15.415x Foundations of Modern Finance

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Lecture 6: Risk and Return

Key Concepts

- Introduction
- Expected Utility
- Risk Aversion
- Mean-Variance Preferences
- Asset Returns
- Other Dimensions of Risk
- Joint Distributions of Returns
- Portfolios
- Portfolios: Risk and Return
- Systematic and Idiosyncratic Risk

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Measurement and management of risk

- Measurement and management of risk is at the core of finance:
- An investor saving for retirement: riskier strategy with more upside vs safer one, with less downside?
    - A hedge fund: how much capital to allocate to various trading strategies?
    - An insurance company: how to manage payout risk?
    - A sovereign wealth fund: how to structure financial investments given the composition of the country's economy?

Fundamental concepts and tools

- Need a systematic framework to making decisions under uncertainty – Expected Utility Theory.
- Develop analytical tools for quantifying risk, and for dealing with portfolios of investments – portfolio analytics.
- Key concept: Diversification -- the only “free lunch” in financial markets!

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Decisions under uncertainty

- Decisions under uncertainty boil down are choices among random payoffs:

$$x = \$1{,}000 \text{ or } \$0, \qquad 50/50 \text{ odds}$$

$$y = \$600 \text{ or } \$200, \qquad 70/30 \text{ odds}$$

- How should we model such choices?
  - Naïve approach: compute expected payoff, choose the higher one.
  - What about randomness? $50/50$ gamble $\$1{,}000$ / $\$0$ vs $\$500$ for sure?

"Rational" vs "Behavioral" approaches

- Two approaches: "rational" and "behavioral."
- Rational approach is prescriptive: a model of choice with internal consistency and basic desired properties.
- Behavioral approach is descriptive: empirically-motivated model of observed individual behavior, captures behavioral biases and inconsistencies.
- We want to make investment decisions consistently.
- Focus on the rational model.

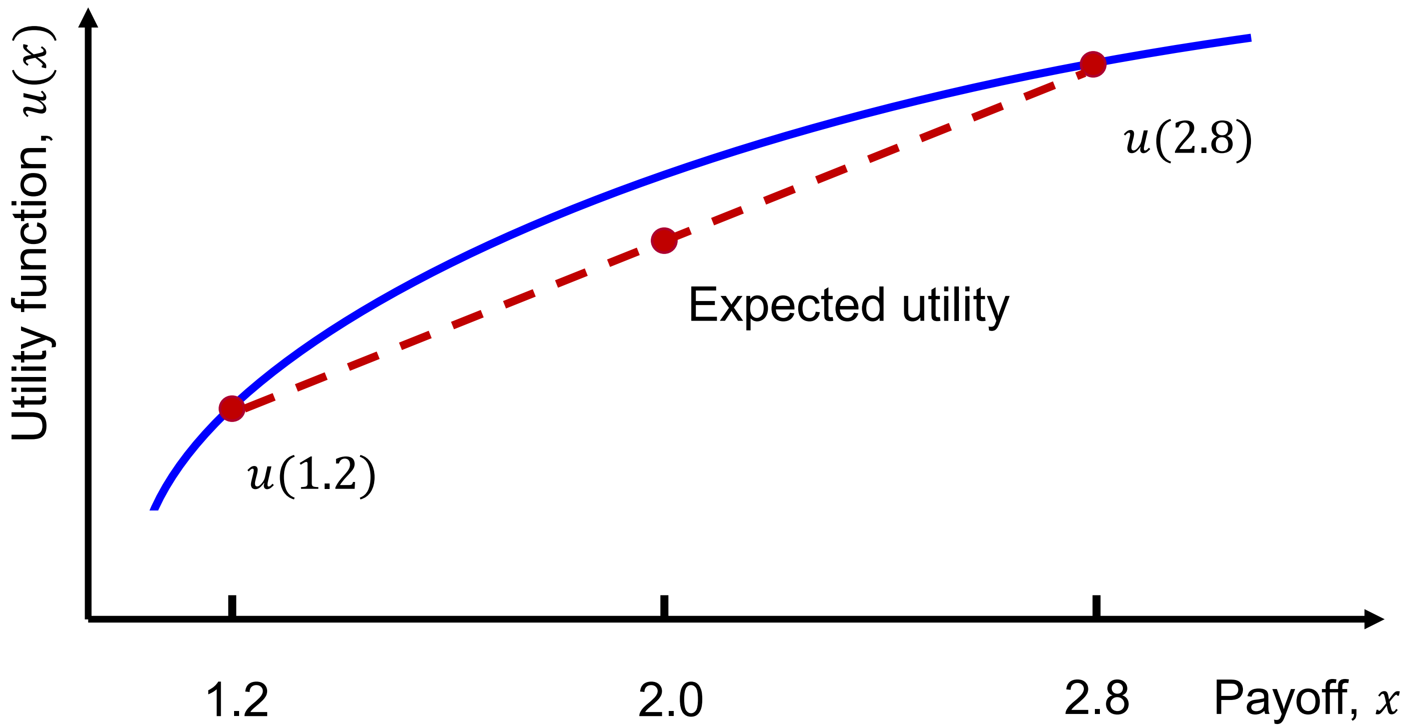
Basic assumptions

- Preferences are over outcomes only (cash flows).
- Our model abstracts from the mechanism by which cash flows are generated.

Expected utility theory

- Expected utility theory is the leading model of consistent decision making under uncertainty.
- Investor evaluates each gamble not by its expected payoff, but by its expected utility.
- A utility is a nonlinear transformation of the payoff: a $\$1{,}000$ payoff may not be twice as valuable as $\$500$.

An illustration



- Payoffs are transformed nonlinearly, with function $u(x)$.
- Use payoff probabilities as weights, linearly.

Choice among risky payoffs

- Utility function $u(\cdot)$ transforms payoffs.

- Investor then compares payoffs based on their expected utility:

$$x \text{ preferred to } y \quad \Leftrightarrow \quad \mathrm{E}[u(x)] > \mathrm{E}[u(y)]$$

- Select among investments consistently.

- For example, choices are transitive: if X is preferred to Y, and Y to Z -- then X is preferred to Z.

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Basic properties

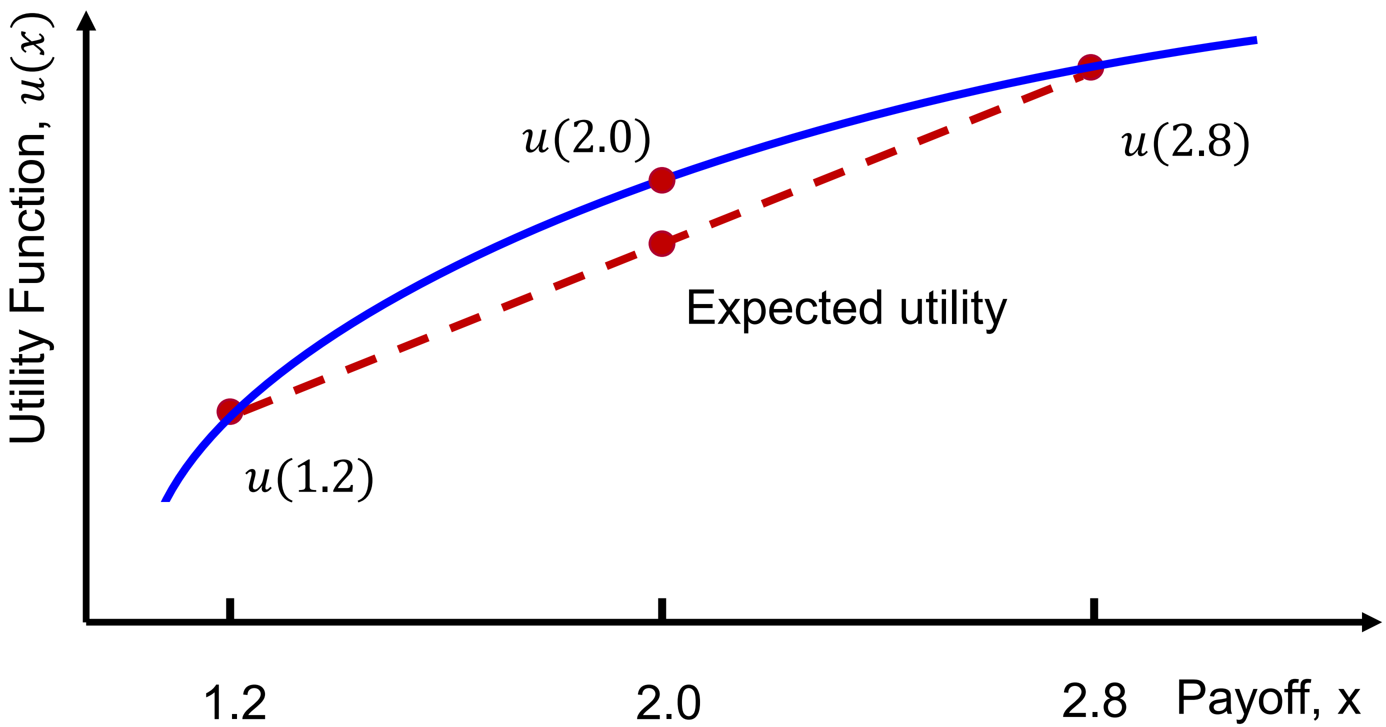
- Prefer more to less: $x + \epsilon \succeq x$ for all $x, \epsilon \geq 0$.
  - Implies utility function is non-decreasing: $u'(x) \geq 0$.

without risk with risk

- Aversion to risk: $u(\mathrm{E}[x]) \geq \mathrm{E}[u(x)]$.
  - Implies $u(\cdot)$ is concave: $u''(x) \leq 0$.

Risk aversion: a graphical illustration

- Increasing, concave utility: $u(\mathrm{E}[x]) \geq \mathrm{E}[u(x)]$ $(\mathrm{E}[x] = 2.0)$.



Risk premium

- Consider an investment with random return x: investor starts with W and ends with $W(1+x)$.
- Expected return is zero: $\mathrm{E}[x]=0$, variance $\sigma_x^2=\mathrm{E}\left[x^2\right]$.
- Risk-averse investor would prefer a zero riskless return:

$$\mathrm{E}[u(W(1+x))] \leq u(W)$$

- Define the risk premium π, such that indifferent between a random return x and losing a fraction π of wealth for sure:

$$\underbrace{\mathrm{E}\left[u(W(1+x))\right]}_{\text{risky}} = \underbrace{u(W(1-\pi))}_{\text{riskless but less}}$$

Derive π: use Taylor expansion

- Want to determine risk premium π based on

$$\mathrm{E}\left[u\left(W(1+x)\right)\right] = u\left(W(1-\pi)\right)$$

- Assume x is close to zero: $x \in (-\epsilon, \epsilon)$, $\epsilon \ll 1$.

- Then risk π is also small in magnitude.

- Use Taylor expansion to simplify the problem: expand both sides of the equation around 0.

Derivation

$$\underbrace{\mathrm{E}\big[u\big(W(1+x)\big)\big]}_{\substack{\text{expected utility}\\ \text{of the risky payoff}}} = \mathrm{E}\big[u(W) + u'(W)Wx + 0.5u''(W)W^2x^2 + \cdots\big]$$

$$= u(W) + u'(W)W\,\underbrace{\mathrm{E}[x]}_{=0} + 0.5u''(W)W^2\sigma_x^2 + \cdots$$

$$= \underbrace{u\big(W(1-\pi)\big)}_{\substack{\text{utility of the}\\ \text{risk-free payoff}}} = u(W) - u'(W)W\pi + \cdots$$ higher order of infini-small

Risk premium and risk aversion

- We find

$$u(W) + 0.5u''(W)W^2\sigma_x^2 = u(W) - u'(W)W\pi$$

- Risk premium π is given by

$$\pi = -\frac{1}{2}\frac{Wu''(W)}{u'(w)}\sigma_x^2$$

- Risk premium is a product of the relative risk aversion coefficient, $RRA(W)$:

$$RRA(W) = -\frac{\overbrace{Wu''(W)}^{\le 0}}{\underbrace{u'(w)}_{\ge 0}}$$

W is not u(W) !!!!!!

and a measure of return risk – return variance σ_x^2.

Examples of utility functions

- Linear utility

$$u(W) = a + bW,\ b > 0$$

  - $RRA(W) = 0$ (agent is risk-neutral).

- Power utility

$$u(W) = \begin{cases} \frac{1}{1-\gamma} W^{1-\gamma}, & \gamma > 0, \neq 1 \\ \ln(W), & \gamma = 1 \end{cases}$$

  - $RRA(W) = \gamma$ (constant relative risk aversion).

have the same risk aversion, regardless of the level of wealth

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Ranking investments

- Assume all returns have a Normal (Gaussian) distribution: $\tilde{r}_i \sim N\left(\bar{r}_i, \sigma_i^2\right)$.
- Investors rank returns based on their expected utility:

evaluate each investment using only these two moments

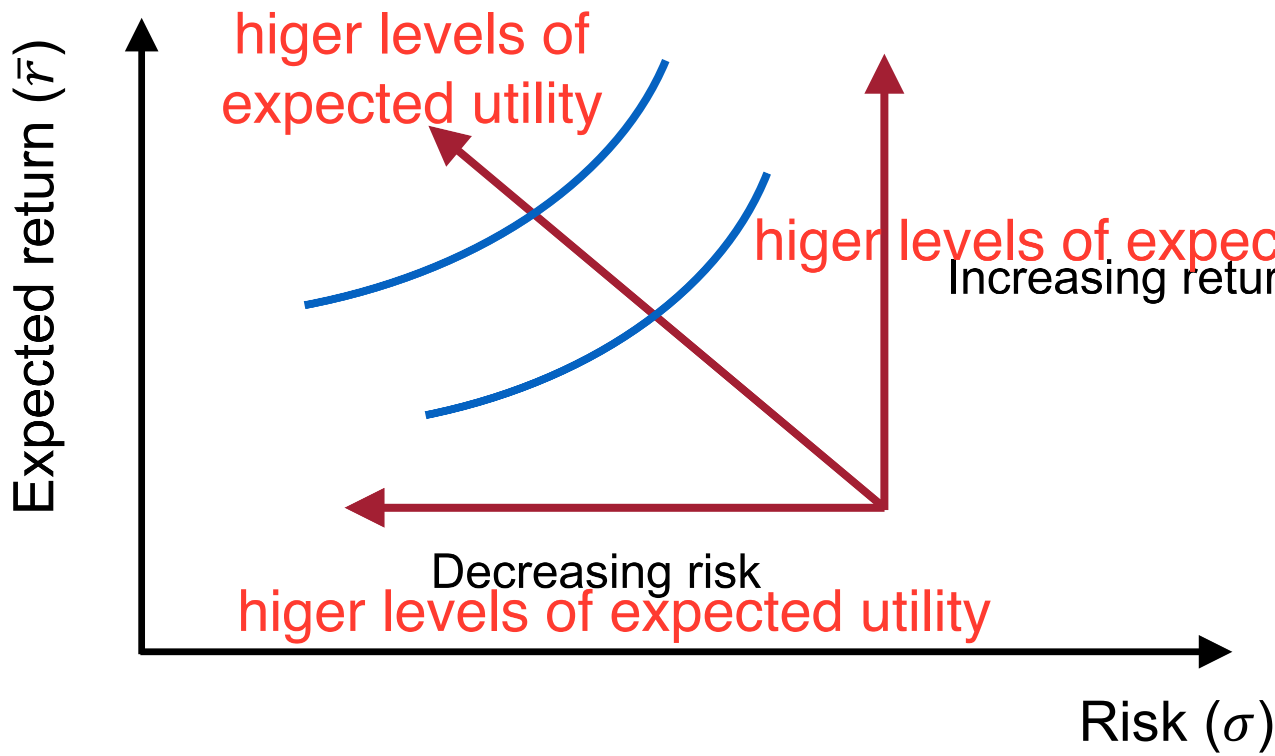
$$\mathrm{E}[u(\tilde{r}_i)] = F(\bar{r}_i, \sigma_i^2)$$

- Investor prefers higher mean return $\bar{r}$.
- Investor dislikes higher variance of return σ^2.
- Mean-variance preferences.
- Variance (or standard deviation) is the only measure risk.

Investor preferences over risk and return

visualize investor preferences

by plotting the expected utility indifference curves in the space of risk and expected return



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Notation

- P_0 — initial price.
- $\tilde{P}_1$ — price at the end of the period (uncertain random variable).
- $\widetilde{D}_1$ — dividend at the end of period.
- Return on an asset over a single period is random:

$$\tilde{r}_1 = \frac{\widetilde{D}_1 + \tilde{P}_1 - P_0}{P_0} = \frac{\widetilde{D}_1 + \tilde{P}_1}{P_0} - 1$$

- Expected return:

$$\mathrm{E}[\tilde{r}_1] = \frac{\mathrm{E}\left[\widetilde{D}_1\right] + \mathrm{E}[\tilde{P}_1]}{P_0} - 1$$

- Excess return:

$$\tilde{r}_1^e = \tilde{r}_1 - \underbrace{r_f}_{\text{risk-free rate}}$$

Basic statistics

- Basic statistics: mean, variance, and standard deviation (volatility).
- Moments not known -- estimate from historical data.

	Return moments	Common sample estimators
Mean	$\bar{r} = \mathrm{E}[\tilde{r}]$	$\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$
Variance	$\sigma^2 = \mathrm{E}\left[(\tilde{r} - \bar{r})^2\right]$	$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{r})^2$
Standard deviation	$\sigma = \sqrt{\sigma^2}$	$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$

covariance 1/(T-1) sum of (r_1t-r_1^-)
(r_2t-r_2^-)

Riskier assets on average earn higher returns

Average annual total returns from 1926 to 2018 (nominal)

higher returns come
with higher levels of risk

Asset	Mean (%)	SD (%)
T-bills	3.4	3.1
Long term T-bonds	5.9	9.8
Long term corp. bonds	6.3	8.4
Large stocks	11.9	19.8
Small stocks	16.2	31.6
Inflation	3.0	4.0

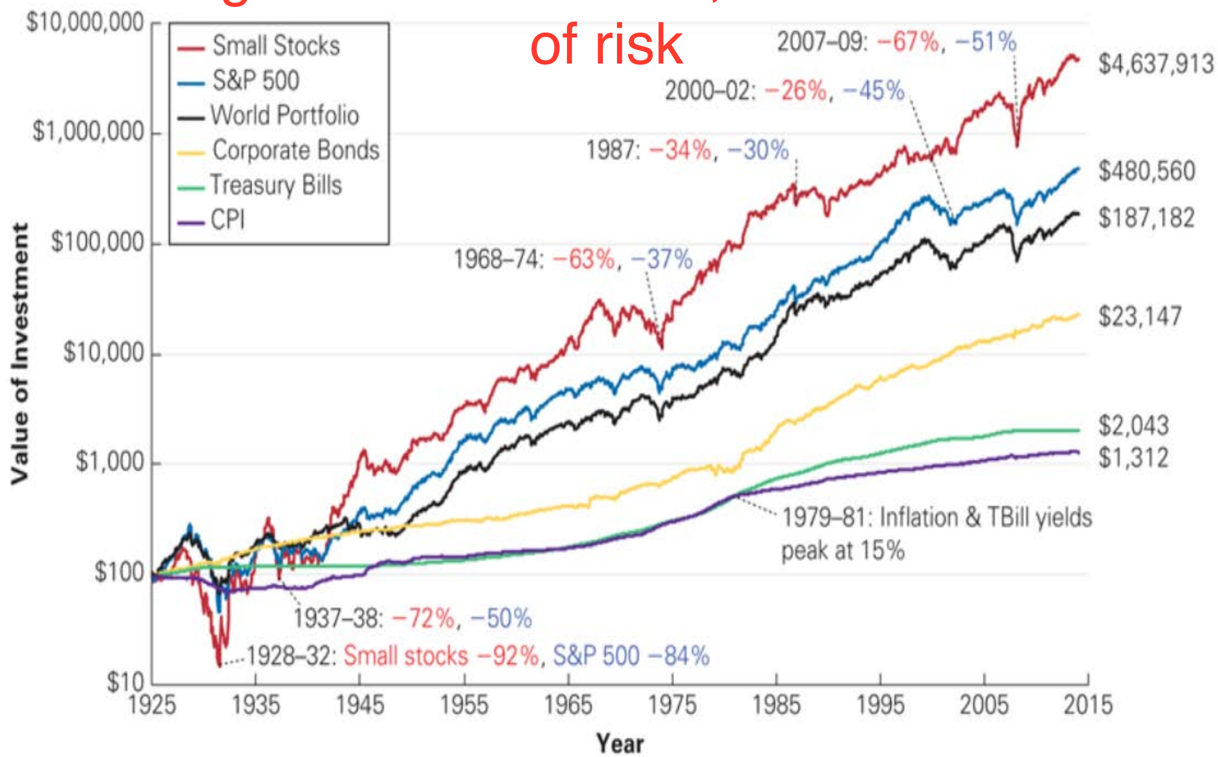
Source: Stocks, bonds, bills and inflation, 2019 Year Book, Ibbotson Associates, Chicago, 2019.

Long-term returns

- Value at the end of 2015 of $100 invested at the end of 1925 in various asset classes.

- Why would any investor buy bonds? It is all about the risk-return tradeoff.

in order to achieve high levels of returns, comes with a much higher level of risk



Source: Stocks, bonds, bills and inflation, 2016 Year Book, Ibbotson Associates, Chicago, 2016.

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Risk is more than variance

think outside of the mean variance framework

- Risk has many dimensions: it is not just variance.

skew to the right: large positive outcomes are more likely than large negative outcomes.

- Skewness: is the distribution symmetric? Negative vs positive outcomes.
- Derivatives may exhibit high skewness in returns (positive or negative).
- Kurtosis: does the distribution have fat tails?

high kurtosis: extreme outcomes on the right or on the left are both relatively likely.
In particular, they're more likely than under the normal distribution

- High kurtosis is common in financial markets: asset returns often have non-Normal distribution.
- Presence of tail risk implies that return risk is hard to estimate.

Example: Exchange-Traded Funds (ETFs)

portfolios, but traded as stocks.

- Monthly returns (exchange-traded funds, CRSP US Stock Database):
- SPY: S&P 500 ETF; equity fund
- AGG: aggregate bond ETF. bond fund

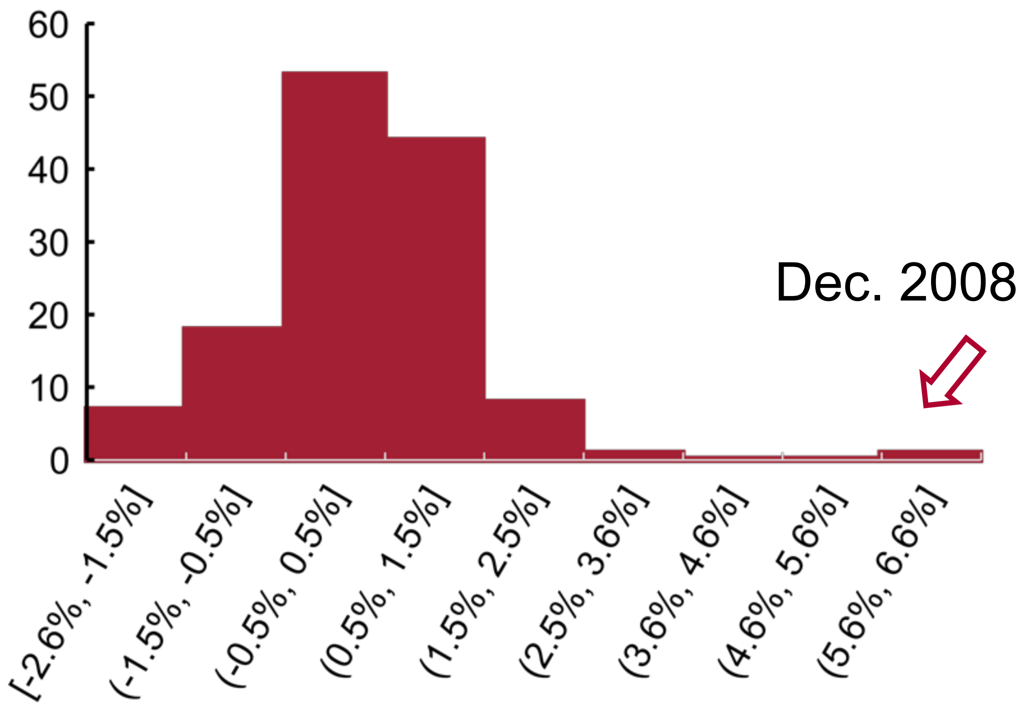
Jan 2008 – Dec 2018

	SPY	AGG
Mean	0.68%	0.29%
Standard deviation	4.34%	1.10%

Heavy tails in returns

- Large returns more common than under normal distribution.
  - Monthly returns on AGG (aggregate bond ETF).
- Return on Dec. 2008 is 5.8 standard deviations above the sample mean.
- Under the normal distribution, this should happen once in 10 Million years.

very far from normal
high kurtosis



Return volatility changes over time

- Properties of returns change over time.

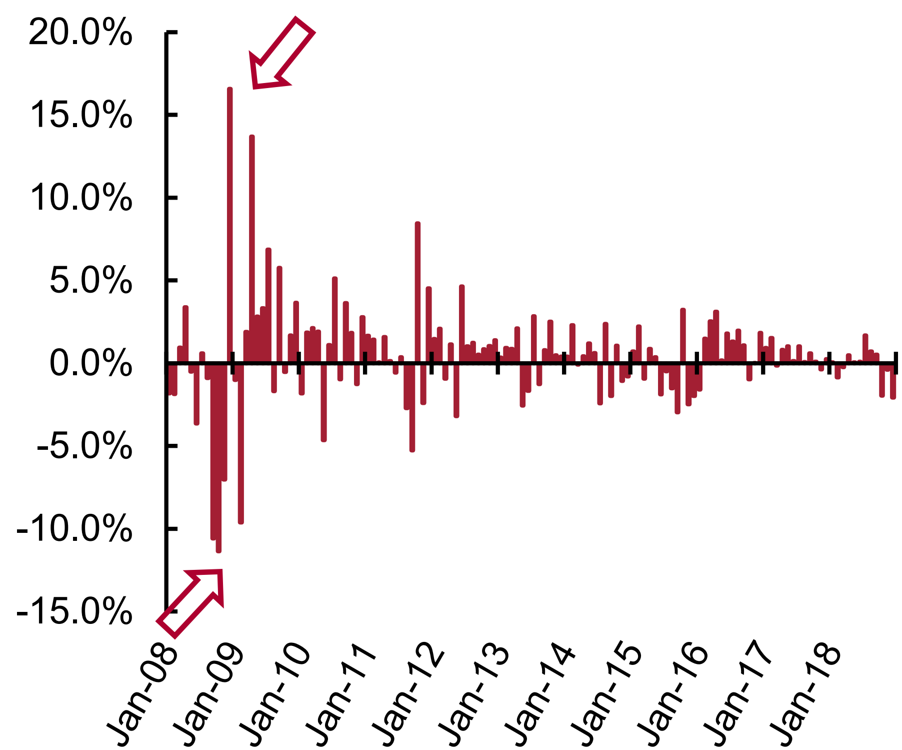
understand high kurtosis relative to the normal distribution

- Extremely high volatility in 2008/2009.
- Return volatility tends to rise during economic distress.
- 5 months within a single year with returns exceeding 3 standard deviations (3-σ events).

extremely unlikely under the normal distribution with constant volatility.

- It is important to model time-variation in return volatility.

Monthly returns on HYG (high yield corporate bond ETF)



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Correlation

- Correlation: How closely do two variables move together?

linear measure

$$\mathrm{Cov}(\tilde{r}_i, \tilde{r}_j) = \mathrm{E}\left[(\tilde{r}_i - \bar{r}_i)\left(\tilde{r}_j - \bar{r}_j\right)\right] = \sigma_{ij} \quad \text{[Covariance]}$$

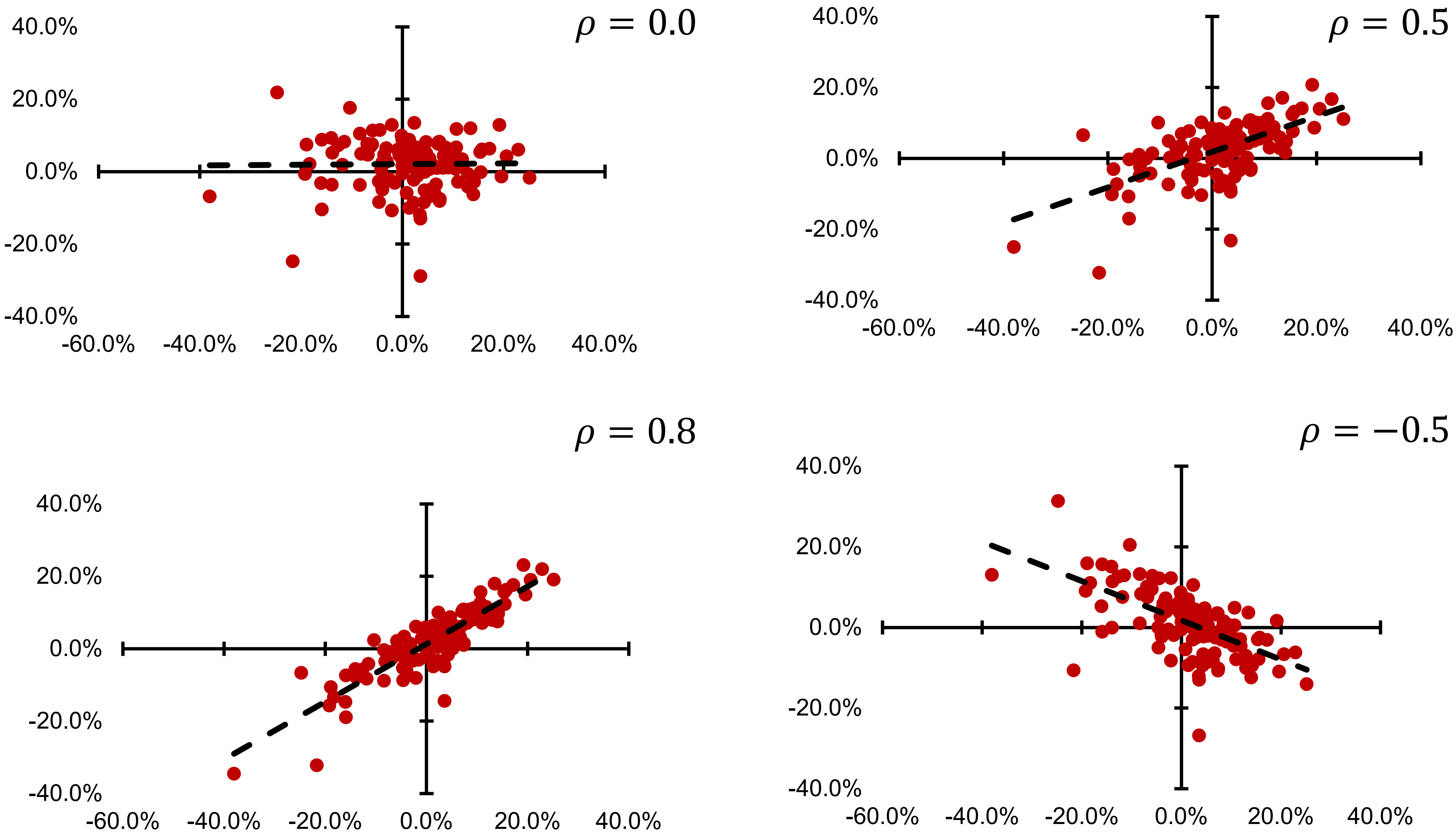
$$\mathrm{Corr}(\tilde{r}_i, \tilde{r}_j) = \frac{\mathrm{E}\left[(\tilde{r}_i - \bar{r}_i)\left(\tilde{r}_j - \bar{r}_j\right)\right]}{\sigma_i \sigma_j} = \rho_{ij} \quad \text{[Correlation]}$$

$$\beta_{ij} = \frac{\sigma_{ij}}{\sigma_j^2} \quad \text{[Beta]}$$

beta: the degree to which returns on asset j affect statistically returns on asset i.

When beta is equal to 1, for each 1% movement in asset j, we would see on average 1% movement in asset i.

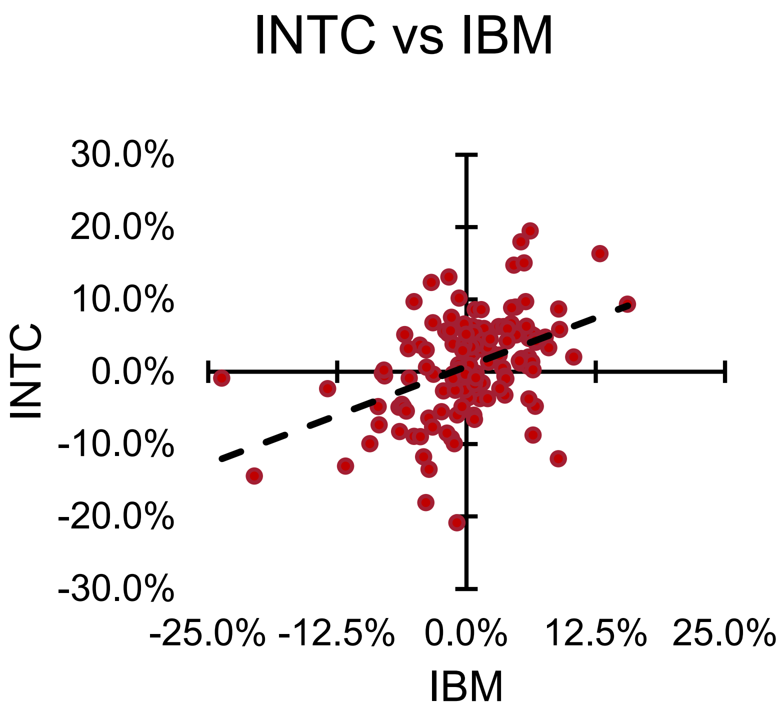
Correlation between two random variables

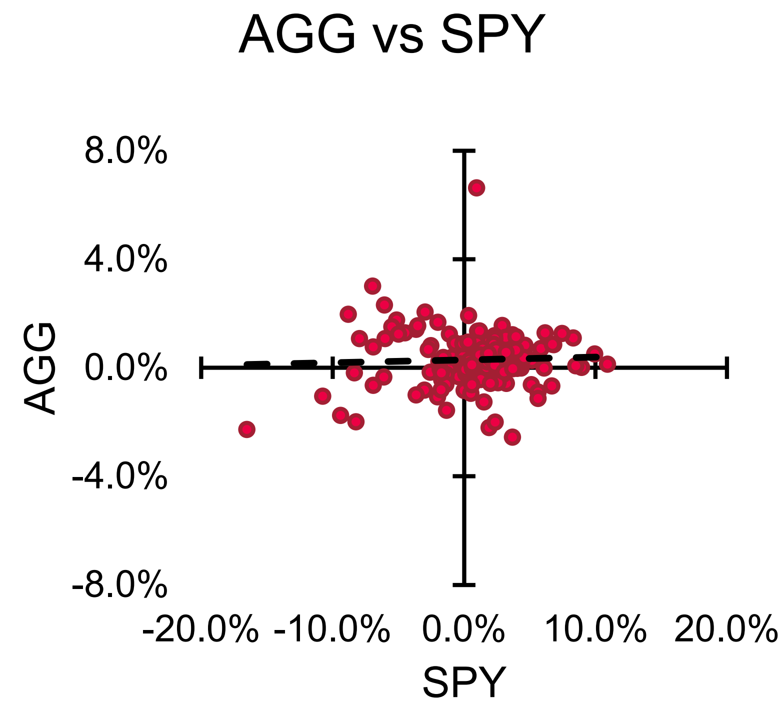


(Slope of the scattered plot gives the beta)

Empirical example of return correlation

- Monthly return correlation, Jan. 2008—Dec. 2018.





- Two technology stocks are positively correlated;
- Stock and bond markets are almost uncorrelated.

Historical correlations across assets

cross-sectional correlations: how returns are related to each other at a point in time

Annual Nominal Returns (1926-2018)						
	Bills	Long-term Treasury bonds	Long-term Corporate bonds	Large stocks	Small stocks	Inflation
T-bills	1.00	0.18	0.16	-0.02	-0.08	0.42
Long-term Treasury bonds		1.00	0.89	0.00	-0.10	-0.13
Long-term corporate bonds			1.00	0.16	0.06	-0.14
Large stocks				1.00	0.79	0.00
Small stocks					1.00	0.05
Inflation						1.00

high degree of co-movement

Source: Stocks, bonds, bills and inflation, 2019 Year Book, Ibbotson Associates, Chicago, 2019.

time-series correlation
or order correlation

Historical serial correlations

how returns of a single asset during different periods are related to each other.

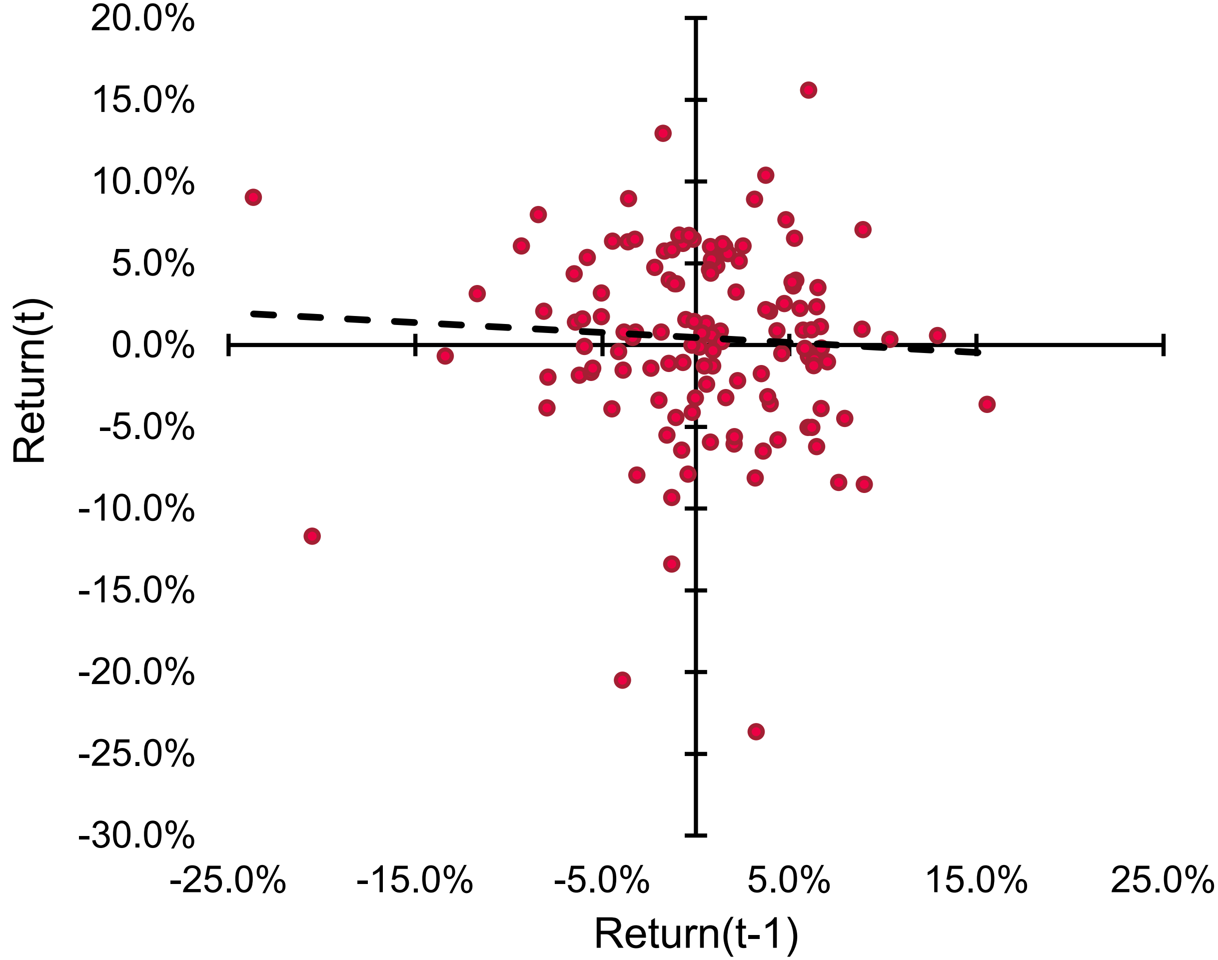
- Returns on risky assets are almost serially uncorrelated.
- High autocorrelation would imply that recent past returns help forecast future returns -- would be easy to profit from this by trading.

Serial Correlations of Annual Asset Returns (1926-2018)		
Asset	Serial Correlation	
	Nominal return	Real return
T-bills	0.92	0.67
Long-term Treasury Bonds	-0.15	-0.06
Long-term corporate Bonds	0.03	0.14
Large stocks	0.01	0.01
Small stocks	0.06	0.03

Source: Stocks, bonds, bills and inflation, 2019 Year Book, Ibbotson Associates, Chicago, 2019.

Historical serial correlations

- Monthly returns on IBM against last-month returns, Jan. 2008 – Dec. 2018.



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Definitions

- Portfolio is a collection of n assets.
- Composition: N_i shares of each asset i, share price P_i.
- Portfolio value equals the sum of values of individual positions:

$$\text{Portfolio Value } V = N_1 P_1 + N_2 P_2 + \cdots + N_n P_n = \sum_{i=1}^{n} N_i P_i$$

Portfolio composition

- Portfolio composition can also be described by its asset weights:

$$w_i = \frac{N_i P_i}{N_1 P_1 + N_2 P_2 + \cdots + N_n P_n} = \frac{N_i P_i}{V}$$

- A typical portfolio has $V > 0$.
- When $V = 0$ (zero net investment), we call this an arbitrage portfolio.
- If $V > 0$, $w_1 + w_2 + \cdots w_n = 1$.

Example: portfolio composition

- Your investment account has $100,000.
- There are 3 positions:

1) 200 shares of stock A;
2) 1,000 shares of stock B;
3) 750 shares of stock C.

Example: portfolio composition

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	10%
B	1,000	$60	$60,000	60%
C	750	$40	$30,000	30%
Total			$100,000	100%

- Asset A: $200 \times \$50 = \$10{,}000$.
- Weight on A: $\$10{,}000/\$100{,}000 = 10\%$.
- Weights sum up to 100%: $10\% + 60\% + 30\% = 100\%$.

Add leverage

- Your broker informs you that you only need to keep $50,000 in your investment account to support the same portfolio.
- You can buy the same stocks on margin, using leverage.
- Withdraw $50,000 to use for other purposes, leave $50,000 in the account.

Portfolio composition with leverage

use margin: borrow money at the risk-free rate to buy risky assets in the portfolio.

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	20%
B	1,000	$60	$60,000	120%
C	750	$40	$30,000	60%
Riskless Bond	–50,000	$1	–$50,000	–100%
Total			$50,000	100%

- New position: riskless bond, -$50,000.
- Total portfolio value is $50,000 compared to $100,000 without leverage.
- Weights of risky assets double; all weights still sum up to 100%.

Example: mortgage and leverage

- Purchase a home for $500,000.
- Pay 20% down + mortgage for 80%.

Asset	Shares	Price per Share	Dollar Investment	Portfolio Weight
Home	1	$500,000	$500,000	500%
Mortgage	-1	$400,000	–$400,000	–400%
Total			$100,000	100%

- Leverage ratio $= \dfrac{\text{asset value}}{\text{net investment}} = 500\,K \,/\, 100\,K = 5.$

Leverage magnifies gains and losses

- Suppose house value declines by 15%.
- Cash buyer loses 15%; a buyer with a mortgage loses 75%.
  - New house value: $\$500{,}000 \times (1 - 0.15) = \$425{,}000$
  - Mortgage value is unchanged: $-\$400{,}000$ portfolio position.
  - New value of the portfolio (house value that belongs to the owner): $\$25{,}000$.
  - This is a 75% decline from initial investment of $\$100{,}000$:

$$\begin{aligned}&\text{Levered investment decline } (75\%)\\ &= \text{ Leverage ratio } (5) \times \text{Original investment decline } (15\%)\end{aligned}$$

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Advantages of forming portfolios

- Why not pick the best asset instead of forming a portfolio?
- Don't know which stock is best.
- Diversification -- reduce unnecessary risks.
- Enhance performance by focusing bets (hedging).
- Portfolios can customize and manage risk/reward trade-offs.

by taking a short position in the market, you are betting on stock specific performance rather than on that plus additional movements in the market.
So forming portfolios could be very useful in order to isolate views on particular elements of the trade.

How to pick the "best" portfolio?

- What does "best" mean?
- What properties of a portfolio do we care about?
- Risk and reward:
  - Higher expected returns are preferred;
  - Higher risks are not desirable.

Portfolio properties

- Properties of a portfolio are determined by the returns of its assets and their weight in the portfolio.
- Start with expected return on the portfolio: it depends on expected returns on individual assets and their portfolio weights.
- Expected returns on portfolio assets

Asset	1	2	⋯	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	…	$\bar{r}_n$

- Expected return on the portfolio is a weighted average of expected returns on individual asset:

$$\bar{r}_p = E[r_p] = w_1\,\bar{r}_1 + w_2\,\bar{r}_2 + \cdots + w_n\,\bar{r}_n = \sum_{i=1}^{n} w_i\,\bar{r}_i$$

Expected return on the portfolio

- Expected returns on portfolio assets

Asset	1	2	⋯	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	…	$\bar{r}_n$

- Expected return on the portfolio is a weighted average of expected returns on individual asset:

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Variance of portfolio return

- Variance of portfolio returns depends on the entire covariance matrix of individual asset returns.
  - Derive the general expression below.

	$\mathbf{1}$	$\mathbf{2}$	$\cdots$	$\boldsymbol{n}$
$\mathbf{1}$	σ_1^2	σ_{12}	$\cdots$	σ_{1n}
$\mathbf{2}$	σ_{21}	σ_2^2	$\cdots$	σ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\boldsymbol{n}$	σ_{n1}	σ_{n2}	$\cdots$	σ_n^2

- Diagonal elements are individual return variances: $\sigma_{nn} = \sigma_n^2$.
- Off-diagonal elements capture pair-wise co-movement of asset returns.

Example: a portfolio with two assets

- Monthly stock returns, Jan. 2008 – Dec. 2018.
- SPY (equity ETF).
- AGG (bond ETF).

Sample mean	
SPY	AGG
0.68%	0.90%

Sample covariance matrix		
	SPY	AGG
SPY	\sigma_1^2 0.00188	0.00002
AGG	0.00002	0.00012

- More intuitive: $\sigma_1 = 4.34\%$, $\sigma_2 = 1.1\%$, $\rho_{12} = 0.04$.

Example: portfolio return with two assets

- Portfolio return is a weighted average of individual returns:

$$\tilde{r}_p = w_1\,\tilde{r}_1 + w_2\,\tilde{r}_2$$

	SPY	AGG
Investment	\$600	\$400
Weight	0.6	0.4

- $r_{SPY} = 2\%;\ r_{AGG} = -1\%$ over the next month.

- Portfolio return:

$$r_p = \frac{(600)(2\%) + (400)(-1\%)}{1{,}000} = (0.6)(2\%) + (0.4)(-1\%) = 0.8\%$$

Portfolio mean and variance, two assets

- Expected portfolio return:

$$\bar{r}_p = w_1 \bar{r}_1 + w_2 \bar{r}_2$$

- Unexpected portfolio return:

$$\tilde{r}_p - \bar{r}_p = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2)$$

- The variance of the portfolio return:

	1	2
1	$w_1^2\sigma_1^2$	$w_1 w_2 \sigma_{12}$
2	$w_1 w_2 \sigma_{12}$	$w_2^2\sigma_2^2$

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \sigma_{12}$$ sum of variance matrix

Example: portfolio return with two assets

Sample mean	
SPY	AGG
0.68%	0.90%

- Equally weighted portfolio: $w_1 = w_2 = 0.5$.
- Mean of the portfolio return (use sample means to estimate expected asset returns):

$$\bar{r}_p = (0.5)(0.68\%) + (0.5)(0.90\%) = 0.79\%$$

- Expected return on a portfolio is a weighted average of expected returns on individual assets.

Example: portfolio variance with two assets

Covariance matrix		
	SPY	AGG
SPY	0.00188	0.00002
AGG	0.00002	0.00012

$$\sigma_p^2 = (0.5)^2(0.00188) + (0.5)^2(0.0002) + (2)(0.5)^2(0.00012) = 0.00051$$

- Portfolio volatility is not a weighted average of individual asset volatilities:

$$\sigma_p = 2.26\% < \text{ Weighted Average}$$

by combining securities
together, one can achieve a significant reduction in risk

$\psi(x)$ is convex, then $E[\psi(X)]$ is also convex: $E[\psi(\theta X+(1-\theta)Y)] \leq \theta E[\psi(X)]+(1-\theta)E[\psi(Y)]$

proof:

$$
\begin{gathered}
E[\psi(\theta X+(1-\theta)Y)]=\sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}\psi(\theta X_i+(1-\theta)Y_j) \\
\leq \sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}(\theta\psi(X_i)+(1-\theta)\psi(Y_j))=\sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}\theta\psi(X_i)+\sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}(1-\theta)\psi(Y_j) \\
=\sum_{i=1}^{n}\theta\psi(X_i)\sum_{j=1}^{m}p_{ij}+\sum_{j=1}^{m}(1-\theta)\psi(Y_j)\sum_{i=1}^{n}p_{ij}=\sum_{i=1}^{n}\theta\psi(X_i)p_i+\sum_{j=1}^{m}(1-\theta)\psi(Y_j)p_j \\
=\theta E[\psi(X)]+(1-\theta)E[\psi(Y)]
\end{gathered}
$$

thus the portfolio variance is smaller

$$
\begin{gathered}
Var(Y)=E[(Y-EY)^2], Var(X)=E[(X-EX)^2] \\
Var(\theta X+(1-\theta Y))=E[((\theta X+(1-\theta)Y)-E(\theta X+(1-\theta)Y))^2] \\
=E[(\theta(X-EX)+(1-\theta)(Y-EY))^2] \\
\leq \theta E[(X-EX)^2]+(1-\theta)E[(Y-EY)^2]=\theta Var(X)+(1-\theta)Var(Y)
\end{gathered}
$$

General expressions for portfolio mean and variance

- We now consider a portfolio of n assets.
- Portfolio weights are $\{w_1, w_2, \dots, w_n\}$, $\sum_i w_i = 1$
- Portfolio return is a weighted average of individual asset returns:

$$\tilde{r}_p = w_1\, \tilde{r}_1 + w_2\, \tilde{r}_2 + \cdots + w_n\, \tilde{r}_n = \sum_{i=1}^{n} w_i\, \tilde{r}_i$$

General expressions for portfolio mean and variance

- Expected return on the portfolio:

$$\bar{r}_p = E\left[r_p\right] = w_1\,\bar{r}_1 + w_2\,\bar{r}_2 + \cdots + w_n\,\bar{r}_n = \sum_{i=1}^{n} w_i\,\bar{r}_i$$

- Variance of the return on the portfolio = weighted sum of all the variances and covariances of its assets:

$$\sigma_p^2 = Var\left[\tilde{r}_p\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}\,, \qquad \sigma_{ii} = \sigma_i^2$$

Key Concepts

- Introduction
- Expected Utility
- Risk Aversion
- Mean-Variance Preferences
- Asset Returns
- Other Dimensions of Risk
- Joint Distributions of Returns
- Portfolios
- Portfolios: Risk and Return
- Systematic and Idiosyncratic Risk

Example: diversification with a two-asset portfolio

- Diversification reduces risk.
- Two assets: INTC and IBM.
- Compare their return volatility to an equally-weighted (50/50) portfolio.

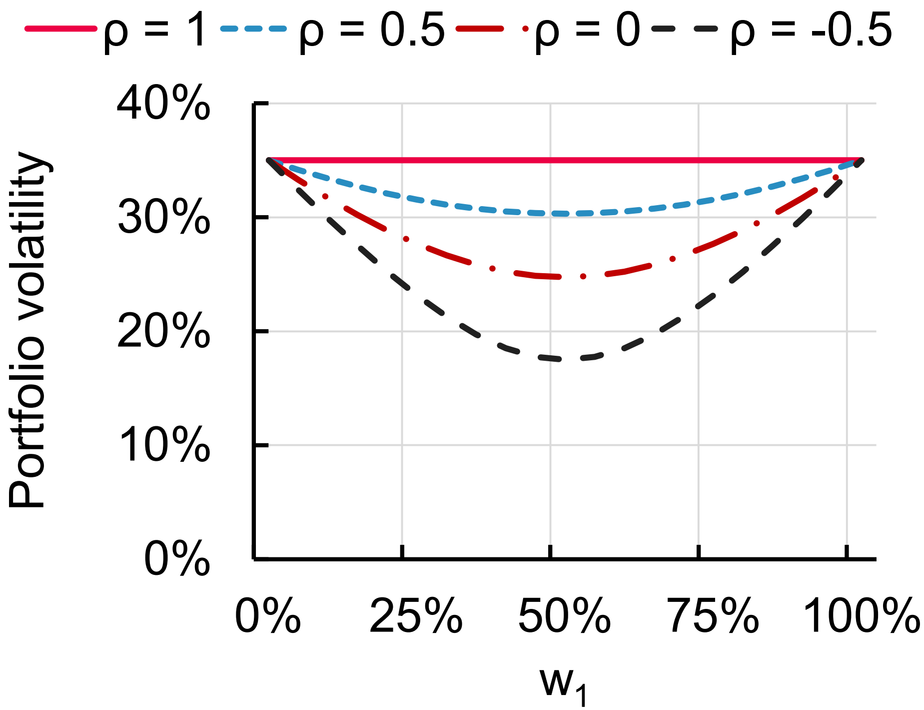
	IBM	INTC	Portfolio
Volatility	5.69%	6.96%	5.38%

- Portfolio is less volatile than either of the two stocks individually!

By combining them into the portfolio,
we are able to reduce the impact of individual
firm-specific risk (asset-specific risk).

Diversification and correlation

- Consider two assets with the same volatility, 35%.
  standard deviation = 35%
- Portfolio with weight w in asset 1 and $1 - w$ in asset 2.
- Vary correlation ρ between the two assets: $\rho = 1, 0.5, 0, -0.5$.
- Volatility of the portfolio return is less than the volatility of each individual asset return.



Certain risks cannot be diversified away

- Diversification is effective up to a certain limit – risk cannot be fully eliminated through diversification.
- Remaining risk is known as non-diversifiable (also called market risk, systematic risk, common risk).
- Risk comes in two kinds:
  - Diversifiable risks;
  - Non-diversifiable risks.
- Sources of non-diversifiable risks include: affect multiple businesses in the same direction
  - Business cycle;
  - Inflation;
  - Liquidity. Many securities decline in value when market liquidity deteriorates. but securities like US Treasuries tend to move in the opposite direction

What determines limits of diversification?

- Consider an equally-weighted portfolio of n assets.
- Portfolio variance is the sum of all the terms in the matrix on the right:

	1	$\cdots$	n
1	$w_1^2\sigma_1^2$	$\cdots$	$w_1 w_n \sigma_{1n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$w_n w_1 \sigma_{n1}$	$\cdots$	$w_n^2\sigma_n^2$

- A typical variance term: $\left(\frac{1}{n}\right)^2 \sigma_{ii}$ -- total number of variance terms is n.
- A typical covariance term: $\left(\frac{1}{n}\right)^2 \sigma_{ij},\ (i \neq j)$ -- total number of covariance terms is $n^2 - n$.
  - With 100 assets, 100 diagonal elements, and 9,900 off-diagonal.

Diversification can reduce portfolio risk without reduction in expected return, e.g., a portfolio of many securities with identical returns and risk, but imperfect correlation.

Decompose portfolio variance

- Add all the terms:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ii} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ij}$$

$$= \left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \left(\frac{n^2-n}{n^2}\right)\left(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}\right)$$

$$= \left(\frac{1}{n}\right)(\text{average variance}) + \left(1-\frac{1}{n}\right)(\text{average covariance})$$

- As n becomes very large:
  - Contribution of variance terms goes to zero.
  - Contribution of covariance terms goes to "average covariance."

Portfolio variance and return correlation

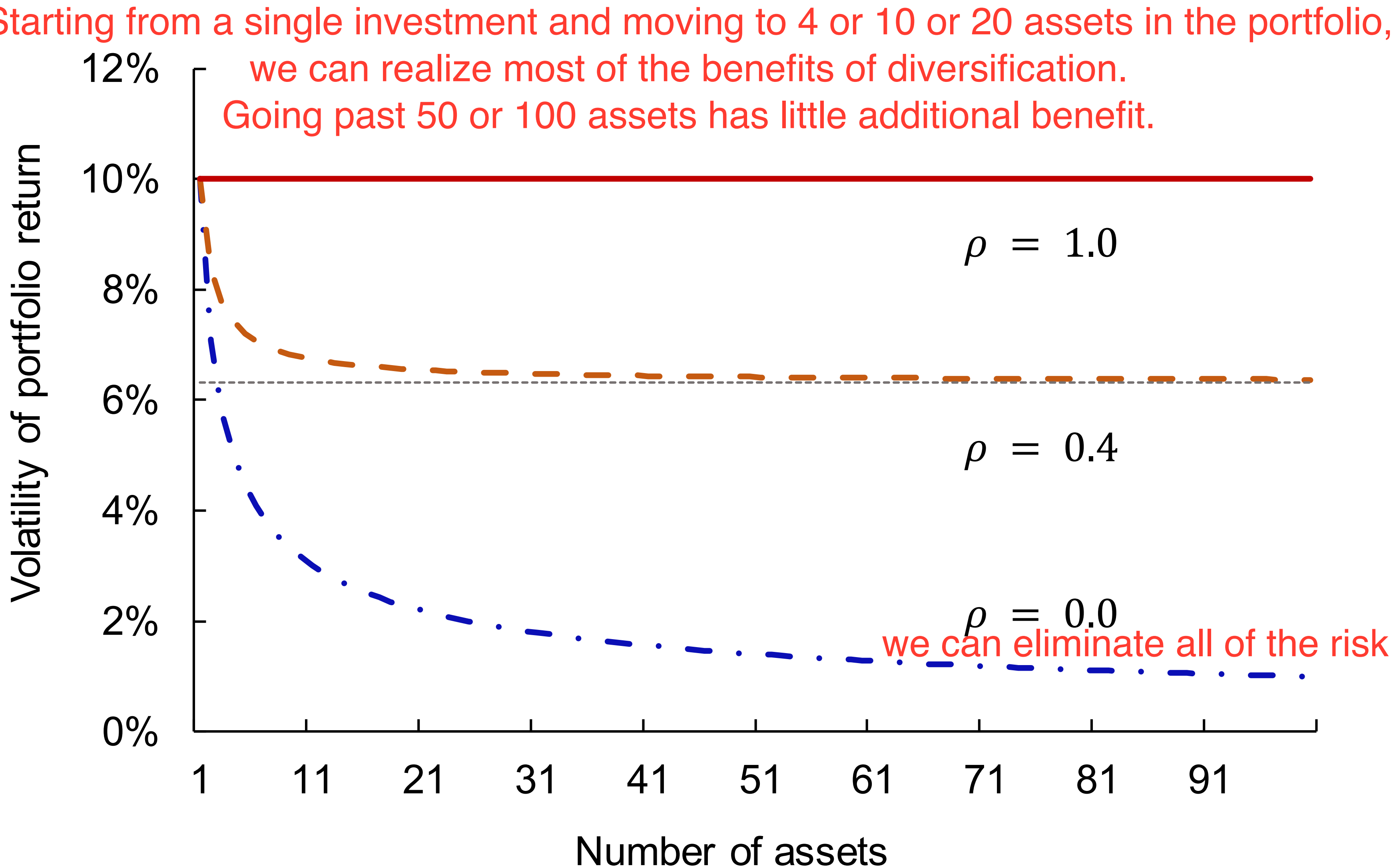
- The average US stock has a monthly standard deviation of 10% and the average correlation between stocks is 40%.
- If you invest the same amount in each stock, what is variance of the portfolio?

$$Cov[R_i, R_j] = \rho_{ij}\sigma_i\sigma_j = 0.40 \times 0.10 \times 0.10 = 0.004$$

$$Var[R_p] = \frac{1}{n}0.10^2 + \frac{n-1}{n}0.004 \approx 0.004 \quad \text{if } n \text{ is large}$$

$$\sigma_p \approx \sqrt{0.004} = 6.3\%$$

Return correlation and limits of diversification



Summary

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