MIT Sloan School of Management
Kogan and Wang **15.415x**

Practice problems, Fin 2, 2021

1. Consider a stock and several put options on this stock. Suppose there are four European put options, maturing in 3 months. Assume that the prices of the put options are given by

Strike	\$40	\$50	\$60	\$70
Price	\$2	\$5	\$9	\$12

Show that there is arbitrage in this market, and construct an arbitrage trade.

2. Suppose that a financial asset ABC currently trades at \$2,000, and the annual dividend yield of 3%. Suppose the one-year risk-free rate is 4%. You observe that the one-year futures contract on one unit of ABC at \$2,040. Does this imply arbitrage? If you answered yes, describe the arbitrage strategy.

3. Consider a state-space model with three states possible at time 1: 1, 2, and 3. Each state is equally likely to occur. For each of the future states, investors can buy a primitive state-contingent claim (Arrow-Debreu claim), paying \$1 in the corresponding state and nothing otherwise. Market prices of these claims are

State	1	2	3
Price	0.5	0.2	0.1

(a) What is the risk-free interest rate between time 0 and time 1?

(b) Suppose an investor buys a portfolio of state-contingent claims with 1 claim on state 1, 1 claim on state 2, and 3 claims on state 3. What is the expected return on the investor's portfolio?

4. The following are some key statistics of GM taken at 05/28/2015.

	GM
Price	36.20
BookValue Per Share	22.71
Return on equity ROE	11.66%
Payout Ratio	56%
Beta	1.02%

(a) What is the growth rate on GM's dividend?

(b) What is the required rate of return on GM's equity according to the Gordon Growth Model?

(c) Assume a 3% interest rate and a 6% market premium, what is the required rate of return on GM's equity according to CAPM?

5. Using the properties of the capital market line (CML) and the security market line (SML), determine which of the following scenarios are consistent or inconsistent with the CAPM. Explain your answers. Let A and B denote arbitrary securities while F and M represent the riskless asset and the market portfolio respectively.

(a) *Scenario I:*

Security	E[R]	β
A	20%	0.8
B	15%	1.2

(b) *Scenario II:*

Security	E[R]	$\sigma(R)$
A	25%	30%
M	20%	30%

(c) *Scenario III:*

Security	E[R]	$\sigma(R)$
A	25%	55%
F	5%	0%
M	15%	30%

(d) *Scenario IV:*

Security	E[R]	β
A	17%	1.5
F	5%	0
M	13%	1.0

(e) *Scenario V:*

Security	E[R]	β
A	40%	2.0
M	15%	1.0

6. The rate of return on short-term government securities (assume these are risk-free) is 0.5%. Suppose the expected excess rate of return required by the market for a portfolio with a beta measure of 1 is 8%. According to the capital asset pricing model (security market line):

(a) What is the expected rate of return on the market portfolio?

(b) What would be the expected rate of return on a stock with $\beta = 0$?

(c) Suppose you consider buying a share of stock at \$40. The stock is expected to pay \$3 dividends next year and you expect it to sell then for \$41. The stock risk has been evaluated by $\beta = -.5$. Is the stock overpriced or underpriced?

7. Security B has a price of \$50 and a beta of 0.8. The risk-free rate is 3% and the market risk premium is 4%.

(a) According to the CAPM, what return do investors expect on the security?

(b) Investors expect the security not to pay any dividend next year. At what price do investors expect the security to trade next year?

(c) At what price do investors expect the security to trade next year, if the expected dividend next year is \$1 instead of zero?

8. Integral Industries, Inc. (III) has three subsidiaries, A, B, and C. You are negotiating to buy subsidiary C. Subsidiary A and B each contribute to 40% of III's market value and have betas of 0.8 and 1.4, respectively. The company as a whole has a beta of 1.0. What is the beta of subsidiary C? If you end up buying it, what would be C's opportunity cost of capital? The current risk-free rate is 6% and the market risk premium is 6%.

9. You are wondering how to invest your money in the financial market. Your old risk-averse neighbor, Richard, is also seeking his retirement investment. You know that he is better off by having a higher mean and a lower standard deviation of returns. You find that Richard invests 25% of his wealth in a portfolio of "large cap" stocks (L), 15% of his wealth in a portfolio of "small cap" stocks (S) and the rest of his wealth in risk-free treasury bills (F). He expects 8% mean and 12% standard deviation of returns.

Since you are younger and less risk-averse than Richard, you decide to lever up: you borrow exactly the amount of your current wealth. The risk-free rate is 2%.

(a) What is the tangent portfolio? What is the expected return and the standard deviation of the tangent portfolio?

(b) What portfolio would you hold? What is the expected return and the standard deviation of your portfolio?

10. Your employer offers two funds for your pension plan, a money market fund and an S&P 500 index fund. The money market fund holds 3-month Treasury bills, which currently offer a 3% safe return per year. The S&P 500 index fund offers an expected return of 10% per year with a standard deviation of 20%.

(a) You want to achieve an expected return of 8% per year for your portfolio. What should be the composition of your portfolio? What is the standard deviation of its returns?

(b) Now your employer adds an emerging-market fund to the two existing funds. The emerging-market fund offers an expected return of 10% per year, the same as the S&P 500 index fund, but with a standard deviation of 30%, higher than the S&P 500 index fund. Would you consider including the emerging-market fund as part of your portfolio? Explain.

(c) The correlation between the S&P 500 index fund and the emerging-market fund is zero. Consider portfolio A, which consists of 80% in the S&P index fund and 20% in the emerging-market fund. Calculate portfolio A's expected return and standard deviation.

(d) If you mix portfolio A with the money-market fund to achieve an expected return of 8%, is it better than the portfolio in part (10a) of this question? Explain.

11. XYZ shares are traded at \$100 now and their price can go up or down by 20% each year for the next two years, with equal probabilities. XYZ will not pay dividends for the next three years. The current risk-free interest rate is 2% per year for all maturities.

(a) Construct a binomial tree model for the stock price of XYZ for the next two years.

(b) Construct a portfolio of XYZ shares and 1-year risk-free bond that replicates a 1-year European put option on 1 share of XYZ with a strike price of \$95. What is the cost of the replicating portfolio?

(c) Compute the risk-neutral probabilities for each node along the binomial tree.

(d) Compute the price of the Arrow-Debreu claim on the time-2 state in which the stock price equals \$144.

(e) Write down the payoff of a 2-year option on 1 XYZ share which pays, at time 2, the maximum price level of XYZ reached during the 2-year period. Use the risk-neutral method to price this option.

three years: 0, 1, 2

12. Rush-and-Cash's stock is currently traded at \$100 per share. The stock price will either go up or go down by 20% in each of the next two years. The annual interest rate is 10%. Assume the stock pays no dividends the next two years.

(a) What is the price of a two-year European call option written on the stock with the strike price \$100? How many shares of stocks do you need for each node to replicate the payoff of the call option? What is the price of a two-year European put option with the same strike price?

(b) What is the price of a two-year American call option written on the stock with the strike price \$100? What is the price of a two-year American put option with the same strike price?

(c) You sold one call option in year 0 for hedging. Assume that you are now in year 1. You have just obtained some analysis on Rush-and-Cash. According to the analysis, Rush-and-Cash is highly likely to announce an unexpected loss next year, making a stock move down more likely. The stock is currently traded at \$120 ("up" state). How do you want to change your hedging position?

13. Consider 1-year European options on XYZ shares, which are now traded at \$100 in the market. XYZ pays no dividends for the next two years. The current risk-free interest rate is 2%.

(a) Suppose that the price of the European call with strike price of \$100 is \$5. What should be the price of the put with the same strike price?

(b) What should be the price of a 1-year American call option with a strike of \$100? Explain.

(c) Draw the payoff diagram for a portfolio of European options consisting of: (1) long two calls with a strike price \$80, (2) short one call with a strike price \$100, and (3) short one call with a strike price of \$120.

14. Suppose company A issued 1-year European digital call and put options at a series of strike prices, K (i.e., the digital calls pay \$1 at maturity if, at expiration, stock price of A is greater than K and the digital put options pay \$1 if stock price of A is less than K). Assume that risk-free rate is 3% per year (EAR).

(a) Under the assumption that no arbitrage exists, what would you expect the total cost of buying a package of one digital call with $K = \$65$ plus one digital put with $K = \$65$ to be? Why?

(b) Relative to your answer in part (a) above, is it more or less expensive to buy a package of one digital call struck at K=\$70 plus one digital put struck at \$65? Explain.

(c) Relative to your answer in part (a) above, is it more or less expensive to buy a package of one digital call struck at \$65 plus one digital put struck at \$70? Explain.

15. Consider two investors, Bob and Susan. The two investors are offered a choice between two lotteries, A and B. A pays \$100 for sure. B pays \$80 or \$130 with equal probabilities. Assume that both Bob and Susan make choices using the expected utility framework. Both of them are risk averse, and Bob is more risk averse than Susan.

(a) Suppose Susan prefers lottery A to lottery B. Which of the two lotteries do you think Bob will choose? Explain.

(b) Susan is offered another choice, lottery A vs lottery C. Lottery C pays \$80 or \$100 or \$130, with equal probabilities. Which of the two lotteries do you think Susan will prefer? Explain.

16. Analog Gold Company owns a gold mine which can produce 1 million oz of gold and it takes one year to extract the gold at a cost of \$1000 per oz. The gold price now is \$1200 per oz and it will either increase to \$1600 per oz or decrease to \$900 per oz with equal probability next year and stay there forever. Assume that the gold price risk is totally diversifiable. The risk-free rate is 2%.

(a) What is the NPV of this gold mine if the company extracts the gold now?

(b) If the company can wait for a year to decide and they can abandon the gold mine, what is the optimal strategy and how much is this abandonment option worth?

17. Consider two companies A and B in the mining industry. Their expected stock returns are 20% and 18%, respectively. The debt to equity ratios (D/E) of these companies are 3.19 for company A and 2.39 for company B. Both firms maintain a constant debt/equity ratio. Assume IID growth of pre-tax cash flows for both firms.

The risk-free rate is 2% and expected return of market portfolio is 6%. There is no tax and all the bonds of A and B have $\beta = 0.1$.

(a) Assume that the CAPM holds. What is the β of assets for these two companies respectively?

(b) Suppose that we need to value the stock of the third company C also in mining industry. We can use company A and B as comparable peer companies (with similar assets) by taking the average of the expected returns on their assets. The debt to equity ratio of company C is 0.8 and the debt is risk-free. What should be the proper discount rate to valuing the stock of company C?

18. A company's assets are worth \$100 (million) in the up state and \$10 in the down state next year. Both states are equally likely, and all investors are risk-neutral. If bankruptcy occurs, e.g., because debt is not repaid in full, a fixed cost of bankruptcy of \$5 is incurred. Assume that the risk-free rate is zero.

(a) What is the value of the un-levered, i.e., all equity financed firm?

(b) What is the value of the company if there is an outstanding debt with a face-value of \$30?

(c) Maintain the debt assumption. Suppose now a hedge exists which allows the firm's total payoffs to be transformed to \$80 in the up state and \$30 in the down state. (That is, hedge pays $-\$20$ in the update and $+\$20$ in the down state.) What is the value of the company if the hedge is used? Will management undertake this hedge?

19. Company ABC has \$110 (million) in total assets now. It has a zero-coupon bond with \$100 face value outstanding, which is traded at \$70. ABC's equity is traded at \$40. Suppose that there are two states next year, a and b (or good and bad), which are equally likely. In the good state, ABC's asset will be worth \$200. In the bad state, ABC's asset will be worth \$50.

(a) Write down the pay-offs for the good and bad state for both the bond and equity of ABC. Assuming no arbitrage pricing, calculate the state price ϕ_a and ϕ_b for these two states and the risk-free interest rate r_F.

(b) Suppose that company ABC has a project which will pay \$100 in the good state and \$50 in the bad state. The initial investment of this project is \$50. What is the NPV of this project?

(c) Suppose that the CEO is acting on behalf of the shareholders. Is it desirable for the CEO (and thus the shareholders) to undertake the project via pure equity financing? Please explain.

(d) Suppose the CEO considers issuing debt of face value \$65.6 instead to finance the project. This debt would have the same seniority as the existing debt, so that in states where there are insufficient funds to pay the total face value outstanding, each debt holder receives a share pro-rated to the face-value of his debt, i.e., if there is \$1 in asset value but \$3 in outstanding face-value, each dollar in face-value receives \$1/3. Write down the pay-offs for the new debt in the good and the bad states, respectively. Is the face-value of \$65.6 enough to raise the required investment of \$50?

(e) Following part (d), assume that the company has issued the equal-seniority debt of \$65.6 face value and undertakes the project. Write down the pay-offs in the good and the bad state for the existing debt, i.e. debt of face-value \$100, and equity. Calculate the value of the existing debt and equity. Have the values changed from their original values of \$70 and \$40, respectively? Will the shareholders approve the investment under equal-seniority debt financing?

20. You currently have a long position of 30,000 units in 5-year Treasury notes (\$100 face value) with the coupon rate of 10%. You want to use STRIPS futures to hedge away the interest rate risk of your T-note position. The STRIPS futures mature in 2 years, and at maturity, each future contract delivers 10 units of the STRIPS (\$100 face value) of 8-year maturity from now (6-year maturity at the point of the delivery). The term structure of interest rates is flat at 4%.

(a) Compute the future price per contract.

(b) Describe an explicit position in the STRIPS futures you would take to avoid the interest rate risk.

21. On 05/25/2015, the JUL 2015 Corn Futures contract is traded at 362 pricing units per contract, and the DEC 2015 contract is traded at 379 pricing units per contract on CME. The contract specification is such that each contract consists of 5,000 bushels, and the pricing unit is cents per bushel.

(a) What are the prices of the two futures contract in terms of dollars?

(b) Suppose interest rate is zero, what is the net convenience yield (annually compounded) of corn?

22. Assume perfect markets: no transaction costs and no constraints. In addition assume that the one-month risk-free interest rate will remain constant over a three-month period. Two futures contracts with two and three months maturity are traded on a financial asset without any intermediate payout. The price for these contracts are $F_2 = \$100$ and $F_3 = \$101$, respectively.

yield = 0 in future price model

(a) What is the implied spot price of the underlying asset today?

(b) Suppose the true risk-free rate is consistent with F_2 and F_3, but the financial asset is traded at \$97 in the market today. Construct an arbitrage strategy by the transactions of the two-month futures contract and the financial asset in the spot market so that you will realize an arbitrage gain of \$100 in two months. Specifiy the number of shares to buy/sell in the spot market and with the futures contract.

European puts:
Strike \$40 \$50 \$60 \$70
Price \$2 \$5. \$9 \$12

Solutions

1. Portfolio $P(70) - 2P(60) + P(50)$ gives a butterfly spread with a non-negative payoff at maturity.

At maturity: when the stock price $S > 70$, the payoff is 0. When $70 \geq S > 60$, the payoff is $70 - S \geq 0$. When $60 \geq S > 50$, the payoff is $70 - S - 2(60 - S) = S - 50 > 0$. When $50 \geq S$, the payoff is $70 - S - 2(60 - S) + 50 - S = 0$. So the payoff at maturity is always non-negative. arbitrage in this market

Today, constructing this portfolio generates $9 \times 2 - 5 - 12 = \$1$ at time 0. So this is an arbitrage strategy.

2. There is an arbitrage. The price of the futures contract should be:

F/(1 + Rf) = P/(1 + D) (1+.04)/(1+.03) future dividend paid according to FV and reinvest

(F + D*P)/(1 + Rf) = P $F = (1 + .04 - .03) * 2000 = 2020$ future dividend paid according to PV

Thus the price of the futures contract is too high. We should sell a futures contract, borrow money at the risk free rate to buy one share of ABC for \$2000, and wait. At the end of the year, we have to pay $2000 * (.04)$ in interest, but we collect our dividend of $.03 * 2000$, and sell the stock for 2040, leaving us with a profit of \$20.

dividend is paid according to PV, not FV

3. (a) Having a dollar with certainty at time 1 is worth $0.5 + 0.2 + 0.1 = 0.8$ today. So the risk-free rate is:

$$r_F = \frac{1}{0.5 + 0.2 + 0.1} - 1 = 0.25.$$

(b) The expected return will be,

$$\hat{r} = \frac{1/3(1 + 1 + 3)}{0.5 + 0.2 + 3 * 0.1} - 1 = 0.66667$$

BookValue(t + 1) = BookValue(t) + ROE * BookValue(t) * PlowBackRate
BookValueGrowRate = ROE * PlowBackRate
BookValueGrowRate = EarnGrowRate = DividendGrowRate = ReinvestGrowRate

4. (a) We have

ROE: Return on equity
Earn = ROE * BookValue

$$\begin{aligned} g &= ROE\,(1 - p) \\ &= 11.66\% \times (1 - 56\%) \\ &= 5.13\% \end{aligned}$$

(b) The model says that $P = \frac{D_1}{k-g}$, where k is the required return on GM's equity. We first need to find next year's dividend D_1:

Dividend = Earn * PayBackRate

$$\begin{aligned} D_1 &= B \cdot ROE \cdot p \\ &= 22.71 \times 11.66\% \times 56\% \\ &= 1.48 \end{aligned}$$

Now, we use the current price P, the growth rate g from part a, and D_1 to find k:

$$\begin{aligned}
P &= \frac{D_1}{k-g} \\
\Rightarrow k &= \frac{D_1}{P} + g \\
&= \frac{1.48}{36.20} + 5.13\% \\
&= 9.22\%
\end{aligned}$$

(c) According to CAPM,

$$\begin{aligned}
k &= r_f + \beta \left(E\left(r_M \right) - r_f \right) \\
&= 3\% + 1.02 \times 6\% \\
&= 9.12\%
\end{aligned}$$

The required rate of return estimated from the Gordon Growth Model and the CAPM are quite close.

5. (a) Inconsistent. A higher beta requires a higher expected return.

(b) Inconsistent. Portfolio A lies above the CML. This would suggest that the market portfolio is inefficient.

(c) Inconsistent. Portfolio A lies above the CML. This would suggest that the market portfolio is inefficient.

(d) Consistent - the expected return on A is the risk-free rate, plus A's β multiplied by the market risk premium 8%.

(e) Inconsistent. The implied risk-free rate would be negative if A lies on the SML. If you assume that the interest rate can be negative, then it's consistent.

6. (a) According to CAPM, the expected return of market portfolio should be $0.5\% + 8\% = 8.5\%$.

(b) If $\beta = 0$, we have expected return 0.5% (same as the risk-free rate).

(c) If the stock's β is -0.5 now, based on CAPM, the expected return should be -3.5%. But, we expect the stock return from the current stock market price to be $\frac{4}{40} = 10\%$. In order to get a -3.5% return, the current stock price should be higher than 40. The stock is therefore undervalued.

7. (a) Using CAPM, $\bar{r}_A = r_F + \beta \times (\bar{r}_M - r_F)$. Thus,

$$\bar{r}_A = 0.03 + 0.8 \times 0.04 = 0.062 = 6.2\%$$

(b) $E[S_A] = 50 \times (1 + 0.062) = \$53.10$

(c) $E[S_A] = 50 \times (1 + 0.062) - 1 = \$52.10$

8. III can be described as a portfolio of 3 assets A, B, C (assuming the three subsidiaries form the entire III activity). The investment in A, B, C is respectively 40%, 40% and 20% of total portfolio value (III's market value).
Therefore, we can write:

$$\beta_{III} = \frac{2}{5} \times \beta_A + \frac{2}{5} \times \beta_B + \frac{1}{5} \times \beta_C$$
$$\Rightarrow \beta_C = 5 \times 1.0 - 2 \times (0.8) - 2 \times (1.4) = 0.6$$

The cost of capital of subsidiary C, using CAPM, is:

$$\overline{r_C} = r_F + \beta_C(\overline{r_M} - r_F) = 6\% + 0.6 \times 6\% = 9.6\%$$

suppose rational investor

9. (a) Richard invests his wealth in a portfolio on the CML. The tangent portfolio is the risky part of Richard's investment. The tangent portfolio allocates $\frac{25\%}{25\%+15\%} =$ 62.5% of total value to L and the rest (37.5% of total value) to S. The expected return of the tangent portfolio can be computed from

$$r_T = 2\% + \frac{1}{0.4}(8\% - 2\%) = 17\%\ ,$$

and the standard deviation is

$$\sigma_T = \frac{1}{0.4} 12\% = 30\%.$$

(b) You invest all your money (your current wealth + money you borrow) in the tangent portfolio. Since you are levered, your portfolio holdings are 125% in L, 75% in S and (-100%) in the risk-free asset. The expected return of your portfolio is

$$r = 2\% + 2 \times (17\% - 2\%) = 32\%,$$

and the standard deviation is

$$\sigma = 2 \times 30\% = 60\%.$$

10. (a) Denote w_S as the weight in stock:

$$w_S \cdot 10\% + (1 - w_S) \cdot 3\% = 8\%$$
$$\Rightarrow w_S = 71.43\%$$

Standard deviation:

$$\sigma = \sqrt{w_S^2 \cdot (20\%)^2 + (1 - w_S)^2 \cdot 0 + 2 w_S (1 - w_S) \cdot 0}$$
$$= w_S \cdot 20\% = 14.29\%$$

(b) Yes. The combination of the S&P 500 index fund and the emergin-market fund could potentially improve Sharpe ratio through diversification.

(c) Expected return:

$$E\left(r_A\right) = 0.8 \cdot E\left(r_{SP}\right) + 0.2 \cdot E\left(r_{EM}\right) = 10\%$$

Standard deviation:

$$\begin{aligned} \sigma_A &= \sqrt{0.8^2 \times (20\%)^2 + 0.2^2 \times (30\%)^2 + 2 \times 0.8 \times 0.2 \times 20\% \times 30\% \times 0} \\ &= 17.09\% \end{aligned}$$

(d) It is better in the sense that it has the same expected return but lower standard deviation. That is because portfolio A has the same expected return as S&P 500 index fund but lower standard deviation.

11. (a)

$$\begin{array}{ccccc}
 & & & & 144 \\
 & & & \nearrow & \\
 & & 120 & & \\
 & \nearrow & & \searrow & \\
 & & & & 96 \\
100 & & & & \\
 & & & & 96 \\
 & \searrow & & \nearrow & \\
 & & 80 & & \\
 & & & \searrow & \\
 & & & & 64
\end{array}$$

(b) Payoff of the put:

$$\begin{array}{ccc}
 & & 0 \\
 & \nearrow & \\
P_0 & & \\
 & \searrow & \\
 & & 15
\end{array}$$

Denote A and B as the shares of stock and bond(with face value \$100) in the replicating portfolio:

$$\begin{cases} A \cdot 120 + B \cdot 100 = 0 & \text{(u node)} \\ A \cdot 80 + B \cdot 100 = 15 & \text{(d node)} \end{cases}$$

$$\Rightarrow \begin{cases} A = -0.375 \\ B = 0.45 \end{cases}$$

Cost of the portfolio:

$$A \times 100 + B \times \frac{100}{1.02} = 6.62$$

(c) The tree is symmetric so that the risk-neutral probability would be the same at each node. Denote the probability of going up as q:

$$1.2q + 0.8\,(1-q) = 1 + r_f$$
$$\Rightarrow q = 55\%$$

(d)

$$\begin{aligned} \psi_{uu}\,(1+r_f)^2 &= 1 \cdot q^2 \\ \Rightarrow \psi_{uu} &= \frac{q^2}{(1+r_f)^2} \\ &= \frac{0.55^2}{1.02^2} \\ &= 0.29 \end{aligned}$$

(e) exotic option: the maximum price level of XYZ reached during the 2-year period
in fact, 0, 1, 2 are three years

$$\begin{array}{ccccc}
 & & & & 144 \\
 & & & \nearrow & \\
 & & P_u = \frac{144\cdot q + 120\cdot(1-q)}{1+r_f} = 130.59 & & \\
 & \nearrow & & \searrow & \\
 & & & & 120 \\
P_0 = \frac{P_u\cdot q + P_d\cdot(1-q)}{1+r_f} = 113.67 & & & & \\
 & & & & 100 \\
 & \searrow & & \nearrow & \\
 & & P_d = \frac{100}{1+r_f} = 98.04 & & \\
 & & & \searrow & \\
 & & & & 100
\end{array}$$

12. (a) First we have the tree

The easiest way to compute the price of call option is to use risk neutral probabilities (RNP henceforth). Due to the repeated substructure of the tree with $u = 1.2$ and $d = 0.8$, the RNP for 'up' and 'down' states are given by

$$q_u = \frac{1 + r - d}{u - d} = \frac{1.1 - 0.8}{1.2 - 0.8} = 0.75\ ,\ q_d = 1 - q_u = 0.25\ .$$

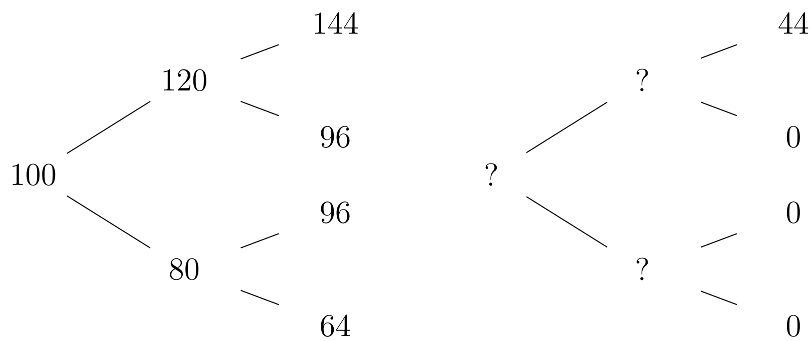


Figure 1: Payoff tree for stock (left) and European call option (right)

Then the payoff at 'up' node in year 1 is given by

$$C_u = \frac{1}{1+r}(q_u \times 44 + q_d \times 0) \; = \; \frac{1}{1.1}(0.75 \times 44) \; = \; 30 \; .$$

The payoff at 'down' node is simply zero as the subsequent nodes have zero payoffs. The payoff at initial node is then

$$C_0 = \frac{1}{1+r}(q_u \times 30 + q_d \times 0) \; = \; \$20.45 \; ,$$

which is the current price of the call option. The price of the European put option can be easily obtained from the put-call parity:

$$P \; = \; C + \frac{K}{(1+r)^2} - S \; = \; 20.45 + \frac{100}{1.1^2} - 100 \; = \; \$3.10 \; .$$

Finally, to obtain the stock positions to replicate the call option, we need to solve

replicate portfolio

$$\begin{aligned} a_u \cdot 144 + b_u \cdot 1.1 = 44 \quad &, \quad a_u \cdot 96 + b_u \cdot 1.1 = 0 \\ a_d \cdot 96 + b_d \cdot 1.1 = 0 \quad &, \quad a_d \cdot 64 + b_d \cdot 1.1 = 0 \\ a_0 \cdot 120 + b_0 \cdot 1.1 = 30 \quad &, \quad a_0 \cdot 80 + b_0 \cdot 1.1 = 0 \; . \end{aligned}$$

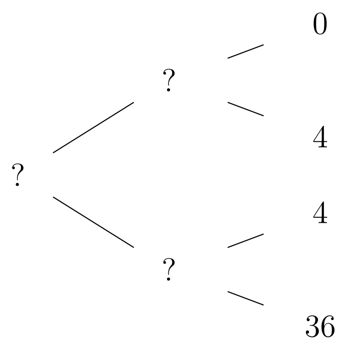
Then

$$a_u = \frac{44}{144-96} = \frac{11}{12} = 0.917 \; , \; a_d = \frac{0}{96-64} = 0 \; , \; a_0 = \frac{30}{120-80} = \frac{3}{4} = 0.75 \; .$$

Hence, the share of stock you need is 0.917 at 'up' node in year 1, is 0 at 'down' node in year 1, and is 0.75 at initial node.

(b) Since there is no dividend payout, the price of American call option should be the same as the price of European call option which is \$20.45. When explicitly computing the payoff tree, you see that it is always optimal not to exercise the call option before the maturity.

The tree for American put option is



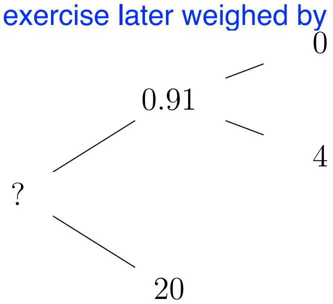
From the put-call parity, the payoff of 'up' node and 'down' node for European put option is given by

$$
\begin{aligned}
P_u &= C_u + \frac{K}{1+r} - S_u = 30 + 90.91 - 120 &= 0.91 \\
P_d &= C_d + \frac{K}{1+r} - S_d = 0 + 90.91 - 80 &= 10.91
\end{aligned}
$$

America option is different from exotic option

The payoff by exercising the American option in year 1 is 0 for 'up' node and 20 for 'down' node. Thus, the American put will be exercised at 'down' node and then the payoff tree becomes

max(exercise now, exercise later weighed by risk-neutral probability)



The price of the American put option at time 0 is

$$P_0 = \frac{1}{1+r}(q_u \times 0.91 + q_d \times 20) = \frac{1}{1.1}(0.75 \times 0.91 + 0.25 \times 20) = \$5.17\ .$$

(c) You don't change your hedging position. As long as the market price does not change, you cannot benefit from your information. In this problem, the stock price has not changed and the RNP for each state in year 2 remains unchanged. Then your hedging strategy does not change.

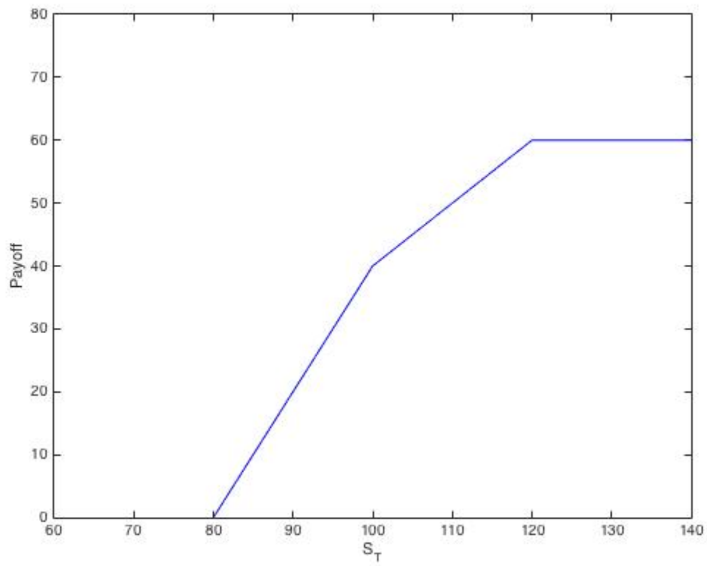
13. (a) By put-call parity,

$$C - P = S - PV\,(K)$$

$$\Rightarrow P = C - S + PV(K)$$
$$= 5 - 100 + \frac{100}{1+2\%}$$
$$= 3.04$$

(b) It should be the same as the European call because the stock does not pay dividend.

(c) The payoff diagram:



14. (a) The combination of put and call with $K = \$65$ will pay \$1 for sure 1 year later which is basically a risk-free bond with \$1 face value. The price should be:

$$\frac{1}{1+r} = \frac{1}{1+3\%} = 0.971$$

The total cost of one digital put plus one digital call option is 0.971.

(b) The payoff in 1 year of one digital call struck at K=\$70 plus one digital put struck at \$65 should be \$0 if the underlying asset's price is between \$65 and \$70 and the payoff is \$1 elsewhere. Compared with part (a), the payoff here is totally dominated by the payoff in part (a). If there is no arbitrage, the price of one digital call struck at K=\$70 plus one digital put struck at \$65 should be less expensive.

(c) The payoff in 1 year of one digital call struck at K=\$65 plus one digital put struck at \$70 should be \$2 if the underlying asset's price is between \$65 and \$70 and the payoff is \$1 elsewhere. Compared with part (a), the payoff here totally dominates the payoff in part (a). If there is no arbitrage, the price of one digital call struck at K=\$65 plus one digital put struck at \$70 should be more expensive.

15. (a) We know that Susan prefers the safe lottery (A) to the risky one (B). As Bob is even more averse to risk than Susan, he would also prefer the safe lottery (A).

(b) Note that lottery C is a composite lottery of lottery A and lottery B, i.e. $C = [A, B; (\frac{1}{3}, \frac{2}{3})]$.

Under the expected utility framework,

$$A \succ B$$

$$A = \left[A, A; \left(\frac{1}{3}, \frac{2}{3}\right)\right] \text{ and } C = \left[A, B; \left(\frac{1}{3}, \frac{2}{3}\right)\right].$$

Then,

$$E[u(A)] = (1/3)E[u(A)] + (2/3)E[u(A)] > (1/3)E[u(A)] + (2/3)E[u(B)] = E[u(C)]$$

Therefore, Susan will prefer lottery A to lottery C.

16. (a) Because the risk of gold price changes is purely idiosyncratic, we discount expected cash flows at the risk-free rate.

$$NPV = \left(-1000 + \frac{E[\text{Gold Price}]}{1 + 2\%} \right) \times 1 \text{ million} \tag{1}$$

$$= -1000 + \frac{0.5 \times (1600 + 900)}{1.02} = 225.49m \tag{2}$$

(b) If the company can wait for a year, they will abandon the gold mine if the gold price is \$900. We have the new NPV:

$$NPV' = \frac{1}{1.02} \times 0.5 \times (-1000 + 1600/1.02) = 278.74m \tag{3}$$

The value of the abandonment option is $NPV' - NPV = 278.74m - 225.49m = 53.25m$

abandon option value benchmark: without abandon = max(must extract now, must extract next year)

17. (a)

$$\beta_{asset} = \beta_E \times \frac{E}{V} + \beta_D \times \frac{D}{V} \tag{4}$$

We have $\beta_D = 0.1$ and we can calculate the β_E from CAPM

$$\beta_E = \frac{R_i - R_f}{R_m - R_f} \tag{5}$$
$$\beta_E^A = \frac{20\% - 2\%}{6\% - 2\%} = 4.5 \tag{6}$$
$$\beta_E^B = \frac{16\% - 2\%}{6\% - 2\%} = 4 \tag{7}$$
$$\beta_{asset}^A = 1.15 \tag{8}$$
$$\beta_{asset}^A = 1.25 \tag{9}$$

(b) The average β_{asset} of A and B are $\frac{1.15+1.25}{2} = 1.2$ so that we have:

$$\beta_E^C = \beta_A^C \times \frac{V}{E} = 2.16 \tag{10}$$
$$R_C = R_f + 2.16 \times (6\% - 2\%) = 10.64\% \tag{11}$$

18. (a) Since the investor's are risk-neutral, we should use the risk-free rate to discount. The value of un-leveled firm should be $V_U = \frac{0.5\times(100+10)}{1+0\%} = 55$.

(b) If we have \$30 face value debt, in the bad state, there is only \$10-\$5=\$5 to pay the debt holder and 0 for equity. The value of the company:

$$V = D + E \tag{12}$$
$$D = \frac{0.5 \times (30 + 5)}{1} = 17.5 \tag{13}$$
$$E = \frac{0.5 \times (70 + 0)}{1} = 35 \tag{14}$$
$$V = 52.5m \tag{15}$$

(c) If the hedge is used, the debt holder can get \$30 at both state and equity holder will get \$50 in the upper state and 0 in the down state.

$$V = D + E \tag{16}$$
$$D = \frac{0.5 \times (30 + 30)}{1} = 30 \tag{17}$$
$$E = \frac{0.5 \times (50 + 0)}{1} = 25 \tag{18}$$
$$V = 55m \tag{19}$$

although overall is better (55>52.5), but equity is worse (25<35)

The value of equity holder decreases after the hedge so that the management won't take it.

19. (a) The payoff for equity is $\{100, 0\}$, and the payoff for debt is $\{100, 50\}$. Thus, we need to solve the following system of equations for q, r_f:

$$
\begin{aligned}
70 &= \frac{1}{1+r_f}\left[100q + 50\left(1-q\right)\right] \\
40 &= \frac{1}{1+r_f}100q
\end{aligned}
$$

Solving, we find $q = \frac{2}{5}$ and $r_f = 0$.

(b) The projects NPV is given by

$$NPV = -50 + \frac{1}{1+r_f}\left[100q + 50\left(1-q\right)\right] = -50 + 70 = 20$$

An alternative way is to recognize that the payoffs are exactly the ones of debt above and thus we immediately have the result.

(c) After the project is undertaken, the payoffs for equity are $\{200, 0\}$, and thus the incremental payoffs to undertaking the project are $\{100, 0\}$. The present value of these payouts is

$$\frac{1}{1+r_f}100q = 40$$

or the students recognized that the payout profile was the same as that of the original equity. Thus, the net present value to shareholders is $NPV = -50 + 40 = -10$. The CEO should not undertake this project with equity financing.

(d) The total face-value of debt of the firm increases to \$166. With the project undertaken, the payoffs of the new debt are given by $\left\{66, 100 \cdot \frac{66}{100+66}\right\}$ where the payoff in the bad state is the pro-rated share of face-value, $\frac{66}{100+66}$, times the payoff in that state, 100. The value of the new debt is given by

$$D_{new} = \frac{1}{1+r_f}\left[66q + 100\frac{66}{100+66}\left(1-q\right)\right] = 50.26$$

and thus is sufficient to raise the investment costs of \$50.

(e) The payoffs to the old debt are simply $D_{old} = \frac{100}{66}D_{new} = \frac{100}{66}50 = 75.76$ (remember we are rounding to the nearest dollar value) which is more than the initial value of \$70, and as such the debtholders are willing to allow equal seniority debt financing. Equity holders, equivalently, have payoffs of $\{134, 0\}$ and thus equity has a value of $\frac{134}{100}E_{initial} = 53.6$ which is higher than their original value of \$40, and thus equity holders are also willing to allow equiaty seniority debt financing of the project.

Note that without rounding, we would have to account for the \$.26 dollars raised beyond the investment cost in the total value of the company, i.e. it is worth $\{300 + .26, 100 + .26\}$. This would not change any of the payoff in any meaningful way.

20. (a) The current value of the underlying asset (8-year STRIPS) is

$$P_S = 10 \times \frac{100}{1.04^8} \approx \$730.69 \ ,$$

PS = P/(1+R)^8
P = (1+R)^8 * PS

and the future price is

$$F = (1 + 4\%)^2 \times P_S \approx \$790.31$$

F/(1+Rf)^2 = PV(F) = PV(ST) = PS
F = (1+Rf)^2 * PS

per contract.

(b) The price of the 5-year T-note is

$$P_{5TN} = \frac{10}{1.04} + \frac{10}{1.04^2} + \frac{10}{1.04^3} + \frac{10}{1.04^4} + \frac{110}{1.04^5} \approx 126.71 \ ,$$

and the duration of the 5-year T-note is

$$D_{5TN} = \frac{\frac{10}{1.04} + \frac{2\times10}{1.04^2} + \frac{3\times10}{1.04^3} + \frac{4\times10}{1.04^4} + \frac{5\times110}{1.04^5}}{P_{5TN}} \approx 4.27 \ .$$

Hence, the value change of the current position per unit rate change is

not face value, but PV

$$\Delta V = -30000 \times P_{5TN} \times \frac{D_{5TN}}{1+r} \approx -\$15.61M \ .$$

The duration of the 8-year STRIPS, which is the underlying asset of STRIPS futures, is 8 year. The duration of the fixed payment for STRIPS futures at maturity is 2 year. Hence, the value change of a unit STRIPS future contract per unit rate change is

earn: normal direction cost: reverse direction

$$\Delta V_{SF} = -P_S \times \frac{8}{1+r} + P_S \times \frac{2}{1+r} = -P_S \times \frac{6}{1+r} \approx -\$4.22K \ .$$

r: return of asset r: risk free rate not F (future price)

Hedging the interest rate risk of the current position requires a short position of $\frac{\Delta V}{\Delta V_{SF}} \approx 3702$ units of STRIPS future contracts.

21. (a)

$$F_{JUL} = 362/100 \times 5000 = \$18,100$$

$$F_{DEC} = 379/100 \times 5000 = \$18,950$$

(b) Denote S_{MAY} as the spot price of corn in May.

today is May

$$\begin{cases} F_{JUL} = S_{MAY}(1 + r - \hat{y})^{2/12} \\ F_{DEC} = S_{MAY}(1 + r - \hat{y})^{7/12} \end{cases}$$

e^(r*T)
OR
(1+r)^T
NOT
(1+r*T)

$$
\begin{aligned}
\Rightarrow \frac{F_{DEC}}{F_{JUL}} &= (1 + r - \hat{y})^{5/12} \\
\Rightarrow \hat{y} &= 1 + r - \left(\frac{F_{DEC}}{F_{JUL}}\right)^{12/5} \\
&= 1 + 0 - \left(\frac{379}{362}\right)^{12/5} \\
&= -11.64\%
\end{aligned}
$$

22. (a) a financial asset without any intermediate payout: yield = 0 in future price model

$$
\begin{cases}
F_2 = S_0^{implied}(1+r)^2 \\
F_3 = S_0^{implied}(1+r)^3
\end{cases}
$$

$$
\Rightarrow 1 + r = \frac{F_3}{F_2}
$$

Therefore,

$$
\begin{aligned}
S_0^{implied} &= \frac{F_2}{(1+r)^2} = \frac{F_2}{(F_3/F_2)^2} \\
&= \frac{100}{(101/100)^2} = \$98.03
\end{aligned}
$$

(b) The spot price is cheaper than the implied spot price. So you would long the asset the spot market and short it with the futures contract. Denote x as the number of shares you buy today. To be neutral to the risk of the asset, it would also be the number of shares that you would short with the two-month futures contract. So you would borrow $\$97 \cdot x$ to buy x shares in the market today. And you would receive $F_2 \cdot x$ buy shorting x shares of the two-month futures contract. The total payoff by the end of the two-month period is $F_2 x - 97x \cdot (1+r)^2$. To realized an arbitrage gain of \$10, we have:

$$
\begin{aligned}
F_2 x - 97x \cdot (1+r)^2 &= 100 \\
x &= \frac{100}{F_2 - 97(1+r)^2} \\
&= \frac{100}{F_2 - 97(F_3/F_2)^2} \\
&= \frac{100}{100 - 97 \times (101/100)^2} \\
&= 95.21
\end{aligned}
$$