \left(S(x) - S(x^*)\right) \\
&\qquad - \left(y - y^*\right)^T (\eta E_m) \left(y - y^*\right) \\
&\qquad - \left(S(y) - S(y^*)\right)^T (\eta E_n) \left(S(y) - S(y^*)\right) \\
&\qquad + \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \begin{bmatrix} 0 & \xi T \\ \eta R & 0 \end{bmatrix} \begin{pmatrix} S(x) - S(x^*) \\ S(y) - S(y^*) \end{pmatrix} \\
&\qquad + \begin{pmatrix} S(x) - S(x^*) \\ S(y) - S(y^*) \end{pmatrix}^T \begin{bmatrix} 0 & R^T \eta \\ T^T \xi & 0 \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&\quad = \begin{pmatrix} x - x^* \\ y - y^* \\ S(x) - S(x^*) \\ S(y) - S(y^*) \end{pmatrix}^T \begin{bmatrix} -\xi E_r & 0 & 0 & \xi T \\ 0 & -\eta E_m & \eta R & 0 \\ 0 & R^T \eta & -\xi E_n & 0 \\ T^T \xi & 0 & 0 & -\eta E_m \end{bmatrix} \\
&\qquad \times \begin{pmatrix} x - x^* \\ y - y^* \\ S(x) - S(x^*) \\ S(y) - S(y^*) \end{pmatrix} \\
&\quad \leqslant -\lambda \left[\sum_{i=1}^{n} \left(x_i - x_j^*\right)^2 + \sum_{j=1}^{m} \left(y_j - y_j^*\right)^2 + \sum_{j=1}^{m} \left(S(y) - S(y_j^*)\right)^2 \right]
\end{aligned}$$

$$< 0 \quad \text{for } x \neq x^* \text{ or } y \neq y^*, \tag{10.6.7}$$

$$\begin{aligned}
\left.\frac{de^{\lambda t/\mu} V(x,y)}{dt}\right|_{(10.4.2)} &\leqslant e^{\lambda/\mu} \left\{ \frac{\lambda}{\mu} \mu \left[\sum_{i=1}^{n} \big(x_i - x_i^*\big)^2 + \sum_{j=1}^{m} \big(y_j - y_j^*\big)^2 \right] \right. \\
&\quad \left. - \lambda \left[\sum_{i=1}^{n} \big(x_i - x_i^*\big)^2 + \sum_{j=1}^{m} \big(y_j - y_j^*\big)^2 \right] \right\} \\
&\leqslant 0.
\end{aligned} \tag{10.6.8}$$

Integrating both sides of equation (10.6.8) from 0 to t yields

$$e^{\lambda t/\mu} V\big(x(t), y(t)\big) - V\big(x(0), y(0)\big) \leqslant 0,$$

i.e.,

$$e^{\lambda t/\mu} V\big(x(t), y(t)\big) \leqslant V\big(x(0), y(0)\big) < \infty.$$

Therefore, let $\underline{\mu} = \min_{i,j}[\xi_i, \eta_j]$. Then, we have

$$\begin{aligned}
\underline{\mu} \left[\sum_{i=1}^{n} \big(x_i - x_i^*\big)^2 + \sum_{j=1}^{m} \big(y_j - y_j^*\big)^2 \right] &\leqslant V\big(x(t), y(t)\big) \\
&\leqslant e^{-\lambda t/\mu} V\big(x(0), y(0)\big),
\end{aligned} \tag{10.6.9}$$

which implies immediately that the equilibrium position (x^*, y^*) is globally exponentially stable, and λ/μ is the Lyapunov exponent.

The proof of the theorem is complete. □

EXAMPLE 10.6.3. Consider the following system:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{pmatrix} S_1(y_1) \\ S_2(y_2) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \\
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{pmatrix} S_1(x_1) \\ S_2(x_2) \end{pmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}.
\end{aligned} \tag{10.6.10}$$

By choosing $\xi = \operatorname{diag}(1, 1)$ and $\eta = \operatorname{diag}(1, 1)$, one can verify that Ω is negative definite, and therefore the equilibrium position (x^*, y^*) of system (10.6.10) is globally exponentially stable.

10.7. Stability of BAM neural networks with variable delays

In this section, we consider more general bidirectional associative memory (BAM) neural networks with time delays as follows:

$$\frac{dx_i}{dt} = -\alpha_i x_i + \sum_{j=1}^{m} a_{ij} g_j\big(y_j\big(t - \tau_{ij}(t)\big)\big) + I_i, \quad i = 1, 2, \dots, n, \tag{10.7.1}$$

$$\frac{dy_i}{dt} = -\alpha_i y_i + \sum_{j=1}^{m} b_{ij} h_j\bigl(x_k\bigl(t - \sigma_{jk}(t)\bigr)\bigr) + I_j, \quad j = 1, 2, \ldots, m, \quad (10.7.2)$$

where α_i, β_j are some positive constants; a_{ij}, b_{jk} are weight coefficients, we admit $a_{ij} \neq b_{ij}$; $\tau_{ij}(t) > 0$ and $\sigma_{jk}(t) > 0$ are variable time delays; satisfying $0 \leqslant \tau_{ij}(t) \leqslant \tau_{ij} =$ constant, $\dot{\tau}_{ij}(t) \leqslant 0$ and $0 \leqslant \sigma_{jk}(t) \leqslant \sigma_{jk} =$ constant, $\dot{\tau}_{jk}(t) \leqslant 0$; the constant inputs I_i and J_j can be interpreted as the sustained environmental stimuli or as the stable reverberation from an adjoining neural network; $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

We assume that output functions g_j, h_k satisfy

$$0 < \frac{dg_j(\xi)}{d\xi} \leqslant 1, \quad 0 < \frac{dh_k(\xi)}{d\xi} \leqslant 1,$$
$$j = 1, 2, \ldots, m, \ k = 1, 2, \ldots, n.$$

We first investigate the existence, uniqueness and globally asymptotic stability of the equilibrium points.

THEOREM 10.7.1. *Assume that the matrix*

$$\Omega = \begin{bmatrix} \alpha & -|A| \\ -|B| & \beta \end{bmatrix}_{(n+m)\times(n+m)}$$

is an M matrix. Then the equilibrium point $(x, y) = (x^*, y^*)$ of systems (10.7.1) and (10.7.2) exists uniquely, and is globally asymptotically stable. Here,

$$\alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_n), \qquad \beta = \operatorname{diag}(\beta_1, \ldots, \beta_m),$$
$$|A| = \bigl(|a_{ij}|\bigr)_{n\times m}, \qquad |B| = \bigl(|b_{ij}|\bigr)_{m\times n}.$$

PROOF. First, we prove the existence and uniqueness of the equilibrium point of systems (10.7.1) and (10.7.2). Consider the following equations:

$$x_i = \sum_{j=1}^{m} \frac{a_{ij}}{\alpha_i} g_j(y_j) + I_i, \quad i = 1, 2, \ldots, n, \quad (10.7.3)$$

$$y_j = \sum_{k=1}^{n} \frac{b_{ij}}{\sigma_j} h_k(x_k) + I_j, \quad j = 1, 2, \ldots, m. \quad (10.7.4)$$

Then $\forall x^{(1)}, x^{(2)} \in R^n$ and $\forall y^{(1)}, y^{(2)} \in R^m$, we have

$$\sum_{j=1}^{m} \frac{a_{ij}}{\alpha_i} g_j\bigl(y_j^{(1)}\bigr) + I_i - \sum_{j=1}^{m} \frac{a_{ij}}{\alpha_i} g_j\bigl(y_j^{(2)}\bigr) - I_i$$
$$\leqslant \sum_{j=1}^{m} \frac{|a_{ij}|}{\alpha_i} \bigl|y_j^{(1)} - y_j^{(2)}\bigr|, \quad i = 1, 2, \ldots, n, \quad (10.7.5)$$

$$\sum_{k=1}^{n}\frac{b_{jk}}{\beta_j}h_k\big(x_k^{(1)}\big)+I_j-\sum_{k=1}^{n}\frac{b_{jk}}{\beta_k}h_k\big(x_k^{(2)}\big)-I_j$$
$$\leqslant\sum_{j=1}^{m}\frac{|b_{jk}|}{\beta_j}\big|x_k^{(1)}-x_k^{(2)}\big|,\quad j=1,2,\dots,m. \tag{10.7.6}$$

By the property of M matrix, we know that Ω being an M matrix implies that $\rho(H)<1$, where $\rho(H)$ is the spectral radius of matrix H, and

$$H=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}.$$

Note that $\rho(H)<1$ implies that the fixed point $(x,y)=(x^*,y^*)$ uniquely exists, which satisfies equations (10.7.3) and (10.7.4), i.e.,

$$\alpha_i x_i^*=\sum_{j=1}^{m}a_{ij}g_j\big(y_j^*\big)+I_I, \tag{10.7.7}$$

$$\beta_j y_j^*=\sum_{k=1}^{n}b_{jk}h_k\big(x_k^*\big)+I_j, \tag{10.7.8}$$

Therefore, $(x,y)=(x^*,y^*)$ is the unique equilibrium point of systems (10.7.1) and (10.7.2).

Next, we prove that $(x,y)=(x^*,y^*)$ is globally asymptotically stable. According to the property of M matrix, the conditions in the theorem imply that there exist $n+m$ positive constants $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m$ such that

$$\xi_k\alpha_k-\sum_{j=1}^{m}|b_{jk}|\eta_j>0,\quad k=1,2,\dots,n, \tag{10.7.9}$$

$$\eta_j\beta_j-\sum_{i=1}^{n}|a_{ij}|\xi_i>0,\quad j=1,2,\dots,m. \tag{10.7.10}$$

We choose the Lyapunov functional as follows:

$$\begin{aligned}V(x,y,t)={}&\sum_{i=1}^{n}\xi_i\big|x_i-x_i^*\big|+\sum_{j=1}^{m}\eta_j\big|y_j-y_j^*\big|\\&+\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{t-\tau_{ij}(t)}^{t}\xi_i|a_{ij}|\big|g_j\big(y_j(s)\big)-g_j\big(y_j^*\big)\big|\,ds\\&+\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{t-\sigma_{jk}(t)}^{t}\eta_j|b_{jk}|\big|h_k\big(x_k(s)\big)-h_k\big(x_k^*\big)\big|\,ds.\end{aligned} \tag{10.7.11}$$

Obviously $V(x^*, y^*, t) = 0$, $V(x^*, y^*, t) > 0$ for $x \neq x^*$, $y \neq y^*$ and $V \to +\infty$ as $|x|^2 + |y|^2 \to \infty$.

Calculating the right-upper Dini derivative D^+V of V along the solution of systems (10.7.1) and (10.7.2) yields

$$\begin{aligned}
&\frac{dx_i}{dt} = -\alpha_i\left(x_i - x_i^*\right) + \sum_{j=1}^{m} a_{ij}\left[g_j\left(y_j(t-\tau_{ij})\right) - g_j\left(y_j^*\right)\right],\\
&\quad i = 1, \ldots, n, && (10.7.12)\\
&\frac{dy_j}{dt} = -\beta_j\left(y_j - y_j^*\right) + \sum_{k=1}^{n} b_{jk}\left[h_k\left(x_k(t-\sigma_{jk})\right) - h_k\left(x_k^*\right)\right],\\
&\quad j = 1, \ldots, m. && (10.7.13)
\end{aligned}$$

By using the facts $\dot{\tau}_{ij}(t) \leqslant 0$ and $\dot{\sigma}_{jk}(t) \leqslant 0$, we have

$$\begin{aligned}
&D^+V|_{(10.7.12)\cup(10.7.13)}\\
&\quad\leqslant -\sum_{i=1}^{n} \xi_i\alpha_i\left|x_i - x_i^*\right| + \sum_{i=1}^{n}\sum_{j=1}^{m} \xi_i|a_{ij}|\left|g_j\left(y_j(t)\right) - g_j\left(y_j^*\right)\right|\\
&\qquad - \sum_{j=1}^{m} \eta_j\beta_j\left|y_j - y_j^*\right| + \sum_{j=1}^{m}\sum_{k=1}^{n} \eta_j|b_{jk}|\left|h_k\left(x_k(t)\right) - h_k\left(x_k^*\right)\right|\\
&\quad\leqslant \sum_{k=1}^{n}\left[-\xi_k\alpha_k + \sum_{j=1}^{m} \eta_j|b_{jk}|\right]\left|x_k(t) - x_k^*\right|\\
&\qquad + \sum_{j=1}^{m}\left[-\eta_j\beta_j + \sum_{i=1}^{n} \xi_i|a_{ij}|\right]\left|y_j(t) - y_j^*\right|\\
&\quad< 0 \quad \text{for } x \neq x^* \text{ or } y \neq y^*.
\end{aligned}$$

So, the conclusion of the theorem is true. □

Now, we consider another BAM model with time delays:

$$\begin{aligned}
\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} &= \begin{bmatrix}-\alpha & 0\\ 0 & -\beta\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\begin{pmatrix}g(y(t-\tau(t)))\\ h(x(t-\sigma(t)))\end{pmatrix}\\
&\quad + \begin{pmatrix}I\\ J\end{pmatrix}, \qquad (10.7.14)
\end{aligned}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{m\times n}$ are two positive definite matrices; $\tau(t) = (\tau_1(t), \ldots, \tau_n(t))^T$ satisfying $0 \leqslant \tau_i(t) \leqslant \tau_i = \text{constant}$, $\dot{\tau}_i(t) \leqslant 0$, $i = 1, 2, \ldots, n$; $\sigma(t) = (\sigma_1(t), \ldots, \sigma_m(t))^T$ satisfying $0 \leqslant \sigma_j(t) \leqslant \sigma_j = \text{constant}$, $\dot{\sigma}_j(t) \leqslant 0$, $j = 1, 2, \ldots, m$. Generally,

$A \neq B^T$, $x = (x_1, \dots, x_n)^T$, $y = (y_1, \dots, y_m)^T$, and

$$g\big(y\big(t-\tau(t)\big)\big) = \big(g_1\big(y_1\big(t-\tau_1(t)\big)\big), \dots, g_m\big(y_m\big(t-\tau_m(t)\big)\big)\big)^T,$$
$$h\big(x\big(t-\sigma(t)\big)\big) = \big(h_1\big(x_1\big(t-\sigma_1(t)\big)\big), \dots, h_n\big(x_n\big(t-\sigma_n(t)\big)\big)\big)^T,$$
$$0 < \frac{dg_j(\xi)}{d\xi} \leqslant 1, \quad j = 1, 2, \dots, n,$$
$$0 < \frac{dh_i(\xi)}{d\xi} \leqslant 1, \quad i = 1, 2, \dots, m.$$

Let $(x, y) = (x^*, y^*)$ be a known equilibrium point. We rewrite equation (10.7.14) as

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} + \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \\ \times \begin{pmatrix} h(x(t-\sigma(t))) - h(x^*) \\ g(y(t-\tau(t))) - g(y^*) \end{pmatrix}.\end{aligned} \tag{10.7.15}$$

THEOREM 10.7.2. *If there exist four positive definite matrices $\xi = \operatorname{diag}(\xi_1, \dots, \xi_n)$, $\eta = \operatorname{diag}(\eta_1, \dots, \eta_n)$, $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $\mu = \operatorname{diag}(\mu_1, \dots, \mu_m)$ such that*

$$Q = \begin{bmatrix} -2\xi\alpha + \lambda & 0 & 0 & \xi A \\ 0 & -2\eta\beta + \mu & \eta B & 0 \\ 0 & B^T\eta & -\lambda & 0 \\ A^T\xi & 0 & 0 & -\mu \end{bmatrix}$$

is negative definite, then the equilibrium point $x = x^*$, $y = y^*$ is globally asymptotically stable.

PROOF. We construct the Lyapunov function:

$$\begin{aligned} V(x, y, t) = {} & \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\ & + \sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} \big(g_j\big(y_j(s)\big) - g_j\big(y_j^*\big)\big)^2 \, ds \\ & + \sum_{i=1}^{m} \int_{t-\sigma_i(t)}^{t} \big(h_i\big(x_i(s)\big) - h_i\big(x_i^*\big)\big)^2 \, ds. \end{aligned} \tag{10.7.16}$$

Since

$$\big|g_j(y_j) - g_j\big(y_j^*\big)\big| \leqslant \big|y_j - y_j^*\big|, \quad j = 1, 2, \dots, n,$$

$$\left|h_i(x_i) - h_i\left(x_i^*\right)\right| \leqslant \left|x_i - x_i^*\right|, \quad i = 1, 2, \ldots, m,$$

we calculate the derivative V along the solution of (10.7.15) to obtain

$$\begin{aligned}
\frac{dV}{dt}\bigg|_{(10.7.15)} \leqslant{}& \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \left\{ \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \right. \\
&\left. + \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{pmatrix} h(x(t-\sigma)) - h(x^*) \\ g(y(t-\tau)) - g(y^*) \end{pmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} + \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{pmatrix} h(x(t-\sigma)) - h(x^*) \\ g(y(t-\tau)) - g(y^*) \end{pmatrix} \right\}^T \\
&\times \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} + \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&- \begin{pmatrix} h(x(t-\sigma)) - h(x^*) \\ g(y(t-\tau)) - g(y^*) \end{pmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{pmatrix} h(x(t-\sigma)) - h(x^*) \\ g(y(t-\tau)) - g(y^*) \end{pmatrix} \\
={}& 2 \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{bmatrix} -\alpha\xi & 0 \\ 0 & -\beta\eta \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&+ \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&+ \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{bmatrix} 0 & \xi A \\ \eta B & 0 \end{bmatrix} \begin{pmatrix} h(x(t-\sigma(t))) - h(x^*) \\ g(y(t-\tau(t))) - g(y^*) \end{pmatrix} \\
&+ \begin{pmatrix} h(x(t-\sigma(t))) - h(x^*) \\ g(y(t-\tau(t))) - g(y^*) \end{pmatrix}^T \begin{bmatrix} 0 & \eta B^T \\ \xi A^T & 0 \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \\
&- \begin{pmatrix} h(x(t-\sigma(t))) - h(x^*) \\ g(y(t-\tau(t))) - g(y^*) \end{pmatrix}^T \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix} \begin{pmatrix} h(x(t-\sigma(t))) - h(x^*) \\ g(y(t-\tau(t))) - g(y^*) \end{pmatrix} \\
={}& \begin{pmatrix} x - x^* \\ y - y^* \\ h(x(t-\sigma(t))) - h(x^*) \\ g(y(t-\tau(t))) - g(y^*) \end{pmatrix}^T \begin{bmatrix} -2\xi\alpha + \lambda & 0 & 0 & \xi A \\ 0 & -2\eta\beta + \mu & \eta B & 0 \\ 0 & B^T\eta & -\lambda & 0 \\ A^T\xi & 0 & 0 & -\mu \end{bmatrix} \\
&\times \begin{pmatrix} x - x^* \\ y - y^* \\ h(x(t-\sigma(t))) - h(x^*) \\ g(y(t-\tau(t))) - g(y^*) \end{pmatrix} < 0 \quad \text{for } x \neq x^* \text{ or } y \neq y^*. \qquad (10.7.17)
\end{aligned}$$

This means that the equilibrium point $x = x^*$, $y = y^*$ is globally asymptotically stable. The proof of the theorem is complete. □

EXAMPLE 10.7.3. Let us consider a 4-dimensional BAM neural network with delays:

$$\begin{aligned}
\frac{dx_1}{dt} &= -2x_1 + g_1\big(y_1(t-\tau_{11})\big) + \frac{1}{2}g_2\big(y_2(t-\tau_{12})\big) + I_1,\\
\frac{dx_2}{dt} &= -3x_2 + \frac{3}{2}g_1\big(y_1(t-\tau_{21})\big) - g_2\big(y_2(t-\tau_{22})\big) + I_2,\\
\frac{dy_1}{dt} &= -3y_1 - \frac{3}{2}h_1\big(x_1(t-\sigma_{11})\big) + h_2\big(x_2(t-\sigma_{22})\big) + J_1,\\
\frac{dy_2}{dt} &= -2y_2 - h_1\big(x_1(t-\sigma_{21})\big) + \frac{1}{2}h_2\big(x_2(t-\sigma_{22})\big) + J_2.
\end{aligned} \tag{10.7.18}$$

Assume that g_i and h_j satisfy

$$0 < \frac{dg_i(\xi)}{d\xi} \leqslant 1, \quad 0 < \frac{dh_j(\xi)}{d\xi} \leqslant 1, \quad i, j = 1, 2,$$

and $\tau_{ij} > 0,\ \sigma_{ij} > 0,\ i, j = 1, 2$, are time delays. By Theorem 10.7.1, we have

$$\alpha = \mathrm{diag}(\alpha_1,\ \alpha_2) = \mathrm{diag}(2, 3), \qquad \beta = \mathrm{diag}(\beta_1,\ \beta_2) = \mathrm{diag}(3, 2),$$

$$A = \begin{bmatrix} 1 & 0.5 \\ 1.5 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1.5 & 1 \\ -1 & 0.5 \end{bmatrix}.$$

Obviously,

$$\Omega = \begin{bmatrix} \alpha & -|A| \\ -|B| & \beta \end{bmatrix}_{4\times 4} = \begin{bmatrix} 2 & 0 & -1 & -0.5 \\ 0 & 3 & -1.5 & -1 \\ -1.5 & -1 & 3 & 0 \\ -1 & -0.5 & 0 & 2 \end{bmatrix}$$

is an M matrix.

According to Theorem 10.7.1, we know that the equilibrium point $x = x^*$, $y = y^*$ of system (10.7.18) exists uniquely, and is globally asymptotically stable.

EXAMPLE 10.7.4. For a comparison, we consider another 4-dimensional BAM neural network with delays:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}\begin{pmatrix} g_1(y_1(t-\tau_1)) \\ g_2(y_2(t-\tau_2)) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},\\
\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{pmatrix} h_1(x_1(t-\sigma_1)) \\ h_2(x_2(t-\sigma_2)) \end{pmatrix}\\
&\quad + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix},
\end{aligned} \tag{10.7.19}$$

where

$$\alpha = \beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}.$$

Since

$$\Omega = \begin{bmatrix} \alpha & -|A| \\ -|B| & \beta \end{bmatrix}_{4\times 4} = \begin{bmatrix} 1 & 0 & -0.5 & -0.5 \\ 0 & 1 & -0.5 & -0.5 \\ -0.5 & -0.5 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix}$$

is not an M matrix, Theorem 10.7.1 cannot be applied here. However, the conditions in Theorem 10.7.2 are satisfied. In fact, if we take

$$\xi = \eta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then it is easy to verify that Q is negative definite. Hence, the equilibrium point of system (10.7.10) exists uniquely and is globally asymptotically stable.

10.8. Exp. stability and exp. periodicity of DNN with Lipschitz type activation function

In this section, we consider the model of neural networks with delays, described by the following DDEs:

$$\begin{aligned} \frac{dy_i}{dt} &= -d_i y_i + \sum_{j=1}^{n} a_{ij} f_j\big(y_j\big(t - \tau_j(t)\big)\big) + I_i, \quad t \geqslant 0, \\ y_i(t) &= \phi_i(t), \quad -\tau \leqslant t \leqslant 0, \end{aligned} \tag{10.8.1}$$

where $0 \leqslant \tau_j(t) \leqslant \tau_j = \text{constant}$, $\dot{\tau}_j(t) \leqslant 0$, $j = 1, 2, \dots, n$, and

$$f_i(\cdot) \in L := \{y_i \mid 0 \leqslant \dot{y}_i \leqslant L\}.$$

DEFINITION 10.8.1. An equilibrium point $y^* \in R^n$ of system (10.8.1) is said to be globally exponentially stable if there exist two positive constants $\alpha \geqslant 1$ and $\beta > 0$ such that for any $y_0 \in R^n$ and $t \in [0, +\infty)$,

$$\|y(t) - y^*\| \leqslant \alpha \|\phi - y^*\| e^{-\beta t}.$$

THEOREM 10.8.2. *Suppose $g \in L$ and $DL^{-1} - |A|$ is an M matrix, where $D = \operatorname{diag}(d_1, \dots, d_n)$. Then, the equilibrium point y^* of the delayed neural system (10.8.1) is globally exponentially stable.*

PROOF. Since $DL^{-1} - |A|$ is an M matrix, the neural system (10.8.1) has unique equilibrium point y^*. Let $x_i = y_i - y_i^*$ and $g_j(x_j) = f_j(x_j + y_j^*) - f_j(y_j^*)$, then system (10.8.1) can be rewritten as

$$\frac{dx_i}{dt} = -d_i x_i + \sum_{j=1}^{n} a_{ij} g_j\big(x_j\big(t - \tau_j(t)\big)\big), \quad t \geqslant 0, \tag{10.8.2}$$

and now $x^* = 0$ is the unique equilibrium point of system (10.8.2).

Since $DL^{-1} - |A|$ is an M matrix, there exists a positive definite matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ such that

$$\lambda_i d_i - L_i \sum_{j=1}^{n} \lambda_j |a_{ji}| > 0 \quad \text{for } i = 1, 2, \dots, n.$$

Thus, we can choose a small $\varepsilon > 0$ such that

$$\lambda_i(d_i - \varepsilon) - L_i \sum_{j=1}^{n} \lambda_j e^{\varepsilon\tau} |a_{ji}| > 0 \quad \text{for } i = 1, 2, \dots, n,$$

where $\tau = \max_{1 \leqslant i \leqslant n} \tau_i$. We use the following positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n} \lambda_i \left(|x_i| e^{\varepsilon\tau} + \sum_{j=1}^{n} |a_{ij}| \int_{t-\tau_j}^{t} \big|g_j\big(x_j(s)\big)\big| e^{s+\varepsilon\tau_j}\, ds \right).$$

Computing the right-upper derivative along the solution of (10.8.2), we have

$$\begin{aligned}
&D^+ V\big(x(t)\big)\big|_{(10.8.2)} \\
&\quad = \sum_{i=1}^{n} \lambda_i \operatorname{sign}\big(x_i(t)\big) \left[-d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j\big(x_j(t - \tau_j)\big) \right] e^{\varepsilon t} \\
&\qquad + \sum_{i=1}^{n} \lambda_i \big|x_i(t)\big| \varepsilon e^{\varepsilon t} + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i |a_{ij}| \Big(\big|g_j\big(x_j(t)\big)\big| e^{t+\varepsilon\tau_j} \\
&\qquad - \big|g_j\big(x_j(t - \tau_j)\big)\big| e^{\varepsilon t} \big(1 - \dot{\tau}_j(t)\big) \Big) \\
&\quad \leqslant \sum_{i=1}^{n} \lambda_i \left[(\varepsilon - d_i)|x_i| + \sum_{j=1}^{n} |a_{ij}| e^{\varepsilon\tau} \big|g_j(x_j)\big| \right] e^{\varepsilon t} \\
&\quad \leqslant -\sum_{i=1}^{n} \left[\lambda_i(\varepsilon - d_i) - \sum_{j=1}^{n} e^{\varepsilon\tau} |a_{ji}| L_i \lambda_j \right] |x_j| e^{\varepsilon t} \\
&\quad \leqslant 0.
\end{aligned} \tag{10.8.3}$$

Therefore,

$$V(t) \leqslant V(0), \quad t \geqslant 0,$$

from which we obtain

$$e^{\varepsilon t}\Big(\min_{1\leqslant i\leqslant n} \lambda_i\Big)\sum_{i=1}^{n}\big|y_i - y_i^*\big| \leqslant V(t)$$

and

$$\begin{aligned} V(0) &= \sum_{i=1}^{n}\lambda_i\left[\big|x_i(0)\big| + \sum_{j=1}^{n}|a_{ij}|\int\limits_{-\tau_j(t)}^{0}\big|g_j\big(x_j(s)\big)\big|e^{s+\varepsilon\tau_j}\,ds\right] \\ &\leqslant \left[\max_{1\leqslant i\leqslant n}\lambda_i + L\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\lambda_i\max_{1\leqslant i\leqslant n}|a_{ij}|\right]\|\phi - y^*\|, \end{aligned}$$

where $L = \max_{1\leqslant i\leqslant n} L_i$ is a constant. Therefore, we obtain

$$\big\|y(t) - y^*\big\|_1 \leqslant \alpha\big\|\phi - y^*\big\|_1 e^{-\varepsilon t} \quad \forall t \geqslant 0, \tag{10.8.4}$$

in which

$$\alpha = \frac{\max_{1\leqslant i\leqslant n}\lambda_i + L\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\lambda_i\max_{1\leqslant i\leqslant n}|a_{ij}|}{\min_{1\leqslant i\leqslant n}\lambda_i} \geqslant 1.$$

It then follows from (10.8.4) that the equilibrium point of (10.8.1) is globally exponentially stable for any delays, in view of the equivalence of the norms $\|x\|_1$ and $\|x\|$. □

REMARK 10.8.3. Theorem 10.8.2 improves and generalizes the result in [349] (see Theorem 2 therein). Here, we have used variable time delays to replace the constant delays used in [349].

THEOREM 10.8.4. *(See [266].) Assume that there exist constants $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2, \ldots, n$, such that the matrix*

$$W = \begin{bmatrix} \mathrm{diag}(-2\beta_1 d_1 + \alpha_1 L_1^2, \ldots, -2\beta_1 d_1 + \alpha_n L_n^2) & \beta_i a_{ij} + \beta_j a_{ji} \\ (\beta_i a_{ij} + \beta_j a_{ji})^T & \mathrm{diag}(-\alpha_1, \ldots, -\alpha_n) \end{bmatrix}_{2n\times 2n}$$

is negative definite. Then the equilibrium $y = y^*$ of system (10.8.1) is globally exponentially stable. Here, $\beta = \mathrm{diag}(\beta_1, \ldots, \beta_n)$.

PROOF. Construct the Lyapunov functional

$$V\big(x(t),t\big)=\sum_{i=1}^{n}\beta_i x_i^2(t)+\sum_{i=1}^{n}\alpha_i\int\limits_{t-\tau_j(t)}^{t} g_j^2\big(x(s)\big)\,ds. \tag{10.8.5}$$

Obviously, $V(x(t),t)$ is positive definite and radially unbounded. Calculating the derivative of $V(x(t),t)$ along the solution of (10.8.2) results in

$$\begin{aligned}
\frac{dV}{dt}\bigg|_{(10.8.2)} &= -2\sum_{i=1}^{n}\beta_i d_i x_i^2(t)+2\sum_{i=1}^{n}\beta_i x_i(t)\sum_{j=1}^{n}a_{ij}g_j\big(x_j\big(t-\tau_j(t)\big)\big)\\
&\quad+\sum_{i=1}^{n}\alpha_i\big[g_i^2\big(x_i(t)\big)-g_i^2\big(x_i\big(t-\tau_i(t)\big)\big)\big(1-\dot\tau_j(t)\big)\big]\\
&\leqslant-\sum_{i=1}^{n}\big(2\beta_i d_i-\alpha_i L_i^2\big)x_i^2(t)\\
&\quad+2\sum_{i=1}^{n}\beta_i x_i(t)\sum_{j=1}^{n}a_{ij}g_j\big(x_j\big(t-\tau_j(t)\big)\big)\\
&\quad-\sum_{i=1}^{n}\alpha_i g_i^2\big(x_i\big(t-\tau_i(t)\big)\big)\\
&=\begin{pmatrix}x(t)\\ g(x(t-\tau(t)))\end{pmatrix}^T W\begin{pmatrix}x(t)\\ g(x(t-\tau(t)))\end{pmatrix}\\
&\leqslant-\mu\big[x^T(t)x(t)+g^T\big(x\big(t-\tau(t)\big)\big)g\big(x\big(t-\tau(t)\big)\big)\big],
\end{aligned} \tag{10.8.6}$$

where W is defined in the theorem, and $-\mu$ is the maximum eigenvalue of matrix W. Note that W is the coefficient matrix of (10.8.6).

Let ε be the unique positive solution to the equation

$$\max_{1\leqslant i\leqslant n}\big(\varepsilon\beta_i+\varepsilon\alpha_i\tau_i e^{\varepsilon\tau_i}L_i^2\big)=\mu.$$

Then the derivative of $e^{\varepsilon t}V(x(t),t)$ along the solution of system (10.7.2) is given by

$$\begin{aligned}
\frac{de^{\varepsilon t}V(x(t),t)}{dt}\bigg|_{(10.8.2)} &\leqslant e^{\varepsilon t}\varepsilon\left[\sum_{i=1}^{n}\beta_i x_i^2(t)+\sum_{i=1}^{n}\alpha_i\int\limits_{t-\tau_i(t)}^{t}g_i^2\big(x_i(s)\big)\,ds\right]\\
&\quad-e^{\varepsilon t}\mu\left[\sum_{i=1}^{n}x_i^2(t)+\sum_{i=1}^{n}g_i^2\big(x_i\big(t-\tau_i(t)\big)\big)\right]
\end{aligned}$$

$$\leqslant e^{\varepsilon t}\mu\Bigg[\varepsilon\sum_{i=1}^{n}\alpha_i\int\limits_{t-\tau_i}^{t} g_i^2\big(x_i(s)\big)\,ds - \sum_{i=1}^{n}(\mu-\varepsilon\beta_i)x_i^2(t)\Bigg]. \tag{10.8.7}$$

Integrating both sides of (10.8.7) from 0 to t_1 yields

$$e^{\varepsilon t_1}V\big(x(t_1),t_1\big) \leqslant V\big(x(0),0\big) + \int\limits_0^{t_1} e^{\varepsilon t}\sum_{i=1}^{n}\alpha_i\int\limits_{t-\tau_i}^{t} g_i^2\big(x_i(s)\big)\,ds\,dt - \int\limits_0^{t_1}(\mu-\varepsilon\beta_i)x_i^2(s)e^{\varepsilon t}\,ds. \tag{10.8.8}$$

Now, we estimate the integral

$$\int\limits_0^{t_1} e^{\varepsilon t}\varepsilon\alpha_i\int\limits_{t-\tau_i}^{t} f_i^2\big(x_i(s)\big)\,ds\,dt.$$

By the condition $|g_i(x_i)| \leqslant L_i|x_i|$ and changing the order of integration, we have

$$\begin{aligned}
&\int\limits_0^{t_1} e^{\varepsilon t}\varepsilon\alpha_i\int\limits_{t-\tau_i}^{t} g_i^2\big(x_i(s)\big)\,ds\,dt \\
&\quad\leqslant \varepsilon\alpha_i\int\limits_{-\tau_i}^{t_1}\Bigg(\int\limits_{\max[s,0]}^{\min[s+\tau_i,t_1]} e^{\varepsilon t}\,dt\Bigg)L_i^2\big|x_i(s)\big|^2\,ds \\
&\quad\leqslant \varepsilon\alpha_i\int\limits_{-\tau_i}^{t_1}\tau_i e^{\varepsilon(s+\tau_i)}L_i^2\big|x_i(s)\big|^2\,ds \\
&\quad\leqslant \varepsilon\alpha_i\int\limits_{-\tau_i}^{0}\tau_i e^{\varepsilon(s+\tau_i)}L_i^2\big|x_i(s)\big|^2\,ds + \varepsilon\alpha_i\int\limits_{0}^{t_1}\tau_i e^{\varepsilon(s+\tau_i)}L_i^2\big|x_i(s)\big|^2\,ds \\
&\quad\leqslant \varepsilon\alpha_i\tau_i e^{\varepsilon\tau_i}\Bigg[\int\limits_{-\tau_i}^{0}L_i^2\big|x_i(s)\big|^2\,ds + L_i^2\int\limits_0^{t_1}L_i^2 e^{\varepsilon s}\big|x_i(s)\big|^2\,ds\Bigg].
\end{aligned} \tag{10.8.9}$$

Now, substituting the inequalities (10.8.9) and (10.8.8) yields

$$\begin{aligned}
e^{\varepsilon t_1} V\big(x(t_1), t_1\big) &\leqslant V\big(x(0), 0\big) + \sum_{i=1}^{n} \varepsilon\alpha_i\tau_i e^{\varepsilon\tau_i} \int_{-\tau_i}^{0} L_i^2 \big|x_i(s)\big|^2\, ds \\
&\quad - \int_{-\tau_i}^{0} \sum_{i=1}^{n} \big(\mu - \varepsilon\beta_i - \varepsilon\alpha_i\tau_i e^{\varepsilon\tau_i} L_i^2\big) e^{\varepsilon s} \big|x_i(s)\big|^2\, ds \\
&\leqslant V\big(x(0), 0\big) + \sum_{i=1}^{n} \varepsilon\alpha_i\tau_i e^{\varepsilon\tau_i} \int_{-\tau_i}^{0} L_i^2 \big|x_i(s)\big|^2\, ds \\
&\leqslant \bar{M} = \text{some positive constant.}
\end{aligned}$$

Therefore, the following estimation is satisfied for any $t_1 > 0$:

$$e^{\varepsilon t_1} \sum_{i=1}^{n} \beta_i x_i^2(t) \leqslant e^{\varepsilon t_1} V\big(x(t_1), t_1\big) \leqslant \bar{M},$$

which implies that the equilibrium point of system (10.8.1) is globally exponentially stable, and the proof is complete. □

REMARK 10.8.5. It should be noted that the method used in proving Theorem 10.8.4 is different from that of Theorem 10.8.2 because the two functionals employed in the two theorems are different.

In the following, we consider the model of continuous-time neural network with delays described by the following differential equations with delay [373]:

$$\begin{aligned}
\frac{dx_i(t)}{dt} &= -d_i x_i(t) + \sum_{i=1}^{n} d_i a_{ij} g_j\big(x_i(t)\big) + \sum_{i=1}^{n} b_{ij} g_j\big(x_j(t-\tau_j)\big) + I_i(t), \\
&\qquad t \geqslant 0, \\
x_i(t) &= \phi_i(t), \quad -\tau \leqslant t \leqslant 0,\ i = 1, 2, \ldots, n.
\end{aligned} \tag{10.8.10}$$

Define $x_t(\theta) = x(t+\theta),\ \theta \in [-\tau, 0],\ t \geqslant 0$. Let

$$\|x_t\| = \sup_{-\tau\leqslant\theta\leqslant 0} \sum_{i=1}^{n} \big|x_i(t+\theta)\big|. \tag{10.8.11}$$

DEFINITION 10.8.6. The neural system (10.8.10) is said to be exponentially periodic if there exists one ω-periodic solution of the system and all other solutions of the system converge exponentially to it as $t \to +\infty$.

Let $C = C([-\tau, i], R^n)$ be the Banach space of all continuous functions from $[-\tau, 0]$ to R^n with the topology of uniform convergence. For any $\phi \in C$, let

$$\|\phi\| = \sup_{-\tau \leqslant t \leqslant 0} \sum_{i=1}^{n} \|\phi_i(t)\|.$$

Given any $\phi, \psi \in C$, let $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi))^T$ and $x(t, \phi) = (x_1(t, \psi), x_2(t, \psi), \ldots, x_n(t, \psi))^T$ be the solutions of (10.8.10) starting from ϕ and ψ, respectively.

THEOREM 10.8.7. *Suppose $g \in L$. Let $M = (m_{ij})_{n\times n}$, where $m_{ij} = |a_{ij}| + |b_{ij}|$. If $DL^{-1} - M$ is an M matrix, where $D = \operatorname{diag}(d_1, \ldots, d_n)$, then for every periodic input $I(\cdot)$, the delayed neural system (10.8.10) is exponentially periodic.*

PROOF. Define $x_t(\phi) = x(t + \theta, \phi), \theta \in [-\tau, 0]$, then $x_t(\phi) \in C$ for all $t \geqslant 0$. Thus, it follows from (10.8.10) that

$$\begin{aligned}
&\frac{d}{dt}\big(x_i(t, \phi) - x_i(t, \phi)\big) = -\big(x_i(t, \phi) - x_i(t, \phi)\big) \\
&\quad + \sum_{j=1}^{n} a_{ij}\big(g_j\big(x_j(t, \phi)\big) - g_j\big(x_j(t, \phi)\big)\big) \\
&\quad + \sum_{j=1}^{n} b_{ij}\big(g_j\big(x_j(t - \tau_j, \phi)\big) - g_j\big(x_j(t - \tau_j, \phi)\big)\big) \qquad (10.8.12)
\end{aligned}$$

for $t \geqslant 0$, $i = 1, 2, \ldots, n$. By noticing $g \in L$, it is easy to deduce that there exist k_i $(0 \leqslant k_i \leqslant L_i)$ such that $|g_i(x_i(t, \phi)) - g_i(x_i(t, \psi))| \leqslant k_i|x_i(t, \phi) - x_i(t, \psi)|$.

Since $DL^{-1} - M$ is an M matrix, there exists a positive diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ such that

$$\begin{aligned}
&\lambda_i d_i - k_i\lambda_i|a_{ii}| - k_i \sum_{j=1, j\neq i}^{n} \lambda_j|a_{ji}| - k_i \sum_{j=1}^{n} \lambda_j|b_{ji}| > 0, \\
&\quad i = 1, 2, \ldots, n, \qquad (10.8.13)
\end{aligned}$$

and there exists a constant $\varepsilon > 0$ such that

$$\begin{aligned}
F_i(\varepsilon) = \lambda_i(d_i - \varepsilon) - k_i\lambda_i|a_{ii}| - k_i \sum_{j=1, j\neq i}^{n} \lambda_j|a_{ji}| \\
- k_i e^{\varepsilon\tau} \sum_{j=1}^{n} \lambda_j|b_{ji}| > 0, \quad i = 1, 2, \ldots, n. \qquad (10.8.14)
\end{aligned}$$

Now, define a positive definite and radially unbounded Lyapunov function as follows:

$$V(t) = \sum_{i=1}^{n} \lambda_i \left(\left| x_i(t,\phi) - x_i(t,\psi) \right| e^{\varepsilon t} + \sum_{j=1}^{n} |b_{ji}| \int_{t-\tau_j}^{t} \left| g_j\bigl(x_j(s,\phi)\bigr) \right.\right.$$
$$\left.\left. - g_j\bigl(x_j(s,\psi)\bigr) \right| e^{\varepsilon(s+\tau_j)}\, ds \right). \tag{10.8.15}$$

Then computing the right-upper derivative of $V(t)$ along the solution of (10.8.12) for $t \geqslant 0$ yields

$$\begin{aligned}
\frac{dV}{dt}\bigg|_{(10.8.12)} &= \sum_{i=1}^{n} \lambda_i \operatorname{sign}\bigl(x_i(t,\phi) - x_i(t,\psi)\bigr) \Biggl[-\bigl(d_i x_i(t,\phi) - d_i x_i(t,\psi)\bigr) \\
&\quad + \sum_{i=1}^{n} a_{ij} \Bigl(g_j\bigl(x_j(t,\phi)\bigr) - g_j\bigl(x_j(t,\psi)\bigr) \Bigr) \\
&\quad + \sum_{j=1}^{n} b_{ij} \Bigl(g_j\bigl(x_j(t-\tau_j,\phi)\bigr) - g_j\bigl(x_j(t-\tau_j,\psi)\bigr) \Bigr) \Biggr] e^{\varepsilon t} \\
&\quad + \sum_{i=1}^{n} \lambda_i \left| x_i(t,\phi) - x_i(t,\psi) \right| \varepsilon e^{\varepsilon t} \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i |b_{ij}| \Bigl[\left| g_j\bigl(x_j(t,\phi)\bigr) - g_j\bigl(x_j(t,\psi)\bigr) \right| e^{\varepsilon(s+\tau_j)} \\
&\quad - \left| \Bigl(g_j\bigl(x_j(t-\tau_j,\phi)\bigr) - g_j\bigl(x_j(t-\tau_j,\psi)\bigr) \Bigr) \right| e^{\varepsilon t} \Bigr] \\
&= \sum_{i=1}^{n} \lambda_i \Biggl[-d_i \left| x_i(t,\phi) - x_i(t,\psi) \right| \\
&\quad + \sum_{i=1}^{n} |a_{ij}| \left| g_j\bigl(x_j(t,\phi)\bigr) - g_j\bigl(x_j(t,\psi)\bigr) \right| \\
&\quad + \sum_{j=1}^{n} |b_{ij}| \left| g_j\bigl(x_j(t,\phi)\bigr) - g_j\bigl(x_j(t,\psi)\bigr) \right| \Biggr] e^{\varepsilon t} \\
&\quad + \sum_{i=1}^{n} \lambda_i \left| x_i(t,\phi) - x_i(t,\psi) \right| \varepsilon e^{\varepsilon t} \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i |b_{ij}| \Bigl[\left| g_j\bigl(x_j(t,\phi)\bigr) - g_j\bigl(x_j(t,\psi)\bigr) \right| e^{\varepsilon(s+\tau_j)}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \left|\left(g_j\left(x_j(t-\tau_j,\phi)\right) - g_j\left(x_j(t-\tau_j,\psi)\right)\right)\right| e^{\varepsilon t}\Big] \\
&\leqslant \sum_{i=1}^{n} \lambda_i \Bigg[-d_i \left|x_i(t,\phi) - x_i(t,\psi)\right| \\
&\quad + \sum_{j=1}^{n} |a_{ij}| \left|g_j\left(x_j(t,\phi)\right) - g_j\left(x_j(t,\psi)\right)\right|\Big] e^{\varepsilon t} \\
&\quad + \sum_{i=1}^{n} \lambda_i \left|x_i(t,\phi) - x_i(t,\psi)\right| \varepsilon e^{\varepsilon t} \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i |b_{ij}| \Big[\left|g_j\left(x_j(t,\phi)\right) - g_j\left(x_j(t,\psi)\right)\right| e^{\varepsilon(s+\tau_j)}\Big] \\
&\leqslant -\sum_{i=1}^{n} \Bigg[\lambda_i (d_i - \varepsilon) - k_i \lambda_i |a_{ii}| - k_i \sum_{j=1, j\neq i}^{n} \lambda_j |a_{ji}| \\
&\quad - k_i e^{\varepsilon\tau} \sum_{j=1}^{n} \lambda_j |b_{ji}| \Bigg] \left|x_i(t,\phi) - x_i(t,\psi)\right| \leqslant 0.
\end{aligned}
$$

Therefore,

$$V(t) \leqslant V(0), \quad t \geqslant 0. \tag{10.8.16}$$

From (10.8.16), we obtain

$$e^{\varepsilon t}\Big(\min_{1\leqslant i\leqslant n} \lambda_i\Big) \sum_{i=1}^{n} \left|x_i(t,\phi) - x_i(t,\psi)\right| \leqslant V(t)$$

and

$$
\begin{aligned}
V(0) &= \sum_{i=1}^{n} \lambda_i \Bigg[\left\|x_i(0,\phi) - x_i(0,\psi)\right\| \\
&\quad + \sum_{j=1}^{n} |b_{ij}| \int_{-\tau_j}^{0} \left|g_j\left(x_j(s,\phi)\right) - g_j\left(x_j(s,\psi)\right)\right| e^{s+\varepsilon\tau_j}\, ds \Bigg] \\
&\leqslant \Bigg[\max_{1\leqslant i\leqslant n} \lambda_i + L_{\max} \tau e^{\varepsilon\tau} \sum_{i=1}^{n} \lambda_i \max_{1\leqslant i\leqslant n} |b_{ij}| \Bigg] \|\phi - \psi\|,
\end{aligned}
$$

where $L_{\max} = \max_{1\leqslant i\leqslant n} L_i$ is a constant. Therefore, from (10.8.16) we obtain

$$\sum_{i=1}^{n} \left|x_i(t,\phi) - x_i(t,\psi)\right| \leqslant \alpha \|\phi - \psi\|_1 e^{-\varepsilon t} \quad \forall\, t \geqslant 0, \tag{10.8.17}$$

in which

$$\alpha = \frac{\max_{1\leqslant i\leqslant n}\lambda_i + L_{\max}\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\lambda_i \max_{1\leqslant i\leqslant n}|b_{ij}|}{\min_{1\leqslant i\leqslant n}\lambda_i} \geqslant 1.$$

It then follows from (10.8.17) that

$$\alpha e^{-\varepsilon(m\omega-\tau)} \leqslant \frac{1}{4}. \tag{10.8.18}$$

Define a Poincaré mapping $H : C \to C$ by $H\phi = x_\omega(\phi)$. Then it follows from (10.8.10) that

$$\left\| H^m\phi - H^m\psi \right\| \leqslant \frac{1}{4}\|\phi - \psi\|. \tag{10.8.19}$$

This implies that H^m is a contraction mapping. Therefore, there exists a unique fixed point $\phi^* \in C$ such that $H^m\phi^* = \phi^*$. So, $H^m(H\phi^*) = H(H^m\phi^*) = H\phi^*$. This shows that $H\phi^* \in C$ is also a fixed point of H^m, hence, $H^m\phi^* = \phi^*$, i.e., $x_\omega(\phi^*) = \phi^*$. Let $x(t,\phi^*)$ be the solution of (10.8.12) through $(0,\phi^*)$. By using $I(t+\omega) = I(t)$ for $t \geqslant 0$, $x(t+\omega,\phi^*)$ is also a solution of (10.8.12). Note that $x_{t+\omega}(\phi^*) = x_t(x_\omega(\phi^*)) = x_t(\phi^*)$ for $t \geqslant 0$, then $x(t+\omega,\phi^*) = x(t,\phi^*)$ for $t \geqslant 0$. This shows that $x(t,\phi^*)$ is a periodic solution of (10.8.12) with period ω. From (10.8.18), it is easy to see that all other solutions of (10.8.12) converge to this periodic solution exponentially as $t \to +\infty$.

The proof is finished. □

10.9. Stability of general ecological systems and neural networks

Stability of general neural networks and ecological systems has been investigated by using the Lyapunov function method and the LaSalle invariant principle [87]. However, it has been noted that the LaSalle invariant principle can only be applied to orbits which tend to a maximal invariant set. Because the structure of a maximal invariant set is very complex in general, for any known equilibrium $x = x^*$, its stability cannot be determined by using these methods.

In this section, we present a number of methods and results concerning the Lyapunov asymptotic stability of a specific equilibrium position for a class of general neural networks and ecological systems.

Consider the following general dynamical system [87]:

$$\frac{dx}{dt} = A(x)\big[B(x) - CD(x)\big], \tag{10.9.1}$$

where

$$A(x) = \mathrm{diag}\big(a_1(x_1),\ldots,a_n(x_n)\big),$$

$$B(x) = \big(b_1(x_1), \ldots, b_n(x_n)\big)^T,$$
$$C = (c_{ij})_{n\times n},$$
$$D(x) = \big(d_1(x_1), \ldots, d_n(x_n)\big)^T,$$

and $a_i(x_i), b_i(x_i) \in [R_+, R]$, $a_i(x_i) > 0$ for $x_i > 0$, c_{ij} are constants.

In [87] it has been shown that system (10.9.1) includes the Volterra–Lotka systems, the Gilpin and Ayala models of competition, the Eigen and Schuster equation, and many other nonlinear neural networks as specific examples.

Let $x = x^* > 0$ (i.e., $x_i^* > 0$, $i = 1, 2, \ldots, n$) be an equilibrium position of (10.9.1). Thinking of real biological systems, we only study the stability of system (10.9.1) in $R_+^n = \{x\colon x_i > 0,\ i = 1, 2, \ldots, n\}$.

Rewrite system (10.9.1) as

$$\frac{dx(t)}{dt} = A(x)\big[\big(B(x) - B\big(x^*\big)\big) - \big(CD(x) - CD\big(x^*\big)\big)\big]. \tag{10.9.2}$$

Cohen and Grossberg [87] used the LaSalle invariant principle and the Lyapunov function of the form:

$$V(x) = -\sum_{i=1}^{n} \int_0^{x_i} b_i(\xi_i)\dot{d}_i(\xi_i)\, d\xi_i + \frac{1}{2}\sum_{j,k=1}^{n} c_{jk}d_j(x_j)d_k(x_k), \tag{10.9.3}$$

to analyze the stability of (10.9.1). However, for a specific equilibrium position $x = x^*$, this method cannot be used to determine its Lyapunov stability. A further study on this topic is given below [259].

THEOREM 10.9.1. *Assume that $c_{ij} = c_{ji}$, $i, j = 1, 2, \ldots, n$, $d_i(x_i) \in C^2$, $b_i(x_i) \in C^1$, $i = 1, 2, \ldots, n$, $\dot{d}_i(x_i) > 0$, $i = 1, 2, \ldots, n$. If the matrix $H(h_{ij})_{n\times n}$ is positive definite, then $x = x^*$ is asymptotically stable, where*

$$h_{ij} = \frac{\partial^2 V}{\partial x_i \partial x_j}\bigg|_{x=x^*} = \frac{\partial^2 V}{\partial x_j \partial x_i}\bigg|_{x=x^*} = h_{ji}.$$

PROOF. We construct a new Lyapunov function:

$$W(x) = V(x) - V\big(x^*\big),$$

where $V(x)$ is defined in (10.9.3) and $V(x^*)$ is the value of $V(x)$ at $x = x^*$. Obviously, $W(x^*) = 0$ and

$$\begin{aligned}\frac{\partial W}{\partial x_i}\bigg|_{x=x^*} &= -b_i\big(x_i^*\big)\dot{d}_i\big(x_i^*\big) + \sum_{j=1}^{n} c_{ij}\dot{d}_i\big(x_i^*\big)d_j\big(x_j^*\big) \\ &= -\dot{d}_i\big(x_i^*\big)\left[b_i\big(x_i^*\big) - \sum_{j=1}^{n} c_{ij}d_j\big(x_i^*\big)\right] = 0, \quad i = 1, 2, \ldots, n,\end{aligned}$$

$$\begin{aligned}
\frac{\partial W}{\partial x_j}\bigg|_{x=x^*} &= -\dot{d}_j\big(x_j^*\big)\left[b_j\big(x_j^*\big) - \sum_{i=1}^n c_{ij} d_i\big(x_i^*\big)\right] = 0, \quad j = 1, 2, \dots, n, \\
\frac{\partial^2 W}{\partial x_i^2}\bigg|_{x=x^*} &= -\dot{d}_i'(x_i)\left[b_i(x_i) - \sum_{j=1}^n c_{ij} d_j(x_j)\right]\bigg|_{x=x^*} \\
&\quad - \dot{d}_i(x_i)\big[b_i'(x_i) - c_{ii}\dot{d}_i(x_i)\big]\big|_{x=x^*} \\
&= -\dot{d}_i\big(x_i^*\big)\big[g_i'\big(x_i^*\big) - c_{ii}\dot{d}_i\big(x_i^*\big)\big] = h_{ii}, \quad i = 1, 2, \dots, n, \\
h_{ij} &= \frac{\partial^2 W}{\partial x_i \partial x_j}\bigg|_{x=x^*} = c_{ij}\dot{d}_j\big(x_j^*\big)\dot{d}_i\big(x_i^*\big) = c_{ji}\dot{d}_i\big(x_i^*\big)\dot{d}_j\big(x_j^*\big) \\
&= \frac{\partial^2 W}{\partial x_j \partial x_i}\bigg|_{x=x^*} = h_{ij}, \quad i \neq j,\ i, j = 1, 2, \dots, n.
\end{aligned}$$

According to the minimax principle of multivariate function we know that the positive (negative) definiteness of the matrix $H(h_{ij})_{n\times n}$ implies $x = x^*$ to be the minimal (maximal) value point of $W(x)$, i.e., $W(x)$ is positive (negative) definite in some neighborhood of x^*. In addition,

$$\frac{dW}{dt}\bigg|_{(10.9.1)} = -\sum_{i=1}^n a_i(x_i)\dot{d}_i(x_i)\left[b_i(x_i) - \sum_{j=1}^n c_{ij} d_j(x_j)\right]^2 < 0$$

for $x \neq x^*$.

By Lyapunov asymptotic stability, the conclusion is true. □

Now, we apply the Lyapunov first method to study the stability of $x = x^*$ of system (10.9.1).

Let

$$y_i = \int_{x_i^*}^{x_i} \frac{d\xi_i}{\alpha_i(\xi_i)}, \qquad y_i^* = 0.$$

Since $\dot{y}_i = \frac{1}{\alpha_i(x_i)} > 0$, the inverse function, $x_i = \varphi_i(y_i)$, exists. Note that $x_i^* = \varphi_i(0)$. Set

$$\begin{aligned}
\tilde{b}_i(y_i) &:= b_i\big(\varphi_i(y_i)\big) = b_i(x_i), \\
\tilde{d}_i(y_i) &:= d_i\big(\varphi_i(y_i)\big)) = d_i(x_i).
\end{aligned}$$

Then (10.9.2) can be transformed into

$$\frac{dy_i}{dt} = \tilde{b}_i(y_i) - \tilde{b}_i(0) - \sum_{j=1}^n c_{ij}\big[\tilde{d}_j(y_i) - \bar{d}_j(0)\big], \quad i = 1, \dots, n. \tag{10.9.4}$$

Applying the Lyapunov first method we study the stability of the zero solution, $y = 0$, of (10.9.4).

Assume that by the first method, system (10.9.4) can be written as

$$\frac{dy_i}{dt} = \dot{\tilde{b}}_i(0)y_i - \sum_{j=1}^{n} c_{ij}\dot{\tilde{d}}_j(0)y_j := \sum_{j=1}^{n} \sigma_{ij}y_j, \quad i, 2, \dots, n, \tag{10.9.5}$$

where

$$\sigma_{ij} = \begin{cases} -c_{ii}\dot{\tilde{d}}_i(0) + \dot{\tilde{b}}_i(0), & \text{for } i = j = 1, 2, \dots, n, \\ -c_{ij}\dot{\tilde{d}}_j(0), & \text{for } i \neq j,\ i, j = 1, 2, \dots, n. \end{cases}$$

THEOREM 10.9.2. *Suppose that* $((-1)^{\delta_{ij}}|\sigma_{ij}|)_{n\times n}$ *is a Hurwitz matrix. Then* $\sigma_{ii} < 0$ $(i = 1, 2, \dots, n)$ *imply the zero solution of* (10.9.5) *to be asymptotically stable. If there are at least some* $\sigma_{i_i i_0} > 0$ $(1 \leqslant i_0 \leqslant n)$*, then the zero solution of* (10.9.5) *is unstable.*

PROOF. Since $((-1)^{\delta_{ij}}|\sigma_{ij}|)_{n\times n}$ is a Hurwitz matrix, there exist constants $p_i > 0$ $(i = 1, 2, \dots, n)$ such that

$$p_j|\sigma_{jj}| - \sum_{\substack{i=1\\ i\neq j}}^{n} p_i|\sigma_{ij}| > 0, \quad j = 1, 2, \dots, n,$$

or

$$|s_{jj}| - \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{p_i}{p_j}|\sigma_{ij}| > 0, \quad j = 1, 2, \dots, n.$$

Let $z_i = p_i y_i$, $i = 1, 2, 3, \dots, n$. Then system (10.9.5) can be transformed into

$$\frac{dz_i}{dt} = \sum_{j=1}^{n} \frac{p_i}{p_j}\sigma_{ij}z_j, \quad j = 1, 2, \dots, n. \tag{10.9.6}$$

The eigenvalues of $(\sigma_{ij})_{n\times n}$ and $(\sigma_{ij}p_i/p_j)_{n\times n}$ are the same. From Gershgorin's theorem, $\sigma_{ij} < 0$ $(i = 1, 2, \dots, n)$ imply that all eigenvalues of $(\sigma_{ij})_{n\times n}$ are in the open left half of the complex plane. Moreover, if there exist some $\sigma_{i_0 i_0} > 0$, then at least there is an eigenvalue in the open right-half of the complex plane. Hence, according to the first Lyapunov method we know that the conclusion holds. □

In the following, we further study the asymptotic stability and globally asymptotic stability for a given equilibrium position of (10.9.1) [259].

DEFINITION 10.9.3. A real matrix $F(f_{ij})_{n\times n}$ is said to be Lyapunov–Volterra (L-V) stable (quasi Lyapunov–Volterra stable) if there exists a positive definite matrix $P = \operatorname{diag}(p_i, \ldots, p_n)$ such that $PF + F^T P$ is negative definite (negative semi-definite). Here, F^T is the transpose of F.

In the following, let $x = x^* > 0$ be an equilibrium position of (10.9.1).

THEOREM 10.9.4. *Suppose that*

(1) $(d_j(x_j) - d_j(x_j^*))(x_j - x_j^*) > 0$ *for* $x_j \neq x_j^*$ $(j = 1, 2, \ldots, n)$ *and* $(b_j(x_j) - b_j(x_j^*))(x_j - x_j^*) < 0$ *for* $x_j \neq x_j^*$ $(j = 1, 2, \ldots, n)$;
(2) *the matrix* $-C = (-c_{ij})_{n\times n}$ *is quasi Lyapunov–Volterra stable.*

Then $x = x^*$ *of* (10.9.2) *is asymptotically stable.*

PROOF. Since the matrix $-C$ is quasi L-V stable, there exists $P = \operatorname{diag}(p_1, \ldots, p_n) > 0$ such that

$$-\left[PC + C^T P\right] \leqslant 0.$$

Choose a Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i \int_{x_i^*}^{x_i} \left(d_i(x_i) - d_i\left(x_i^*\right)\right) \frac{dx_i}{a_i(x_i)}. \tag{10.9.7}$$

Obviously, $V(x^*) = 0$. Due to condition (10.9.1) we known that $V(x)$ is positive definite in some neighborhood of x^*.

Computing the derivative $D^+V(x)$ of $V(x)$ along the solution of (10.9.2) and simplifying the result yields

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(10.9.2)} &= -\frac{1}{2}\left(d(x) - d\left(x^*\right)\right)^T \left(PC + C^T P\right)\left(d(x) - d\left(x^*\right)\right) \\
&\quad + \sum_{i=1}^{n} p_i \left(d_i(x_i) - d_i\left(x_i^*\right)\right)\left(b_i(x_i) - b_i\left(x_i^*\right)\right) && (10.9.8) \\
&\leqslant \sum_{i=1}^{n} p_i \left(b_i(x_i) - b_i\left(x_i^*\right)\right) \\
&< 0 \quad \text{for } x,\ x \neq x^*, && (10.9.9)
\end{aligned}$$

i.e., $\frac{dV}{dt}|_{(10.9.2)}$ is negative definite in the same neighborhood of x^*. So the equilibrium point x^* of (10.9.1) is asymptotically stable. □

COROLLARY 10.9.5. *Suppose condition* (1) *of Theorem* 10.9.4 *holds, and moreover one of the following conditions is satisfied*:

(1) $-C$ *is Lyapunov–Volterra stable*;
(2) $C + C^T$ *is positive semi-definite*;
(3) $C = C^T$ *is positive semi-definite*;
(4) $C = -C^T$.

Then $x = x^*$ *of* (10.9.1) *is asymptotically stable.*

PROOF. Since any one of the conditions (1)–(4) implies $-C$ to be quasi Lyapunov–Volterra stable, condition (2) of Theorem 10.6.1 is satisfied. □

THEOREM 10.9.6. *If the conditions of Theorem* 10.9.4 *hold, and*

$$\int_{x_i^*}^{0} \left(d_i(x_i) - d_i\left(x_i^*\right)\right) \frac{dx_i}{a_i(x_i)} = \int_{x_i^*}^{+\infty} \left(d_i(x_i) - d_i\left(x_i^*\right)\right) \frac{dx_i}{a_i(x_i)} = +\infty$$

is true, then $x = x^*$ *of* (10.9.1) *is globally asymptotically stable in* R_+^n.

PROOF. Choose the positive definite Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i \int_{x_i^*}^{x_i} \left(d_i(x_i) - d_i\left(x_i^*\right)\right) \frac{dx_i}{a_i(x_i)}.$$

Obviously, $V(t) \to +\infty$ as $x_i \to +\infty$ or $x_i \to 0^+$. Thus, the proof is similar to that of Theorem 10.9.4. □

THEOREM 10.9.7. *Assume that*

(1) *condition* (1) *of Theorem* 10.9.4 *holds*;
(2) $c_{ii} > 0$ $(i = 1, 2, \ldots, n)$, $((-1)^{\delta_{ij}}|c_{ij}|)_{n\times n}$ *is a Hurwitz matrix, where*

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Then $x = x^*$ *is* (10.9.1) *is asymptotically stable.*

PROOF. According to the property of M matrix, condition (1) is equivalent to the statement that there exist constants $p_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$-p_j c_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} p_i |c_{ij}| < 0, \quad j = 1, 2, \ldots, n.$$

Choose the positive definite Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i \left| \int_{x_i^*}^{x_i} \frac{dx_i}{a_i(x_i)} \right|, \tag{10.9.10}$$

and then compute the Dini derivative, $D^+V(x)$, along the solution of (10.9.1) to obtain

$$\begin{aligned}
D^+V(x)|_{(10.9.2)} &= \overline{\lim_{h\to 0^+}} \frac{V(x(t+h)) - V(x(t))}{h} \\
&= \sum_{i=1}^{n} p_i \operatorname{sgn}(x_i - x_i^*) \frac{1}{a_i(x_i)} \frac{dx_i}{dt} \\
&= \sum_{i=1}^{n} p_i \operatorname{sgn}(x_i - x_i^*) \Bigg[b_i(x_i) - b_i(x_i^*) \\
&\qquad - \sum_{j=1}^{n} c_{ij} \big(d_j(x_j) - d_j(x_j^*)\big) \Bigg] \\
&\leqslant \sum_{j=1}^{n} \Bigg[-p_j c_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} p_i |x_{ij}| \Bigg] \big| d_j(x_j) - d_j(x_j^*) \big| \\
&\qquad + \sum_{i=1}^{n} p_i \big(b_i(x_i) - b_i(x_i^*)\big) \operatorname{sgn}(x_1 - x_i^*) \\
&< 0 \quad \text{for } x \neq x^*,
\end{aligned}$$

which clearly indicates that $x = x^*$ is asymptotically stable. □

THEOREM 10.9.8. *If the conditions in Theorem 10.9.7 are satisfied, and in addition,*

$$\left| \int_{x_i^*}^{0} \frac{dx_i}{a_i(x_i)} \right| = \int_{x_i^*}^{+\infty} \frac{dx_i}{a_i(x_i)} = +\infty,$$

then $x = x^*$ of (10.9.2) is globally asymptotically stable in R_+^n.

PROOF. Again using the Lyapunov function (10.9.6), by $V(x) \to +\infty$ as $x_i \to 0^+$ (or $x_i \to +\infty$), one can follow the proof of Theorem 10.9.7 to complete the proof. □

COROLLARY 10.9.9. *If condition* (1) *of Theorem* 10.9.7 *holds, and in addition one of the following conditions is satisfied*:

(1) $c_{jj} > \sum_{\substack{i=1\\ i\neq j}}^{n} |c_{ij}|,\ j = 1, 2, \ldots, n$;

(2) $c_{ii} > \sum_{\substack{j=1\\ j\neq i}}^{n} |c_{ij}|,\ i = 1, 2, \ldots, n$;

(3) $c_{ii} > \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{n} (|c_{ij}| + |c_{ji}|),\ j = 1, 2, \ldots, n$;

(4) $\sum_{i,j=1}^{n} (\frac{(1-\delta_{ij})c_{ij}}{c_{ii}})^2 < 1, c_{ii} > 0,\ i = 1, 2, \ldots, n$;

then $x = x^*$ of (10.9.2) is asymptotically stable. Furthermore, if

$$\left|\int_{x_i^*}^{0} \frac{dx_i}{a_i(x_i)}\right| = \int_{x_i^*}^{+\infty} \frac{dx_i}{a_i(x_i)} = +\infty, \tag{10.9.11}$$

then $x = x^*$ is globally asymptotically stable in R_+^n.

PROOF. By the property of M matrix, any of the conditions (1)–(4) implies condition (2) of Theorem 10.9.7. So the conclusion of Corollary 10.9.9 is true. □

THEOREM 10.9.10. *Suppose that*

(1) *condition* (1) *of Theorem* 10.9.4 *holds*;
(2) *there exist constants $p_i > 0$ $(i = 1, 2, \ldots, n)$ such that*

$$p_j c_{jj} - \sum_{\substack{i=1\\ i\neq j}}^{n} p_i |c_{ij}| \geqslant 0.$$

Then $x = x^*$ is asymptotically stable. Furthermore, if

$$\left|\int_{x_i^*}^{0} \frac{dx_i}{a_i(x_i)}\right| = \int_{x_i^*}^{+\infty} \frac{dx_i}{a_i(x_i)} = +\infty,$$

then $x = x^*$ is globally asymptotically stable in R_+^n.

PROOF. Construct the positive definite Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i \left|\int_{x_i^*}^{x_i} \frac{dx_i}{a_i(x_i)}\right|.$$

Then, we have

$$
\begin{aligned}
D^+V(x) &= \sum_{i=1}^{n} p_i \frac{dx_i}{dt} \operatorname{sgn}\big(x_i - x_i^*\big) \\
&\leqslant \sum_{j=1}^{N} \Bigg[-p_j c_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} p_i |c_{ij}| \Bigg] \big| d_j\big(x_j - d_j\big(x_j^*\big)\big) \big| \\
&\quad + \sum_{i=1}^{n} p_i \big(b_i(x_i) - b_i\big(x_i^*\big)\big) \operatorname{sgn}\big(x_i - x_i^*\big) \\
&\leqslant \sum_{i=1}^{n} p_i \big(b_i(x_i) - b_i\big(x_i^*\big)\big) \operatorname{sgn}\big(x_i - x_i^*\big) \\
&< 0 \quad \text{for } x \neq x^*.
\end{aligned} \tag{10.9.12}
$$

So the conclusion is true. □

THEOREM 10.9.11. *Suppose the following conditions are satisfied*:

(1) *condition* (1) *of Theorem* 10.9.4 *holds with* $c_{ii} > 0$, $i = 1, 2, \dots, n$;

(2)

$$
\int_{x_i^*}^{0} \big(b_i(x_i) - b_i\big(x_i^*\big)\big) \frac{dx_i}{a_i(x_i)} = \int_{x_i^*}^{+\infty} \big(b_i(x_i) - b_i\big(x_i^*\big)\big) \frac{dx_i}{a_i(x_i)} = -\infty;
$$

(3) *there exist constants* $p_i > 0$ $(i = 1, 2, \dots, n)$ *such that the matrix* $G = (g_{ij})_{n\times n}$ *is negative semi-definite, where*

$$
g_{ii} = \begin{cases} -p_i, \quad i = j = 1, 2, \dots, n, \\ \dfrac{1}{2}\left(\dfrac{p_j(c_{ji}(d_i(x_i) - d_i(x_i^*)))}{b_i(x_i) - b_i(x_i^*)} + \dfrac{p_i(c_{ij}(d_j(x_j) - d_j(x_j^*)))}{b_j(x_j) - b_j(x_j^*)} \right), \\ \quad i \neq j, i, \ j = 1, 2, \dots, n. \end{cases}
$$

Then $x = x^*$ *is globally asymptotically stable in* R^n_+.

PROOF. We take the positive definite and radially unbounded Lyapunov function:

$$
V(x) = -\sum_{i=1}^{n} p_i \int_{x_i^*}^{x_i} \big(b_i(x_i) - b_i\big(x_i^*\big)\big) \frac{dx_i}{a_i(x_i)}. \tag{10.9.13}
$$

Then differentiating V with respect to time t along the solution of (10.9.2) yields

$$\begin{aligned}
\frac{dV}{dt}\bigg|_{(10.9.2)} &= -\sum_{i=1}^{n} p_i\big[b_i(x_i) - b_i\big(x_i^*\big)\big]^2 \\
&\quad + \sum_{i=1}^{n} c_{ii} p_i\big[b_i(x_i) - b_i\big(x_i^*\big)\big]\big[d_i(x_i) - d_i\big(x_i^*\big)\big] \\
&\quad + \sum_{i,j=1}^{n} \frac{1}{2}\big\{p_i\big[b_i(x_i) - b_i\big(x_i^*\big)\big]c_{ij}\big[d_j(x_i) - d_j\big(x_j^*\big)\big] \\
&\quad + p_j\big[b_j(x_j) - b_j\big(x_j^*\big)\big]c_{ji}\big[d_i(x_i) - d_i\big(x_i^*\big)\big]\big\} \\
&\leqslant -\sum_{i=1}^{n} p_i\big[b_i(x_i) - b_i\big(x_i^*\big)\big]^2 \\
&\quad + \sum_{i=1}^{n} c_{ii} p_i\big[b_i(x_i) - b_i\big(x_i^*\big)\big]\big[d_i(x_i) - d_i\big(x_i^*\big)\big] \\
&\quad + \sum_{i,j=1}^{n} \frac{1}{2}\Bigg[\bigg|\frac{p_i c_{ij}(d_j(x_j) - d_j(x_j^*))}{b_j(x_j) - b_j(x_j^*)} \\
&\quad + \frac{p_j c_{ji}(d_i(x_i) - d_i(x_i^*))}{b_i(x_i) - b_i(x_i^*)}\bigg| \\
&\quad \times \big|b_j(x_j) - b_j\big(x_j^*\big)\big|\big|b_i(x_i) - b_i\big(x_i^*\big)\big|\Bigg] \\
&= \sum_{i=1}^{n} c_{ii} p_i\big[b_i(x_i) - b_i\big(x_i^*\big)\big]\big[d_i(x_i) - d_i\big(x_i^*\big)\big] \\
&\quad + \begin{pmatrix} b_1(x_1) - b_1(x_1^*) \\ \cdots \\ b_n(x_n) - b_n(x_n^*) \end{pmatrix}^T G \begin{pmatrix} b_1(x_1) - b_1(x_1^*) \\ \cdots \\ b_n(x_n) - b_n(x_n^*) \end{pmatrix} \\
&\leqslant \sum_{i=1}^{n} c_{ii} p_i\big[b_i(x_i) - b_i\big(x_i^*\big)\big]\big[d_i(x_i) - d_i\big(x_i^*\big)\big] \\
&< 0 \quad \text{for } x \neq 0.
\end{aligned} \tag{10.9.14}$$

Consequently, $x = x^*$ is globally asymptotically stable in R_+^n. □

THEOREM 10.9.12. *Suppose that*

(1) *condition* (1) *of Theorem* 10.9.4 *holds, and*

$$\left|\int_{x_i^*}^{0} \frac{dx_i}{a_i(x_i)}\right| = \int_{x_i^*}^{+\infty} \frac{dx_i}{a_i(x_i)} = +\infty;$$

(2) $\left|\dfrac{c_{ij}(d_j(x_j) - d_j(x_j^*))}{b_j(x_j) - b_j(x_j^*)}\right| \leqslant a_{ij} = const.,\ i \neq j,\ i, j = 1, 2, \ldots, n;$

(3) *there exist constants* $p_i > 0$ $(i = 1, 2, \ldots, n)$ *such that*

$$p_j - \sum_{i=1, i\neq j}^{n} p_i a_{ij} \geqslant 0.$$

Then the solution $x = x^*$ *is globally asymptotically stable in* R_+^n.

PROOF. Choosing the Lyapunov function as given in (10.9.9), we have

$$\begin{aligned}
&D^+V(x)|_{(10.9.1)} \\
&\quad = \sum_{i=1}^{n} p_i \frac{dx_i}{dt} \operatorname{sgn}\bigl(x_i - x_i^*\bigr) \\
&\quad \leqslant -\sum_{i=1}^{n} p_i c_{ii} \bigl|d_i(x_i) - d_i\bigl(x_i^*\bigr)\bigr| - \sum_{i=1}^{n} p_i \bigl|b_i(x_i) - b_i\bigl(x_i^*\bigr)\bigr| \\
&\qquad - \sum_{i=1}^{n} p_i \sum_{\substack{i=1\\ i\neq j}}^{n} c_{ij} \bigl|d_j(x_j) - d_j\bigl(x_j^*\bigr)\bigr| \\
&\quad \leqslant -\sum_{i=1}^{n} p_i c_{ii} \bigl|d_i(x_i) - d_i\bigl(x_i^*\bigr)\bigr| \\
&\qquad - \sum_{j=1}^{n} \left[p_j - \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{p_i |c_{ij}(d_j(x_j) - d_j(x_j^*))|}{|b_j(x_j - b_j(x_j^*))|}\right] \bigl[\bigl|b_j(x_j) - b_j\bigl(x_j^*\bigr)\bigr|\bigr] \\
&\quad \leqslant -\sum_{i=1}^{n} p_i c_{ii} \bigl|d_i(x_i) - d_i\bigl(x_i^*\bigr)\bigr| \\
&\qquad - \sum_{j=1}^{n} \left[p_j - \sum_{\substack{i=1\\ i\neq j}}^{n} p_i |a_{ij}|\right] \bigl|b_j(x_j) - b_j\bigl(x_j^*\bigr)\bigr| \\
&\quad \leqslant -\sum_{i=1}^{n} p_i c_{ii} \bigl|d_i(x_i) - d_i\bigl(x_i^*\bigr)\bigr| < 0 \quad \text{for } x = x^*.
\end{aligned} \tag{10.9.15}$$

Hence, the conclusion is true. □

THEOREM 10.9.13. *Assume that*

(1) *condition* (1) *of Theorem* 10.9.4 *holds, with* $c_{ii} \geqslant 0,\ i = 1, 2, \ldots, n$;
(2) *it holds that*

$$\left| \int_{x_i^*}^{0} \left[c_{ii}\left(d_i(x_i) - d_i\left(x_i^*\right)\right) + b_i(x_i) - b_i\left(x_i^*\right)\right] \frac{dx_i}{a_i(x_i)} \right| = \int_{x_i^*}^{+\infty} \left[c_{ii}\left(d_i(x_i) - d_i\left(x_i^*\right)\right) + b_i(x_i) - b_i\left(x_i^*\right)\right] \frac{dx_i}{a_i(x_i)} = +\infty;$$

(3) *there exist constants* $p_i > 0$ $(i = 1, 2, \ldots, n)$ *such that the matrix* $W(\omega_{ij})_{n\times n}$ *is negative definite, where*

$$\omega_{ij} = \begin{cases} -p_i, \quad i = j = 1, 2, \ldots, n, \\ \dfrac{1}{2}\left| \dfrac{p_i x_{ij}(d_j(x_j) - d_j(x_j^*))}{c_{jj}(d_j(x_j) - d_j(x_j^*)) + b_j(x_j) - b_j(x_j^*)} \right. \\ \quad \left. + \dfrac{p_j x_{ji}(d_i(x_i) - d_i(x_i^*))}{c_{ii}(d_i(x_i) - d_i(x_i^*)) + b_i(x_i) - b_i(x_i^*)} \right|, \\ \quad i \neq j,\ i, j = 1, 2, \ldots, n. \end{cases}$$

Then $x = x^*$ *is globally asymptotically stable.*

PROOF. Choosing the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i \int_{x_i^*}^{x_i} \left[c_{ii}\left(d_i(x_i) - d_i(x_i^*)\right) + b_i(x_i) - b_i\left(x_i^*\right)\right] dx_i,$$

we obtain

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(10.9.1)} = {} & \sum_{i=1}^{n} p_i \left[c_{ii}\left(d_i(x_i) - d_i\left(x_i^*\right)\right) + b_i(x_i) - b_i\left(x_i^*\right)\right]^2 \\ & + \sum_{i=1}^{n} p_i \left[c_{ii}\left(d_i(x_i) - d_i\left(x_i^*\right)\right) + b_i(x_i) - b_i\left(x_i^*\right)\right] \\ & \times \sum_{\substack{j=1 \\ j\neq i}}^{n} c_{ij}\left[d_j(x_j) - d_j\left(x_j^*\right)\right] \end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{i=1}^{n} p_i\big[c_{ii}\big(d_i(x_i) - d_i\big(x_i^*\big)\big) + b_i(x_i) - b_i\big(x_i^*\big)\big]^2 \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 2\omega_{ij}\big|c_{ii}\big(d_i(x_i) - d_i\big(x_i^*\big)\big) + b_i(x_i) - b_i\big(x_i^*\big)\big| \\
&\quad \times \big|c_{jj}\big(d_j(x_j) - d_j\big(x_j^*\big)\big) + b_j(x_j) - b_j\big(x_j^*\big)\big| \\
&< 0 \quad \text{for } x \neq x^*.
\end{aligned} \tag{10.9.16}$$

Therefore, the proof of Theorem 10.9.13 is complete. □

THEOREM 10.9.14. *If the following conditions are satisfied*:

(1) *condition* (1) *of Theorem* 10.8.7 *holds*;

(2)

$$\frac{c_{ij}(dj(x_j) - d_j(x_j^*))}{c_{jj}(dj(x_j) - d_j(x_j^*)) + b_j(x_j) - b_j(x_j^*)} \leqslant \eta_{ij} = const.,$$
$$i \neq j,\ i, j = 1, 2, \dots, n;$$

(3) *there exist constants* $p_i > 0$ $(i = 1, 2, \dots, n)$ *such that*

$$p_j - \sum_{\substack{i=1\\ i\neq j}}^{n} p_j \eta_{ij} > 0$$

and

$$\left|\int_{x_i^*}^{0} \frac{dx_i}{a_i(x_i)}\right| = \int_{x_i^*}^{+\infty} \frac{dx_i}{a_i(x_i)} = +\infty;$$

then $x = x^*$ *is globally asymptotically stable in* R_+^n.

PROOF. Taking the positive definite Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i\left|\int_{x_i^*}^{x_i} \frac{dx_i}{a_i(x_i)}\right|,$$

we have

$$D^+V(x)|_{(10.9.1)} = \sum_{i=1}^{n} p_i \frac{dx_i}{dt}\operatorname{sign}\big(x_i - x_i^*\big)$$

$$
\begin{aligned}
&\leqslant -\sum_{i=1}^{n} p_i \left| c_{ii}\left(d_i(x_i) - d_i\left(x_i^*\right)\right) + b_i(x_i) - b_i\left(x_i^*\right)\right| \\
&\quad + \sum_{j=1}^{n} p_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \left| c_{ij}\left(d_j(x_j) - d_j\left(x_j^*\right)\right)\right| \\
&\leqslant -\sum_{j=1}^{n} p_j \left| c_{jj}\left(d_j(x_j) - d_j\left(x_j^*\right)\right) + b_j(x_j) - b_j\left(x_j^*\right)\right| \\
&\quad + \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i\neq j}}^{n} p_i \left| c_{ij}\left(d_j(x_j) - d_j\left(x_j^*\right)\right)\right| \\
&= -\sum_{j=1}^{n} \left| p_j - \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{p_i |c_{ij}| (d_j(x_j) - d_j(x_j^*))}{|c_{jj}(d_j(x_j) - d_j(x_j^*)) + b_j(x_j) - b_j(x_j^*)|} \right| \\
&\quad \times \left| c_{jj}\left(d_j(x_j) - d_j\left(x_j^*\right)\right) + b_j(x_j) - b_j\left(x_j^*\right)\right| \\
&\leqslant -\sum_{j=1}^{n} \left(p_j - \sum_{\substack{i=1 \\ i\neq j}}^{n} p_i \eta_{ij} \right) \left| c_{ij}\left(d_j(x_j) - d_j\left(x_j^*\right)\right) + b_j(x_j) - b_j\left(x_j^*\right)\right| \\
&< 0 \quad \text{for } x = x^*.
\end{aligned} \tag{10.9.17}
$$

Thus, $x = x^*$ is globally asymptotically stable in R_+^n. □

10.10. Cellular neural network

The cellular neural network (CNN) is a new type of analog circuits proposed by Chua and Yang [82,83], which has many desirable properties. Cellular neural network possesses some of key features of neural networks and has important potential applications in image processing, pattern recognition, optimal computation, etc.

A cellular neural network may be described by the following set of differential equations:

$$
\begin{aligned}
C\frac{dV_{x_{ij}}}{dt} &= -\frac{1}{R_x} V_{x_{ij}}(t) + \sum_{(k,l)\in N_r(i,j)} A(i,j,k,l) V_{y_{kl}}(t) \\
&\quad + \sum_{(k,l)\in N_r(i,j)} B(i,j,k,l) V_{u_{kl}} + I, \quad 1 \leqslant i \leqslant M,\ 1 \leqslant j \leqslant N,
\end{aligned} \tag{10.10.1}
$$

where the physical meanings of C, $V_{x_{ij}}$, $V_{y_{kl}}$ and $V_{u_{kl}}$ and I can be found in [82, 83]. The output equations of the CNN is given by

$$V_{y_{ij}} := \frac{1}{2}\big[|V_{x_{ij}} + 1| - |V_{x_{ij}} - 1|\big], \quad 1 \leqslant i \leqslant M,\ 1 \leqslant j \leqslant N,$$

while the input equations are

$$V_{u_{ij}} = E_{ij}, \quad |E_{ij}| \leqslant 1,\ 1 \leqslant i \leqslant M,\ 1 \leqslant j \leqslant N.$$

Since the CNN is easy to be realized using electronic circuits, it has wide applications in computational optimization, signal processing, pattern recognition, etc. Up to now, CNN theory, methodology and applications have been the main topics in many international conferences and symposiums. Many scientific articles have been published in the literature so far. In this section, we briefly introduce the basic stability theory of CNN. Because CNN is a special type of Hopfield neural network, the Lagrange stability theory discussed in Section 10.2, related to general neural network (with time delay), is certainly applicable to CNN, and thus not repeated here.

By using appropriate transformation and simplified notations, we can transform system (10.10.1) to the following system:

$$\frac{dx_i}{dt} = -d_i x_i \sum_{j=1}^{n} a_{ij} f_j(x_j) + I_i, \quad i = 1, 2, \ldots, n, \tag{10.10.2}$$

where

$$f_j(x_j) = \frac{1}{2}\big[|x_i + 1| - |x_1 - 1|\big].$$

Let

$$\sigma_j(x_j) = \begin{cases} 0 & \text{for } |x_j| \geqslant 1, \\ 1 & \text{for } |x_j| < 1; \end{cases}$$

$$w_j(x_j) = \begin{cases} -1 & \text{for } x_j \leqslant -1, \\ 0 & \text{for } |x_j| \leqslant 1, \\ 1 & \text{for } x_j \geqslant 1, \end{cases} \qquad j = 1, \ldots, n.$$

Then, we can further rewrite (10.10.2) as

$$\frac{dx_i}{dt} = -d_i x_i + \sum_{j=1}^{n} a_{ij}\sigma_j(x_j) f_j(x_j) + \sum_{j=1}^{n} a_{ij} w_j(x_j) + I_i. \tag{10.10.3}$$

Let

$$\overline{A} := \operatorname{diag}(-d_1, \ldots, -d_n) + \big(a_{ij}\sigma_j(x_j)\big)_{n\times n}, \qquad C = (a_{ij})_{n\times n} w(x),$$

$$w(x) := \big(w_1(x), \ldots, w_n(x)\big)^T, \qquad I = (I_1, \ldots, I_n)^T.$$

Now, we consider the existence and number of equilibrium points of the CNN model (10.10.3).

THEOREM 10.10.1.

(1) *The number of the equilibrium points of the CNN* (10.10.3) *is not less than one.*
(2) *If every matrix* $\hat{A}$ *is nonsingular, then the number of the equilibrium points of the CNN* (10.10.3) *is not more than* 3^n.
(3) *All equilibrium points of* (10.10.3) *can be expressed as*

$$\begin{aligned} x &= -\hat{A}^{-1}(I+C) := x^*, \\ \sigma_j(x_j) &\equiv \sigma_j\left(x_j^*\right), \end{aligned} \tag{10.10.4}$$

where $\sigma_j(x_j) \equiv \sigma_j(x_j^*)$ *implies that* x_j *and* x_j^* *belong to the same interval* $(-\infty, -1]$, $(-1, 1)$, *or* $[1, +\infty)$.

PROOF. (1) Let the right-hand side of (10.10.3) be equal to zero and rearrange the resulting equation as

$$x_i = d_i^{-1}\left[\sum_{j=1}^n a_{ij}\sigma_j(x_j)f_j(x_j) + \sum_{j=1}^n a_{ij}w_j(x_j) + I_i\right]. \tag{10.10.5}$$

Consider an operator $\phi(x)$, defined by

$$\begin{aligned} &\phi_i(x_i) = d_i^{-1}\left[\sum_{j=1}^n a_{ij}\sigma_j(x_j)f_j(x_j) + \sum_{j=1}^n a_{ij}w_j(x_j) + I_i\right], \\ &\quad 1 \leqslant i, j \leqslant n. \end{aligned} \tag{10.10.6}$$

Further, let

$$M = \max_{1\leqslant i,j\leqslant n}\left[d_i\sum_{j=1}^n |a_{ij}|r + |I_i|\right].$$

Obviously, the operator ϕ maps the set

$$Q = \left\{x_i \mid |x| \leqslant M,\ 1 \leqslant i, j \leqslant n\right\}$$

into Q, which is a compact convex set. According to Brown fixed point theorem, we know that for the map $\phi : Q \to Q$, there exists at least one fixed point $x = x^*$, i.e., $x = x^*$ is an equilibrium point of (10.10.3). Therefore, the number of equilibrium points of the CNN (10.10.3) is not less than one.

(2) We divide the interval $(-\infty, +\infty)$ into three intervals: $(-\infty, -1]$, $(-1, 1)$ and $[1, +\infty)$. Thus, the whole space R^n is divided into 3^n independent regions, and so $\sigma_j(x_j)$ and $w_j(x_j)$ have and only have 3^n different regions to take values.

(3) For every $\hat{A}$, $\det \hat{A} \neq 0$. Let the right-hand side of (10.10.3) be equal to zero, and rewrite the resulting equation in a matrix form:

$$\hat{A}x + I + C = 0.$$

Then solving the above equation in an arbitrary region D_{i_0}, we have

$$x = -\hat{A}^{-1}(I + C) := x^*. \tag{10.10.7}$$

It may have $x^* \notin D_{i_0}$ if x^* is not a solution of (10.10.4), and $x^* \in D_{i_0}$ only if x^* is a real solution of (10.10.4). Therefore, the number of equilibrium points of (10.10.3) are at most equal to 3^n, and every equilibrium point can be expressed by (10.10.4).

The proof is complete. □

For linear systems with constant coefficients, asymptotic stability is equivalent to globally exponential stability, thus one may apply linear system stability to CNN systems, since though CNN systems are nonlinear in general, they are usually linear if considered in separate regions. So far, since almost all results related to asymptotic stability in the literature are exponential stability, the results presented in this section are only concerned with exponential stability and globally exponential stability [257,258].

THEOREM 10.10.2. *Let* $D^{-1} = \operatorname{diag}(d_1^{-1}, d_2^{-1}, \dots, d_n^{-1})$. *If* $\rho(D^{-1}(A)) < 1$, *where* $\rho(D^{-1}(A))$ *is the spectral radius of the matrix* $(D^{-1}|A|)$, *then the equilibrium point of the CNN* (10.10.3) *is globally exponentially stable.*

PROOF. Theorem 10.10.1 implies the existence of equilibrium point of the CNN (10.10.3). Let $x = x^*$ be an equilibrium point of (10.10.3), then rewrite (10.10.3) as

$$\frac{d(x_i - x_i^*)}{dt} = -d_i\big(x_i - x_i^*\big) + \sum_{j=1}^{n} a_{ij}\big[f_j(x_j) - f_j\big(x_j^*\big)\big]. \tag{10.10.8}$$

Since $\rho(D^{-1}(A)) < 1 \Leftrightarrow (I_n - D^{-1}|A|)$ is an M matrix, where I_n is a $n \times n$ unit matrix, there exist constants $p_i > 0,\ i = 1, 2, \dots, n$, such that

$$-p_j d_j + \sum_{i=1}^{n} p_i |a_{ij}| = -\delta_j < 0.$$

Now, we employ the positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i \big|x_i - x_i^*\big|, \tag{10.10.9}$$

and compute the right-upper Dini derivative D^+V of V along the solution of (10.10.8) and simplifying the result yields

$$\begin{aligned}
D^+V(x)|_{(10.10.9)} &\leqslant \sum_{j=1}^{n}\left[-p_j d_j + \sum_{i=1}^{n} p_i|a_{ij}|\right]\left|x_i - x_i^*\right| \quad (< 0 \text{ for } x \neq x^*)\\
&\leqslant -\sum_{j=1}^{n}\delta_j\left|x_j - x_j^*\right|\\
&\leqslant -\min_{1\leqslant j\leqslant n}\left(\frac{\delta_j}{p_j}\right)\sum_{i=1}^{n} p_i\left|x_i - x_i^*\right|,
\end{aligned}$$

and hence,

$$V\big(x(t, t_0, x_0)\big) \leqslant V\big(x(t_0)\big)e^{-\min_{1\leqslant j\leqslant n}(\frac{\delta_j}{p_j})(t-t_0)}$$

which shows that $x = x^*$ of (10.10.3) is globally exponentially stable. □

THEOREM 10.10.3. *If $a_{ii} < 0$, $i = 1, 2, \ldots, n$, and the matrix*

$$H = \begin{bmatrix} -\operatorname{diag}(d_1, d_2, \ldots, d_n) & \frac{1}{2}(A - \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \\ \frac{1}{2}(A^T - \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn}) & \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \end{bmatrix}$$

is negative definite, then the equilibrium point of (10.10.3) *is globally exponentially stable.*

PROOF. We choose the positive definite and radially unbounded Lyapunov function:

$$V(x) = \frac{1}{2}\sum_{i=1}^{n}\big(x_i - x_i^*\big)^2. \tag{10.10.10}$$

Then using the fact

$$-\big(x_i - x_i^*\big)\big(f_i(x_i) - f_i\big(x_i^*\big)\big) \leqslant -\big[f_i(x_i) - f_i\big(x_i^*\big)\big]^2$$

and computing the derivative of $V(t)$ along the solution of (10.10.3) yields

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(10.10.3)} &= -\sum_{i=1}^{n} d_i\big(x_i - x_i^*\big)^2 + \sum_{i=1}^{n} a_{ii}\big(x_i - x_i^*\big)\big(f_i(x_i) - f_i\big(x_i^*\big)\big)\\
&\quad + \sum_{j=1, j\neq i}^{n} a_{ij}\big(x_i - x_i^*\big)\big(f_i(x_i) - f_i\big(x_i^*\big)\big)\\
&\leqslant \sum_{i=1}^{n} -d_i\big(x_i - x_i^*\big)^2 + \sum_{i=1}^{n} a_{ii}\big(f_j(x_i) - f_i\big(x_i^*\big)\big)^2
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1, j\neq i}^{n} a_{ij}\big(x_i - x_i^*\big)\big(f_j(x_i) - f_j\big(x_i^*\big)\big) \\
&\leqslant \begin{pmatrix} x - x^* \\ f(x) - f(x^*) \end{pmatrix}^T H \begin{pmatrix} x - x^* \\ f(x) - f(x^*) \end{pmatrix} \\
&\leqslant \lambda_{\max}(H)\left[\sum_{i=1}^{n}\big(x_i - x_i^*\big)^2 + \sum_{i=1}^{n}\big(f_i(x_i) - f_i\big(x_i^*\big)\big)^2\right] \\
&\leqslant \lambda_{\max}(H)\left[\sum_{i=1}^{n}\big(x_i - x_i^*\big)^2\right].
\end{aligned}
$$

So

$$
V\big(x(t, t_0, x_0)\big) \leqslant V\big(x(t_0)\big)e^{2\lambda_{\max}(H)(t-t_0)},
$$

where $\lambda_{\max}(H)$ is the largest eigenvalue of matrix H. This shows that the equilibrium point $x = x^*$ of CNN (10.10.3) is globally exponentially stable. □

THEOREM 10.10.4. *If* $a_{ii} < 0,\ i = 1, 2, \ldots, n$, *and there exist* $\varepsilon > 0$ *such that the matrix*

$$
H + \begin{bmatrix} I_n\varepsilon & 0 \\ 0 & 0 \end{bmatrix}
$$

is negative semi-definite, then the equilibrium point of (10.10.3) *is globally exponentially stable.*

PROOF. By employing the same Lyapunov function (10.10.10), we have

$$
\begin{aligned}
\frac{dV}{dt}\bigg|_{(10.10.3)} &= \sum_{j=1}^{n}(-d_j + \varepsilon)\big(x_j - x_j^*\big)^2 \\
&\quad + \sum_{j=1}^{n} a_{jj}\big(x_j - x_j^*\big)\big(f_j(x_j) - f_j\big(x_j^*\big)\big) \\
&\quad + \sum_{j=1, j\neq i}^{n} a_{ij}\big(x_i - x_i^*\big)\big(f_j(x_i) - f_j\big(x_i^*\big)\big) - \varepsilon\sum_{j=1}^{n}\big(x_j - x_j^*\big)^2 \\
&\leqslant -\varepsilon\sum_{j=1}^{n}\big(x_j - x_j^*\big)^2,
\end{aligned}
$$

and thus

$$
V\big(x(t, t_0, x_0)\big) \leqslant V\big(x(t_0)\big)e^{-2\varepsilon(t-t_0)}. \tag{10.10.11}
$$

(10.10.11) clearly indicates that the equilibrium point $x = x^*$ of CNN (10.10.3) is globally exponentially stable. □

COROLLARY 10.10.5. *If*

$$d_i > \sum_{j=1,\, j\neq i}^{n} \frac{1}{2}|a_{ij}|, \quad i = 1, 2, \ldots, n,$$

and

$$-a_{jj} \geqslant \sum_{i=1,\, i\neq j}^{n} \frac{1}{2}|a_{ij}|, \quad j = 1, 2, \ldots, n,$$

then the equilibrium point $x = x^*$ of (10.10.3) is globally exponentially stable.

PROOF. The conditions of this corollary imply that there exists ε such that

$$d_i - \varepsilon \geqslant \sum_{j=1,\, j\neq i}^{n} \frac{1}{2}|a_{ij}|, \quad i = 1, 2, \ldots, n,$$

$$-a_{jj} \geqslant \sum_{i=1,\, i\neq j}^{n} \frac{1}{2}|a_{ij}|, \quad j = 1, 2, \ldots, n.$$

Hence, the matrix

$$H + \begin{bmatrix} I_n\varepsilon & 0 \\ 0 & 0 \end{bmatrix}$$

is negative semi-definite, and so the conditions of Theorem 10.10.4 are satisfied. □

THEOREM 10.10.6. *If $-d_i a_{ii} < 0$, $i = 1, 2, \ldots, n$, and the matrix*

$$\left[-d_i + a_{ii} + \left(1 - \delta_{ij}|a_{ij}|\right)\right]_{n\times n}$$

is a Hurwitz matrix, then the equilibrium point $x = x^*$ of (10.10.3) is globally exponentially stable. Here, δ_{ij} is a Kronecker sign function.

PROOF. The conditions of this theorem imply that $-[-d_i + a_{ii} + (1 - \delta_{ij}|a_{ij}|]_{n\times n}$ is an M matrix. Thus, there exist constants $p_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$(-d_j + a_{jj})p_j + \sum_{i=1,\, i\neq j}^{n} p_i|a_{ij}| < 0, \quad j = 1, 2, \ldots, n,$$

which implies that there exists $\varepsilon > 0$ such that

$$(-d_j + a_{jj} + \varepsilon)p_j + \sum_{i=1,\, i\neq j}^{n} p_i|a_{ij}| \leqslant 0, \quad j = 1, 2, \ldots, n.$$

Next, we take the Lyapunov function:

$$V(x) = \sum_{i=1}^{n} p_i \left|x_i - x_i^*\right|. \tag{10.10.12}$$

Due to that $-|x_i - x_i^*| \leqslant -|f_i(x_i) - f_i(x_i^*)|$, we have

$$\begin{aligned}
&D^+V|_{(10.10.3)} \\
&\quad = \sum_{i=1}^{n} p_i\left[-d_i\left(x_i - x_i^*\right) + \sum_{j=1}^{n} a_{ij}\left(f_j(x_j) - f_j\left(x_j^*\right)\right)\right]\operatorname{sign}\left(x_i - x_i^*\right) \\
&\quad \leqslant \sum_{j=1}^{n}\left\{\left[-p_j d_j\left(x_j - x_j^*\right) + a_{jj}\left|f_j(x_j) - f_j\left(x_j^*\right)\right|\right]\right. \\
&\qquad \left. + \sum_{i=1,i\neq j}^{n} p_i|a_{ij}|\left|f_j(x_j) - f_j\left(x_j^*\right)\right|\right\} \\
&\quad \leqslant -\sum_{j=1}^{n} p_j\varepsilon\left|x_j - x_j^*\right| + \sum_{j=1}^{n}\left\{-p_j d_j + p_j\varepsilon + a_{jj}\right. \\
&\qquad \left. + \sum_{i=1,\, i\neq j}^{n} p_i|a_{ij}|\left|f_j(x_j) - f_j\left(x_j^*\right)\right|\right\} \\
&\quad \leqslant -\sum_{j=1}^{n} p_j\varepsilon\left|x_j - x_j^*\right|
\end{aligned}$$

which, in turn, yields

$$V\left(X(t)\right) \leqslant V\left(X(t_0)\right)e^{-\varepsilon(t-t_0)}. \tag{10.10.13}$$

The inequality (10.10.13) shows that the equilibrium point $x = x^*$ of CNN (10.10.3) is globally exponentially stable. $\square$

Finally, we consider the globally exponential stability of CNNs with time delays [277]. Consider a CNN with multiple varying time delays

$$\frac{dx_i}{dt} = -x_i + \sum_{j=1}^{n} a_{ij} f\left(x_j(t)\right)$$

$$+\sum_{j=1}^{n} b_{ij} f\big(x_j\big(t-\tau_{ij}(t)\big)\big), \quad i=1,2,\dots,n, \tag{10.10.14}$$

where $x=(x_1,\dots,x_n)^T \in R^n$ is the state vector, $A \in R^{n\times n}$ and $B \in R^{n\times n}$ are respectively the connection weight matrices associated with state vectors without and with time delays, $f(x_j)=\frac{1}{2}(|x_j+1|-|x_j-1|)$ $(j=1,\dots,n)$ is the piecewise-linear activation function, $u=(u_1,\dots,u_n)^T$ is the external input vector, and $0 \leqslant \tau_{ij}(t) \leqslant \tau_{ij} = \text{constant}$, $\dot{\tau}_{ij}(t) \leqslant 0,\ i,j=1,2,\dots,n$.

Let x^* be an equilibrium state of (10.10.14). Without loss of generality, let $|x_i^*|<1,\ i=1,2,\dots,m_1,\ x_i^* \geqslant 1,\ i=m_1+1,\dots,m_2,\ x_i^* \leqslant -1,\ i=m_2+1,\dots,n$. Rewrite (10.10.14) as

$$\frac{dx_i}{dt} = -\big(x_i-x_i^*\big)+\sum_{j=1}^{n} a_{ij}\big(f\big(x_j(t)\big)-f\big(x_j^*\big)\big)$$
$$+\sum_{j=1}^{n} b_{ij}\big(f\big(x_j\big(t-\tau_{ij}(t)\big)\big)-f\big(x_j^*\big)\big), \quad i=1,2,\dots,n. \tag{10.10.15}$$

Then the stability of x^* of system (10.10.14) is equivalent to the stability of zero solution of system (10.10.15).

LEMMA 10.10.7. *Define*

$$h\big(x_i-x_i^*\big) := \big(x_i-x_i^*\big)-\big(f(x_i)-f\big(x_i^*\big)\big),$$

where $h(\cdot)$ is a function. Then we have

$$\big|x_i-x_i^*\big| = \big|f(x_i)-f\big(x_i^*\big)\big|+\big|h\big(x_i-x_i^*\big)\big|.$$

PROOF. Because $(x_i-x_i^*)(f(x_i)-f(x_i^*)) \geqslant 0$, so $|h(x_i-x_i^*)| \leqslant x_i-x_i^*|$. □

THEOREM 10.10.8. *The equilibrium state x^* of (10.10.14) is globally exponentially stable if*

$$1-a_{jj} > \sum_{i=1,\, i\neq j}^{n} |a_{ij}|+\sum_{i=1}^{n} |b_{ij}|, \quad j=1,2,\dots,n, \tag{10.10.16}$$

or

$$1-a_{ii} > \sum_{j=1,\, j\neq i}^{n} |a_{ij}|+\sum_{j=1}^{n} |b_{ij}|, \quad i=1,2,\dots,n. \tag{10.10.17}$$

PROOF. First, we consider the conditions in (10.10.16). Let

$$\delta = \min_{1\leqslant j\leqslant n}\left\{1 - a_{jj} > \sum_{i=1,\, i\neq j}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_{ij}|,\ \ j = 1, 2, \ldots, n\right\}.$$

We construct a positive definite and radially unbounded Lyapunov functional:

$$\begin{aligned} &V\big(x(t), t\big) \\ &\quad = \sum_{i=1}^{n} \big|x_i - x_i^*\big| + \sum_{i,j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} |b_{ij}| \big|f(s) - f\big(x_j^*\big)\big|\, ds. \end{aligned} \tag{10.10.18}$$

By $\dot\tau_{ij}(t) \leqslant 0$, we have

$$\begin{aligned} &D^+ V\big(x(t), t\big)\big|_{(10.10.15)} \\ &\quad = \sum_{i=1}^{n} D^+\big|x_i - x_i^*\big| + \sum_{i,j=1}^{n} |b_{ij}| \big|f\big(x_j(t)\big) - f\big(x_j^*\big)\big| \\ &\qquad - \sum_{i,j=1}^{n} |b_{ij}| \big|f\big(x_j\big(t - \tau_{ij}(t)\big)\big) - f\big(x_j^*\big)\big|\big(1 - \dot\tau_{ij}(t)\big) \\ &\quad \leqslant \sum_{i,j=1}^{n} \Bigg[-\big|x_i - x_i^*\big| + a_{ii}\big|f\big(x_i(t)\big) - f\big(x_i^*\big)\big| \\ &\qquad + \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \big|f\big(x_j(t)\big) - f\big(x_j^*\big)\big| \\ &\qquad + \sum_{j=1}^{n} |b_{ij}| \big|f\big(x_j\big(t - \tau_{ij}(t)\big)\big) - f\big(x_j^*\big)\big|\Bigg] \\ &\qquad + \sum_{i,j=1}^{n} |b_{ij}| \big|f\big(x_j(t)\big) - f\big(x_j^*\big)\big| \\ &\qquad - \sum_{i,j=1}^{n} |b_{ij}| \big|f\big(x_j\big(t - \tau_{ij}(t)\big)\big) - f\big(x_j^*\big)\big| \\ &\quad = \sum_{i=1}^{n} (-1 + a_{ii})\big|f\big(x_i(t)\big) - f\big(x_i^*\big)\big| + \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \big|f\big(x_j(t)\big) - f\big(x_j^*\big)\big| \\ &\qquad + \sum_{j=1}^{n} |b_{ij}| \big|f\big(x_j\big(t - \tau_{ij}(t)\big)\big) - f\big(x_j^*\big)\big| - h\big(x_i - x_i^*\big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\Bigg[(-1+a_{ii}) + \sum_{i=1, i\neq j}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_{ij}|\Bigg]\big|f\big(x_j(t)\big) - f\big(x_j^*\big)\big| \\
&\quad - \sum_{j=1}^{n}\big|h\big(x_i - x_i^*\big)\big| \\
&\leqslant -\delta\sum_{j=1}^{n}\big|f\big(x_j(t)\big) - f\big(x_j^*\big)\big| - \sum_{j=1}^{n}\big|h\big(x_i - x_i^*\big)\big| \\
&\leqslant -\min\{\delta, 1\}\Bigg[\sum_{j=1}^{n}\big|f\big(x_j(t)\big) - f\big(x_j^*\big)\big| + \sum_{j=1}^{n}\big|h\big(x_j - x_j^*\big)\big|\Bigg] \\
&= -\min\{\delta, 1\}\sum_{j=1}^{n}\big|x_j - x_j^*\big|.
\end{aligned}
\tag{10.10.19}
$$

Choose $\varepsilon > 0$ to satisfy the following condition:

$$
\min\{\delta, 1\} - \varepsilon - \varepsilon\sum_{i=1}^{n}\tau_{ij}|b_{ij}|e^{\varepsilon\tau_{ij}} \geqslant 0.
\tag{10.10.20}
$$

Deriving the Dini derivative of $e^{\varepsilon t}V(x(t), t)$ and using

$$
\int_{t-\tau_{ij}(t)}^{t} |b_{ij}|\big|f\big(x_j(s)\big) - f\big(x_j^*\big)\big|\,ds \leqslant \int_{t-\tau_{ij}(t)}^{t} |b_{ij}|\big|f\big(x_j(s)\big) - f\big(x_j^*\big)\big|\,ds,
$$

we have

$$
\begin{aligned}
&D^+\big(e^{\varepsilon t}V\big(x(t), t\big)\big) = e^{\varepsilon t}D^+V\big(x(t), t\big) \\
&\quad \leqslant e^{\varepsilon t}\Bigg[\varepsilon\Bigg(\sum_{j=1}^{n}\big|x_j(t) - x_j^*\big| + \sum_{i,j=1}^{n}\int_{t-\tau_{ij}(t)}^{t} |b_{ij}|\big|f\big(x_j(s)\big) - f\big(x_j^*\big)\big|\,ds\Bigg) \\
&\qquad - \min\{\delta, 1\}\sum_{j=1}^{n}\big|x_j - x_j^*\big|\Bigg].
\end{aligned}
\tag{10.10.21}
$$

Integrating both sides of (10.10.21) from t_0 to an arbitrary $t^* > t_0$, we can obtain

$$
\begin{aligned}
&e^{\varepsilon t^*}V\big(x\big(t^*\big), t^*\big) - V\big(x(t_0), t_0\big) \\
&\quad \leqslant \int_{t_0}^{t^*} \varepsilon e^{\varepsilon t}\int_{t-\tau_{ij}}^{t}\sum_{i,j=1}^{n}|b_{ij}|\big|f\big(x_j(s)\big) - f\big(x_j^*\big)\big|\,ds\,dt
\end{aligned}
$$

$$-\sum_{j=1}^{n}\big(\min\{\delta,1\}-\varepsilon\big)\int_{t_0}^{t^*} e^{\varepsilon t}\big|x_j(t)-x_j^*\big|\,dt. \tag{10.10.22}$$

Estimating the first term on the right-hand side of (10.10.22) by exchanging the integrals, we have

$$\begin{aligned}
&\sum_{i,j=1}^{n}\varepsilon|b_{ij}|\int_{t_0}^{t^*} e^{\varepsilon t}\int_{t-\tau_{ij}}^{t}\big|f\big(x_j(s)\big)-f\big(x_j^*\big)\big|\,ds\,dt\\
&\quad\leqslant \sum_{i,j=1}^{n}\varepsilon|b_{ij}|\int_{-\tau_{ij}}^{t^*}\int_{\max\{s,0\}}^{\min\{s+\tau_{ij},t\}} e^{\varepsilon t}\,dt\big|f\big(x_j(s)\big)-f\big(x_j^*\big)\big|\,ds\\
&\quad\leqslant \sum_{i,j=1}^{n}\varepsilon|b_{ij}|\int_{-\tau_{ij}}^{t^*}\tau_{ij}e^{\varepsilon(s+\tau_{ij})}\big|x_j(s)-x_j^*\big|\,ds\\
&\quad= \sum_{i,j=1}^{n}\varepsilon|b_{ij}|\tau_{ij}\int_{-\tau_{ij}}^{0} e^{\varepsilon(s+\tau_{ij})}\big|x_j(s)-x_j^*\big|\,ds\\
&\qquad+\sum_{i,j=1}^{n}\varepsilon|b_{ij}|\tau_{ij}\int_{t_0}^{t^*} e^{\varepsilon(s+\tau_{ij})}\big|x_j(s)-x_j^*\big|\,ds\\
&\quad\leqslant \sum_{i,j=1}^{n}\varepsilon|b_{ij}|\tau_{ij}e^{\varepsilon\tau_{ij}}\int_{-\tau_{ij}}^{t_0}\big|x_j(s)-x_j^*\big|\,ds\\
&\qquad+\sum_{i,j=1}^{n}\varepsilon\tau_{ij}|b_{ij}|e^{\varepsilon\tau_{ij}}\int_{t_0}^{t^*} e^{\varepsilon s}\big|x_j(s)-x_j^*\big|\,ds.
\end{aligned}\tag{10.10.23}$$

Then substituting (10.10.23) into (10.10.22) yields

$$\begin{aligned}
e^{\varepsilon t^*}V\big(x\big(t^*\big),t^*\big)\leqslant V\big(x(t_0),t_0\big)-\sum_{j=1}^{n}\Bigg(\min\{\delta,1\}-\varepsilon-\sum_{i=1}^{n}\varepsilon\tau_{ij}|b_{ij}|e^{\varepsilon\tau_{ij}}\Bigg)\\
\times\int_{t_0}^{t^*} e^{\varepsilon s}\big|x_j(s)-x_j^*\big|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i,j=1}^{n} \varepsilon \tau_{ij} |b_{ij}| e^{\varepsilon \tau_{ij}} \int_{-\tau_{ij}}^{t_0} \left| x_j(s) - x_j^* \right| ds \\
&\leqslant M = \text{constant.}
\end{aligned}
$$

So

$$
V\big(x(t), t\big) \leqslant e^{-\varepsilon t} M \quad \forall t \geqslant t_0. \tag{10.10.24}
$$

According to (10.10.18) and (10.10.24), we have

$$
\sum_{i=1}^{n} \left| x_i - x_i^* \right| \leqslant V\big(x(t), t\big) \leqslant e^{-\varepsilon t} M, \tag{10.10.25}
$$

which implies that the equilibrium state x^* of (10.10.14) is globally exponentially stable.

Now we consider (10.10.17). Let

$$
\eta = \min_{1 \leqslant i \leqslant n} \left\{ 1 - a_{ii} > \sum_{j=1,\, j \neq i}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}|,\ i = 1, 2, \ldots, n \right\}.
$$

Assume

$$
\left| x_\ell - x_\ell^* \right| = \max_{1 \leqslant i \leqslant n} \left| x_i - x_i^* \right|.
$$

We construct a positive definite and radially unbounded Lyapunov function:

$$
\begin{aligned}
V\big(x(t), t\big) &= \max_{1 \leqslant i \leqslant n} \left| x_i - x_i^* \right| + \sum_{j=1}^{n} |b_{ij}| \left| f\big(x_j(s)\big) - f\big(x_j^*\big) \right| \\
&= \left| x_\ell - x_\ell^* \right| + \sum_{j=1}^{n} |b_{ij}| \left| f\big(x_j(s)\big) - f\big(x_j^*\big) \right|.
\end{aligned} \tag{10.10.26}
$$

Then, using Lemma 10.10.7, for any fixed $\ell \in \{1, 2, \ldots, n\}$, we obtain

$$
\begin{aligned}
D^+ V\big(x(t), t\big)\big|_{(10.10.15)} &\leqslant D^+ \left| x_\ell - x_\ell^* \right| + \sum_{j=1}^{n} |b_{\ell j}| \left| f\big(x_j(t)\big) - f\big(x_j^*\big) \right| \\
&\quad - \sum_{j=1}^{n} |b_{\ell j}| \left| f\big(x_j(t - \tau_{ij})\big) - f\big(x_j^*\big) \right| \\
&\leqslant - \left| x_\ell - x_\ell^* \right| + a_{\ell\ell} \left| f\big(x_\ell(t)\big) - f\big(x_\ell^*\big) \right| \\
&\quad + \sum_{j=1, j \neq \ell}^{n} |a_{\ell j}| \left| f\big(x_\ell(t)\big) - f\big(x_\ell^*\big) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{n} |b_{\ell j}| \big| f\big(x_j(t)\big) - f\big(x_j^*\big) \big| \\
&\leqslant \left(-1 + a_{\ell\ell} + \sum_{j=1, j\neq \ell}^{n} |a_{\ell j}| + \sum_{j=1}^{n} |b_{\ell j}| \right) \\
&\quad \times \big| f\big(x_\ell(t)\big) - f\big(x_\ell^*\big) \big| - \big| h\big(x_\ell - x_\ell^*\big) \big| \\
&\leqslant -\eta \big| f\big(x_\ell(t)\big) - f\big(x_\ell^*\big) \big| - \big| h\big(x_\ell - x_\ell^*\big) \big| \\
&\leqslant -\min\{\eta, 1\} \big| \big(x_\ell - x_\ell^*\big) \big| < 0 \\
&\quad \text{for } x_\ell \neq x_\ell^*.
\end{aligned}
\tag{10.10.27}
$$

When ℓ is time varying, suppose $t_0 < t_1 < \cdots$, define $\ell_i := \ell(t_1), i = 1, 2, \ldots$. Since $|x_\ell - x_\ell^*| = |x_{\ell_i} - x_{\ell_i}^*|$ for $t = t_i, i = 1, 2, \ldots, V(x(t_1), t_1) > V(x(t_2), t_2) > V(x(t_3), t_3) > \cdots$. Because $V(x(t), t)$ is monotone decreasing and bounded below, there exist a minimum $\lim_{t\to\infty} V(x(t), t) = \overline{V}$. Now we show that $\overline{V} = 0$ so that $\lim_{t\to\infty} x(t) = x^*$. Suppose that the minimum $\overline{V} > 0$. Then, in the compact level set $\{x \mid V(x(t), t) = \overline{V}\}$, according to the Weierstrass accumulation principle, there exists $\{\tilde{t}_k\}$ such that $x(\tilde{t}_k) \to \hat{x} \neq x^*$. In addition, since $D^+V(\hat{x}(t), t) < 0$, $V(\hat{x}(t), t) < \overline{V}$. This contradicts that $\overline{V}$ is a minimum. Therefore, $\lim_{t\to\infty} x(t) = x^*$.

The remaining proof is similar to the last part (from (10.10.20)–(10.10.25)) of part (i), and is thus omitted. □

REMARK 10.10.9. The result in Theorem 10.10.8 can be easily generalized to the following: The equilibrium state x^* of (10.10.14) is globally exponentially stable, if the matrix

$$
I_n - \operatorname{diag}(a_{11}, \ldots, a_{nn}) - \big((1 - \delta_{ij})|a_{ij}|\big)_{n\times n} - \big(|b_{ij}|\big)_{n\times n}
$$

is an M matrix.

THEOREM 10.10.10. *The equilibrium state x^* of (10.10.14) is globally exponentially stable, if $A + A^T + Q - 2I_n$ is negative definite, where*

$$
Q = \operatorname{diag}\left(\sum_{j=1}^{n} |b_{1j}| + |b_{j1}|, \ldots, \sum_{j=1}^{n} |b_{nj}| + |b_{jn}| \right).
$$

PROOF. We prove Theorem 10.10.10 in two steps. In the first step, we show that the set $\Omega = \{x \mid f(x) - f(x^*) = 0\}$ is globally exponentially attractive and $x_1^*, \ldots, x_m^*$ are globally exponentially stable. In the second step, we prove $x_{m+1}^*, \ldots, x_n^*$ to be globally exponentially stable.

(1) First step, we prove that the set

$$\begin{aligned}\Omega &:= \{x \mid f(x) - f(x^*) = 0\} \\ &\equiv \{x \mid x_i = x_i^*,\ i = 1, \ldots, m_1,\ x_i x_i^* > 0, \\ &\quad |x_i| \geqslant 1,\ |x_i^*| \geqslant 1,\ i = m_1 + 1, \ldots, n\}\end{aligned}$$

to be globally exponentially attractive, so $x_i^*,\ i = 1, 2, \ldots, m_1$, is globally exponentially stable.

Construct a positive definite but not radially unbounded Lyapunov function as follows:

$$\begin{aligned}V(x(t), t) &= \sum_{i=1}^{n} 2 \int_{x_i^*}^{x_i} \big(f(x_i) - f(x_i^*)\big)\, dx_i \\ &\quad + \sum_{i,j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} |b_{ij}| \big(f(x_j(s)) - f(x_j^*)\big)^2\, ds.\end{aligned} \tag{10.10.28}$$

Then,

$$\begin{aligned}\frac{dV(x(t), t)}{dt}\bigg|_{(10.10.15)} &= -2 \sum_{i=1}^{n} \big(x_i(t) - x_i^*\big)\big(f(x_i(t)) - f(x_i^*)\big) \\ &\quad + 2 \sum_{i,j=1}^{n} a_{ij} \big(f(x_i(t)) - f(x_i^*)\big)\big(f(x_j(t)) - f(x_j^*)\big) \\ &\quad + 2 \sum_{i,j=1}^{n} b_{ij} \big(f(x_i(t)) - f(x_i^*)\big)\big(f(x_j(t - \tau_{ij}(t))) - f(x_j^*)\big) \\ &\quad + \sum_{i,j=1}^{n} |b_{ij}| \Big[\big(f(x_i(t)) - f(x_i^*)\big)^2 \\ &\quad - \big(f(x_j(t - \tau_{ij}(t))) - f(x_j^*)\big)^2\Big]\big(1 - \dot{\tau}_{ij}(t)\big) \\ &\leqslant -2 \sum_{i=1}^{n} \big(f(x_i(t)) - f(x_i^*)\big)^2 \\ &\quad + 2 \sum_{i,j=1}^{n} a_{ij} \big(f(x_i(t)) - f(x_i^*)\big)\big(f(x_j(t)) - f(x_j^*)\big) \\ &\quad + \sum_{i,j=1}^{n} |b_{ij}| \Big[\big(f(x_i(t)) - f(x_i^*)\big)^2 + \big(f(x_j(t - \tau_{ij}(t))) - f(x_j^*)\big)^2\Big]\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i,j=1}^{n} |b_{ij}| \big[\big(f\big(x_j(t)\big) - f\big(x_j^*\big)\big)^2 - \big(f\big(x_j\big(t-\tau_{ij}(t)\big)\big) - f(x_j^*)\big)^2 \big] \\
&\quad - 2\sum_{i=1}^{n} \big(x_i(t) - x_i^*\big)\big(f\big(x_i(t)\big) - f\big(x_i^*\big)\big) + 2\sum_{i=1}^{n}\big(f\big(x_i(t)\big) - f\big(x_i^*\big)\big)^2 \\
&= \big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big[-2I_n + A + A^T + Q\big]\big(f(x(t)) - f\big(x^*\big)\big) \\
&\quad - 2\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big[\big(x(t) - x^*\big) - \big(f\big(x(t)\big) - f\big(x^*\big)\big)\big] \\
&\leqslant -\lambda\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big(f\big(x(t)\big) - f\big(x^*\big)\big) \\
&\quad - 2\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big[\big(x(t) - x^*\big) - \big(f\big(x(t)\big) - f\big(x^*\big)\big)\big],
\end{aligned}
$$

where $-\lambda$ is the maximum eigenvalue of $[-2I_n + A + A^T + Q]$. Since $\lambda > 0$, we can choose ε to satisfy

$$
\lambda - 2\varepsilon - \tau_{ij}\varepsilon \sum_{i=1}^{n} |b_{ij}| e^{\varepsilon\tau_{ij}} \geqslant 0, \quad j = 1, 2, \ldots, n. \tag{10.10.29}
$$

Deriving the time derivative of $e^{\varepsilon t}V(x(t), t)$ and by the monotone property of $f(x_i)$, we have

$$
\begin{aligned}
&\frac{de^{\varepsilon t}V(x(t), t)}{dt}\bigg|_{(10.10.15)} = \varepsilon e^{\varepsilon t} V\big(x(t), t\big) + e^{\varepsilon t}\frac{dV(x(t), t)}{dt}\bigg|_{(10.10.15)} \\
&\leqslant \varepsilon e^{\varepsilon t} \sum_{i,j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} |b_{ij}|\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big)^2\, ds \\
&\quad + e^{\varepsilon t}\Bigg[2\varepsilon \sum_{i=1}^{n} \int_{x_i^*}^{x_i} \big(f(x_i) - f\big(x_i^*\big)\big)\, dx_i \\
&\quad + (2-\lambda)\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big(f\big(x(t)\big) - f\big(x^*\big)\big) \\
&\quad - 2\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big(x(t) - x^*\big)\Bigg] \\
&\leqslant \varepsilon e^{\varepsilon t} \sum_{i,j=1}^{n} \int_{t-\tau_{ij}}^{t} |b_{ij}|\big|f\big(x_j(s)\big) - f\big(x_j^*\big)\big|^2\, ds \\
&\quad + e^{\varepsilon t} \sum_{i,j=1}^{n} \big[-(2-2\varepsilon)\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big(x(t) - x^*\big) \\
&\quad + (2-\lambda)\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big(x(t) - x^*\big)\big]
\end{aligned}
$$

$$= \varepsilon e^{\varepsilon t} \sum_{i,j=1}^{n} \int_{t-\tau_{ij}}^{t} |b_{ij}| \big| f\big(x_j(s)\big) - f\big(x_j^*\big)\big|^2 \, ds$$

$$+ e^{\varepsilon t}\big[-(\lambda - 2\varepsilon)\big(f\big(x(t)\big) - f\big(x^*\big)\big)^T \big(x(t) - x^*\big)\big]. \tag{10.10.30}$$

Integrating both sides of (10.10.30) from t_0 to an arbitrary $t^* > t_0$ yields

$$\begin{aligned} & e^{\varepsilon t^*} V\big(x\big(t^*\big), t^*\big) \\ & \quad \leqslant V\big(x(t_0), t_0\big) - \int_{t_0}^{t^*} \varepsilon e^{\varepsilon t} \int_{t-\tau_{ij}}^{t} \sum_{i,j=1}^{n} |b_{ij}| \big| f\big(x_j(s)\big) - f\big(x_j^*\big)\big|^2 \, ds \, dt \\ & \qquad - \sum_{j=1}^{n} (\lambda - 2\varepsilon) \int_{t_0}^{t^*} e^{\varepsilon t} \big(f\big(x_j(t)\big) - f\big(x_j^*\big)\big) \big| x_j(t) - x_j^* \big| \, dt. \end{aligned} \tag{10.10.31}$$

Estimating the first term of (10.10.31) by exchanging the order of integration, we obtain

$$\begin{aligned} & \sum_{i,j=1}^{n} \varepsilon \int_{t_0}^{t^*} e^{\varepsilon t} \int_{t-\tau_{ij}}^{t} |b_{ij}| \big| f\big(x_j(s)\big) - f\big(x_j^*\big)\big|^2 \, ds \, dt \\ & \quad \leqslant \sum_{i,j=1}^{n} \varepsilon |b_{ij}| \int_{-\tau_{ij}}^{t} \int_{\max\{s,0\}}^{\min\{s+\tau_{ij}, t\}} e^{\varepsilon t} \, dt \big(x_j(s) - x_j^*\big)\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big) \, ds \\ & \quad \leqslant \sum_{i,j=1}^{n} \varepsilon |b_{ij}| \int_{-\tau_{ij}}^{t} \tau_{ij} e^{\varepsilon (s+\tau_{ij})} \big(x_j(s) - x_j^*\big)\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big) \, ds \\ & \quad = \sum_{i,j=1}^{n} \varepsilon \tau_{ij} |b_{ij}| \int_{-\tau_{ij}}^{t_0} e^{\varepsilon (s+\tau_{ij})} \big(x_j(s) - x_j^*\big)\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big) \, ds \\ & \qquad + \sum_{i,j=1}^{n} \varepsilon \tau_{ij} |b_{ij}| \int_{t_0}^{t^*} e^{\varepsilon (s+\tau_{ij})} \big(x_j(s) - x_j^*\big)\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big) \, ds \\ & \quad \leqslant \sum_{i,j=1}^{n} \varepsilon \tau_{ij} |b_{ij}| e^{\varepsilon \tau_{ij}} \int_{-\tau_{ij}}^{t_0} \big(x_j(s) - x_j^*\big)\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big) \, ds \end{aligned}$$

$$+\sum_{i,j=1}^{n} \varepsilon\tau_{ij}|b_{ij}|e^{\varepsilon\tau_{ij}} \int_{t_0}^{t^*} e^{\varepsilon s}\big(x_j(s)-x_j^*\big)\big(f\big(x_j(s)\big)-f\big(x_j^*\big)\big)\,ds. \tag{10.10.32}$$

Then substituting (10.10.32) into (10.10.31) yields

$$\begin{aligned}
e^{\varepsilon t^*} V\big(x\big(t^*\big), t^*\big) \leqslant{}& V\big(x(t_0), t_0\big) - \sum_{j=1}^{n}\left(\lambda - 2\varepsilon - \tau_{ij}\sum_{i=1}^{n}|b_{ij}|e^{\varepsilon\tau_{ij}}\right) \\
&\times \sum_{i=1}^{n}\int_{t_0}^{t^*} e^{\varepsilon s}\big(x_j(s)-x_j^*\big)\big(f\big(x_j(s)\big)-f\big(x_j^*\big)\big)\,ds \\
&+\sum_{i,j=1}^{n} \varepsilon\tau_{ij}|b_{ij}|e^{\varepsilon\tau_{ij}} \int_{-\tau_{ij}}^{t_0} \big(x_j(s)-x_j^*\big)\big(f\big(x_j(s)\big)-f\big(x_j^*\big)\big)\,ds \\
\leqslant{}& M = \text{constant}.
\end{aligned} \tag{10.10.33}$$

According to (10.10.28) and (10.10.33), we have

$$\sum_{i=1}^{n}\int_{x_i}^{x_i^*}\big(f(x_i)-f\big(x_i^*\big)\big)\,dx_i \leqslant V\big(x(t),t\big) \leqslant e^{-\varepsilon t}M. \tag{10.10.34}$$

We now prove that the set Ω is globally exponentially attractive by (10.10.28)–(10.10.34), and $x_1^*, \dots, x_n^*$ is globally exponentially stable, $\forall i \in \{1, \dots, m\}$, i.e., $|x_i^*| < 1$. By (10.10.33) and (10.10.34), we have

$$\begin{aligned}
\frac{1}{2}\big|x_i(t)-x_i^*\big|^2 &= \int_{x_i^*}^{x_i(t)} \big(f(x_i)-f\big(x_i^*\big)\big)\,dx_i \\
&\leqslant V\big(x(t),t\big) \leqslant e^{-\varepsilon t}M \quad \text{for } |x_i|<1,
\end{aligned} \tag{10.10.35}$$

$$\begin{aligned}
\big(1-x_i^*\big)\big|x_i(t)-x_i^*\big| &= \int_{x_i^*}^{x_i(t)} \big(f(x_i)-f\big(x_i^*\big)\big)\,dx_i \\
&\leqslant V\big(x(t),t\big) \leqslant e^{-\varepsilon t}M \quad \text{for } |x_i|\geqslant 1,
\end{aligned} \tag{10.10.36}$$

$$\begin{aligned}
\big(-1-x_i^*\big)\big|x_i(t)-x_i^*\big| &= \int_{x_i^*}^{x_i(t)} \big(f(x_i)-f\big(x_i^*\big)\big)\,dx_i \\
&\leqslant V\big(x(t),t\big) \leqslant e^{-\varepsilon t}M \quad \text{for } |x_i|\leqslant 1.
\end{aligned} \tag{10.10.37}$$

Equations (10.10.35)–(10.10.37) show that $x_i^*,\ i = 1, 2, \ldots, m_1$, are globally exponentially stable.

$\forall i \in \{m_1 + 1, \ldots, m_2\}$, i.e., $x_i^* \geqslant 1$, by (10.10.33) and (10.10.34), we can obtain that when $|x_i| \leqslant 1$,

$$\begin{aligned}
\frac{1}{2}\bigl(x_i(t) - 1\bigr)^2 &= \int_{x_i^*}^{x_i(t)} \bigl(f(x_i) - f\bigl(x_i^*\bigr)\bigr)\, dx_i \\
&= \int_{x_i^*}^{1} \bigl(f(x_i) - f\bigl(x_i^*\bigr)\bigr)\, dx_i + \int_{1}^{x_i(t)} \bigl(f(x_i) - f\bigl(x_i^*\bigr)\bigr)\, dx_i \\
&\leqslant V\bigl(x(t), t\bigr) \leqslant e^{-\varepsilon t} M,
\end{aligned} \tag{10.10.38}$$

which shows that $x_i(t)$ moves into $[1, +\infty)$ from $|x_i| \leqslant 1$.

When $x_i \leqslant -1$, we have

$$\begin{aligned}
&\int_{x_i^*}^{x_i(t)} \bigl(f(x_i) - f\bigl(x_i^*\bigr)\bigr)\, dx_i \\
&\quad= \int_{x_i^*}^{1} \bigl(f(x_i) - f\bigl(x_i^*\bigr)\bigr)\, dx_i + \int_{1}^{-1} \bigl(f(x_i) - f\bigl(x_i^*\bigr)\bigr)\, dx_i \\
&\qquad+ \int_{-1}^{t_i(t)} \bigl(f(x_i) - f\bigl(x_i^*\bigr)\bigr)\, dx_i \\
&\quad= 0 + \int_{1}^{-1} (x_i - 1)\, dx_i + \int_{-1}^{t_i(t)} -2\, dx_i \\
&\quad= 2 + (-2)\bigl(x_i(t) + 1\bigr) \\
&\quad= -2x_i(t),
\end{aligned}$$

so,

$$-2x_i(t) \leqslant V\bigl(x(t), t\bigr) \leqslant e^{-\varepsilon t} M. \tag{10.10.39}$$

Equation (10.10.39) implies that $x_i(t)$ moves from $[-\infty, -1]$ into $[-1, 1]$ exponentially, therefore, it further moves into $[1 + \infty)$ by (10.10.38).

Similarly, $\forall i \in \{m_2+1,\dots,n\}$, i.e., $x_i^* \leqslant -1$, by (10.10.33) and (10.10.34), we obtain the following results. When $|x_i| \leqslant 1$,

$$
\begin{aligned}
&\int_{x_i^*}^{x_i(t)} \big(f(x_i) - f\big(x_i^*\big)\big)\,dx_i \\
&\quad = \int_{x_i^*}^{-1} \big(f(x_i) - f\big(x_i^*\big)\big)\,dx_i + \int_{-1}^{t_i(t)} \big(f(x_i) - f\big(x_i^*\big)\big)\,dx_i \\
&\quad = 0 + \int_{-1}^{t_i(t)} (x_i+1)\,dx_i \\
&\quad = \frac{1}{2}(x_i+1)^2,
\end{aligned}
$$

and so

$$
\frac{1}{2}(x_i+1)^2 \leqslant V\big(x(t),t\big) \leqslant e^{-\varepsilon t}M, \tag{10.10.40}
$$

implying that $x_i(t)$ moves into $(-\infty,-1)$ exponentially.

When $x_i \geqslant 1$, we obtain

$$
\begin{aligned}
&\int_{x_i^*}^{x_i(t)} \big(f(x_i) - f\big(x_i^*\big)\big)\,dx_i \\
&\quad = \int_{x_i^*}^{-1} \big(f(x_i) - f\big(x_i^*\big)\big)\,dx_i + \int_{-1}^{1} \big(f(x_i) - f\big(x_i^*\big)\big)\,dx_i \\
&\qquad + \int_{1}^{x_i(t)} \big(f(x_i) - f\big(x_i^*\big)\big)\,dx_i \\
&\quad = 0 + \int_{-1}^{1} (x_i+1)\,dx_i + \int_{1}^{x_i(t)} 2\,dx_i \\
&\quad = 2 + 2\big(x_i(t)-1\big) \\
&\quad = 2x_i(t),
\end{aligned}
$$

and thus,

$$
2x_i(t) \leqslant V\big(x(t),t\big) \leqslant e^{-\varepsilon t}M. \tag{10.10.41}
$$

Equation (10.10.41) indicates that $x_i(t)$ moves from $[1, +\infty]$ into $[-1, 1]$. Therefore, $x_i(t)$ further moves into $(-\infty, 1)$ exponentially due to (10.10.40).

(2) Second step. We prove x_j^* $(j = m_1 + 1, \ldots, n)$ to be also globally exponentially stable. When $t \gg 1$, $f(x_j) = f(x_j^*)$, $j = m_1 + 1, \ldots, n$.

Rewrite (10.10.15) as the following two equations:

$$\begin{aligned}\frac{dx_i}{dy} = {}& -\big(x_i - x_i^*\big) + \sum_{j=1}^{m_1} a_{ij}\big(f\big(x_j(t)\big) - f\big(x_j^*\big)\big)\\&+ \sum_{j=1}^{n} b_{ij}\big(f\big(x_j\big(t - \tau_{ij}(t)\big)\big) - f\big(x_j^*\big)\big),\\&i = 1, 2, \ldots, m_1,\end{aligned} \tag{10.10.42}$$

$$\begin{aligned}\frac{dx_i}{dy} = {}& -\big(x_i - x_i^*\big) + \sum_{j=1}^{m_1} a_{ij}\big(f\big(x_j(t)\big) - f\big(x_j^*\big)\big)\\&+ \sum_{j=1}^{n} b_{ij}\big(f\big(x_j\big(t - \tau_{ij}(t)\big)\big) - f\big(x_j^*\big)\big),\\&i = m_1 + 1, \ldots, n.\end{aligned} \tag{10.10.43}$$

Let $|x_i(t) - x_i^*| \leqslant M_i e^{-\varepsilon(t-t_0)}$, $i = 1, 2, \ldots, m_1$. by substituting the solution of (10.10.42) into (10.10.43) and using the variation of constants, the solution of (10.10.43) can be expressed as

$$\begin{aligned}x_i(t) = {}& e^{-(t-t_0)}\big(x_i(t_0) - x_i^*\big) + \sum_{j=1}^{m_1} a_{ij}\int_{t_0}^{t} e^{-(t-s)}\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big)\,ds\\&+ \sum_{j=1}^{m_1} b_{ij}\int_{t_0}^{t} e^{-(t-s)}\big(f\big(x_j\big(s - \tau_{ij}(s)\big)\big) - f\big(x_j^*\big)\big)\,ds\\&(i = m_1 + 1, \ldots, n),\end{aligned}$$

and thus

$$\begin{aligned}&\big|x_i(t, t_0, x_0)\big|\\&\quad\leqslant e^{-(t-t_0)}\big|x_i(t_0) - x_i^*\big| + \sum_{j=1}^{m_1} |a_{ij}|\int_{t_0}^{t} e^{-(t-s)}\big|f\big(x_j(s)\big) - f\big(x_j^*\big)\big|\,ds\\&\qquad+ \sum_{j=1}^{m_1} |b_{ij}|\int_{t_0}^{t} e^{-(t-s)}\big|x_j\big(s - \tau_{ij}(s)\big) - x_j^*\big|\,ds\, e^{-(t-t_0)}\big|x_i(t_0) - x_i^*\big|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m_1}|a_{ij}|M_j\int_{t_0}^{t} e^{-(t-s)}e^{-\varepsilon(s-t_0)}\,ds \\
&+\sum_{j=1}^{m_1}|b_{ij}|M_j\int_{t_0}^{t} e^{-(t-s)}e^{-\varepsilon(s-\tau_{ij}-t_0)}\,ds\,e^{-(t-t_0)}\big|x_i(t_0)-x_i^*\big| \\
&+\sum_{j=1}^{m_1}|a_{ij}|M_j\frac{e^{-\varepsilon(t-t_0)}}{1-\varepsilon} \\
&+\sum_{j=1}^{m_1}|b_{ij}|M_je^{-\varepsilon\tau_{ij}}\frac{e^{-\varepsilon(t-t_0)}}{1-\varepsilon}, \quad i=m_1+1,\dots,n.
\end{aligned}
\tag{10.10.44}
$$

Equation (10.10.44) shows that the equilibrium $x_i^*, i = m_1 + 1, \dots, n$, is also globally exponentially stable.

The proof for Theorem 10.10.10 is complete. □

REMARK 10.10.11. The condition in Theorem 10.10.10 can be easily generalized to: there exists a positive definite diagonal matrix $P = \operatorname{diag}(p_1, p_2, \dots, p_n)$ such that

$$
\begin{aligned}
&PA + A^TP - 2P + I_n \\
&\quad + \operatorname{diag}\left(\sum_{j=1}^{n}\big(p_1|b_{1j}| + p_j|b_{j1}|\big), \dots, \sum_{j=1}^{n}\big(p_1|b_{1j}| + p_j|b_{j1}|\big)\right)
\end{aligned}
$$

is negative definite. Then the conclusion of Theorem 10.10.8 holds.

COROLLARY 10.10.12. *The equilibrium x^* of (10.10.14) is globally exponentially stable if any one of the following conditions is satisfied*:

(1) $\lambda_{\max}(A + A^T) \leqslant 0$, *and* $\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}(|b_{ij}| + |b_{ji}|) < 2$;
(2) $\lambda_{\max}(A + A^T) < 0$, *and* $\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}(|b_{ij}| + |b_{ji}|) < 2$;
(3) $\lambda_{\max}(A + A^T + Q) < 2$;
(4) $A = 0$, *and* $\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}(|b_{ij}| + |b_{ji}|) < 2$.

PROOF. We only prove (1), the other cases are similar. $\forall y \neq 0$, we have

$$
\begin{aligned}
y^T\big(A + A^T + Q - 2I_n\big)y &= y^T\big(A + A^T\big)y + y^T(Q)y - 2y^Ty \\
&\leqslant \lambda_{\max}y^Ty + \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\big(|b_{ij}| + |b_{ji}|\big)y^Ty - 2y^Ty
\end{aligned}
$$

$$\leqslant \left(\max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \big(|b_{ij}| + |b_{ji}|\big) - 2 \right) y^T y < 0.$$

Thus, $A + A^T + Q - 2I_n$ is negative definite. According to Theorem 10.10.10, the equilibrium x^* of (10.10.14) is globally exponentially stable. □

THEOREM 10.10.13. *Let $\tau_{ij}(t) = \tau_j(t)$. The equilibrium state x^* of (10.10.14) is globally exponentially stable, if $A + A^T + BB^T - I_n$ is negative definite.*

PROOF. Here, we again employ the positive definite but not radially bounded Lyapunov function (10.10.28):

$$V\big(x(t), t\big) = \sum_{i=1}^{n} 2 \int_{x_i^*}^{x_i} \big(f(x_i) - f\big(x_i^*\big)\big)\, dx_i + \sum_{j=1}^{n} \int_{t-\tau_j(t)}^{t} |b_{ij}|\big(f\big(x_j(s)\big) - f\big(x_j^*\big)\big)^2\, ds.$$

Since $A + A^T + BB^T - I_n$ is negative definite, we obtain

$$\begin{aligned}
\frac{dV(x(t), t)}{dt}\bigg|_{(10.10.15)} &= -2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&\quad + 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T A\big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&\quad + 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T B\big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big) \\
&\quad + \big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&\quad - \big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big) \\
&\quad - 2\big(x(t) - \big(x^*\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&\quad + 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&= \big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(A + A^T - I_n\big)\big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&\quad + 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T B\big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big) \\
&\quad - \big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big) \\
&\quad - 2\big(x(t) - \big(x^*\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&\quad + 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&= \big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(A + A^T - I_n\big)\big(f\big(x(t) - f\big(x^*\big)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \big(f\big(x(t) - f\big(x^*\big)\big)\big)^T BB^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&- \big[B^T f\big(x(t) - f\big(x^*\big)\big) - \big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big)\big]^T \\
&\times \big[B^T f\big(x(t) - f\big(x^*\big)\big) - \big(f\big(x(t-\tau) - f\big(x^*\big)\big)\big)\big] \\
&- 2\big(x(t) - \big(x^*\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&+ 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
\leqslant{}& \big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(A + A^T BB^T - I_n\big)\big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&- 2\big(x(t) - \big(x^*\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&+ 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
\leqslant{}& -\mu\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&- 2\big(x(t) - \big(x^*\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big) \\
&+ 2\big(f\big(x(t) - f\big(x^*\big)\big)\big)^T \big(f\big(x(t) - f\big(x^*\big)\big)\big),
\end{aligned}
\tag{10.10.45}
$$

where $-\mu$ is the maximum eigenvalue of $A+A^T BB^T - I_n$. The remaining proof is similar to the last part of the proof for Theorem 10.10.10, and hence is omitted. □

REMARK 10.10.14. The condition in Theorem 10.10.13 can be easily generalized to: If there exist $p_i > 0$, $i = 1, \dots, n$, such that the matrix $PBP - B^T P + PA + A^T P - P$ is negative definite, where $P = \mathrm{diag}(p_1, \dots, p_n)$.

In [233], it is shown that a CNN is asymptotically stable if there is a constant β such that

$$A + A^T + \beta I_n < 0 \quad \text{(i.e., negative definite)}, \tag{10.10.46}$$

$$\|B\|_2^2 \leqslant 1 + \frac{1}{2}\beta. \tag{10.10.47}$$

Subsequently, in [232], the second condition in (10.10.47) is relaxed to

$$\|B\|_2^2 \leqslant 1 + \beta. \tag{10.10.48}$$

The following corollary shows that Theorem 10.10.13 is an extension and improvement of the main results in [232,233].

COROLLARY 10.10.15. *If there is a constant β such that (10.10.46) and (10.10.47) or (10.10.48) are satisfied, then the condition in Theorem 10.10.13 is satisfied.*

PROOF. If there is a constant β such that $A + A^T + \beta I_n < 0$ and $\|B\|_2^2 \leqslant 1 + \frac{1}{2}\beta$, then $A + A^T + \beta I_n < 0$ and $\|B\|_2^2 \leqslant 1 + \beta$ (the conditions in [232]). Since

$\lambda_{\max}(BB^T) \leqslant \|BB^T\|_2 = \|B^2\|_2 \leqslant \|B\|_2^2 \leqslant 1+\beta$, $BB^T - (1+\beta)I_n$ is negative semi-definite, where $\lambda_{\max}(BB^T)$ is the maximum eigenvalue of BB^T. combining this with the first condition (10.10.46), we have $A + A^T BB^T - I_n < 0$. $\square$

COROLLARY 10.10.16. *The equilibrium of* (10.10.14) *is globally exponentially stable if any of the following conditions hods*:

(1) $\lambda_{\max}(A + A^T) \leqslant 0$, *and* $\|B\|_2 < 1$, *where* $\|B\|_2 = \sqrt{\lambda_{\max}(BB^T)}$;
(2) $\lambda_{\max}(A + A^T) < 0$, *and* $\|B\|_2 \leqslant 1$;
(3) $\lambda_{\max}(A + A^T + BB^T) < 0$;
4) $A = 0$, *and* $\|B\|_2 < 1$.

The proof is simple and thus omitted.

In the following, we present five examples to compare the new results with the existing ones.

EXAMPLE 10.10.17. Consider the following two-state CNN:

$$\frac{dx_1}{dt} = -x_1 - 2f\big(x_1(t)\big) + 2f\big(x_2(t)\big) + 0.99f\big(x_1(t)\big) + 0.99f\big(x_2(t-\tau)\big),$$
$$\frac{dx_2}{dt} = -x_2 + 2f\big(x_1(t)\big) - 2f\big(x_2(t)\big),$$

which gives

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.99 & 0.99 \\ 0 & 0 \end{bmatrix}, \qquad BB^2 = \begin{bmatrix} 1.9602 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$A + A^T + BB^T - I_2 = \begin{bmatrix} -3.0398 & 4 \\ 4 & -5 \end{bmatrix}$$

is not negative definite. So the condition in theorem of [232] does not hold, but the condition in (10.10.16) of Theorem 10.10.8 is satisfied.

EXAMPLE 10.10.18. Consider the following two-state CNN:

$$\frac{dx_1}{dt} = -x_1 + 0.89f\big(x_1(t-\tau_1)\big) + 0.1f\big(x_2(t-\tau_2)\big) + u_1,$$
$$\frac{dx_2}{dt} = -x_2 + 0.89f\big(x_1(t-\tau_1)\big) + 0.1f\big(x_2(t-\tau_2)\big) + u_2.$$

For this system, we have

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.89 & 0.1 \\ 0.89 & 0.1 \end{bmatrix}, \qquad BB^2 = \begin{bmatrix} 0.8021 & 0.8021 \\ 0.8021 & 0.8021 \end{bmatrix}.$$

Since $\|B\|_2 = 1.6042 > 0$, the conditions of theorem in [9] are not satisfied. Moreover,

$$A + A^T + BB^T - I_2 = \begin{bmatrix} -0.1979 & 0.8021 \\ 0.8021 & -0.1979 \end{bmatrix}$$

is not negative definite, so the condition of the main theorem in [9] is not satisfied.
However, the condition in (10.10.17) of Theorem 10.10.8 is satisfied.

EXAMPLE 10.10.19. Consider a two-state CNN as follows:

$$\begin{aligned}
\frac{dx_1}{dt} &= -x_1 - f\big(x_1(t)\big) + f\big(x_2(t)\big) + 0.78 f\big(x_1(t-\tau_1)\big) \\
&\quad + 0.2 f\big(x_2(t-\tau_2)\big) + u_1, \\
\frac{dx_2}{dt} &= -x_2 + f\big(x_1(t)\big) - f\big(x_2(t)\big) + 0.78 f\big(x_1(t-\tau_1)\big) \\
&\quad + 0.2 f\big(x_2(t-\tau_2)\big) + u_2.
\end{aligned}$$

For this example,

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.78 & 0.2 \\ 0.78 & 0.2 \end{bmatrix}, \qquad BB^2 = \begin{bmatrix} 0.6484 & 0.6484 \\ 0.6484 & 0.6484 \end{bmatrix}.$$

Then,

$$A + A^T + BB^T - I_2 = \begin{bmatrix} -2.3516 & 2.6484 \\ 2.6484 & -2.3516 \end{bmatrix}$$

is not negative definite, so the condition of the theorems in [7] and [232] are not satisfied. However, it satisfies the condition in (10.10.17) of Theorem 10.10.8.

EXAMPLE 10.10.20. Consider a two-state CNN:

$$\begin{aligned}
\frac{dx_1}{dt} &= -x_1 - 2.1 f(x_1) + 8 f(x_2) + f\big(x_1(t-\tau_1)\big) \\
&\quad - 2 f\big(x_2(t-\tau_2)\big) + u_1, \\
\frac{dx_2}{dt} &= -x_2 - 8 f(x_1) + 3.2 f(x_2) + 2 f\big(x_1(t-\tau_1)\big) \\
&\quad - 2 f\big(x_2(t-\tau_2)\big) + u_2,
\end{aligned}$$

for which

$$A = \begin{bmatrix} -2.1 & 8 \\ -8 & 3.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix},$$

so

$$A + A^T = \begin{bmatrix} -4.2 & 0 \\ 0 & 6.4 \end{bmatrix},$$

$$
\begin{aligned}
Q &= \operatorname{diag}\bigl(|b_{11}| + |b_{11}| + |b_{12}| + |b_{21}| \\
&\qquad + |b_{22}| + |b_{22}| + |b_{21}| + |b_{12}|\bigr) - 2I_2 \\
&= \begin{bmatrix} 2+4 & 0 \\ 0 & 4+4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \\
A + A^T + Q - 2I_2 &= \begin{bmatrix} -0.2 & 0 \\ 0 & -0.4 \end{bmatrix}
\end{aligned}
$$

which is negative definite, so the condition of Theorem 10.10.10 holds. But the conditions in [353] (Theorems 3.1 and 3.2) and that in [8] (Theorem 1) are not satisfied.

EXAMPLE 10.10.21. Let

$$
A = \begin{bmatrix} -1/2 & -5 \\ 5 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix}.
$$

Then, we have

$$
\begin{aligned}
A + A^T &= \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}, \qquad \|B\|_2 = \frac{4}{3}, \\
\bigl(1 - \|B\|_2\bigr) &= -\frac{1}{3} > \lambda_{\max}\bigl(A + A^T\bigr) = -1.
\end{aligned}
$$

Hence, the condition of Theorem 10.10.13 holds but the condition of [8] (Theorem 1), that of the theorem in [7], the conditions of the theorem in [353], and that of the main theorem in [232] are not satisfied.

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Chapter 11

Limit Cycle, Normal Form and Hopf Bifurcation Control

In this chapter, we study nonlinear dynamical systems, with particular attention given to the computation of normal form and limit cycle, Hopf bifurcation control, and their applications. Both mathematical and practical engineering problems are presented, which are described by ordinary differential equations, discrete maps and time-delay differential equations.

11.1. Introduction

Nonlinear dynamics, more grandly called "nonlinear science" or "chaos theory", is a rapidly-growing area, which plays an important role in the study of almost all disciplines of science and engineering, including mathematics, mechanics, aeronautics, electrical circuits, control systems, population problems, economics, financial systems, stock markets, ecological systems, etc. In general, any dynamical system contains certain parameters (usually called bifurcation parameters or control parameters) and thus it is vital to study the dynamic behavior of such systems as the parameters are varied. The complex dynamical phenomena include instability, bifurcation and chaos (e.g., see [12,146,399,67,160,427]). Studies of nonlinear dynamical systems may be roughly divided into two main categories: local analysis and global analysis. For instance, post-critical behavior such as saddle-node bifurcation and Hopf bifurcation can be studied locally in the vicinity of a critical point, while heteroclinic and homoclinic orbits, and chaos are essentially global behavior and have to be studied globally. These two categories need to be treated with different theories and methodologies.

In studying the local behavior of a dynamical system, in particular for qualitative properties, the first step is usually to simplify the system. Such a simplification should keep the dynamical behavior of the system unchanged. Many methodologies have been developed in analyzing local dynamics, such as center manifold theory, normal form theory, averaging method, multiple time scales, Lyapunov–Schmidt reduction, the method of succession functions, the intrinsic harmonic balancing technique, etc. These methods can be used to obtain the so called "simplified" governing differential equations which describe the dynamics of the system in the vicinity of a point of interest. The "simplified" system is topologically equivalent to the original system, and thus it greatly simplifies the analysis of the original system. Usually, center manifold theory is applied first to reduce the system to a low dimensional center manifold, and then the method of normal forms is employed to further simplify the system [146,322]. However, approaches have been developed which combine the two theories into one unified procedure without applying center manifold theory [420,422,423,441]. In this chapter, the local dynamical analysis is mainly based on this unified procedure of normal forms. The normal form of a nonlinear dynamical system is not uniquely defined and computing the explicit formula of a normal form in terms of the original system's coefficients is not easy. In the past few years, efficient methodologies and software based on symbolic computations using Maple and Mathematica have been successfully employed in normal form computations.

The phenomenon of limit cycle was first discovered and studied by Poincaré [333] who presented the break through qualitative theory of differential equations. In order to determine the existence of limit cycles for a given differential equation and the properties of limit cycles, Poincaré introduced the well-known method of Poincaré Map, which is still the most basic tool for studying the stability and bifurcations of periodic orbits. The driving force behind the study of limit cycle theory was the invention of triode vacuum tube which was able to produce stable self-excited oscillations of constant amplitude. It was noted that such kind of oscillation phenomenon could not be described by linear differential equations. At the end of the 1920s Van der Pol [387] developed a differential equation to describe the oscillations of constant amplitude of a triode vacuum tube:

$$\ddot{x} + \mu\big(x^2 - 1\big)\dot{x} + x = 0 \quad (\mu \neq 0), \tag{11.1.1}$$

where the dot denotes a differentiation with respect to time t, and μ is a parameter. Equation (11.1.1) is now called Van der Pol's equation. Later a more general equation called Liénard equation [289] was developed, for which Van der Pol's equation is a special case.

Limit cycles are generated through bifurcations, among which the most popular and important one is Hopf bifurcation [171]. Consider the following general nonlinear system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \quad \boldsymbol{x} \in R^n,\ \mu \in R,\ \boldsymbol{f} : R^{n+1} \to R^n, \tag{11.1.2}$$

where $\boldsymbol{x}$ is an n-dimensional state vector while μ is a scalar parameter, called bifurcation parameter. (Note that in general one may assume that μ is an m-dimensional vector for $m \geqslant 1$.) The function $\boldsymbol{f}$ is assumed to be analytic with respect to both $\boldsymbol{x}$ and μ. Equilibrium solutions of system (11.1.2) can be found by solving the nonlinear algebraic equation $\boldsymbol{f}(\boldsymbol{x}, \mu) = \boldsymbol{0}$ for all real μ. Let $\boldsymbol{x}^*$

be an equilibrium (or fixed point) of the system, i.e., $\boldsymbol{f}(\boldsymbol{x}^*, \mu) \equiv \mathbf{0}$ for any real value of μ. Further, suppose that the Jacobian of the system evaluated at the equilibrium $\boldsymbol{x}^*$ has a pair of complex conjugates, denoted by $\lambda_{1,2}(\mu)$ with $\lambda_1 = \bar{\lambda}_2 = \alpha(\mu) + i\omega(\mu)$ such that $\alpha(\mu^*) = 0$ and $\frac{d\alpha(\mu^*)}{d\mu} \neq 0$. Here, the second condition is usually called transversality condition, implying that the crossing of the complex conjugate pair at the imaginary axis is not tangent to the imaginary axis. Then Hopf bifurcation occurs at the critical point $\mu = \mu^*$, giving rise to bifurcation of a family of limit cycles. Other local bifurcations such as double-zero, Hopf-zero, double Hopf, etc. bifurcations may result in more complex dynamical behaviors. In this chapter, Hopf bifurcations will be particularly studied.

The normal form of Hopf bifurcation can be used to analyze bifurcation and stability of limit cycles in the vicinity of a Hopf critical point. To determine the existence of multiple limit cycles in the neighborhood of a Hopf critical point, one needs to compute the coefficients of the normal form, or the focus values of the critical point [420,291,62,290,63]. Many methodologies have been developed for computing normal forms and focus values. To find maximal number of multiple limit cycles, one needs to compute high-order normal forms or high-order focus values, which must be found in explicit symbolic expressions. This raises a crucial problem—computation efficiency, since a nonefficient computer program would quickly cause a computer system to crash. Therefore, it is important to study the existing methods of computing normal forms and focus values. In this chapter, we will present three typical methods: the Takens method [377,378], a perturbation method based on multiple time scales [322,420], and the singular point value method [291,62,290,63]. Examples will be presented to show that these three methods yield the same computational complexity.

Global bifurcation analysis, on the other hand, is more difficult than local analysis. Besides homoclinic and heteroclinic bifurcations [146], the most exciting discovery in nonlinear dynamics is chaos. Since the discovery of the Lorenz attractor [294], which has led to a new era in the study of nonlinear dynamical systems, many researchers from different disciplines such as mathematics, physics and engineering extensively investigated the dynamical property of chaotic systems. For a quite long period, people thought that chaos was not predictable nor controllable. However, the OGY method [328] developed in 90s of the last century has completely changed the situation, and the study of bifurcation and chaos control begun. The general goal of bifurcation control is to design a controller such that the bifurcation characteristics of a nonlinear system undergoing bifurcation can be modified to achieve certain desirable dynamical behavior, such as changing a Hopf bifurcation from subcritical to supercritical [427,66,70]. The main goal of chaos control is to simply eliminate chaotic motion so that the system's trajectories eventually converge to an equilibrium point or a periodic motion [67].

Bifurcation and chaos control, which as an emerging research field has become challenging, stimulating, and yet quite promising. Unlike classical control theory, bifurcation control refers to the task of designing a controller to modify the bifurcation properties of a given nonlinear system, thereby achieving some desirable dynamical behaviors. Particularly, Hopf bifurcation control in both discrete and continuous time-delay dynamical systems is studied. Recently, feedback controllers using polynomial functions have been developed for controlling Hopf bifurcations in both continuous systems and discrete maps [430,73,74], which have been shown to have superior properties than traditional controllers such as the washout filter controller [393]. The new controller is easy to be implemented, which not only preserves the system's equilibrium points, but also keeps the dimension of the system unchanged. This approach can be extended to study higher dimensional dynamical systems.

The rest of the chapter is organized as follows. In the next section, three typical methods for computing normal forms (focus values) are presented. In Section 11.3, the simplest normal form (SNF) is introduced, and the SNFs for codimension-one bifurcations are particularly discussed. Section 11.4 is devoted to study Hopf bifurcation control. Both mathematical and practical problems are presented in each section to illustrate the application of theories.

11.2. Computation of normal forms and focus values

In this section, we present three typical methods for computing normal forms and the focus value, which are widely used in applications associated with Hopf bifurcation. We first introduce the Takens method [377,378], then discuss a perturbation method using multiple time scales [322,420] and finally present the singular point value method [291,62,290,63]. Note that the Takens method and the singular point value method are only applicable to 2-dimensional systems, while the perturbation method can be applied to general n-dimensional systems. In other words, if one wants to apply the Takens method or the singular point value method to compute normal forms (focus values), one must first apply center manifold theory, while with the perturbation method one does not need applying center manifold theory since it combines normal form theory and center manifold theory in one unified approach.

11.2.1. The Takens method

Consider the following general 2-dimensional system

$$\dot{\boldsymbol{x}} = L\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}) \equiv \boldsymbol{v}_1 + \boldsymbol{f}_2(\boldsymbol{x}) + \boldsymbol{f}_3(\boldsymbol{x}) + \cdots + \boldsymbol{f}_k(\boldsymbol{x}) + \cdots, \tag{11.2.1}$$

where $\boldsymbol{x} = (x_1, x_2)^T \in \boldsymbol{R}^2$, and

$$L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{11.2.2}$$

Let $\boldsymbol{v}_1 = L\boldsymbol{x} \equiv J\boldsymbol{x}$. (Usually J is used to denote the Jacobian matrix. Here L is used in consistent with the Lie bracket notation.) It is assumed that all eigenvalues of L have zero real parts, implying that the dynamics of system (11.2.1) is described on a 2-D center manifold. $\boldsymbol{f}_k(\boldsymbol{x})$ denotes the kth-order vector homogeneous polynomials of $\boldsymbol{x}$.

The basic idea of the Takens normal form theory is to find a near-identity nonlinear transformation, given by

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}) \equiv \boldsymbol{y} + \boldsymbol{h}_2(\boldsymbol{y}) + \boldsymbol{h}_3(\boldsymbol{y}) + \cdots + \boldsymbol{h}_k(\boldsymbol{y}) + \cdots \tag{11.2.3}$$

such that the resulting system

$$\dot{\boldsymbol{y}} = L\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}) \equiv L\boldsymbol{y} + \boldsymbol{g}_2(\boldsymbol{y}) + \boldsymbol{g}_3(\boldsymbol{y}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y}) + \cdots \tag{11.2.4}$$

becomes as simple as possible. Here, both $\boldsymbol{h}_k(\boldsymbol{y})$ and $\boldsymbol{g}_k(\boldsymbol{y})$ are the kth-order vector homogeneous polynomials of $\boldsymbol{y}$.

To apply normal form theory, first define an operator as follows:

$$\begin{aligned} &L_k : \mathcal{H}_k \mapsto \mathcal{H}_k, \\ &U_k \in \mathcal{H}_k \mapsto L_k(U_k) = [U_k, \boldsymbol{v}_1] \in \mathcal{H}_k, \end{aligned} \tag{11.2.5}$$

where $\mathcal{H}_n$ denotes a linear vector space consisting of the kth-order vector homogeneous polynomials. The operator $[U_k, \boldsymbol{v}_1]$ is called Lie bracket, defined as

$$[U_k, \boldsymbol{v}_1] = D\boldsymbol{v}_1 \cdot U_k - DU_k \cdot \boldsymbol{v}_1. \tag{11.2.6}$$

Next, define the space $\mathcal{R}_k$ as the range of L_k, and the complementary space of $\mathcal{R}_k$ as $\mathcal{K}_k = Ker(L_k)$. Thus,

$$\mathcal{H}_k = \mathcal{R}_k \oplus \mathcal{K}_k, \tag{11.2.7}$$

and we can then choose bases for $\mathcal{R}_k$ and $\mathcal{K}_k$. Consequently, a vector homogeneous polynomial $\boldsymbol{f}_k \in \mathcal{H}_k$ can be split into two parts: one is spanned by the basis of $\mathcal{R}_k$ and the other by that of $\mathcal{K}_k$. Normal form theory shows that the part of $\boldsymbol{f}_k$ belonging to $\mathcal{R}_k$ can be eliminated while the part belonging to $\mathcal{K}_k$ must be retained, which is called normal form.

By applying the Takens normal form theory [378], one can find the kth-order normal form $\boldsymbol{g}_k(\boldsymbol{y})$, while the part belonging to $\mathcal{R}_k$ can be removed by appropriately choosing the coefficients of the nonlinear transformation $\boldsymbol{h}_k(\boldsymbol{y})$. The "form" of the normal form $\boldsymbol{g}_k(\boldsymbol{y})$ depends upon the basis of the complementary space $\mathcal{K}_k$, which is determined by the linear vector $\boldsymbol{v}_1$. We may apply the matrix method [146] to find the basis of the space $\mathcal{R}_k$ and then determine the basis of the complementary space $\mathcal{K}_k$. Once the basis of $\mathcal{K}_k$ is chosen, the form of $\boldsymbol{g}_k(\boldsymbol{y})$ can be determined, which actually represents the normal form.

In general, when one applies normal form theory to a system, one can find the "form" of the normal form (i.e., the basis of the complementary space $\mathcal{K}_k$), but not the explicit expressions. However, in practical applications, the solutions for the normal form and the nonlinear transformation need to be found explicitly. To achieve this, one may assume a general form of the nonlinear transformation and substitute it back into the original differential equation, with the aid of normal form theory, to obtain the kth-order algebraic equations by balancing the coefficients of the homogeneous polynomial terms. These algebraic equations are then used to determine the coefficients of the normal form and the nonlinear transformation. Thus, the key step in the computation of the kth-order normal form is to find the kth-order algebraic equations.

The following theorem [438] gives the recursive formula for computing the exact k-order algebraic equations, from which normal form and associated nonlinear transformation can be explicitly obtained.

THEOREM 11.2.1. *The recursive formula for computing the coefficients of the k-order normal form and nonlinear transformation is given by*

$$\begin{aligned}\boldsymbol{g}_k &= \boldsymbol{f}_k + [\boldsymbol{h}_k, \boldsymbol{v}_1] + \sum_{i=2}^{k-1} \bigl\{ [\boldsymbol{h}_{k-i+1}, \boldsymbol{f}_i] + D\boldsymbol{h}_i (\boldsymbol{f}_{k-i+1} - \boldsymbol{g}_{k-i+1}) \bigr\} \\ &\quad + \sum_{m=2}^{[\frac{k}{2}]} \frac{1}{m!} \sum_{i=m}^{k-m} D^m \boldsymbol{f}_i \sum_{\substack{l_1+l_2+\cdots+l_m j=k-(i-m) \\ 2\leqslant l_1,l_2,\ldots,l_m\leqslant k-(i-m)-2(m-1)}} \boldsymbol{h}_{l_1}\boldsymbol{h}_{l_2}\cdots\boldsymbol{h}_{l_m}\end{aligned} \tag{11.2.8}$$

for $k = 2, 3, \ldots,$ where $\boldsymbol{f}_k$, $\boldsymbol{h}_k$, and $\boldsymbol{g}_k$ are the kth-order vector homogeneous polynomials of $\mathbf{y}$ (where $\mathbf{y}$ has been dropped for simplicity). $\boldsymbol{f}_k$ represents the kth-order terms of the original system, $\boldsymbol{h}_k$ is the kth-order nonlinear transformation, and $\boldsymbol{g}_k$ denotes the kth-order normal form.

When the eigenvalues of a Jacobian involve one or more pure imaginary pairs, a complex analysis may simplify the solution procedure. It has actually been noted that the real analysis given in [421] yields the coupled algebraic equations, while it will be seen that the complex analysis can decouple the algebraic equations.

Thus, introduce the linear transformation:

$$\begin{cases} x_1 = \frac{1}{2}(z+\bar{z}), \\ x_2 = \frac{i}{2}(z-\bar{z}), \end{cases} \quad \text{i.e.,} \quad \begin{cases} z = x_1 - i x_2, \\ \bar{z} = x_1 + i x_2, \end{cases} \tag{11.2.9}$$

where i is the unit of imaginary number, satisfying $i^2 = -1$, and $\bar{z}$ is the complex conjugate of z. Then the linear part of system (11.2.1), $\boldsymbol{v}_1$, becomes

$$\boldsymbol{v}_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \partial x_1 = (iz - i\bar{z})^T.$$

Applying the transformation (11.2.9) into system (11.2.1) yields

$$\dot{z} = iz + f(z, \bar{z}), \qquad \dot{\bar{z}} = -iz + \overline{f}(z, \bar{z}), \tag{11.2.10}$$

where f is a polynomial in z and $\bar{z}$ starting from the second order terms, and $\overline{f}$ is the complex conjugate of f. Here, for convenience we use the same notation $\boldsymbol{f} = (f, \overline{f})^T$ for the complex analysis.

To find the normal form of Hopf singularity, one may assume a nonlinear transformation, given by

$$z = y + \sum h_k(y, \bar{y}), \qquad \bar{z} = \bar{y} + \sum \bar{h}_k(y, \bar{y}), \tag{11.2.11}$$

and determines the basis, $\boldsymbol{g}_k$, for the complementary space of $\mathcal{K}_k$, or employ Poincaré normal form theory to determine the so called "resonant" terms. It is well known that the "resonant" terms are given in the form of $z^j \bar{z}^{j-1}$ (e.g., see [146]), and the kth-order normal form is

$$\boldsymbol{g}_k(y, \bar{y}) = \begin{pmatrix} (b_{1k} + ib_{2k}) y^{(k+1)/2} \bar{y}^{(k-1)/2} \\ (b_{1k} - ib_{2k}) \bar{y}^{(k+1)/2} y^{(k-1)/2} \end{pmatrix}, \tag{11.2.12}$$

where b_{1k} and b_{2k} are real coefficients to be determined. It is obvious to see from equation (11.2.12) that the normal form contains odd order terms only, as expected. In normal form computation, the two kth-order coefficients b_{1k} and b_{2k} should be, in general, retained in the normal form.

Finally, based on equations (11.2.4), (11.2.10)–(11.2.12), one can use the algebraic equation (11.2.9) order by order starting from $k = 2$, and then apply normal form theory to solve for the coefficients b_{1k} (k is odd) explicitly in terms of the original system coefficients. Summarizing the above results gives the following theorem.

THEOREM 11.2.2. *For system (11.2.1) with L given by (11.2.2), the normal form is given by equation (11.2.12), where b_{1k} is the kth-order focus value.*

11.2.2. A perturbation method

In this subsection we present a perturbation technique based multiple time scales to compute the normal form associated with Hopf singularity [420]. This approach has been used to develop a unified approach to directly compute the normal forms of Hopf and degenerate Hopf bifurcations for general n-dimensional systems without the application of center manifold theory [420]. In the following, we briefly describe the perturbation approach.

Consider the general n-D differential equation:

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in R^n, \ \boldsymbol{f}: R^n \to R^n, \tag{11.2.13}$$

where $J\boldsymbol{x}$ represents the linear terms of the system, and the nonlinear function $\boldsymbol{f}$ is assumed to be analytic; and $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium point of the system, i.e., $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. Further, assume that the Jacobian of system (11.2.13) evaluated at the equilibrium point $\boldsymbol{0}$ contains a pair of purely imaginary eigenvalues $\pm i$, and thus the Jacobian of system (11.2.13) may be assumed in the Jordan canonical form:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \quad A \in R^{(n-2)\times(n-2)}, \tag{11.2.14}$$

where A is *stable* (i.e. all of its eigenvalues have negative real parts).

The basic idea of the perturbation technique based on multiple scales is as follows: Instead of a single time variable, multiple independent variables or scales are used in the expansion of the system response. To achieve this, introducing the new independent time variables $T_k = \epsilon^k t,\ k = 0, 1, 2, \ldots$, yields partial derivatives with respect to T_k as follows:

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial T_0}{\partial t}\frac{\partial}{\partial T_0} + \frac{\partial T_1}{\partial t}\frac{\partial}{\partial T_1} + \frac{\partial T_2}{\partial t}\frac{\partial}{\partial T_2} + \cdots \\ &= D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots \end{aligned} \tag{11.2.15}$$

where $D_k = \frac{\partial}{\partial T_k}$ denotes a differentiation operator. Then, assume that the solutions of system (11.2.13) in the neighborhood of $\boldsymbol{x} = \boldsymbol{0}$ are expanded in the series:

$$\boldsymbol{x}(t;\epsilon) = \epsilon \boldsymbol{x_1}(T_0, T_1, \ldots) + \epsilon^2 \boldsymbol{x_2}(T_0, T_1, \ldots) + \cdots. \tag{11.2.16}$$

Note that the same perturbation parameter, ϵ, is used in both the time and space scalings (see equations (11.2.15) and (11.2.16)). In other words, this perturbation approach treats time and space in a unified scaling.

Applying the formulas (11.2.15) and (11.2.16) into system (11.2.13), and solving the resulting ordered linear differential equations finally yields the normal form, given in polar coordinates (the detailed procedure can be found in (see [420]):

$$\begin{aligned} \frac{dr}{dt} &= \frac{\partial r}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial r}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots \\ &= D_0 r + D_1 r + D_2 r + \cdots, \end{aligned} \tag{11.2.17}$$

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial \phi}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial \phi}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial \phi}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots \\ &= D_0 \phi + D_1 \phi + D_2 \phi + \cdots, \end{aligned} \tag{11.2.18}$$

where $D_i r$ and $D_i \phi$ are uniquely determined, implying that the normal form given by equations (11.2.17) and (11.2.18) is unique. Further, it has been shown that [420] the derivatives $D_i r$ and $D_i \phi$ are functions of r only, and only $D_{2k} r$

and $D_{2k}\phi$ are nonzero, which can be expressed as $D_{2k}r = v_{2k+1}r^{2k+1}$ and $D_{2k}\phi = t_{2k+1}r^{2k}$, where both v_{2k+1} and t_{2k+1} are expressed in terms of the original system's coefficients. The coefficient v_{2k+1} is called the kth-order focus value of the Hopf-type critical point (the origin). Summarizing the above results gives the following theorem.

THEOREM 11.2.3. *For the general n-dimensional system (11.2.13), which has a Hopf-type singular point at the origin, i.e., the linearized system of (11.2.13) has a pair of purely imaginary eigenvalues and the remaining eigenvalues have negative real parts, the normal form for the Hopf or generalized Hopf bifurcations up to $(2k+1)$th order term is given by*

$$\dot{r} = r\big(v_1 + v_3r^2 + v_5r^4 + \cdots + v_{2k+1}r^{2k}\big), \tag{11.2.19}$$

$$\dot{\theta} = 1 + \dot{\phi} = 1 + t_3r^2 + t_5r^4 + \cdots + t_{2k+1}r^{2k}, \tag{11.2.20}$$

where the constants $v_{2k+1} = D_{2k}r/r^{2k+1}$ and $t_{2k+1} = D_{2k}\phi/r^{2k+1}$ are explicitly expressed in terms of the original system parameters, and $D_{2k}r$ and $D_{2k}\phi$ are obtained recursively using multiple time scales.

11.2.3. The singular point value method

This iterative method computes focus value via the computation of singular point quantities (see [62,290,63] for details).

To introduce this method, consider the planar polynomial differential system:

$$\dot{x} = \delta x - y + \sum_{k=2}^{\infty} X_k(x, y), \qquad \dot{y} = x + \delta y + \sum_{k=2}^{\infty} X_k(x, y), \tag{11.2.21}$$

where $X_k(x, y)$ and $Y_k(x, y)$ are homogeneous polynomials of x, y with degree k. The origin $(x, y) = (0, 0)$ is a singular point of system (11.2.21), which is either a focus or a center (when $\delta = 0$). Since we are interested in the computation of focus value, we assume $\delta = 0$ in the following analysis. Introducing the transformations, given by

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{11.2.22}$$

into system (11.2.21) results in

$$\begin{aligned} \frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z, w) = Z(z, w), \\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z, w) = -W(z, w), \end{aligned} \tag{11.2.23}$$

where z, w and T are complex variables, and

$$Z_k(z,w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta, \qquad W_k = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta. \tag{11.2.24}$$

System (11.2.21) and (11.2.23) are said to be concomitant.

If system (11.2.21) is a real planar, differential system, then the coefficients of system (11.2.23) must satisfy the conjugate conditions:

$$\overline{a_{\alpha\beta}} = b_{\alpha\beta}, \quad \alpha \geqslant 0,\ \beta \geqslant 0,\ \alpha+\beta \geqslant 2. \tag{11.2.25}$$

By the following transformations:

$$z = re^{i\theta}, \qquad w = re^{-i\theta}, \qquad T = it, \tag{11.2.26}$$

system (11.2.23) can be transformed into

$$\begin{aligned} \frac{dr}{dt} &= \frac{ir}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} - b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m, \\ \frac{d\theta}{dt} &= 1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} + b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m. \end{aligned} \tag{11.2.27}$$

For a complex constant h, $|h| \ll 1$, we may write the solution of (11.2.27) satisfying the initial condition $r|_{\theta=0} = h$ as

$$r = \widetilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} v_k(\theta) h^k. \tag{11.2.28}$$

Evidently, if system (11.2.21) is a real system, then $v_{2k+1}(2\pi)$ $(k = 1, 2, \ldots)$ is the kth-order focal (or focus) value of the origin.

For system (11.2.21), we can uniquely derive the following formal series:

$$\begin{aligned} \varphi(z,w) &= z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \\ \psi(z,w) &= w + \sum_{k+j=2}^{\infty} d_{k,j} w^k z^j, \end{aligned} \tag{11.2.29}$$

where

$$c_{k+1,k} = d_{k+1,k} = 0, \quad k = 1, 2, \ldots, \tag{11.2.30}$$

such that

$$\frac{d\varphi}{dt} = \varphi + \sum_{j=1}^{\infty} p_j \varphi^{j+1} \psi^j, \qquad \frac{d\psi}{dt} = -\psi - \sum_{j=1}^{\infty} q_j \psi^{j+1} \varphi^j. \tag{11.2.31}$$

Let $\mu_0 = 0,\ \mu_k = p_k - q_k,\ k = 1, 2, \ldots.\ \mu_k$, is called the kth-order singular point quantity of the origin of system (11.2.23) [62]. Based on the singular quantities, we can define the order of the singular critical point and extended center as follows.

If $\mu_0 = \mu_1 = \cdots = \mu_{k-1} = 0$ and $\mu_k \neq 0$, then the origin of system (11.2.23) is called the *kth-order weak critical singular point*. In other words, k is the multiplicity of the origin of system (11.2.23).

If $\mu_k = 0$ for $k = 1, 2, \ldots,$ then the origin of system (11.2.23) is called an *extended center* (*complex center*).

If system (11.2.21) is a real autonomous differential system with the concomitant system (11.2.23), then for the origin, the kth-order focus value v_{2k+1} of system (11.2.21) and the kth-order quantity of the singular point of system (11.2.23) have the relation [291,290], given in the following theorem.

THEOREM 11.2.4. *Given system* (11.2.21) ($\delta = 0$) *or* (11.2.23)*, for any positive integer m, the following assertion holds*:

$$v_{2k+1}(2\pi) = i\pi\left(\mu_k + \sum_{j=1}^{k-1} \xi_k^{(j)} \mu_j\right), \quad k = 1, 2, \ldots, \tag{11.2.32}$$

where $\xi_m^{(j)}$ $(j = 1, 2, \ldots, k-1)$ *are polynomial functions of coefficients of system* (11.2.23).

Furthermore, we have the following results [62].

THEOREM 11.2.5. *The recursive formulas for computing the singular point quantities of system* (11.2.23) *is given by* $c_{11} = 1,\ c_{20} = c_{02} = c_{kk} = 0,\ k = 2, 3, \ldots,$ *and* $\forall(\alpha, \beta),\ \alpha \neq \beta$*, and* $m \geqslant 1$:

$$C_{\alpha\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} \big[(\alpha - k + 1)a_{k,j-1} - (\beta - j + 1)b_{j,k-1}\big] \times C_{\alpha-k+1,\beta-j+1} \tag{11.2.33}$$

and

$$\mu_m = \sum_{k+j=3}^{2m+4} \big[(m - k + 2)a_{k,j-1} - (m - j + 2)b_{j,k-1}\big] \times C_{m-k+2,m-j+2}, \tag{11.2.34}$$

where $a_{kj} = b_{kj} = C_{kj} = 0$ *for* $k < 0$ *or* $j < 0$.

It is clearly seen from equation (11.2.32) that

$$\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0 \quad \Longleftrightarrow \quad v_3 = v_5 = \cdots = v_{2k-1} = 0.$$

Therefore, when determining the conditions such that $v_1 = v_2 = \cdots = v_{k-1} = 0$, one can instead use the equations: $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$. If the μ_k's are simpler than the v_k's then this method is better than the method of directly computing v_k. However, in general such μ_k are not necessarily simpler than v_k. We shall see this in the next two subsections.

It should be pointed out that since the normal form (focus value) is not unique, the focus values obtained using different methods are not necessarily the same. However, the first nonzero focus value must be identical (ignoring a constant multiplier). This implies that for different focus values obtained using different approaches, the solution to the equations $v_3 = v_5 = \cdots = v_{2k-1} = 0$ (or $\mu_1 = \mu_3 = \cdots = \mu_{k-1} = 0$) must be identical.

For the three methods described above, symbolic programs have been developed using Maple, which will be used in the following two subsections.

11.2.4. Applications

The Brusselator model

We first use the well-known Brusselator model to consider the application of the three methods described in the previous subsection. The model is described by [325]

$$\begin{aligned} \dot{w}_1 &= A - (1 + B)w_1 + w_1^2 w_2, \\ \dot{w}_2 &= Bw_1 - w_1^2 w_2, \end{aligned} \tag{11.2.35}$$

where $A, B > 0$ are parameters. The system has a unique equilibrium point:

$$w_{1e} = A, \qquad w_{2e} = \frac{B}{A}. \tag{11.2.36}$$

Evaluating the Jacobian of system (11.2.35) at the equilibrium point (11.2.36) shows that a Hopf bifurcation occurs at the critical point $B = 1 + A^2$. Letting

$$B = 1 + A^2 + \mu, \tag{11.2.37}$$

where μ is a perturbation parameter, then the Jacobian has eigenvalues $\lambda = \pm Ai$. Suppose $A = 1$, and then introduce the transformation: $w_1 = w_{1e} + x_1$, $w_2 = w_{2e} - x_1 + x_2$ into (11.2.35) to obtain the new system:

$$\begin{aligned} \dot{x}_1 &= x_2 + \mu x_1 + \mu x_1^2 + 2x_1 x_2 - x_1^3 + x_1^2 x_2, \\ \dot{x}_2 &= -x_1. \end{aligned} \tag{11.2.38}$$

Now at the critical point defined by $\mu = 0$, we apply the three methods described in the previous subsection to compute the first-order focus value. The Maple programs are employed to obtain the following results.

The Takens method: $b_{13} = -\frac{3}{8}$;
The perturbation method: $v_3 = -\frac{3}{8}$;
The singular point value method: $\mu_1 = \frac{3}{4}i$.

It is seen that $b_{13} = v_3 = \frac{i}{2}\mu_1$. Ignoring the constant factor $\frac{i}{2}$, the three methods give the identical result for the first-order focus value: $-\frac{3}{8}$. This shows that the limit cycles bifurcating from the critical point $\mu = 0$ in the vicinity of the equilibrium point (w_{1c}, w_{2c}) is supercritical, i.e., the bifurcating limit cycles are stable since the first-order focus value is negative.

Further, computing the second order focus value yields

The Takens method: $b_{15} = -\frac{1}{96}$;
The perturbation method: $v_5 = -\frac{1}{96}$;
The singular point value method: $\mu_2 = -\frac{67}{48}i$.

This indicates that the Takens method and the perturbation method still give same second-order focus value, but the singular point value method yields a different μ_2. This is not surprising since the second-order singular point value is a combination of μ_1 and μ_2.

One more further step computation shows that

The Takens method: $b_{17} = -\frac{2695}{36864}$;
The perturbation method: $v_5 = -\frac{4543}{36864}$;
The singular point value method: $\mu_2 = \frac{6239}{2304}i$.

For the third-order focus value, even the Takens method and the perturbation method give different answers.

Numerical simulation results obtained from the original system (11.2.35) are shown in Figure 11.2.1. This figure clearly indicates that when $A = 1.0$, $B = 1.95$, the trajectory converges to the stable equilibrium point $(w_{1c}, w_{2c}) = (1, 1.95)$ (see Figure 11.2.1(a)); while when $A = 1.0$, $B = 2.05$, the equilibrium point becomes unstable and a stable limit cycle bifurcates from the equilibrium point (see Figure 11.2.1(b)).

The induction machine model

In this subsection, we present a model of induction machine to demonstrate the application of the results obtained in the previous subsections. The model is based on the one discussed in [212] and the same notations are adopted here. Since in this chapter we are mainly interested in the application of focus value computation, we will not give the detailed derivation of the model.

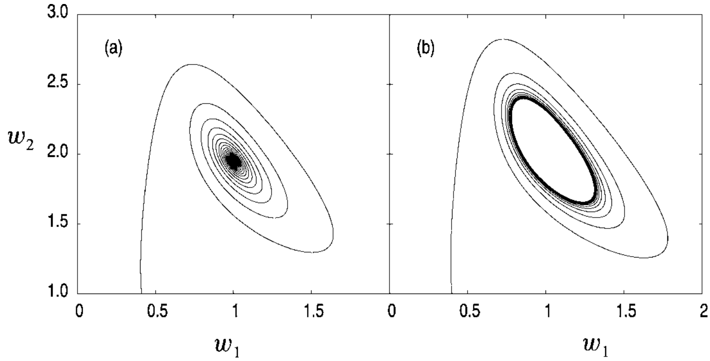


Figure 11.2.1. Simulated trajectories of the Brusselator model (11.2.35) for $A = 1.0$ with the initial point $(w_1, w_2) = (2, -1)$: (a) convergent to the stable equilibrium point $\boldsymbol{w}^+$ when $B = 1.95$; and (b) convergent to a stable limit cycle when $B = 2.05$.

An induction machine (or asynchronous machine) is one of the electrical machines which is widely used in industry. The behavior of induction machine was studied for years, but the main attention has been focused on steady state solutions due to the complexity of the model (even with simplifying assumptions). In order to study dynamical behavior of the model such as instability and bifurcations, it needs to determine the conditions of the bifurcation (critical) points.

The model is described by a system of seven ordinary differential equations, given by

$$\begin{aligned}
\dot{\phi}_{qs} &= \omega_b \left\{ u_q - \phi_{ds} + \frac{r_s}{X_{1s}} \left[X_{aq} \left(\frac{\phi_{qs}}{X_{1s}} + \frac{\phi'_{qr}}{X'_{1r}} \right) - \phi_{qs} \right] \right\}, \\
\dot{\phi}_{ds} &= \omega_b \left\{ u_d + \phi_{qs} + \frac{r_s}{X_{1s}} \left[X_{aq} \left(\frac{\phi_{ds}}{X_{1s}} + \frac{\phi'_{dr}}{X'_{1r}} \right) - \phi_{ds} \right] \right\}, \\
\dot{\phi}_{0s} &= \omega_b \left\{ \frac{r_s}{X_{1s}} (-\phi_{0s}) \right\}, \\
\dot{\phi}'_{qr} &= \omega_b \left\{ -(1-\omega_r)\phi'_{dr} + \frac{r'_r}{X'_{1r}} \left[X_{aq} \left(\frac{\phi_{qs}}{X_{1s}} + \frac{\phi'_{qr}}{X'_{1r}} \right) - \phi'_{qr} \right] \right\}, \\
\dot{\phi}'_{dr} &= \omega_b \left\{ (1-\omega_r)\phi'_{qr} + \frac{r'_r}{X'_{1r}} \left[X_{aq} \left(\frac{\phi_{ds}}{X_{1s}} + \frac{\phi'_{dr}}{X'_{1r}} \right) - \phi'_{dr} \right] \right\}, \\
\dot{\phi}_{0r} &= \omega_b \left\{ \frac{r'_r}{X'_{1r}} (-\phi'_{0r}) \right\}, \\
\dot{\omega}'_r &= \frac{1}{2H} \left\{ \frac{X_{ad}}{X_{1s} X'_{1r}} \left(\phi_{qs}\phi'_{dr} - \phi_{ds}\phi'_{qr} \right) - T_L \right\},
\end{aligned} \tag{11.2.39}$$

where, except for the state variables, all variables denote the system parameters. Letting

$$w_1 = \phi_{qs}, \quad w_2 = \phi_{ds}, \quad w_3 = \phi_{0s}, \quad w_4 = \phi'_{qr}, \quad w_5 = \phi'_{dr},$$
$$w_6 = \phi'_{0r}, \quad w_7 = \omega_r,$$

and substituting proper parameter values to equation (11.2.39) finally yields a model of a 3hp induction machine as $\dot{\boldsymbol{w}} = \boldsymbol{f}(\boldsymbol{w}, V)$:

$$\begin{aligned}
\dot{w}_1 &= -\frac{3}{10}w_1 - w_2 + \frac{3}{10}w_4 + V,\\
\dot{w}_2 &= w_1 - \frac{3}{10}w_2 + \frac{3}{10}w_5,\\
\dot{w}_3 &= -\frac{3}{5}w_3,\\
\dot{w}_4 &= \frac{1}{2}w_1 - \frac{1}{2}w_4 - w_5 + w_5w_7,\\
\dot{w}_5 &= \frac{1}{2}w_2 + w_4 - \frac{1}{2}w_5 - w_4w_7,\\
\dot{w}_6 &= -w_6,\\
\dot{w}_7 &= \frac{7}{120\pi^3}(14w_1w_5 - 14w_2w_4 - 1),
\end{aligned} \tag{11.2.40}$$

where $V > 0$ is a bifurcation parameter, representing the input voltage of the motor.

Setting $\dot{w}_i = 0,\ i = 1, 2, \ldots, 7$, results in two equilibrium solutions (fixed points):

$$\begin{aligned}
w_1^{\pm} &= -\frac{3(-350V^2 + 15 \pm 10S)}{7630V},\\
w_2^{\pm} &= \frac{7315V^2 - 150 \pm 9S}{7360V},\\
w_3^{\pm} &= 0,\\
w_4^{\pm} &= -\frac{1}{14V},\\
w_5^{\pm} &= \frac{35V^2 \pm S}{70V},\\
w_6^{\pm} &= 0,\\
w_7^{\pm} &= \frac{-350V^2 + 124 \pm 10S}{109},
\end{aligned} \tag{11.2.41}$$

where $S = \sqrt{1225V^4 - 105V^2 - 25}$, indicating that the equilibrium solutions exist when $V^2 \geqslant \frac{3+\sqrt{109}}{70}$. The Jacobian of equation (11.2.40) is given by

$$J(\boldsymbol{w}) = \begin{bmatrix} -\frac{3}{10} & -1 & 0 & \frac{3}{10} & 0 & 0 & 0 \\ 1 & -\frac{3}{10} & 0 & 0 & \frac{3}{10} & 0 & 0 \\ 0 & 0 & -\frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 + w_7 & 0 & w_5 \\ 0 & \frac{1}{2} & 0 & 1 - w_7 & -\frac{1}{2} & 0 & -w_4 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{49w_5}{60\pi^3} & -\frac{49w_4}{60\pi^3} & 0 & -\frac{49w_2}{60\pi^3} & \frac{49w_1}{60\pi^3} & 0 & 0 \end{bmatrix}. \tag{11.2.42}$$

The conditions for various singularities of the system have been obtained in [428], but the stability of bifurcating limit cycles was not discussed. Here, we will consider the stability of limit cycles generated from a Hopf bifurcation for which the Jacobian of (11.2.42) has a pair of purely imaginary eigenvalues, which requires that $V \geqslant V_0 = ((3+\sqrt{109})/70)^{1/2} \approx 0.4381830425$ [428]. Since $\boldsymbol{w}^-$ is always unstable when $V > V_0$, we only consider $\boldsymbol{w}^+$ which can be shown to be stable when $V \in (0.4381830425, 6.2395593195) \cup (7.75369242394, \infty)$ and unstable when $V \in (6.2395593195, 7.35369242394)$. The point $V_0 = 0.4381830425$ is a static critical point. Furthermore, we can employ the criterion given in [428] to show that

$$V_{h1} = 6.2395593195 \quad \text{and} \quad V_{h2} = 7.35369242394 \tag{11.2.43}$$

are two solutions at which Hopf bifurcations occur. When $V = V_{h1}$, the eigenvalues of $J(\boldsymbol{x})$ are:

$$\pm 0.7905733366i, \quad -1, \quad -0.6, \quad -0.5630004665,$$
$$-0.5184997667 \pm 1.0893171380i,$$

at which the equilibrium solution $\boldsymbol{w}^+$ becomes (see equation (11.2.41))

$$\begin{aligned} &w_1^+ = 0.0000063079, \qquad w_2^+ = 6.2361231187, \\ &w_4^+ = -0.0114476949, \\ &w_5^+ = 6.2361020924, \qquad w_7^+ = 0.9990816376, \\ &w_3^+ = w_6^+ = 0. \end{aligned} \tag{11.2.44}$$

To find the focus values associated with the Hopf critical point $V_{h1} = 6.2395593195$, introduce the following transformation:

$$\boldsymbol{w} = \boldsymbol{w}^+ + T\boldsymbol{x}, \tag{11.2.45}$$

where T is given by

$$T = \begin{bmatrix} 0.10862467 & 0.46795879 & 0 & 0 & -0.34351003 & -0.06005005 & 0.41854500 \\ 0.50584433 & -0.00073468 & 0 & 0 & -0.16247093 & 0.33909080 & -0.01044227 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.56158663 & 0.75176240 & 0 & 0 & -0.24042543 & -0.34572171 & -0.55769267 \\ 0.14569817 & -0.22757384 & 0 & 0 & 1.28746653 & -0.00888757 & -0.15628647 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.05896448 & 0.09391575 & 0 & 0 & 0.03016055 & 0.10325630 & -0.09231719 \end{bmatrix}$$

under which the transformed system is given in the canonical form:

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}_2(\boldsymbol{x}), \tag{11.2.46}$$

where J is

$$J = \begin{bmatrix} 0 & 0.79057334 & 0 & 0 & 0 & 0 & 0 \\ -0.79057334 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.56300047 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.51849977 & 1.08931714 \\ 0 & 0 & 0 & 0 & 0 & -1.08931714 & -0.51849979 \end{bmatrix}.$$

Then applying the Maple program [420] results in the following focus values:

$$v_3 = -0.17753379 \times 10^{-2}, \qquad v_5 = -0.93206291 \times 10^{-5},$$
$$v_7 = -0.15369758 \times 10^{-6}, \tag{11.2.47}$$

which indicates that the family of limit cycles bifurcating from the critical point V_{h1} in the neighborhood of $\boldsymbol{w}^+$ is stable.

Simulation results for this example using system (11.2.40) for $V = 6.0$ and $V = 6.5$ are depicted in Figure 11.2.2. The initial point is chosen as

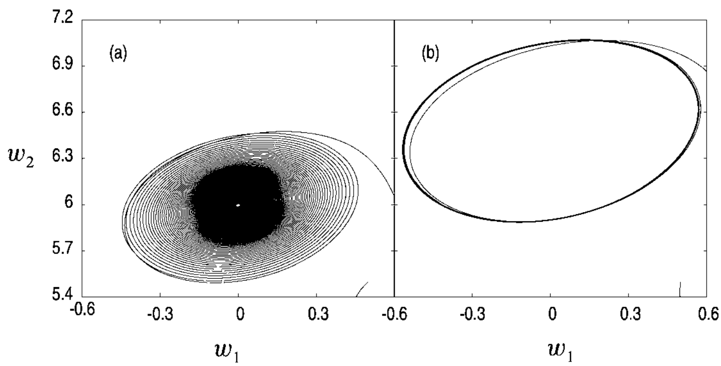


Figure 11.2.2. Simulated trajectories of the induction machine model (11.2.40) projected on the $w_1 - w_2$ plane with the initial point $\boldsymbol{w}_0 = (0.5, 5.5, 2.0, -3.0, 1.0, 4.0, -5.0)^T$: (a) convergent to the stable equilibrium point $\boldsymbol{w}^+ = (0.0000070946,\ 5.9964264430,\ 0,\ -0.0119047619,\ 5.9964027942,\ 0,\ 0.9990067495)^T$ when $V = 6.0$; and (b) convergent to a stable limit cycle when $V = 6.5$.

$\boldsymbol{w}_0 = (0.5,\ 5.5,\ 2.0,\ -3.0,\ 1.0,\ 4.0,\ -5.0)^T$. It can be seen from this figure, as expected, that when $V = 6.0 < V_{h1}$ the trajectory converges to the stable equilibrium point:

$$\boldsymbol{w}^+ = (0.00000709,\ 5.99642644,\ 0,\ -0.01190476,\\ 5.99640279,\ 0,\ 0.99900675)^T,$$

as shown in Figure 11.2.2(a). When $V = 6.5 > V_{h1}$, the equilibrium point becomes unstable and a supercritical Hopf bifurcation occurs, giving rise to a stable limit cycle (see Figure 11.2.2(b)).

It should be pointed out that the perturbation method can be applied to the 7-D system (11.2.46) without employing center manifold theory (more precisely, the center manifold theory is incorporated in the unified approach), while the Takens method and the singular point value method cannot be directly applied to system (11.2.46).

Note that in the above two examples, except for the bifurcation parameters, all system parameters are fixed. In the next section, we will consider a system with free parameters and want to find maximal number of limit cycles by appropriately choosing parameter values.

Hilbert's 16th problem

The well-known 23 mathematical problems proposed by Hilbert in 1990 [167] have significant impact on the mathematics of the 20th century. Two of the 23 problems remain unsolved, one of them is the 16th problem. This problem includes two parts: the first part studies the relative positions of separate branches of algebraic curves, while the second part considers the upper bound of the number of limit cycles and their relative locations in polynomial vector fields. Generally, the second part of this problem is what usually meant when talking about Hilbert's 16th problem. The recent developments on Hilbert's 16th problem may be found in the survey articles [227,429]. A simplified version—the Liénard equation—of the second part of Hilbert's 16th problem has been recently chosen by Smale [364] as one of the 18 most challenging mathematical problems for the 21st century. Although it is still far away from completely solving the problem, the research on this problem has made great progress with contributions to the development of modern mathematics.

Roughly speaking, the second part of Hilbert's 16th problem is to consider the planar vector fields, described by the following polynomial differential equations:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{11.2.48}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent the nth-degree polynomials of x and y. The problem is to find the upper bound, known as the Hilbert number $H(n)$, on the number of limit cycles that the system can have. In general, this is a very difficult technical problem, in particular, for determining global (large) limit cycles. Although it has not been possible to obtain a precise number for $H(n)$, a great deal of efforts have been made in finding the maximal number of limit cycles and raising the lower bound of Hilbert number $H(n)$ for general planar polynomial systems of certain degree, hoping to be close to the real upper bound of $H(n)$.

If the problem is restricted to a neighborhood of isolated fixed points, then the question reduces to studying degenerate Hopf bifurcations, which gives rise to fine focus points. Alternatively, this is equivalent to computing the normal form of differential equations associated with Hopf or degenerate Hopf bifurcation. Suppose the normal form associated with this Hopf singularity is given in polar coordinates (obtained using, say, the method given in [420]) described by equation (11.2.19). The basic idea of finding k small limit cycles around the origin is as follows: First, find the conditions such that $v_1 = v_3 = \cdots = v_{2k-1} = 0$, but $v_{2k+1} \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. In 1952 Bautin [22] proved that a quadratic planar polynomial vector field can have maximal 3 small limit cycles, i.e., $H(2) \geqslant 3$. Later, it was shown that $H(2) \geqslant 4$. For cubic systems, the best result obtained so far is $H(3) \geqslant 12$ [424,431,432].

In the following, we will particularly show that a cubic-order, Z_2-equivariant vector field can have 12 limit cycles. To achieve this, we apply the standard complex formula [227]:

$$\dot{z} = F_2(z, \bar{z}), \qquad \dot{\bar{z}} = \overline{F}_2(z, \bar{z}), \tag{11.2.49}$$

where

$$F_2(z, \bar{z}) = \left(A_0 + A_1|z|^2\right)z + \left(A_2 + A_3|z|^2\right)\bar{z} + A_4 z^3 + A_5 \bar{z}^3. \tag{11.2.50}$$

Let $z = w_1 + i w_2$, $\bar{z} = w_1 - i w_2$, $A_j = a_j + i b_j$, where w_1, w_2 and a_j, b_j are all real. Then system (11.2.49) is transformed to the following real form:

$$\begin{aligned}
\dot{w}_1 &= (a_0 + a_2)w_1 - (b_0 - b_2)w_2 + (a_1 + a_3 + a_4 + a_5)w_1^3 \\
&\quad - (b_1 - b_3 + 3b_4 - 3b_5)w_1^2 w_2 + (a_1 + a_3 - 3a_4 - 3a_5)w_1 w_2^2 \\
&\quad - (b_1 - b_3 - b_4 + b_5)w_2^3, \\
\dot{w}_2 &= (b_0 + b_2)w_1 + (a_0 - a_2)w_2 + (b_1 + b_3 + b_4 + b_5)w_1^3 \\
&\quad + (a_1 - a_3 + 3a_4 - 3a_5)w_1^2 w_2 + (b_1 + b_3 - 3b_4 - 3b_5)w_1 w_2^2 \\
&\quad + (a_1 - a_3 - a_4 + a_5)w_2^3.
\end{aligned} \tag{11.2.51}$$

The two eigenvalues of the Jacobian of system (11.2.51) evaluated at the origin $(w_1, w_2) = (0, 0)$ are $a_0 \pm \sqrt{a_2^2 + b_2^2 - b_0^2}$, indicating that the origin $(0, 0)$ is a saddle point or a node when $a_2^2 + b_2^2 - b_0^2 \geqslant 0$, or a focus point or a center if $a_2^2 + b_2^2 - b_0^2 < 0$. When $a_2^2 + b_2^2 - b_0^2 = 0$, the origin is either a node or a double zero singular point. By a parametric transformation, rename the coefficients of the resulting system to yield the following system:

$$\dot{x} = ax + by + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$
$$\dot{y} = \pm bx + ay + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \tag{11.2.52}$$

where $b > 0$.

For a vector field with Z_2-symmetry, naturally the best way is to have two symmetric focus points about the origin. Thus, if N small limit cycles are found in the neighborhood of one focus point, the whole system would have $2N$ limit cycles. Without loss of generality, the two symmetric focus points can be assumed on the y-axis (or the x-axis), and further assumed to be located at $(0, \pm 1)$ with a proper scaling, leading to the conditions $a_{03} = -b$ and $b_{03} = -a$. Another condition $a_{12} = a$ comes from making the two focus points be Hopf type. Furthermore, by applying proper parameter scaling and time scaling, we obtain the following new system [432]:

$$\begin{aligned}\frac{du}{d\tau} &= v + 2\bar{a}_{21}u^2 + 4auv - \frac{3}{2}v^2 + 4b\bar{a}_{30}u^3 - 2\bar{a}_{21}u^2v - 2auv^2 + \frac{1}{2}v^3,\\ \frac{dv}{d\tau} &= -u - 4\bar{b}_{21}u^2 + 2\big(2a^2 \mp 2b^2 + 1\big)uv - 8\bar{b}_{30}u^3 + 4\bar{b}_{21}u^2v\\ &\quad - \big(2a^2 \mp b^2 + 1\big)uv^2,\end{aligned} \tag{11.2.53}$$

where the coefficients $\bar{a}_{21}, \bar{b}_{21}, \bar{a}_{30}$ and $\bar{b}_{30}$ are expressed in terms of the original parameters $a, b, a_{21}, b_{21}, a_{30}, b_{30}$. Thus, based on equation (11.2.53), we can compute focus values and consider the existence of small limit cycles.

Note that system (11.2.53) contains 6 free parameters, which suggests that we may set 6 focus values zero and obtain 7 small limit cycles for system (11.2.53), and therefore, the original system may have 14 small limit cycles. However, it has been shown in [432] that the existence of 14 limit cycles is not possible. The maximal number of small limit cycles that a cubic-order system with Z_2 symmetry can have is 12.

We now apply the three methods given in the previous subsection to compute the focus values of system (11.2.53) to obtain the following results:

```
With the Takens method:
b13:=1/2*a-2*a^2*b21b-2*a21b*b21b-a21b*a+2*b^2*b21b+3/2*b*a30b
     -1/2*b21b:
b15:=-11/36*a-7/3*a*a21b*b^2-25/9*b21b*a21b*a^2-1/6*a21b*b*a30b
     +92/9*b21b*b^2*a^2-11/9*b21b*a21b*b^2-8/9*b21b*a21b^2
     -14/3*b21b*a^4-50/9*b21b*b^4+11/9*a*a21b^2+2/3*b^3*a30b
     +11/36*b21b+1/9*a^3*a21b-5/3*a*b30b+2*b21b^2*a+23/3*b21b*b30b
     -2/9*a*b^4+4/9*a^3*b^2-2/9*a^5-40/9*b21b^3+152/9*a^3*b21b^2
     -2/3*b^5*a30b+40/9*b^6*b21b-160/9*a^2*b21b^3-16/3*a^3*b30b
     +2/9*a21b^2*a^3-1/9*a^3-6*b21b*a*b*a30b-8*a21b*b21b*b^2*a^2
     -1/3*a21b*a^2*b*a30b-8/9*a21b*a^3*b^2+4*a21b*b21b*a^4
     +4*a21b*b21b*b^4-2/9*a21b^2*b^2*a-4*a21b^2*b^2*b21b
     +4*a21b^2*a^2*b21b-2/3*a^4*b*a30b-152/9*b^2*b21b^2*a
```

```
     +4/3*a^2*b^3*a30b+40/3*b21b^2*b*a30b+10/3*a21b^2*b*a30b
     +8*a21b*b21b^2*a+4/9*a21b*a*b^4+52/3*a^2*b21b*b30b
     -40/3*a^2*b21b*b^4-52/3*b^2*b21b*b30b+16/3*b^2*a*b30b
     +40/3*b^2*b21b*a^4-3*b*a30b*b30b+1/3*a21b*b^3*a30b
     +52/3*a21b*b21b*b30b-10/3*a21b*a*b30b+23/18*a21b*b21b
     -11/12*b*a30b+11/9*b^2*a+5/9*b^2*b21b+5/9*a21b*a+7/9*a^2*b21b
     -40/9*a21b^3*b21b-20/9*a21b^3*a+4/9*a21b*a^5-40/9*a^6*b21b
     +160/9*b^2*b21b^3-160/9*a21b*b21b^3:
b17 := ...   (87 lines)
b19 := ...  (355 lines)
v111:= ... (1180 lines)

With the perturbation method:
v3 :=-2*a21b*b21b-a21b*a+1/2*a-1/2*b21b+3/2*b*a30b+2*b21b*b^2
     -2*b21b*a^2:
v5 :=4*a21b^2*b21b*a^2-7/3*a21b*a*b^2-8/9*a21b*b^2*a^3
     -11/9*a21b*b21b*b^2-2/9*a21b^2*a*b^2-11/36*a+16/3*a*b^2*b30b
     +4/3*b^3*a30b*a^2-25/9*a21b*b21b*a^2-4*a21b^2*b21b*b^2
     -1/6*a21b*b*a30b+40/3*b*a30b*b21b^2-152/9*a*b21b^2*b^2
     -10/3*a21b*a*b30b-40/3*b21b*b^4*a^2+4/9*a21b*a*b^4
     -2/3*b*a30b*a^4+92/9*b21b*b^2*a^2-3*b*a30b*b30b
     -11/12*b*a30b+11/36*b21b+4*a21b*b21b*b^4+4*a21b*b21b*a^4
     -52/3*b21b*b^2*b30b+52/3*a21b*b21b*b30b+52/3*b21b*a^2*b30b
     +1/3*a21b*b^3*a30b+40/3*b21b*b^2*a^4+160/9*b21b^3*b^2
     -16/3*a^3*b30b+23/3*b21b*b30b+11/9*a*b^2+4/9*b^2*a^3+5/9*a21b*a
     -40/9*a21b^3*b21b-20/9*a21b^3*a+40/9*b21b*b^6-40/9*b21b*a^6
     +152/9*b21b^2*a^3+1/9*a21b*a^3-5/3*a*b30b+2*a*b21b^2
     -8/9*a21b^2*b21b+11/9*a21b^2*a+7/9*b21b*a^2+5/9*b21b*b^2
     -2/3*b^5*a30b-160/9*a21b*b21b^3-160/9*b21b^3*a^2+2/9*a21b^2*a^3
     +2/3*b^3*a30b+4/9*a21b*a^5-2/9*a*b^4-14/3*b21b*a^4-50/9*b21b*b^4
     +23/18*a21b*b21b-1/3*a21b*b*a30b*a^2-8*a21b*b21b*b^2*a^2
     -6*a*b*a30b*b21b+10/3*a21b^2*b*a30b+8*a21b*a*b21b^2-40/9*b21b^3
     -1/9*a^3-2/9*a^5:
v7 := ...   (83 lines)
v9 := ...  (344 lines)
v11:= ... (1173 lines)

With the singular point value method:
mu1:=I*(-3*b*a30b-a+b21b+4*b21b*a^2-4*b21b*b^2+4*a21b*b21b+2*a21b*a):
mu2:=-1/48*I*(736*b21b*b30b-17*a+1664*a21b*b21b*b30b+264*a21b*b*a30b
     -960*b21b^2*b*a30b-2544*a21b^2*b*a30b-352*a^3*a21b+64*b^2*a
     +1152*a^4*b21b-96*b^3*a30b-384*b^4*b21b-160*a*b30b
     -320*a*a21b*b30b+1664*a^2*b21b*b30b-1664*b^2*b21b*b30b
     -928*a^2*b*a30b-768*b^2*b21b*a^2+2976*b^3*a21b*a30b
     +864*b^2*a*a21b-384*b^4*a+768*b^2*a^3-1152*b^5*a30b+17*b21b
     -51*b*a30b-1152*a^4*b*a30b-1664*b21b^2*a+496*a21b^2*b21b
     -1024*a21b^2*a-20*a21b*b21b+122*a*a21b+1444*a^2*b21b
     +156*b^2*b21b+3392*a21b^3*b21b+1280*a^2*b21b^3+1280*b21b^3*a21b
     -1024*b^6*b21b+1024*a^6*b21b+768*a^5*a21b-2816*a^3*b21b^2
     +1984*a^3*a21b^2-512*a^3*b30b+1696*a*a21b^3-1280*b^2*b21b^3
     +2304*b^3*a30b*a^2-320*a^3-384*a^5+5760*a^4*a21b*b21b
     +512*a*b30b*b^2+5760*b^4*a21b*b21b+320*b21b^3+768*b^4*a*a21b

     -1536*b^2*a^3*a21b-3072*b^2*a^4*b21b+3072*b^4*a^2*b21b
     -512*b^2*a21b*b21b-288*b*a30b*b30b-11520*b^2*a^2*a21b*b21b
     +2752*a*b21b*b*a30b-2976*a^2*a21b*b*a30b-640*a^2*a21b*b21b
     -1984*b^2*a*a21b^2-8128*b^2*a21b^2*b21b-2176*a*b21b^2*a21b
     +8128*a^2*b21b*a21b^2+2816*b^2*a*b21b^2):
mu3:= ...   (85 lines)
mu4:= ...  (355 lines)
mu5:= ... (1156 lines)
```

The numbers given in the brackets denote the number of lines in the computer output files.

It is easy to see that

$$b_{13} = v_3 = \frac{i}{2}\mu_1,$$

which shows that the three different methods give the same first-order focus value (at most by a difference of a constant fact), as expected. For the second-order focus values, it can be shown that $b_{15} = v_5$, but $v_5 \neq \mu_2$ (within a difference of a constant fact). Further, for the third-order focus values, $b_{17} \neq v_7$. However, setting $b_{13} = v_3 = \mu_1 = 0$ results in

$$\bar{a}_{30} = \frac{a(2\bar{a}_{21} - 1) + \bar{b}_{21}(4\bar{a}_{21} + 4a^2 - 4b^2 + 1)}{3b},$$

and then

$$b_{15} = v_5 = \frac{i}{2}\mu_2.$$

Further letting $b_{15} = v_5 = \mu_2 = 0$ (from which one can obtain $\bar{b}_{30} = \bar{b}_{30}(a, b, \bar{a}_{21}, \bar{b}_{21})$) yields

$$b_{17} = v_7 = \frac{i}{2}\mu_3.$$

This process can be carried out to higher-order focus values, i.e., when $b_{1(2i+1)} = v_{2i+1} = \mu_i = 0,\ i = 1, 2, \ldots, k-1$, we have

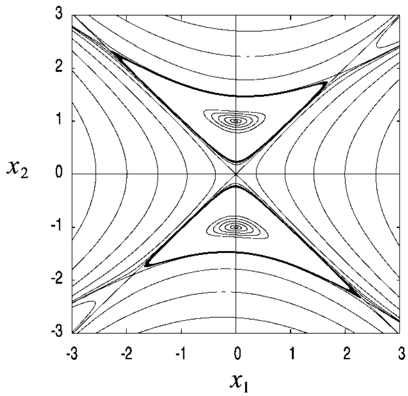
$$b_{1(2k+1)} = v_{2k+1} = \frac{i}{2}\mu_k.$$

For generic case, it has been shown [432] that one can find the following parameter values such that $v_i = 0,\ i = 1, 2, \ldots, 5$, but $v_6 \neq 0$; there are two solutions for which the origin is a saddle point:

$$\begin{aligned}
b &= \pm 15.7264394069a,\\
\bar{b}_{21} &= -1.1061229255a,\\
\bar{a}_{21} &= 0.7000000000 + 103.3880431509a^2,\\
\bar{b}_{30} &= \frac{0.2564102564a^2(0.0089196607 + 0.0982810312a^2 + 0.2527227926a^4)}{0.0661528794 + 0.2704669566a^2},\\
\bar{a}_{30} &= \mp\bigl(0.0806130156 - 17.7870588470a^2\bigr),
\end{aligned} \tag{11.2.54}$$

and other two solutions for which the origin is a node:

$$\begin{aligned}
b &= \pm 0.4765747114a,\\
\bar{b}_{21} &= 0.2033343806a,\\
\bar{a}_{21} &= 0.7000000000 + 1.0149654014a^2,
\end{aligned}$$



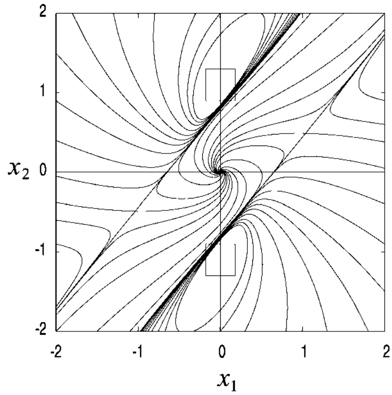


Figure 11.2.3. The phase portraits of system (11.2.52) having 12 limit cycles: (a) when the origin is a saddle point for $a = a_{12} = -b_{03} = -0.0883176873$, $b = -a_{03} = 1.3898788398$, $a_{30} = 0.0282840060$, $a_{21} = 1.1005496998$, $b_{30} = 0.7230834412$, $b_{21} = -0.1483513686$, $b_{12} = -1.0189112324$; and (b) when the origin is a focus point for $a = a_{12} = -b_{03} = -0.7$, $b = -a_{03} = -0.5889218635$, $a_{30} = 2.2161956860$, $a_{21} = -2.5682071892$, $b_{30} = -0.7072960219$, $b_{21} = 2.2961669830$, $b_{12} = -3.1019886923$, where the two square boxes denote the locations of the 12 limit cycles.

$$\bar{b}_{30} = \frac{a^2(0.0481488581 + 65.9546167690a^2 - 9379.2591506305a^4)}{0.0008286738 - 0.1076372236a^2},$$
$$\bar{a}_{30} = \pm\big(0.8202076319 + 2.4368685248a^2\big). \tag{11.2.55}$$

Here, a is an arbitrary real number. It has also been shown that when the origin is a focus point, there still exist 12 small limit cycles [432]. The main result for cubic systems is given in the following theorem.

THEOREM 11.2.6. *For the cubic system (11.2.52), suppose it has property of Z_2-symmetry, then the maximal number of small limit cycles that the system can exhibit is* 12, *i.e.,* $H(3) \geqslant 12$.

We shall not discuss further the procedures and formulas to obtain the 12 limit cycles. Interested readers can find more details from references [431,432]. We give two numerical simulation results below, as shown in Figure 11.2.3, one for the case of the origin being saddle point and the other for the origin being focus point. The case of the origin being a node can be found in [432].

11.3. Computation of the SNF with parameters

The computation of normal forms has been mainly restricted to systems which do not contain perturbation parameters. However, in practice a physical system or a control problem always involves some parameters, usually called perturbation parameters or unfolding. Such normal forms are very important in applications. A conventional normal form (CNF considered in Section 11.2) with unfolding is usually obtained in two steps: First ignore the perturbation parameter and compute the normal form for the corresponding "reduced" system (by setting the parameters zero), and then add an unfolding to the resulting normal form. This way it greatly reduces the computation effort, with the cost that it does not provide the transformation between the original system and the normal form. For the simplest normal form (SNF), on the other hand, since Ushiki [386] introduced the method of infinitesimal deformation in 1984 to study the SNF of vector fields, although many researchers have considered several cases of singularities (for example, see [421,386,17,80,81,358,390,68,443]), no single application using the SNF has been reported. This is because that the main attention in this area has been focused on the computation of the SNF without perturbation parameters. Recently, single zero and Hopf singularities have been considered and the explicit SNFs with unfolding have been obtained by introducing time and parameter rescalings [425,435]. In this section, after general formulas presented, a brief summary for the SNFs associated with dimension-one singularities (single zero and Hopf) will be given.

For a general nonlinear physical or engineering system, which may include stable manifold, normal form theory is usually employed together with center manifold theory [59] in order to take the contribution from the stable manifold. In general, given a nonlinear system, center manifold theory is applied before employing normal form theory. The idea of center manifold theory is similar to normal form theory—simplify the system by applying successive nonlinear transformations. It reduces the original system to a center manifold which has smaller dimension than that of the original system. Different methods have been developed to combine center manifold theory with normal form theory in one unified procedure (e.g., see [420,426,434]). In [426] an efficient computation method and Maple programs are developed for general systems associated with semi-simple cases. However, the normal form computation presented in [426] does not contain perturbation parameters (unfolding), and thus is not directly applicable in solving practical problems.

11.3.1. General formulation

In this section, we shall present an efficient approach for computing the SNF with perturbation parameters (unfolding) directly from general n-D systems which are not necessarily described on center manifold, and apply the method to consider controlling bifurcations. An explicit, recursive formula will be derived for computing the normal forms associated with general semi-simple cases. The approach is efficient since it reduces the computation to minimum at each step of finding ordered algebraic equations. Based on the general recursive formula, the SNFs for single zero and Hopf singularities are obtained.

In this section, we shall derive the explicit formulas for computing the normal forms associated with semi-simple cases. We restrict to autonomous systems. In order for the formulas derived in this section to be used in the next section for controlled systems. Consider the following general control system, given by

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\mu}) + \boldsymbol{u}, \quad \boldsymbol{x}, \boldsymbol{u} \in R^n, \ \boldsymbol{\mu} \in R^s, \ \boldsymbol{F} : R^{n+s} \to R^n, \tag{11.3.1}$$

where $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{\mu}$ are state variable, control variable and system parameter, respectively. $\boldsymbol{\mu}$ may be considered as control parameters. Usually, $\boldsymbol{\mu}$ is not explicitly shown in a control system. In this chapter, $\boldsymbol{\mu}$ is explicitly shown for the convenience of bifurcation analysis. The control function $\boldsymbol{u}$ can be, in general, any kind of function of the parameter $\boldsymbol{\mu}$ as well as time t, which renders system (11.3.1) nonautonomous. However, when a control law is determined, system (11.3.1) may be transformed to an autonomous one. For instance, suppose the feedback, given by

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{11.3.2}$$

is chosen, then system (11.3.1) becomes autonomous, and the bifurcation theory for differential equations can be applied with the $\boldsymbol{\mu}$ as control parameter. Then, system (11.3.1) can be rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\mu}) + \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\mu}) \stackrel{\triangle}{=} J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \ \boldsymbol{\mu} \in \boldsymbol{R}^s, \tag{11.3.3}$$

where $J\boldsymbol{x}$ denotes the linear terms. Further, without loss of generality, it is assumed that $\boldsymbol{x} = \boldsymbol{0}$ is an equilibrium point of the system for any real values of $\boldsymbol{\mu}$, i.e., $\boldsymbol{f}(\boldsymbol{0}, \boldsymbol{\mu}) \equiv \boldsymbol{0}$. The nonlinear function $\boldsymbol{f}$ is assumed to be analytic with respect to $\boldsymbol{x}$ and $\boldsymbol{\mu}$. J is the Jacobian matrix of the system evaluated at the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$, when the parameter $\boldsymbol{\mu}$ reaches its critical point $\boldsymbol{\mu} = \boldsymbol{0}$, given in the form of

$$J = \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix}, \tag{11.3.4}$$

where both J_0 and J_1 are assumed in diagonal form, indicating that all eigenvalues of the Jacobian are semi-simple. J_0 includes the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n_0}$ with zero real parts, while J_1 has the eigenvalues $\lambda_{n_0+1}, \lambda_{n_0+1}, \ldots, \lambda_n$ with negative real parts. In other words, system (11.3.3) does not contain unstable manifold in the vicinity of $\boldsymbol{x}$.

To find the normal form of system (11.3.3), one may expand the dimension of system (11.3.3) from n to $n + s$, by adding the equation $\dot{\boldsymbol{\mu}} = \boldsymbol{0}$ to system (11.3.3) to obtain a new system:

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \quad \dot{\boldsymbol{\mu}} = \boldsymbol{0}, \ \boldsymbol{x} \in \boldsymbol{R}^n, \ \boldsymbol{\mu} \in \boldsymbol{R}^s. \tag{11.3.5}$$

Then a general near-identity transformation may be assumed either in the form of

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{\nu}), \qquad \boldsymbol{\mu} = \boldsymbol{\nu}, \tag{11.3.6}$$

or

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}_1(\boldsymbol{y}, \boldsymbol{\nu}), \qquad \boldsymbol{\mu} = \boldsymbol{\nu} + \boldsymbol{h}_2(\boldsymbol{y}, \boldsymbol{\nu}), \tag{11.3.7}$$

where $\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{\nu})$, $\boldsymbol{h}_1(\boldsymbol{y}, \boldsymbol{\nu})$ and $\boldsymbol{h}_2(\boldsymbol{y}, \boldsymbol{\nu})$ are nonlinear analytic functions of $\boldsymbol{y}$ and $\boldsymbol{\nu}$. The equation $\boldsymbol{\mu} = \boldsymbol{\nu}$ given in equation (11.3.6) emphasizes that the parameter $\boldsymbol{\mu}$ is not changed under the transformation (11.3.6), i.e., reparametrization is not applied. For convenience, we may call transformation (11.3.6) as *state transformation* since it only changes state variable $\boldsymbol{x}$, while call equation (11.3.7) as *state-parameter transformation* because the parameter $\boldsymbol{\mu}$ is also expressed in terms of both $\boldsymbol{y}$ and $\boldsymbol{\nu}$. The state transformation is a natural way from the physical point of view since the parameter $\boldsymbol{\nu}$ is not a function of time. The state-parameter transformation however contains time variation in parameter $\boldsymbol{\mu}$ since it involves the state variable $\boldsymbol{y}$. In this chapter we only consider the near-identity state transformation or simply near-identity (nonlinear) transformation (11.3.6) but with reparametrization $\boldsymbol{\mu} = \boldsymbol{\nu} + \boldsymbol{p}(\boldsymbol{\nu})$. Thus, transformation (11.3.6) becomes

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{\nu}), \qquad \boldsymbol{\mu} = \boldsymbol{\nu} + \boldsymbol{p}(\boldsymbol{\nu}). \tag{11.3.8}$$

For the transformation (11.3.8), we can show that it is not necessary to extend the n-D system (11.3.3) to $(n+s)$-D system (11.3.5). In fact, directly applying normal form theory to system (11.3.3) is equivalent to using system (11.3.5). To prove this, we assume that the transformed system (normal form) is given by

$$\dot{\boldsymbol{y}} = J\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{\nu}), \qquad \dot{\boldsymbol{\nu}} = \boldsymbol{0}. \tag{11.3.9}$$

Then differentiating the first equation of (11.3.9) with respect to t results in

$$\frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{y}}{dt} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{y}}\frac{d\boldsymbol{y}}{dt} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\nu}}\frac{d\boldsymbol{\nu}}{dt}$$

and then substituting equations (11.3.5) and (11.3.9) into the resulting equation yields

$$\left(1 + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{y}}\right)\boldsymbol{g}(\boldsymbol{y}, \boldsymbol{\nu}) = L\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{\nu}) - \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{y}}L\boldsymbol{y} + \boldsymbol{f}\big(\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{\nu}), \boldsymbol{\nu}\big). \tag{11.3.10}$$

Hence, the computation of the normal form of system (11.3.5) completely depends upon equation (11.3.10). However, it is easy to see that equation (11.3.10) can be also directly derived from equation (11.3.3) with the aid of the first equations of (11.3.6) and (11.3.9). Therefore, in this chapter we shall use equation (11.3.3).

Now back to the original system (11.3.3), and let $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)^T$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are variables associated with the eigenvalues of J_0 and J_1, respectively. Then, equation (11.3.3) can be rewritten as

$$\dot{\boldsymbol{x}}_1 = J_0\boldsymbol{x}_1 + \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{\mu}), \qquad \dot{\boldsymbol{x}}_2 = J_1\boldsymbol{x}_2 + \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{\mu}). \tag{11.3.11}$$

By center manifold theory [59], $\boldsymbol{x}_2$ can be expressed in terms of $\boldsymbol{x}_1$ as

$$\boldsymbol{x}_2 = \boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu}), \quad \text{satisfying} \quad \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}, \ \frac{\partial \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{0})}{\partial \boldsymbol{x}_1 \partial \boldsymbol{\mu}} = 0, \tag{11.3.12}$$

under which the second equation of (11.3.11) can be rewritten as

$$\begin{aligned} &D_{\boldsymbol{x}_1}\boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu})\big[J_0\boldsymbol{x}_1 + \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu}), \boldsymbol{\mu})\big] \\ &\quad = J_1\boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu}) + \boldsymbol{f}_2\big(\boldsymbol{x}_1, \boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu}), \boldsymbol{\mu}\big). \end{aligned} \tag{11.3.13}$$

Having found $\boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu})$ from the above equation, the first equation of (11.3.11) becomes

$$\dot{\boldsymbol{x}}_1 = J_0\boldsymbol{x}_1 + \boldsymbol{f}_1\big(\boldsymbol{x}_1, \boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu}), \boldsymbol{\mu}\big), \tag{11.3.14}$$

which governs the dynamics of the original system (11.3.3) in the vicinity of $(\boldsymbol{x}, \boldsymbol{\mu}) = (\boldsymbol{0}, \boldsymbol{0})$.

In order to further simplify equation (11.3.14), introduce the following nonlinear transformation

$$\boldsymbol{x}_1 = \boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{\nu}) \overset{\triangle}{=} \boldsymbol{w} + \sum_{m=2}^{\infty} \boldsymbol{H}_m(\boldsymbol{w}, \boldsymbol{\nu}), \tag{11.3.15}$$

and the time rescaling

$$t = \big(T_0 + T(\boldsymbol{w}, \boldsymbol{\nu})\big)\tau \overset{\triangle}{=} \tau + \sum_{m=1}^{\infty} T_m(\boldsymbol{w}, \boldsymbol{\nu})\tau, \tag{11.3.16}$$

where $\boldsymbol{\nu}$ indicates the parameter rescaling, given in the form of

$$\boldsymbol{\mu} = \boldsymbol{\nu} + \boldsymbol{p}(\boldsymbol{\nu}) \overset{\triangle}{=} \boldsymbol{\nu} + \sum_{m=2}^{\infty} \boldsymbol{p}_m(\boldsymbol{\nu}). \tag{11.3.17}$$

Note that unlike the transformation (11.3.7), here $\boldsymbol{\mu}$ given in equation (11.3.17) does not involve the time-variant variable $\boldsymbol{w}$. Also, note that T_0 has been taken as 1 for convenience.

Further, assume that the normal form of system (11.3.14) is given by

$$\frac{d\boldsymbol{w}}{d\tau} = J_0\boldsymbol{w} + \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{\nu}) \overset{\triangle}{=} J_0\boldsymbol{w} + \sum_{m=2}^{\infty} \boldsymbol{C}_m(\boldsymbol{w}, \boldsymbol{\nu}). \tag{11.3.18}$$

Here, $\boldsymbol{H}_m(\boldsymbol{w}, \boldsymbol{\nu})$ and $\boldsymbol{C}_m(\boldsymbol{w}, \boldsymbol{\nu})$ are the mth-degree, n_0-D vector homogeneous polynomials of $\boldsymbol{w}$ and $\boldsymbol{\nu}$, and $\boldsymbol{p}_m(\boldsymbol{\nu})$ is the mth-degree, s-D vector homogeneous polynomials of $\boldsymbol{v}$, while $T_m(\boldsymbol{w}, \boldsymbol{v})$ is the mth-degree, scalar homogeneous polynomials of its components.

To find the normal form, first differentiating equation (11.3.15) and substituting it into equation (11.3.14) yields

$$\begin{aligned}
&\big(I + D_{\boldsymbol{w}}\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big)\frac{d\boldsymbol{w}}{d\tau} \\
&\quad = \frac{dt}{d\tau}\big[J_0\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big) + \boldsymbol{f}_1\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \\
&\qquad N\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big)\big],
\end{aligned} \tag{11.3.19}$$

and then using equation (11.3.16) and substituting equation (11.3.10) into the above equation and rearranging results in

$$\begin{aligned}
&D_{\boldsymbol{w}}\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})J_0\boldsymbol{w} - J_0\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \\
&\quad = \boldsymbol{f}_1\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big) \\
&\qquad - D_{\boldsymbol{w}}\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) - \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) \\
&\qquad + T(\boldsymbol{w}, \boldsymbol{v})\big[J_0\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big) \\
&\qquad + \boldsymbol{f}_1\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big)\big],
\end{aligned} \tag{11.3.20}$$

where $\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) \equiv \boldsymbol{N}(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v}))$.

Next, one may substitute equation (11.3.15) into equation (11.3.13), and use equation (11.3.20) to find the following equation:

$$\begin{aligned}
&D_{\boldsymbol{x}_1}\boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{\mu})\big\{\big(I + D_{\boldsymbol{w}}\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big)\big(J_0\boldsymbol{w} + \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v})\big) \\
&\quad - T(\boldsymbol{w}, \boldsymbol{v})\big[J_0\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big) \\
&\quad + \boldsymbol{f}_1\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big)\big]\big\} \\
&\qquad = J_1\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) + \boldsymbol{f}_2\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big).
\end{aligned} \tag{11.3.21}$$

By chain rule, $D_{\boldsymbol{x}_1}\boldsymbol{N}(\boldsymbol{x}_1, \boldsymbol{v})(I + D_{\boldsymbol{w}}\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})) = D_{\boldsymbol{w}}\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})$, one can rewrite equation (11.3.21) as

$$\begin{aligned}
&D_{\boldsymbol{w}}\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})J_0\boldsymbol{w} - J_1\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{w}) \\
&\quad = \boldsymbol{f}_2\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big) - D_{\boldsymbol{w}}\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) \\
&\qquad + T(\boldsymbol{w}, \boldsymbol{v})D_{\boldsymbol{w}}\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})\big[I + D_{\boldsymbol{w}}\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big]^{-1}\big[J_0\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big) \\
&\qquad + \boldsymbol{f}_1\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big)\big].
\end{aligned} \tag{11.3.22}$$

Finally, combining equations (11.3.20) and (11.3.22) yields the following compact form:

$$D\begin{pmatrix} \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \\ \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) \end{pmatrix} J_0\boldsymbol{w} - \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix}\begin{pmatrix} \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \\ \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})) \\ \boldsymbol{f}_2(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})) \end{pmatrix} \\
&\quad - D \begin{pmatrix} \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \\ \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) \end{pmatrix} \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) - \begin{pmatrix} \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) \\ \boldsymbol{0} \end{pmatrix} \\
&\quad + T(\boldsymbol{w}, \boldsymbol{v}) \begin{bmatrix} I \\ D\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})[I + D\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})]^{-1} \end{bmatrix} \big[J_0\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})\big) \\
&\quad + \boldsymbol{f}_1\big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v})\big)\big],
\end{aligned} \tag{11.3.23}
$$

where the differential operator $D \equiv D_{\boldsymbol{w}}$.

Equation (11.3.23) is all what we need for computing the normal form $\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v})$, the nonlinear transformations $\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})$ and $\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})$, the time rescaling $T(\boldsymbol{w}, \boldsymbol{v})$, and the reparametrization $\boldsymbol{p}(\boldsymbol{v})$. Note that all $\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v})$, $\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})$ and $\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})$ start from second order terms and can be expressed in terms of vector homogeneous polynomials of $\boldsymbol{w}$ and $\boldsymbol{v}$. $\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v})$ and $\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})$ are n_0-D vectors while $\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})$ is a $(n - n_0)$-D vector. $T(\boldsymbol{w}, \boldsymbol{v})$ is a scalar function while $\boldsymbol{p}(\boldsymbol{v})$ is a s-D vector.

Since, in general, it is not possible to find the closed-form solutions for $\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v})$, $\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v})$, $\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v})$, $T(\boldsymbol{w}, \boldsymbol{v})$ and $\boldsymbol{p}(\boldsymbol{v})$ from equation (11.3.22), we may assume the approximate solutions given by

$$
\begin{aligned}
\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) &= \sum_{m=2}^{\infty} \boldsymbol{C}_m(\boldsymbol{w}, \boldsymbol{v}) = \sum_{m=2}^{\infty} \sum_{m} \boldsymbol{C}_m w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}}, \\
\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) &= \sum_{m=2}^{\infty} \boldsymbol{H}_m(\boldsymbol{w}, \boldsymbol{v}) = \sum_{m=2}^{\infty} \sum_{m} \boldsymbol{H}_m w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}}, \\
\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) &= \sum_{m=2}^{\infty} \boldsymbol{h}_m(\boldsymbol{w}, \boldsymbol{v}) \\
&= \sum_{m=2}^{\infty} \sum_{m} \boldsymbol{h}_m w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}},
\end{aligned} \tag{11.3.24}
$$

and

$$
\begin{aligned}
T(\boldsymbol{w}, \boldsymbol{v}) &= \sum_{m=1}^{\infty} T_m(\boldsymbol{w}, \boldsymbol{v}) = \sum_{m=1}^{\infty} \sum_{m} T_m w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}}, \\
\boldsymbol{p}(\boldsymbol{v}) &= \sum_{m=2}^{\infty} \boldsymbol{p}_m(\boldsymbol{v}) = \sum_{m=2}^{\infty} \sum_{m} \boldsymbol{p}_m v_1^{m_1} v_2^{m_2} \cdots w_s^{m_s},
\end{aligned} \tag{11.3.25}
$$

where $\boldsymbol{C}_m$, $\boldsymbol{H}_m$, $\boldsymbol{h}_m$, T_m and $\boldsymbol{p}_m$ represent the mth-order coefficients. The subscript m means that for all possible nonnegative integers, $m_1, m_2, \ldots, m_{n_0+s}$ satisfy $m_1 + m_2 + \cdots + m_{n_0+s} = m$ (or $m_1 + m_2 + \cdots + m_s = m$ for $\boldsymbol{p}_m$).

Further, for an arbitrary $m \geqslant 2$, one can show that

$$
\begin{aligned}
D &\begin{pmatrix} \boldsymbol{H}_m(\boldsymbol{w}, \boldsymbol{v}) \\ \boldsymbol{h}_m(\boldsymbol{w}, \boldsymbol{v}) \end{pmatrix} J_0 \boldsymbol{w} \\
&= \sum_m D \begin{pmatrix} \boldsymbol{H}_m \\ \boldsymbol{h}_m \end{pmatrix} w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}} (J_0 \boldsymbol{w}) \\
&= \sum_m \left[\sum_{i=1}^{n_0} \frac{\partial}{\partial w_i} \begin{pmatrix} \boldsymbol{H}_m \\ \boldsymbol{h}_m \end{pmatrix} w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}} \lambda_i w_i \right] \\
&= \sum_m (m_1 \lambda_1 + \cdots + m_{n_0} \lambda_{n_0}) \begin{pmatrix} \boldsymbol{H}_m \\ \boldsymbol{h}_m \end{pmatrix} w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}} \\
&= \sum_m \lambda_0 \begin{pmatrix} \boldsymbol{H}_m \\ \boldsymbol{h}_m \end{pmatrix} w_1^{m_1} \cdots w_{n_0}^{m_{n_0}} v_1^{m_{n_0+1}} \cdots v_s^{m_{n_0+s}} \\
&= \lambda_0 \begin{pmatrix} \boldsymbol{H}_m(\boldsymbol{w}, \boldsymbol{v}) \\ \boldsymbol{h}_m(\boldsymbol{w}, \boldsymbol{v}) \end{pmatrix},
\end{aligned}
\tag{11.3.26}
$$

where

$$
\lambda_0 = m_1 \lambda_1 + m_2 \lambda_2 + \cdots + m_{n_0}. \tag{11.3.27}
$$

Thus, one can obtain the following equation from equation (11.3.23) for solving the mth-order coefficients: $\boldsymbol{C}_m$, $\boldsymbol{H}_m$, $\boldsymbol{h}_m$, T_m and $\boldsymbol{p}_m$:

$$
\begin{pmatrix} [\lambda_0 I - J_0] \boldsymbol{H}_m \\ [\lambda_0 I - J_1] \boldsymbol{h}_m \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{f}}_{1m} \\ \tilde{\boldsymbol{f}}_{2m} \end{pmatrix} - \begin{pmatrix} \boldsymbol{C}_m \\ 0 \end{pmatrix}, \tag{11.3.28}
$$

where the mth-order coefficients $\tilde{\boldsymbol{f}}_{1m}$ and $\tilde{\boldsymbol{f}}_{2m}$ are extracted from

$$
\begin{aligned}
\tilde{\boldsymbol{f}}_1 = \boldsymbol{f}_1\big(&\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{w})\big) - D\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) \\
&+ T(\boldsymbol{w}, \boldsymbol{v}) \big[J_0 \big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \big) \\
&+ \boldsymbol{f}_1 \big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v}) \big) \big]
\end{aligned}
\tag{11.3.29}
$$

and

$$
\begin{aligned}
\tilde{\boldsymbol{f}}_2 = \boldsymbol{f}_2\big(&\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{w})\big) - D\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{v}) \\
&+ T(\boldsymbol{w}, \boldsymbol{v}) D\boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}) \big[I + D\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \big]^{-1} \big[J_0 \big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}) \big) \\
&+ \boldsymbol{f}_1 \big(\boldsymbol{w} + \boldsymbol{H}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{h}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{v} + \boldsymbol{p}(\boldsymbol{v}) \big) \big],
\end{aligned}
\tag{11.3.30}
$$

respectively. Note that $\tilde{\boldsymbol{f}}_1$ and $\tilde{\boldsymbol{f}}_2$ contain T_m and $\boldsymbol{p}_m$.

Now we can determine the mth-order normal form coefficients $\boldsymbol{C}_m$, and the nonlinear transformation coefficients $\boldsymbol{H}_m$ and $\boldsymbol{h}_m$ as well as the rescalings T_m and $\boldsymbol{p}_m$ from equation (11.3.28) order by order starting from $m = 2$. Firstly, note from equation (11.3.23) that the mth-order coefficients $\tilde{\boldsymbol{f}}_{1m}$ and $\tilde{\boldsymbol{f}}_{2m}$ contain $\boldsymbol{C}$, $\boldsymbol{H}$, $\boldsymbol{h}$, T and $\boldsymbol{p}$ coefficients whose orders are lower than m. Therefore, the undetermined lower order coefficients may be involved in the two coefficients $\tilde{\boldsymbol{f}}_{1m}$ and $\tilde{\boldsymbol{f}}_{2m}$. Secondly, since λ_0 only contains the eigenvalues of J_0 (with zero real parts) and all eigenvalues of J_1 have nonzero real parts, $\lambda_0 I - J_1$ cannot equal zero for any of its components. This suggests that $\boldsymbol{h}_m$ can be uniquely determined from equation (11.3.28) as

$$\boldsymbol{h}_m = [\lambda_0 I - J_1]^{-1} \tilde{\boldsymbol{f}}_{2m}, \tag{11.3.31}$$

or, by noting that $[\lambda_0 I - J_1]$ is a diagonal matrix,

$$h_m^{(k)} = \frac{\tilde{f}_{2m}^{(k)}}{\lambda_0 - \lambda_{n_0+k}} \quad \text{for } k = 1, 2, \ldots, n - n_0, \tag{11.3.32}$$

where $h_m^{(k)}$ and $\tilde{f}_{2m}^{(k)}$ are the kth components of $\boldsymbol{h}_m$ and $\tilde{\boldsymbol{f}}_{2m}$, respectively.

Finally, we need to solve the equation:

$$[\lambda_0 I - J_0]\boldsymbol{H}_m = \tilde{\boldsymbol{f}}_{1m} - \boldsymbol{C}_m \tag{11.3.33}$$

to determine $\boldsymbol{C}_m$ and $\boldsymbol{H}_m$. Note that $\tilde{\boldsymbol{f}}_{1m}$ contains the lower-order coefficients of $\boldsymbol{C}$, $\boldsymbol{H}$, T and $\boldsymbol{p}$, and thus unlike the CNF computation, we may use the lower order $\boldsymbol{H}$, $\boldsymbol{h}$, T and $\boldsymbol{p}$ coefficients to eliminate $\boldsymbol{C}_m$, leading to the SNF. Similarly, due to the semi-simple property, the matrix $[\lambda_0 I - J_0]$ is a diagonal matrix, one can rewrite equation (11.3.33) in the component form:

$$(\lambda_0 - \lambda_k) H_m^{(k)} = \tilde{f}_{1m}^{(k)} - C_m^{(k)} \quad \text{for } k = 1, 2, \ldots, n_0, \tag{11.3.34}$$

where $H_m^{(k)}$, $\tilde{f}_{1m}^{(k)}$ and $C_m^{(k)}$ are the kth components of $\boldsymbol{H}_m$, $\tilde{\boldsymbol{f}}_{1m}$ and $\boldsymbol{C}_m$, respectively. Then when $\lambda_0 - \lambda_k \neq 0$, we may uniquely determine

$$C_m^{(k)} = 0 \quad \text{and} \quad H_m^{(k)} = \frac{\tilde{f}_{1m}}{\lambda_0 - \lambda_k}. \tag{11.3.35}$$

However, when $\lambda_0 - \lambda_k = 0$, we may use the lower order $\boldsymbol{H}$, $\boldsymbol{h}$, T and $\boldsymbol{p}$ coefficients involved in $\tilde{f}_{1m}$ to possibly eliminate $C_m^{(k)}$. If there are no such lower-order coefficients which can be used at this order, then $C_m^{(k)} = \tilde{f}_{1m}$. The rule determining how to choose the lower-order coefficients depends upon the singularity under consideration.

Having found the explicit formulas (11.3.32) and (11.3.34), it seems that the computation of the coefficients of the normal form and nonlinear transformation is straightforward. However, it has been noted that directly employing these formulas can cause computation problem: A computer may quickly run out of its memory due to enormous algebraic manipulations. As we know that in the computation of normal forms, higher order computations do not affect lower order results, but lower order results influence all higher order calculations. In general, when one finishes $k < m$ order computations, one substitutes the lower order solutions into the original nonlinear function $\boldsymbol{f}$ to obtain the equation for computing the mth-order normal form. The expression of the resulting equation includes all order ($< m$ and $\geqslant m$) expressions and one needs to extract the exact mth-order part from the enormous large expression. In fact, the semi-simple case has been considered with the "extract" method. It has been found that such an approach is not efficient and can easily cause a computer "crash" even for a not very complicated problem. In order to overcome this difficulty, it needs to directly find the expression which only belongs to the mth-order equation. This can greatly reduce the computation time and computer memory demanding. The detailed efficient computation approach will not be discussed in this chapter. Interested readers are referred to the references [438,425,426,434,439].

In the above, we have developed an efficient computation method and derived recursive formulas for computing the coefficients of the SNF and associated transformations (see equations (11.3.31) and (11.3.34)). It has been shown that the transformation for the noncritical variables, $\boldsymbol{h}$, is uniquely determined by equation (11.3.31). However, computing the center manifold part is not straightforward. (The computation of this part for the CNF is straightforward, uniquely determined by equation (11.3.33), see [426].) To find the SNF from equation (11.3.34) one must carefully consider not only the coefficients of $\boldsymbol{H}$, but also that of T and $\boldsymbol{p}$ which are implicitly involved in $\tilde{\boldsymbol{f}}_1$. It should be emphasized that equation (11.3.34) does not contain $\boldsymbol{h}$ coefficients since the kth-order coefficients $\boldsymbol{h}_k$ are solved and only solved from the kth-order algebraic equation (11.3.31). This implies that equation (11.3.34) only contains $\boldsymbol{H}$, T and $\boldsymbol{p}$ which are associated with the center manifold variables, $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore, the final step in computing the SNF is to solve equation (11.3.34), which is similar to finding the SNF of a system which is described on center manifold. However, we cannot obtain a general form or procedure applicable for all semi-simple cases. One has to deal with the singularities case by case. In this chapter, we focus on codimension-one singularities: single zero and Hopf bifurcation. The SNFs for the two singularities based on center manifold (i.e., the original system (11.3.3) is not a general n-D system, but described on center manifold) have been obtained in [425,435]. In the following, we outline the SNF computation rules for the two singularities.

11.3.2. The SNF for single zero

To find the computation rules of the SNF of single zero singularity, we may assume that the original system is described on 1-D center manifold as follows:

$$\frac{dy}{dt} = f(y, \mu) = \sum_{i=1}^{\infty} a_{1i}\mu^i y + \sum_{i=0}^{\infty} a_{2i}\mu^i y^2 + \sum_{i=0}^{\infty} a_{3i}\mu^i y^3 + \cdots \quad (11.3.36)$$

which has an equilibrium $x = 0$ for any real values of μ. The near-identity nonlinear transformation and the time scaling are, respectively, given by

$$y = x + H(x, \mu) = x + \sum_{i=1}^{\infty} b_{1i}\mu^i x + \sum_{i=0}^{\infty} b_{2i}\mu^i x^2 + \cdots \tag{11.3.37}$$

and

$$\begin{aligned} T_0 + T(x, \mu) = 1 &+ \sum_{i=1}^{\infty} T_{0i}\mu^i + \sum_{i=0}^{\infty} T_{1i}\mu^i x \\ &+ \sum_{i=0}^{\infty} T_{3i}\mu^i x^2 + \cdots . \end{aligned} \tag{11.3.38}$$

As shown in [425], the case of zero singularity does not need parameter scaling (reparametrization). Thus, instead of ν, the original parameter μ is used in equations (11.3.37) and (11.3.38).

It has been proved [425] that the SNF of system (11.3.36) is given by the following theorem.

THEOREM 11.3.1. *Under the conditions*: $a_{11} \neq 0$ *and* $a_{k0} \neq 0$ $(k \geqslant 2)$, *where* a_{k0} *is the first nonzero coefficients of* a_{j0}*'s, the SNF of system* (11.3.36) *is given by*

$$\frac{dx}{d\tau} = a_{11}\mu x + a_{k0}x^k \quad (k \geqslant 2), \tag{11.3.39}$$

up to any order.

Note that the coefficients a_{11} (for the 2nd-order equation) and a_{k0} (for the kth-order equation) are known coefficients of the original system, indicating that the 2nd-order equation cannot be reduced. The detailed procedure for computing the coefficients of b_{ij} and T_{ij} can be found in [425]. The above results are based on the assumption $a_{11} \neq 0$, which results in the unfolding $a_{11}\mu x$. Other possible unfolding may not be so simple if $a_{11} = 0$. However, they can be easily obtained by executing the Maple program we developed to find the SNF. For example, suppose $a_{11} = a_{12} = 0$, but $a_{13} \neq 0$ and $a_{21} \neq 0$, then the SNF is found to be

$$\frac{dx}{d\tau} = a_{13}\mu^3 x + a_{21}\mu x^2 + a_{k0}x^k \quad (k \geqslant 2). \tag{11.3.40}$$

The above rules obtained based on center manifold can be applied to solve the key equation (11.3.34) for the general original system (11.3.3). However, it should be noted that the coefficient a_{k0} cannot be directly observed from the original equation (e.g., usually the first equation of the system) since noncenter manifold equations may have contributions to these coefficients. This can be easily handled in symbolic computation.

11.3.3. The SNF for Hopf bifurcation

We now turn to Hopf bifurcation. We discuss the system given on a 2-D center manifold to find the rules of computing the coefficients of the SNF and transformations. Suppose the system is described in complex form:

$$\begin{aligned}\frac{d\boldsymbol{z}}{dt} &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \boldsymbol{z} + \boldsymbol{f}(\boldsymbol{z}, \mu) \\ &\equiv \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \begin{pmatrix} f(z, \bar{z}, \mu) \\ \bar{f}(z, \bar{z}, \mu) \end{pmatrix},\end{aligned} \tag{11.3.41}$$

where $\boldsymbol{z} = (z, \bar{z})^T$ and $\boldsymbol{f} = (f, \bar{f})^T$, T represents a transpose. $\bar{z}$ and $\bar{f}$ are complex conjugates of z and f, respectively.

Further, assume that

$$\begin{aligned}
f_k &= \sum_{j+l+m=k} (a_{1jlm} + i a_{2jlm}) z^j \bar{z}^l \mu^m, \\
H_k &= \sum_{j+l+m=k} (b_{1jlm} + i b_{2jlm}) x^j \bar{x}^l \nu^m, \\
C_2 &= \big[(\alpha_1 + i\beta_1)x + (\alpha_2 + i\beta_2)\bar{x}\big]\nu, \\
C_k &= (c_{1k} + i c_{2k}) x^{(k+1)/2} \bar{x}^{(k-1)/2} \quad (k \geqslant 3,\ \text{odd integer}), \\
t &= \big(T_0 + T\big(x, \bar{x}, \nu\big)\big)\tau \left(1 + \sum_{k=1} \sum_{j+m=k} t_{jm} \left[\frac{1}{2}\big(x + \bar{x}\big)\right]^j \nu^m\right)\tau, \\
\mu &= p_0 \nu + p(\nu) = \nu + \sum_{j=2} p_j \nu^j,
\end{aligned} \tag{11.3.42}$$

where C_2 represents the linear unfolding.

Applying the 2nd-order ($k = 2$) equations of (11.3.34) yields the following solutions:

$$\begin{aligned}
&b_{1200} = a_{2200}, \quad b_{2200} = -a_{12001}, \quad b_{1020} = -\frac{1}{3} a_{2020}, \\
&b_{2020} = \frac{1}{3} a_{1020}, \quad b_{1110} = -a_{2110}, \quad b_{2110} = a_{1110},
\end{aligned} \tag{11.3.43}$$

and

$$\begin{aligned}
&\alpha_1 = a_{1101}, \quad t_{01} = -a_{2101}, \quad b_{1011} = -\frac{1}{2} a_{2011}, \\
&b_{2011} = \frac{1}{2} a_{1011},
\end{aligned} \tag{11.3.44}$$

which results in

$$\beta_1 = \alpha_2 = \beta_2 = 0. \tag{11.3.45}$$

Next, for $k = 3$, similarly we can find the following solutions:

$$\begin{aligned}
&c_{13} = a_{1210} - A_{1210}, \qquad t_{20} = 2(b_{23} - a_{2210} + A_{2210}),\\
&p_2 = -\frac{1}{a_{1101}}(a_{1102} + A_{1102}),\\
&t_{02} = \frac{a_{2101}}{a_{1101}}(a_{1102} + A_{1102}) - (a_{2102} + A_{2102}),\\
&b_{2300} = -\frac{1}{2}(a_{1300} + A_{1300}), \qquad b_{1300} = \frac{1}{2}\Big(a_{2300} + A_{2300} + \frac{1}{4}t_{20}\Big),\\
&b_{2030} = \frac{1}{4}(a_{1030} - A_{1030}), \qquad b_{1030} = -\frac{1}{4}(a_{2030} - A_{2030})\\
&b_{2120} = \frac{1}{2}(a_{1120} - A_{1120}), \qquad b_{1120} = -\frac{1}{2}\Big(a_{2120} - A_{2120} + \frac{1}{4}t_{20}\Big),\\
&b_{2201} = -(a_{1201} + A_{1201}), \qquad b_{1201} = a_{2201} + A_{2201},\\
&b_{2021} = \frac{1}{3}(a_{1021} - A_{1021}), \qquad b_{1021} = -\frac{1}{3}(a_{2021} - A_{2021}),\\
&b_{2111} = a_{1111} - A_{1111}, \qquad b_{1111} = -(a_{2111} - A_{2111}),\\
&b_{2012} = \frac{1}{2}(a_{1012} - A_{1012} + p_2 a_{1011}),\\
&b_{1012} = -\frac{1}{2}(a_{2012} - A_{2012} + p_2 a_{2011}),\\
&b_{1101} = c_{2101} = t_{11} = 0,
\end{aligned} \tag{11.3.46}$$

where A_{jkl}'s are known expressions, given in terms of a_{ijlm}'s. We can now apply the above procedure to solve higher order equations using the general rules given in [435]. Note that most of the equations are uniquely solved using the coefficients b_{ijlm}.

Therefore, the complex SNF of Hopf bifurcation is given by

$$\begin{aligned}
\frac{dx}{d\tau} &= ix + a_{1101}x\nu + (a_{1210} - a_{1200}a_{2110} - a_{2200}a_{1110})x^2\bar{x}\\
&\quad + i\sum_{m=1}^{\infty} c_{2(2m+1)}x^{m+1}x^m,
\end{aligned} \tag{11.3.47}$$

where c_{2j} are explicitly obtained in terms of the original system coefficients a_{ijlm}'s.

Let $x = Re^{i\Theta}$, where R and Θ are, respectively, the amplitude and phase of motion. Then the SNF for the Hopf bifurcation of system (11.3.41) is given as follows.

THEOREM 11.3.2. *The SNF for system (11.3.41) associated with Hopf bifurcation, given in polar coordinates, is*

$$\frac{dR}{d\tau} = a_{1101}\nu R + (a_{1210} - a_{1200}a_{2110} - a_{2200}a_{1110})R^3, \tag{11.3.48}$$

$$\frac{d\Theta}{d\tau} = 1 + c_{23}R^2 + c_{25}R^4 + \cdots + c_{2(2m+1)}R^{2m} + \cdots \tag{11.3.49}$$

up to arbitrary order.

Note that when we derive the SNF of Hopf bifurcation it has been assumed that $a_{1101} \neq 0$. This is clear from equation (11.3.48) that no linear universal unfolding will be present if $a_{1101} = 0$. The bifurcation and stability analysis can be carried out using equation (11.3.48). The steady-state solutions are given by

$$\begin{aligned} &\text{(I)} \quad R = 0, \\ &\text{(II)} \quad R^2 = -\frac{a_{1101}\nu}{a_{1210} - a_{1200}a_{2110} - a_{2200}a_{1110}}, \end{aligned} \tag{11.3.50}$$

where solution (I) actually represents the original equilibrium, while solution (II) denotes a family of limit cycles. The stability of the steady-state solutions can be easily determined by using the Jacobian of equation (11.3.48) as follows: Solution (I) is stable (unstable) if $a_{1101}\nu < 0$ (> 0). Solution (II) is stable (unstable) if $S_{LC} \equiv a_{1210} - a_{1200}a_{2110} - a_{2200}a_{1110} < 0$ (> 0). If $S_{LC} < 0$, then the existence of the limit cycles for $a_{1101}\nu > 0$ implies that the original equilibrium and the periodic solution exchange their stabilities at the critical point $\nu = 0$. In this case, the Hopf bifurcation is supercritical. Otherwise, it is called subcritical Hopf bifurcation.

The above analysis seems like a typical Hopf bifurcation analysis using the CNF. However, it should be noted that all higher order terms (> 3) have been removed from the SNF while the CNF has infinite higher order terms. Thus, Hopf bifurcation analysis based on the CNF up to 3rd-order terms means that all higher order terms in the CNF are neglected. For the SNF, however, the exact 3rd-degree polynomial is used for the analysis.

The above procedure can be directly applied to the general original n-D system (11.3.3). Symbolic program has been coded using Maple.

An application of using the SNF to solve Hopf bifurcation control problem will be given in the next section.

11.4. Hopf bifurcation control

In the past two decades, there has been rapidly growing interest in bifurcation dynamics of control systems, including controlling and anti-controlling bifurcations and chaos. Such bifurcation and chaos control techniques have been widely applied to solve physical and engineering problems (e.g., see [66,1,433,28,61,77,144,189,323,327]). The general goal of bifurcation control is to design a controller such that the bifurcation characteristics of a nonlinear system undergoing bifurcation can be modified to achieve certain desirable dynamical behavior, such as changing a Hopf bifurcation from subcritical to supercritical, eliminating chaotic motions, etc.

In this section, we consider bifurcation control using nonlinear state feedback. A general explicit formula is derived for the control strategy, given in the form of simple homogeneous polynomials. The formula keeps the equilibria of the original system unchanged. The linear part of the formula can be used to modify the system's linear stability, in order to eliminate or delay an existing bifurcation. The nonlinear part, on the other hand, can change the stability of bifurcation solutions, for example, converting a subcritical Hopf bifurcation to supercritical.

Here we want to particularly study Hopf bifurcation since the limit cycles generated by Hopf bifurcation is the most popular phenomenon exhibited in nonlinear dynamical systems. In the following sections, we first consider continuous systems, then study discrete maps and finally discuss time delay differential equations.

11.4.1. Continuous-time systems

For convenience, we will use the Lorenz system and an electrical circuit as examples to illustrate the theory and methodology of Hopf bifurcation control for continuous-time systems, described by ordinary differential equations. In particular, for the Lorenz system, we will apply the CNF to find the stability of bifurcating limit cycles; while for the electrical circuit, we will employ the SNF to analyze the stability of periodic motions.

Consider the general nonlinear system (11.1.2):

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \quad \boldsymbol{x} \in R^n,\ \mu \in R,\ \boldsymbol{f} : R^{n+1} \to R^n, \tag{11.4.1}$$

where $\boldsymbol{x}$ is an n-D state vector while μ is a scalar parameter, called bifurcation parameter. Suppose that at the critical point $\mu = \mu^*$ on an equilibrium solution $\boldsymbol{x} = \boldsymbol{x}^*$, the Jacobian of the system has a complex pair of eigenvalues to first cross the imaginary axis. Then Hopf bifurcation occurs at the critical point and a family of limit cycles bifurcate from the equilibrium solution $\boldsymbol{x}^*$.

Suppose system (11.4.1) has k equilibria, given by

$$\boldsymbol{x}_i^*(\mu) = \left(x_{1i}^*, x_{2i}^*, \ldots, x_{ni}^*\right), \quad i = 1, 2, \ldots, k. \tag{11.4.2}$$

Feedback controller using polynomial function

A general nonlinear state feedback control is applied so that system (11.4.1) becomes

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu) + \boldsymbol{u}(\boldsymbol{x}, \mu). \tag{11.4.3}$$

In order for the controlled system (11.4.3) to keep all the original k equilibria unchanged under the control $\boldsymbol{u}$, it requires that the following conditions be satisfied:

$$\boldsymbol{u}\big(\boldsymbol{x}_i^*, \mu\big) \equiv (u_1, u_2, \dots, u_n)^T = \boldsymbol{0} \tag{11.4.4}$$

for $i = 1, 2, \dots, k$. Then we have the following result [430].

THEOREM 11.4.1. *For system* (11.4.3)*, the feedback control can take the following polynomial function*:

$$\begin{aligned}
&u_q\big(\boldsymbol{x}, \boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \dots, \boldsymbol{x}_k^*, \mu\big) \\
&\quad = \sum_{i=1}^{n} A_{qi} \prod_{j=1}^{k} \big(x_i - x_{ij}^*\big) + \sum_{i=1}^{n} \sum_{j=1}^{k} B_{qij} \big(x_i - x_{ij}^*\big) \prod_{p=1}^{k} \big(x_i - x_{ip}^*\big) \\
&\qquad + \sum_{i=1}^{n} \sum_{j=1}^{k} C_{qij} \big(x_i - x_{ij}^*\big)^2 \prod_{p=1}^{k} \big(x_i - x_{ip}^*\big) \\
&\qquad + \sum_{i=1}^{n} \sum_{j=1}^{k} D_{qij} \big(x_i - x_{ij}^*\big)^2 \prod_{p=1}^{k} \big(x_i - x_{ip}^*\big)^2 + \cdots \\
&\quad (q = 1, 2, \dots, n).
\end{aligned} \tag{11.4.5}$$

It is easy to verify that $u_q(\boldsymbol{x}_i^*,\ \boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \dots, \boldsymbol{x}_k^*, \mu) = 0$ for $i = 1, 2, \dots, k$. Usually, terms given in equation (11.4.5) up to D_{qij} are enough for controlling a bifurcation if the singularity of the system is not highly degenerate. The coefficients A_{qi}, B_{qij}, C_{qij} and D_{qij}, which may be functions of μ, are determined from the stabilities of an equilibrium under consideration and that of the associated bifurcation solutions. More precisely, linear terms are determined by the requirement of shifting an existing bifurcation (e.g., delaying an existing Hopf bifurcation). The nonlinear terms, on the other hand, can be used to change the stability of an existing bifurcation or create a new bifurcation (e.g., changing an existing subcritical Hopf bifurcation to supercritical). Note that not just A_{qi} terms may involve linear terms; B_{qij} terms, etc. may also contain linear terms.

It is not necessary to take all the components $u_q, i = 1, 2, \dots, n$, in the controller. In most cases, using fewer components or just one component may be enough to satisfy the pre-designed control objectives. It is preferable to have a simplest possible design for engineering applications. If $x_{i1}^* = x_{i2}^* = \cdots = x_{ik}^*$

for some i, then one only needs to use these terms and omits the remaining terms in the control law. Moreover, lower-order terms related to these equilibrium components can be added. This greatly simplify the control formula. For example, if $i = 1$, then the general controller can be taken as

$$u_q = \sum_{i=1}^{k-1} a_{qi}\left(x_1 - x_{11}^*\right)^i + A_{q1}\left(x_1 - x_{11}^*\right)^k + B_{q11}\left(x_1 - x_{11}^*\right)^{k+1} + C_{q11}\left(x_1 - x_{11}^*\right)^{k+2},$$

where a_{qi}'s denote the added lower-order terms.

The goals of Hopf bifurcation control are:

(i) to move the critical point $(\boldsymbol{x}^*, \mu^*)$ to a designated position $(\tilde{\boldsymbol{x}}, \tilde{\mu})$;
(ii) to stabilize all possible Hopf bifurcations.

Goal (i) only requires linear analysis, while goal (ii) must apply nonlinear systems theory. In general, if the purpose of the control is to avoid bifurcations, one should employ linear analysis to maximize the stable interval for the equilibrium. The best result is to completely eliminate possible bifurcations using a feedback control. If this is not feasible, then one may have to consider stabilizing the limit cycles by using a nonlinear state feedback.

The Lorenz system

It is well known that the Lorenz system can exhibit very rich periodic and chaotic motions. In this subsection, we use a different version of Lorenz equation, which contains only two parameters, given below [70,393]:

$$\begin{aligned} \dot{x} &= -p(x - y), \\ \dot{y} &= -xz - y, \\ \dot{z} &= xy - z - r, \end{aligned} \tag{11.4.6}$$

where p and r are positive constants, which are considered as control parameters. One can easily show that system (11.4.6) is a special case of the general system [430].

System (11.4.6) has three equilibrium solutions, C_0, C_+ and C_-, given by

$$\begin{aligned} &\mathrm{C}_0\colon\ x_e^0 = y_e^0 = 0,\ z_e^0 = -r, \\ &\mathrm{C}_\pm\colon\ x_e^\pm = y_e^\pm = \pm\sqrt{r-1},\ z_e^\pm = -1. \end{aligned} \tag{11.4.7}$$

Suppose the parameters p and r are positive. Then C_0 is stable for $0 \leqslant r < 1$, and a pitchfork bifurcation occurs at $r = 1$, where the equilibrium C_0 looses its stability and bifurcates into either C_+ or C_-. The two equilibria C_+ and C_- are stable for $1 < r < r_H$, where

$$r_H = \frac{p(p+4)}{p-2} \quad (p > 2), \tag{11.4.8}$$

and at this critical point C_+ and C_- loose their stabilities, giving rise to Hopf bifurcation. We fix $p = 4$, which was used in [70,393]. Then $r_H = 16$, and the Lorenz system (11.4.6) exhibits chaotic motion when $r > 16$. In fact, one can employ numerical simulation to show the coexistence of locally stable equilibria $C_\pm$ and (global) chaotic attractors at a same value of r, with different initial conditions [430].

(A) Without control. We first consider system (11.4.6) without control. The critical point is $p = 4, r_H = 16$ at which the Jacobian of system (11.4.6) evaluated at C_+ and C_- has a real eigenvalue -6 and a purely imaginary pair $\pm 2\sqrt{5}$. Using the shift, given by

$$x = \pm\sqrt{r-1} + \tilde{x}, \qquad y = \pm\sqrt{r-1} + \tilde{y}, \qquad z = -1 + \tilde{z}, \tag{11.4.9}$$

to move $C_\pm$ to the origin and then applying an appropriate linear transformation to system (11.4.6), we obtain the following system:

$$\begin{aligned}
\dot{\tilde{x}} &= 2\sqrt{5}\tilde{y} + \frac{1}{84}\big(\tilde{x} + 4\sqrt{5}\tilde{y} - 6\tilde{z}\big)\mu - \frac{\sqrt{15}}{21}\big(\tilde{x} - 2\sqrt{5}\tilde{y}\big)\big(\tilde{x} - 2\tilde{z}\big) + \cdots, \\
\dot{\tilde{y}} &= -2\sqrt{5}\tilde{x} - \frac{\sqrt{5}}{2100}\big(155\tilde{x} - 10\sqrt{5}\tilde{y} - 6\tilde{z}\big)\mu \\
&\quad - \frac{\sqrt{3}}{105}\big(55\tilde{x} - 5\sqrt{5}\tilde{y} + 42\tilde{z}\big)\big(\tilde{x} - 2\tilde{z}\big) + \cdots, \\
\dot{\tilde{z}} &= -6\tilde{z} + \frac{1}{168}\big(\tilde{x} + 4\sqrt{5}\tilde{y} - 6\tilde{z}\big)\mu - \frac{\sqrt{15}}{42}\big(\tilde{x} - 2\sqrt{5}\tilde{y}\big)\big(\tilde{x} - 2\tilde{z}\big) + \cdots,
\end{aligned}$$

where $\mu = r - 16$ is a bifurcation parameter.

Employing the Maple programs developed in [420] for computing the normal forms of Hopf and generalized Hopf bifurcations yields the following normal form:

$$\begin{aligned}
\dot{\rho} &= \rho\left(\frac{1}{56}\mu + \frac{31}{3248}\rho^2\right) + \ldots, \\
\dot{\theta} &= 2\sqrt{5}\left(1 + \frac{17}{560}\mu - \frac{851}{48720}\rho^2\right) + \cdots,
\end{aligned} \tag{11.4.10}$$

where ρ and θ represent the amplitude and phase of the motion, respectively. The first equation of (11.4.10) clearly shows that the Hopf bifurcation is subcritical since the coefficient of ρ^3 is $\frac{31}{3248} > 0$.

(B) With control. Now, we apply a feedback control to stabilize system (11.4.6). A washout filter control has been used by Wang and Abed [393] for the Lorenz system (11.4.6). The disadvantage of this method is that it increases the dimension of the original system by one, unnecessarily increases the complexity of the controlled system and difficulty in analysis. Here we apply the control formula (11.4.8) to control the Hopf bifurcation. Due to the symmetry of the system and $z_\pm = -1$, we may use a control law with one variable only:

$$u_3 = -k_{31}(z+1) - k_{33}(z+1)^3. \tag{11.4.11}$$

The closed-loop system is now given by

$$\begin{aligned}
\dot{x} &= -p(x-y), \\
\dot{y} &= -xz - y, \\
\dot{z} &= xy - z - r - k_{31}(z+1) - k_{33}(z+1)^3,
\end{aligned} \tag{11.4.12}$$

where the negative signs are used for k_{ij}'s in consistence with that of the controller based on the washout filter. Introducing the transformation (11.4.9) into equation (11.4.12) results in

$$\begin{aligned}
\dot{x} &= -p(\tilde{x} - \tilde{y}), \\
\dot{y} &= -\tilde{x}\tilde{z} + \tilde{x} - \tilde{y} \mp \sqrt{r-1}\,\tilde{z}, \\
\dot{z} &= \tilde{x}\tilde{y} \pm \sqrt{r-1}\big(\tilde{x} + \tilde{y}\big) - \tilde{z} - k_{31}\tilde{z} - k_{33}\tilde{z}^3.
\end{aligned} \tag{11.4.13}$$

Then $O_e = (\tilde{x}, \tilde{y}, \tilde{z}) = (0,0,0)$ is an equilibrium of system (11.4.13), corresponding to the equilibria C_+ and C_- of the original system (11.4.6). The characteristic polynomial of system (11.4.13) for the equilibrium point O_c is

$$P(\lambda) = \lambda^3 + (p + 2 + k_{31})\lambda^2 + (p + r + k_{31} + pk_{31})\lambda + 2p(r-1),$$

which shows that only the linear term of the controller u_3 affects the linear stability. The stability conditions for O_e (under the assumption that $p, r > 0$) can be obtained as

$$\begin{aligned}
p + 2 + k_{31} &> 0, \\
p + r + k_{31}(p+1) &> 0, \\
2p(r-1) &> 0, \\
p(p+4) - r(p - 2 - k_c) + k_{31}^2(p+1) + k_{31}\big(p^2 + 4p + 2\big) &> 0.
\end{aligned}$$

If choosing $k_{31} > 0$, then it only requires $r > 1$ to satisfy the above first 3 inequalities. The last condition implies a critical point at which the controlled system has a Hopf bifurcation emerging from the equilibrium O_e, defined by

$$r_H = \frac{p(p+4) + k_{31}^2(p+1) + k_{31}(p^2 + 4p + 2)}{(p - 2 - k_{31})}, \tag{11.4.14}$$

for $0 < k_{31} < p - 2$.

Setting $k_{31} = 0$ yields $r_H = \frac{p(p+4)}{p-2}$ $(p > 2)$ which is the condition given in equation (11.4.8) for the system without control. It can be seen from equation (11.4.14) that the parameter r_H for the controlled system can reach very large values as long as k_{31} is chosen close to $p-2$. For example, when $p = 4$, choosing $k_{31} = 1.5$ gives $r_H = 188.5$ (and $r_H = 71$ if $k_{31} = 1$). These values of r_H are much larger than $r_H = 16$ for the uncontrolled system. If we choose $r > 1$ and $0 < p - 2 < k_{31}$, then the equilibria C_+ and C_- are always stable, and no Hopf bifurcation occurs from the two equilibria.

Next, we perform a nonlinear analysis to determine the stability of Hopf bifurcation. If $p = 4$, then $k_{31} \in (0, 2)$, and for determination we choose $k_{31} = \frac{2\sqrt{1006}-58}{5} \approx 1.087$, thus $r_H = 82$. Let $r = r_H + \mu = 82 + \mu$, where μ is a perturbation from the critical point. Then, we have the closed-loop system

$$\begin{aligned}
\dot{\tilde{x}} &= -8(\tilde{x} - \tilde{y}),\\
\dot{\tilde{y}} &= -\tilde{x}\tilde{z} + \tilde{x} - \tilde{y} \mp \sqrt{81+\mu}\,\tilde{z},\\
\dot{\tilde{z}} &= \tilde{x}\tilde{y} \pm \sqrt{81+\mu}(\tilde{x}+\tilde{y}) - \frac{2\sqrt{1006}-53}{5}\tilde{z} - k_{33}\tilde{z}^3.
\end{aligned} \tag{11.4.15}$$

The eigenvalues of the Jacobian of system (11.4.15), when evaluated at the equilibrium O_e, are: $\lambda_{1,2} = \pm\sqrt{2\sqrt{1006}+28}\,i \approx 9.5621i$ and $\lambda_3 = -\frac{2\sqrt{1006}-28}{5} \approx -7.0870$. To apply the method of normal forms [146,420,422], we introduce the following transformation:

$$\begin{aligned}
\tilde{x} &= u - \frac{24+\sqrt{1006}}{43}w,\\
\tilde{y} &= u + \frac{2\sqrt{1006}+28}{4}v + w,\\
\tilde{z} &= \pm\frac{\sqrt{1006}+14}{18}u \mp \frac{5(2\sqrt{1006}+28)}{36}v \pm \frac{9\sqrt{1006}-171}{215}w,
\end{aligned}$$

to equation (11.4.15), and then employ the Maple program [420] to obtain an identical CNF for the system associated with the two equilibria C_+ and C_-, given in polar coordinates as

$$\begin{aligned}
\dot{\rho} = \rho\Bigg[&\frac{1249-34\sqrt{1006}}{52942}\mu\\
&+\left(\frac{4646315818-102399253\sqrt{1006}}{358010321904}\right.\\
&\left.-\frac{5746272+187233\sqrt{1006}}{4235360}k_{33}\right)\rho^2\Bigg] + \cdots,
\end{aligned} \tag{11.4.16}$$

$$\dot{\theta} = \sqrt{2\sqrt{1006}+28}\Bigg[1 + \frac{122602 - 773\sqrt{1006}}{17153208}\mu - \left(\frac{21706679417 + 211691192\sqrt{1006}}{6444185794272} + \frac{34871 + 1594\sqrt{1006}}{16941440}k_{33}\right)\rho^2\Bigg] + \cdots. \tag{11.4.17}$$

Approximations up to 3rd-order for the steady-state solutions and their stabilities can be found from equation (11.4.16): The solution $\rho = 0$ represents the initial equilibrium solution O_e (or $C_\pm$ for the original system (11.4.6)), which is stable when $\mu < 0$ (i.e., $r < r_H = 82$) and unstable when $\mu > 0$ ($r > 82$). The supercritical Hopf bifurcation solution can be obtained, if

$$\frac{4646315818 - 102399253\sqrt{1006}}{358010321904} - \frac{5746272 + 187233\sqrt{1006}}{4235360}k_{33} < 0,$$

i.e.,

$$k_{33} > \frac{3672843514\sqrt{1006} - 115816173526}{478327912875} \approx 0.001416.$$

Choosing $k_{33} = 0.01$, we have the controller:

$$u = -1.087(z+1) - 0.01(z+1)^3. \tag{11.4.18}$$

So the controlled system described in the original states is given by

$$\begin{aligned}
\dot{x} &= -4(x - y),\\
\dot{y} &= -xz - y,\\
\dot{z} &= xy - z - r - 1.087(z+1) - 0.01(z+1)^3.
\end{aligned} \tag{11.4.19}$$

The corresponding normal form then becomes

$$\begin{aligned}
\dot{\rho} &= \rho(0.003222\mu - 0.023683\rho^2) + \cdots,\\
\dot{\theta} &= 9.562165 + 0.054678\mu - 0.046512\rho^2 + \cdots
\end{aligned}$$

and the solution for the family of bifurcating limit cycles is obtained as

$$\rho = 0.136070\sqrt{\mu} = 0.136070\sqrt{r - 82}. \tag{11.4.20}$$

Some numerical simulation results, obtained from the controlled system (11.4.19), are given in Figures 11.4.1 and 11.4.2. Figure 11.4.1 depicts that the trajectories converge to the equilibria C_+ and C_- for $1 < r < 82$, while Figure 11.4.2 demonstrates the stable limit cycles bifurcating from the system when $r > 82$. By using equation (11.4.20), one can estimate the amplitudes of the three limit cycles shown in Figure 11.4.2 as 0.136, 0.385 and 0.593, respectively. These

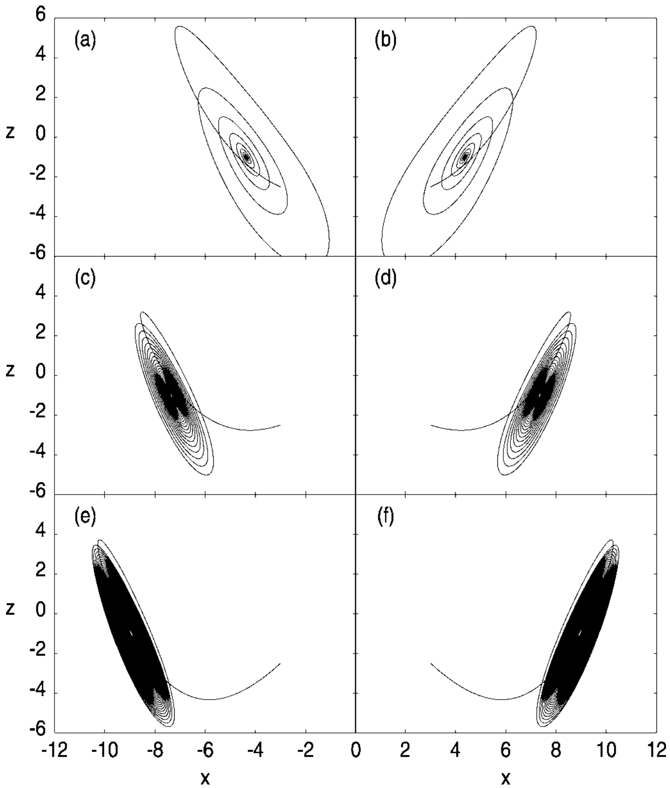


Figure 11.4.1. Stable equilibria $C_{\pm}$ of the controlled Lorenz system (11.4.19) with the control law (11.4.18) with the initial conditions $(x_0, y_0, z_0) = (\pm 3.0, \pm 12.0, -2.5)$ when (a) and (b) $r = 20$; (c) and (d) $r = 55$; and (e) and (f) $r = 81$.

approximations give a good prediction, confirmed by the numerical simulation results. It can be seen from Figures 11.4.1 and 11.4.2 that the symmetry of the two equilibria C_+ and C_- remain unchanged before and after the Hopf bifurcation generated by using the simple control (11.4.18).

A nonlinear electrical circuit

Now we use a nonlinear electrical circuit to demonstrate the use of the SNF to consider Hopf bifurcation. The electrical circuit, shown in Figure 11.4.3, consists of an inductor, L, two capacitors C_1 and C_2, two resistors R_1 and R_2, a tunnel-diode and a conductance. Suppose L, C_1, C_2, R_1 and R_2 are linear components, and in addition, R_1 may be varied. The tunnel-diode and the conductance are nonlinear elements, and they are voltage-controlled. The conductance is a combination of a tunnel-diode and a current-reversing device. The characteristics of the

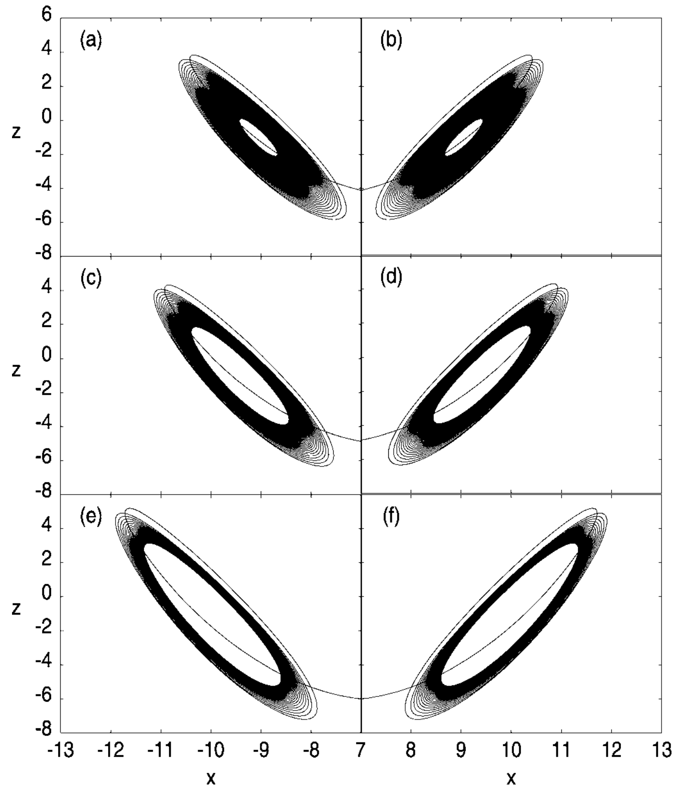


Figure 11.4.2. Stable limit cycles around $C_{\pm}$ of the controlled Lorenz system (11.4.19) with the control law (11.4.18) with the initial conditions $(x_0, y_0, z_0) = (\pm 3.0, \pm 12.0, -2.5)$ when (a) and (b) $r = 83$; (c) and (d) $r = 90$; and (e) and (f) $r = 101$.

tunnel-diode is given by [79] by

$$\begin{aligned} i_d = f(V_d) &\triangleq 0.01776V_d - 0.10379V_d^2 + 0.22962V_d^3 \\ &\quad - 0.22631V_d^4 + 0.08372V_d^5. \end{aligned} \tag{11.4.21}$$

Thus, the characteristics of the conductance is $i_G = -f(V_G)$. The current in the inductor and the voltages across the capacitors are chosen as the state variables (as shown in Figure 11.4.3), leading to the following differential equations:

$$\begin{aligned} L\frac{di_L}{dt} &= -R_1 i_L - V_{C_1}, \\ C_1\frac{dV_{C_1}}{dt} &= -i_G + i_L - \frac{1}{R_2}(V_{C_1} - V_{C_2}), \end{aligned}$$

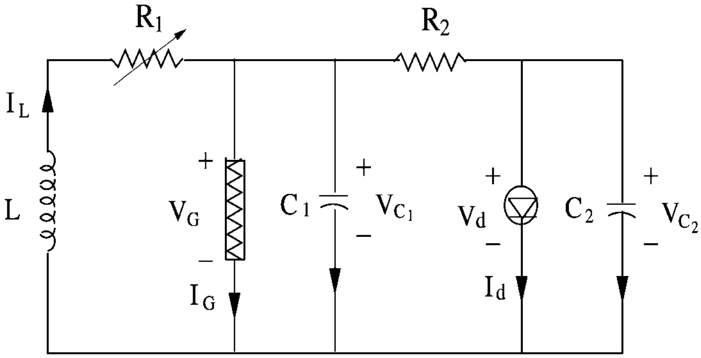


Figure 11.4.3. A nonlinear electrical circuit.

$$C_2 \frac{dV_{C_2}}{dt} = -i_d + \frac{1}{R_2}(V_{C_1} - V_{C_2}). \tag{11.4.22}$$

Denoting the state variables i_{L_1}, V_{C_1} and V_{C_2} by x, y and z, respectively, we may rewrite equation (11.4.22) as

$$\begin{aligned}
\frac{dx}{dt} &= -R_1 x - y, \\
\frac{dy}{dt} &= x - 0.001(y - z) + 0.01776y - 0.10379y^2 + 0.22962y^3 \\
&\quad - 0.22631y^4 + 0.08372y^5, \\
\frac{dz}{dt} &= 0.001(y - z) - 0.01776z + 0.10379z^2 - 0.22962z^3 \\
&\quad + 0.22631z^4 - 0.08372z^5,
\end{aligned} \tag{11.4.23}$$

where L, C_1, C_2 and R_2 have been chosen respectively, the values 1, 1, 1 and 1000 in the corresponding units, while R_1 is treated as a control parameter.

System (11.4.23) has multiple equilibrium solutions obtained from $\dot{x} = \dot{y} = \dot{z} = 0$. Here, we only consider bifurcations from the trivial solution $x = y = z = 0$, and pay particular attention to Hopf bifurcation. It is easy to obtain the characteristic polynomial of system (11.4.23) evaluated at the trivial equilibrium solution as

$$\begin{aligned}
P(\lambda) = \lambda^3 &+ (0.002 + R_1)\lambda^2 + (0.9996845824 + 0.002R_1)\lambda \\
&+ 0.01876 - 0.0003154176R_1.
\end{aligned}$$

Applying the Hurwitz criterion yields the stability condition for the trivial equilibrium:

$$0.0167600020 < R_1 < 59.4767064362. \tag{11.4.24}$$

Further, it can be shown that a static bifurcation occurs at $R_1 = 59.4767064362$ while a Hopf bifurcation emerges at $R_1 = 0.0167600020$. Suppose the current state of the system is under the selection of $R_1 = 30$, and we decrease R_1 until $R_1 = 0.0167600020$ at which the trivial equilibrium solution becomes unstable, and a family of limit cycles bifurcates as R_1 further decreases.

(A) Without control. First, consider the case without control. To obtain the stability condition using the SNF described in Section 11.3, let

$$R_1 = 0.0167600020 - \mu. \tag{11.4.25}$$

Then, introduce the transformation $(x, y, z)^T = Q(\tilde{x}, \tilde{y}, \tilde{z})$, where Q is

$$Q = \begin{bmatrix} 1.0000431639 & 0.0107620497 & -0.0009999305 \\ -0.0060001928 & -1.0000825710 & -0.0000019999 \\ -0.0009999841 & -0.0000127613 & 1.0000014998 \end{bmatrix}, \tag{11.4.26}$$

to system (11.4.23) to obtain

$$\begin{aligned}
\frac{d\tilde{x}}{dt} &= 0.9998590413\tilde{y} + \big(1.0000655704\tilde{x} + 0.0107622908\tilde{y} \\
&\quad - 0.0009999529\tilde{z}\big)\mu - 0.0000000401\tilde{x}^2 - 0.0011171091\tilde{y}^2 \\
&\quad + 0.0001037875\tilde{z}^2 - 0.0000134046\tilde{x}\tilde{y} - 0.0000002076\tilde{x}\tilde{z} \\
&\quad - 0.0000000071\tilde{y}\tilde{z} + \cdots, \\
\frac{d\tilde{y}}{dt} &= -0.9998590413\tilde{x} - \big(0.0060000928\tilde{x} + 0.0000645705\tilde{y} \\
&\quad - 0.0000059994\tilde{z}\big)\mu + 0.0000037366\tilde{x}^2 + 0.1038052724\tilde{y}^2 \\
&\quad - 0.0000008302\tilde{z}^2 + 0.0012456004\tilde{x}\tilde{y} + 0.0000000042\tilde{x}\tilde{z} \\
&\quad + 0.0000004152\tilde{y}\tilde{z} + \cdots, \\
\frac{d\tilde{z}}{dt} &= -0.0187600020\tilde{z} + \big(0.0009999716\tilde{x} + 0.0000107613\tilde{y} \\
&\quad - 0.0000009999\tilde{z}\big)\mu + 0.0000001038\tilde{x}^2 + 0.0000002076\tilde{y}^2 \\
&\quad + 0.1037902594\tilde{z}^2 + 0.0000000051\tilde{x}\tilde{y} - 0.0002075769\tilde{x}\tilde{z} \\
&\quad - 0.0000026490\tilde{y}\tilde{z} + \cdots,
\end{aligned} \tag{11.4.27}$$

where $\cdots$ represents higher order terms. System (11.4.27) clearly shows that its Jacobian evaluated at the origin $\tilde{x} = \tilde{y} = \tilde{z} = 0$ is in Jordan canonical form.

Executing the Maple program yields the following SNF:

$$\frac{dR}{d\tau} = R\big(0.5000005000\nu + 0.0861699786R^2\big),$$

$$\frac{d\Theta}{d\tau} = 0.9998590408 + 0.0083811918\nu - 0.3163637547R^2 + \cdots \tag{11.4.28}$$

up to arbitrary order, which clearly indicates that a subcritical Hopf bifurcation occurs at the critical point $\nu = 0$, and thus the bifurcating limit cycles of the uncontrolled system (11.4.23) are unstable.

(B) With control. Now, consider adding a feedback control to system (11.4.23). It is required that the control does not change the equilibrium $x = y = z = 0$, but converts the subcritical Hopf bifurcation to supercritical. There exist many ways to design the feedback control. We take a simple one, given in the form of

$$u_2 = -k_n y^3, \tag{11.4.29}$$

which is added to the second equation of equation (11.4.23). Then, under the same transformation used in the case of no control, employing the Maple program to obtain the following SNF for the controlled system:

$$\begin{aligned} \frac{dR}{d\tau} &= R\big[0.5000005000\nu + (0.0861699786 - 0.3750754316k_n)R^2\big], \\ \frac{d\Theta}{d\tau} &= 0.9998590408 \\ &\quad + \frac{0.0104785841k_n^2 - 0.0126007750k_n + 0.0726815346}{k_n - 0.2297403970}R^2 \\ &\quad + \cdots. \end{aligned} \tag{11.4.30}$$

Thus, as long as

$$\left.\begin{array}{l} 0.0861699786 - 0.3750754316k_n < 0 \\ k_n - 0.2297403970 \neq 0 \end{array}\right\} \quad \text{i.e., } k_n > 0.2297403971, \tag{11.4.31}$$

the Hopf bifurcation of the controlled system is supercritical.

The numerical simulation results of the electrical circuit are shown in Figures 11.4.4 and 11.4.5, respectively for the uncontrolled and controlled systems. It is seen from Figure 11.4.4(a) and Figure 11.4.5(a) that the trajectories converge to the origin $x = y = z = 0$ when $R_1 = 1.0$ (i.e., $\mu = -0.983239998$) for both controlled and uncontrolled systems. This indicates that the origin $x = y = z = 0$ is stable when $\mu < 0$. However, when $R_1 = 0.012$ (i.e., $\mu = 0.004760002$) the trajectory of the uncontrolled system diverges to infinity even from an initial point close to the origin (see Figure 11.4.4(b)) implying that the origin is unstable and the Hopf bifurcation is subcritical; while the trajectory of the controlled system converges to a stable limit cycle (see Figure 11.4.5(b) where only the final steady-state of the limit cycle is shown and the initial point is marked by +). This indeed verifies that the Hopf bifurcation of the controlled system becomes supercritical.

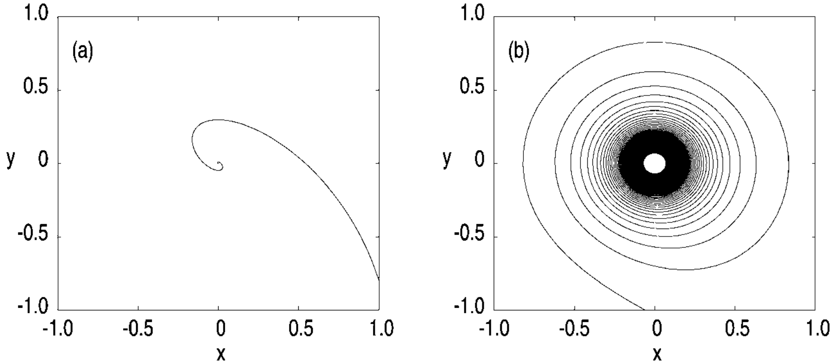


Figure 11.4.4. Simulated trajectories of the uncontrolled electrical circuit (11.4.23): (a) $R_1 = 1$, convergent to the origin $x = y = z = 0$ from the initial point $(1.0, -0.8, 1.0)$; and (b) $R_1 = 0.012$, divergent to infinity from the initial point $(0.05, -0.05, 0.08)$.

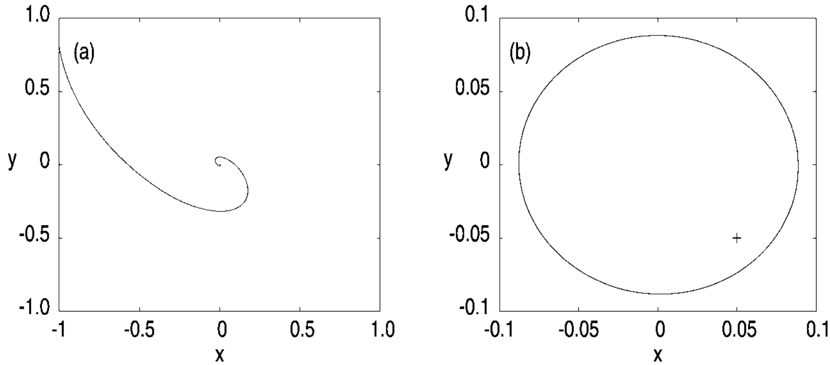


Figure 11.4.5. Simulated trajectories of the controlled electrical circuit (11.4.23) with the controller (11.4.29) with $k_n = 0.5$: (a) $R_1 = 1$, convergent to the origin $x = y = z = 0$ from the initial point $(-1.0, 0.8, -1.0)$; and (b) $R_1 = 0.012$, convergent to a limit cycle from the initial point $(0.05, -0.05, 0.08)$.

Note that the feedback control (11.4.29) changes the second equation of (11.4.23) to

$$\frac{dy}{dt} = x - 0.001(y - z) + 0.01776y - 0.10379y^2 - 0.27038y^3 \\ - 0.22631y^4 + 0.08372y^5.$$

It is seen from the above equation that the sign of the third order term has been changed from positive to negative, which renders the subcritical Hopf bifurcation to supercritical.

11.4.2. Discrete maps

There are wide potential applications in controlling and anti-controlling bifurcations. For bifurcation control, the potential applications include delaying or avoiding voltage collapse in electric power systems, controlling pathological heart rhythms, and enhancing the operability of the compression system in jet engines. Anti-controlling bifurcations, on the other hand, can serve as a warning signal. For example, the occurrence of a saddle node bifurcation in an electric system has been linked to the incipient instability, a stable limit cycle can be used as the warning signal of impending collapse or catastrophe by introducing a supercritical Hopf bifurcation near the bifurcation point. Also, anti-control of Hopf bifurcation is viewed as an approach of generating limit cycles in dynamical systems [397]. In this section, the polynomial type of controller will be extended to consider control and anti-control of Hopf bifurcations in discrete maps.

2-D discrete maps

Consider the following general 2-dimensional parametrized map:

$$\begin{aligned} x_{n+1} &= F(x_n, y_n, \mu), \\ y_{n+1} &= G(x_n, y_n, \mu), \end{aligned} \tag{11.4.32}$$

which exhibits Hopf bifurcation if a simple pair of complex conjugate eigenvalues of the linearized map crosses the unit circle [146,136,214]. One can divide the Hopf bifurcation into resonant and nonresonant cases depending on whether these eigenvalues cross at the roots of unity. Further, if the pair of eigenvalues λ_0 and $\bar{\lambda}_0$ do not satisfy $\lambda_0^n = 1$ for $n = 1, 2, 3, 4$, then the Hopf bifurcation is called week resonance, otherwise, it is strong resonance. In the case of strong resonance, the dynamical motion is complex and the bifurcation could generate a stable or unstable fixed point of order-4 subharmonic solution, or a Hopf circle. While in the case of nonresonance or week resonance, a Hopf bifurcation occurs [136]. The main results are summarized below.

THEOREM 11.4.2. *Suppose that the discrete system (11.4.32) satisfies $F(0, 0, \mu) = G(0, 0, \mu) = 0$ on some neighborhood of $\mu = 0$ and that when $\mu = 0$ the Jacobian matrix of the map at the fixed point $(x, y) = (0, 0)$ is given by*

$$J\big(\theta_0(\mu)\big) = \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix}, \tag{11.4.33}$$

where $e^{ni\theta_0} \neq 1$ for $n = 1, 2, 3, 4$. Further, if the following conditions:

$$\cos\theta_0(F_{\mu x} + G_{\mu y}) + \sin\theta_0(G_{\mu x} - F_{\mu y}) \neq 0 \tag{11.4.34}$$

and

$$\mathrm{Re}\, A(0) \neq 0 \tag{11.4.35}$$

are satisfied, then an invariant circle bifurcates for either $\mu > 0$ or $\mu < 0$, depending upon the sign of $\mathrm{Re}\, A(0)$. *Here, the partial derivatives are evaluated at the critical point $(x, y, \mu) = (0, 0, 0)$ and $A(\mu)$ is a complex function of μ, and* $\mathrm{Re}\, A(0) = \cos\theta_0 \,\mathrm{Re}(\alpha_1(0)) + \sin\theta_0 \,\mathrm{Im}(\alpha_1(0))$, *where*

$$\alpha_1(0) = \gamma_{32}'(0) = \gamma_{32} + \frac{|\gamma_{21}|^2}{1-\bar{\lambda}_0} + \frac{2|\gamma_{20}|^2}{\lambda_0^2-\bar{\lambda}_0} + \gamma_{21}\gamma_{22}\frac{2\lambda_0-1}{\lambda_0(1-\lambda_0)}, \tag{11.4.36}$$

in which

$$\begin{aligned}
\gamma_{22} &= \frac{1}{8}\big[(F_{xx} + 2G_{xy} - F_{yy}) + i(G_{xx} - 2F_{xy} - G_{yy})\big],\\
\gamma_{21} &= \frac{1}{4}\big[(F_{xx} + F_{yy}) + i(G_{xx} + G_{yy})\big],\\
\gamma_{20} &= \frac{1}{8}\big[(F_{xx} - 2G_{xy} - F_{yy}) + i(G_{xx} + 2F_{xy} - G_{yy})\big],\\
\gamma_{32} &= \frac{1}{16}\big[(3F_{xxx} + F_{xyy} + G_{xxy} + 3G_{yyy})\\
&\quad + i(3G_{xxx} + G_{xyy} - F_{xxy} - 3F_{yyy})\big].
\end{aligned} \tag{11.4.37}$$

This invariant circle is attracting (a supercritical bifurcation) if it bifurcates into the region of μ for which the trivial equilibrium point (origin) is unstable and repelling (a subcritical bifurcation) if it bifurcates into the region for which the origin is stable.

Note that the above theorem assumes that the bifurcation point is $(x, y, \mu) = (0, 0, 0)$. In general case, if the bifurcation point is not at the origin, one can introduce a linear transformation to shift the bifurcation point to the origin. The condition (11.4.33) guarantees that system (11.4.32) has a pair of complex conjugate eigenvalues crossing the unit circle, while condition (11.4.34) implies a nonzero transversality condition, i.e., $\frac{d|\lambda|}{d\mu}(0) \neq 0$. Also, note that an invariant circle bifurcates from the origin when $\mu = 0$ in system (11.4.32) provided condition (11.4.35) holds, in which the value of $\mathrm{Re}\, A(0)$ is obtained via a change of the system's coordinates.

3-D discrete maps

The critical condition at which a Hopf bifurcation occurs in high dimensional discrete maps is usually derived from the Schur–Cohn criterion [397,223]. Unlike a 2-D system, in this case, center manifold theory is applied first to reduce the high dimensional system to a 2-D center manifold, and then normal formal theory is employed to determine the stability of the Hopf bifurcation.

The Schur–Cohn criterion, similar to the well-known Hurwitz criterion for the continuous-time system, can be used to determine the conditions that the roots of characteristic polynomial lie inside the unit circle, which is described as follows.

THEOREM 11.4.3 *(Schur–Cohn criterion [223]). Suppose that the characteristic polynomial, $P(\lambda)$, of an $n \times n$ Jacobian matrix J, is given by*

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0, \tag{11.4.38}$$

then the necessary and sufficient conditions for $P(\lambda)$ to have all its roots inside the unit circle are:

(i) $P(1) > 0$ *and* $(-1)^n P(-1) > 0$;
(ii) *the two* $(n-1) \times (n-1)$ *matrices*:

$$\Delta_{n-1}^{\pm} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ a_{n-1} & 1 & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_3 & & & & \vdots \\ a_2 & a_3 & \cdots & a_{n-1} & 1 \end{bmatrix} \pm \begin{bmatrix} 0 & \cdots & \cdots & 0 & a_0 \\ \vdots & & & a_0 & a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & & & & a_{n-1} \\ a_0 & a_1 & \cdots & a_{n-1} & a_{n-2} \end{bmatrix}$$

are both positive innerwise.

For example, when $n = 3$, $P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Its roots lie inside the unit circle if and only if

$$|a_0 + a_2| < 1 + a_1 \quad \text{and} \quad |a_1 - a_0 a_2| < 1 - a_0^2.$$

When $n = 4$, the roots of $P(\lambda)$ are all lie inside the unit circle if and only if

$$\begin{aligned} &|a_0| < 1, \quad |a_1 + a_3| < 1 + a_2 + a_0, \quad \text{and} \\ &\left|a_2(1 - a_0) + a_0\left(1 - a_0^2\right) + a_3(a_0 a_3 - a_1)\right| \\ &\quad < a_0 a_2(1 - a_0) + \left(1 - a_0^2\right) + a_1(a_0 a_3 - a_1). \end{aligned}$$

The conditions for a Hopf bifurcation require that at a critical parameter value, a pair of complex conjugate eigenvalues lie on the unit circle and other eigenvalues are still inside the unit circle. The critical conditions for a Hopf bifurcation to occur in high dimensional discrete maps can be derived using the Schur–Cohn criterion. Recently, Wen et al. obtained the criterion of Hopf bifurcation for 3-D [397] and 4-D discrete maps [398].

For the 3-D case, the result obtained by Wen et al. in [397] is stated below.

THEOREM 11.4.4. *(See [397].) For a matrix $M = (m_{ij})_{3\times 3}$, the necessary and sufficient conditions for M to have a pair of complex conjugate eigenvalues located on the unit circle and the remaining eigenvalues inside the unit circle are*

(i) $|a_0| < 1$,
(ii) $|a_0 + a_2| < 1 + a_1$,
(iii) $a_1 - a_0 a_2 = 1 - a_0^2$,

where $a_k = a_k(m_{ij}) \in R$, $k = 0, 1, 2$, are the coefficients of the characteristic polynomial of the matrix M.

Denote the pair of complex conjugate eigenvalues of matrix M by $\lambda_1(\epsilon)$ and $\bar{\lambda}_1(\epsilon)$, where ϵ is a bifurcation parameter. Then the critical condition for the Hopf bifurcation together with the transversality condition $(\frac{\partial|\lambda_1(\epsilon)|}{\partial\epsilon}|_{\epsilon=0} \neq 0)$ and the nonresonance condition $(\lambda_1^n(0) \neq 1)$ can determine the occurrence of a Hopf bifurcation.

To determine the stability of a Hopf bifurcation in a high dimensional discrete map, the center manifold reduction and Iooss's Hopf bifurcation theory are used [397,181]. To achieve this, consider a 3-D map, given by

$$X_{k+1} = G(X_k; \mu), \tag{11.4.39}$$

where $X_k = (x_k, y_k, w_k)$, and μ is a bifurcation parameter. Suppose that a Hopf bifurcation occurs at the desired location $X_0 = (x_0, y_0, w_0)$, and the critical value of μ for the occurrence of the Hopf bifurcation at X_0 is μ_0. Let $\epsilon = \mu_0 - \mu$ and $\Delta X_k = (x_k - x_0, y_k - y_0, w_k - w_0)$. Then system (11.4.39) can be written as

$$\Delta X_{k+1} = \tilde{G}(\Delta X_k; \epsilon). \tag{11.4.40}$$

The Jacobian $D\tilde{G}(0; 0)$ satisfies the necessary conditions for Hopf bifurcation. Let P be the eigenmatrix corresponding to $D\tilde{G}(0; \epsilon)$. Then under the transformation $\Delta X_k = PY_k$, system (11.4.40) becomes

$$Y_{k+1} = F(Y_k; \epsilon), \tag{11.4.41}$$

where $Y_k = (y_{1k}, y_{2k}, y_{3k})^T$, $F(Y_k, \epsilon) = (F_1, F_2, F_3)^T$, and $DF(0; \epsilon)$ takes the form:

$$DF(0; \epsilon) = \begin{bmatrix} \mathrm{Re}\,\lambda_1(\epsilon) & -\mathrm{Im}\,\lambda_1(\epsilon) & 0 \\ \mathrm{Im}\,\lambda_1(\epsilon) & \mathrm{Re}\,\lambda_1(\epsilon) & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{11.4.42}$$

Under the complex transformation $z_k = y_{1k} + iy_{2k}$, $\bar{z_k} = y_{1k} - iy_{2k}$, $W_k = y_{3k}$, system (11.4.41) can be rewritten as

$$z_{k+1} = \lambda_1(\epsilon)z_k + G\big(z_k, \bar{z}_k, W_k; \epsilon\big), \tag{11.4.43}$$

$$W_{k+1} = \lambda_3(\epsilon)W_k + H\big(z_k, \bar{z}_k, W_k; \epsilon\big), \tag{11.4.44}$$

where $G(z_k, \bar{z}_k, W_k; \epsilon) = F_1 + iF_2 - \lambda_1(\epsilon)z_k$, $H(z_k, \bar{z}_k, W_k; \epsilon) = F_3 - \lambda_3(\epsilon)W_k$. There exists a local center manifold for equations (11.4.43) and (11.4.44), given in a series form:

$$W_k\big(z_k, \bar{z}_k; \epsilon\big) = \sum_{i+j=2}^{m} w_{ij}(\epsilon)z_k^i\bar{z}_k^j + \mathrm{o}\big(|z_k|^{m+1}\big), \tag{11.4.45}$$

where w_{ij} can be determined by substituting (11.4.45) into (11.4.44). Thus, we obtain a center manifold (a 2-D map) by substituting (11.4.45) (where w_{ij} is known) into (11.4.43), which can be expressed as

$$\tilde{\Phi}_\epsilon\big(z_k, \bar{z_k}\big) = \lambda_1(\epsilon)z_k + \sum_{i+j=2}^{3} \xi_{ij}(\epsilon)z_k^i\bar{z}_k^j + \mathrm{o}\big(|z_k|^4\big). \tag{11.4.46}$$

Based on Iooss's Hopf bifurcation theory [181], we can make a smooth ϵ-dependent transformation of coordinates from $\tilde{\Phi}_\epsilon(z_k, \bar{z_k})$ to the normal form, denoted by $\Phi_\epsilon(z_k, \bar{z_k})$:

$$\Phi_\epsilon\big(z_k, \bar{z_k}\big) = \lambda_1(\epsilon)z_k + \alpha_1(\epsilon)z_k^2\bar{z}_k + \mathrm{O}\big(|z_k|^5\big), \tag{11.4.47}$$

where $\alpha_1(\epsilon)$ is the coefficient of the normal form, satisfying

$$\begin{aligned}\alpha_1(0) = \xi_{21} &+ \frac{2|\xi_{02}|^2}{\lambda_1^2(0) - \bar{\lambda}_1(0)} + \frac{|\xi_{11}|^2}{1 - \bar{\lambda}_1(0)} \\ &+ \frac{(1 - 2\lambda_1(0))\xi_{11}\xi_{20}}{\lambda_1^2(0) - \lambda_1(0)},\end{aligned} \tag{11.4.48}$$

where $\xi_{ij} = \xi_{ij}(\epsilon)|_{\epsilon=0}$. The stability of the Hopf bifurcation is then determined by the sign of

$$\operatorname{Re} A(0) = \operatorname{Re}\big(\lambda_1(0)\big)\operatorname{Re}\big(\alpha_1(0)\big) + \operatorname{Im}\big(\lambda_1(0)\big)\operatorname{Im}\big(\alpha_1(0)\big). \tag{11.4.49}$$

If $\operatorname{Re} A(0)$ is positive (negative), a stable (unstable) Hopf bifurcation is formed.

It should be noted that in analogous to continuous-time systems [420], the above two steps in the Schur–Cohn process can be combined to develop one unified approach. This approach directly reduces the original system to the normal form on a 2-D center manifold. The detailed formulation for this general method is out of the scope of the chapter, and thus will not be further discussed here.

Polynomial bifurcation controller for discrete maps

In this subsection, we introduce a feedback controller using polynomial functions for discrete maps. This method has been discussed in Section 4.1.1 for continuous-time systems and successfully applied to study the Lorenz system and a nonlinear electrical circuit.

Consider a general discrete map, given by

$$x_{n+1} = f(x_n, \mu), \quad x \in R^n, \ \mu \in R^m, \ f : R^{n+m} \to R^n, \tag{11.4.50}$$

with k equilibria, determined from the equation $x = f(x, \mu)$. Suppose the equilibria are given by

$$x_i^*(\mu) = \left(x_{1i}^*, x_{2i}^*, \ldots, x_{ni}^*\right), \quad i = 1, 2, \ldots, k.$$

The goal of Hopf bifurcation control is to design a controller, given in the form of

$$h = h(x, \mu),$$

such that the original equilibrium point x^* is unchanged under the control h. Thus, it requires

$$h\left(x_i^*, \mu\right) = (h_1, h_2, \ldots, h_n)^T = 0.$$

Similar to the control formula (11.4.5) used for continuous-time systems [430], we propose a general polynomial control function for discrete maps as follows:

$$\begin{aligned}
h_q(x, \mu) = {} & \sum_{i=1}^{n} A_{qi} \prod_{j=1}^{k} \left(x_i - x_{ij}^*\right) \\
& + \sum_{i=1}^{n} \sum_{j=1}^{k} B_{qij} \left(x_i - x_{ij}^*\right) \prod_{p=1}^{k} \left(x_i - x_{ip}^*\right) \\
& + \sum_{i=1}^{n} \sum_{j=1}^{k} C_{qij} \left(x_i - x_{ij}^*\right)^2 \prod_{p=1}^{k} \left(x_i - x_{ip}^*\right) \\
& + \sum_{i=1}^{n} \sum_{j=1}^{k} D_{qij} \left(x_i - x_{ij}^*\right)^3 \prod_{p=1}^{k} \left(x_i - x_{ip}^*\right) + \cdots \\
& (q = 1, 2, \ldots, n).
\end{aligned} \tag{11.4.51}$$

Then controlled system is then given by

$$x_{n+1} = f(x_n, \mu) + h(x_n; \mu) \equiv F(x_n, \mu), \tag{11.4.52}$$

and the two goals of the control are the same as that of continuous-time systems.

Examples

In this subsection, we use the polynomial feedback controller (11.4.51) to control and anti-control Hopf bifurcations in discrete maps. First, we consider the classical delay logistic map and show that using a simple polynomial feedback controller can not only delay the onset of an existing Hopf bifurcation at the nontrivial fixed point (controlling Hopf bifurcation), but also create a new Hopf bifurcation at the trivial fixed point (anti-controlling Hopf bifurcation). In both cases, the structure of the original system is preserved. That is, the polynomial law will not change the stability properties of the equilibrium point which is not the control object. Then, we study anti-controlling Hopf bifurcations at a period-1 fixed point in Hénon map, which was considered in [397] using a washout filter controller. For comparison, we choose the same parameter values used in [397]. It will be shown that with a simple polynomial controller, we can effectively create a Hopf bifurcation at the period-1 fixed point in Hénon map without increasing the dimension of the system and keep the stability property of the other fixed point. Compared to the washout filter controller, our method is easier to be implemented, and the analysis is simpler. Finally, a polynomial controller is applied to a high dimensional discrete map (a 3-D Hénon map). Anti-control of Hopf bifurcation is considered at a critical point, leading to a strange attractor. Central manifold theory and normal form theory are applied to determine the stability of the Hopf bifurcation. It will be shown that our approach is much easier to apply and the derivation is less involved compared to using the traditional washout filter controller [397].

(A) Delay logistic map. We begin with the design of a polynomial controller to control and anti-control Hopf bifurcation in the classical delay logistic map, given by

$$y_{n+1} = \mu y_n (1 - y_{n-1}). \tag{11.4.53}$$

Letting $x_n = y_{n-1}$, equation (11.4.53) is transformed into a system of 2-D discrete map:

$$\begin{aligned} x_{n+1} &= y_n, \\ y_{n+1} &= \mu y_n (1 - x_n). \end{aligned} \tag{11.4.54}$$

System (11.4.54) has a trivial fixed point $(x, y) = (0, 0)$ and a nontrivial fixed point $(x, y) = (\frac{\mu-1}{\mu}, \frac{\mu-1}{\mu})$. It is easy to show that when $\mu > 1$, the trivial fixed point is a saddle node. At the nontrivial fixed point, the Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ 1-\mu & 1 \end{bmatrix},$$

which has eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\big(1 \pm \sqrt{5-4\mu}\big).$$

For $\mu > \frac{5}{4}$, the eigenvalues are complex conjugate and can be written as

$$\lambda,\ \bar{\lambda} = (\mu - 1)e^{\pm ic}, \quad c = \tan^{-1}\sqrt{4\mu - 5}.$$

Thus, at $\mu = 2$, $\lambda, \bar{\lambda} = e^{\pm i\pi/3}$ are the sixth roots of the unity, and

$$\frac{d}{d\mu}\big|\lambda(\mu)\big|_{\mu=2} = 1 \neq 0.$$

Thus, system (11.4.54) undergoes a Hopf bifurcation when $\mu = 2$, at which the system has a trivial fixed point (the origin) and a nontrivial fixed point: $(x, y) = (\frac{1}{2}, \frac{1}{2})$. When $\mu > 2$, $|\lambda| = |\bar{\lambda}| = \mu - 1 > 1$, the nontrivial fixed point is an unstable node while the trivial fixed point is a saddle point.

To determine the stability coefficient $\operatorname{Re} A(0)$, we first transform system (11.4.54) into a normal form. Note that when $\mu = 2$, the complex eigenvector corresponding to the eigenvalue $e^{i\pi/3}$ is given by

$$\begin{bmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + i\begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} \equiv e_1 + ie_2. \tag{11.4.55}$$

Applying a change of the coordinates at the nontrivial fixed point $(x, y) = (\frac{1}{2}, \frac{1}{2})$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + [e_2\ e_1]\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \tag{11.4.56}$$

into system (11.4.54) yields the normal form:

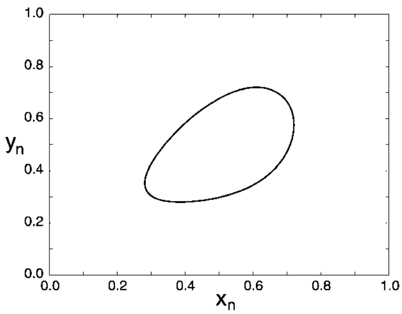
$$\begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 2uv + 2v^2 \\ 0 \end{bmatrix}. \tag{11.4.57}$$

Using formulas (11.4.36) and (11.4.37), we can easily obtain that $\gamma_{22} = \frac{1}{2} + \frac{1}{2}i$, $\gamma_{21} = -1$, $\gamma_{20} = \frac{1}{2} - \frac{1}{2}i$, $\gamma_{32} = 0$, $\alpha_1(0) = \frac{\sqrt{3}}{2} + \frac{1}{4} - \frac{5\sqrt{3}}{4}i$. Thus,

$$\operatorname{Re} A(0) = \frac{\sqrt{3} - 7}{4} < 0, \tag{11.4.58}$$

indicating that a stable Hopf bifurcation occurs at $\mu = 2$. The numerical simulation result is shown in Figure 11.4.6(a), confirming the existence of a stable closed orbit in the phase space.

Now, consider adding a polynomial controller to system (11.4.54) so that we may delay the Hopf bifurcation from the nontrivial fixed point. For the sake of simplification, we may only choose a control component for the second equation of system (11.4.54). By using formula (11.4.51), we can explicitly write the



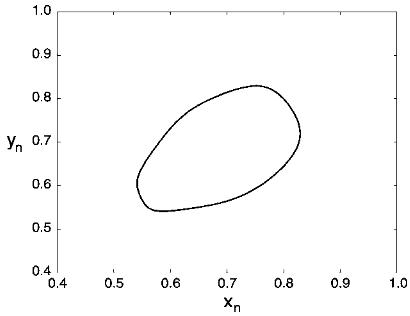


Figure 11.4.6. The Hopf bifurcation in the delay logistic equation (4.2): (a) without control; and (b) with a feedback controller (11.4.59) with $\mu = 3.05,\ A_{11} = \mu$ and $A_{12} = 0$.

controller as

$$h_n = A_{11}(x_n - 0)\left(x_n - \frac{\mu - 1}{\mu}\right) + A_{12}(y_n - 0)\left(y_n - \frac{\mu - 1}{\mu}\right), \quad (11.4.59)$$

which preserves the two equilibria of system (11.4.54). Higher order terms are neglected for simplicity. The controlled system is then given by

$$\begin{aligned} x_{n+1} &= y_n, \\ y_{n+1} &= \mu y_n(1 - x_n) + A_{11}(x_n - 0)\left(x_n - \frac{\mu - 1}{\mu}\right) \\ &\quad + A_{12}(y_n - 0)\left(y_n - \frac{\mu - 1}{\mu}\right). \end{aligned} \quad (11.4.60)$$

The Jacobian matrix of (11.4.60) evaluated at the nontrivial fixed point $(x, y) = (\frac{\mu-1}{\mu}, \frac{\mu-1}{\mu})$ is

$$\begin{bmatrix} 0 & 1 \\ 1 - \mu + \frac{\mu-1}{\mu}A_{11} & 1 + \frac{\mu-1}{\mu}A_{12} \end{bmatrix}. \quad (11.4.61)$$

Denote the eigenvalues of (11.4.61) by λ_1, λ_2. In order that the controlled system has a Hopf bifurcation, the conditions $\lambda_1 = \bar{\lambda}_2$ and $|\lambda_1| = 1$ must be satisfied. Under the choice of $A_{11} = \frac{1}{2}\mu$, the necessary conditions for (11.4.61) to have a pair of complex conjugate eigenvalues are

$$\begin{aligned} &1 - \mu + \frac{\mu - 1}{\mu}A_{11} = -1, \\ &\quad \left(1 + \frac{2}{3}A_{12}\right)^2 < 4. \end{aligned} \quad (11.4.62)$$

Thus, we may further choose $\mu = 3,\ A_{12} = 0$. Then the Jacobian matrix evaluated at the nontrivial fixed point of the controlled system becomes

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \tag{11.4.63}$$

which has a pair of complex conjugate eigenvalues: $\lambda, \bar{\lambda} = e^{\pm i\pi/3}$ and

$$\frac{d}{d\mu}\left|\lambda(\mu)\right|_{\mu=3} = \frac{1}{2} \neq 0,$$

indicating that a Hopf bifurcation occurs at $\mu = 3$ for the controlled system (11.4.60) instead of a Hopf bifurcation at $\mu = 2$ for the uncontrolled system (11.4.54). This clearly shows that the Hopf bifurcation at the nontrivial fixed point is delayed by using a polynomial controller. While at the trivial fixed point, the Jacobian matrix for the controlled system is

$$\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}, \tag{11.4.64}$$

which has eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ and $\lambda_2 = \frac{3-\sqrt{5}}{2} < 1$. So the trivial fixed point is still a saddle point. That means, when we delay the Hopf bifurcation at the nontrivial point, we do not change the stability of the trivial point. In other words, the stability of the trivial fixed point is still under control. To determine the stability of the Hopf bifurcation in the controlled system, we apply a similar change of the coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{11.4.65}$$

to bring system (11.4.60) into the normal form:

$$\begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 3uv \\ 0 \end{bmatrix}. \tag{11.4.66}$$

Using formulas (11.4.36) and (11.4.37), we obtain that $\gamma_{22} = \frac{3}{4}i$, $\gamma_{20} = -\frac{3}{4}i$, $\gamma_{21} = \gamma_{32} = 0$, $\alpha_1(0) = -\frac{9}{32} - \frac{9\sqrt{3}}{32}i$. Thus,

$$\operatorname{Re} A(0) = -\frac{9}{16} < 0. \tag{11.4.67}$$

Since $\operatorname{Re} A(0) < 0$, the Hopf bifurcation of the controlled system is stable. The phase portrait obtained from a numerical simulation is depicted in Figure 11.4.6(b) to show a stable closed orbit.

Next, we show that the polynomial feedback controller can also be used to anti-control the Hopf bifurcation, that is, to generate a Hopf bifurcation when it is desirable. Consider the same delay logistic equation. Now our aim is to create a

Hopf bifurcation at the trivial fixed point $(x, y) = (0, 0)$. As we know, for $\mu > 1$, the trivial fixed point is a saddle node. Physically, the saddle node is linked to incipient instability in an electric system. In this case, the artificially generated stable limit cycle can be served as a warning signal. Introducing a supercritical Hopf bifurcation may be beneficial in this case.

For convenience, we employ the same form of the controller used in delaying the Hopf bifurcation at the nontrivial fixed point. The controlled system is

$$\begin{aligned} x_{n+1} &= y_n, \\ y_{n+1} &= \mu y_n(1-x_n) + A_{11}(x_n - 0)\left(x_n - \frac{\mu-1}{\mu}\right) \\ &\quad + A_{12}(y_n - 0)\left(y_n - \frac{\mu-1}{\mu}\right), \end{aligned} \tag{11.4.68}$$

whose Jacobian matrix evaluated at $(x, y) = (0, 0)$ is

$$\begin{bmatrix} 0 & 1 \\ \frac{1-\mu}{\mu}A_{11} & \mu + \frac{1-\mu}{\mu}A_{12} \end{bmatrix}. \tag{11.4.69}$$

Using a similar analysis as above, we may choose $A_{12} = \mu$ and $A_{11} = \frac{3}{2}$. Then a Hopf bifurcation occurs at the trivial fixed point when $\mu = 3$. The Jacobian matrix at the trivial fixed point for the controlled system when $\mu = 3$ is given by (11.4.63). Thus, the eigenvalues of (11.4.69) are $\lambda, \bar{\lambda} = e^{\pm i\pi/3}$ and

$$\frac{d}{d\mu}\left|\lambda(\mu)\right|_{\mu=3} = \frac{1}{6} \neq 0.$$

Further, introducing

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{11.4.70}$$

into system (11.4.68) results in the normal form:

$$\begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \sqrt{3}u^2 + \sqrt{3}v^2 \\ 0 \end{bmatrix}. \tag{11.4.71}$$

Then, using formulas (11.4.36) and (11.4.37), we find $\gamma_{22} = \gamma_{20} = \gamma_{32} = 0$, $\gamma_{21} = \sqrt{3}$, $\alpha_1(0) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$. Thus,

$$\operatorname{Re} A(0) = -\frac{3}{2} < 0, \tag{11.4.72}$$

implying that a stable Hopf bifurcation occurs at $\mu = 3$ for the controlled system from the trivial fixed point $(0, 0)$. See the numerical simulation result shown in Figure 11.4.7.

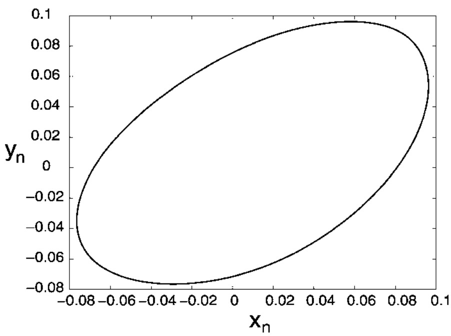


Figure 11.4.7. The Hopf bifurcation from the trivial equilibrium of the delay logistic equation (4.2) under a feedback controller (11.4.59) with $\mu = 3.05$, $A_{11} = 1.5$ and $A_{12} = \mu$.

It should be noted that at $\mu = 3$, the nontrivial fixed point is an unstable node for the uncontrolled delay logistic map. For the controlled system (11.4.68), its Jacobian matrix evaluated at the nontrivial fixed point is

$$\begin{bmatrix} 0 & 1 \\ -\frac{9}{4} & 3 \end{bmatrix}, \tag{11.4.73}$$

which gives the eigenvalues $\lambda_1 = \lambda_2 = \frac{3}{2} > 1$. This indicates that under the control, the nontrivial fixed point is still an unstable node at the critical point $\mu = 3$. Note that the above design procedure is an illustration, by no means the polynomial controller is unique. We can design different polynomial controllers for a specified system to reach different types of bifurcation control. For example, for the above anti-bifurcation control, we may want to design a controller such that a Hopf bifurcation occurs from the trivial fixed point while the nontrivial fixed point is stabilized, or another Hopf bifurcation emerges from the nontrivial fixed point, etc. Such an control strategy will be demonstrated in the next example.

(B) 2-D Hénon map. Now, we turn to use a polynomial feedback controller to anti-control Hopf bifurcations in a 2-dimensional Hénon map. The results are compared with that obtained by Wen et al. with the widely used washout filter controller [397]. For a consistent comparison, we apply the polynomial feedback control to reach the same objective considered in [397], that is, to create a Hopf bifurcation at the period-1 fixed point of the 2-D Hénon map. The 2-D Hénon system can be described by a 2-D discrete map:

$$\begin{aligned} x_{n+1} &= \rho - x_n^2 + 0.3 y_n, \\ y_{n+1} &= x_n, \end{aligned} \tag{11.4.74}$$

which has a classical period-doubling cascade to chaos when the bifurcation parameter varies from 0.1 to 4. The period-1 fixed points (x^0, y^0) and (x^1, y^1) are given by

$$\left(x^0, y^0\right) = \left(\frac{1}{2}\left(-0.7 + \sqrt{0.49 + 4\rho}\right), \frac{1}{2}\left(-0.7 + \sqrt{0.49 + 4\rho}\right)\right) \quad (11.4.75)$$

and

$$\left(x^1, y^1\right) = \left(\frac{1}{2}\left(-0.7 - \sqrt{0.49 + 4\rho}\right), \frac{1}{2}\left(-0.7 - \sqrt{0.49 + 4\rho}\right)\right). \quad (11.4.76)$$

Our objective is to create a Hopf bifurcation at the desired location (x^0, y^0) for different values of the parameter ρ. Note that the fixed point (x^1, y^1), which is not the anti-bifurcation control object, is a saddle point when $0.1 < \rho < 4.0$. In [397], a controller with washout filter is designed to generate a Hopf bifurcation at the period-1 fixed point (x^0, y^0). The original 2-D system becomes three-dimension due to introducing the washout filter. This not only changes the structure of the 2-D Hénon map, but also increases the complexity of dynamical analysis.

In the following, we shall show that with a simple polynomial feedback controller, we can achieve the same goal in [397] without changing the structure of the 2-D Hénon map. Moreover, we shall show that besides using a controller to generate a Hopf bifurcation from the fixed point (x^0, y^0), we can use the same control to keep the stability of the fixed point (x^1, y^1) unchanged (i.e., it is still a saddle point), or change it from a saddle point to a stable node. This will clearly demonstrate the advantage of our control method using polynomial functions.

To achieve this, we use a polynomial feedback controller in the form of

$$\begin{aligned}
h_{n1} &= A_{11}\left(x_n - x^0\right)\left(x_n - x^1\right) + A_{12}\left(y_n - y^0\right)\left(y_n - y^1\right) \\
&\quad + A_{21}\left(x_n - x^0\right)^2\left(x_n - x^1\right) + A_{22}\left(x_n - x^0\right)\left(x_n - x^1\right)^2, \\
h_{n2} &= B_{11}\left(x_n - x^0\right)^2\left(x_n - x^1\right).
\end{aligned} \quad (11.4.77)$$

The linear terms in the controller are used to modify the Jacobian matrix of the linearized system, thus to control the appearance of Hopf bifurcation. On the other hand, the nonlinear terms can be used to control the stability of the Hopf bifurcation. The Hopf bifurcation may be supercritical or subcritical depending upon the choice of appropriate nonlinear coefficients. For Hopf bifurcation, nonlinear terms up to third order are enough. With the controller defined in (11.4.77), the controlled 2-D Hénon map is given by

$$\begin{aligned}
x_{n+1} &= \rho - x_n^2 + 0.3y_n + h_{n1}, \\
y_{n+1} &= x_n + h_{n2}.
\end{aligned} \quad (11.4.78)$$

The Jacobian matrix of (11.4.78) evaluated at (x^0, y^0) is

$$\begin{bmatrix} -2x^0 + A_{11}(x^0 - x^1) + A_{22}(x^0 - x^1)^2 & 0.3 + A_{12}(y^0 - y^1) \\ 1 & 0 \end{bmatrix}. \quad (11.4.79)$$

By a similar analysis as that for the delay logistic map, choosing $A_{11} = 1$, we obtain a Hopf bifurcation when $A_{12} = -\frac{13}{\sqrt{49+400\rho}}$ and $A_{22} = \frac{30}{49+400\rho}$, for which the Jacobian matrix (11.4.79) has a pair of complex conjugate eigenvalues $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Applying the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{11.4.80}$$

into the controlled 2-D Hénon map (11.4.78) yields the normal form:

$$\begin{aligned}
u_{n+1} = {} & -\frac{\sqrt{3}}{60}\big[-40A_{21}x^0x^1v_n - 40A_{22}x^0x^1v_n - 20\sqrt{3}A_{11}u_nv_n \\
& + 20\sqrt{3}A_{11}x^1u_n + 20\sqrt{3}A_{11}x^0u_n - 15\sqrt{3}A_{21}u_nv_n^2 - 20\sqrt{3}A_{21}\big(x^0\big)^2u_n \\
& - 15\sqrt{3}A_{22}u_nv_n^2 - 20\sqrt{3}A_{22}\big(x^1\big)^2u_n + 10v_n^2 - 10A_{11}v_n^2 - 40A_{12}v_n^2 \\
& + 20A_{22}x^1v_n^2 + 20A_{11}x^1v_n + 20A_{11}x^0v_n - 40A_{11}x^0x^1 + 40A_{12}y^1v_n \\
& - 40A_{12}y^0v_n - 40A_{12}y^0y^1 + 10A_{21}x^1v_n^2 + 20A_{21}x^0v_n^2 - 20A_{21}\big(x^0\big)^2v_n \\
& + 40A_{21}\big(x^0\big)^2x1 - 20A_{22}\big(x^1\big)^2v_n + 10A_{22}x^0v_n^2 + 40A_{22}x^0\big(x^1\big)^2 \\
& + 20\sqrt{3}u_nv_n - 45A_{21}u_n^2v_n + 60A_{21}x^0u_n^2 - 45A_{22}u_n^2v_n + 60A_{22}x^1u_n^2 \\
& + 30A_{22}x^0u_n^2 - 15\sqrt{3}A_{21}u_n^3 - 15\sqrt{3}A_{22}u_n^3 - 5A_{21}v_n^3 - 5A_{22}v_n^3 \\
& - 30A_{11}u_n^2 + 30u_n^2 + 20\sqrt{3}A_{21}x^1u_nv_n + 40\sqrt{3}A_{21}x^0u_nv_n \\
& - 40\sqrt{3}A_{21}x^0x^1u_n + 40\sqrt{3}A_{22}x^1u_nv_n + 20\sqrt{3}A_{22}x^0u_nv_n \\
& - 40\sqrt{3}A_{22}x^0x^1u_n + 10\sqrt{3}u_n - 40\rho - 2v_n + 30A_{21}x^1u_n^2\big] \\
& + \frac{\sqrt{3}}{120}B_{21}\big[-40x^0x^1v_n - 15\sqrt{3}u_nv_n^2 - 20\sqrt{3}\big(x^0\big)^2u_n + 10x^1v_n^2 \\
& + 20x^0v_n^2 - 20\big(x^0\big)^2v_n + 40\big(x^0\big)^2x^1 - 15\sqrt{3}u_n^3 - 45u_n^2v_n + 30x^1u_n^2 \\
& + 60x^0u_n^2 - 5v_n^3 + 20\sqrt{3}x^1u_nv_n \\
& + 40\sqrt{3}x^0u_nv_n - 40\sqrt{3}x^0x^1u_n\big],
\end{aligned} \tag{11.4.81}$$

$$\begin{aligned}
v_{n+1} = {} & \frac{\sqrt{3}}{2}u_n + \frac{1}{2} + v_n + B_{21}\bigg[-\frac{1}{2}x^0v_n^2 + \frac{3\sqrt{3}}{8}u_n^3 + \frac{9}{8}u_n^2v_n - \frac{3}{4}x^1u_n^2 \\
& + \frac{3\sqrt{3}}{8}u_nv_n^2 - \frac{\sqrt{3}}{2}x^1u_nv_n - \frac{3}{2}x^0u_n^2 - \sqrt{3}x^0u_nv_n + \sqrt{3}x^0x^1u_n \\
& + \frac{1}{8}v_n^3 - \frac{1}{4}x^1v_n^2 - \big(x^0\big)^2x^1 + x^0x^1v_n + \frac{\sqrt{3}}{2}\big(x^0\big)^2u_n \\
& + \frac{1}{2}\big(x^0\big)^2v_n\bigg].
\end{aligned} \tag{11.4.82}$$

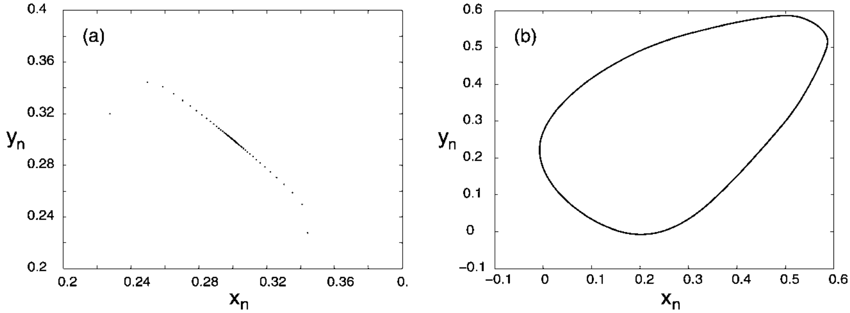


Figure 11.4.8. Simulated phase portraits when $\rho = 0.3$ for: (a) a period-1 fixed point of the 2-D Hénon map (11.4.74) without control; and (b) a Hopf bifurcation in the controlled 2-D Hénon map (11.4.78) with $\rho = 0.305$, $A_{11} = A_{21} = 1$, $A_{12} = -1$, $A_{22} = \frac{30}{169}$, $B_{21} = 0$.

It is easy to show that the coefficients A_{21} and B_{21} can be used to modify Re $A(0)$ (and thus to control the stability of Hopf bifurcation generated at (x^0, y^0)), and also to control the stability of the fixed point (x^1, y^1)). For example, when $\rho = 0.3$, we may set $A_{21} = 1$ and $B_{21} = 0$, then the Jacobian matrix for the controlled 2-D Hénon map (11.4.78) evaluated at the fixed point (x^1, y^1) has eigenvalues 2.935 and -0.545, indicating that it is a saddle point, same as the original system. For the fixed point (x^0, y^0), it follows from equations (11.4.36) and (11.4.37) that $\gamma_{22} = 0.225\sqrt{3} - 0.175i$, $\gamma_{21} = -0.099\sqrt{3}$, $\gamma_{20} = -0.225\sqrt{3} + 0.175i$, $\gamma_{32} = -1.104 - 0.221\sqrt{3}i$, $\alpha_1(0) = 0.975 - 0.683i$. Thus,

$$\operatorname{Re} A(0) = -0.104, \tag{11.4.83}$$

implying that a stable Hopf bifurcation occurs at the fixed point (x^0, y^0). For a consistent comparison with the results obtained in [397], we choose $\rho = 0.3,\ 0.4,\ 1.4$, corresponding to the uncontrolled 2-D Hénon map to have a stable period-1 orbit, a period-2 orbit and a chaotic attractor, respectively. The numerical simulation results are shown in Figures 11.4.8–11.4.10, where a Hopf bifurcation occurs from the period-1 orbit (see Figure 11.4.8) and period-2 orbit (see Figure 11.4.9) under the feedback control. In Figure 11.4.10, the chaotic motion becomes periodic under the control.

For this example, we can also choose different values of the coefficients A_{21} and B_{21} to obtain different bifurcation controls for different control objectives. For example, if one, besides generating a stable Hopf bifurcation from (x^0, y^0), also wants to stabilize the saddle point (x^1, y^1). This can be achieved, for $\rho = 0.3$, by choosing $A_{21} = -1$ and $B_{21} = \frac{100}{169}$. Indeed, under these choices, a stable Hopf bifurcation occurs at the fixed point (x^0, y^0), and the fixed point (x^1, y^1) is stabilized (see Figure 11.4.11).

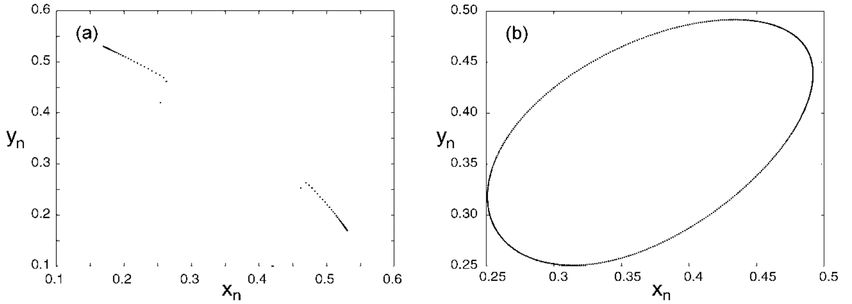


Figure 11.4.9. Simulated phase portraits when $\rho = 0.4$ for: (a) a period-2 fixed point of 2-D Hénon map (11.4.74) without control; and (b) a Hopf bifurcation in the controlled 2-D Hénon map (11.4.78) with $\rho = 0.405$, $A_{11} = 1$, $A_{12} = -\frac{13}{\sqrt{209}}$, $A_{21} = 0$, $A_{22} = \frac{30}{209}$, $B_{21} = 0$.

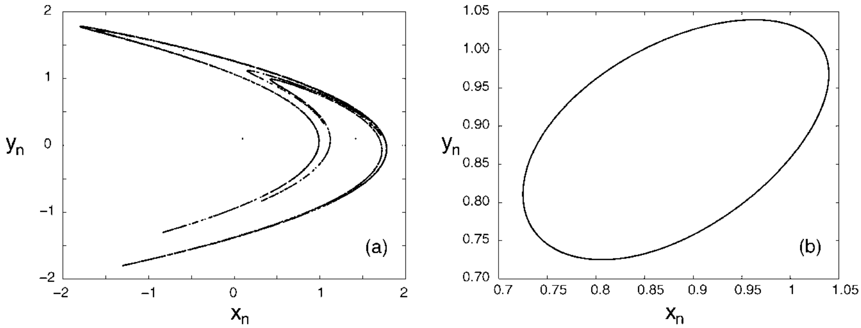


Figure 11.4.10. Simulated phase portraits when $\rho = 1.4$ for: (a) a chaos of 2-D Hénon map (11.4.74) without control; and (b) a Hopf bifurcation in the controlled 2-D Hénon map (11.4.78) with $\rho = 1.405$, $A_{11} = 1$, $A_{12} = -\frac{13}{\sqrt{609}}$, $A_{21} = 0$, $A_{22} = \frac{30}{609}$, $B_{21} = 0$.

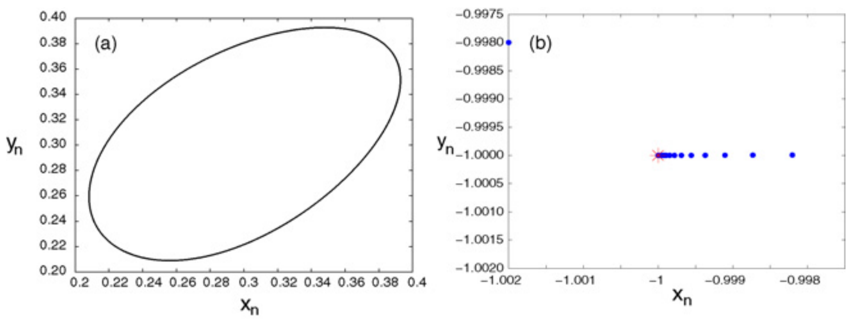


Figure 11.4.11. Simulated phase portraits when $\rho = 0.3$ for: (a) a Hopf bifurcation at (x^0, y^0) in the controlled 2-D Hénon map (11.4.78) with $\rho = 0.305$, $A_{11} = 1$, $A_{12} = -1$, $A_{21} = -1$, $A_{22} = \frac{30}{169}$, $B_{21} = \frac{100}{169}$; and (b) a stable node at that fixed point (x^1, y^1).

(C) 3-D Hénon map. Finally, we consider an example of anti-control of Hopf bifurcation in a higher dimensional discrete map using polynomial function. We use the modified 3-dimensional Hénon map [398] to illustrate chaos control by creating a Hopf bifurcation. The modified 3-D Hénon map is given by

$$\begin{aligned} x_{n+1} &= \rho - y_n^2 - 0.3 z_n, \\ y_{n+1} &= x_n, \\ z_{n+1} &= y_n, \end{aligned} \tag{11.4.84}$$

where ρ is a bifurcation parameter. The 3-D Hénon map without control exhibits a strange attractor at $\rho = \rho^0 = 1.4$. In [398], a washout filter controller was used to generate a Hopf bifurcation at the period-1 fixed point, given by

$$\begin{aligned} \left(x^0, y^0, z^0\right) = \big(&-0.65 + 0.5\sqrt{1.69 + 4\rho},\ -0.65 + 0.5\sqrt{1.69 + 4\rho}, \\ &- 0.65 + 0.5\sqrt{1.69 + 4\rho}\big), \end{aligned}$$

when $\rho = \rho^0 = 1.4$. As we pointed out before, the dimension of the original system is increased by one when a washout filter controller is introduced. The increase of the dimension of a discrete-time dynamical system complicates the system and analysis. Here, a simple polynomial controller is used to generate a Hopf bifurcation from the period-1 fixed point, meanwhile keeping the local stability property of the fixed point:

$$\begin{aligned} \left(x^1, y^1, z^1\right) = \big(&-0.65 - 0.5\sqrt{1.69 + 4\rho}, \\ &- 0.65 - 0.5\sqrt{1.69 + 4\rho},\ -0.65 - 0.5\sqrt{1.69 + 4\rho}\big). \end{aligned}$$

Due to the similarity between the 2-D Hénon map and the 3-D Hénon map, we employ the same form of the polynomial controller used in the 2-D Hénon map for the first equation of the 3-D Hénon map (for convenience, we do not add controls to the second and third equations), which is then in the form of

$$\begin{aligned} h_n = {}& A_{11}\left(x_n - x^0\right)\left(x_n - x^1\right) + A_{12}\left(y_n - y^0\right)\left(y_n - y^1\right) \\ &+ A_{21}\left(x_n - x^0\right)^2\left(x_n - x^1\right) + A_{22}\left(x_n - x^0\right)\left(x_n - x^1\right)^2. \end{aligned} \tag{11.4.85}$$

Thus, the controlled system is

$$\begin{aligned} x_{n+1} &= \rho - y_n^2 - 0.3 z_n + h_n, \\ y_{n+1} &= x_n, \\ z_{n+1} &= y_n. \end{aligned} \tag{11.4.86}$$

The Jacobian matrix of (11.4.86) evaluated at (x^0, y^0, z^0) when $\rho = 1.4$ is

$$\begin{bmatrix} \frac{27}{10}A_{11} + \frac{729}{100}A_{22} & -\frac{7}{5} + \frac{27}{10}A_{12} & -\frac{3}{10} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

By Theorem 11.4.4, it is easy to use A_{11} and A_{12} to determine the critical conditions of Hopf bifurcation. From the condition (iii) in Theorem 11.4.4, we obtain the critical values of A_{11} and A_{12} as

$$A_{11} = -\frac{27}{10}A_{22} - 4981 \quad \text{and} \quad A_{12} = 0,$$

where $|a_0| = 0.3 < 1$ and $|a_0 + a_2| = 1.933 < 2.4 = 1 + a_1$.

Further, it is easy to verify that

$$\left.\frac{\partial|\lambda_1(\epsilon)|}{\partial\epsilon}\right|_{\epsilon=0} = 0.264 \quad \text{and} \quad \lambda_1^n(0) \neq 1, \tag{11.4.87}$$

indicating that both the transversality condition and the nonresonant condition are satisfied. A_{21} and A_{22} can be chosen arbitrarily to determine the stability of the Hopf bifurcation, and also to control the stability of the other fixed point, (x^1, y^1, z^1). For example, when $A_{22} = 0$ and $A_{21} = -\frac{490}{2187}$, the eigenvalues for the Jacobian matrix of the controlled system (11.4.86) evaluated at (x^1, y^1, z^1) are -2.036, 1.961 and 0.075, which are the same as that of the original 3-D Hénon map when $\rho = 1.4$. Thus, the stability of the fixed point (x^1, y^1, z^1) is unchanged under the control when $\rho = 1.4$. The stability of the Hopf bifurcation is determined by the sign of $\mathrm{Re}\,A(0)$ (see equation (11.4.49)). By applying the center manifold and normal form theories, it can be shown that

$$\mathrm{Re}\,A(0) = 4.414, \tag{11.4.88}$$

and thus a stable Hopf bifurcation is obtained under the control. The numerical simulation results are shown in Figure 11.4.12.

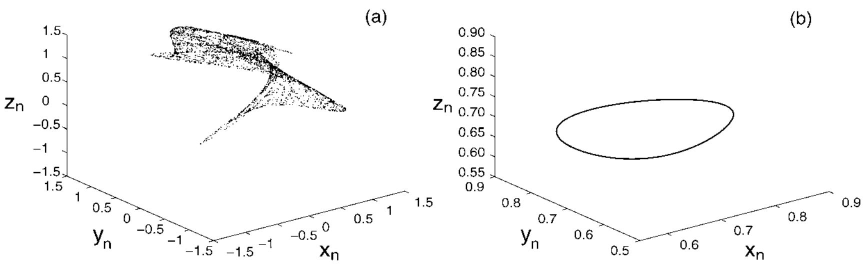


Figure 11.4.12. Simulated 3-D phase portraits when $\rho = 1.4$ for: (a) a chaos of the 3-D Hénon map (11.4.84) without control; and (b) a Hopf bifurcation in the controlled 3-D Hénon map (11.4.86) with $\rho = 1.41$, $A_{11} = -\frac{49}{81}$, $A_{12} = 0$, $A_{21} = -\frac{490}{2187}$ and $A_{22} = 0$.

11.4.3. 2-D lifting surface

The final application of Hopf bifurcation control is for a nonlinear aeroelastic problem related to 2-D supersonic lifting surface. Due to its evident practical importance, the study of the flutter instability of flight vehicle constitutes an essential prerequisite in their design process. The flutter instability can jeopardize aircraft performance and dramatically affect its survivability. Moreover, the tendency of increasing structural flexibility and maximum operating speed increase the likelihood of the flutter occurrence within the aircraft operational envelope. In order to prevent such events to occur, two principal issues have been discussed [306]: (i) increase, without weight penalties, of the flutter speed, and (ii) possibilities to convert unstable limit cycles into stable ones. Both two issues are related to controlling Hopf bifurcations. In particular, issue (i) implies increase of the stability of an equilibrium and delay of the occurrence of Hopf bifurcations [146,420,305]; while issue (ii) is related to controlling Hopf bifurcations once a periodic vibration has been initiated [66].

This study primarily deals with the determination and control of the flutter speed of supersonic/hypersonic lifting surfaces, based on the character of the flutter boundary. This implies the determination of the conditions generating the catastrophic type of flutter boundary, and implementation of an active control capability enabling one to convert this type of flutter boundary into a benign one. This issue is of a considerable importance toward the expansion, without catastrophic failures, of the flight envelope of the vehicle. In contrast to the issue of the determination of the flutter boundary that requires a linearized analysis, the problem of the determination of the character of the flutter boundary, requires a nonlinear analysis. As it has been shown [288,287] at hypersonic speeds the aerodynamic nonlinearities play a detrimental role, in the sense that they contribute to conversion of the benign flutter boundary to a catastrophic one. Therefore, an active control capability enabling one to prevent conversion of the flutter boundary into a catastrophic one should be implemented.

The investigation is based on a nonlinear model of a wing section of the high speed aircraft incorporating active control in [306]. The geometry of the model is shown in Figure 11.4.13. Structural, aerodynamic and control nonlinearities have been included in the present aeroelastic model. In this context, the parameter B represents a measure of the degree of the structural nonlinearity of the system, in the sense that, corresponding to $B < 0$ or $B > 0$, the structural nonlinearities are soft or hard, respectively, while for $B = 0$, the system is structurally linear. The linear and nonlinear active controls are given in terms of two normalized control gain parameters Ψ_1 and Ψ_2, respectively. Based on Piston Theory Aerodynamics (PTA), the nonlinear unsteady aerodynamic lift and moment are obtained through the integration of the pressure difference on the upper and lower surfaces of the airfoil. Notice also that the aerodynamic correction factor $\gamma = M_\infty/\sqrt{M_\infty^2 - 1}$

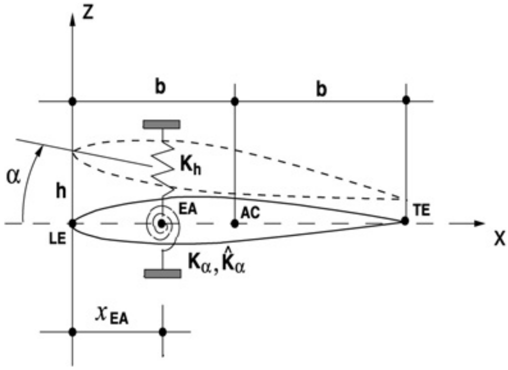


Figure 11.4.13. Geometry of the cross-section of lifting surface.

which enables one to extend the applicability of the PTA to the low supersonic flight speed range, has been included in the present model. In the context previously described, the system of dimensionless aeroelastic governing equations including linear and nonlinear time-delay feedback controls can be described by

$$\begin{aligned}
&\ddot{\xi} + \chi_\alpha \ddot{\alpha} + 2\zeta_h\left(\frac{\varpi}{V}\right)\dot{\xi} + \left(\frac{\varpi}{V}\right)^2 \xi = \mathcal{L}(t),\\
&\frac{\chi_\alpha}{r_\alpha^2}\ddot{\xi} + \ddot{\alpha} + \frac{2\zeta_\alpha}{V}\dot{\alpha} + \frac{1}{V^2}\alpha + \frac{1}{V^2}B\alpha^3\\
&\quad = \mathcal{M}(t) - \frac{\Psi_1}{V^2}\alpha(t-\tau) - \frac{\Psi_2}{V^2}\alpha^3(t-\tau),
\end{aligned} \tag{11.4.89}$$

where

$$\begin{aligned}
\mathcal{L}(t) &= -\frac{\gamma}{12\mu M_\infty}\left\{12\alpha + \delta_A M_\infty^2(1+\kappa)\gamma^2\alpha^3 + 12\left[\dot{\xi} + \dot{\alpha}(b - x_{ea})/b\right]\right\},\\
\mathcal{M}(t) &= -\frac{\gamma}{12\mu M_\infty}\frac{1}{r_\alpha^2 b}\left\{12(b - x_{ea})\alpha + \delta_A M_\infty^2(b - x_{ea})(1+\kappa)\gamma^2\alpha^3\right.\\
&\qquad \left. + 4\left[3(b - x_{ea})\dot{\xi} + \dot{\alpha}\left(4b^2 - 6bx_{ea} + 3x_{ea}^2\right)/b\right]\right\},
\end{aligned} \tag{11.4.90}$$

and $\xi(t) = h(t)/b$ (h is the plunging displacement), $\alpha(t)$ is the twist angle about the pitch axis, $\mathcal{L}(t)$ and $\mathcal{M}(t)$ denote the dimensionless aerodynamic lift and moment, respectively, while τ is the time delay. The meaning of the remaining parameters can be found in a nomenclature (see [306,288,444]).

Mathematical model is generally the first approximation of the considered real system. More realistic models should include some of the past states of the system, that is, the model should include time delay. The time delay in control can occur either beyond our will or it can be designed as to increase the performance of the system [307]. For this reason, as a necessary prerequisite, a good understanding of its effects on the flutter instability boundary and its character (benign or catastrophic) is required.

In order to capture the effect of time delay, τ, introduced in the related terms Ψ_1 and Ψ_2, let $\xi = x_1$, $\alpha = x_2$, $\dot{\xi} = x_3$, $\dot{\alpha} = x_4$ and $x_{2t} = x_2(t-\tau)$. Then, one can rewrite equation (11.4.89) as a set of four first-order differential equations:

$$\begin{aligned}
\dot{x}_1 &= x_3,\\
\dot{x}_2 &= x_4,\\
\dot{x}_3 &= a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_2^3 + e_1x_{2t} + e_2x_{2t}^3,\\
\dot{x}_4 &= b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_2^3 + f_1x_{2t} + f_2x_{2t}^3,
\end{aligned} \tag{11.4.91}$$

where all the coefficients are explicitly expressed in terms of the parameters of equation (11.4.89), which can be found in [444].

For convenience in the following analysis, rewrite equation (11.4.91) in the vector form:

$$\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t-\tau) + \boldsymbol{F}\big(\boldsymbol{x}(t), \boldsymbol{x}(t-\tau)\big), \tag{11.4.92}$$

where $\boldsymbol{x}, \boldsymbol{F} \in R^4$, A_1 and A_2 are 4×4 matrices. A_1, A_2 and $\boldsymbol{F}$ are given by

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & e_1 & 0 & 0\\ 0 & f_1 & 0 & 0 \end{bmatrix}, \quad \text{and}\\
\boldsymbol{F} &= \begin{pmatrix} 0\\ 0\\ a_5x_2^3(t) + e_2x_2^3(t-\tau)\\ b_5x_2^3(t) + f_2x_2^3(t-\tau) \end{pmatrix},
\end{aligned} \tag{11.4.93}$$

respectively.

Hopf bifurcation has been extensively studied using many different methods [146,305], for example, Lyapunov's quantity used in the context of the supersonic panel flutter where the effects of structural, aerodynamical and physical nonlinearities have been incorporated [287]. In [306,307], the dynamic behavior of the system without time delay in the control was studied in the vicinity of a Hopf-type critical point. In particular, the effect of the active control on the character of the flutter boundary (where the Jacobian has a purely imaginary pair) is investigated. It is shown that for different flight speeds, stable (unstable) equilibrium and stable (unstable) limit cycles exist.

In this subsection, we will consider the effect of the time delay involved in the feedback control. Nonlinear systems involving time delay have been studied by many authors (e.g., see [70,349,24,156,108,440]). The main attention here will be focused on Hopf bifurcation. Th results obtained from the model (11.4.89) reveal an important fact: time delay in the linear control can contribute to the expansion of the flight envelope, while in nonlinear control it can contribute to the conversion of the unstable limit cycles into a stable one.

Linearized system

As the first step, we analyze the stability of the trivial solution of the linearized system of (11.4.92), which is given by

$$\dot{\boldsymbol{x}}(t) = A_1\boldsymbol{x}(t) + A_2\boldsymbol{x}(t-\tau), \quad \boldsymbol{x} \in R^4. \tag{11.4.94}$$

The characteristic function can be obtained by substituting the trial solution, $\boldsymbol{x}(t) = \boldsymbol{c}e^{\lambda\tau}$, where $\boldsymbol{c}$ is a constant vector, into the linear part to find

$$\begin{aligned} D(\lambda) &= \det\bigl(\lambda I - A_1 - A_2 e^{-\lambda\tau}\bigr) \\ &= \lambda^4 - (a_3+b_3)\lambda^3 + (a_3b_4 - a_4b_3 - b_2 - a_1)\lambda^2 \\ &\quad + (b_2a_3 - b_3a_2 + a_1b_4 - b_1a_4)\lambda + a_1b_2 - a_2b_1 \\ &\quad - \bigl[f_1\lambda^2 + (b_3e_1 - a_3f_1)\lambda + (b_1e_1 - a_1f_1)\bigr]e^{-\lambda\tau}, \end{aligned} \tag{11.4.95}$$

where I denotes the identify matrix. Based on equation (11.4.95), it can be shown [440] that *The number of the eigenvalues of the characteristic equation* (11.4.95) *with negative real parts, counting multiplicities, can change only when the eigenvalues become pure imaginary pairs as the time delay τ and the components of A_1 and A_2 are varied.*

It is seen from equation (11.4.95) that when $a_1(b_2+f_1) \neq b_1(a_2+e_1)$, none of the roots of $D(\lambda)$ is zero. Thus, the trivial equilibrium $\boldsymbol{x} = \boldsymbol{0}$ becomes unstable only when equation (11.4.95) has at least a pair of purely imaginary roots $\lambda = \pm i\omega$ (i is the imaginary unit), at which a Hopf bifurcation occurs. The critical value for a Hopf bifurcation to occur can be found from the following equation:

$$\begin{aligned} D(i\omega) = {} & \bigl[\bigl(f_1\omega^2 + a_1f_1 - b_1e_1\bigr)\cos(\omega\tau) + \omega(f_1a_3 - b_3e_1)\sin(\omega\tau) + \omega^4 \\ & + (b_2 + a_1 - a_3b_4 + a_4b_3)\omega^2 + a_1b_2 - b_1a_2\bigr] \\ & + \bigl[\omega(f_1a_3 - b_3e_1)\cos(\omega\tau) - \bigl(a_1f_1 + f_1\omega^2 - b_1e_1\bigr)\sin(\omega\tau) \\ & + (a_4+b_3)\omega^3 + (b_2a_3 - b_3a_2 + a_1b_4 - b_1a_4)\omega\bigr]i. \end{aligned} \tag{11.4.96}$$

Setting the real and imaginary parts of $D(i\omega)$ zero results in

$$\cos(\omega\tau) = P_1/P \quad \text{and} \quad \sin(\omega\tau) = P_2/P, \tag{11.4.97}$$

where

$$\begin{aligned} P_1 = {} & -f_1\omega^6 + \bigl(a_3b_3e_1 + b_4b_3e_1 - f_1a_4b_3 - a_3^2f_1 + b_1e_1 \\ & - 2f_1a_1 - f_1b_2\bigr)\omega^4 + \bigl(a_1b_4b_3e_1 + b_1a_4f_1a_3 + b_2a_3b_3e_1 - b_3^2a_2e_1 \end{aligned}$$

$$
\begin{aligned}
&- b_2a_3^2 f_1 - b_1e_1a_3b_4 + b_3a_2 f_1a_3 - a_1 f_1a_4b_3 - 2 f_1a_1b_2 + f_1b_1a_2 \\
&+ b_1e_1b_2 + b_1e_1a_1 - a_1^2 f_1\big)\omega^2 + (a_1 f_1 - b_1e_1)(b_1a_2 - a_1b_2), \\
P_2 = {}& \omega\big[(f_1b_4 + b_3e_1)\omega^4 + \big(f_1b_4a_3^2 - f_1a_3a_4b_3 - f_1b_1a_4 - f_1b_3a_2 \\
&+ 2 f_1a_1b_4 - b_1e_1a_3 - a_3b_4b_3e_1 + b_2b_3e_1 \\
&- b_1e_1b_4 + a_1b_3e_1 + a_4b_3^2e_1\big)\omega^2 + b_1a_4(b_1e_1 - a_1 f_1)\big] \\
&+ a_1 f_1(a_1b_4 - a_2b_3) + a_3b_1(f_1a_2 - e_1b_2) + a_1e_1(b_2b_3 - b_1b_4), \\
P = {}& f_1^2\omega^4 + \big[(b_3e_1 - f_1a_3)^2 + 2 f_1(f_1a_1 - b_1e_1)\big]\omega^2 \\
&+ (b_1e_1 - a_1 f_1)^2.
\end{aligned}
\tag{11.4.98}
$$

With the aid of equations (11.4.97) and (11.4.98), one may apply the identity $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$ to obtain the following 8th-degree characteristic polynomial of ω:

$$
\omega^8 + q_1\omega^6 + q_2\omega^4 + q_3\omega^2 + q_4 = 0, \tag{11.4.99}
$$

where

$$
\begin{aligned}
q_1 &= a_3^2 + b_4^2 + 2(b_2 + a_4b_3 + a_1), \\
q_2 &= (a_1 + b_1)^2 + (a_3b_4 - a_4b_3)^2 - f_1^2 \\
&\quad + 2\big[a_3(b_2a_3 - a_2b_3) + a_4(a_1b_4 - b_1a_3)\big] \\
&\quad + 2\big[b_3(a_4b_2 - a_2b_4) + b_4(a_1b_4 - b_1a_4)\big], \\
q_3 &= (a_1b_4 - a_4b_1)^2 + (a_2b_3 - a_3b_2)^2 - (a_3 f_1 - b_3e_1)^2 \\
&\quad + 2(a_4b_2 - a_2b_4)(a_1b_3 - a_3b_1) \\
&\quad + 2\big[(a_1 + b_1)(a_1b_2 - a_2b_1) + f_1(b_1e_1 - f_1a_1)\big], \\
q_4 &= (a_1b_2 - a_2b_1)^2 - (a_1 f_1 - b_1e_1)^2.
\end{aligned}
\tag{11.4.100}
$$

If equation (11.4.99) has no positive real roots (for ω^2), then system (11.4.94) does not contain center manifold, but only stable and unstable manifolds. On the other hand, if equation (11.4.99) has at least one positive solution for ω, one may substitute the solution(s) into equation (11.4.97) to find the smallest $\tau_{\min}$, at which the system undergoes a Hopf bifurcation.

Although closed-form solution exists for the roots of a general 4th-degree polynomial (we can consider equation (11.4.99) as a 4th-degree polynomial of ω^2), it is not useful here in finding the relations between the parameters since the expressions are too involved to be treated analytically. In this chapter, we will use a numerical approach to find the relations among the flutter speed V_F ($\equiv U_F/b\omega_\alpha$), flight Mach Number M_∞, time delay τ and control gains Ψ_1, Ψ_2. More computation results will be given later in Section 4.3.3.

Center manifold reduction

In order to obtain the explicit analytical expressions for the stability condition of Hopf bifurcation solutions (limit cycles), we need to reduce system (1) to its center manifold [156]. While studying the critical infinite dimensional problem on a 2-D center manifold, we express the delay equation as an abstract evolution equation on Banach space H of continuously differentiable function $\boldsymbol{u}$: $[-\tau, 0] \to R^2$ as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}_t + \boldsymbol{F}(t, \boldsymbol{x}_t), \tag{11.4.101}$$

where $\boldsymbol{x}_t(\theta) = \boldsymbol{x}(t+\theta)$ for $-\tau \leqslant \theta \leqslant 0$, and A is a linear operator for the critical case, expressed by

$$A\boldsymbol{u}(\theta) = \begin{cases} \frac{d\boldsymbol{u}(\theta)}{d\theta} & \text{for } \theta \in [-\tau, 0), \\ A_1\boldsymbol{u}(0) + A_2\boldsymbol{u}(-\tau) & \text{for } \theta = 0. \end{cases} \tag{11.4.102}$$

The nonlinear operator F is in the form of

$$\boldsymbol{F}(\boldsymbol{u})(\theta) = \begin{cases} 0 & \text{for } \theta \in [-\tau, 0) \\ \boldsymbol{F}(\boldsymbol{u}(0), \boldsymbol{u}(-\tau)) & \text{for } \theta = 0. \end{cases} \tag{11.4.103}$$

Similarly, we can define the dual/adjoint space H^* of continuously differentiable function $\boldsymbol{v} : [0, \tau] \to R^2$ with the dual operator

$$A^*\boldsymbol{v}(\sigma) = \begin{cases} -\frac{d\boldsymbol{v}(\sigma)}{d\sigma} & \text{for } \sigma \in (0, \tau], \\ A_1^*\boldsymbol{v}(0) + A_2^*\boldsymbol{v}(\tau) & \text{for } \sigma = 0. \end{cases} \tag{11.4.104}$$

From the discussion given in the previous subsection, we know that the characteristic equation (11.4.95) has single pair of purely imaginary eigenvalues $\Lambda = \pm i\omega$. Therefore, H can be split into two subspaces as $H = P_\Lambda \oplus Q_\Lambda$, where P_Λ is a 2-D space spanned by the eigenvectors of the operator A associated with the eigenvalues Λ, while Q_Λ is the complementary space of P_Λ. Then for $\boldsymbol{u} \in H$ and $\boldsymbol{v} \in H^*$, we can define a bilinear operator:

$$\begin{aligned} \langle \boldsymbol{v}, \boldsymbol{u} \rangle &= \bar{\boldsymbol{v}}^T(0)\boldsymbol{u}(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \bar{\boldsymbol{v}}^T(\xi - \theta)\big[d\eta(\theta)\big]\boldsymbol{u}(\xi)\, d\xi \\ &= \bar{\boldsymbol{v}}^T(0)\boldsymbol{u}(0) + \int_{-\tau}^{0} \bar{\boldsymbol{v}}^T(\xi + \theta)A_2(\xi)\boldsymbol{u}(\xi)\, d\xi. \end{aligned} \tag{11.4.105}$$

Corresponding to the critical characteristic root $i\omega$, the complex eigenvector $\boldsymbol{q}(\theta) \in H$ satisfies

$$\frac{d\boldsymbol{q}(\theta)}{d\theta} = i\omega\boldsymbol{q}(\theta), \quad \text{for } \theta \in [-\tau, 0),$$

$$A_1\boldsymbol{q}(0) + A_2\boldsymbol{q}(-\tau) = i\omega\boldsymbol{q}(0), \quad \text{for } \theta = 0. \tag{11.4.106}$$

The general solution of equation (11.4.106) is

$$\boldsymbol{q}(\theta) = \boldsymbol{C}e^{i\omega\theta}. \tag{11.4.107}$$

From the boundary conditions we find the following matrix equation:

$$\begin{bmatrix} -i\omega & 0 & 1 & 0 \\ 0 & -i\omega & 0 & 1 \\ a_1 & a_2 + e_1e^{-i\omega\tau} & a_3 - i\omega & a_4 \\ b_1 & b_2 + f_1e^{-i\omega\tau} & b_3 & b_4 - i\omega \end{bmatrix} \boldsymbol{C} = \boldsymbol{0}. \tag{11.4.108}$$

By letting $\boldsymbol{C} = (C_1, C_2, C_3, C_4)^T$ and choosing $C_1 = 1$, we uniquely determine C_2, C_3 and C_4. Then the eigenvector $\boldsymbol{q}(\theta) = \boldsymbol{C}e^{i\omega\theta}$ is found. Thus, the real basis for P_Λ is obtained as $\Phi(\theta) = (\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2}) = (\mathrm{Re}(\boldsymbol{q}(\theta)), \mathrm{Im}(\boldsymbol{q}(\theta)))$, that is,

$$\Phi(\theta) = \begin{bmatrix} \cos(\omega\theta) & \sin(\omega\theta) \\ \frac{L_1\cos(\omega\theta)+\omega L_2\sin(\omega\theta)}{L_0} & \frac{L_1\sin(\omega\theta)-\omega L_2\cos(\omega\theta)}{L_0} \\ -\omega\sin(\omega\theta) & \omega\cos(\omega\theta) \\ \frac{\omega(\omega L_2\cos(\omega\theta)-L_1\sin(\omega\theta))}{L_0} & \frac{\omega(\omega L_2\sin(\omega\theta)+L_1\cos(\omega\theta))}{L_0} \end{bmatrix}, \tag{11.4.109}$$

where L_i $(i = 0, \ldots, 2)$ are explicitly expressed in terms of the original system parameters.

Similarly, from the equation

$$A^*\boldsymbol{q}^*(\sigma) = -i\omega\boldsymbol{q}^*(\sigma)$$

or

$$-\frac{d\boldsymbol{q}^*(\sigma)}{d\sigma} = -i\omega\boldsymbol{q}^*(\sigma) \quad \text{for } \sigma \in [0, \tau),$$

$$A_1^*\boldsymbol{q}^*(0) + A_2^*\boldsymbol{q}^*(\tau) = -i\omega\boldsymbol{q}^*(0) \quad \text{for } \sigma = 0, \tag{11.4.110}$$

one can choose the real basis for the dual space Q_Λ as

$$\begin{aligned} \Psi(\sigma) &= (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = \big(\mathrm{Re}\big(\boldsymbol{q}^*(\sigma)\big), \mathrm{Im}\big(\boldsymbol{q}^*(\sigma)\big)\big) \\ &= \begin{bmatrix} \frac{L_3\cos(\omega\sigma)+L_4\sin(\omega\sigma)}{M} & \frac{L_3\sin(\omega\sigma)-L_4\cos(\omega\sigma)}{M} \\ \frac{L_5\cos(\omega\sigma)+L_6\sin(\omega\sigma)}{M} & \frac{L_5\sin(\omega\sigma)-L_6\cos(\omega\sigma)}{M} \\ \frac{L_7\cos(\omega\sigma)+L_8\sin(\omega\sigma)}{M} & \frac{L_7\sin(\omega\sigma)-L_8\cos(\omega\sigma)}{M} \\ N_1\cos(\omega\sigma) - N_2\sin(\omega\sigma) & N_1\sin(\omega\sigma) + N_2\cos(\omega\sigma) \end{bmatrix}, \end{aligned} \tag{11.4.111}$$

where the explicit expressions of L_i ($i = 3, \dots, 8$) and M are expressed explicitly in terms of the original system's parameters [444], and N_1 and N_2 can be obtained from the relation $\langle \Psi, \Phi \rangle = I$, given in terms of ω, τ and the coefficients a_i, b_i, e_i and f_i in equation (11.4.107).

Next, by defining $\boldsymbol{w} \equiv (w_1, w_2)^T = \langle \Psi, \boldsymbol{u}_t \rangle$ (which actually represents the local coordinate system on the 2-D center manifold, induced by the basis Ψ), then with the aid of equations (11.4.109) and (11.4.111), one can decompose $\boldsymbol{u}_t$ into two parts to obtain

$$\boldsymbol{u}_t = \boldsymbol{u}_t^{P_\Lambda} + \boldsymbol{u}_t^{Q_\Lambda} = \Phi \langle \Psi, \boldsymbol{u}_t \rangle + \boldsymbol{u}_t^{Q_\Lambda} = \Phi \boldsymbol{w} + \boldsymbol{u}_t^{Q_\Lambda}, \tag{11.4.112}$$

which implies that the projection of $\boldsymbol{u}_t$ on the center manifold is $\Phi \boldsymbol{w}$. Then, applying equations (11.4.101) and (11.4.112) results in

$$\left\langle \Psi, \Phi \dot{\boldsymbol{w}} + \dot{\boldsymbol{u}}_t{}^Q \right\rangle = \left\langle \Psi, A\left(\Phi \boldsymbol{w} + \boldsymbol{u}_t^Q\right)\right\rangle + \left\langle \Psi, F\left(t, \Phi \boldsymbol{w} + \boldsymbol{u}_t^Q\right)\right\rangle, \tag{11.4.113}$$

and therefore,

$$\langle \Psi, \Phi \rangle \dot{\boldsymbol{w}} = \langle \Psi, A\Phi \rangle \boldsymbol{w} + \left\langle \Psi, \boldsymbol{F}\left(\boldsymbol{t}, \Phi \boldsymbol{w} + \boldsymbol{u}_t^Q\right)\right\rangle$$

which can be written as

$$I \dot{\boldsymbol{w}} = D_\Lambda \boldsymbol{w} + \mathcal{N}(\boldsymbol{w}).$$

Finally, we obtain the equation of the center manifold, as given in the following theorem.

THEOREM 11.4.5. *The 2-D center manifold of system (11.4.91) associated with Hopf bifurcation is given by*

$$\dot{\boldsymbol{w}} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \boldsymbol{w} + \mathcal{N}(\boldsymbol{w}), \tag{11.4.114}$$

where $\mathcal{N}(\boldsymbol{w})$ represents the nonlinear terms contributed from the original system to the center manifold.

The lowest order nonlinear terms of the center manifold, needed to determine the solutions, are:

$$\begin{aligned} \mathcal{N}_3(\boldsymbol{w}) &= \Psi^T(0) F(\Phi \boldsymbol{w}) = \Psi^T(0) \begin{pmatrix} 0 \\ 0 \\ a_5(\Phi(0)\boldsymbol{w})_2^3 + e_2(\Phi(-\tau)\boldsymbol{w})_2^3 \\ b_5(\Phi(0)\boldsymbol{w})_2^3 + f_2(\Phi(-\tau)\boldsymbol{w})_2^3 \end{pmatrix} \\ &= \begin{pmatrix} C_{30}^1 w_1^3 + C_{21}^1 w_1^2 w_2 + C_{12}^1 w_1 w_2^2 + C_{03}^1 w_2^3 \\ C_{30}^1 w_2^3 + C_{21}^2 w_1^2 w_2 + C_{12}^2 w_1 w_2^2 + C_{03}^2 w_2^3 \end{pmatrix}, \end{aligned} \tag{11.4.115}$$

where "$(\cdots)_2^3$" denotes the cubic order terms extracted from the second component of the vector $(\cdots)$. In fact, since Φ is a 4×2 matrix and $\boldsymbol{w}$ is a 2×1 vector, $\Phi \boldsymbol{w}$ is a 4×1 vector which may include higher order terms in the components, we just intercept the third order terms. Therefore, we obtain the normal form as follows.

THEOREM 11.4.6. *The normal form of system (11.4.91) associated with Hopf bifurcation is*

$$\begin{aligned} \dot{r} &= Lr^3, \\ \dot{\theta} &= \omega + br^2, \end{aligned} \tag{11.4.116}$$

where L is a Lyapunov coefficient, also called as Lyapunov First Quantity (LFQ), given by

$$L = \frac{1}{8}\left(3C_{30}^1 + C_{12}^1 + C_{21}^2 + 3C_{03}^2\right). \tag{11.4.117}$$

When $L < 0$ (> 0), the Hopf bifurcation is supercritical (subcritical).

Results

In this subsection, some numerical results are presented to investigate the stability with respect to the choices of the time delay, τ, and the linear and nonlinear control gains, Ψ_1 and Ψ_2, using the formulas presented in the previous sections.

In order to compare the results with those given in [306] where the approach [287] was used and no time delay is presented, we shall take the same parameter values used in [306]. The main chosen varying parameters are M_∞, Ψ_1, Ψ_2 and time delay τ, while other parameters given in equation (11.4.107) are fixed:

$$\begin{aligned} &b = 1.5, \quad \mu = 50, \quad \bar{\omega} = 1.0, \quad r_\alpha = 0.5, \\ &\chi_\alpha = 0.25, \quad \zeta_h = \zeta_\alpha = 0, \\ &\gamma = 1, \quad \kappa = 1.4, \quad \delta_A = 1, \quad B = 1, \\ &x_0 = 0.5, \quad \omega_\alpha = 60. \end{aligned} \tag{11.4.118}$$

The stability of the aeroelastic system in the vicinity of the flutter boundary is analyzed on the basis of equations (11.4.109) and (11.4.111).

We know from [306] that when either the linear or the nonlinear control gain is added, at relatively moderate supersonic flight Mach numbers the flutter boundary is benign, while with the increase of the flight Mach number, due to the built-up aerodynamic nonlinearities that become prevalent, the flutter boundary becomes catastrophic. Here, we will show how the stability changes when the time delay is introduced.

Four typical cases are discussed below. Note that in the discussions, V_F denotes the flutter velocity at which Hopf bifurcation (due to flutter instability) is initiated, leading to periodic motions. The stability of bifurcating limit cycles is determined by the sign of L—the Lyapunov coefficient. The computation of L is based on the center manifold and equation (11.4.117) described in the previous subsection.

CASE 1. $\tau = 0$ (i.e., no delay), Ψ_1 is varied, while $\Psi_2 = 0$ (nonlinear feedback control is not applied).

Consider the linear feedback control with gain Ψ_1, but without the time delay. The results of the flutter velocity with respect to flight Mach number for different values of Ψ_1 and the corresponding Lyapunov coefficients recover what obtained in [306]. This constitutes an excellent validation of our methodology since the results in [306] were produced with a different method (see [287]). The effect of the linear control on V_F is depicted in Figure 11.4.14(b) (as solid lines), while the effect on L has the similar trend as the case with time delay, τ (see Figure 11.4.15(a)). It is shown that the flutter speed monotonically increases with increases of flight Mach numbers, M_∞, and/or the control gain, Ψ_1. When $L < 0$ (> 0), the corresponding motion is stable (unstable) in the sense of Hopf bifurcation. We may define the value of the Mach number at which $L = 0$ as the critical value, M_{TR}, where TR means transitory indicating L is crossing the zero critical value. It can be seen that in general the motions are stable for smaller Mach numbers, and unstable for larger Mach numbers. Moreover, it is interesting to observe that the slopes of the curves are slightly decreasing as the Ψ_1 is increasing, suggesting that the M_{TR} is larger for larger values of Ψ_1, and physically giving a measure of the rapidity of transition of the aeroelastic system, from the

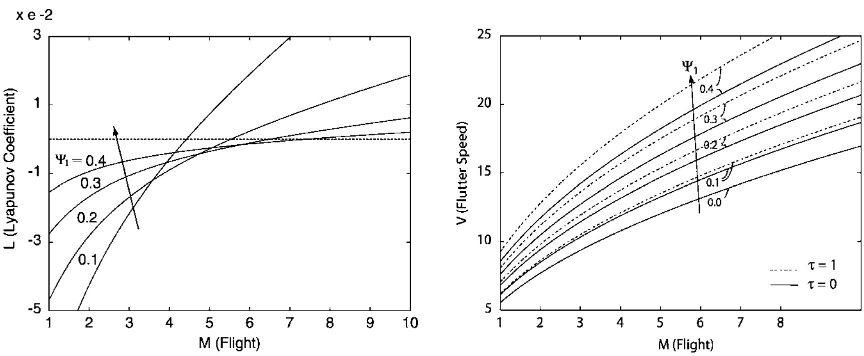


Figure 11.4.14. (a) The LFQ corresponding to $\Psi_2 = 10\Psi_1$ for $\tau = 0$; and (b) effects of the linear control with or without time delay on the flutter boundary.

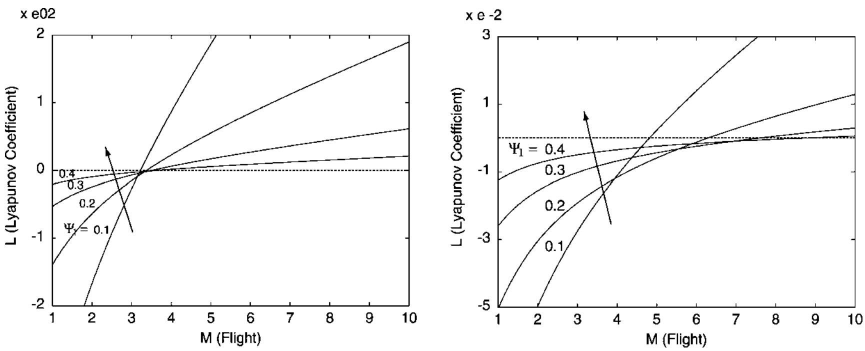


Figure 11.4.15. (a) The LFQ corresponding to $\Psi_2 = 0$, $\tau = 1$ for $\Psi_1 = 0.1,\ 0.2,\ 0.3,\ 0.4$; and (b) the LFQ corresponding to $\Psi_2 = 10\Psi_1$, $\tau = 1$ for $\Psi_1 = 0.1,\ 0.2,\ 0.3,\ 0.4$.

benign state to the catastrophic one, i.e., an idea of the occurrence of a mild or explosive type of flutter.

CASE 2. $\tau = 0$ (i.e., no delay), but $\Psi_2 = 10\Psi_1$.

Here, for convenience in comparison, we take $\Psi_2 = 10\Psi_1$. The presence of Ψ_2 does not change the relation between the flutter velocity and Mach number since the flutter speed is only determined by linear terms. The Lyapunov coefficients for this case are depicted in Figure 11.4.14(a). This result is also in good agreement with that in [306]. It clearly shows that the nonlinear feedback control is more effective than the linear feedback control in rendering the flutter boundary a benign one.

CASE 3. Time delay τ is fixed, Ψ_1 is varied while $\Psi_2 = 0$.

It should be noted that the time delay τ given in equation (11.4.107) is nondimensionalized. The real time delay is $\hat{\tau} = \tau\omega_\alpha$. We fix the time delay ($\tau = 1$ is selected in this chapter) and investigate the effects of the linear and nonlinear control gains on the flutter stability boundary.

The results for considering the linear control only are shown in Figures 11.4.14(b) and 11.4.15(a). It is noted from Figure 11.4.14(b) that the trends are similar to that of the case without time delay. V_F is a monotonically increasing function of the linear control gain, Ψ_1, and the flight Mach number, M_∞. However, it should be noted that the value of V_F with the time delay, for any particular point (Ψ_1, M_∞), has an increase, compared to the case without time delay. Compared with the case without time delay, it is seen that the effect of τ becomes more prominent for larger values of Ψ_1. This suggests that employing time delay in the feedback control is beneficial in controlling flutter instability, and a better control may be obtained using a proper combination of time delay with a larger linear control gain. Similar trends can also be observed from Figure 11.4.15(a) where the values of Lyapunov coefficient are shown. Again, comparing this figure with that in [306] indicates that the time delay helps stabilize vibrating motions.

CASE 4. Time delay τ is fixed, with $\Psi_2 = 10\Psi_1$.

The results obtained for this case are shown in Figure 11.4.15(b). The effect of the nonlinear control combined with the linear control can be clearly observed from this figure. Further, a comparison between Figures 11.4.14(a), 11.4.15(a), and 11.4.15(b) again confirm that time delay and nonlinear control lay much more stress on the stability.

It is seen from the above results that introducing a time delay into the feedback control can have a profound effect on the stability of the bifurcating motions. It can transfer subcritical Hopf bifurcations (occurring in the presence of aerodynamic nonlinearities), to supercritical. To obtain the best controller in controlling both the initiation of Hopf bifurcation and the stability of bifurcating motions, further parametric study is needed.

In this chapter, we have studied normal form computations, bifurcation of limit cycles, and Hopf bifurcation control. Several recently developed methods are introduced. Illustrative examples chosen from both mathematical and practical problems are presented, and numerical results are given to confirm the analytical predictions. It has been shown that the phenomenon of limit cycle exists in many real problems, and the importance of Hopf bifurcation control is seen from both theoretical development and practical applications.

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