15.455x – Mathematical Methods for Quantitative Finance

Recitation Notes #2

Exercise: Let z and z' be two independent, normalized random variables. Define the stochastic process

$$X_t \equiv z\cos(\omega t) + z' \sin(\omega t),$$

where ω is a constant. Show that X_t is weakly stationary.

Solution: All we have to do is compute and apply linearity (and a pinch of trigonometry). $\mathrm{E}\left[X_t\right] = 0$ since z, z' have zero mean. For the variance,

$$\begin{aligned}\mathrm{Var}(X_t) = \mathrm{E}\left[X_t^2\right] &= \mathrm{E}\left[z^2\cos^2(\omega t) + 2zz'\cos(\omega t)\sin(\omega t) + z'^2\sin^2(\omega t)\right] \\ &= \cos^2(\omega t)\,\mathrm{E}\left[z^2\right] + 2\cos(\omega t)\sin(\omega t)\,\mathrm{E}\left[zz'\right] + \sin^2(\omega t)\,\mathrm{E}\left[z'^2\right] \\ &= \cos^2(\omega t) + \sin^2(\omega t) = 1.\end{aligned}$$

Now for the autocovariance, or two-point function, observe that for any $t \neq s$

$$\begin{aligned}\mathrm{E}\left[X_t X_s\right] &= \mathrm{E}\left[\left(z\cos(\omega t) + z'\sin(\omega t)\right)\left(z\cos(\omega s) + z'\sin(\omega s)\right)\right] \\ &= \cos(\omega t)\cos(\omega s) + \sin(\omega t)\sin(\omega s) \\ &= \cos\left(\omega(t-s)\right),\end{aligned}$$

which is a function of the time difference $t - s$ only.

weakly stationary: the first moments, the means, should be time invariant, and the second moments, the variances and two point functions like the auto covariance functions, should depend only on the difference in time indices, not on the exact location of a particular point in time

Exercise:

Let a stochastic process be defined by

$$X_t = z_t + \theta z_{t-2},$$

where z_t are independent, normalized random variables. Is X_t stationary? Find the mean, variance, and autocovariance function.

Solution:

The process is stationary because the defining equation has the same form if all the t are shifted by a constant. The first two moments can be computed by using linearity.

$$\begin{aligned} \mathrm{E}\left[X_t\right] &= \mathrm{E}\left[z_t\right] + \theta\ \mathrm{E}\left[z_{t-2}\right] = 0, \\ \mathrm{E}\left[X_t^2\right] &= \mathrm{E}\left[\left(z_t^2 + 2\theta z_t z_{t-2} + \theta^2 z_{t-2}^2\right)\right] = 1 + \theta^2. \end{aligned}$$

What about the autocovariance function (ACF)? From its definition,

$$\begin{aligned} \gamma_k = \mathrm{E}\left[X_t X_{t-k}\right] &= \mathrm{E}\left[\left(z_t + \theta z_{t-2}\right)\left(z_{t-k} + \theta z_{t-k-2}\right)\right] \\ &= \theta\, \mathrm{E}\left[z_{t-2} z_{t-k}\right] + \theta\, \mathrm{E}\left[z_t z_{t-k-2}\right] \\ &= \begin{cases} \theta, & k = \pm 2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Exercise:

Define a random variable A as the average of the next four observations of the X_t defined above,

$$A = \frac{1}{4}\left(X_1 + X_2 + X_3 + X_4\right).$$

What are the mean and variance of A?

Solution:

The mean is zero since each X_t is mean zero. In computing the variance, we can do the algebra to express results in terms of the expectations we just computed. It is non-zero only for expectations of two X_t that are either zero or two time steps apart.

$$\begin{aligned} \mathrm{Var}(A) = \mathrm{E}\left[A^2\right] &= \frac{1}{16}\,\mathrm{E}\left[\Big(X_1 + X_2 + X_3 + X_4\Big)^2\right] \\ &= \frac{1}{16}\Big(4\,\mathrm{E}\left[X_1^2\right] + 2\,\mathrm{E}\left[X_1 X_3\right] + 2\,\mathrm{E}\left[X_2 X_4\right]\Big) \\ &= \frac{1}{16}\Big[4(1+\theta^2) + 4\theta\Big] = \frac{1}{4}(1 + \theta + \theta^2). \end{aligned}$$

Exercise:

Use recursion to show that the AR(1) process can be expressed as an MA process...of infinite order.

only z's on the right hand side and no R's

Solution:

Let's simplify the defining equation

$$R_t - \mu = -\lambda(R_{t-1} - \mu) + \sigma z_t$$

by introducing $Y_t = (R_t - \mu)/\sigma$, in terms of which we can continue to substitute:

$$\begin{aligned} Y_t &= z_t - \lambda Y_{t-1} & &= z_t - \lambda\big[z_{t-1} - \lambda Y_{t-2}\big] \\ &= z_t - \lambda z_{t-1} + \lambda^2 Y_{t-2} & &= z_t - \lambda z_{t-1} + \lambda^2\big[z_{t-2} - \lambda Y_{t-3}\big] \\ &= z_t - \lambda z_{t-1} + \lambda^2 z_{t-2} - \lambda^3 Y_{t-3} & &= z_t - \lambda z_{t-1} + \lambda^2 z_{t-2} - \lambda^3\big[z_{t-3} - \lambda Y_{t-4}\big] \\ &= \cdots \end{aligned}$$

If we were to continue the substitutions indefinitely (and recalling that $|\lambda| < 1$), we would obtain

$$Y_t = \sum_{k=0}^{\infty} (-\lambda)^k z_{t-k}.$$

Because this is a semi-infinite sum of z's extending into the past, we see immediately that $\mathrm{E}\left[z_s Y_t\right] = 0$ for every $t < s$. In particular, it means that

future is uncorrelated with past

$$\mathrm{E}\left[z_t \left(R_{t-1} - \mu\right)\right] = 0.$$