Recitation 8

Pricing a Callable Bond

Suppose that you have estimated the following binomial model for one-year interest rates, where the probability of rates rising or falling equals 50%:

$$\begin{tikzcd}[row sep=small] & 5.5\% \\ 4\% \arrow[ur] \arrow[dr] & \\ & 3.9\% \end{tikzcd}$$

(a) Using this model, what is the theoretical price of a two-year 6% coupon bond with no options attached? Assume the coupon payments are paid annually and the face value is \$100.

Solution: As always, let's move backwards through the binomial tree.

At the end of two years, the holder of the bond will receive the face value of \$100 plus a coupon payment of \$6. Since the diagram above shows the evolution of one-year interest rates, the appropriate discount rate for the payoff received at the end of two years is $(1.04)(1.055) = 1.0972$ in the "up" node and $(1.04)(1.039) = 1.08056$ in the "down" node.

Since there is a 50% chance of rates rising or falling, the expected discounted payoff received at the end of two years is:

$$0.5 \times \frac{106}{(1.04)(1.039)} + 0.5 \times \frac{106}{(1.04)(1.055)} = 97.35.$$

Now, the holder of the bond will also receive a coupon payment of \$6 at the end of one year, which has a present value of $\frac{6}{1.04} = 5.77$. Thus, the price of the two-year 6% coupon bond with no options attached is $5.77 + 97.35 = 103.12$.

(b) Consider the same bond as in Part (a), but with a call option that allows the issuer to call the bond at the end of the first year for \$101. What is the value of the call option? What is the theoretical value of the callable bond?

Solution: Recall from the Week 8 lecture that the value of a callable bond is equal to the value of a non-callable bond minus the value of the embedded call option. We know from Part (a) that the value of the non-callable bond is 103.12. So, we just need to find the value of the call option using the same binomial tree.

The payoff of the call option in each node $i = \{u, d\}$ at the end of the first year is equal to $C_i = \max(P_i - K, 0)$, where P_i is the value of the non-callable bond in each node and $K = 101$ is the strike price of the call option.

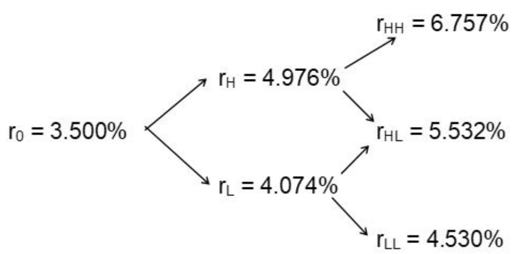
Since $P_u = \frac{106}{1.055} = 100.47$, $C_u = \max(100.47 - 101, 0) = 0$. Similarly, $P_d = \frac{106}{1.039} = 102.02$, and $C_d = \max(102.02 - 101, 0) = 1.02$. The value of the call option is the present value of the expected payoffs of the option at the end of the first year, or

$$C = 0.5 \times \frac{0}{1.04} + 0.5 \times \frac{1.02}{1.04} = 0.49.$$

Finally, the value of the callable bond is \$103.12 − \$0.49 = \$102.63.

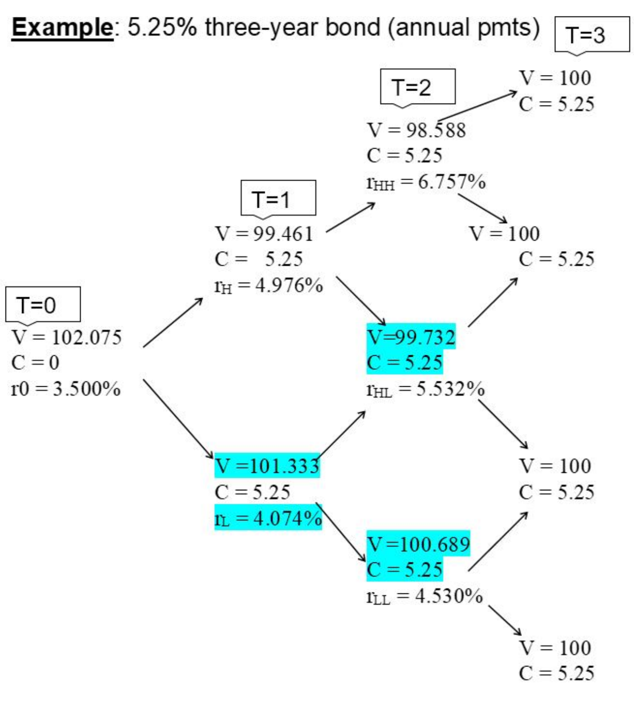
Pricing an American Option

Assume the following annual binomial tree is correct for risk-neutral pricing of bonds and bond options, where Pr(up) = Pr(down) = 0.5:



In the Week 8 lecture, we saw how to price a European call option on a bond using the binomial tree above. Now, let's see how we can use the same tree to price an American call option, in which a three-year, 5.25% annual coupon bond is callable at the end of either one or two years for \$99.50.

As in the Week 8 lecture, we can use the tree to price a *non-callable* three-year, 5.25% annual coupon bond as shown in the diagram below:



To price an American call option with strike price $K = 99.50$ that can be exercised at the end of either one or two years, we work backwards through the tree.

At the end of two years (i.e., when $T = 2$), the value of the call option is:

1. $\max(V - K, 0) = \max(98.588 - 99.5, 0) = 0$ in the "up-up" node, since the value of the bond, V, is equal to 98.588 in the "up-up" node;

2. $\max(V - K, 0) = \max(99.732 - 99.5, 0) = 0.232$ in the "up-down"/"down-up" node, since $V = 99.732$;

3. $\max(V - K, 0) = \max(100.689 - 99.5, 0) = 1.189$ in the "down-down" node, since $V = 100.689$.

Things are more complicated at the end of one year (i.e., when $T = 1$), since we now have to compare the value of exercising the call option at $T = 1$ to the value of waiting to exercise the call option at $T = 2$:

1. In the "up" node at $T = 1$, the value of waiting to exercise the call option at $T = 2$ is $(0.5 \times 0 + 0.5 \times 0.232)/1.04976 = 0.110$, since the risk-neutral probability of rates rising or falling is 0.5, and the prevailing interest rate in the "up" node is 4.976%. The value of exercising the call option is $\max(V - K, 0) = \max(99.461 - 99.5, 0) = 0$, since $V = 99.461$.

   Thus, the value of the call option in the "up" node is $\max(0.110, 0) = 0.110$, as it is optimal to wait to exercise the call option at $T = 2$.

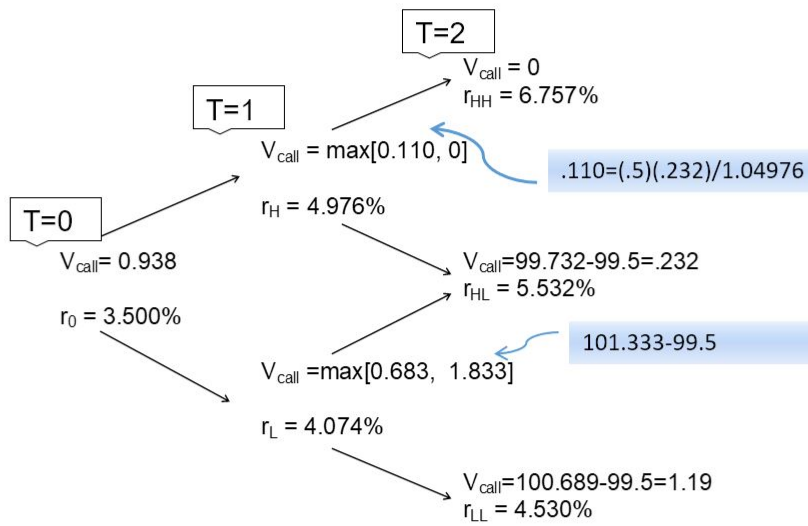
2. In the "down" node at $T = 1$, the value of waiting to exercise the call option at $T = 2$ is $(0.5 \times 0.232 + 0.5 \times 1.19)/1.04074 = 0.683$, since the prevailing interest rate in the "down" node is 4.074%. The value of exercising the call option is $\max(V - K, 0) = \max(101.333 - 99.5, 0) = 1.833$, since $V = 101.333$.

   Thus, the value of the call option in the "down" node is $\max(0.683, 1.833) = 1.833$, as it is optimal to exercise the call option at $T = 1$.

In the final step, we can find the value of the American call option at $T = 0$ to be:

$$V_{\text{call}} = (0.5 \times 0.110 + 0.5 \times 1.833)/1.035 = 0.938$$

since the prevailing interest rate at $T = 0$ is 3.5%. The following diagram summarizes the calculations performed above:



Effective Duration and Convexity

Recall the definitions of **modified duration** and **convexity** for an option-free bond from the Week 3 lecture:

- Modified duration is related to the first derivative of the price-yield relationship:

$$D_m = -\frac{dP}{Pdy}.$$

- Convexity is related to the second derivative of the price-yield relationship:

$$C = \frac{d^2P}{Pdy^2}.$$

- Together, modified duration and convexity can be used to approximate the price change of a bond associated with a small change in yields:

$$dP \cong -D_m(P)(dy) + \frac{1}{2}C(P)(dy)^2.$$

How can we adjust our definitions of modified duration and convexity for bonds with embedded options?

Since embedded options affect the sensitivity of bond prices to yields, we can use **effective duration** and **effective convexity**. These measures are estimated using a model of how prices move with changes in yields; for option-free bonds, they are the same as modified duration and standard convexity.

Effective duration and effective convexity are commonly used in delta- and gamma-hedging strategies. For example, hedging mortgage portfolios requires using effective duration and effective convexity because the prepayment option is a type of American call option.

Formally, we can define effective duration as:

$$D_{\text{eff}} = \frac{1}{P_{\text{initial}}} \times \frac{P_{\text{rates fall}} - P_{\text{rates rise}}}{2s}, \tag{1}$$

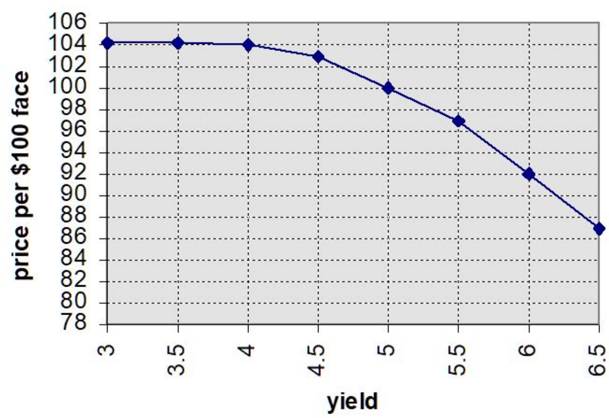
where s is the amount that interest rates rise or fall. Effective convexity is defined as:

$$C_{eff} = \frac{1}{P_{\text{initial}}} \times \frac{(P_{\text{rates fall}} - P_{\text{initial}}) - (P_{\text{initial}} - P_{\text{rates rise}})}{s^2}. \tag{2}$$

Just like before, we can use effective duration and effective convexity to approximate the price change of a bond with embedded options for a small change in yields:

$$dP \cong -D_{\text{eff}}(P)(dy) + \frac{1}{2}C_{\text{eff}}(P)(dy)^2. \tag{3}$$

As an example, suppose that a mortgage-backed security (MBS) has an estimated price/yield relationship summarized in the following figure:



(a) If yields are currently at 5.5%, what is the effective duration and effective convexity for the MBS?

Solution: According to the figure, the price of the MBS is \$97 when yields are at 5.5%. To apply Equations (1) and (2) to find the effective duration and effective convexity of the MBS, we consider a 0.5% increase or decrease in yields, so $s = 0.005$. From the figure, the price of the MBS is \$100 when yields fall to 5% and \$92 when yields rise to 6%.

From Equation (1), the effective duration of the MBS is:

$$D_{\text{eff}} = \frac{1}{P_{\text{initial}}} \times \frac{P_{\text{rates fall}} - P_{\text{rates rise}}}{2s} = \frac{1}{97} \times \frac{100 - 92}{2(0.005)} = 8.247.$$

From Equation (2), the effective convexity of the MBS is:

$$\begin{aligned} C_{eff} &= \frac{1}{P_{\text{initial}}} \times \frac{(P_{\text{rates fall}} - P_{\text{initial}}) - (P_{\text{initial}} - P_{\text{rates rise}})}{s^2} \\ &= \frac{1}{97} \times \frac{(100 - 97) - (97 - 92)}{0.005^2} = -824.74. \end{aligned}$$

In this case, the MBS has *negative convexity*. This is typical for mortgage bonds, as well as for callable bonds when yields are low. That is when they are more likely to be called