MIT Sloan School of Management
Kogan and Wang

15.415.2x Sample Exam

Grade Sheet

1.	________	/	15
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6.	________	/	15
7.	________	/	20
Total	________	/	120

1. (15 points) Suppose that the price of XYZ goes up or down by a factor of 1.4 and 0.9 respectively, each period. Up and down moves are equally likely, and independent across time. The risk-free rate is 5% per period. The initial price of XYZ is \$100.

Consider a barrier option on XYZ. This option matures at $t = 2$ and pays off as a European put with a strike of \$125, unless the price of the option before $t = 2$ reaches above \$130, in which case the option pays zero at maturity.

(a) (3 points) Compute the risk-neutral probability of the stock price going up.

(b) (3 points) What is the time-0 price of the state-contingent claim paying \$1 in the state with the highest possible stock price at time $t = 2$ and nothing otherwise?

(c) (3 points) Compute the arbitrage-free price of the barrier option at time $t = 0$.

(d) (3 points) What is the expected return on the barrier option between $t = 0$ and $t = 1$?

(e) (3 points) What is the expected return on the replicating portfolio for the barrier option between $t = 0$ and $t = 1$?

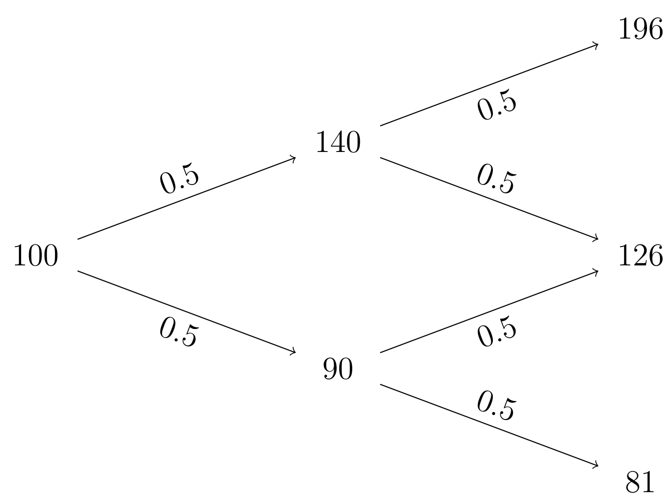
Solutions

(a) We have $u = 1.4$. The risk-neutral probability of the stock price going up q_u satisfies:

$$1 = \frac{1.4q_u + 0.9(1 - q_u)}{1.05}$$
$$q_u = \frac{1.05 - 0.9}{1.4 - 0.9} = 0.3$$

So $q_u = 0.3$.

(b) The binomial tree of the XYZ's price is



The time 0 price of the state-contingent claim described is:

$$\phi_{uu} = \frac{q_u^2}{1.05^2} = \$0.0816327$$

(c) The barrier option only pays out in the down-down state, since in all the other time 2 states, XYZ's share price is above 125. In the down-down state, the price of XYZ is 81, so the option payoff is $125 - 81 = 44$. We end up in this state with risk-neutral probability $(1 - q_u)^2$. So the price of the barrier option at time $t = 0$ is:

$$P_0 = \frac{(1 - q_u)^2 \times (125 - 81)}{1.05^2} = \$19.555556$$

(d) The option price at time 1 is $P_{1,u} = 0$ in the up state, since the payoff at time 2 is then guaranteed to be 0. In the down state, the option price is $P_{1,d} = \frac{(1-q_u)\times(125-81)}{1.05} = \$29.333333$. So the expected return on the barrier option between $t = 0$ and $t = 1$ is

$$\bar{r} = \frac{0.5 \times 0 + 0.5 \times 29.333333}{19.555556} - 1 = -25\%$$

(e) The replicating portfolio yields the same results as the risk-neutral pricing approach: the expected return is -25%.

2. (20 points) You are given the following data on stocks A and B:

	Asset A	Asset B
Expected Return	8%	10%
Standard Deviation	15%	20%

The correlation between stocks A and B is zero. The risk-free rate is 5%.

(a) (3 points) Compute the Sharpe ratios of returns on stocks A and B, respectively.

(b) (3 points) Consider a portfolio P, with 20% invested in stock A, and 80% invested in stock B. What is the Sharpe ratio of this portfolio?

(c) (4 points) What is the highest Sharpe ratio one can achieve using stocks A and B?

(d) (3 points) Is portfolio P mean-variance efficient? Yes/No.

(e) (4 points) What is the lowest possible standard deviation of a portfolio using stocks A and B, and the risk-free asset, with the expected return of 15%?

(f) (3 points) Suppose you can also invest in stock C. This stock has expected return of 6%, standard deviation of 20%, zero correlation with stock A, and correlation of 0.6 with stock B. Can you construct a portfolio using stocks A, B, and C and the risk-free asset with the expected return of 15% and a lower standard deviation than the optional portfolio in item (2e)?

Solutions

(a) Recall the formula for Sharpe ratio:

$$SR = \frac{r_i - r_f}{\sigma_i}$$

So, $SR_A = \frac{8\%-5\%}{15\%} = 0.2$; $SR_B = \frac{10\%-5\%}{20\%} = 0.25$.

(b) The portfolio's expected return is $0.2 \times 8\% + 0.8 \times 10\% = 9.6\%$.

Recall that the correlation between the stocks is zero. So the portfolio's variance is $\sigma_P = \sqrt{0.2^2 \times .15^2 + 0.8^2 \times 0.2^2} = 16.278821\%$.

The portfolio Sharpe ratio is $SR = \frac{9.6\% - 5\%}{16.278821\%} = 0.282576$.

(c) Suppose the weights on A and B are w and $1 - w$ respectively. Then

$$SR = \frac{w \times 8\% + (1 - w) \times 10\% - 5\%}{\sqrt{w^2 \times 0.15^2 + (1 - w)^2 \times 0.2^2}}$$

The solution of $\max_w SR$ is $w^\star = 0.516130$, and $SR_{max} = 0.320156$

(d) No - the portfolio P Sharpe ratio is less than the maximum Sharpe ratio.

(e) For a given expected return, the portfolio with the lowest possible standard deviation is given by a combination of the risk-free asset and the tangency portfolio. Such a portfolio has the maximum Sharpe ratio. So, $\frac{15\% - 5\%}{\sigma} = 0.320156 \Rightarrow \sigma = 31.234752\%$

(f) Yes.

3. (15 points) You are given the following information on three stocks:

Stock number	1	2	3
Expected return	10%	12.5%	9%
Standard deviation of return	15%	30%	25%

Pair-wise correlations between the three stocks are equal to zero. The risk-free interest rate is 4%.

Suppose that the Market portfolio consists of the three stocks above, with equal weights.

(a) (4 points) What is the Sharpe ratio of the Market portfolio?

(b) (4 points) Compute the beta of stock 1 on the market portfolio.

(c) (4 points) Compute the CAPM alpha of stock 1 with respect to the Market portfolio.

(d) (3 points) Is the Market portfolio mean-variance efficient? (Yes/No).

Solutions

(a) The market portfolio expected return is $\bar{r}_M = \frac{1}{3}(10\% + 12.5\% + 9\%) = 10.5\%$.

The market portfolio standard deviation is $\sigma_M = \sqrt{\frac{1}{3^2}0.15^2 + \frac{1}{3^2}0.3^2 + \frac{1}{3^2}0.25^2} = 0.1394$.

So the Sharpe ratio is

$$SR = \frac{\bar{r}_M - r_f}{\sigma_M} = \frac{10.5\% - 4\%}{13.94\%} = 0.4663$$

(b) Stock 1's beta is

$$\beta_1 = \frac{Cov(r_1, r_M)}{Var(r_M)}$$

And we know $Cov(r_1, r_M) = \frac{1}{3}Var(r_1)$. So,

$$\begin{aligned}\beta_1 &= \frac{\frac{1}{3}Var(r_1)}{Var(r_M)} \\ &= \frac{1}{3} \times \frac{0.15^2}{0.1394^2} \\ &= 0.3860\end{aligned}$$

(c) Using the CAPM, the stock 1 alpha is:

$$\alpha_1 = (\bar{r}_1 - r_f) - \beta_1(\bar{r}_M - r_f) = 6\% - 0.3860 \cdot 6.5\% = 3.4910\%$$

(d) No, because stock 1 has positive alpha.

4. (20 points) In 1992, the Western Company was facing the same situation as the Southern Company in the Acid Rain case. The Western Company decided to stay with its existing technology and purchase the needed allowances during phase one of the Clear Air Act. It is now in 1999 (year 0) and phase two of the Clean Air Act will start next year (2000). Western Company is considering the following options:

(A) Stick with the default policy of burning high-sulfur coal without scrubbers and purchasing allowances.

(B) Install scrubbers now (year 0) to cut down emissions from burning high-sulfur coal and buy or sell allowances when necessary.

You also have the following information:

- The current plant will emit 200,000 tons of sulfur dioxide each year. It will run for another 10 years, from 2000 to 2009 (i.e., year 1 to year 10).
- Scrubbers can be installed now at the cost of \$800 million, which will start operating next year (2000, year 1). The scrubbers will remove 90% of the sulfur dioxide emission.
- The price of the allowances is now trading at \$450/ton and is expected to grow at 10% per year forever.
- You are allowed to emit 20,000 tons of sulfur dioxide every year from the 2000 to 2011.
- The cost of scrubbers can be depreciated linearly over the life-time of the plant. It has zero salvage value after 2010.
- Allowances can be traded and held without constraints or frictions.
- Assume a 10% cost of capital. The current risk-free interest rate is 2%.
- Assume the corporate tax rate is 30%.

(a) (10 points) Compute the NPV of installing scrubbers in 1999:

i. (4 points) What is the present value of depreciation from the scrubbers?

ii. (4 points) What is the present value of after-tax savings from scrubbers?

iii. (2 points) What is the NPV of this option, that is, installing scrubbers in 1999 (year 0)?

(b) (10 points) Suppose that in 2000 (year 1), the government may introduce special tax incentives to reduce pollution. Under the new policy, the cost of installing scrubbers can be deducted from firm profits immediately (as opposed to the current ten-year depreciation period). The likelihood of this new policy being introduced is 50%, and this event can be treated as a purely idiosyncratic risk. Assume that all other assumptions remain the same, in particular, the price of allowances, in year 0 and year 1, will not be affected. Compute the market value of the option to install scrubbers in 2000 (year 1):

i. (4 points) What is the NPV in 2000 for installing scrubbers if the government does introduce the special tax incentive by the end of 1999?

ii. (4 points) What is the NPV in 2000 for installing scrubbers if the government does not introduce the incentive?

iii. (2 points) What is the NPV today (1999) of the option to install scrubbers in 2000.

Solutions

(a) The depreciation is $800/10 = 80M$ every year. The total present value of depreciation is

$$PVD = \frac{80 \times 30\%}{1.1} + \frac{80 \times 30\%}{1.1^2} + ... \frac{80 \times 30\%}{1.1^{10}} = 147.4696M$$

The current plant emits 200,000 tons, or 0.2mm tons, of sulfur dioxide each year. Scrubbers remove 90% of this emission. So, the scrubbers remove 0.2×0.9 million tons of sulfur dioxide each year. The value of the savings is the price of allowances per ton each year, multiplied by the tons of sulfur dioxide emission removed, then summed and discounted to present value after tax. So, the total present value of after-tax savings is

$$\begin{aligned} PVS &= \frac{0.2 \times 0.9 \times 450 \times 1.1 \times (1 - 30\%)}{1.1} + ... + \frac{0.2 \times 0.9 \times 450 \times 1.1^{10} \times (1 - 30\%)}{1.1^{10}} \\ &= 567M \end{aligned}$$

Then the NPV of installing scrubbers is

$$NPV = 147.4696 + 567 - 800 = -85.5304M$$

This means the option value of installing scrubbers is zero, since we won't install them.

(b) If the policy is introduced in 2000, then the NPV in 2000 is (remember now the expected year 1 price is 450×1.1)

$$
\begin{aligned}
NPV_{1,u} &= -800 + 800 \times 30\% + \\
&\frac{0.2 \times 0.9 \times 450 \times 1.1^2 \times (1-30\%)}{1.1} + ... + \frac{0.2 \times 0.9 \times 450 \times 1.1^{11} \times (1-30\%)}{1.1^{10}} \\
&= -560 + 623.7M \\
&= 63.7M
\end{aligned}
$$

Note that the present value of depreciation is now $800 \times 30\%$ because the cost of installing scrubbers can be deducted from firm profits immediately.

If the policy is not introduced in 2000, then the NPV in 2000 is:

$$
\begin{aligned}
NPV_{1,d} &= -800 + 147.4696 + \\
&\frac{0.2 \times 0.9 \times 450 \times 1.1^2 \times (1-30\%)}{1.1} + ... + \frac{0.2 \times 0.9 \times 450 \times 1.1^{11} \times (1-30\%)}{1.1^{10}} \\
&= -652.5304 + 623.7M \\
&= -28.8304M
\end{aligned}
$$

We'll install only when the policy is introduced. Recall the probability of the policy being introduced is 0.5. So, the value of the option is

$$NPV = \frac{0.5 \times 63.7}{1.1} = 28.9545M$$

5. (15 points) Still consider the decision Western Company faces in the previous question, Question 4. Let us consider the situation described in part (b). Now, suppose that the government's special tax incentives will change the price of the allowances in 2000, but not its current price. In particular, the price of allowances will drop to \$390/ton if the tax incentives are introduced in 2000, and it will increase to \$600/ton otherwise. The price is expected to grow at 10% per year after that forever. In this case, the risk concerning the introduction of tax incentives can no longer be treated as idiosyncratic. We now have to re-evaluate option (B).

(a) (5 points) Compute the risk-neutral probabilities for the two states concerning the tax incentive, introduced or not, respectively.

(b) (5 points) Compute the NPV of installing scrubbers in 2000 if the tax incentive is introduced.

(c) (5 points) Compute the NPV in 1999 for the option of installing scrubbers in 2000.

Solutions

(a) The risk neutral probability of introducing the policy q satisfies

$$450 = \frac{q \times 390 + (1-q) \times 600}{1.02} \Rightarrow q = 0.6714$$

Then the risk neutral probability of not introducing the policy is $1 - 0.6714 = 0.3286$.

(b) If the policy is introduced, the total present value of after-tax saving in year 1 is

$$
\begin{aligned}
PVS &= \frac{0.2 \times 0.9 \times 390 \times 1.1 \times (1-30\%)}{1.1} + ... + \frac{0.2 \times 0.9 \times 390 \times 1.1^{10} \times (1-30\%)}{1.1^{10}} \\
&= 491.4M
\end{aligned}
$$

So the NPV of installing scrubbers in 2000 if the tax incentive is introduced is

$$NPV_{1,u} = -800 + 800 \times 30\% + 491.4 = -68.6M$$

(c) When the policy is not introduced and price increases to 600 in year 1, the total present value of after-tax saving in year 1 is

$$
\begin{aligned}
PVS &= \frac{0.2 \times 0.9 \times 600 \times 1.1 \times (1-30\%)}{1.1} + ... + \frac{0.2 \times 0.9 \times 600 \times 1.1^{10} \times (1-30\%)}{1.1^{10}} \\
&= 756M
\end{aligned}
$$

Then the NPV of installing scrubbers in 2000 if the tax incentive is not introduced is

$$NPV_{1,d} = 147.4696 + 756M - 800 = 103.4696M$$

So with this option, scrubbers will be installed only when the policy is not introduced in 2000. Without this option, the NPV of installing scrubbers in year 0 is

$$NPV_0 = \frac{0.6714 \times (-68.6) + 0.3286 \times 103.4696}{1.02} = -11.8215M < 0$$

So we'll not install scrubbers in year 0. Then the option value of installing scrubbers in year 1 is

$$NPV = \frac{0.3286 \times 103.4696}{1.02} - 0 = 33.3334M$$

6. (15 points) Consider a setting with two dates, $t = 0, 1$. At time $t = 1$, the value of assets of company ABC is given by the following table:

State	Probability	Asset value
Good	80%	\$90
Bad	20%	\$40

Company ABC has two bonds outstanding: Bond 1 has face value of \$40; Bond 2 has face value of \$10. Both bonds mature at time $t = 1$. Bond 2 is junior to bond 1: in the event of default, holders of Bond 1 must be paid first, and only then do holders of Bond 2 gain a claim on any remaining firm assets.

In addition to the two bonds, the firm has equity. The risk free rate is 5%. There are no taxes.

(a) (3 points) Tabulate the payoffs of the two bonds and equity in the two states at time $t = 1$.

(b) (3 points) Suppose that the market value of ABC's equity at time $t = 0$ is \$28.57. Compute the risk-neutral probability of the "Good" state.

(c) (3 points) Compute the prices of Bond 1 and Bond 2, implied by absence of arbitrage.

(d) (3 points) Compute the promised yields on Bond 1 and Bond 2 at time 0.

(e) (3 points) Compute the time-0 expected returns on Bond 1, Bond 2 and equity.

Solutions

(a) In the good state, both bondholders are paid the full face value of the bond. The remaining value goes to equity holders. In the bad state, bond 1 holders get paid the full amount (40, which is the value of the company's assets in the bad state), but bond 2 holders get paid 0, and equity holders get 0.

	P	A	B1	B2	E
Good	0.8	90	40	10	40
Bad	0.2	40	40	0	0

(b) The risk-neutral probability of the good state q satisfies $28.57 = \frac{q \times 40}{1.05} \Rightarrow q = 0.749963$

(c) B1 is risk free, so it can be discounted and the risk-free rate. Its price is $P_{B1} = \frac{40}{1.05} = \$38.095238$.
Notice that B2 is $\frac{1}{4}$ unit of equity (it is worth 1/4 of the equity payoff in both the good and bad states of the world), so it must be worth 1/4 of the equity value today. So, $P_{B2} = \frac{1}{4} \times 28.57 = \$7.1425$

(d) Using the definition of yields, we have: $y_1 = \frac{40}{P_{B1}} - 1 = 5\%$; $y_2 = \frac{10}{7.1425} - 1 = 40.007\%$

(e) The expected return is the expectation of the value of the bond or equity, divided by its value today (then minus 1). We have: $\bar{r}_{B1} = \frac{40}{38.095238} - 1 = 5\%$; $\bar{r}_{B2} = \frac{0.8 \times 10}{7.1425} - 1 = 12.006\%$; $\bar{r}_E = \bar{r}_{B2} = 12.006\%$

7. (20 points) Your firm is considering investing in a new project. According to the cash flow projections, the project will generate a free cash flow of -\$10M in year 0 (now), followed by \$2M in year 1. From that point, the free cash flow is expected to grow at a rate of 4%. Currently, the term structure of interest rates is flat at 3%. Free cash flow from your project will be taxed at an effective tax rate of 30%.

As a comparable to your project, you've identified a firm ABC. ABC has publicly traded equity and debt. You are given the following market data on ABC:

- Dividend yield (D_1/P_0) of 3%;
- Expected dividend growth rate of 6%;
- Expected return on debt of 4%;
- Debt/equity ratio of 25%.

ABC faces an effective tax rate of 25%.

(a) (4 points) Assuming the Gordon model (constant growth model) for equity of ABC, determine the expected return on ABC's stock.

(b) (4 points) Assume that ABC will maintain a constant debt/equity ratio of 25% forever. Compute ABC's WACC using the data provided.

(c) (4 points) If ABC was financed 100% by equity, what average return would its equity earn in the market?

(d) (4 points) Compute the NPV of the project if it is 100% financed.

(e) (4 points) Suppose that in year 1 (next year) your firm will issue $500,000 in debt backed by the cash flows of the project. This debt will be perpetual, and free of default risk.

i. Compute the tax shield in year 2 from the debt.

ii. Compute the present value of the tax shield from the debt.

iii. Compute the present value of the project, including the tax shield of the debt.

Solutions

(a) From the Gordon Growth model, $P_0 = \frac{D_1}{\bar{r_E} - g} \Rightarrow \bar{r}_E = g + \frac{D_1}{P_0} = 0.06 + 0.03 = 9\%$

(b) Since we assume ABC will maintain a constant debt to equity ratio, we compute WACC using the regular formula, plugging in ABC's current leverage ratio.
$WACC_{ABC} = \frac{0.25}{1+0.25} \times (1 - 0.25) \times 4\% + \frac{1}{1+0.25} \times 9\% = 7.8\%$

(c) If ABC is 100% equity financed, its equity return would be its expected return on assets: $r_A = \frac{0.25}{1+0.25} \times 4\% + \frac{1}{1+0.25} \times 9\% = 8\%$.

(d) If the project is fully financed, then we discount cash flows at the unlevered cost of capital, which is the same as r_A from firm ABC.
$V_U = -10 + \frac{2}{8\% - 4\%} \times (1 - 30\%) = \$25M$

(e) Since the debt is risk free, the interest payment is $500,000 \times 3\% = \$0.015M$ every year. The tax shield in year 2 is $0.015 \times 30\% = \$0.0045M$.
The present value of the tax shield from the debt is

$$PVTS = \frac{1}{1.03} \times \frac{0.0045M}{0.03} = \$0.1456M$$

The present value of the project becomes

$$V_L = V_U + PVTS = 25 + 0.1456 = \$25.1456M$$