15.455x Mathematical Methods of Quantitative Finance

Week 4: Continuous-Time Finance

Paul F. Mende
MIT Sloan School of Management

Finance at MIT
Where ingenuity drives results

Continuous-time finance

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From discrete to continuous time

Discrete-time processes

- Exact
- Complete
- Useful
- Computable
- Extensible

Continuous-time

- Distinguish continuity of time from continuity of price processes
- Applicability: pros and cons
- Limit from discrete to continuous is instructive

Scaling the random walk

Basic building block is **elementary random walk** with multiple steps of unit size

$$\mathbb{E}_t[z_s] = 0, \quad t < s$$
$$\mathrm{Var}(z_t) = 1$$
$$\mathrm{Cov}(z_t, z_s) = \delta_{ts}$$

Define basic RW random variable as sum of many steps

- Shift in origin t0
- Conditional expectation
- Difference of paths

1 represents the size of the time step and big T represents the overall length of the path

$$B_{1,T} = \sum_{t=t_0+1}^{t_0+T} z_t$$

$$\mathbb{E}_t[B_{1,T}] = 0, \quad t \le t_0$$

no realization of random var, only future

$$\mathrm{Var}_t(B_{1,T}) = T$$
$$\mathrm{Var}_{t_1}(B_{1,T}) = T - (t_1 - t_0), \quad t_0 \le t_1 \le T$$

$$B_{1,T'} - B_{1,T} = f(T - T')$$

- Time-translation invariance

only depend on time differences

only things that contribute to the variance are things that are ahead of my observation point. Those are the points that have not yet been observed.

Take it to the limit

Whenever constructing any limiting process, always ask these questions:

- Does the limit converge?

- In what sense does the result represent the converging process?

- Is the process unique?
  - Can multiple processes posses the same limit?
  - Can there be different limits by varying the limiting process?

multiple variables: taking limits in different ways

Scaling the random walk

Study the sum of many steps as scale changes for **time step** and **step size**

Can we subdivide interval while preserving distribution of terminal values?

- **<u>Case I</u>**: rescale time step only:

$$\text{Let } \Delta t = T/n,$$

$$B_{\Delta t,T} \equiv \sum_{t=1}^{n} z_t.$$

$$\mathbb{E}[B_{\Delta t,T}] = 0$$

$$\mathrm{Var}(B_{\Delta t,T}) = n\mathrm{Var}(z_t) = n$$

$$\lim_{n\to\infty} \mathrm{Var}(B_{\Delta t,T}) = \infty$$

Scaling the random walk

Study the sum of many steps as scale changes for **time step** and **step size**

Can we subdivide interval while preserving distribution of terminal values?

- **<u>Case II</u>**: rescale time step and step size:

$$\text{Let } \Delta t = T/n, \quad \epsilon_t \equiv \lambda z_t,$$

$$B_{\Delta t,T} = \sum_{t=1}^{n} \epsilon_t = \lambda \sum_{t=1}^{n} z_t$$

$$\mathbb{E}[B_{\Delta t,T}] = 0,$$

$$\mathrm{Var}(B_{\Delta t,T}) = n\mathrm{Var}(\epsilon_t) = n\lambda^2 \mathrm{Var}(z_t) = n\lambda^2$$

$$\text{Suppose } \lambda = 1/n \to 0 \text{ as } n \to \infty.$$

$$\text{Then } \lim_{n\to\infty} \mathrm{Var}(B_{\Delta t,T}) = n\lambda^2 \to 0.$$

Scaling the random walk

Study the sum of many steps as scale changes for **time step** and **step size**

Can we subdivide interval while preserving distribution of terminal values?

- **Case III**: rescale time step and step size **simultaneously** in specific relationship:

$$\text{Let } \Delta t = T/n, \quad \lambda = \sqrt{\Delta t} = \sqrt{T/n}, \quad \epsilon_t \equiv \lambda z_t$$

$$B_{\Delta t, T} = \sum_{t=1}^{n} \epsilon_t = \sqrt{\Delta t} \sum_{t=1}^{n} z_t$$

$$\mathbb{E}[B_{\Delta t, T}] = 0,$$

$$\text{Var}(B_{\Delta t, T}) = n \text{Var}(\epsilon_t) = n \Delta t \, \text{Var}(z_t) = T$$

$$\lim_{\Delta t \to 0} B_{\Delta t, T} \sim \mathcal{N}(0, T)$$

Properties of the limit

Brownian motion paths

- Everywhere continuous, nowhere differentiable
- Convergence in distribution

Why construct as limit of discrete process?

- Historical
- Conceptual
- Computational
- LLN, CLT, universality

Issues of uniqueness, completeness more subtle

Alternative limiting processes

- Causal structure
- Non-anticipating
- Jumps
- Cadlag, caglad

Full circle: limit of the limit

Consider behavior of Brownian paths that are separated by finite vs. infinitesimal times

- Finite: terminal values normally distributed with variance proportional to time

$$X(t_1, t_2) = B(t_2) - B(t_1),$$
$$X \sim \mathcal{N}(0, t_2 - t_1), \quad t < t_1 \leq t_2$$

doesn't matter what the origin of time is

- Infinitesimal: use process as fundamental building block, analogous to unit step RW

$$\mathrm{d}B_t \sim \mathcal{N}(0, \mathrm{d}t)$$

variance equal to time interval

$$\mathrm{Cov}(\mathrm{d}B_t, \mathrm{d}B_{t'}) = \begin{cases} 0, & t \neq t' \\ \mathrm{d}t, & t = t' \end{cases}$$

$$B(T) = B(0) + \int_0^T \mathrm{d}B_t$$

split non–Gaussian into infinitesimal steps and take sum will yield Gaussian

Stochastic integrals and SDE's

Differential form useful in developing closed-form analytical models

- Stochastic differential equations (SDE) reduce to partial differential equations (PDE)

  don't have any randomness

Integral form useful in Monte Carlo simulations

- Example: generate ensemble of time-dependent price paths, compute solutions as risk-neutral expectations of discounted payoffs

$$V(0) = e^{-rT}\mathbb{E}^Q[V(T)]$$

Scales for drift and volatility

- Use elementary Brownian motion to build prices processes. Recall that for Δt=1,

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) = \mu_0 + \sigma_0 z_t \sim \mathcal{N}(\mu_0, \sigma_0^2)$$

$$\log\left(\frac{S_T}{S_0}\right) = \log\left(\frac{S_T}{S_{T-1}}\frac{S_{T-1}}{S_{T-2}}\cdots\frac{S_1}{S_0}\right) \sim \mathcal{N}(\mu_0 T, \sigma_0^2 T)$$

- So take limit as time step shrinks, holding scaling parameters fixed:

$$\begin{aligned}\log\left(\frac{S_T}{S_0}\right) &= \lim_{\Delta t \to 0}\left[\sum_{t=1}^{T/\Delta t}(\mu \Delta t) + \sum_{t=1}^{T/\Delta t}(\sigma z_t \sqrt{\Delta t})\right] \\ &= \mu T + \sigma \int \mathrm{d}B_t \sim \mathcal{N}(\mu T, \sigma^2 T)\end{aligned}$$

$$\mathrm{d}B_t \sim \mathcal{N}(0, \mathrm{d}t)$$

Scales for drift and volatility

More generally, if drift and volatility depend on time deterministically,

$$\log\left(\frac{S_{t_2}}{S_{t_1}}\right) = \int_{t_1}^{t_2} \mu(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \sigma(t)\,\mathrm{d}B_t$$

Itô processes and Itô's lemma

Itô process

Define an **Itô process** as stochastic process of the form

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t$$

ordinary ways of integrating will fail: dB is random and not differentiable, X won't be differentiable, and we can't take derivatives in the usual way.

How do functions behave?

$$\text{If } \mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t,\ \text{ then what is } \mathrm{d}\left(F(X)\right)?$$

- The usual chain rule would say

$$\mathrm{d}F(t,X) = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X$$

- However since *X* is **nowhere differentiable**, this **does not** hold.

Itô's lemma

Ideas behind proof:

- Although *X* is not differentiable, the function *F* is, so use Taylor's theorem to expand it
- Identify leading order terms in d*t*
- Look for **convergence in probability**, and identify terms with vanishing variance as non-stochastic

$$F'(x) = \lim_{\Delta x \to 0} \left(\frac{F(x+\Delta x) - F(x)}{\Delta x} \right)$$

Replace standard limit with distributional one

introduce randomness

$$\lim_{\Delta x \to 0} \text{Prob} \left[\left(\frac{F(x+\Delta x) - F(x)}{\Delta x} - F'(x) \right)^2 > 0 \right] = 0,$$

$$\lim_{\Delta x \to 0} \mathbb{E} \left[\left(\frac{F(x+\Delta x) - F(x)}{\Delta x} - F'(x) \right)^2 \right] = 0$$

Itô's lemma

Expanding,

take expectation on both side

$$\text{If } \mathrm{d}X_t = a\ \mathrm{d}t + b\ \mathrm{d}B_t, \ \text{then}$$

$$\mathrm{d}F = \left(\frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial X}\,\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\,\mathrm{d}X)^2 \right.$$

$$\left. + \frac{\partial^2 F}{\partial t \partial X}\,\mathrm{d}t\,\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\,\mathrm{d}t)^2 + \mathcal{O}\left((\,\mathrm{d}t)^3, (\,\mathrm{d}X)^3, \cdots\right) \right)$$

$$= \left(\frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial X}\left[a\ \mathrm{d}t + b\ \mathrm{d}B_t\right] + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\left[a\ \mathrm{d}t + b\ \mathrm{d}B_t\right]^2 + \cdots \right)$$

Moments of truth

Compute moments up to fourth order:

$$\mathbb{E}[\mathrm{d}B_t] = 0, \quad \mathbb{E}[(\mathrm{d}B_t)^2] = \mathrm{d}t$$
$$\mathbb{E}[(\mathrm{d}B_t)^3] = 0, \quad \mathbb{E}[(\mathrm{d}B_t)^4] = 3(\mathrm{d}t)^2$$

So for the Itô process,

$$\begin{aligned}
\mathbb{E}[\mathrm{d}X_t] &= \mathbb{E}[a\ \mathrm{d}t + b\ \mathrm{d}B_t] = a\ \mathrm{d}t, \\
\mathbb{E}[(\mathrm{d}X_t)^2] &= \mathbb{E}[(a\ \mathrm{d}t + b\ \mathrm{d}B_t)^2] = a^2\,(\mathrm{d}t)^2 + b^2\ \mathrm{d}t, \\
\mathrm{Var}(\mathrm{d}X_t) &= b^2\ \mathrm{d}t, \\
\mathrm{Var}\left((\mathrm{d}X_t)^2\right) &= \mathbb{E}[(\mathrm{d}X_t)^4] - \mathbb{E}[(\mathrm{d}X_t)^2]^2 = 2b^4(\mathrm{d}t)^2 + \mathcal{O}((\mathrm{d}t)^3)
\end{aligned}$$

Itô's lemma

ignore higher order terms

Since variance of higher terms vanish to order d*t*, treat them as non-stochastic.

$$
\begin{aligned}
\mathrm{d}F &= \left(\frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial X}\,\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\mathrm{d}X)^2 + \cdots \right) \\
&= \left(\frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial X}\left[a\ \mathrm{d}t + b\ \mathrm{d}B_t\right] + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}\left[b^2\ \mathrm{d}t\right] \right) \\
&= \left[\frac{\partial F}{\partial t} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2} \right] \mathrm{d}t + \left[\frac{\partial F}{\partial X} \right] \mathrm{d}X \\
&= \left[\frac{\partial F}{\partial t} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2} + a\frac{\partial F}{\partial X} \right] \mathrm{d}t + \left[b\frac{\partial F}{\partial X} \right] \mathrm{d}B
\end{aligned}
$$

This is the desired result for d*F*, which is therefore also an Itô process.

the differential of F when F is a function of an Ito processis also an Ito process

Itô's lemma

Heuristic: expand and replace
$$(\mathrm{d}B_t)^2 \to \mathrm{d}t,$$
$$(\mathrm{d}X_t)^2 \to b^2\ \mathrm{d}t$$

Then the differential has one additional term beyond the usual chain rule,

$$\boxed{\mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\mathrm{d}t}$$

X is an Ito process

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t$$

not independent of a: dX will expand with a

The **differential** of a function of an Itô process **is itself an Itô process.**

Itô's lemma

$$\mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\mathrm{d}t$$

Examples:

$$\text{Let } \frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}B.$$

$$\text{Then } \mathrm{d}S = (\mu S)\mathrm{d}t + (\sigma S)\mathrm{d}B, \text{ and}$$

$$\mathrm{d}F = \mathrm{d}(\log S) = \left[\mu - \frac{\sigma^2}{2}\right]\mathrm{d}t + \sigma \mathrm{d}B$$

- Lognormal variable
- Geometric Brownian motion
- Same volatility, lower drift

$$\begin{aligned} dF &= d(logS) \\ &= \frac{\partial log(S)}{\partial t} * dt + \frac{\partial log(S)}{\partial S} dS + \frac{(\sigma * S)^2}{2} \frac{\partial^2 log(S)}{\partial S^2} * dt \\ &= 0 + \frac{dS}{S} - \frac{\sigma^2}{2} * dt \\ &= \mu dt + \sigma dB - \frac{\sigma^2}{2} * dt \end{aligned}$$

Itô process dynamics

Generalized random walk: variable scale and variable drift

- General process with coefficients that are integrable functions depending on *X* and *t*.

$$\mathrm{d}X = a(X,t)\mathrm{d}t + b(X,t)\mathrm{d}B$$

- Integrating the differential form gives the distribution from which the path segment is drawn. Constant or time-varying coefficients give **normally distributed** paths

$$\mathrm{d}X = \mu \mathrm{d}t + \sigma \mathrm{d}B, \quad X_{t_2} - X_{t_1} \sim \mathcal{N}\left((\mu(t_2 - t_1), \sigma^2(t_2 - t_1)\right)$$

$$\mathrm{d}X = a(t)\mathrm{d}t + b(t)\mathrm{d}B, \quad X_{t_2} - X_{t_1} \sim \mathcal{N}\left(\int_{t_1}^{t_2} a(t)\mathrm{d}t, \int_{t_1}^{t_2} b(t)^2\mathrm{d}t\right)$$

- Integrate more general differentials, reversing Itô formula

Stochastic differential equations

Insights from **form** of equations (without solving), from **solutions** to equations, or from **transformation** into new equations (e.g., PDE)

Itô processes

Brownian motion with drift

$$\mathrm{d}S_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t$$ integrate both sides

$$S_T = S_0 + \mu T + \sigma(B_T - B_0)$$

unknown

drawn from a Gaussian distribution

- Allows possible negative prices. generally rejected for stock prices
- Is this a problem in practice (i.e., if probability is sufficiently low)?
- Could a large enough drift drift term and sufficiently positive initial value prevent negative prices?

Itô processes

Geometric Brownian motion with drift continuous–time version of our log–normal process, standard model for stock prices

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

refer to Page 21

$$d(\log S_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB_t$$

integrate both sides

$$S_T = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma(B_T - B_0)}$$

$$\int_t d(logS) = \int_t (\mu - \frac{\sigma^2}{2})dt + \int_t \sigma dB$$

$$log(S_T) - log(S_0) = log(S_T/S_0) = (\mu - \frac{\sigma^2}{2})T + \sigma(B_T - B_0)$$

- Excludes negative prices. exponential is always positive
- Drift and variance go to zero as S approaches zero. multiply by S on drift and variance quiet out, thus keep positive (avoid reach origin)
- As a model for asset prices, does not allow for bankruptcy, credit defaults,... stock prices that go to 0
- Is it empirically a good fit for data?

Itô processes

AR 1 process continuous–time analog of the mean reversion process

Ornstein–Uhlenbeck process

deterministic term: restoring to mean

$$\mathrm{d}S_t = \lambda(\bar{S} - S_t)\mathrm{d}t + \sigma \mathrm{d}B_t$$

stochastic term: ordinary random walk

- Unbounded process infinitely large or negative
- Sign of lambda
- **Mean-reversion** dynamics
- Drift term moves *S* toward mean value
- Symmetric around mean value
- Restoring force **proportional** to distance from mean value
- Random shocks unbiased
- Constructed out of simple Brownian plus simple deterministic piece
- Can generalize to let the mean value itself be slowly varying

models for interest rates

Itô processes

Cox–Ingersoll–Ross process Another model used for interest rates

$$\mathrm{d}\rho_t = \lambda(\bar{\rho} - \rho_t)\mathrm{d}t + \sigma\sqrt{\rho_t}\mathrm{d}B_t$$

$$\text{Let } F = \sqrt{\rho}, \quad \frac{\partial F}{\partial \rho} = \frac{1}{2\sqrt{\rho}}, \quad \frac{\partial^2 F}{\partial \rho^2} = -\frac{1}{4}\rho^{-3/2}$$

$$\mathrm{d}F = \left(\frac{4\lambda\bar{\rho} - \sigma^2}{8F} - \frac{1}{2}\lambda F\right)\mathrm{d}t + \frac{1}{2}\sigma\mathrm{d}B_t$$

- **Mean-reversion** dynamics
- **Avoids origin** for $2\lambda\bar{\rho} > \sigma^2$
- Interest rates and term structure

From SDE to PDE: The Black-Scholes equation

Finance at MIT

Where ingenuity drives results

Itô's lemma applied to a special portfolio

V: value of derivative

Consider: Let $V = V(t, S)$ and $\mathrm{d}S = (\mu S)\mathrm{d}t + (\sigma S)\mathrm{d}B$.

S: underlying asset of derivative

$$\mathrm{d}V = \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S}\right)\mathrm{d}S$$

Now combine the two previous in a portfolio:

short \Delta shares of stocks

portfolio value $\pi \equiv V - \Delta S,\quad \Delta$ constant.

$\mathrm{d}\pi = \mathrm{d}V - \Delta \mathrm{d}S$ wrong! from next slide \Delta is not const

$$= \left(\frac{\partial V}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S} - \Delta\right)\mathrm{d}S$$

Only last term is stochastic...and its **coefficient vanishes** for special choice of delta.

Black-Scholes equation

- The **stochastic term vanishes** for the evolution of this portfolio if $\Delta = \frac{\partial V}{\partial S}$

- Since the right-hand side is in general **time-varying**, so is quantity of shares held in hedging portfolio. dynamic hedging: as the market price changes, the exact number of shares delta that we need changes.

- Because there is **no risk** remaining, the portfolio growth rate is risk-free r

$$\mathrm{d}\pi = (r\pi)\mathrm{d}t = r(V - \Delta S)\mathrm{d}t = \left(rV - rS\frac{\partial V}{\partial S}\right)\mathrm{d}t$$

- Equating coefficients of dt

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Black-Scholes equation

This gives the non-stochastic partial differential equation

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0}$$

Black-Scholes equation

What do we know about the solution from the equation alone?

homogeneous equation, linear in V, of first and second partial derivatives

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0}$$

linear: superposition: any linear combination of solutions is also a solution.

to get the exact ones of interest to us for finance, we'll need to apply boundary conditions.

- Explicit parameters:
  - Volatility
  - Risk-free rate
  - Independent of drift rate. (Where did μ go?)

- Implicit parameters: strike price, expiration date, type (call/put/exotic) will be set by boundary conditions