15.415x Foundations of Modern Finance

Leonid Kogan and Jiang Wang
MIT Sloan School of Management

Lecture 3: Discounting and Compounding

Key Concepts

- Historical asset returns and discount rates
- Special cash flows
- Compounding
- Extensions

Key Concepts

- Historical asset returns and discount rates
- Special cash flows
- Compounding
- Extensions

Discount rate and asset return

- The present value of a given cash flow depends on the proper discount rate/rate of return expected by the financial market.
- The expected rate of return depends on the timing and risk of the cash flow.
  - We will study in detail the relationship between risk and return later on.
- As a starting point, we can use past returns on assets with different timing and risk characteristics to form rough estimates of expected rates of return on these assets as well as their risks.

Discount rate and asset return

- P_0 is the price of an asset at the beginning of period.
- P_1 is its price at the end of period – uncertain.
- D_1 is the dividend at the end of period – uncertain.
- Return (rate of return) from the asset over the period:

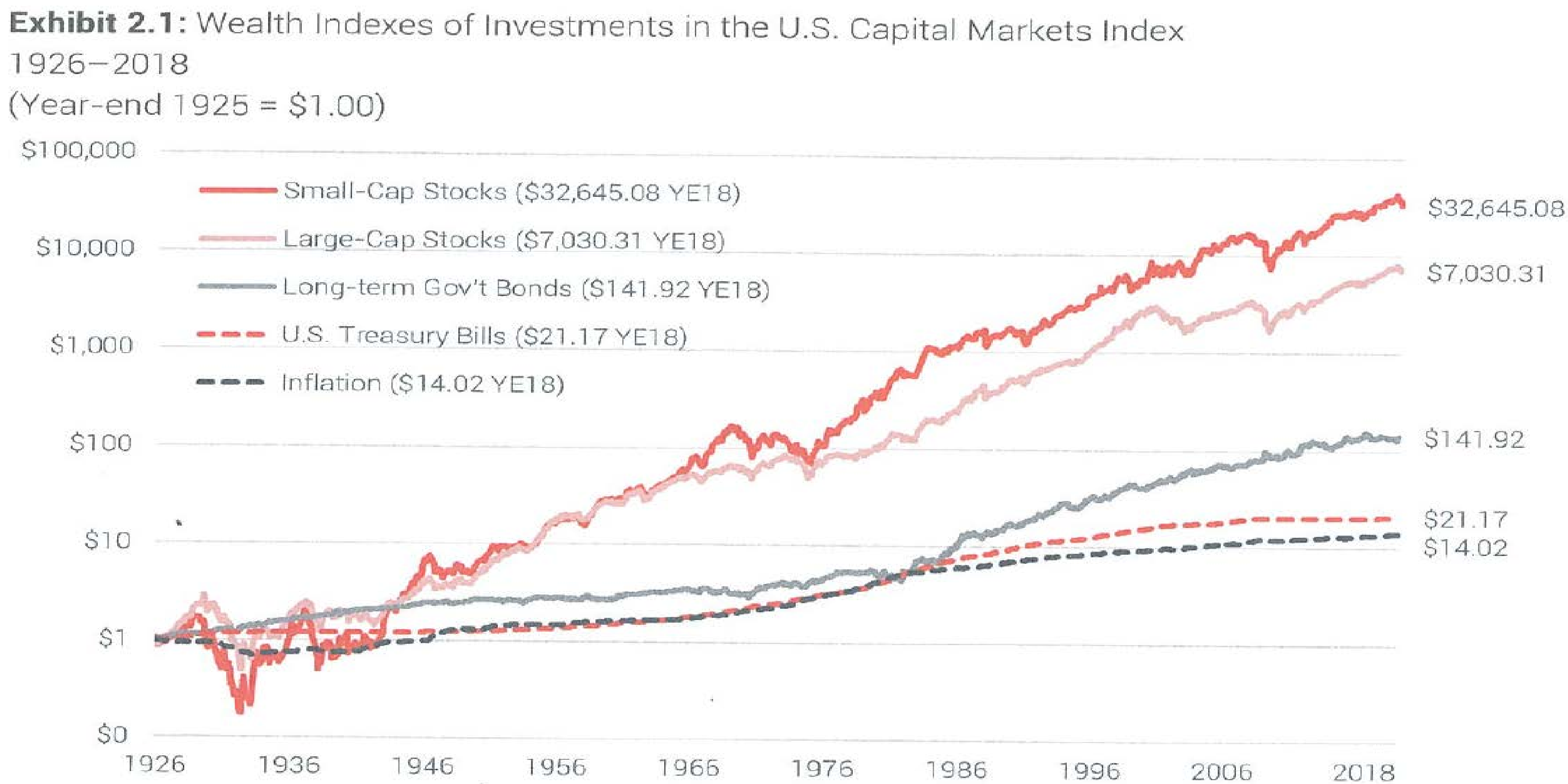
$$r = \frac{D_1 + P_1 - P_0}{P_0} = \frac{D_1 + P_1}{P_0} - 1.$$

- Expected return: $\bar{r} = E[r]$.
- For the riskless asset, its return r_F, the risk-free rate, reflects the time value of money.
- For a risky asset, its return net of the risk-free rate, $r - r_F$, is the excess return.
- The expected excess return, $\pi = \bar{r} - r_F$, gives the risk premium on the asset or the price of risk.

Historical return and risk

Return Indices of Investments in the U.S. Capital Markets (1926-2018)

(2019 SBBI Yearbook)



Historical return and risk

SD: standard deviation

Average Annual Total Returns from 1926 to 2018 (Nominal)

Asset	return Mean (%)	risk SD (%)	Premium (%)
T-bills	3.4	3.1: interest rates are moving 3.1	
Long term Treasury bonds	5.9	9.8	2.5
Long term corporate bonds	6.3	8.4	2.9
Large stocks	11.9	19.8	8.5
Small stocks	16.2	31.6	12.8
Inflation	3.0	4.0	

real return: minus inflation

Key Concepts

- Historical asset returns and discount rates
- Special cash flows
- Compounding
- Extensions

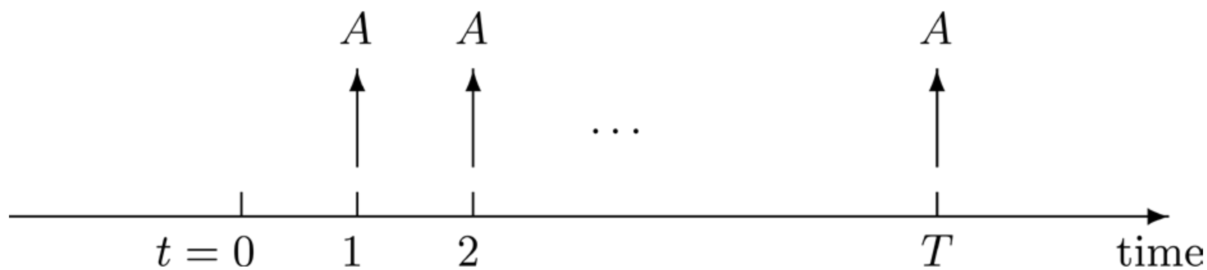
Special cash flows

We now consider several special cash flows:

- Annuity
- Annuity with constant growth
- Perpetuity
- Perpetuity with constant growth

Special cash flows

Annuity A constant cash flow for T periods (starting in period 1)



$$\mathrm{PV} = \frac{A}{1+r} + \frac{A}{(1+r)^2} + \cdots + \frac{A}{(1+r)^T}$$

$$(1+\mathrm{r}) \times \mathrm{PV} = A + \frac{A}{(1+r)} + \cdots + \frac{A}{(1+r)^{T-1}}$$

$$r \times \mathrm{PV} = A - \frac{A}{(1+r)^T}$$

$$\mathrm{PV}\ (\mathrm{Annuity}) = A \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

$$\mathrm{FV}\ (\mathrm{Annuity}) = (1+r)^T \times \mathrm{PV}\ (\mathrm{Annuity})$$

Special cash flows

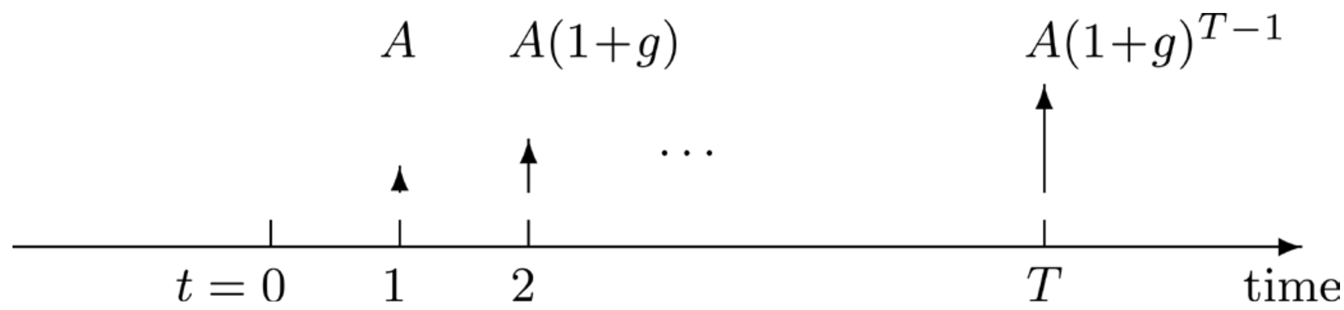
Example.

- An insurance company sells an annuity of $10,000 per year for 20 years.
- Suppose $r = 5\%$.
- What should the company sell it for?

$$\mathrm{PV} = 10{,}000 \times \frac{1}{0.05} \times \left(1 - \frac{1}{1.05^{20}}\right) = 10{,}000 \times 12.46$$
$$= 124{,}622.1$$

Special cash flows

Annuity with constant growth rate g



$$\text{PV (Annuity with growth)} = A \times \left[\frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \cdots + \frac{(1+g)^{T-1}}{(1+r)^T}\right]$$

$$= A \times \begin{cases} \dfrac{1}{r-g}\left[1 - \left(\dfrac{1+g}{1+r}\right)^T\right] & \text{if } r \neq g \\ \dfrac{T}{1+r} & \text{if } r = g \end{cases}$$

Special cash flows

Example. Saving for retirement.

- Suppose that you are now 30 and need \$2 million at age 65 for your retirement.
- At the end of each year, you can save an amount that grows by 5% each year.
- How much should you start saving now, assuming that $r = 8\%$?

$$\frac{A}{0.08 - 0.05}\left[1 - \left(\frac{1.05}{1.08}\right)^{35}\right] = \frac{2{,}000{,}000}{(1.08)^{35}}$$

$$\Rightarrow A = \frac{\frac{2{,}000{,}000}{(1.08)^{35}}}{20.898} = 6{,}472.96$$

Special cash flows

Example. You just won the lottery and it pays \$100,000 a year for 20 years. Are you a millionaire? Suppose that $r = 10\%$.

$$\mathrm{PV} = 100{,}000 \times \frac{1}{0.10}\left(1 - \frac{1}{1.10^{20}}\right) = 100{,}000 \times 8.514$$
$$= 851{,}356$$

- What if the payments last for 50 years?

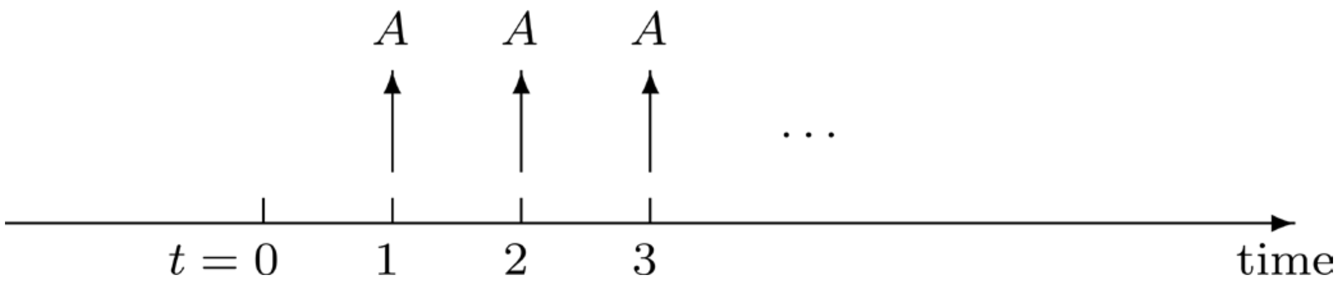
$$\mathrm{PV} = 100{,}000 \times \frac{1}{0.10}\left(1 - \frac{1}{1.10^{50}}\right) = 100{,}000 \times 9.915$$
$$= 991{,}481$$

- How about forever - a perpetuity?

$$\mathrm{PV} = \frac{100{,}000}{0.10} = 1{,}000{,}000$$

Special cash flows

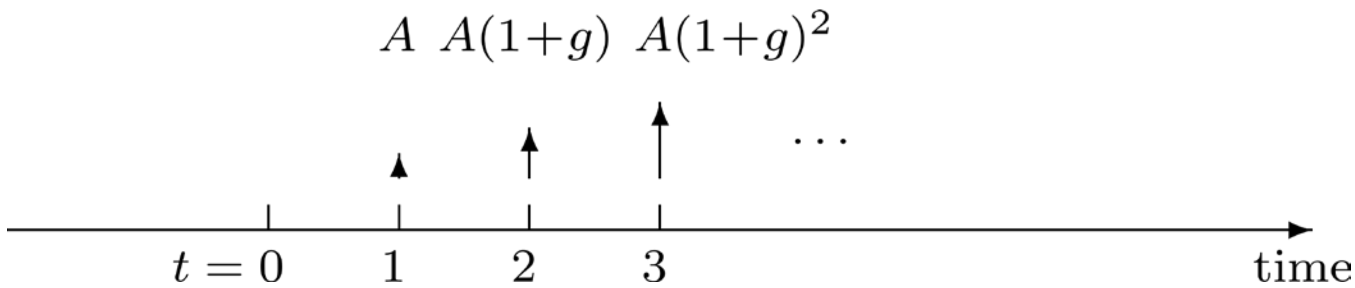
Perpetuity An annuity with infinite maturity



$$\text{PV (Perpetuity)} = \frac{A}{r}$$

Special cash flows

Perpetuity with constant grow g



$$\text{PV (Perpetuity with growth)} = \frac{A}{r-g}, \qquad r > g$$

Example. Super Growth Inc. will pay an annual dividend next year of \$3. The dividend is expected to grow 5% per year forever. For companies of this risk class, the expected return is 10%. What should be Super Growth's price per share?

discount rate

$$\mathrm{PV} = \frac{3}{1.10} + \frac{3(1+0.05)}{1.10^2} + \frac{3(1+0.05)^2}{1.10^3} + \cdots = \frac{3}{0.10-0.05} = 60$$

Key Concepts

- Historical asset returns and discount rates
- Special cash flows
- Compounding
- Extensions

Compounding

Interest may be credited/charged more often than annually.

- Bank accounts and credit cards: daily
- Loans and leases: monthly
- Bonds: semi-annually ...

For the same quoted interest rate, the effective annual rate may differ, depending on how frequent interests are received/paid.

Compounding

Typical quote convention:

- Annual Percentage Rate (APR),
- k periods of compounding,
- Interest per period is APR/k,
- Actual annual rate differs from APR.

10% APR Compounded Annually, Semi-Annually, Quarterly, and Monthly

Month	$1,000	$1,000	$1,000	$1,000
1				$1,008
2				$1,017
3			$1,025	$1,025
4				$1,034
5				$1,042
6		$1,050	$1,051	$1,051
7				$1,060
8				$1,069
9			$1,077	$1,078
10				$1,087
11				$1,096
12	$1,100	$1,103	$1,104	$1,105

Compounding

Example. Bank of America's one-year CD offers 5% APR, with semi-annual compounding. If you invest \$10,000, how much money do you have at the end of one year? What is the actual annual rate of interest you earn?

- Quoted APR of 5% is not the actual annual rate.
- It is only used to compute the 6-month interest rate:

$$(5\%)(\frac{1}{2}) = 2.5\%$$

- Investing \$10,000, at the end of one year you have:

$$\$10{,}000(1 + 0.025)(1 + 0.025) = \$10{,}506.25$$

- In the second 6-month period, you earn interest on interest.
- The actual annual rate, the Effective Annual Rate (EAR), is:

$$r_{\text{EAR}} = (1 + 0.025)^2 - 1 = 5.0625\%$$

APR

- Let r_{APR} be the APR and k be the number of compounding intervals per year. In one year, one dollar invested today yields:

$$\left(1+\frac{r_{APR}}{k}\right)^{k}$$

- Effective annual rate, r_{EAR} is given by:

$$(1+r_{EAR})=\left(1+\frac{r_{APR}}{k}\right)^{k} \quad \Leftrightarrow \quad \boxed{r_{EAR}=\left(1+\frac{r_{APR}}{k}\right)^{k}-1}$$

APR & EAR

Example. Suppose $r_{APR} = 5\%$. Here, $e \approx 2.71828$

k	Value of \$1 in a year	r_{EAR}
1	1.050000	5.0000%
2	1.050625	5.0625%
12	1.051162	5.1162%
365	1.051268	5.1267%
8,760	1.051271	5.1271%
$\vdots$	$\vdots$	$\vdots$
∞	$e^{0.05} = 1.051271$	5.1271%

Mortgage example

Example. Fixed rate mortgage calculation in the U.S.

- 20% down payment, and borrow the rest from bank using property as collateral;
- Pay a fixed monthly payment for the life of the mortgage;
- Have the option to prepay. the borrower can borrow money from the market at a lower rate and then repay the previous mortgage

Suppose that you bought a house for $500,000 with $100,000 down payment and financed the rest with a thirty-year fixed rate mortgage at 8.5% APR compounded monthly.

- The monthly payment M is determined by:

$$400{,}000 = \sum_{t=1}^{360} \frac{M}{[1 + (0.085/12)]^t}$$

$$= \frac{M}{(0.085/12)}\left(1 - 1/[1 + (0.085/12)]^{360}\right) = \frac{M}{(0.085/12)}(0.9212)$$

$$\Rightarrow \quad M = 3{,}075.65$$

- Effective annual interest rate (EAR):

$$[1 + 0.085/12]^{12} - 1 = 1.08839 - 1 = 8.839\%$$

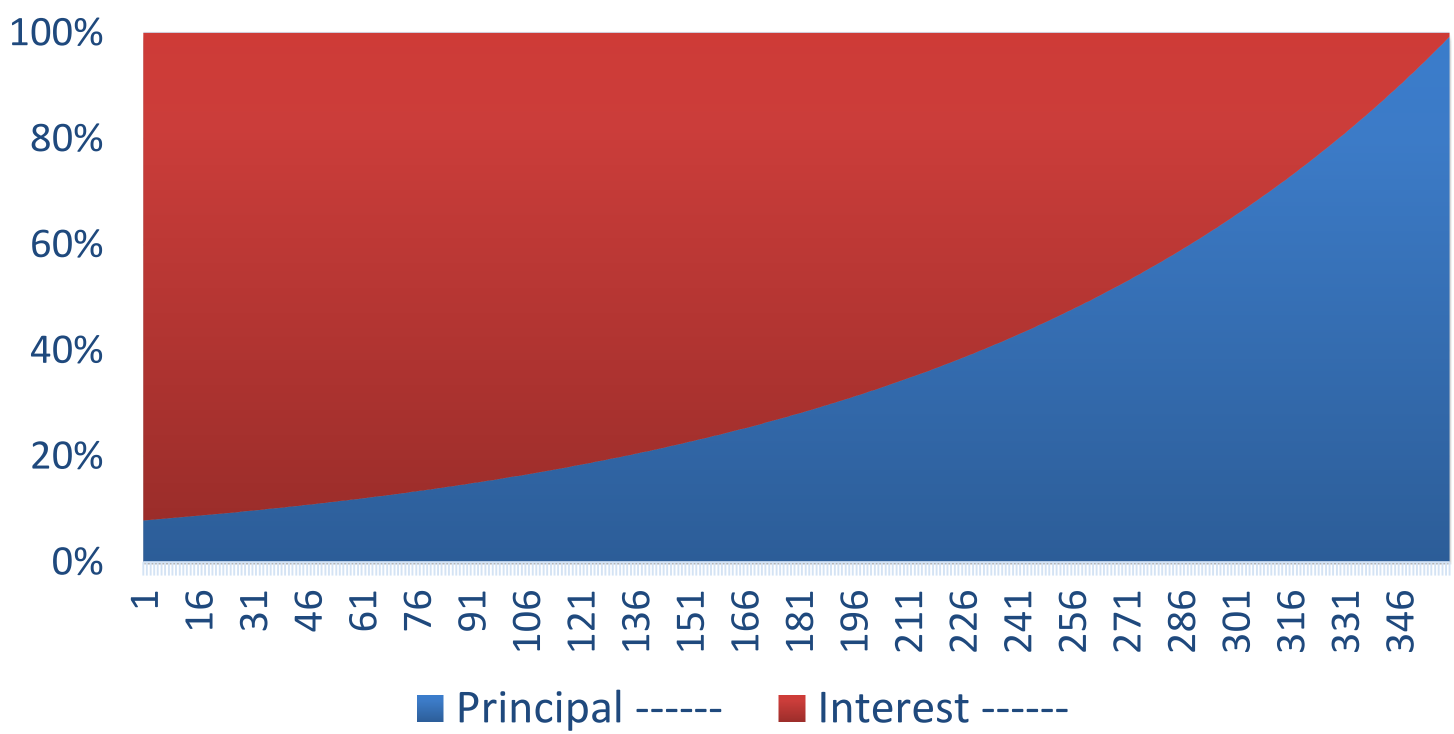
Mortgage payments

- Monthly payments

t (month)	Payment	Interest	Principal	Remaining Principal
1	3,075.65	2,833.33	242.32	399,757.68
2	3,075.65	2,831.62 =399757 *(0.085/12)	244.04	399757-244= 399,513.64
3	3,075.65	2,829.89	245.77	399,267.88
⋮	⋮	⋮	⋮	⋮
120	3,075.65	2,514.38	561.27	354,410.19
121	3,075.65	2,510.41	565.25	353,850.23
⋮	⋮	⋮	⋮	⋮
240	3,075.65	1,766.40	1,309.25	248,065.24
241	3,075.65	1,757.13	1,318.53	246,746.71
⋮	⋮	⋮	⋮	⋮
359	3,075.65	43.11	3,032.54	3,054.02
360	3,075.65	21.63	3,054.02	0.00

Mortgage payments

- Total monthly payment is the same for each month;
- The percentage of principal payment increases over time;
- The percentage of interest payment decreases over time.



Key Concepts

- Historical asset returns and discount rates
- Special cash flows
- Compounding
- Extensions

Extensions

- Taxes
- Currencies
- Forecasting cash flows
- Choosing the right discount rate for each cash flow:
  - Risk-free interest rate (time value)
  - Risk adjustment (risk premium)

$$\mathrm{PV}(CF_1, CF_2, \ldots, CF_T) = \frac{E[CF_1]}{(1+r_1)} + \frac{E[CF_2]}{(1+r_2)^2} + \cdots + \frac{E[CF_T]}{(1+r_T)^T}$$

discount rate at year 1 r1 may be very different from discount rate at year T

Summary

- Historical asset returns and discount rates
- Special cash flows
- Compounding
- Extensions

