15.415x Foundations of Modern Finance

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Lecture 11: Options, Part 1

Key concepts

- Introduction: option types
- Payoffs of European options
- Option strategies
- Corporate securities as options
- Pricing bounds for European options
- Put-Call parity for European options
- Early exercise of American options
- Binomial model: single period
- Binomial model: multiple periods

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Introduction to option types

- Option types:
  - Call: The right to buy an asset (the underlying asset) for a given price (strike price, or exercise price) on or before a given date (expiration date, or maturity date).
  - Put: The right to sell an asset for a given price on or before the expiration date.

- Exercise styles:
  - European: Owner can exercise the option only on expiration date.
  - American: Owner can exercise the option on or before expiration date.

Introduction to option types

- Key elements in defining an option:
  - Underlying asset and its price S,
  - Exercise price (strike price) K,
  - Expiration date (maturity date) T (today is 0),
  - European or American.

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Example: a European call option

- A European call option on IBM with exercise price $100.
- It gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at $100 on the expiration date.
- The option's payoff depends on the share price of IBM on the expiration date.

European options on stocks are typically settled by physical delivery: the buyer of the option receives shares of the stock if the option is exercised. We convert the value of this transaction into dollars and think about the payoff of the option at expiration as the corresponding cash flow.

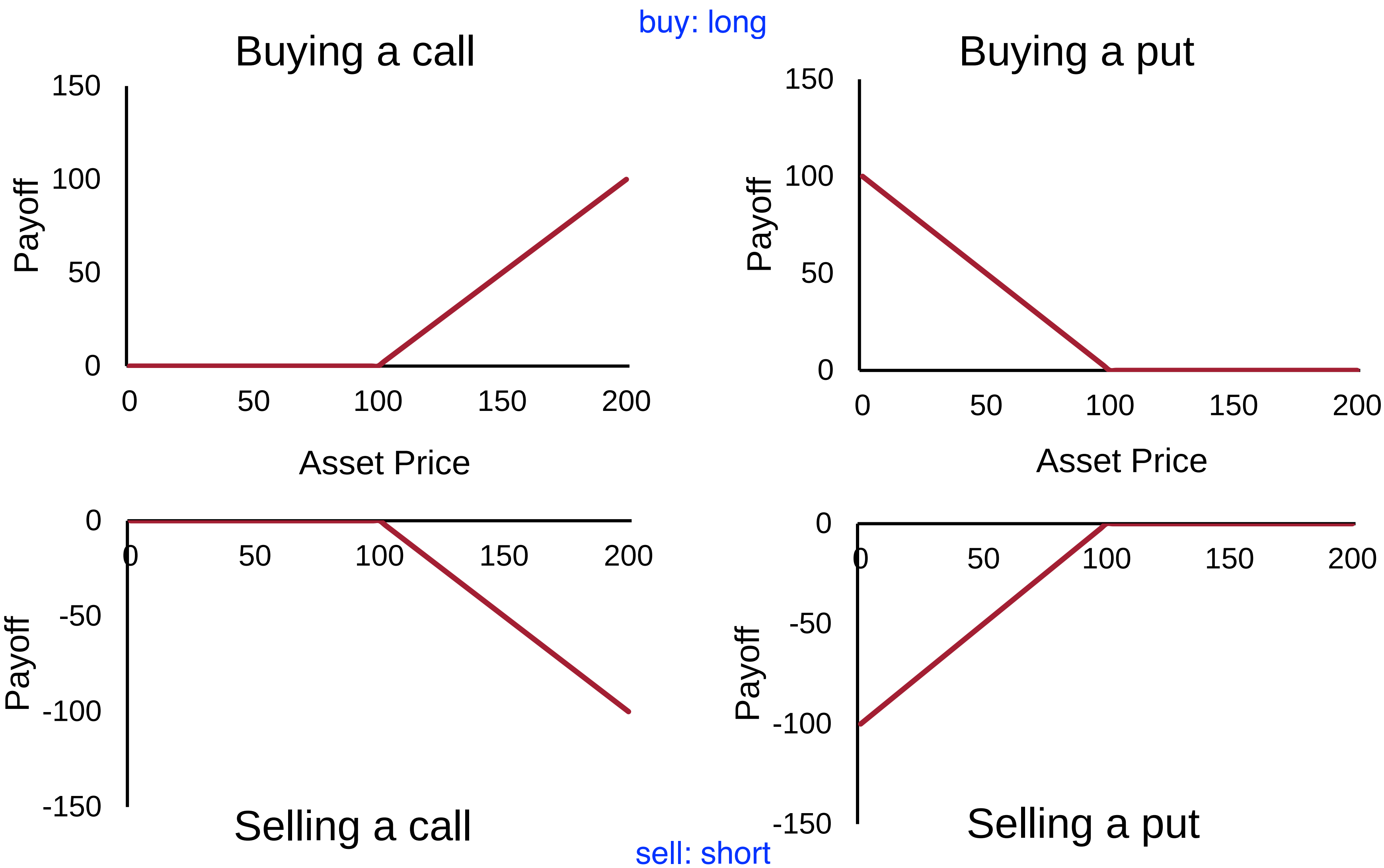
Example: a European call option

IBM Price at T	Action	Payoff
< 80	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
S_T	Exercise	$S_T - 100$

- Observations:
  - The payoff of an option is never negative; sometimes, it is positive.
  - Actual payoff depends on the price of the underlying asset:

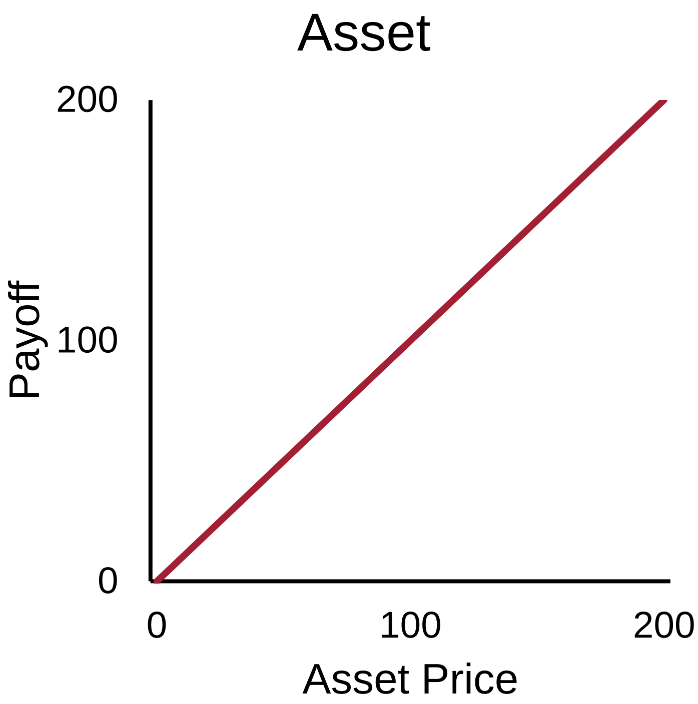
$$CF_T(\text{call}) = \max[0, S_T - K]$$

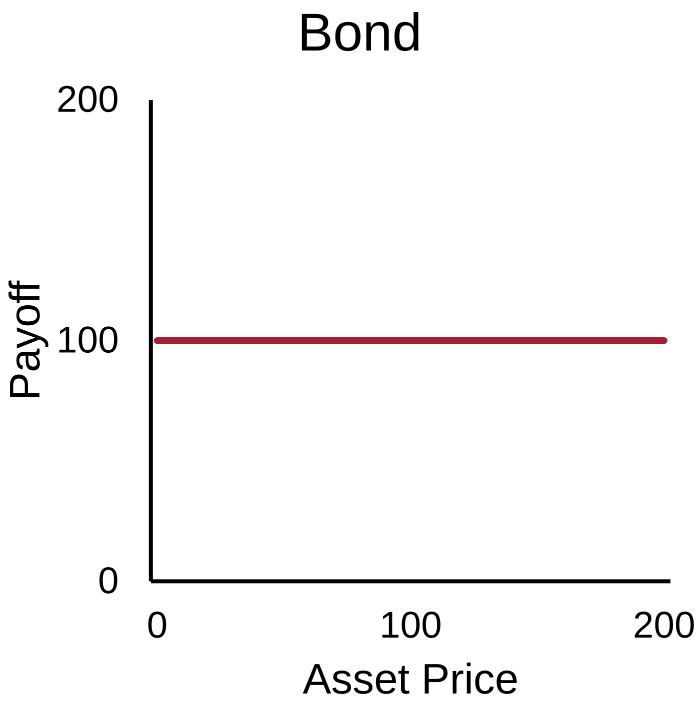
Payoff diagrams (strike price = 100)



Payoff diagrams

- The underlying asset and the bond (with face value $100) have the following payoff diagrams





Net option payoff

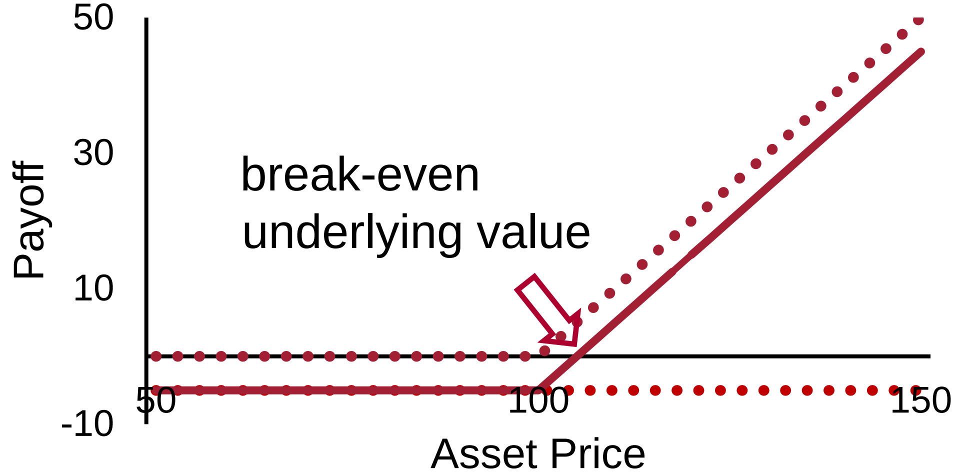
If the option expires unexercised, its net payoff is negative.

- The net payoff of an option must include its cost.
- Example: a European call on IBM shares with an exercise price of \$100 and maturity of three months is trading at \$5.
- The 3-month interest rate, not annualized, is 1%.
- At maturity, the call's net payoff is as follows.

At maturity, the call's net payoff is equal to its payoff minus the future value of the option price, which is \$5.05.

IBM Price	Action	Payoff	Net payoff
< 80	Not Exercise	0	−5.05
80	Not Exercise	0	−5.05
90	Not Exercise	0	−5.05
100	Not Exercise	0	−5.05
110	Exercise	10	4.95
120	Exercise	20	14.95
S_T	Exercise	$S_T - 100$	$S_T - 100$ −5.05

Net option payoff



The break-even point is given by S_T at which Net Payoff is zero:

$$\text{Net payoff} = \max[S_T - K, 0] - C(1+r)^T$$
$$= S_T - 100 - (5)(1+0.01)$$
$$= 0$$

or

The price of IBM must be above this threshold, above \$105.05, in order for the holder of the option to make a profit.

$$S_T = \$105.05$$

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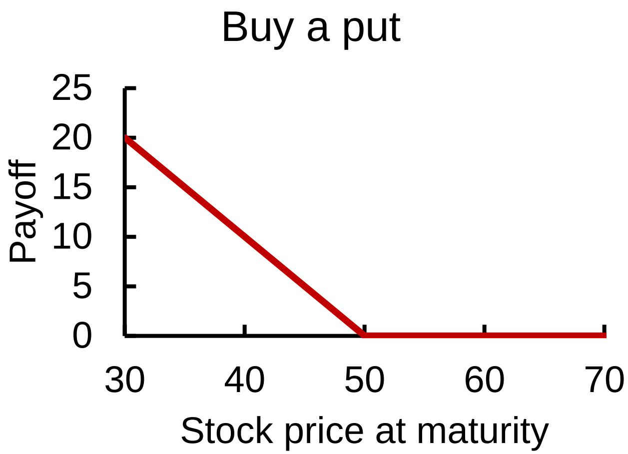
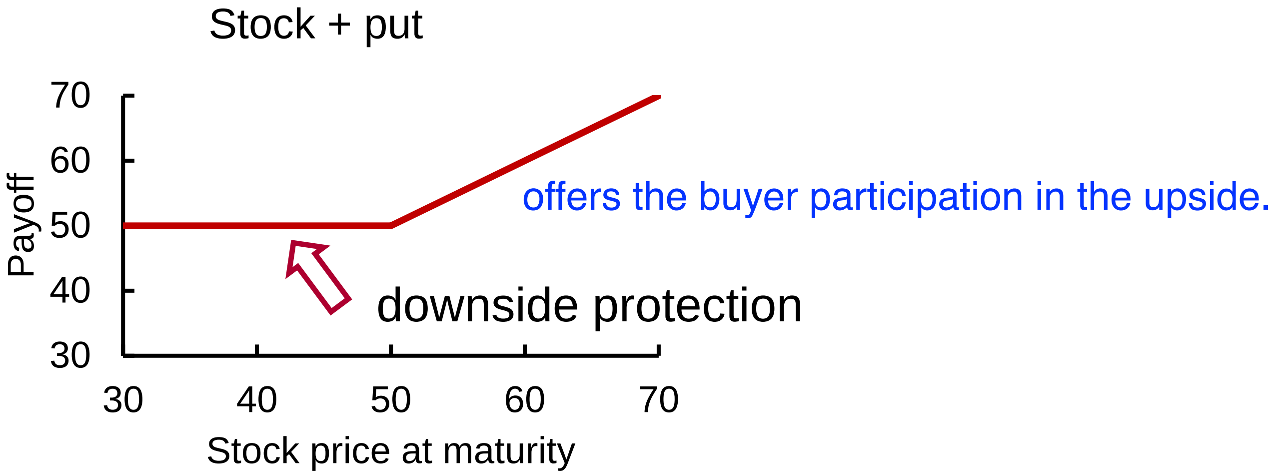
Option strategies: protective put

Keep in mind that what we are looking at are the payoffs, not the net payoffs. In other words, we are not taking into account the initial cost of setting up these positions

Buy the underlying stock, and buy a put with a strike price of $50:

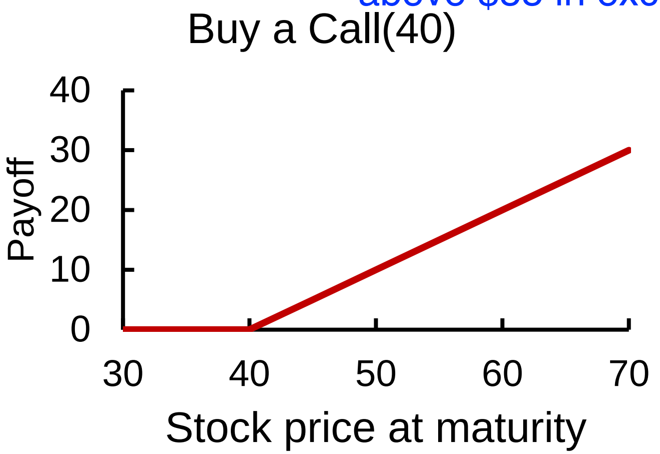


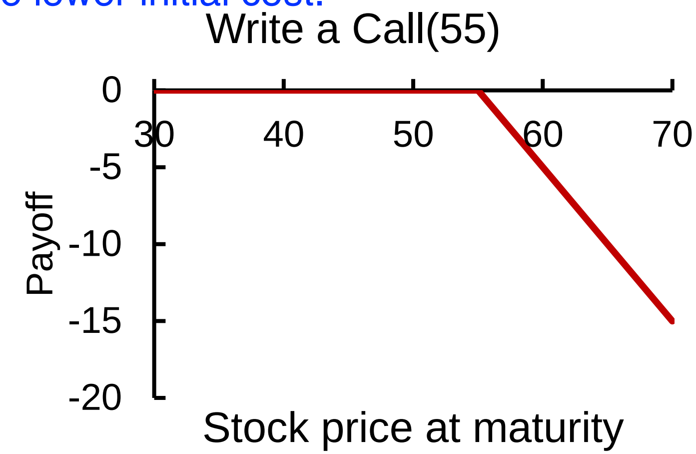


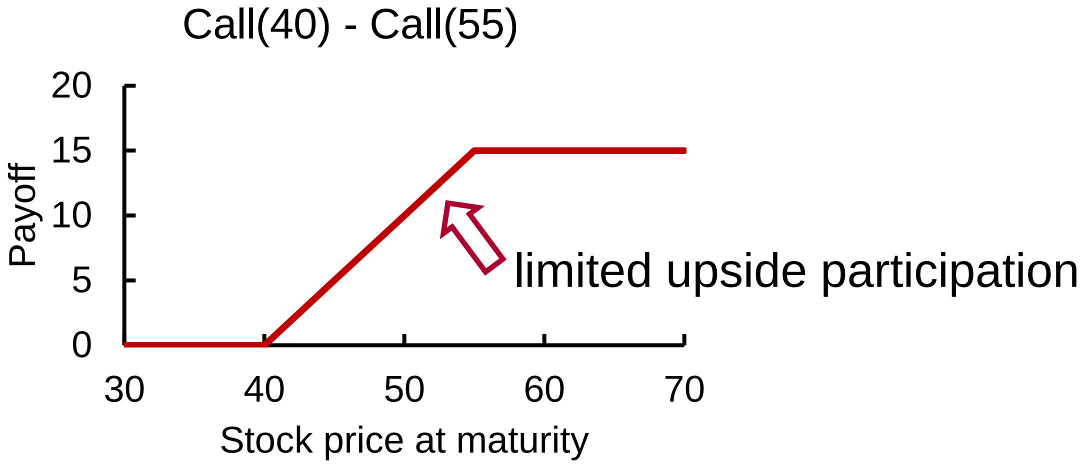
Option strategies: bull spread

Buy a call with a strike price of $45, and write a call with a strike price of $55:

This position sacrifices participation in further upside above $55 in exchange for the lower initial cost.

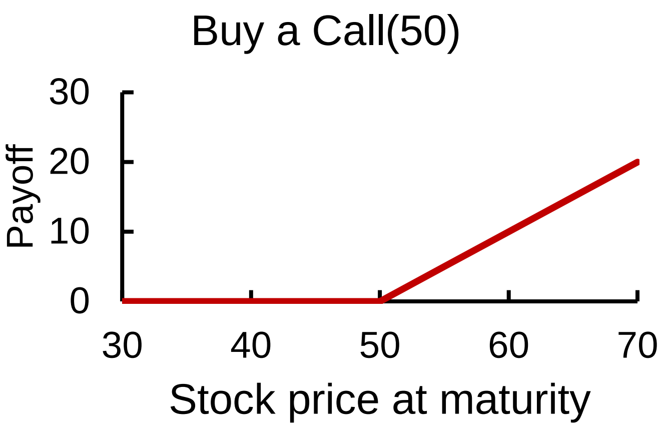


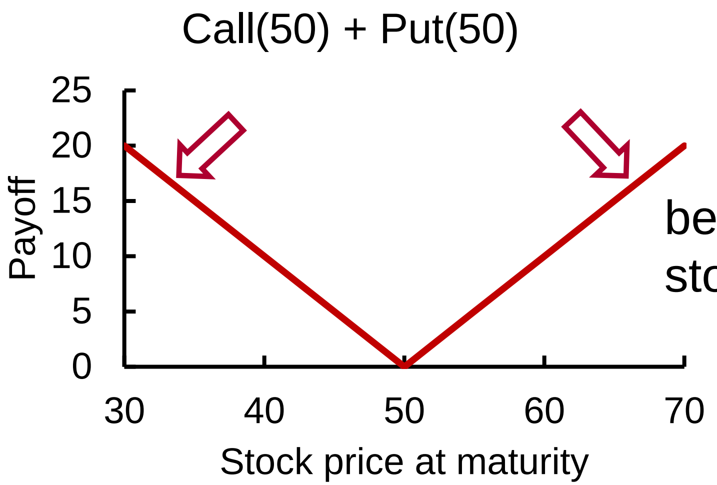




Option strategies: straddle

Buy a call with a strike price of $50, and buy a put with a strike price of $50:

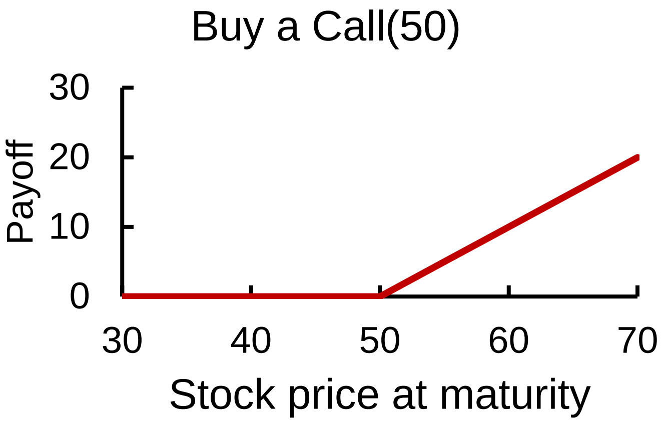


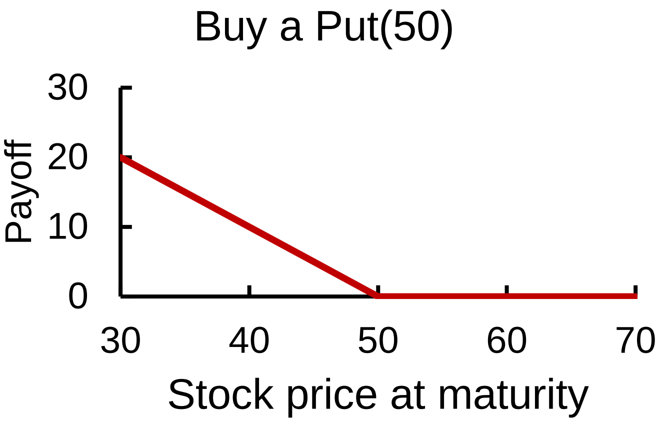


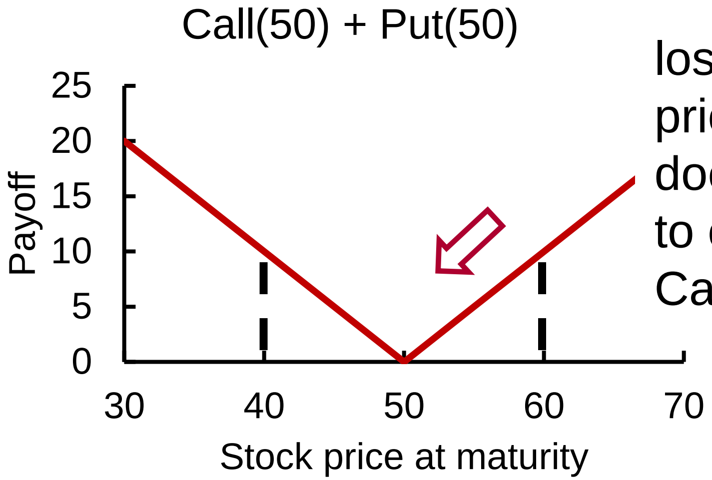
benefit from large stock price moves

Option strategies: straddle

Buy a call with a strike price of $50, and buy a put with a strike price of $50:







lose money if the stock price
does not move far enough to cover the initial cost of Call(50) + Put(50)

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Corporate securities as options

- Consider a firm with debt in its capital structure.

Balance sheet, market values			
Assets	$30	$25	Bond
		$5	Equity
Total	$30	$30	

- Firm bond has a face value of $50.
- Default is likely: if the firm's assets are worth less than $50 when the bond matures, the firm will be unable to afford its debt.
- In that event, the assets are turned over to the bondholders, and the equity is worth zero.

Corporate securities as options

- Consider the value of the stock, and a call on the underlying assets of the firm with an exercise price of $50:

Asset value	Value of the stock	Value of a call on the assets, strike = $50
⋮	⋮	⋮
$20	$0	$0
$40	$0	$0
$50	$0	$0
$60	$10	$10
$80	$30	$30
$100	$50	$50
⋮	⋮	⋮

- The stock gives the same payoff as a call option written on firm's assets.

- Equity is essentially a call option on the firm's assets, strike price equal to face value of debt.

Corporate securities as options

- Many corporate securities can be viewed as options.
- Equity (E): A call option on the firm's assets (A) with the exercise price equal to its bond's redemption value.
- Debt (D): A portfolio combining the firm's assets (A) and a short position in the call with the exercise price equal to its bond's face value (F):

Consider the payoff of the firm's debt. If the assets of the firm fall below the face value of debt at debt maturity (the firm defaults), the payoff of debt equals the value of the firm's assets. If the firm's assets are worth more than the debt value, then the payoff of debt equals its face value.

$$A = D + E \;\Rightarrow\; D = A - E$$

$$E \equiv \max[0, A - F]$$

$$D = A - E = A - \max[0, A - F]$$

There are more than one way to replicate debt with a portfolio:

D = A - E = A - max[0, A-F], so we can replicate debt with a portfolio combining A and a short call on A with exercise price = F.

At the same time, D = min[A, F] = F + min[0, A-F] = F - max[0, F-A], so we can also replicate debt with a portfolio combining a riskless bond and a short put on A with exercise price = F.

Corporate securities as options

new shares are issued

- Warrant: Call options on the firm's stock, with stock dilution as a result of exercise.

Convertible bonds are bonds that can be converted into equity at a prespecified conversion ratio.

- Convertible bond: A portfolio combining straight bonds and a call on the firm's stock with the exercise price related to the conversion ratio.

- Callable bond: A portfolio combining straight bonds and a short position in a call on these bonds.

A callable bond gives the issuer the right to buy the bond back at a specified price.

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Preliminaries

principal: higher payoff imply higher price

- For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.
- Notation:
  - S : Price of stock now;
  - S_T : Price of stock at T;

    paying $1 at maturity
  - B : Price of discount bond of par $1 and maturity T ($B \leq 1$);
  - C : Price of a European call with strike K and maturity T;
  - P : Price of a European put with strike K and maturity T.
- For our discussion:
  - Consider only European options, exercised only at maturity.
  - Assume the underlying stock pays no dividends.

Basic properties of options

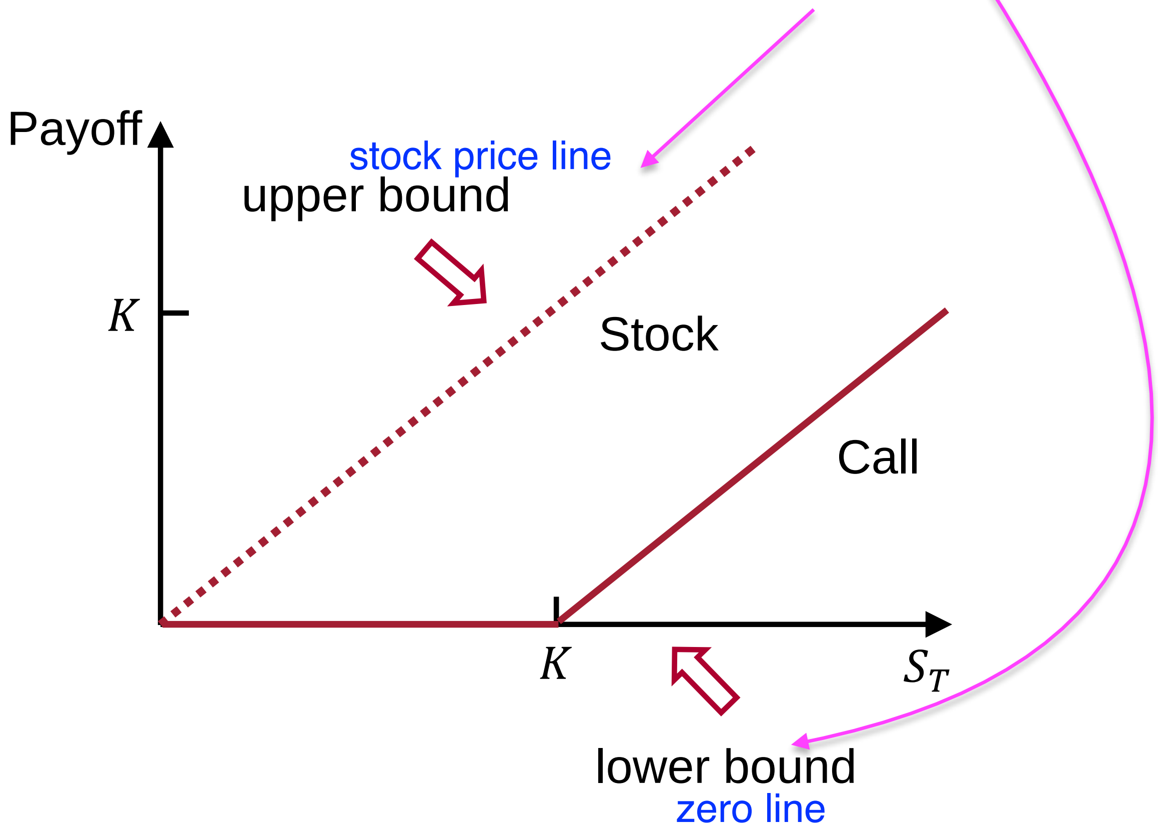
- If an option is exercised now, the resulting cash flow is called its exercise value.
  - For a call, its exercise value is $S - K$, where S is the current stock price;
  - For a put, its exercise value is $K - S$.

- An option is deemed to be:
  - In the money (ITM) if its exercise value is positive;
  - At the money (ATM) if its exercise value is zero;
  - Out of the money (OTM) if its exercise value is negative.

Price bounds

- European options on a non-dividend paying stock.

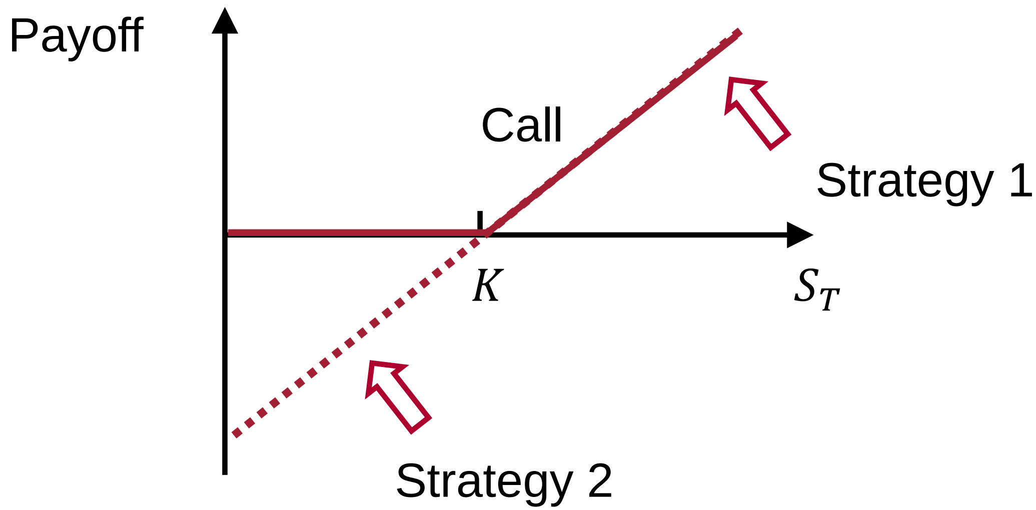
1. The payoff of call can never be negative: $C \geq 0$.
2. The payoff of stock dominates that of call: $C \leq S$.



Price bounds (cont'd)

3. Lower bound: $C \geq S - K B$

- Strategy 1: Buy a call;
- Strategy 2: ~~Buy a share of stock by borrowing K B.~~ buy a share of the stock and borrow the present value of the strike: S - KB payoff at maturity T: S_T - K
- The payoff of Strategy 1 dominates that of Strategy 2.



- Since $C \geq 0$, we have:

$$C \geq \max[0, S - KB]$$

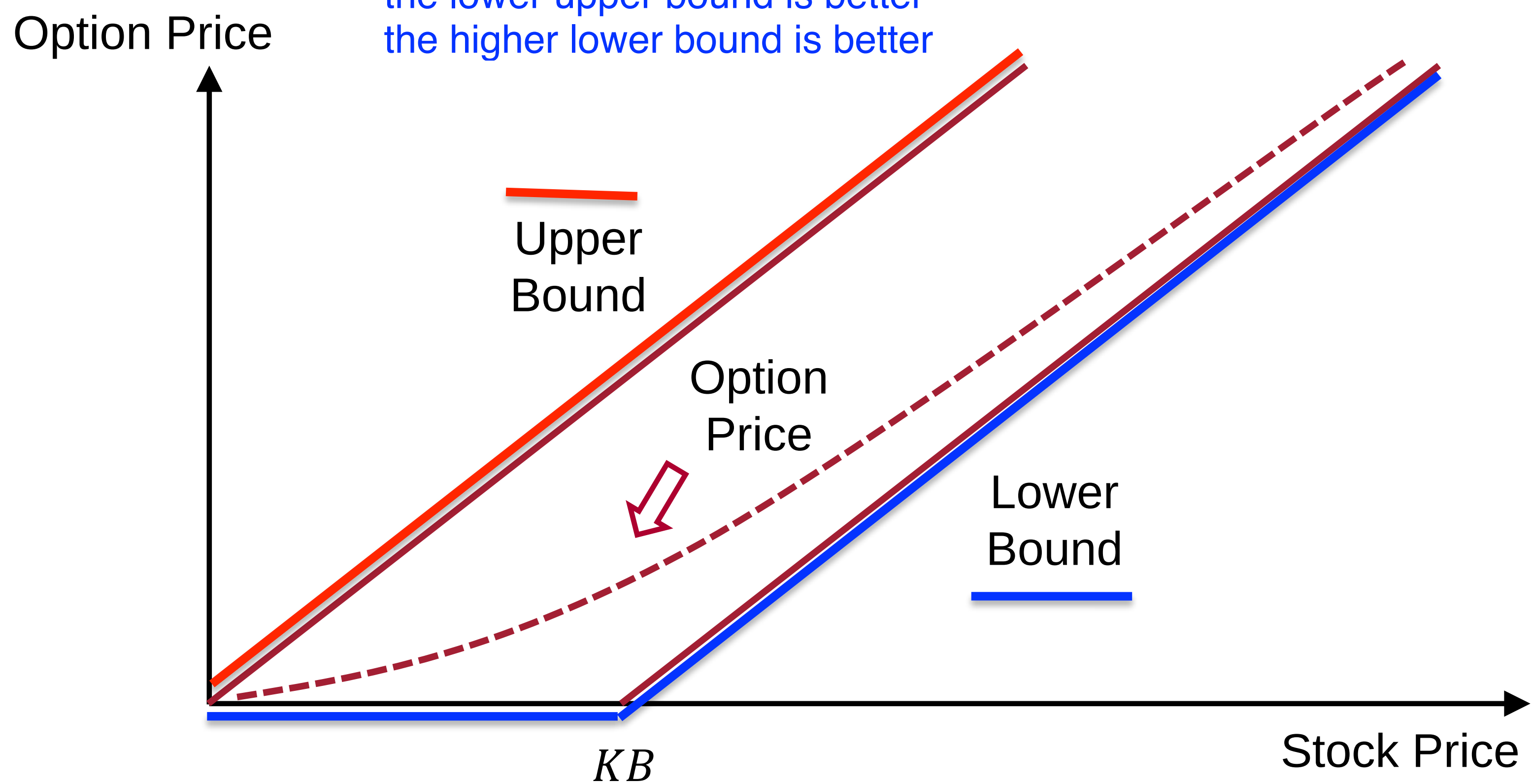
Price bounds (cont'd)

sometimes, you can not duplicate the final payoff of the asset exactly, you can only construct portfolios whose payoffs are the bounds of asset final payoff, then you can not deduce the PV of the asset, you can only deduce some bounds of PV

4. Combining the above, we have:

$$\max[S - KB, 0] \leq C \leq S$$

the lower upper bound is better
the higher lower bound is better



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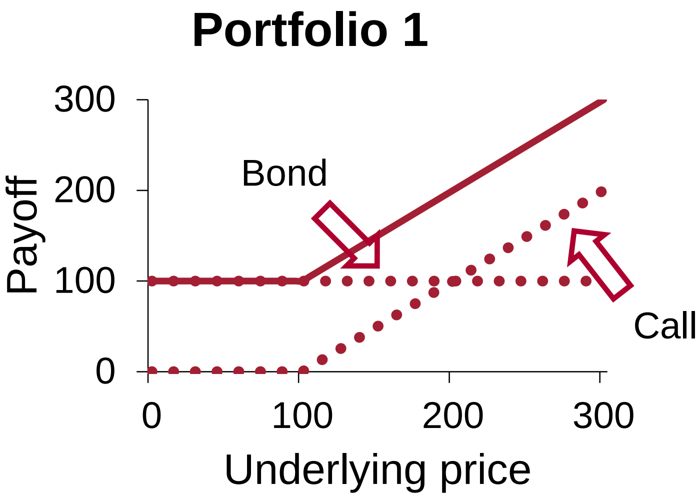
Normally, if the put call parity is violated, any rational investor will immediately make a set of trades that exploits it until the violation disappears (almost immediately). This may involve short selling a call option and a bond or a put option and a stock depending on which way the put call parity is violated.

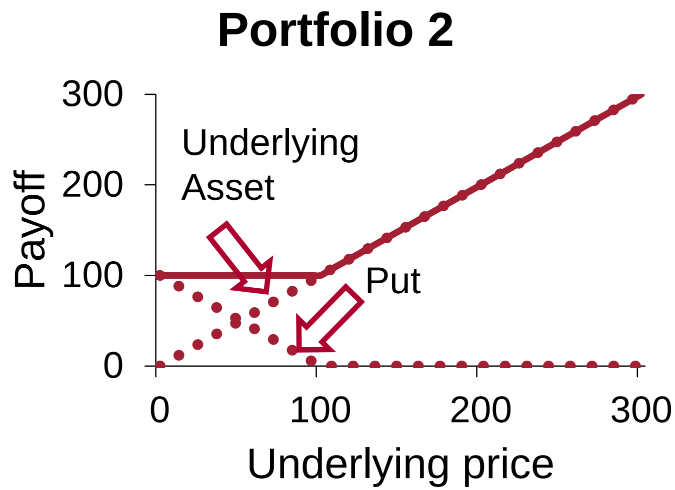
If there are short sale restrictions where you can not short the stock, then even if the put call parity is violated in that direction, it wont immediately disappear as investors wont be able to take advantage of this by short selling the stock. Thus, it wont immediately disappear, and the violation of the put call parity may continue to exist.

Put-call parity for European options

Portfolio 1: A call with strike K=$100 and a bond with par of $100;

Portfolio 2: A put with strike $100 and a share of the underlying asset.





$$C + BK = P + S$$

Their payoffs are identical, so must be their prices:

strike price should be equal

This is called the put-call parity.   infer put from call, or infer call from put

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American options, early exercise

- American options are worth more than their European counterparts.
- Without dividends and with positive interest rates ($B_t \leq 1$), never exercise an American call early.
- If exercise at $t < T$, collect $S_t - K$ payoff from immediate exercise of the American call
- If sell the option at t instead, collect at least the price of a European call, which is

sell the option at t $$\underbrace{C(S_t, K, T-t) \geq}_{\text{American > European}} \underbrace{\max[0, S_t - KB_t] \geq}_{\text{European lower bound}} \underbrace{S_t - K}_{\text{payoff of exercise}}$$

- Better to sell than exercise, thus early exercise is never optimal!
- By the law of one price:

$$c(S_t, K, T-t) = C(S_t, K, T-t)$$

If it is not optimal to exercise an American call before maturity, then its market value must be the same as the value of the corresponding European call.

American options, early exercise

- Without dividends, it can be optimal to exercise an American put early.
- Example. A put with a strike of $10 on a stock with price of zero.
  - Exercise now gives $10 today;
  - Exercise later gives $10 later. $10: receive no more than $10
- Better to exercise now (assuming positive interest rate).

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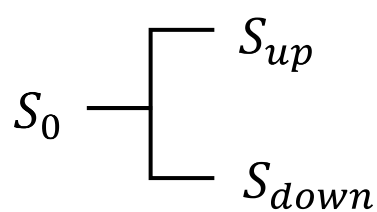
Option pricing models

- In order to have a complete option pricing model, we need to make additional assumptions about the price process of the underlying asset (stock).
- We assume that prices do not allow arbitrage.
- A benchmark model – price follows a binomial process.



Binomial option pricing model

0 ———————————————— 1

$S,\ B_T;\ \ C_0 = ?$ $\qquad\qquad$ $S_1,\ 1;\ C_1 = \text{Max}\ [S_1 - K,\ 0]$

- Determinants of option value:
  1. price of underlying asset S,
  2. strike price K,
  3. time to maturity T,
  4. interest rate r,
  5. volatility of underlying asset σ.

A European call on a stock

- Current stock price is \$50;
- There is one period to go;
- Stock price will either go up to \$75 or go down to \$25;
- There are no cash dividends;
- The strike price is \$50;
- One period borrowing and lending rate is 10%. risk free rate

A European call on a stock

- The stock and bond present two investment opportunities:

Stock	Bond
$50 \begin{cases} 75 \\ 25 \end{cases}$	$1 \begin{cases} 1.1 \\ 1.1 \end{cases}$

- The option's payoff at expiration is:

$$C_0 \begin{cases} 25 \\ 0 \end{cases}$$

- What is C_0, the value of the option today?

Replicating portfolio

and use the law of one price

- Form a portfolio of stock and bond that replicates the call's payoff:
  - a shares of the stock;
  - b dollars in the riskless bond.

such that:

$$75a + 1.1b = 25 \quad 1$$

$$25a + 1.1b = 0 \quad 2$$

- Unique solution: $a = 0.5$ and $b = -11.36$.

option's delta

note that option's delta is irrelevant to risk free rate,
which is canceled by equation 1 minus equation 2

Replicating portfolio

- Replication strategy:
  - borrow \$11.36
  - buy half a share of stock and sell \$11.36 worth of bond;
  - payoff of this portfolio is identical to that of the call;
  - market value of the call must equal the current cost of this "replicating portfolio" which is

$$(50)(0.5) - 11.36 = 13.64$$

- Number of shares needed to replicate one call option is called the option's hedge ratio or delta.
- In the above problem, the option's delta is $a = 0.5$.

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model free results are restrictions on option prices that must hold, regardless of the properties of the underlying asset.

with model

Binomial option pricing model

- Multiple periods:

$$
S = 50 \;\begin{cases} 75 & \begin{cases} 112.5 \\ 37.5 \end{cases} \\[2ex] 25 & \begin{cases} 37.5 \\ 12.5 \end{cases} \end{cases}
$$

- European call price process (strike price = 50):

$$
C_0 \;\begin{cases} C_u & \begin{cases} C_{uu} = 62.5 \\ C_{ud} = 0 \end{cases} \\[2ex] C_d & \begin{cases} C_{du} = 0 \\ C_{dd} = 0 \end{cases} \end{cases}
$$

- The terminal value of the call is known.

- C_u and C_d denote the option value next period when the stock price goes up and goes down, respectively.

- Compute the time-0value working backwards: first C_u and C_d and then C_0.

Period 1, "up" node

Start with Period 1:

- Suppose the stock price goes up to \$75 in period 1.
- Construct the replicating portfolio at node ($t = 1$, up):

$$112.5a + 1.1b = 62.5$$

$$37.5a + 1.1b = 0$$

- Unique solution: $a = 0.833, b = -28.4$.
- The cost of this portfolio: $(0.833)(75) - 28.4 = 34.075$.
- By Law of One Price, $C_u = 34.075$ – same as the initial cost of the replicating portfolio.

Period 1, "down" node

- Suppose the stock price goes down to \$25 in period 1. Repeat the above for node ($t = 1$, down):

$$112.5a + 1.1b = 0$$

$$37.5a + 1.1b = 0$$

- The replicating portfolio: $a = 0$, $b = 0$.
- The call value at the lower node next period is $C_d = 0$.

Period 0

- Now go back one period, to period 0:
- The option's value next period is either 34.075 or 0:

$$C_0 \left\{ \begin{array}{l} C_u = 34.075 \\ \\ C_d = 0 \end{array} \right.$$

- If we can construct a portfolio of the stock and bond to replicate the value of the option next period, then the cost of this replicating portfolio must equal the option's present value.

Period 0

$$C_0 \left\{ \begin{array}{l} C_u = 34.075 \\ \\ C_d = 0 \end{array} \right.$$

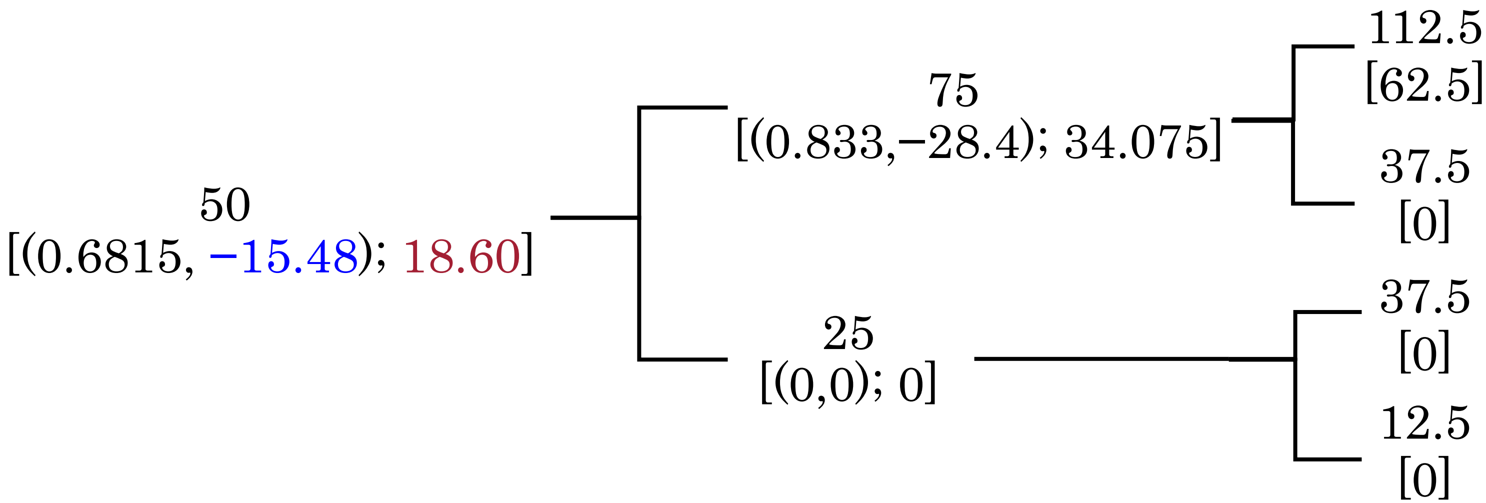
- Find a and b so that:

$$75a + 1.1b = 34.075$$

$$25a + 1.1b = 0$$

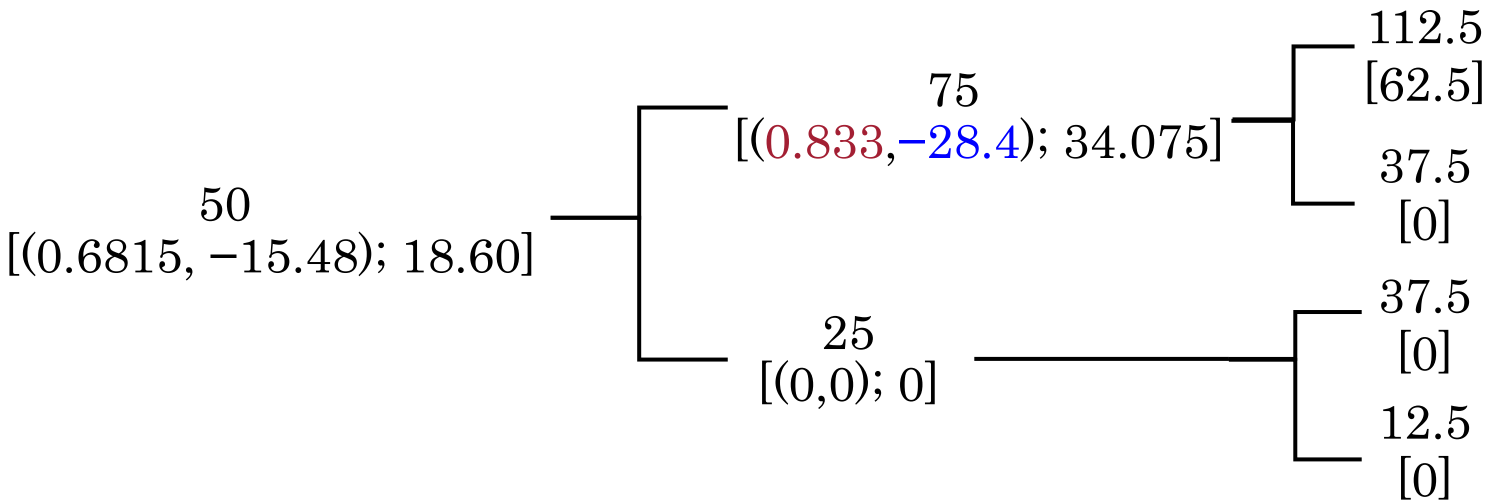
- Unique solution: $a = 0.6815, b = -15.48$.
- The cost of this portfolio: $(0.6815)(50)$ – $15.48 = 18.60$.
- The present value of the option must be $C_0 = 18.60$.

Option replication



- Period 0: Spend \$18.60 and borrow \$15.48 at 10% interest rate to buy 0.6815 shares of the stock.

Option replication



- Period 1, "up node": the portfolio value is 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%.
  - One period later, the payoff of this portfolio exactly matches that of the call: 62.5 for $S_{uu} = 112.5$ and 0 for $S_{ud} = 37.5$.
- Period 1, "down node": the portfolio becomes worthless. Close out the position.
  - The portfolio payoff one period later is zero.

Binomial option pricing model

- Bottom line: law of one price
  - Replication strategy gives payoffs identical to those of the call.
  - Initial cost of the replication strategy must equal the call price.

Binomial option pricing model

- What we have used to calculate option's value:
  - current stock price,
  - magnitude of possible future changes of stock price — volatility,
  - interest rate,
  - strike price,
  - time to maturity.

Binomial option pricing model

- What we have not used:
  - probabilities of upward and downward movements,
  - investor's attitude towards risk. risk tolerance
- Questions on the Binomial Model:
  - What is the length of a period?
  - Price can take more than two possible values.
  - Trading takes place continuously. The binomial model has discrete time steps

  All of these concerns can be addressed by shortening the time step in the model
- Response: The length of a period can be arbitrarily small.

As we reduce the time step in the binomial model, we will derive the celebrated Black-Scholes-Merton option pricing model.

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