Week 5 – Black-Scholes-Merton and the Greeks

MIT Sloan School of Management

Finance at MIT

Where ingenuity drives results

Topic outline

- Calibrating and implementing multistep binomial trees
- The Black-Scholes-Merton model
- The Greeks
- Financial engineering application: Dynamic delta hedging of portfolio protection products

Multi-step trees

- Multi-step trees make it easy to numerically price derivatives under realistic assumptions about the distribution of future prices for the underlying security
- Stock returns are often modeled as having a constant expected return μ and constant volatility σ
  - That is, over a small interval of time h:

sigma squared represents the annualized variance on a continuous basis.

$$E\left[\frac{S_{t+h}}{S_t}\right] = e^{\mu \times h}; \quad \text{and} \quad E\left[\left(\frac{S_{t+h}}{S_t} - e^{\mu} h\right)^2\right] = \sigma^2 \times h$$

variance of sum = sum of variance

- Consider an option with maturity T
- We chop off the time interval $[0, T]$ into n little intervals

independently and identically distributed over time

$$\left[0, \frac{T}{n}, \frac{2T}{n}, \frac{3T}{n}, \cdots, \frac{(n-1)T}{n}, T\right]$$

  - Each interval is of size $h = T/n$. We denote the steps $i = 0, 1, ..., n$ $\Longrightarrow i \times h =$ calendar time on the tree

Multi-step trees

- Assume that the stock process follows:

$$S_{i+1} = \begin{cases} S_i \times u & \text{with probability} \quad q \\ S_i \times d & \text{with probability} \quad 1-q \end{cases} \tag{1}$$

Notice: q is the actual probability!

- We now choose u, d and q to approximate the expected return μ and variance σ^2 on the tree
- The following conditions must be satisfied:
  1. **Expected Return:** $E\left(\frac{S_{i+1}}{S_i}\right) = e^{\mu \times h} \Longrightarrow q \times u + (1-q) \times d = e^{\mu \times h}$
  2. **Variance:** $E\left(\left(\frac{S_{i+1}}{S_i} - e^{\mu \times h}\right)^2\right) = \sigma^2 \times h \Longrightarrow$

$$q\left(u - e^{\mu \times h}\right)^2 + (1-q)\left(d - e^{\mu \times h}\right)^2 = \sigma^2 \times h$$

- Two equations, three parameters $\Longrightarrow$ we have one degree of freedom left. The following assumption gives symmetry to the problem.
  3. **Percentage increase in stock = percentage decrease in stock**

$$u = 1/d$$

Multi-step trees

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- These three equations in three unknowns give the solution:

consider dividend: mu*h ==> (mu–div_yield)*h

$$u = e^{\sigma \times \sqrt{h}} \; ; \quad d = 1/u \; ; \qquad \text{and} \;\; q = \frac{e^{\mu \times h} - d}{u - d}.$$

used to solve sigma

- We can then use these parameters and the procedure (1) to construct our tree relatively easily.
- Next figure shows an example from *BinomialTree.xls*

The spreadsheet is available with this week' course materials

Multi-step trees: Example

BINOMIAL TREE MODEL

Stock Assumption		Option Assumption			Tree		Risk Neutral Prob	
mu	0.1	T	0.1		n	10	q*	0.495834
sigma	0.3	K	100		h	0.01		
r	0.02	Call or Put	1	(=1 for call	u	1.030455	Price Binomial	3.787
div	0				d	0.970446	Delta Binomial	0.527
S0	100				q	0.509173		

time ==>	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
i==>	0	1	2	3	4	5	6	7	8	9	10
0	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125	130.996	134.986
1		97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125
2			94.176	97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722
3				91.393	94.176	97.045	100.000	103.045	106.184	109.417	112.750
4					88.692	91.393	94.176	97.045	100.000	103.045	106.184
5						86.071	88.692	91.393	94.176	97.045	100.000
6							83.527	86.071	88.692	91.393	94.176
7								81.058	83.527	86.071	88.692
8									78.663	81.058	83.527
9										76.338	78.663
10											74.082

times u
current → up
times d
→ down

1. Time index: $i = 0, 1, ..., n$. Node index: $j = 0, 1, ..., n$
2. Because we start from up-left corner
   - an *up* movement in S between i and $i+1$ correspond to same index j;
   - a *down* movement in S between i and $i+1$ corresponds to an *increase* in the index j.

Multi-step trees: derivative pricing

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How do we price a derivative security on this binomial tree?

- Risk neutral pricing provides an immediate answer

consider dividend: r*h ==> (r−div_yield)*h

$$\frac{q^* S_0^u + (1 - q^*) S_0^d}{e^{rh}} = S_0 * e^{-\delta h}$$

save and buy later = buy now

1. Compute the risk neutral probability:

$$q^* = \frac{e^{r \times h} - d}{u - d}$$

Since u, d, h are the same for each step, q^* remains constant.

2. Move backward on the tree one step at a time:

For European options, $V_{i,j} = e^{-r \times h} \times E^* \left[V_{i+1} | (i,j) \right]$

where i, j is the (time, node) state on the tree.

Multi-step trees: derivative pricing of European call

- For instance, for call options, start from the end of the tree with the final condition

$$c_{n,j} = \max(S_{n,j} - K, 0) \quad \text{for} \quad j = 0, 1, ..., n$$

- Then move backward applying

$$c_{i,j} = e^{-r \times h} \times (q^* c_{i+1,j} + (1 - q^*) c_{i+1,j+1})$$

Multi-step trees: Example

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BINOMIAL TREE MODEL

Stock Assumption		Option Assumption			Tree		Risk Neutral Prob	
mu	0.1	T	0.1		n	10	q*	0.495834
sigma	0.3	K	100		h	0.01		
r	0.02	Call or Put	1	(=1 for call	u	1.030455	Price Binomial	3.787
div	0				d	0.970446	Delta Binomial	0.527
So	100				q	0.509173		

time ==>	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
i==>	0	1	2	3	4	5	6	7	8	9	10
0	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125	130.996	134.986
1		97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125
2			94.176	97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722
3				91.393	94.176	97.045	100.000	103.045	106.184	109.417	112.750
4					88.692	91.393	94.176	97.045	100.000	103.045	106.184
5						86.071	88.692	91.393	94.176	97.045	100.000
6							83.527	86.071	88.692	91.393	94.176
7								81.058	83.527	86.071	88.692
8									78.663	81.058	83.527
9										76.338	78.663
10											74.082

34 = 134 – 100

Option Price

time ==>	0.000	0.010	0.020	0.030	0.040	0.050	0.060	0.070	0.080	0.090	0.100
i==>	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
0	3.787	5.381	7.440	9.982	12.965	16.283	19.802	23.428	27.165	31.016	34.986
1		2.221	3.358	4.943	7.053	9.707	12.830	16.243	19.762	23.388	27.125
2			1.103	1.801	2.871	4.445	6.640	9.477	12.790	16.203	19.722
3				0.418	0.749	1.323	2.289	3.851	6.224	9.437	12.750
4					0.092	0.185	0.373	0.753	1.520	3.065	6.184
5						0.000	0.000	0.000	0.000	0.000	0.000
6							0.000	0.000	0.000	0.000	0.000
7								0.000	0.000	0.000	0.000
8									0.000	0.000	0.000
9										0.000	0.000
10											0.000

Multi-step trees: Example with 250 steps instead of 10

take the number of steps to infinity gives the Black–Scholes–Merton price

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As the strike price gets closer to the actual stock price, the value of the option is more sensitive to movements in the stock price and hence it takes more steps for the value to converge.

BINOMIAL TREE MODEL

Stock Assumption		Option Assumption			Tree		Risk Neutral Prob	
mu	0.1	T	0.1		n	250	q*	0.499167
sigma	0.3	K	100		h	0.0004		
r	0.02	Call or Put	1	(=1 for call	u	1.006018	Price Binomial	3.876
div	0				d	0.994018	Delta Binomial	0.527
So	100				q	0.501833		

time ==>	0	0.0004	0.0008	0.0012	0.0016	0.002	0.0024	0.0028	0.0032	0.0036	0.004
i==>	0	1	2	3	4	5	6	7	8	9	10
0	100.000	100.602	101.207	101.816	102.429	103.045	103.666	104.289	104.917	105.548	106.184
1		99.402	100.000	100.602	101.207	101.816	102.429	103.045	103.666	104.289	104.917
2			98.807	99.402	100.000	100.602	101.207	101.816	102.429	103.045	103.666
3				98.216	98.807	99.402	100.000	100.602	101.207	101.816	102.429
4					97.629	98.216	98.807	99.402	100.000	100.602	101.207
5						97.045	97.629	98.216	98.807	99.402	100.000
6							96.464	97.045	97.629	98.216	98.807
7								95.887	96.464	97.045	97.629
8									95.313	95.887	96.464
9										94.743	95.313
10											94.176

time ==>	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.003	0.003	0.004	0.004
i==>	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
0	3.876	4.193	4.527	4.879	5.247	5.633	6.036	6.457	6.894	7.348	7.818
1		3.561	3.861	4.177	4.511	4.863	5.231	5.618	6.021	6.442	6.879
2			3.262	3.545	3.845	4.161	4.495	4.847	5.216	5.602	6.005
3				2.980	3.246	3.529	3.828	4.145	4.479	4.831	5.200
4					2.715	2.964	3.230	3.513	3.812	4.129	4.463
5						2.466	2.699	2.949	3.214	3.497	3.796
6							2.233	2.451	2.684	2.933	3.198
7								2.016	2.218	2.436	2.668
8									1.814	2.002	2.204
9										1.628	1.801
10											1.455

Multi-step trees: Probability distribution of S_T

- What is the probability distribution of the stock price at maturity S_T?

- Without going into details, consider an n-step tree.
- There are $n+1$ nodes at maturity, $j = 0, 1, ..., n$.
  - The top node $j = 0$ can only be reached with n ups u and 0 downs d
  - Node $j = 1$ can only be reached with $n-1$ ups u and 1 down d

$$\vdots \qquad\qquad\qquad \vdots$$

  - Node $j = n$ can only be reached with 0 ups u and n downs d

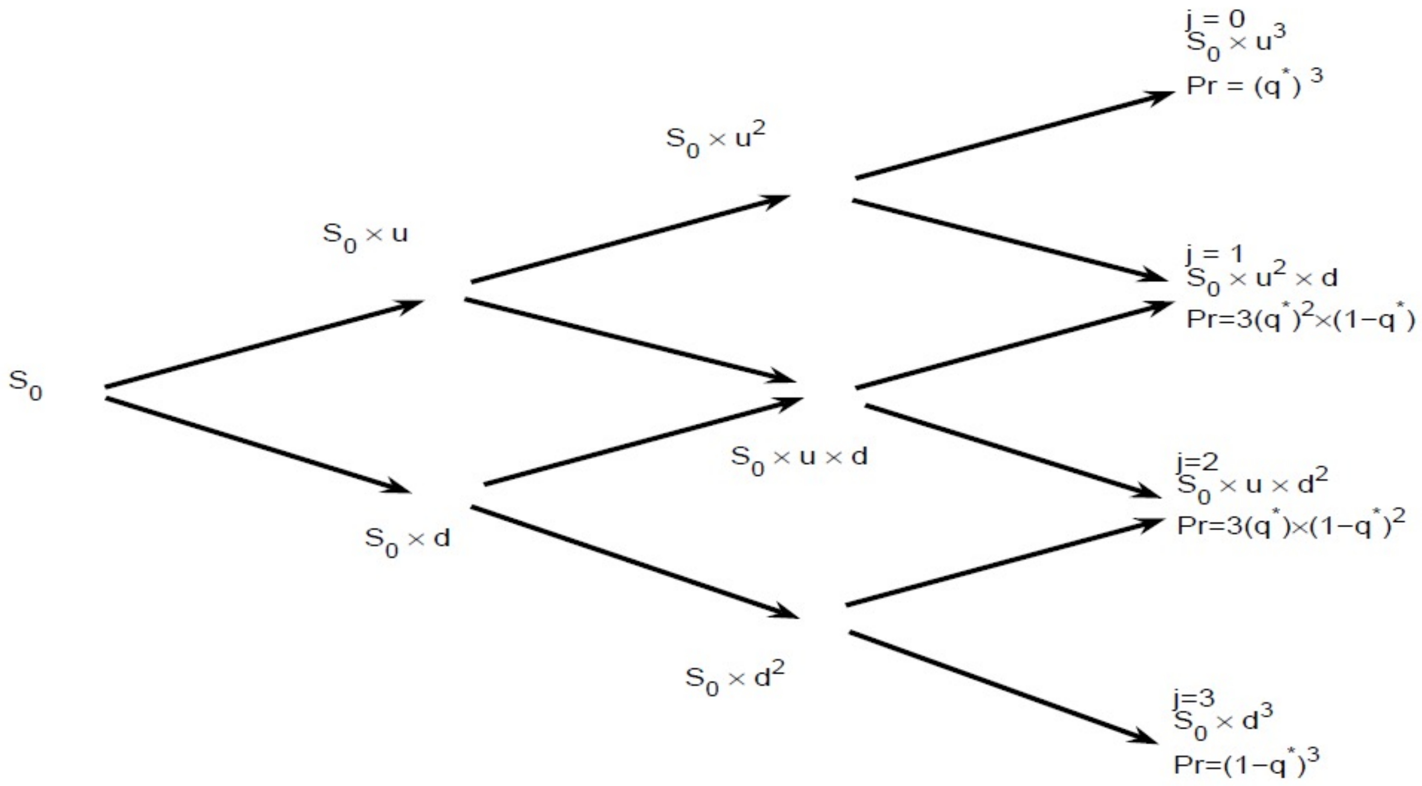
risk–neutral

  - If each up u has probability q^*, what is the probability of reaching exactly node j?

$$\Pr(\text{node} = j \text{ at } T = n \times h) = \left(\frac{n!}{j!(n-j)!}\right) \times (q^*)^{n-j} \times (1-q^*)^j$$

where $n! = 1 \times 2 \times 3 \times ... \times n$ (note: 0!=1)

For instance, if $n = 3$

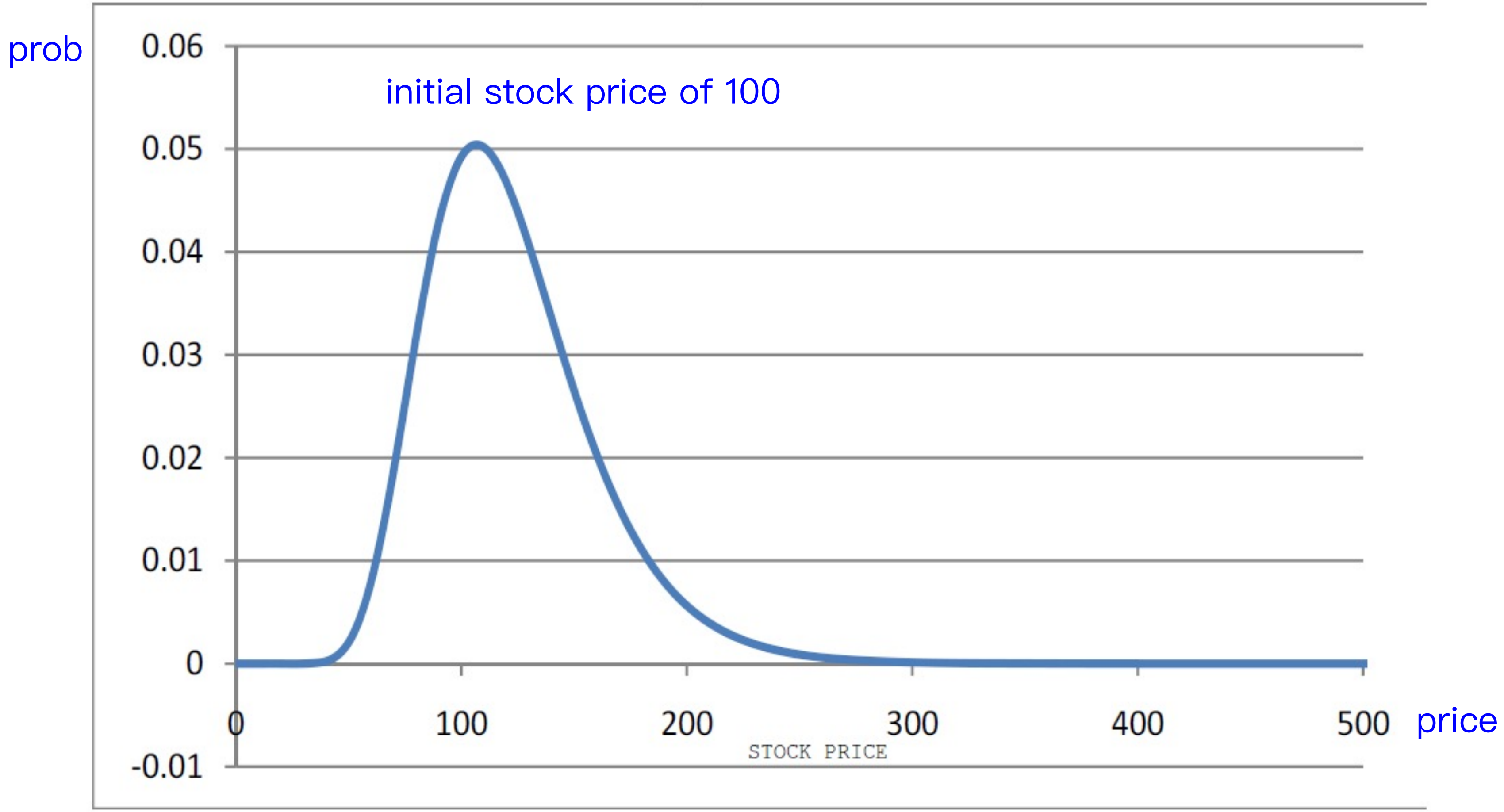


$$\Pr\left(j = 0 \text{ at } T = n \times h\right) = \left(\frac{3!}{0! \times 3!}\right) \times (q^*)^3 \times (1 - q^*)^0 = (q^*)^3$$

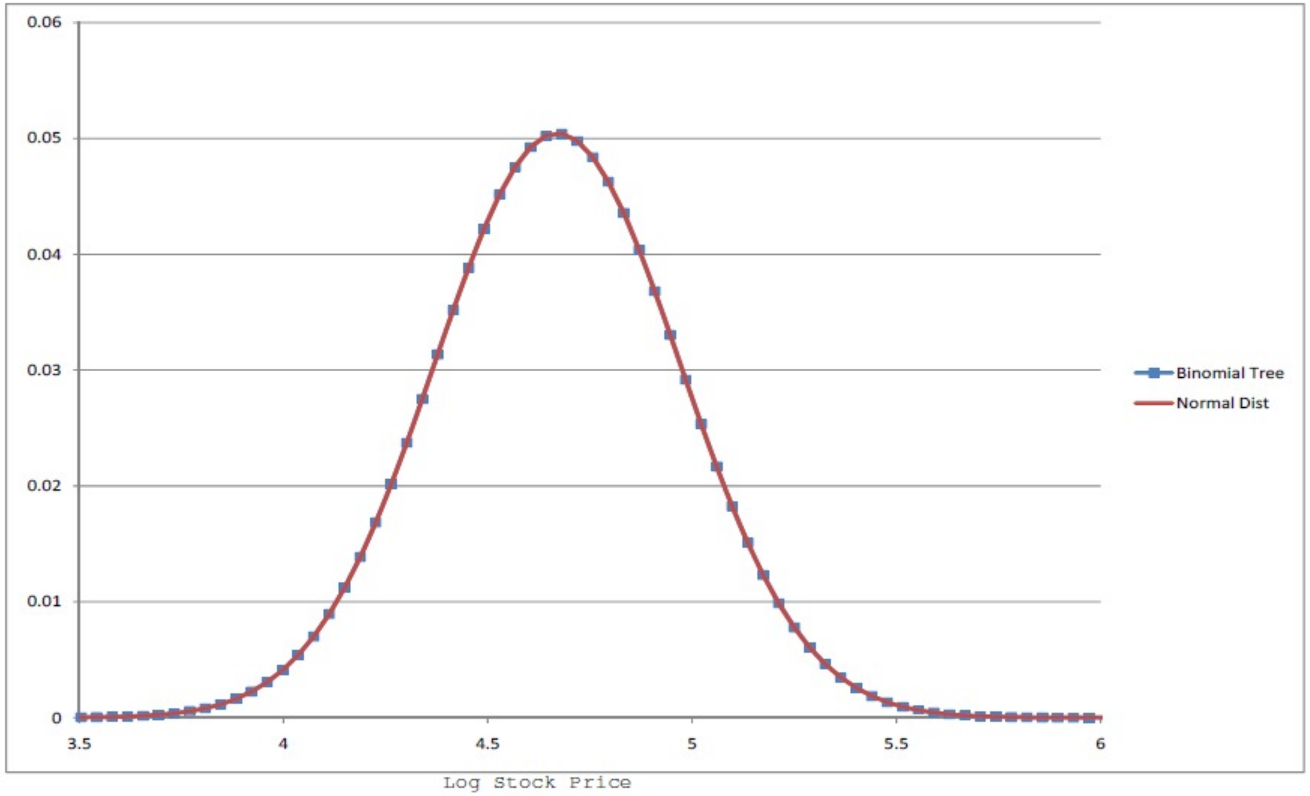
$$\Pr\left(j = 1 \text{ at } T = n \times h\right) = \left(\frac{3 \times 2}{2 \times 1}\right) \times (q^*)^2 \times (1 - q^*) = 3(q^*)^2(1 - q^*)$$

As n increases to infinity, the distribution at $T = n \times h$ of the stock price S_T converges to a log-normal distribution.

Probabilities of S_T for $n = 250$ and $T = 1$



Or, plotting the probability density with respect to $\log(S_T)$, we obtain



Also plotted is the normal density with mean $\mu_T = log(S_0) + (r - \delta - \frac{\sigma^2}{2})T$ and standard deviation $\sigma_T = \sigma\sqrt{T}$

Multi-step trees and the Black-Scholes-Merton (BSM) model

- It is evident that the probability distribution of $log(S_T)$ implied by the binomial tree is the same as the normal distribution.
- The option price from the binomial tree is given by

$$c = E^* \left[e^{-rT} \max(S_T - K, 0) \right] = \sum_{j=1}^{n} e^{-rT} \max(S_{n,j} - K, 0) \boxed{Q^*_{n,j}}$$

where $Q^*_{n,j}$ is the risk neutral probability of $S_{n,j}$.[1]

- Using the approximation that S_T is log-normally distributed, denote

$f(S_T) =$ Log normal distribution with mean μ_T, standard deviation σ_T

- Then, as n increases, the call option is given by:

$$c = \int_0^\infty e^{-rT} \max(S_T - K, 0) \boxed{f(S_T) dS_T}$$

This equation is mathematically equivalent to the Black-Scholes-Merton formula, but unlike that model can't be used directly to find option prices.

Assumptions for BSM Model

(1) Financial markets are frictionless:

- No taxes or transactions costs
- Assets are perfectly divisible
- No restrictions on short sales

(2) The interest rates for risk-free borrowing and lending are the same and constant. The annualized continuously compounded risk-free interest rate is r

(3)The stock pays no dividends over the life of the option

(4) Stock prices conform to the log-normal model:

- Stock prices follow a continuous path
- The mean and variance of the log return are constant
- The log return over any period is independent of the log return over any other period
- The log return is normally distributed

Idea behind BSM: Replicating dynamic trading strategy

Black and Scholes (1973) and Merton (1974) show that under the assumptions, one can replicate the payoff on a European call option by dynamically rebalancing a portfolio of the stock and a risk-free security.

By no arbitrage, the value of the European call must equal the initial cost of the replicating portfolio

The replicating portfolio has the properties:

- The stock position is always positive short position in the risk–free bond.
- The fraction of stock in the replicating portfolio is the **hedge ratio** or the delta:

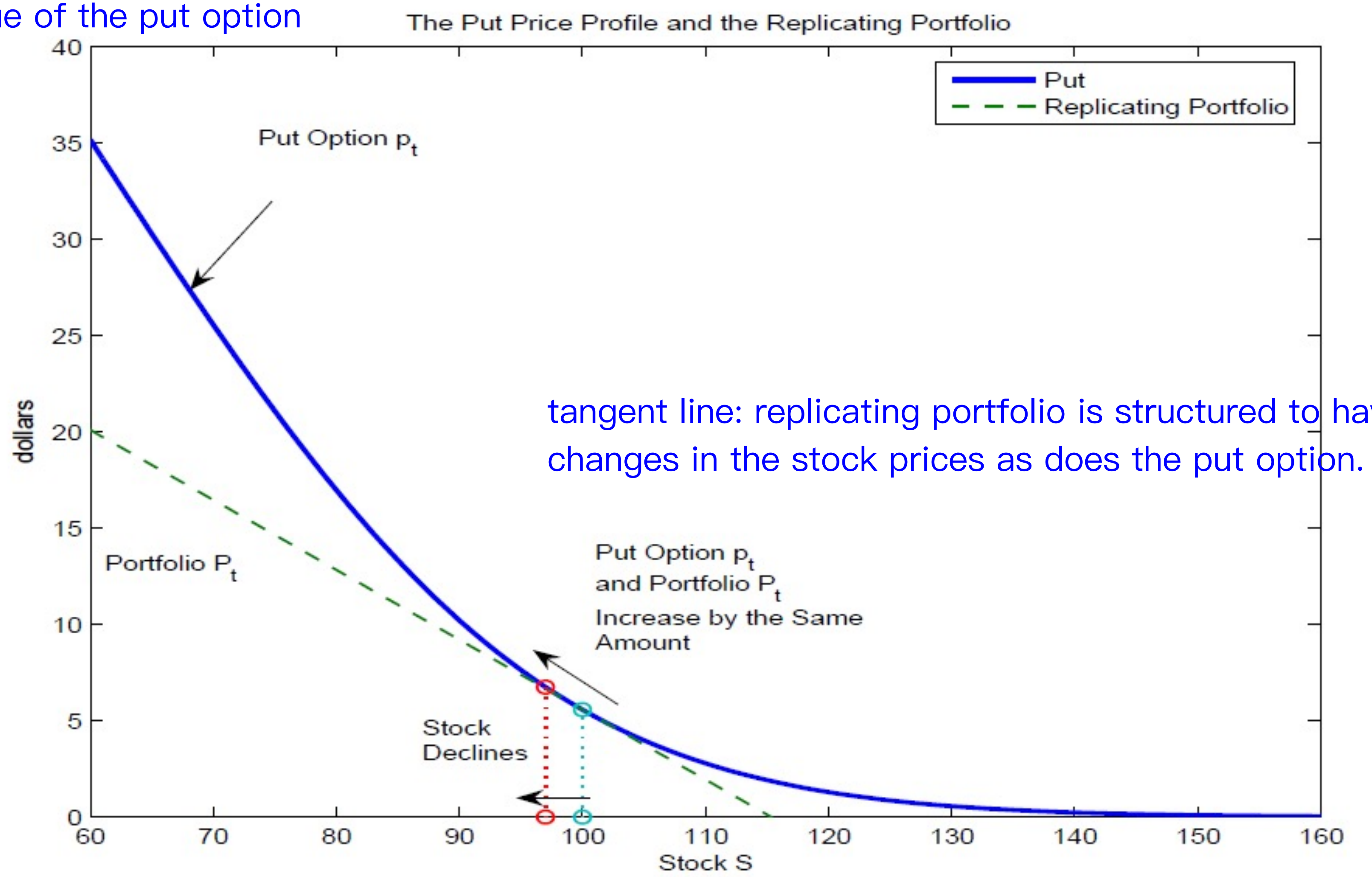
$$\Delta_c = \frac{\partial C}{\partial S}$$

- Since the Δ of the call changes constantly, the fraction of the stock in the replicating portfolio is constantly adjust to replicate the payoff of the call.
- Similarly for a put option the hedge ratio is:

$$\Delta_p = \frac{\partial P}{\partial S}$$

Graphical intuition behind dynamic replication

value of the put option



tangent line: replicating portfolio is structured to have the same sensitivity to changes in the stock prices as does the put option.

put option when the current stock price is at 1 0 0 and the strike price is fixed at 1 0 0

Black-Scholes-Merton formula

- The inputs into the original BSM formula are:
  - S = current stock price
  - K = exercise price
  - $T - t$ = time to expiration
  - r = continuously compounded and annualized risk-free interest rate
  - σ = annualized standard deviation of log returns (annualized “volatility”)

- The most commonly used version of the formula also incorporates the dividend yield.
  - We’ll look at this and other generalizations of BSM next week

Basic BSM formula

Black, Scholes and Merton show that the value of a European call option on a non-dividend paying stock is:

$$\mathbf{C}(\mathbf{S}, \mathbf{K}, \mathbf{T} - \mathbf{t}, \mathbf{r}, \sigma) = \mathbf{S}\,\mathcal{N}(\mathbf{d_1}) - \mathbf{K}\,\mathrm{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\,\mathcal{N}(\mathbf{d_2})$$

where $\mathcal{N}(\cdot)$ is the *cumulative density function* of a *standard normal* random variable, and:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

Using put-call-parity, the value of a European style put option on the same stock with the same strike price is:

$$\mathbf{P}(\mathbf{S}, \mathbf{K}, \mathbf{T} - \mathbf{t}, \mathbf{r}, \sigma) = \mathbf{K}\,\mathrm{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\,\mathcal{N}(-\mathbf{d_2}) - \mathbf{S}\,\mathcal{N}(-\mathbf{d_1})$$

Important: The expected return μ is *not* an input to the formula. Of course, we would expect this from our discussion of the binomial model.

Basic BSM formula: Example

- Consider an at-the-money option: Current stock price $S = 100$, strike price $K = 100$, (continuously compounded) interest rate $r = 5\%$, maturity $T = 1$, return volatility $\sigma = 30\%$
- We then have

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100}{100}\right) + (.05 + (0.30)^2/2) \times 1}{0.30\sqrt{1}} = 0.3167$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.3167 - .3\sqrt{1} = 0.0167$$

- Therefore $\mathcal{N}(d_1) = 0.62425$ and $\mathcal{N}(d_2) = 0.50665$
- The value of the call option is

$$c_0 = S\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) = 100 \times 0.62425 - 100 \times e^{.05\times 1} \times 0.50665 = 14.2312$$

- The value of a put option can be computed from these data by recalling that

$$\mathcal{N}(-d_1) = 1 - \mathcal{N}(d_1) = 0.37575; \quad \mathcal{N}(-d_2) = 1 - \mathcal{N}(d_2) = 0.49335$$

so that

$$p_0 = Ke^{-rT}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) = -100 \times 0.37575 + 100 \times e^{.05\times 1} \times 0.49335 = 9.3542$$

BSM for dividend-paying stocks

The basic BSM formula prices European put and call options on an underlying security that pays no dividends and whose price is log-normally distributed

When there is a known dividend or dividend yield the formula can be easily adjusted:

more dividend, less call price

1. **Options with *known* dividend**
   - Define $S^* = S - PV(D)$
     where $PV(D)$ = Present Value of Dividends before Expiration
   - Use Black-Scholes Formula with S^* instead of S

For more on these variations see Hull Chapter 17

2. **Options with *known* dividend yield δ**
   - Define $S^* = S \times e^{-\delta \times T}$ and use

future appreciation of the value of the index is reduced by the dividends that are paid

$$c = Se^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2); \quad p = Ke^{-rT}\mathcal{N}(-d_2) - Se^{-\delta T}\mathcal{N}(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}; \quad d_2 = d_1 - \sigma\sqrt{T}$$

BSM formula: Interpretation

The Black-Scholes formula expresses the option as a portfolio of stocks and bonds:

- $\mathcal{N}(d_1)$ is the fraction of the share we hold in the replicating portfolio today (at t). It is equivalent to Δ in the binomial model. In fact, for a call we can show that:

$$\Delta_c = \frac{\partial C}{\partial S} = \mathcal{N}(d_1) > 0$$

See Hull problem 15.17 for proofs

Similarly, for a put we can show that:

$$\Delta_p = \frac{\partial P}{\partial S} = -\mathcal{N}(-d_1) < 0$$

- For a call, $K e^{-r(T-t)} \mathcal{N}(d_2)$ is the amount of initial borrowing in the replicating portfolio
- Therefore, the value of the call is again the cost of the replicating portfolio:

$$C = \Delta_c \times S - B = \mathcal{N}(d_1)S - K e^{-r(T-t)} \mathcal{N}(d_2)$$

to calculating profit/loss of hedged position, you should consider both the bond yield and stock yield in replicating portfolio and compare it with option price change

Deriving the BSM formula

- There are at least three different ways to derive the BSM formula:
  (1) As *n* => ∞, the log normal binomial tree model for stock prices converges to the BSM price
  (2) Black-Scholes partial differential equation (PDE)
  (3) Risk-neutral pricing

  - All are helpful for developing intuition.
  - You can read about them in Hull or other books.
  - In the interest of time we won't go through those derivations here.

Black and Scholes decided to try out their theory and started trading based on it. If they found a stock option price that was too low based on the model, they bought the option, and they hedged the position by shorting the replicating portfolio, and vise versa of the option price was too high.
Black and Scholes were making a lot of money!

Risk in options and the Greeks

- Risk managers and options traders need to know the sensitivity of their options holdings to various factors
- Those sensitivities are known as “The Greeks.” We’ll discuss:
  1. Delta the first order sensitivity to the underlying asset's price
  2. Gamma the second order sensitivity to the underlying asset's price;
  3. Theta the sensitivity to the passage of time
  4. Rho the sensitivity to interest rates
  5. vega the sensitivity to the underlying's volatility
- For options that can be priced using BSM, they often take a simple form

Risk in options and the Greeks

- Terminology:
  1. **Delta**: Sensitivity of option to changes in underlying

$$\Delta = \frac{d \text{ Option Price}}{d\ S} = \begin{cases} \mathcal{N}(d_1) & \text{for Calls} \\ -\mathcal{N}(-d_1) & \text{for Puts} \end{cases}$$

  2. **Gamma**: Sensitivity of Delta (Δ) to changes in the underlying. For both calls and puts:

$$\Gamma = \frac{d\,\Delta}{d\,S} = \frac{N'(d_1)}{S\sigma\sqrt{T}} \quad \text{with} \quad N'(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$$

$$\Longrightarrow \Gamma = \text{curvature of option price with respect to stock } S$$

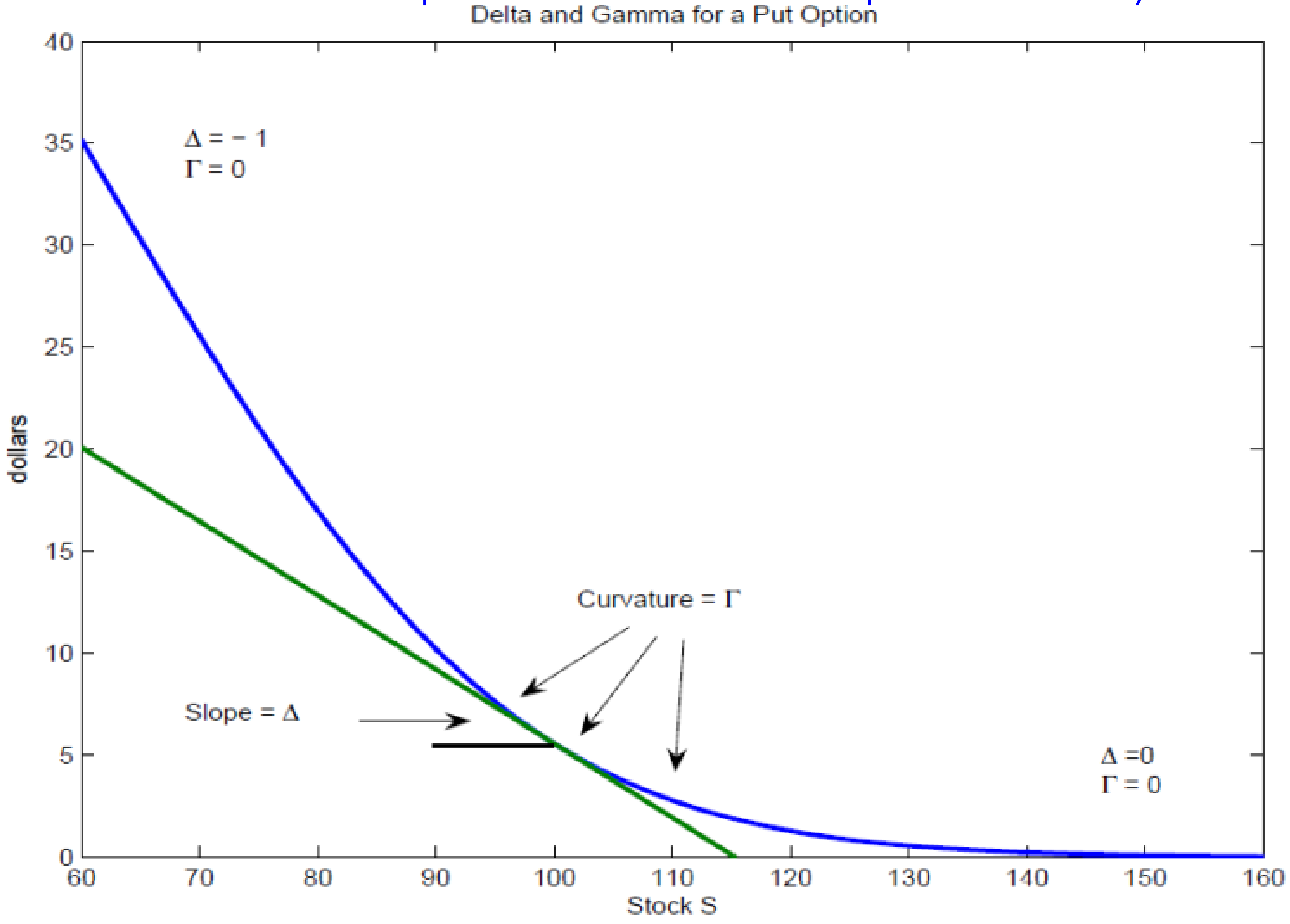
Note: Expressions with T instead of T-t assume t=0

$\mathcal{Q}$: Try to derive Δ and Γ via the Black-Scholes formula

Risk in options and the Greeks

assume that the current stock price is 100 and that the option is currently at–the–money

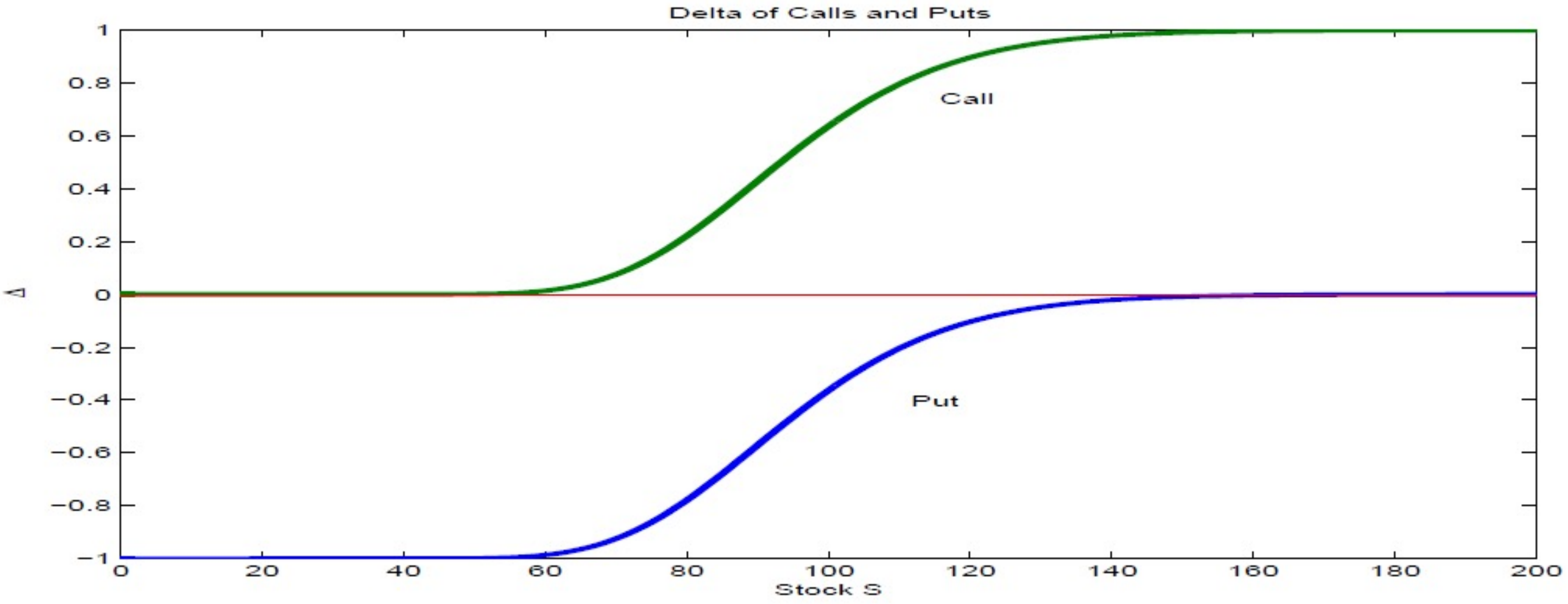
Option value



Risk in options and the Greeks

$K = 100$

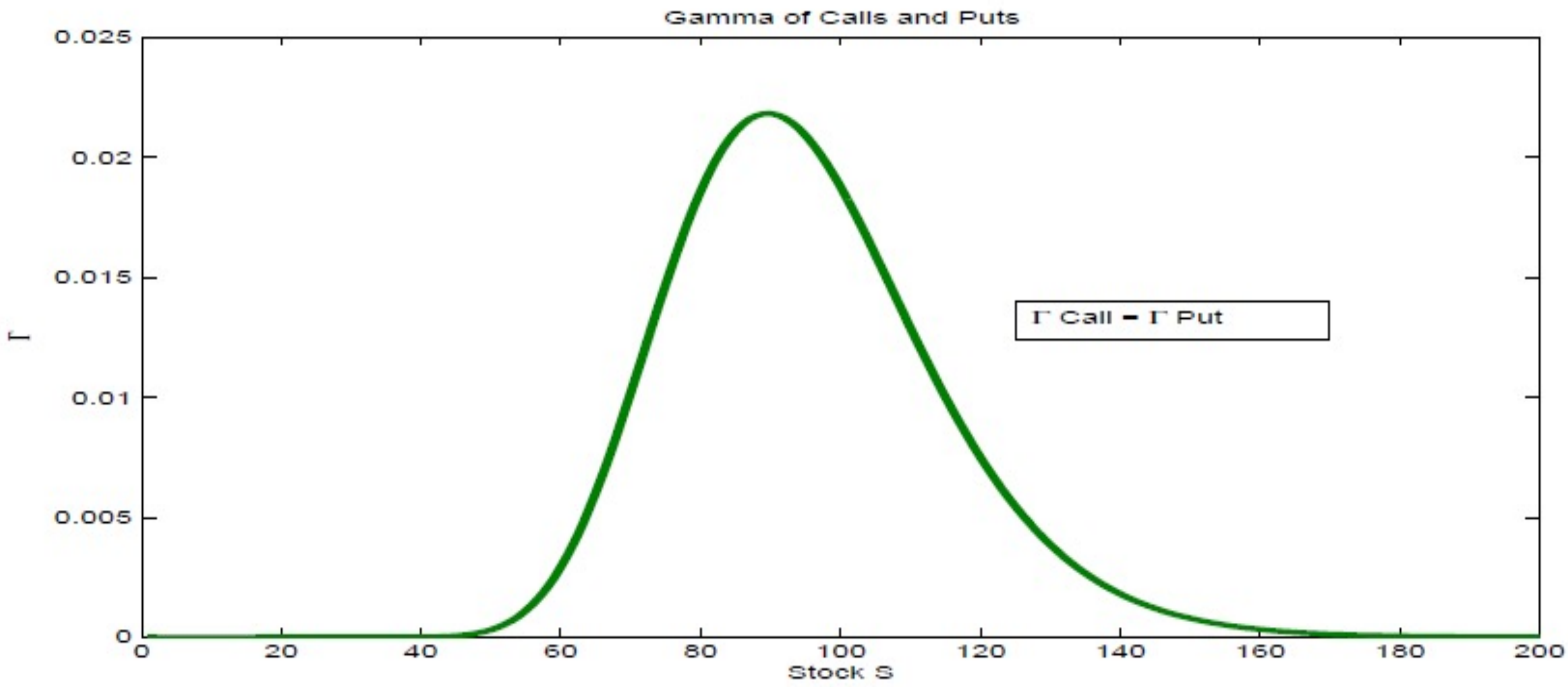
large variation in Deltas with stock prices that causes the replicating portfolios for options to change over time: necessitate frequent portfolio rebalancing and dynamic hedging strategies



Note: This holds all other parameters (K, T, r, σ) constant

Risk in options and the Greeks

peaks near the strike price of the option



Note: This holds all other parameters (K, T, r, σ) constant

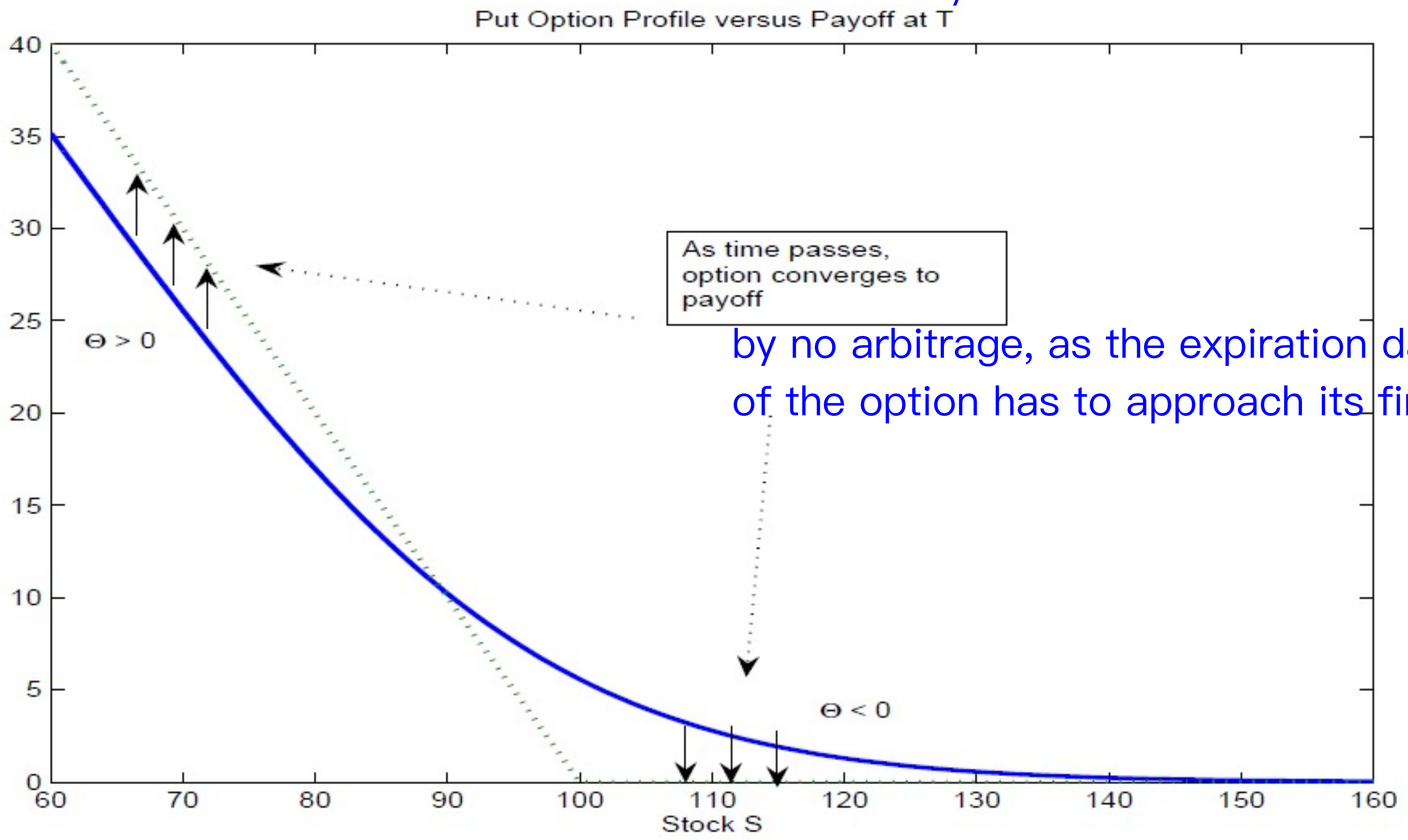
Risk in options and the Greeks

3. **Theta**: Sensitivity of option to passage of time t

Theta is telling us about what happens to option value as its remaining maturity gets smaller

$$\Theta = \frac{d\ \text{Option Price}}{d\ t} = \text{Long ugly formula}$$

T – t: maturity – now



by no arbitrage, as the expiration date approaches,the current value of the option has to approach its final payoff.

Note: Remember we are holding *T* constant

Risk in options and the Greeks

- For a put option, $\Theta > 0$ for low S and $\Theta < 0$ for high S. Why?
  - For S high, payoff is zero, but put price is positive
    $\Longrightarrow$ As time passes (but keeping S fixed), the put price must decline
  - For S low, why is $\Theta > 0$?
  - Simple example:
    - If the firm goes bankrupt $\Longrightarrow S = 0$
    - Then, the put payoff at T is K
    - Value at time $t < T$ is $p = e^{-r(T-t)}K$, which increases with t

that's not always the case: The present value will be higher, the sooner the payoff can be realized

common: for both puts and calls, a shorter time to expiration tends to reduce option value because it shortens the time over which the option provides insurance value

Risk in options and the Greeks

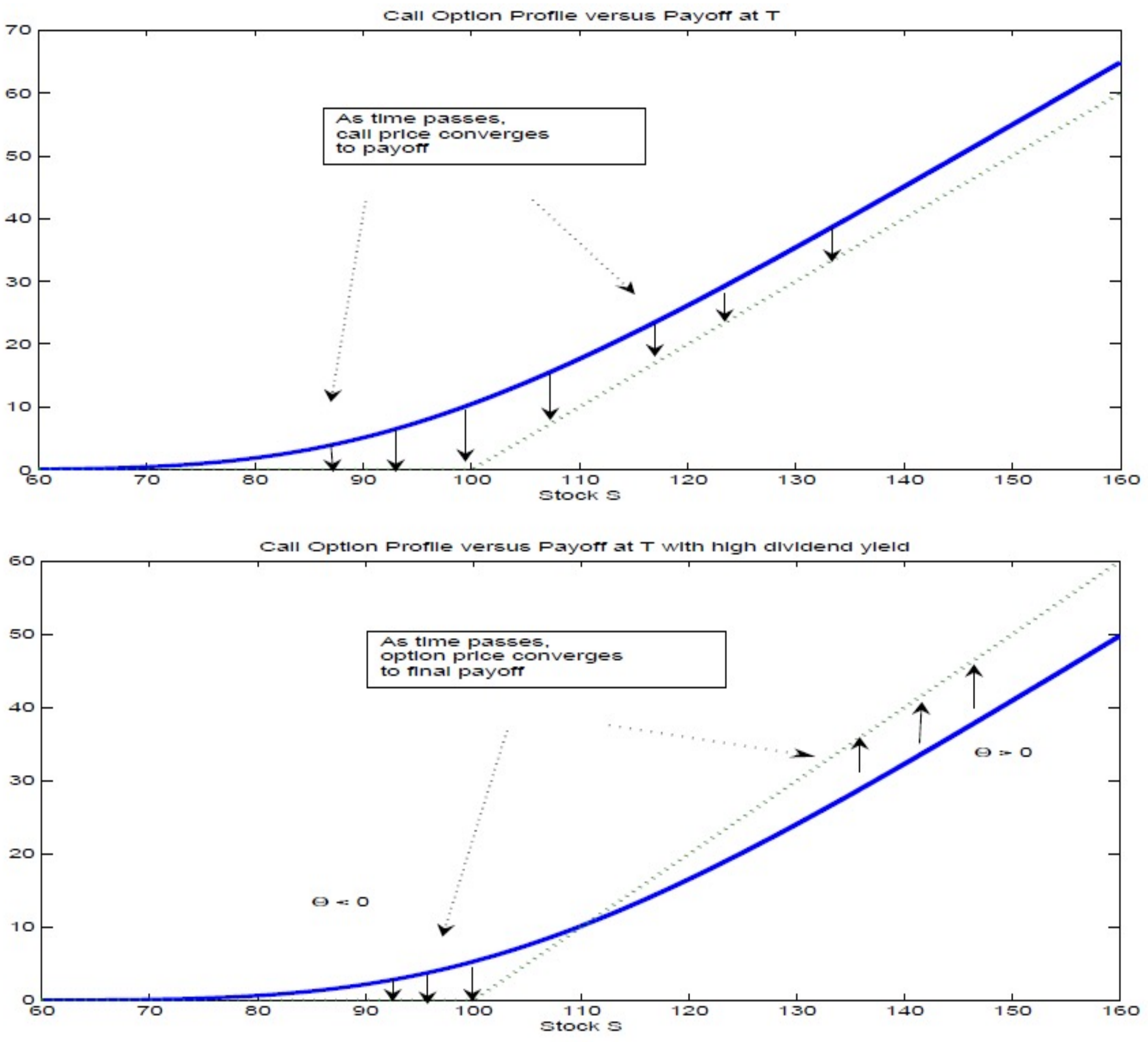
- What about call options?

  when exercised, it's better to postpone paying the strike price

  - For $\boxed{\text{non-dividend}}$ paying stock, $\Theta < 0$ for call options
  - If S is very high, option holder will receive $S - K$ at maturity (with high probability)
  - This is the payoff of a long forward, so present value at $t < T$ is $c = S - Ke^{-rT} > S - K$
    - Intuition: for calls, we will pay K, whose present value today is smaller than K, pushing up the price of the call
  - What if S is low? When the stock price is low, the option provides downside protection for longer, if it has a longer time to maturity
  - If the stock pays (a lot of) dividends, then $\Theta > 0$ for high S, as the option holder misses the dividend payout during the life of the option (compared to a stock holder)

    because the dividend causes the stock price to drop, it can be optimal to early exercise a call option to capture that dividend.

Risk in options and the Greeks



Risk in options and the Greeks

In BSM model, the volatility and risk-free interest rate are constant

Still, it is interesting to ask how a change in σ or *r* change the value of the option

4. **Rho**: Change in option price due to a change in interest rate r

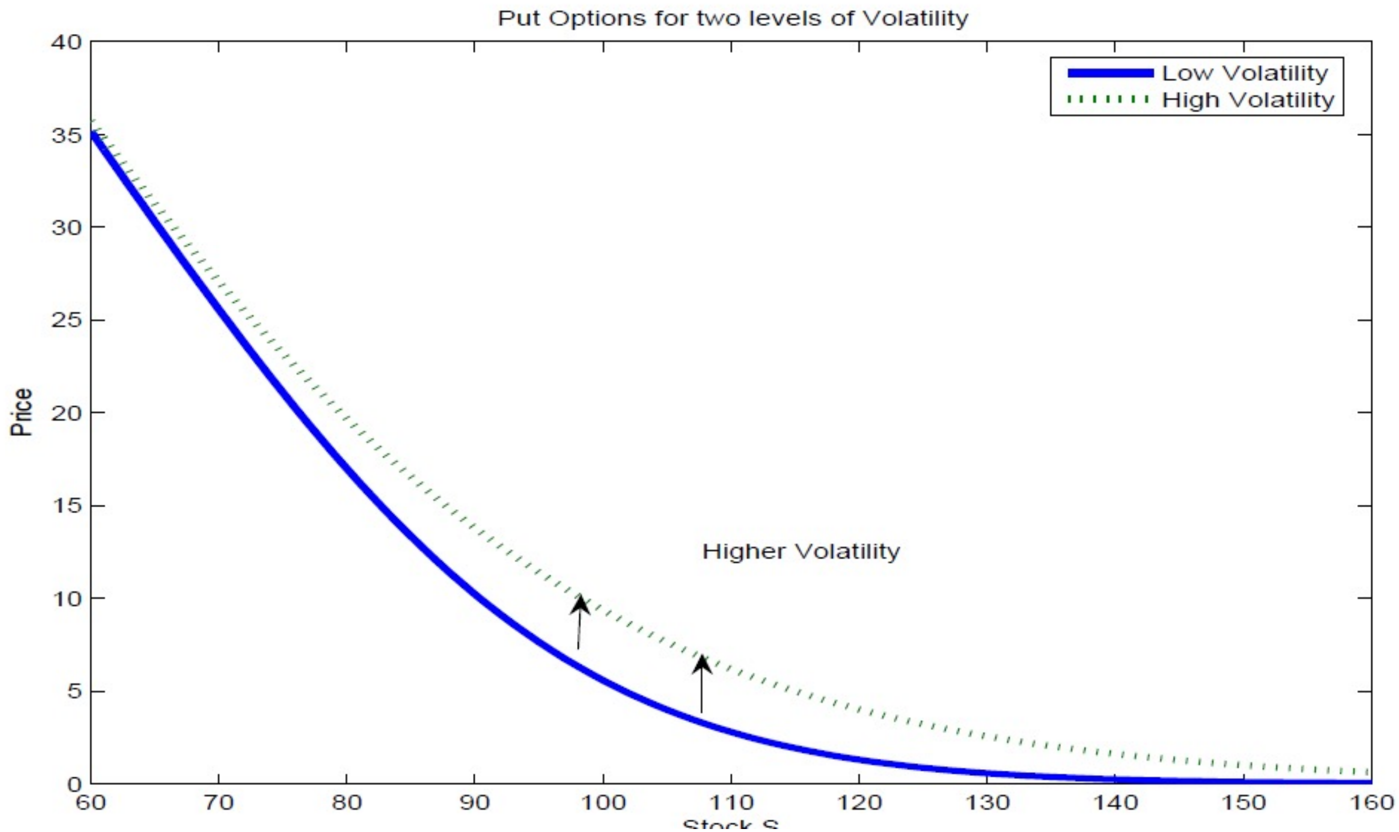
$$\rho = \frac{d \text{ Option Price}}{d\, r} = \begin{cases} KTe^{-rT}N(d_2) > 0 & \text{for Calls} \\ -KTe^{-rT}N(-d_2) < 0 & \text{for Puts} \end{cases}$$

Intuition: It depends on whether the option holder will pay *K* (call) or receive *K* (put). The PV of K declines as *r* increases, making the payment made smaller for the long call and payment received smaller for the long put.

Risk in options and the Greeks

5. **Vega**: Change in option price due to a change in volatility σ

$$\nu = \frac{d \text{ Option Price}}{d\,\sigma} = S\sqrt{T}N'(d_1) > 0$$



Both put and call options provide more protection and more potential upside when volatility increases.

Using options for financial engineering

- Investors have a taste for securities where their principal is protected but they share in the upside if the market does well
- Investment banks often offer such securities, and hedge the short position with options or dynamic trading strategies
  - See examples of MITTS and SUN securities with class materials
- The example we will study is a “Capital Protected Note” loosely based on a security issued in 2008 by Morgan Stanley

Using options for financial engineering: Example

It's Feb 22, 2008 and you sold a Capital Protected Note with:

- Maturity February 20, 2015
- Issue price \$10
- Principal \$10
- Interest 0%
- Principal protection 100%
- Payoff at maturity = principal + Supplemental Redemption Amount (SRA) if positive

$$SRA = \$10 \times 116\% \times \frac{\text{Final Index Value} - \text{Initial Index Value}}{\text{Initial Index Value}}$$

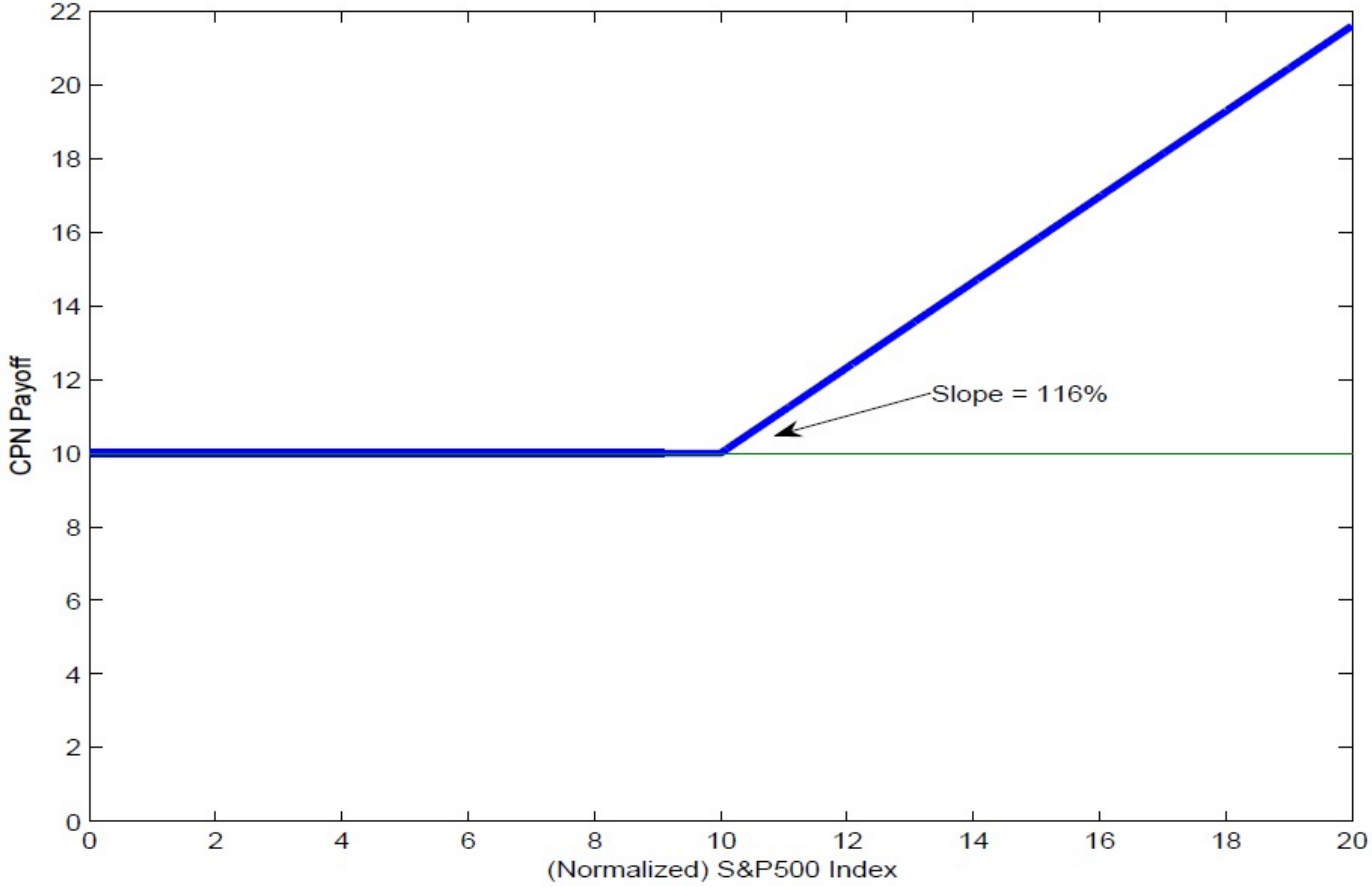
- Index is S&P 500 normalized to have Initial Index Value = \$10

You want to protect your short position against increases in the stock price index

How to do this? Delta-hedge by setting up a replicating portfolio and rebalancing it over time as market conditions change.

As the issuer of this security, you want to lock in the gain from selling the security

Payoff on Capital Protected Note



Payoff on Capital Protected Note

The payoff on the note can be decomposed into:

(1) A zero coupon bond with principal \$10 and maturity T = 7

(2) 1.16 at-the-money call options on the normalized S&P 500 with maturity T = 7

- The reference index is normalized so that S_0 = β x S&P 500 = \$10
- On 2/28/08, S&P 500 = 1353.1 => β = 10/1353.1

- Other data on 2/28/08
  - Interest rate r = 3.23% (continuously compounded)
  - Dividend yield on S&P 500 δ = 2%
  - Forecast of market volatility over the 7 years σ = 15%

The value of the security using BSM for dividend-paying stock is:

$$e^{-rT}\$10 + 1.16 \times Call(10, 10, r, \delta, \sigma, T)$$
$$= \$7.9764 + 1.16 \times \$1.7$$
$$= \$9.9483$$

- A little less than \$10 The bank makes \$0.05 per share that they issue.
- **Investors give up interest on principal in exchange for a call option**

Delta hedging at t=0

- You have a **short** position in the Capital Protected Note
- Hedge with an offsetting **long** position =>
  - Buy a zero coupon bond for $7.9764 to hedge the bond component
  - Buy 1.16 units of the replicating portfolio for the embedded call option

- **Setting up the replicating portfolio for each call:**
  - We can calculate the call $\Delta = N(d_1) = 0.5747$
  - Then the bond position = Call(10, 10, r, δ, σ, T) – $\Delta \times S_0$ = 1.7 - .05747 x 10 = -4.047
  - In sum, for each call option, invest 0.5747 x $ 10 = $5.747 in the S&P 500 and borrow $4.047
- **Value of replicating portfolio = $5.747 - $4.047 = 1.7** which is the same as the value of the call option.
- **Multiply both positions by 1.16 to scale up to the replicating portfolio for the Capital Protected Bond**

Dynamic delta hedging

Theoretically we need to frequently rebalance the portfolio as the Δ changes.

- It will change with the stock price
- It will also change the passage of time, and any changes in r and σ

Question: Which Greeks tells us about these?

- With regard to stock prices:
  - If S_t increases then Δ increases
  - If S_t decreases then Δ decreases
- Recalculate Δ and new value of call.
- Adjust holdings of stocks and bonds in replicating portfolio to match new option value

you can think of the $0.05 that the bank earns as compensation for doing this delta hedging on behalf of investors, who wouldn't be able to do it as inexpensively on their own.

Q: Why do dynamic replication here instead of initially just buying call options on the S&P 500 at the CBOE?

The answer is that call options on the S&P 500 are actively traded for shorter maturities, but there is unlikely to be adequate liquidity at longer maturities like seven years. When liquid options aren't available in the market, it can be cheaper, particularly for a large bank, to hedge using a dynamic delta hedging strategy.

How well does dynamic replication work in practice?

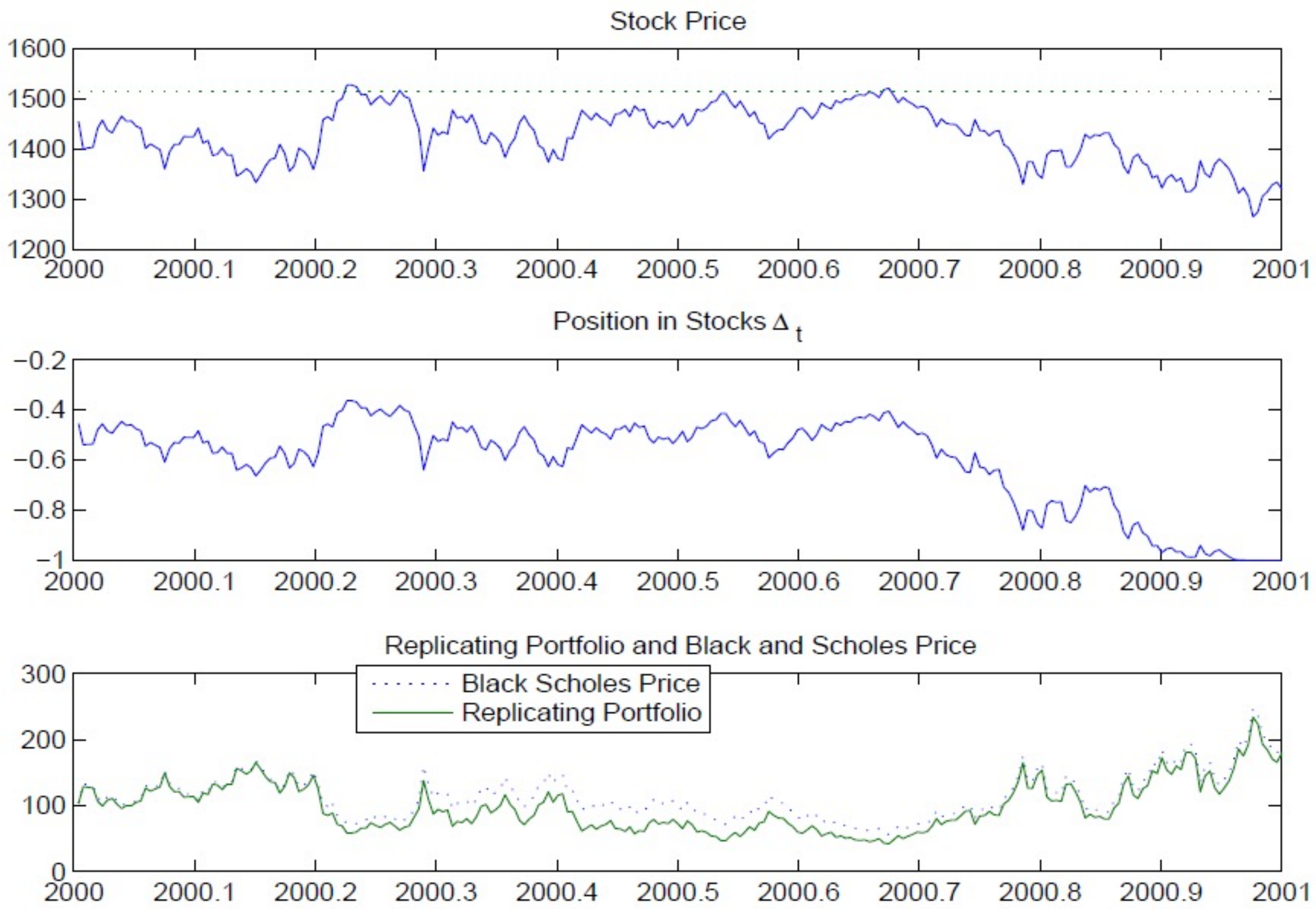
- Depends on a number of factors that include
  - Frequency of rebalancing
  - Stability of parameters
  - Accuracy of volatility estimate
  - Whether jumps in stock prices

- Well enough that may financial institutions are willing to offer products like Capital Protected Notes

How well does dynamic replication work in practice?

A put option example:

Let's look at Jan – Dec 2000. We replicate S&P500 option with $T = 1$ year. $\sigma =$ standard deviation of returns in 1999.



Portfolio Insurance

In 1981 UC-Berkeley Profs Hayne Leland and Mark Rubinstein joined forces with John O'Brien and formed Leland O'Brien and Rubinstein Associates, Incorporated (LOR).

<u>Business Idea</u>: Use the latest techniques from derivative security pricing and dynamic replication to offer investors insurance on their portfolios.

- For example, a pension plan fully invested in equity could purchase insurance to insure the portfolio against losses of value

LOR did not directly sell insurance. Rather they advised clients on the dynamic asset allocation that would insure against a drop in value of the portfolio.

Large potential demand from both mutual funds and pension plans

Business started slow but took off in 1984-86

By 1987 an estimated $100 billion in assets were covered by portfolio insurance products

- Total market cap of NYSE/Amex/Nasdaq of 2.2 trillion

Portfolio insurance

Various forms of portfolio insurance were developed.

- E.g. using out-of-the money put options where the investor could suffer a limited loss which is like a “deductible” that lowers the premium

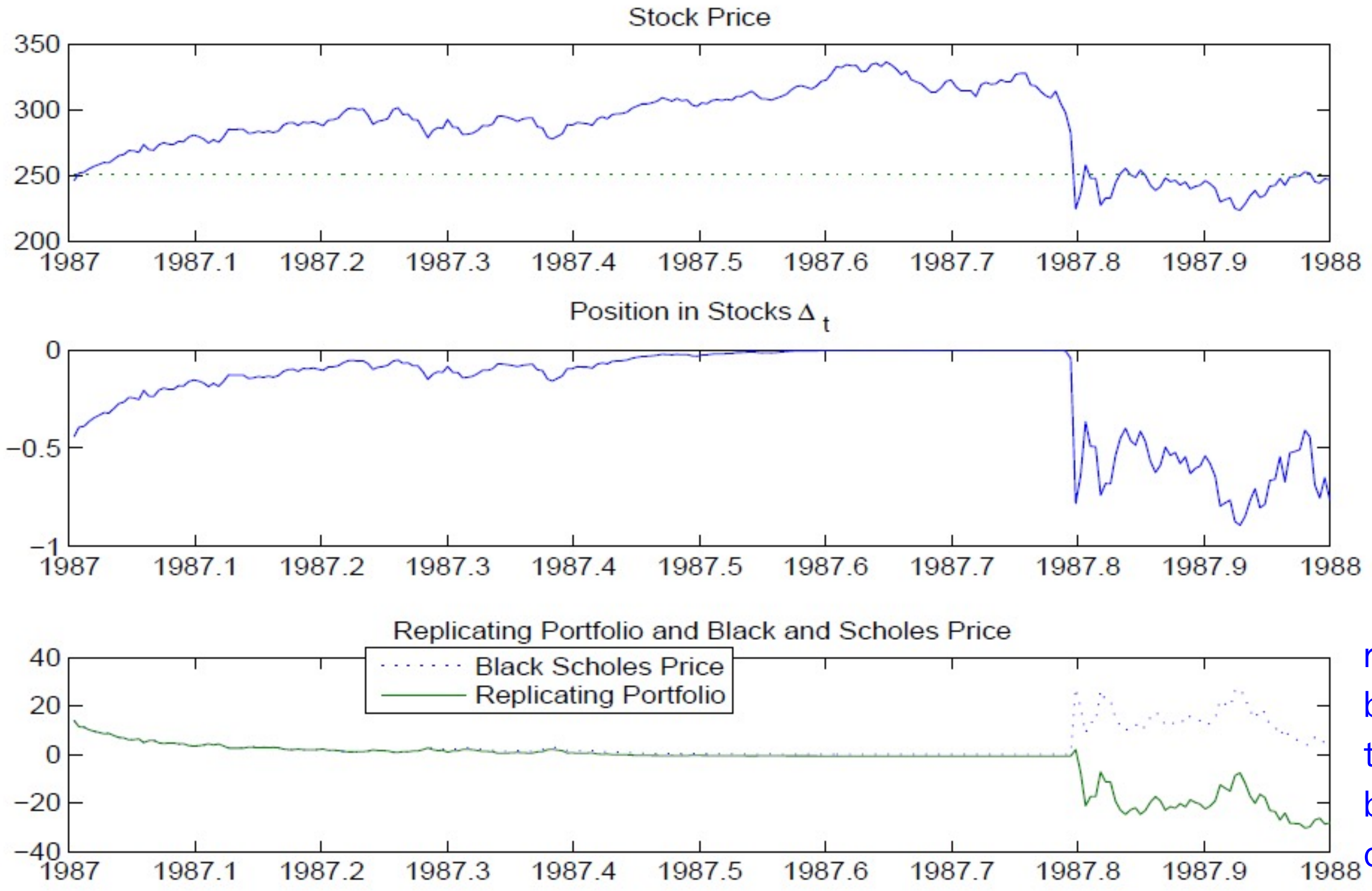
The most significant development in this period was **“perpetual” insurance**

- In a standard fixed term contract, the investor was covered only for some period of time (typically, 3 years)
- Most pension funds have long term liabilities, limiting the value of term insurance
- Perpetual insurance allowed the investor (insured) to decide to exercise its option at any time in the future

But then came the **1987 Crash ...**

How well does dynamic replication work in practice?

This example is for call options around the time of the 1987 market crash.



replicating portfolio becoming too small at the time of the crash, and hence, benefiting too little from the drop in stock prices.

failed to protect investors against losses

Conclusions on delta hedging

- This example illustrates in practice what we learned in theory: delta-hedging strategies only work well when stock price movement are fairly smooth
  - With large jumps in stock prices, it is impossible to rebalance fast enough to eliminate all risk
- Next week we will modify the hedging strategy to incorporate gamma hedging.
  - We'll see that as in fixed income strategies, incorporating this additional protection leads to a more robust hedge