Recitation 4

Options Strategies

As mentioned during the Week 4 lecture, a **bull spread** is an options trading strategy designed to profit from a rise in the price of a stock. We saw an example of a **bull call spread**, which involved the simultaneous purchase and sale of call options with different strike prices. In this case, an investor pays money upfront, and profits later when the call options expire.

We could have also used a **bull put spread**, which instead involves the simultaneous purchase and sale of put options. Specifically, in a bull put spread, the investor purchases an out-of-the-money (OTM) put option with strike price K_1 and sells an in-the-money (ITM) put option with strike price K_2, where $K_2 > K_1$.

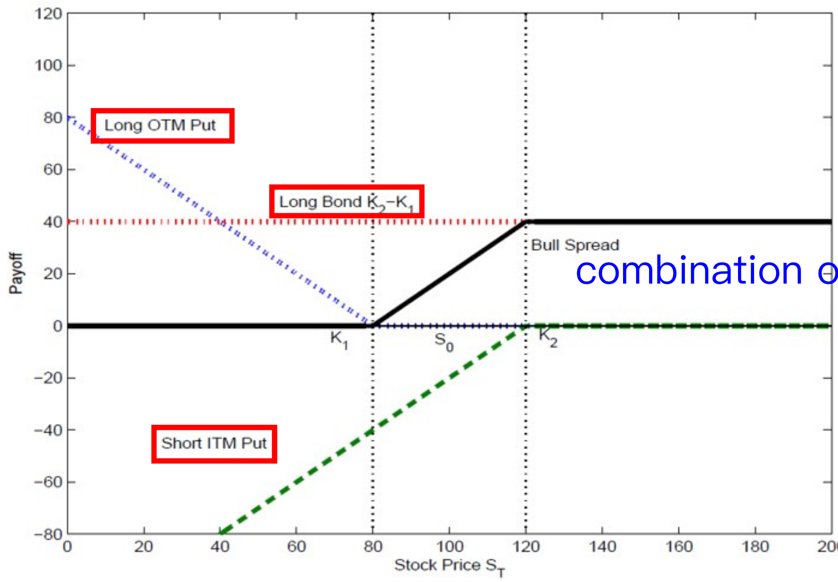
Recall that a **bear spread** involves selling an out-of-the-money (OTM) put option with strike price K_1 and purchasing an in-the-money (ITM) put option with strike price K_2, where $K_2 > K_1$. Thus, a bull put spread has payoffs like that of a short position in a bear spread.

In order to yield the same payoff as the bull call spread, a bull put spread also involves the purchase a zero-coupon bond with face value equal to $K_2 - K_1$. Why? Recall from put-call parity that, given the price of a European call option, we can always find the price of an equivalent European put option as follows:

$$put + S_0 = call + e^{-r\times T} \times K,$$

where S_0 is the spot price of the underlying stock. In other words, the payoff of a portfolio consisting of a call option and a zero-coupon bond with face value K and interest rate r is the same as that of a portfolio consisting of a put option and the underlying stock.

The figure below displays the payoff of the bull put spread as a function of the underlying stock price at maturity:



combination of three components

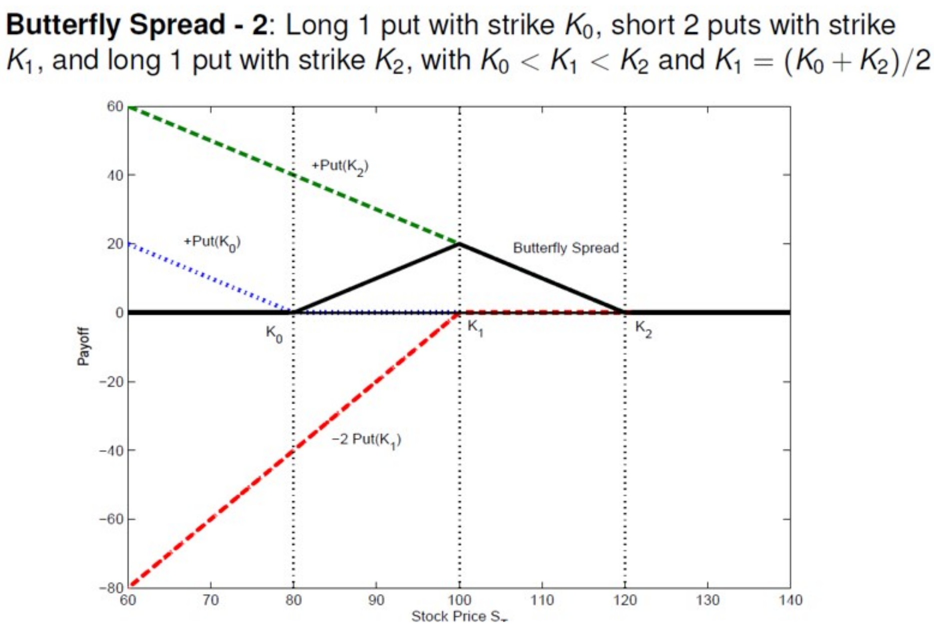
Recall that a **butterfly spread** is an options trading strategy that combines bull and bear spreads to create a market-neutral position with capped profits and losses. In the Week 4 lecture, we saw how to create a butterfly spread using a combination of four call options. Equivalently, we can construct the same butterfly spread using a combination of four put options:

1. A long position in one put option with strike price K_0;
2. A short position in two put options with strike price K_1;
3. A long position in one put option with strike price K_2;

where $K_0 < K_1 < K_2$, and $K_1 = \frac{(K_0+K_2)}{2}$.

Specifically, the above strategy is a **long put butterfly spread** created by buying one put with a lower strike price K_0, selling two at-the-money puts with strike price K_1, and buying one put with a higher strike price K_2. This position achieves a maximum payoff when the underlying stock remains at the strike price K_1 of the intermediate options.

The figure below displays the payoff of the long put butterfly spread as a function of the underlying stock price at maturity:



What if we reversed the positions in the long put butterfly spread? In particular, say we enter:

1. A short position in one put option with strike price K_0;
2. A long position in two put options with strike price K_1;
3. A short position in one put option with strike price K_2.

The payoff from this strategy would look similar to the payoff from the long put butterfly spread, but "flipped" about the horizontal axis in the figure above. This strategy is referred to as a **short put butterfly spread**, and it realizes its maximum payoff when the price of the underlying stock is either above the upper strike price or below the lower strike price at the maturity date.

In other words, entering into a short put butterfly spread is a good idea when you expect the volatility of the stock price to increase, implying a higher likelihood of moving beyond the range of strike prices of the butterfly.

Risk Neutral Pricing

Recall from the Week 4 lecture that there are two main conceptual frameworks for pricing derivatives: (i) replicating portfolios and (ii) risk neutral pricing.

Risk neutral pricing states that the price of any derivative is equal to the expectation of its discounted future payoffs, where the expectation is computed using risk-neutral probabilities and discounting is at the risk-free rate:

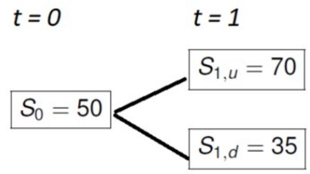
$$Price = E^* \left[e^{-r \times T} \times Payoff \right].$$

Here, $E^* [\cdot]$ denotes the risk neutral expectation, and $Payoff$ is the payoff of the option at maturity date T.

We saw an example of how risk neutral pricing can be used to find the price of a call option.

Can we similarly use it to find the price of a put option?

Suppose that the price of a stock evolves as follows:



Assume that the risk-free rate is 11%. Using risk neutral pricing, what is the price of a put option on the stock with maturity $T = 1$ and strike price $K = 50$?

First, we can find the risk neutral probability q^* that the "up" node occurs at $t = 1$ as:

$$S_0 = q^* \times S_{1,u} \times e^{-r\times T} + (1 - q^*) \times S_{1,d} \times e^{-r\times T}.$$

Solving for q^*:

$$q^* = \frac{S_0 \times e^{r\times T} - S_{1,d}}{S_{1,u} - S_{1,d}}. \tag{1}$$

Plugging in $S_0 = 50$, $S_{1,d} = 35$, $S_{1,u} = 70$, $r = 0.11$, and $T = 1$ into Equation (1), we see that $q^* = \frac{50\times e^{0.11} - 35}{70-35} = 0.5947$. According to the diagram provided, the payoff of the put option is equal to 0 in the "up" node at $t = 1$ and $max(K - S_{1,d}, 0) = 15$ in the "down" node.

Finally, to compute the price of the put option, we simply apply the risk neutral pricing formula:

$$Price = e^{-r\times T} \times E^*\left[Payoff\right] = e^{-0.11} \times \left[q^* \times 0 + (1 - q^*) \times 15\right] = 5.4462.$$

We said that risk neutral pricing can be used to price any derivative, not just options. One example we've seen before is a forward contract.

Recall that a forward contract is an agreement between two parties to buy or sell an underlying asset on an agreed-upon date and price. In the Week 1 lecture, we saw that the forward price of a non-dividend paying stock is given by:

$$F_{0,T} = S_0 \times e^{r\times T}$$

where $F_{0,T}$ is the forward price, S_0 is the current price of the stock, and T is the time to maturity of the forward contract. Can we derive the same formula using risk neutral pricing?

The profit at maturity date T from a long position in the forward contract is given by $S_T - F_{0,T}$. Since it costs nothing to enter into the forward contract, the value of the contract is equal to 0 at initiation. Thus,

$$\begin{aligned} &e^{-rT} \times E^*\left[S_T - F_{0,T}\right] = 0 \Rightarrow \\ &F_{0,T} = E^*\left[S_T\right]. \end{aligned}$$

In a risk neutral world, the return on any stock is equal to the risk-free rate. In the case of a non-dividend paying stock, this implies that:

$$S_0 = e^{-r\times T} E^*\left[S_T\right].$$

Combining these two equations, we arrive at the same expression for the forward price of a non-divided paying stock as in the Week 1 lecture:

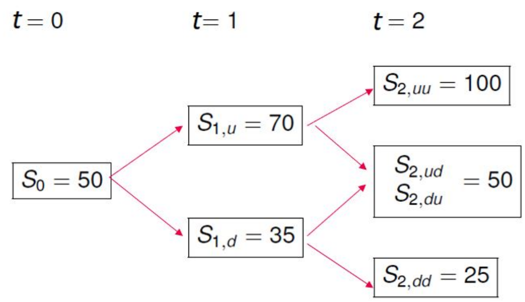
$$F_{0,T} = E^* [S_T] = S_0 \times e^{r \times T}.$$

In other words, *the forward price is the risk neutral expectation of the underlying asset value at T.*

Dynamic Replication

As discussed in the Week 4 lecture, the replicating portfolio approach to pricing an option should imply the same price as the risk neutral pricing approach. In a multi-period setting, however, constructing the replicating portfolio may be cumbersome.

As an example, consider a stock price that evolves according to the binomial tree below:



Using the replicating portfolio approach, we can price a call option with maturity $T = 2$ and strike price $K = 50$ in five steps.

1. Determine the payoff from the call option at each node of the tree at $t = 2$.

Let $c_{2,uu}$ be the payoff of the call option in the "up-up" node at $t = 2$, $c_{2,ud} = c_{2,du}$ be the payoffs in the "up-down" and "down-up" nodes, and $c_{2,dd}$ be the payoff in the "down-down" node. Since the strike price of the call option is 50, $c_{2,uu} = 50$, $c_{2,ud} = c_{2,du} = 0$, and $c_{2,dd} = 0$.

2. Find the position "delta" to invest in stocks for the replicating portfolio at each node of the tree at $t = 1$.

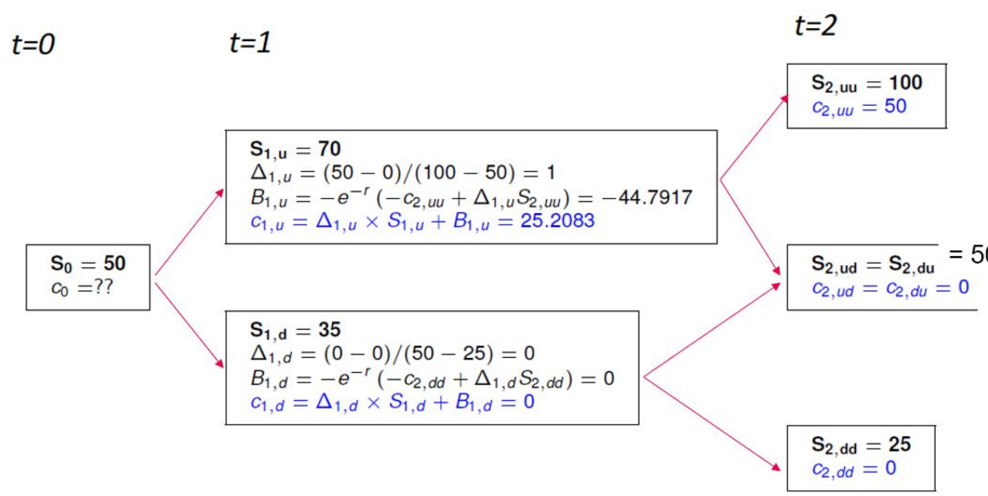
At $t = 1$, $\Delta_{1,u} = \frac{c_{2,uu} - c_{2,ud}}{S_{2,uu} - S_{2,ud}} = \frac{50-0}{100-50} = 1$ in the "up" node, and $\Delta_{1,d} = \frac{c_{2,du} - c_{2,dd}}{S_{2,du} - S_{2,dd}} = \frac{0-0}{50-25} = 0$ in the "down" node.

3. Find the amount of risk-free bonds in the replicating portfolio at each node of the tree at $t = 1$.

Once we know $\Delta_{1,u}$ and $\Delta_{1,d}$, we can compute the amount of bonds as $B_{1,u} = -e^{-r} \left(-c_{2,uu} + \Delta_{1,u} S_{2,uu} \right) = -44.7917$ in the "up" node and $B_{1,d} = -e^{-r} \left(-c_{2,dd} + \Delta_{1,d} S_{2,dd} \right) = 0$ in the "down" node.

4. Find the value of the replicating portfolio at each node of the tree at $t = 1$. By no arbitrage, this is also the value of the call option at each node.

The value of the replicating portfolio—and call option—at $t = 1$ is $c_{1,u} = \Delta_{1,u} \times S_{1,u} + B_{1,u} = 25.2083$ in the "up" node and $c_{1,d} = \Delta_{1,d} \times S_{1,d} + B_{1,d} = 0$ in the "down" node. The following binomial tree summarizes Steps 1-4:



5. Repeat Steps 2-4 for the $t = 0$ node, using the fact that the value of the call option is $c_{1,u} = 25.2083$ in the "up" node at $t = 1$ and $c_{1,d} = 0$ in the "down" node.

The binomial tree below summarizes Step 5. The price of the call option is $c_0 = 13.4294$.