15.415x Foundations of Modern Finance

Leonid Kogan and Jiang Wang

MIT Sloan School of Management

Lecture 7: Arbitrage Pricing Theory (APT)

Key Concepts

- The Main Idea of APT
- Factor Models
- Well Diversified Portfolios
- Expected Returns on Diversified Portfolios
- Factor Risk Prices / Risk Premia
- Factor-Mimicking Portfolios
- APT for Individual Securities
- Implementation of APT (Macro Factor Model)
- Implementation of APT (Portfolio Factor Model)

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Main steps of APT

- Factor model of returns in which risk can be decomposed into two components:
  - Systematic risks (common to many assets);
  - Non-systematic risks (specific to individual assets).
- Diversification eliminates risk → For diversified portfolios, $\bar{r}_p$ depends only on systematic factors (arbitrage otherwise).
- Portfolios come from risky assets → For "almost all" risky assets.
  - Expected return $\bar{r}_i$ depends only on systematic factors.
- End result: Model to price risky assets by their exposure to systematic risks.

Systematic vs. idiosyncratic risk

- Uncertainty in asset returns has two sources: Common factors and firm-specific shocks.
- Common factors:
  - Proxy for economic conditions or events that affect all firms and investors.
  - Such factors may include interest rates, price of oil, government policy shocks, etc.
  - Example: If return on an asset increases when inflation increases, it can be used to hedge uncertainty in future inflation rate → smaller risk premium as a result of investors' extra demand for this asset.
  - Represent systematic risks that cannot be diversified away.

Systematic vs. idiosyncratic risk

- Firm-specific events:
  - Such events may include new product innovations, lawsuits, changes in management, labor strikes, …
  - These firm-specific or idiosyncratic risks can be diversified away.

Example: a 2-factor model

- Suppose that the only two systematic sources of risk are:
  - Unanticipated changes in economic growth; and
  - Unanticipated changes in energy prices.

expected value is zero.

- The return on any stock respond to both sources of macro shocks and to firm-specific shocks:

expected return

factor loadings

firm-specific shocks

$$\tilde{r}_i = \bar{r}_i + b_{i,GR}\, \tilde{f}_{GR} + b_{i,EN}\, \tilde{f}_{EN} + \tilde{\epsilon}_i$$

systematic shocks

Example: a 2-factor model

$$\tilde{r}_i = \bar{r}_i + b_{i,GR}\,\tilde{f}_{GR} + b_{i,EN}\,\tilde{f}_{EN} + \tilde{\epsilon}_i$$

- A solar panel installer.
- Cash flows have moderate exposure to economic growth → b_{GR} is positive.
- Benefits from rising energy costs → b_{EN} is likely positive and large.

- A long-distance trucking firm.
- Cash flows are very sensitive to economic activity → b_{GR} is likely positive and large.
- Sensitive to energy costs → b_{EN} is negative and large.

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A single-factor model

- A large number of risk assets, $i = 1, 2, 3, ...$
  - $\tilde{r}_i$ is the (random) return.
  - $\bar{r}_i$ is the expected return.
- Returns are driven by a common, systematic factor, and idiosyncratic shocks.
- $\tilde{F}$ is a systematic factor that affects most asset returns (e.g., return on the market portfolio). OR economic growth
- $\tilde{f}$ is the news component of this common factor: $\tilde{f} = \tilde{F} - \overline{F}$.
- Idiosyncratic shock to asset i: $\tilde{\epsilon}_i$ with zero mean, $E[\tilde{\epsilon}_i] = 0$.
- A key assumption: $\tilde{\epsilon}_i$ are asset-specific, i.e., $\tilde{\epsilon}_i$ are uncorrelated across assets:

$$Cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0 \text{ for } i \neq j$$

A single-factor model

- Describe asset returns as

$$\tilde{r}_i = \underbrace{\bar{r}_i}_{\substack{\text{expected} \\ \text{return}}} + \underbrace{b_i \tilde{f} + \tilde{\epsilon}_i}_{\text{risk}}$$

- Return variance

$$\sigma_i^2 = \underbrace{b_i^2 \sigma_f^2}_{\substack{\text{systematic} \\ \text{risk}}} + \underbrace{\mathrm{Var}(\tilde{\epsilon}_i)}_{\substack{\text{idiosyncratic} \\ \text{risk}}}$$

- Return covariance

$$\mathrm{Cov}\left(\tilde{r}_i, \tilde{r}_j\right) = \mathrm{Cov}(\bar{r}_i + b_i \tilde{f} + \tilde{\epsilon}_i, \bar{r}_j + b_j \tilde{f} + \tilde{\epsilon}_j) = b_i b_j \sigma_f^2$$

because $\mathrm{Cov}\left(\tilde{f}, \tilde{\epsilon}_i\right) = \mathrm{Cov}(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0$.

  - Factor exposure determines how much asset returns co-move.
  - Idiosyncratic risk affects individual return variance.

Multifactor models

you can detract factor expectation from factor variable, thus the factor variable has zero mean. factor model also works without detract (zero mean factor)

- A multifactor model specifies

$$\tilde{r}_i = \bar{r}_i + \underbrace{b_{i,1}\,\tilde{f}_1 + b_{i,2}\,\tilde{f}_2 + \cdots + b_{i,K}\,\tilde{f}_K}_{\substack{\text{systematic} \\ \text{component}}} + \tilde{\epsilon}_i$$

- The $\tilde{f}_1$, $\tilde{f}_2$, …, $\tilde{f}_K$ are the common factors.
  - Common factors may be correlated with each other.
  - The $b_{i,1}$, $b_{i,2}$,…, $b_{i,K}$, are the asset's factor sensitivities (or factor loadings or factor betas).
- The residuals are firm-specific:

$$\mathrm{Cov}\left(\tilde{\epsilon}_i, \tilde{\epsilon}_j\right) = 0 \quad \text{for all } i \neq j$$

- We assume that all factor shocks have zero mean, $\mathrm{E}\left[\tilde{f}_k\right] = 0$, $k = 1,2, \ldots, K$.

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Portfolio return

- The return process of a portfolio is

$$\tilde{r}_p = \bar{r}_p + b_{p,1}\,\tilde{f}_1 + b_{p,2}\,\tilde{f}_2 + \cdots + b_{p,K}\,\tilde{f}_K + \tilde{\epsilon}_p$$

where

$$\bar{r}_p = \sum_{i=1}^{N} w_i\,\bar{r}_i\,, \quad b_{p,k} = \sum_{i=1}^{N} w_i\,b_{i,k}\,, \quad \tilde{\epsilon}_p = \sum_{i=1}^{N} w_i\,\tilde{\epsilon}_i$$

- Because $\tilde{\epsilon}_i$'s are uncorrelated, the non-systematic variance of a portfolio is

$$\mathrm{Var}(\tilde{\epsilon}_p) = \mathrm{Var}\left(\sum_{i=1}^{N} w_i\tilde{\epsilon}_i\right) = \sum_{i=1}^{N} w_i^2\mathrm{Var}(\tilde{\epsilon}_i)$$

Well diversified portfolios

- Consider an equally-weighted portfolio with $w_i = 1/N$.
- Let $\overline{\sigma_i^2}$ denote the average non-systematic variance:

$$\overline{\sigma_i^2} = \frac{1}{N}\sum_{i=1}^{N} \mathrm{Var}(\tilde{\epsilon}_i)$$

- Then, idiosyncratic portfolio variance is

$$\mathrm{Var}\left(\tilde{\epsilon}_p\right) = \frac{1}{N}\overline{\sigma_i^2}$$

- When N goes to infinity → non-systematic variance goes to zero!
  - This result does not require that portfolios have equal weights. The conclusion holds as long as portfolio weights are relatively evenly distributed across the assets.

Well diversified portfolios

- Asset-specific risk is uncorrelated across assets, it can be diversified away by holding large diversified portfolios.

- A well-diversified portfolio is a portfolio that distributes holdings over a large number of securities so that the non-systematic variance $\mathrm{Var}(\tilde{\epsilon}_p)$ is negligible.

- In a well-diversified portfolio, firm-specific effects average out:

$$\tilde{\epsilon}_p \approx 0$$

- For a well-diversified portfolio, only systematic (factor) risk is present:

$$\tilde{r}_p = \bar{r}_p + b_{p,1}\,\tilde{f}_1 + b_{p,2}\,\tilde{f}_2 + \cdots + b_{p,K}\,\tilde{f}_K$$

Example

- Consider two portfolios:
  - Annual returns, starting in 1926. Source: Kenneth R. French's Data Library.
  - P_1 contains the largest US stocks, top 30% relative to NYSE stocks by size (~500 securities in recent years).
  - P_2 contains mid-cap US stocks, next 40% relative to NYSE stocks by size (~1,000 recently).
- These portfolios are well-diversified, and do not overlap in holdings.
- If return distribution was described by a single-factor model, we would observe an approximate linear relation between the two portfolios

$$\tilde{r}_{P_i} = \bar{r}_{P_i} + b_{P_i} \tilde{f}, \qquad i = 1,2$$

$$\tilde{r}_{P_1} = \bar{r}_{P_1} + b_{P_1} \tilde{f}$$

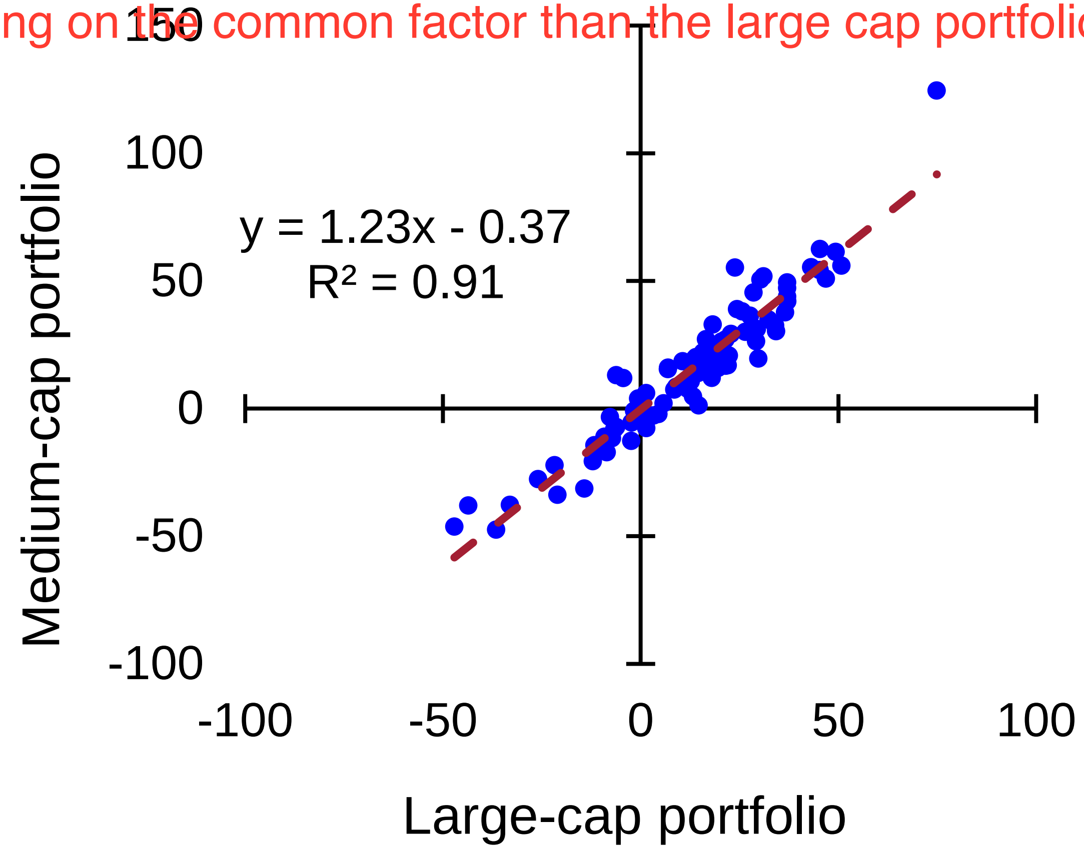
$$\tilde{r}_{P_2} = \bar{r}_{P_2} + b_{P_2} \tilde{f}$$

$$then \quad \frac{\tilde{r}_{P_1} - \bar{r}_{P_1}}{\tilde{r}_{P_2} - \bar{r}_{P_2}} = \frac{b_{P_1}}{b_{P_2}}$$

Example: returns of size-sorted portfolios

- Both portfolios are exposed to the market-wide shocks, which account for most of return variation for each portfolio.

Observe that the slope of the line that captures the linear relation between these two portfolios is higher than 1. What this means is that the mid cap portfolio has a higher loading on the common factor than the large cap portfolio.



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An arbitrage argument

- Example: single systematic factor, two well-diversified portfolios.

excess returns are measured relative to the risk-free rate

	Expected excess return	Factor loading
Portfolio A	5%	1.0
Portfolio B	8%	2.0

- There is arbitrage in this market!
- Arbitrage strategy:

this strategy doesn't require any capital

  - Borrow \$1;
  - Short \$1 of Portfolio B;
  - Invest \$2 of Portfolio A.
  - No risk (zero factor loading), zero investment, and positive payoff.

$2

no exposure to the factor

$$
\begin{aligned}
\text{Payoff} = & -(1 + r_f) \times \$1 \quad \text{risk-free} \\
& -(1 + r_f + 8\% + 2.0\tilde{f}) \times \$1 \\
& +(1 + r_f + 5\% + 1.0\tilde{f}) \times \$2 \\
= & \ \$0.02
\end{aligned}
$$

APT pricing relation

- Expected excess returns and factor loading must be linearly related.
- For a single-factor model, expected excess returns on diversified portfolios must be proportional to the factor loading:

for any portfolio, p $\bar{r}_p - r_f = \lambda\, b_p$

\lambda has to be common to all portfolios.

- Suppose this is not the case:

$$\bar{r}_q - r_f = \lambda' b_q, \quad \lambda' \neq \lambda, \quad b_q \neq 0.$$

- Create an arbitrage trade:
  - Short \$1 of portfolio p;
  - Buy $\$(b_p/b_q)$ of portfolio q;
  - Borrow $\$(b_p/b_q - 1)$

$$\begin{aligned}\text{Payoff} = &-\left(1 + r_f + \lambda\, b_p + b_p \tilde{f}\right) \times 1 \\ &+\left(1 + r_f + \lambda' b_q + b_q \tilde{f}\right) \times (b_p/b_q) \\ &-\left(1 + r_f\right) \times (b_p/b_q - 1) \quad \text{risk-free} \\ = &\ (\lambda' - \lambda) b_p \neq 0 \Rightarrow \text{arbitrage}\end{aligned}$$

For a well-diversified portfolio, only systematic (factor) risk is present:

$$\tilde{r}_p = \bar{r}_p + b_{p,1}\, \tilde{f}_1 + b_{p,2}\, \tilde{f}_2 + \cdots + b_{p,K}\, \tilde{f}_K$$

A zero-investment portfolio without risk cannot produce non-zero payoffs

APT pricing relation

- Arbitrage opportunities cannot exist in a frictionless market.
- To avoid arbitrage, expected excess returns (risk premia) on all well-diversified portfolios must satisfy

$$\bar{r}_p - r_f \quad = \quad \lambda \quad \times \quad b_p$$

magnitude of the factor risk exposure

Risk premium = Price of risk × Quantity of risk

- λ tells us how much compensation one earns in the market for a unit of factor risk exposure.
- λ is called the market price of risk of the factor, or the factor risk premium.

APT relation for multi-factor models

- APT pricing relation generalizes to multi-factor models

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

- Expected excess return on a diversified portfolio is determined by its loadings on the common factors:
  - Factor exposures measure portfolio risk;
  - Multi-dimensional nature of risk: each factor exposure carries its own risk premium.

p21, b_p = b_q

proof

- Intuition: can construct multiple portfolios with the same factor loadings – these all must have the same risk premium to avoid arbitrage.
- Therefore, portfolio risk premium is determined by its factor loadings.

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Factor risk premia

- We can use the APT relation

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

  to recover prices of risk for each factor as implied by expected returns on other assets.

Recovering risk prices from portfolio returns

- Consider an example with $K = 2$ factors: economic growth shock (GR) and energy price shock (EN).

- Start with the general APT relation

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

- Observe risk premia on two well-diversified portfolios, A and B:

	Expected Return	Factor Loadings	
		GR	EN
Portfolio A	12%	1.0	1.25
Portfolio B	10%	2.0	-0.50
Risk-free asset	2%		

- Want to recover factor risk premia for GR and EN.

Recovering risk prices from portfolio returns

- APT relation implies two equations for expected excess returns on portfolios A and B:

$$\underbrace{12\% - 2\%}_{\text{risk premium}} = \underbrace{1.0}_{\text{factor loading}} \times \underbrace{\lambda_{GR}}_{\text{price of risk}} + \underbrace{1.25}_{\text{factor loading}} \times \underbrace{\lambda_{EN}}_{\text{price of risk}} \quad (A)$$

$$10\% - 2\% = 2.0\,\lambda_{GR} - 0.50\,\lambda_{EN} \quad (B)$$

- Solving these equations, we find

$$\lambda_{GR} = 5\%,$$

$$\lambda_{EN} = 4\%.$$

To avoid arbitrage

- All other portfolios must have expected returns consistent with these factor premia, e.g., if portfolio C has factor loadings $b_{GR} = 1.0$, and $b_{EN} = 0.5$, then

$$\bar{r}_C - r_f = 1.0\,\lambda_{GR} + 0.5\,\lambda_{EN} = 7\%$$

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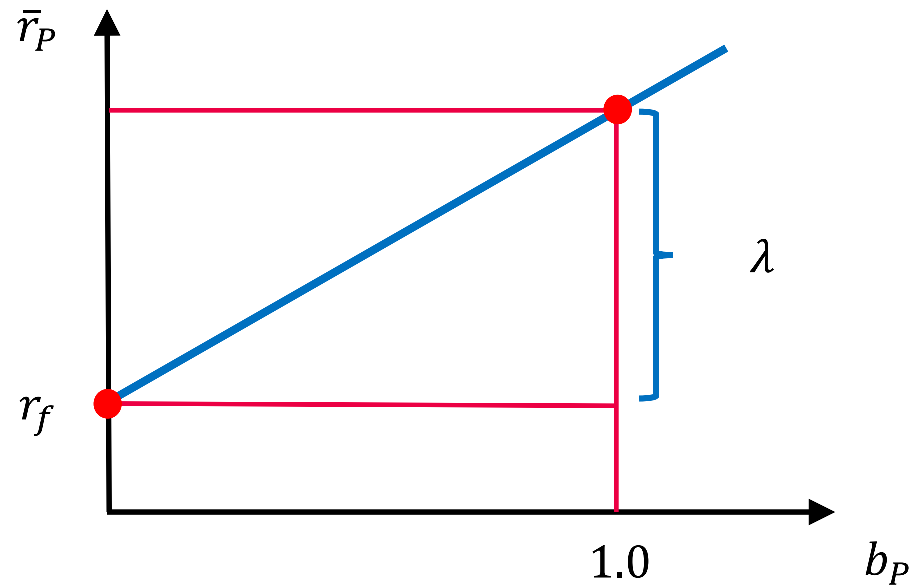
Factor mimicking portfolios

we can replace any common shock with a corresponding return on Factor mimicking portfolios

- Consider a special case of a single-factor model.
- Factor mimicking portfolios are portfolios with unit factor exposure, $b_P = 1$.
- Risk premium on the factor-mimicking portfolio equals the factor risk premium.
- This portfolio is perfectly correlated with the factor – can use it instead of the factor in the APT relation.



we can replace the factor by its mimicking portfolio in our factor model

Multiple factors

- A model with K factors and linearly independent portfolios $P_1, P_2, \ldots, P_K$.
- Construct factor-mimicking portfolios: risk premium of each factor equals the expected excess return on the factor-mimicking portfolio.

loading on factor j is equal to one. loadings on all the other factors are equal to zero

- A factor-mimicking portfolio for factor j is a well-diversified portfolio with a beta of 1 on factor j and a beta of 0 on any other factor.
- A factor-mimicking portfolio for factor k with weights $(w_0, w_1, w_2, \ldots, w_K)$, w_0 in the risk-free asset, satisfies

$$\underbrace{w_1}_{\substack{\text{portfolio weight}\\ \text{of } P_1}} \times \underbrace{b_{P_1,1}}_{\substack{\text{factor loading}\\ \text{of } P_1 \text{ on factor } 1}} + w_2 b_{P_2,1} + \cdots + w_K b_{P_K,1} = 0 \qquad (1)$$

$$\ldots$$

$$w_1 b_{P_1,j} + w_2 b_{P_2,j} + \cdots + w_K b_{P_K,j} = 1 \qquad (j)$$

$$\ldots$$

$$w_1 b_{P_1,K} + w_2 b_{P_2,K} + \cdots + w_K b_{P_K,K} = 0 \qquad (K)$$

Example

- Mimic the Energy shock (EN) using portfolios A and B: weights w_A, w_B.

	Expected Return	Factor Loadings	
		GR	EN
Portfolio A	12%	1.0	1.25
Portfolio B	10%	2.0	-0.50
Risk-free asset	2%		

$$1.0\, w_A + 2.0\, w_B = 0 \qquad (GR)$$
$$1.25\, w_A - 0.50\, w_B = 1 \qquad (EN)$$

- Result: $w_A = 0.67, \;\; w_B = -0.33.$ 0.67 is invested in risk-free, which loading is 0

- The risk premium on the Energy factor is then

$$\lambda_{EN} = 10\%\, w_A + 8\%\, w_B = 4\%$$

mimic: expected excess return = factor risk premium

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APT for individual securities

- For any well-diversified portfolio p with factor sensitivities $b_{p,1}, \ldots, b_{p,K}$, the risk premium equals

$$\bar{r}_p - r_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

where λ_n is the risk premium on the n^{th} factor.

- This result also applies to almost all individual securities.

- This is the Arbitrage Pricing Theory (APT), developed by Stephen Ross in 1976.

APT for individual securities: intuition

- Suppose that many assets violate the APT relation.
- Then can find many assets for which $\alpha_i \neq 0$ in

$$\bar{r}_i - r_f = \alpha_i + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \cdots + \lambda_K b_{i,K}$$

- Suppose many assets have a positive alpha (negative values work analogously).
- Combine them in a well-diversified, equally-weighted portfolio p^*:

$$\bar{r}_{p^*} - r_f = \overline{\alpha} + \lambda_1 b_{p^*,1} + \lambda_2 b_{p^*,2} + \cdots + \lambda_K b_{p^*,K}$$

portfolio loading is AVG of individual assets

where $\overline{\alpha}$ is the average alpha across assets in p^*.

- This contradicts the APT results for diversified portfolios, so we cannot find many assets that violate APT.

$$\tilde{r}_p = \bar{r}_p + b_{p,1}\,\tilde{f}_1 + b_{p,2}\,\tilde{f}_2 + \cdots + b_{p,K}\,\tilde{f}_K + \tilde{\epsilon}_p$$

$$\bar{r}_p = \sum_{i=1}^{N} w_i\,\bar{r}_i\,, \quad b_{p,k} = \sum_{i=1}^{N} w_i\,b_{i,k}\,, \quad \tilde{\epsilon}_p = \sum_{i=1}^{N} w_i\,\tilde{\epsilon}_i$$

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Implementation of APT

- Three steps:
  - Identify/choose the factors.
    - Economic variables that are thought to affect asset returns.
    - How many and which?
  - Estimate factor loadings of assets.
    - Usually by a time-series regression of diversified portfolio returns on factors.
  - Estimate factor premia.
    - Usually by a cross-sectional regression of excess returns on factor loadings.
- End up with an assessment of which factors matter and how much.

A Macro-factor model

- Chen, Roll, and Ross (1986, *Journal of Business*).
- In addition to the market factor, use economic variables to represent systematic factors explaining the returns of financial assets.
- Monthly growth rate of industrial production (MP).
- Changes in expected inflation (DEI).
    - Measured by changes in T-Bill rates.
- Unexpected inflation (UI).

A Macro-factor model

- Unexpected changes in risk premium (UPR), measured as the difference between returns on bonds rated Baa (or lower), and long-term government bonds.
- Unexpected changes in the term premium (UTS), measured as the difference between returns on long-term government bonds and T-Bills.
- Data: Monthly observations from 1953 to 1984.

let's take the S&P 500 Total Return Index and the NASDAQ Composite Total Return Index as proxies for well-diversified portfolios. we'll assume that we know Rf (the risk-free return) is 2%. We'll also assume that the annual expected return of the portfolios are 7% for the S&P 500 Total Return Index and 9% for the NASDAQ Composite Total Return Index

Step 1: Determine Systematic Factors: Let's assume that the real gross domestic product (GDP) growth rate and the 10-year Treasury bond yield change are the factors that we need

Step 2: Obtain Betas: We ran a regression on historical quarterly data of each index against quarterly real GDP growth rates and quarterly T-bond yield changes.

	GDP	Bond
SP500	3.45	0.033
NASDAQ	4.74	0.098

Step 3: Obtain Factor Prices or Factor Premiums

7% = 2% + 3.45*f_1 + 0.033*f_2

9% = 2% + 4.74*f_1 + 0.098*f_2

Solving these equations we get: f_1= 1.43% and f_2= 2.47%. Therefore, E(R i)=2%+1.43%⋆β 1 +2.47%⋆β 2

Estimation: betas factor loadings

- Group stocks into 20 size portfolios (5% smallest to 5% largest).
- Run time-series regressions (5 years of monthly data) to obtain factor sensitivities.
  - For each size portfolio i, estimate β's in

$$R_{i,t} = a_i + \beta_{i,RM} RM_t + \beta_{i,MP} MP_t + \beta_{i,DEI} DEI_t$$
$$+\beta_{i,UI} UI_t + \beta_{i,UPR} UPR_t + \beta_{i,UTS} UTS_t + \epsilon_{i,t}$$

    where RM is for the return on the market index (e.g., value-weighted stock index).

- Result: estimates $\beta_{i,RM}, \beta_{i,MP}, \beta_{i,DEI}, \beta_{i,UI}, \beta_{i,UPR}, \beta_{i,UTS}$ for each portfolio i.

Estimation: risk premia

risk price

- Run cross-sectional regressions to get factor risk premia and determine if they are statistically significant.
  - Using the β's of the 20 portfolios, estimate λ's in a cross-sectional regression of monthly returns on the betas

portfolio return

$$R_i = a_i + \lambda_{RM}\beta_{i,RM} + \lambda_{MP}\beta_{i,MP} + \lambda_{DEI}\beta_{i,DEI}$$
$$+\lambda_{UI}\beta_{i,UI} + \lambda_{UPR}\beta_{i,UPR} + \lambda_{UTS}\beta_{i,UTS} + \epsilon_i$$

  - Average $\hat{\lambda}_k$ over time to estimate the risk premium for factor k.
  - Result: estimates of the risk premium (λ) for each of the factors.

Results

- Factors are not very highly correlated.
- All economic factors are priced.
- Market factor is not priced separately from other factors.

	VWNY	MP	DEI	UI	UPR	UTS	Constant
1958–84	−2.403	11.756	−.123	−.795	8.274	−5.905	10.713
	(−.633)	(3.054)	(−1.600)	(−2.376)	(2.972)	(−1.879)	(2.755)
1958–67	1.359	12.394	.005	−.209	5.204	−.086	9.527
	(.277)	(1.789)	(.064)	(−.415)	(1.815)	(−.040)	(1.984)
1968–77	−5.269	13.466	−.255	−1.421	12.897	−11.708	8.582
	(−.717)	(2.038)	(−3.237)	(−3.106)	(2.955)	(−2.299)	(1.167)
1978–84	−3.683	8.402	−.116	−.739	6.056	−5.928	15.452
	(−.491)	(1.432)	(−.458)	(−.869)	(.782)	(−.644)	(1.867)

NOTE.—VWNY = return on the value-weighted NYSE index; EWNY = return on the equally weighted NYSE index; MP = monthly growth rate in industrial production; DEI = change in expected inflation; UI = unanticipated inflation; UPR = unanticipated change in the risk premium (Baa and under return − long-term government bond return); UTS = unanticipated change in the term structure (long-term government bond return − Treasury-bill rate); and YP = yearly growth rate in industrial production. *t*-statistics are in parentheses.

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- Factor-Mimicking Portfolios
- APT for Individual Securities
- Implementation of APT (Macro Factor Model)
- Implementation of APT (Portfolio Factor Model)

The Fama-French factor model

- Fama and French (1993 Journal of Financial Economics, 1996 Journal of Finance).
- Factors do not necessarily have to be macroeconomic variables.
- Sufficient that they correlate with changes in the macroeconomy.

Size: market capitalization of the stock.

- Generate factor-mimicking portfolios by sorting firms by size and book-to-market ratio.
  - Intuition: small firms and high B/M firms are exposed differently to macroeconomic factors.

The Fama-French factor model

- Factor portfolios:
  - $R_M - R_f$: Return on the value-weighted market minus the T-Bill rate.
  - SMB (“Small minus Big”): Return on small-cap stocks minus return on large-cap stocks.
  - HML (“High minus Low”): Return on stocks with high B/M ratio (value firms) minus return on stocks with low B/M ratio (growth firms).
- Factors carry significant risk premia: $\lambda_{R_M - R_f} = 0.43\%$, $\lambda_{SMB} = 0.27\%$, $\lambda_{HML} = 0.40\%$ (per month) (Fama and French 1993).

The Fama-French factor model

- Data: Monthly returns on all NYSE, AMEX, and NASDAQ stocks from 1963 to 1991.

- Methodology:

  - Form 25 stock portfolios based on size and book-to-market equity.

  - Run time-series regressions of monthly excess returns on the returns to the market portfolio and mimicking portfolios for size and book-to-market equity (see next page).

$$R_i - R_f = \alpha_i + b_i\left(R_M - R_f\right) + s_i SMB + h_i HML + \epsilon_i$$

  - Evaluate factor loadings and the intercepts (APT alphas).

Results

- B/M and Size portfolios exhibit a large spread in average returns.
- Small stocks ("Small" row) outperform large stocks ("Big" row) on average.
- Value stocks ("High" column) outperform growth stocks ("Low" column) on average.

	Book-to-Market Equity (BE/ME) Quintiles									
Size	Low	2	3	4	High	Low	2	3	4	High
	Panel A: Summary Statistics									
	Means					Standard Deviations				
Small	0.31	0.70	0.82	0.95	1.08	7.67	6.74	6.14	5.85	6.14
2	0.48	0.71	0.91	0.93	1.09	7.13	6.25	5.71	5.23	5.94
3	0.44	0.68	0.75	0.86	1.05	6.52	5.53	5.11	4.79	5.48
4	0.51	0.39	0.64	0.80	1.04	5.86	5.28	4.97	4.81	5.67
Big	0.37	0.39	0.36	0.58	0.71	4.84	4.61	4.28	4.18	4.89

Results

- Factors do a good job explaining the cross-section of returns.

	b					t(b)				
Small	1.03	1.01	0.94	0.89	0.94	39.10	50.89	59.93	58.47	57.71
2	1.10	1.04	0.99	0.97	1.08	52.94	61.14	58.17	62.97	65.58
3	1.10	1.02	0.98	0.97	1.07	57.08	55.49	53.11	55.96	52.37
4	1.07	1.07	1.05	1.03	1.18	54.77	54.48	51.79	45.76	46.27
Big	0.96	1.02	0.98	0.99	1.07	60.25	57.77	47.03	53.25	37.18

	s					t(s)				
Small	1.47	1.27	1.18	1.17	1.23	39.01	44.48	52.26	53.82	52.65
2	1.01	0.97	0.88	0.73	0.90	34.10	39.94	36.19	32.92	38.17
3	0.75	0.63	0.59	0.47	0.64	27.09	24.13	22.37	18.97	22.01
4	0.36	0.30	0.29	0.22	0.41	12.87	10.64	10.17	6.82	11.26
Big	−0.16	−0.13	−0.25	−0.16	−0.03	−6.97	−5.12	−8.45	−6.21	−0.77

	h					t(h)				
Small	−0.27	0.10	0.25	0.37	0.63	−6.28	3.03	9.74	15.16	23.62
2	−0.49	0.00	0.26	0.46	0.69	−14.66	0.34	9.21	18.14	25.59
3	−0.39	0.03	0.32	0.49	0.68	−12.56	0.89	10.73	17.45	20.43
4	−0.44	0.03	0.31	0.54	0.72	−13.98	0.97	9.45	14.70	17.34
Big	−0.47	0.00	0.20	0.56	0.82	−18.23	0.18	6.04	18.71	17.57

Results

- Factors do a good job explaining the cross-section of returns: high R^2
- Portfolios are well diversified, returns are well explained by the common factors.

	R^2					s(e)				
Small	0.93	0.95	0.96	0.96	0.96	1.97	1.49	1.18	1.13	1.22
2	0.95	0.96	0.95	0.95	0.96	1.55	1.27	1.28	1.16	1.23
3	0.95	0.94	0.93	0.93	0.92	1.44	1.37	1.38	1.30	1.52
4	0.94	0.92	0.91	0.88	0.89	1.46	1.47	1.51	1.69	1.91
Big	0.94	0.92	0.87	0.89	0.81	1.19	1.32	1.55	1.39	2.15

Results

- Intercepts (α_i) from 3-factor regressions are close to 0.
  - Estimates of alphas are statistically indistinguishable from 0 for most portfolios.
- Some violations for small stocks (extreme B/M quintiles).

	Book-to-Market Equity (BE/ME) Quintiles									
Size	Low	2	3	4	High	Low	2	3	4	High
Panel B: Regressions: $R_i - R_f = a_i + b_i(R_M - R_f) + s_i SMB + h_i HML + e_i$										
	a					t(a)				
Small	−0.45	−0.16	−0.05	0.04	0.02	$\boxed{-4.19}$	−2.04	−0.82	0.69	0.29
2	−0.07	−0.04	0.09	0.07	0.03	−0.80	−0.59	1.33	1.13	0.51
3	−0.08	0.04	−0.00	0.06	0.07	−1.07	0.47	−0.06	0.88	0.89
4	0.14	−0.19	−0.06	0.02	0.06	1.74	−2.43	−0.73	0.27	0.59
Big	0.20	−0.04	−0.10	−0.08	−0.14	$\boxed{3.14}$	−0.52	−1.23	−1.07	−1.17

Conclusion

- We showed how risk premia on common risk factors can be inferred from historical returns on financial assets.
- Multiple techniques available, including cross-sectional and time-series regression methods.
- The main weakness of APT: the theory does not tell us what the common factors are, this is an empirical question.
- APT model is a flexible and general valuation framework.
- Absence of arbitrage imposes internal consistency, APT connects expected returns to measures of risk – loadings on the common factors.

Summary

- The Main Idea of APT
- Factor Models
- Well Diversified Portfolios
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