Week 6 – Volatility; and other options pricing models

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Topic outline

1. Delta-gamma hedging

2. Other options pricing models
   - Stock indices
   - Currency options
   - Black's model for options on futures

3. Volatility
   - Empirical shortcomings of BSM
   - Implied volatility and the VIX
   - Models incorporating time-varying volatility

Delta-gamma hedging

- We have seen that there are some issues with delta hedging

  (1) We need to rebalance the portfolio frequently, which is expensive with transactions costs

  (2) The hedge can break down when there are large changes in stock prices

- The problems can be alleviated by "delta-gamma" hedging
- This involves adding to the hedge portfolio a security with a positive gamma
  - E.g., a short-term traded option a long position in a liquid short term traded option

- Consider a portfolio *i* which is short the *T*-dated call *Call(S,T)* (like the one embedded in the Capital Protected Note), long *N* stocks, and long N^C of T_1-dated calls, $Call(S, T_1)$

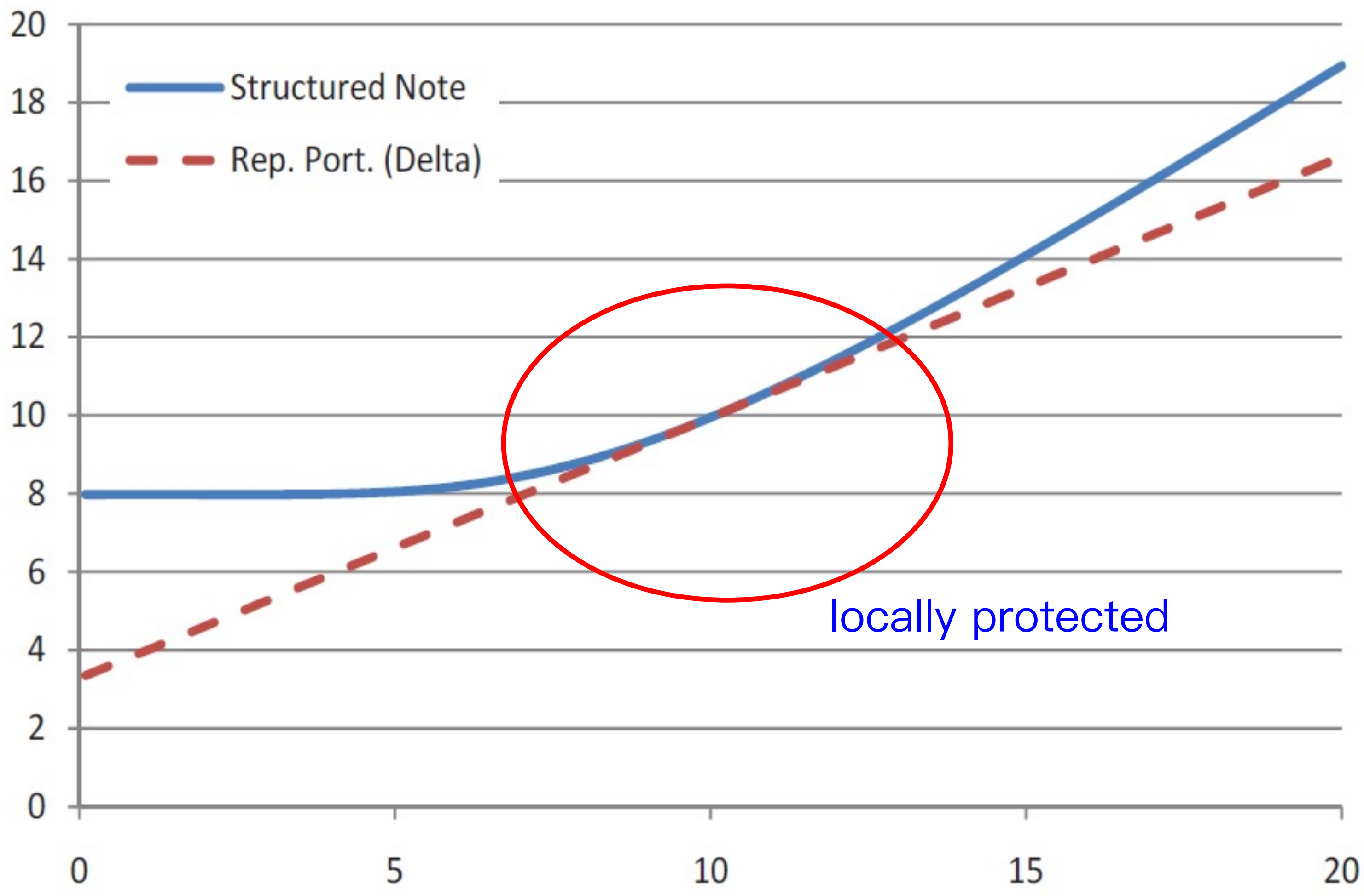
$$i = -Call(S,T) + N \times S + N^c \times Call(S,T_1)$$

short position in the call is being hedged by these two long positions.

We want to hedge both the sensitivity of *i* to changes in the stock price (di/dS = 0) and the change in that sensitivity to changes in the stock price, i.e., the convexity, so that

$$\frac{d\left(\frac{di}{dS}\right)}{dS} = \frac{di^2}{dS^2} = 0$$

Delta hedging: Capital Protected Note



Delta-gamma hedging

- The delta-gamma hedge then requires:

$$\frac{di}{dS} = 0 \Longrightarrow -\frac{dCall(S,T)}{dS} + N + N^c \times \frac{dCall(S,T_1)}{dS} = 0 \quad \text{(Delta Hedging)}$$

$$\frac{d^2 i}{dS^2} = 0 \Longrightarrow -\frac{d^2 Call(S,T)}{dS^2} + N^c \times \frac{d^2 Call(S,T_1)}{dS^2} = 0 \quad \text{(Gamma Hedging)}$$

- Using the notation $\Delta(S,T)$ and $\Gamma(S,T)$ to indicate the Delta and Gamma of the option with maturity T, solving the two equations in two unknowns we obtain:

larger the gamma of the position being hedged relative to the option we're using to hedge it, the larger the size of the hedge position needs to be.

$$N^c = \frac{\Gamma(S,T)}{\Gamma(S,T_1)}; \quad N = \Delta(S,T) - N^c \times \Delta(S,T_1)$$

Note that the position in stocks is smaller (if $N^c > 0$) than in the case of only Delta-hedging, as we now have to also hedge the position in the short-term call option, which is used to hedge against Gamma.

delta hedge is being accomplished by the delta of the short term call option

Delta-gamma hedging: The Capital Protected Note

- For instance, using a 1−year option to hedge the CPN, we have

$$Call(S,T) = 1.7; \quad \Gamma(S,T) = 0.08016; \quad \Delta(S,T) = 0.5747$$
$$Call(S,T_1) = 0.6443; \quad \Gamma(S,T_1) = 0.2575; \quad \Delta(S,T_1) = 0.5512$$

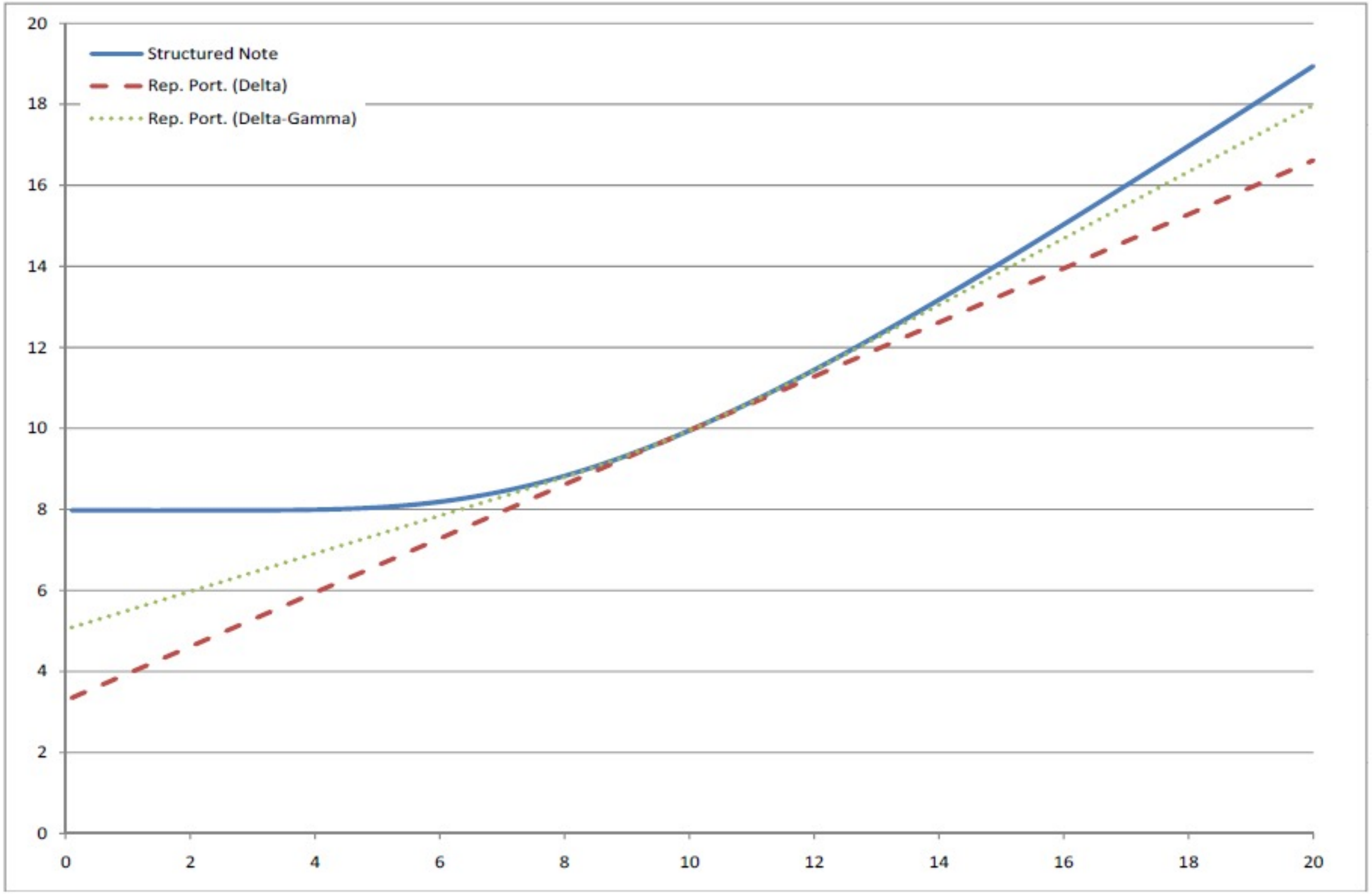
we obtain

$$\text{Position in short-term call} = N^c = 0.3113$$
$$\text{Position in stock} = N = 0.4031;$$
$$\text{Position in bonds} = 1.7 - N \times S - N^c \times Call(S,T_1) = -2.5315$$

The next figure plots the Capital Protected Note for various stock prices, along with the delta hedge portfolio and the delta-gamma hedge portfolio.

you could have accomplished the same delta– gamma hedge using a put option in place of the short term call option and as the source of positive convexity, because the put option has a negative delta, it would have implied that the position in the stock in the replicating portfolio would have actually been larger than it would have been in the case of pure delta hedging

Delta-gamma hedging: The Capital Protected Note



Delta-gamma hedging: The Capital Protected Note

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- The Delta-Gamma hedging strategy allows for larger swings in the stock price before calling for a rebalancing
  - For instance, under Delta hedging, we need to rebalance when $S < 9$ or $S > 11$, as the dashed line starts diverging from solid line
  - Instead, Delta-Gamma requires rebalancing only when $S < 8$ or $S > 13$, as the dotted line is very close to the solid line for a much wider range of prices
- Less frequent rebalancing implies lower transaction costs
  - Of course, now we have more transaction costs because we have to rebalance also the T_1 – option positions
  - We need to use very liquid, exchange traded securities to minimize transaction costs on options

using options as part of the hedge can be more expensive than just trading in the stock for a delta hedging strategy

- Note that the additional benefit of the strategy is that large sudden changes in the stock price (plus/minus 20%) are hedged
- A curiosity: From the figure, the CPN is valued at \$8 for S low. In what sense is this a "capital protected note"? If the investor sells the security when S is low, he/she would not recover \$10.

while the capital protected note does protect the initial capital at the horizon of seven years, it doesn't really offer capital protection along the way.

Other options pricing models

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Stock price indices

A small change in the standard BSM pricing formula for European options accounts for the fact that the dividend stream depresses stock price growth

Rests on observation that stock price distribution at time *T* is the same if:

- The stock starts at price S_0 and pays a dividend at a yield *q*, or
- The stock starts at $S_0 \times e^{-qT}$ and pays no dividend

Pricing formula is given by:

$$c = S_0 \boxed{e^{-qT}} N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / K) + (r \boxed{-q} + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- This formula is often used or pricing options on stock price indices (e.g., S&P 500)

Put-call parity formula is similarly modified to: $c + Ke^{-rT} = p + \boxed{S_0 e^{-qT}}$

- This can be used to solve for the “implied dividend yield q” if you know the price of puts and calls

European currency options

- A similar variation on the BSM formula accounts for the implicit dividend stream on the foreign currency because it can be invested in a risk-free bond paying a rate r_f f stands for foreign
- Here define S_0 is the current spot exchange rate

  S_0 is the value of one unit of foreign currency in U.S. dollars, e.g., 1 euro per 1.1 dollar => $S_0 = 1.1$

- Pricing formula for call is given by:

$$c = S_0 \boxed{e^{-r_f T}} N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / K) + (r \boxed{- r_f} + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- ~~This formula is often used or pricing options on stock price indices (e.g., S&P 500)~~

Put-call parity formula is similarly modified to: $c + Ke^{-rT} = p + S_0 e^{-r_f T}$

Futures options

the dividend yield in this case will be set equal to the risk– free rate

- Options on futures (futures options) also can be valued using a variant of BSM called **Black's Model**
- Futures options have potential advantages over spot options
  - Futures contracts may be easier to trade and more liquid than the underlying asset.
  - Exercise of option does not lead to delivery of underlying asset.
  - Futures options and futures usually trade on same exchange.

- European futures options and European spot options are equivalent when futures contract matures at the same time as the option.

because the futures price always converges to the cash price at the futures contract expiration date

Futures options mechanics

Futures call option

- Right to enter into a long futures contract at a pre-specified futures price
- If exercised holder gets long position in currently priced futures contract plus cash difference between most recent settlement price on futures and strike price on futures option
- Effective payoff is max($F - K$, 0)

just reflects the second item because the currently priced futures contract is always worth 0

Futures put option

- Right to enter into a short futures contract at a pre-specified futures price
- If exercised holder get currently priced short position in futures contract and receives cash difference between strike price and most recent settlement price
- Effective payoff is max($K - F$, 0)

- Popular contracts include agricultural commodities, energy, metals and interest rates
- Most futures options are American

for European options and log normal distributed

Black's Model for valuing futures options

$$c = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2); \quad p = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}; \quad d_2 = d_1 - \sigma\sqrt{T}$$

BSM with dividend, replace S = F_0, delta = r, produce Black's Model

Fischer Black derived this model in 1976

by focusing on the forward price at expiration

- Avoids need to calculate convenience yield or income on underlying asset
- Often the underlying is a forward rather than a futures price
- Very useful in applications beyond futures options

- Risk neutral pricing provides an immediate answer
  1. Compute the risk neutral probability:

$$q^* = \frac{e^{r \times h} - d}{u - d}$$

consider dividend: r*h ==> (r−div_yield)*h

$$\frac{q^* S_0^u + (1 - q^*)S_0^d}{e^{rh}} = S_0 * e^{-\delta h}$$

save and buy later = buy now

Since u, d, h are the same for each step, q^* remains constant.

Black's model for puts and calls:

current future price

$$c = e^{-rT}[\boxed{F_0}N(d_1) - KN(d_2)]$$

$$p = e^{-rT}[KN(-d_2) - F_0N(-d_1)]$$

$$d_1 = \frac{\ln(F_0/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

risk−neutral prob of future:

for future mature at T, long in future equals saving now and purchase later

save amount now is equal to save amount next year and discounted

$$\frac{F_0}{e^{rT}} = \frac{p(uF_0) + (1-p)(dF_9)}{e^{r(T-1)}} * \frac{1}{e^r}$$

$$p = \frac{1-d}{u-d}$$

Portfolio A: a European call futures option plus an amount of cash equal to Ke^{-rT}

Portfolio B: a European put futures option plus a long futures contract plus an amount of cash equal to F_0e^{-rT}, where F_0 is the futures price

$$max(F_T - K, 0) + K = max(K - F_T, 0) + (F_T - F_0) + F_0$$

a futures contract is like a forward contract and settled at the end of its life rather than on a day−to−day basis

put and call worth the same today: $c + Ke^{-rT} = p + F_0e^{-rT}$

note that futures contract in portfolio B is worth zero today

the risk−neutral probability equation and put−call parity imply this model is equivalent to the model for pricing stock options with a constant dividend yield of risk−free rate.

American put-call parity: $S_0e^{-qT} - K < C - P < S_0 - Ke^{-rT}$

left-hand: portfolio 1 at time 0 with value of $S_0e^{-qT} + P_0^{Am}$ has value $S_te^{-qT}e^{qt} + K - S_t <= K$ for $t < T$ if exercised at time t, portfolio 2 at time 0 with value of $K + C_0^{Eu}$ has value $Ke^{rT} + C^{Eu} >= K$ for $t < T$, so the init price should keep the inequality $S_0e^{-qT} + P_0^{Am} <= C_0^{Eu} + K <= C_0^{Am} + K$

right-hand: portfolio 1 at time 0 with value of $Ke^{-rT} + C_0^{Am}$ has value $Ke^{-rT}e^{-rt} + S_t - K <= S_t$ for $t < T$ if exercised at time t, portfolio 2 at time 0 with value of $S_0 + P_0^{Eu}$ has value $S_te^{qt} + P^{Eu} >= S_t$ for $t < T$, so the init price should keep the inequality $Ke^{-rT} + C_0^{Am} <= S_0 + P_0^{Eu} <= S_0 + P_0^{Am}$

For American futures options, the put–call relationship is $F_0e^{-rT} - K < C - P < F_0 - Ke^{-rT}$

constant volatility assumption is a leading candidate for why the Black– Scholes– Merton model often fails to explain the options prices observed in the market.

Volatility

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Does BSM yield option prices similar to the market price of traded options?

The table below compares BSM options prices to market prices at a point in time

- The pattern shown is quite typical
- The data is from May 3, 2007 (a calm period shortly before the global financial crisis that began later that year)
  - The S&P 500 index was at S = 1502.39
  - The one-month risk-free rate was at r = 4.713% (c.c.)
  - The dividend yield on the S&P 500 was about q = 1.91%
- We estimate the volatility using the previous 3 months of returns and find that:

63 trading days in 3 months and 252 trading days in a year

$$\sigma = \sqrt{\frac{1}{63}\sum_{i=1}^{63}(R_{t-i} - \overline{R})^2} \times \sqrt{252} = 12.36\%$$

  - Note the estimate is based only on observations from trading days

procedure is intrinsically backwards– looking, and perhaps traders that were forward– looking came to a different conclusion about the appropriate volatility parameter for the life of the option.

traders' valuations in fact seem conservative or pessimistic: higher value on protection put, lower value on upside call

Example: Comparing BSM predictions to market prices

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time to expiration

SPX (S&P 500 INDEX)	Today	cc rate	div yield	volatility
1502.39	5/3/2007	0.04713	0.0191	0.1236

				CALLS			PUTS		
Maturity	Time to T	Strike	Moneyness K/S	Mkt Price	B/S	BSC/Mkt	Mkt Price	B/S	BSP/Mkt
6/15/2007	0.12	1430	0.952	83.9	80.12	0.955	6.2	3.19	0.514
6/15/2007	0.12	1435	0.955	79.4	75.74	0.954	6.7	3.78	0.564
6/15/2007	0.12	1440	0.958	75	71.44	0.953	7.3	4.46	0.610
6/15/2007	0.12	1445	0.962	70.6	67.24	0.952	7.9	5.23	0.662
6/15/2007	0.12	1450	0.965	66.3	63.14	0.952	8.7	6.10	0.701
6/15/2007	0.12	1455	0.968	62.1	59.15	0.952	9.3	7.08	0.761
6/15/2007	0.12	1460	0.972	57.9	55.27	0.955	10.1	8.17	0.809
6/15/2007	0.12	1465	0.975	53.8	51.52	0.958	10.9	9.39	0.862
6/15/2007	0.12	1470	0.978	49.8	47.89	0.962	11.9	10.74	0.902
6/15/2007	0.12	1475	0.982	45.9	44.40	0.967	12.6	12.22	0.970
6/15/2007	0.12	1480	0.985	42.1	41.05	0.975	14.1	13.84	0.982
6/15/2007	0.12	1485	0.988	38.4	37.84	0.986	15.4	15.61	1.014
6/15/2007	0.12	1490	0.992	34.8	34.79	1.000	17.05	17.52	1.028
6/15/2007	0.12	1495	0.995	31.4	31.88	1.015	18.55	19.59	1.056
6/15/2007	0.12	1500	0.998	28.05	29.13	1.039	20.35	21.82	1.072
6/15/2007	0.12	1505	1.002	24.55	26.54	1.081	21.95	24.19	1.102
6/15/2007	0.12	1510	1.005	22	24.10	1.095	24	26.73	1.114
6/15/2007	0.12	1515	1.008	19.3	21.81	1.130	26.2	29.41	1.123
6/15/2007	0.12	1520	1.012	16.6	19.68	1.186	28.6	32.25	1.128
6/15/2007	0.12	1525	1.015	14.8	17.70	1.196	31.2	35.24	1.130
6/15/2007	0.12	1530	1.018	12.3	15.86	1.290	34	38.38	1.129
6/15/2007	0.12	1535	1.022	10.3	14.17	1.376	37	41.66	1.126
6/15/2007	0.12	1540	1.025	8.6	12.61	1.467	40.3	45.07	1.118
6/15/2007	0.12	1545	1.028	7.05	11.19	1.587	43.7	48.62	1.113
6/15/2007	0.12	1550	1.032	5.95	9.89	1.663	47.4	52.30	1.103
6/15/2007	0.12	1555	1.035	4.5	8.72	1.937	51.2	56.09	1.096
6/15/2007	0.12	1560	1.038	3.7	7.65	2.068	55.2	60.00	1.087
6/15/2007	0.12	1565	1.042	2.9	6.69	2.308	59.4	64.01	1.078
6/15/2007	0.12	1570	1.045	2.325	5.83	2.509	63.7	68.13	1.070
6/15/2007	0.12	1575	1.048	1.9	5.07	2.667	68.2	72.33	1.061

Put out of money when K/S low

large discrepancy between mkt and BSM

Call out of money when K/S high

What could explain the discrepancies?

Terminology

- "**Moneyness**" is the ratio of the strike price to the current stock price

$$\Longrightarrow \; K/S < 1 \Longrightarrow K < S \Longrightarrow \text{puts are OTM, calls are ITM}$$
$$\Longrightarrow \; K/S > 1 \Longrightarrow K > S \Longrightarrow \text{puts are ITM, calls are OTM}$$

- In the table (and in general)
  - For low K/S, BS formula seems to *underprice* both calls and put
  - For high K/S, BS formula seems to *overprice* both calls and put

What could explain the discrepancies?

Could it be the wrong inputs?

- Interest rates and dividends yields are observable and not too variable
  - Moreover, their variation has an opposite impact on calls and puts, but we saw that mispricing went in the same direction for puts and calls
- **Volatility is a more likely suspect**
  - Harder to predict
  - Changes over time
  - Average level of mispricing may be related to the volatility estimate, but that alone cannot explain why BSM underprices OTM puts and overprices ITM puts (and vice versa for calls)

What else could explain mispricing?

- Limits on dynamic hedging
  - Discontinuous stock price paths deviations from log normality
  - Transactions costs
  - Limits to shorting
  - These all mean that pricing cannot be purely by no-arbitrage. Options prices have risk premiums.

Comparing BSM predictions to market prices; volatility revisited

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at the money

Same table as earlier, but volatility chosen to match BSM prices with ATM options prices

- σ = 11.5% accomplishes this, but still have under/overpricing of options at other moneyness

instead of the 12.4

SPX (S&P 500 INDEX)	Today	cc rate	div yield	volatility
1502.39	5/3/2007	0.04713	0.0191	0.115

				CALLS			PUTS		
Maturity	Time to T	Strike	Moneyness K/S	Mkt Price	B/S	BSC/Mkt	Mkt Price	B/S	BSP/Mkt
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6/15/2007	0.12	1455	0.968	62.1	57.95	0.933	9.3	5.88	0.633
6/15/2007	0.12	1460	0.972	57.9	53.99	0.932	10.1	6.89	0.682
6/15/2007	0.12	1465	0.975	53.8	50.15	0.932	10.9	8.03	0.736
6/15/2007	0.12	1470	0.978	49.8	46.45	0.933	11.9	9.29	0.781
6/15/2007	0.12	1475	0.982	45.9	42.88	0.934	12.6	10.70	0.849
6/15/2007	0.12	1480	0.985	42.1	39.47	0.937	14.1	12.26	0.869
6/15/2007	0.12	1485	0.988	38.4	36.21	0.943	15.4	13.97	0.907
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6/15/2007	0.12	1515	1.008	19.3	20.06	1.039	26.2	27.66	1.056
6/15/2007	0.12	1520	1.012	16.6	17.95	1.081	28.6	30.52	1.067
6/15/2007	0.12	1525	1.015	14.8	15.99	1.081	31.2	33.54	1.075
6/15/2007	0.12	1530	1.018	12.3	14.20	1.154	34	36.71	1.080
6/15/2007	0.12	1535	1.022	10.3	12.56	1.219	37	40.04	1.082
6/15/2007	0.12	1540	1.025	8.6	11.06	1.286	40.3	43.52	1.080
6/15/2007	0.12	1545	1.028	7.05	9.70	1.376	43.7	47.13	1.079
6/15/2007	0.12	1550	1.032	5.95	8.47	1.424	47.4	50.88	1.073
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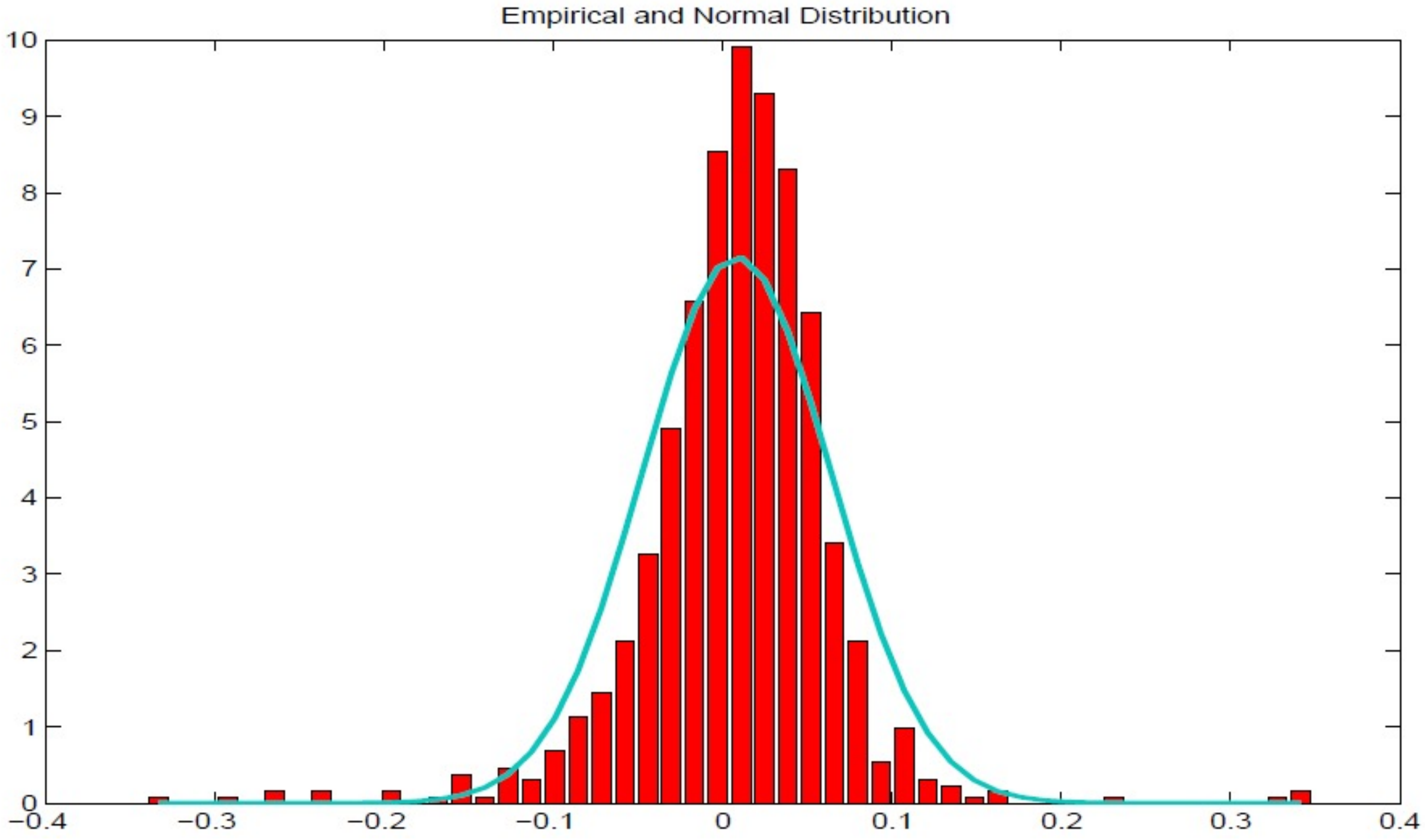
6/15/2007	0.12	1570	1.045	2.325	4.74	2.037	63.7	67.03	1.052
6/15/2007	0.12	1575	1.048	1.9	4.05	2.132	68.2	71.32	1.046

Stock prices deviate from log-normality

- Black and Scholes assume log-normal returns
  - That is, $r_S = \log\left(\left(S_{t+h} + \text{dividends}\right)/S_t\right)$ is normally distributed
- Figure plots the empirical distribution of monthly returns and normal distribution
  $\Longrightarrow$ **Fat Tails**: extreme observations more likely than implied by normal



A closer look at the deviations from log-normality in the tails

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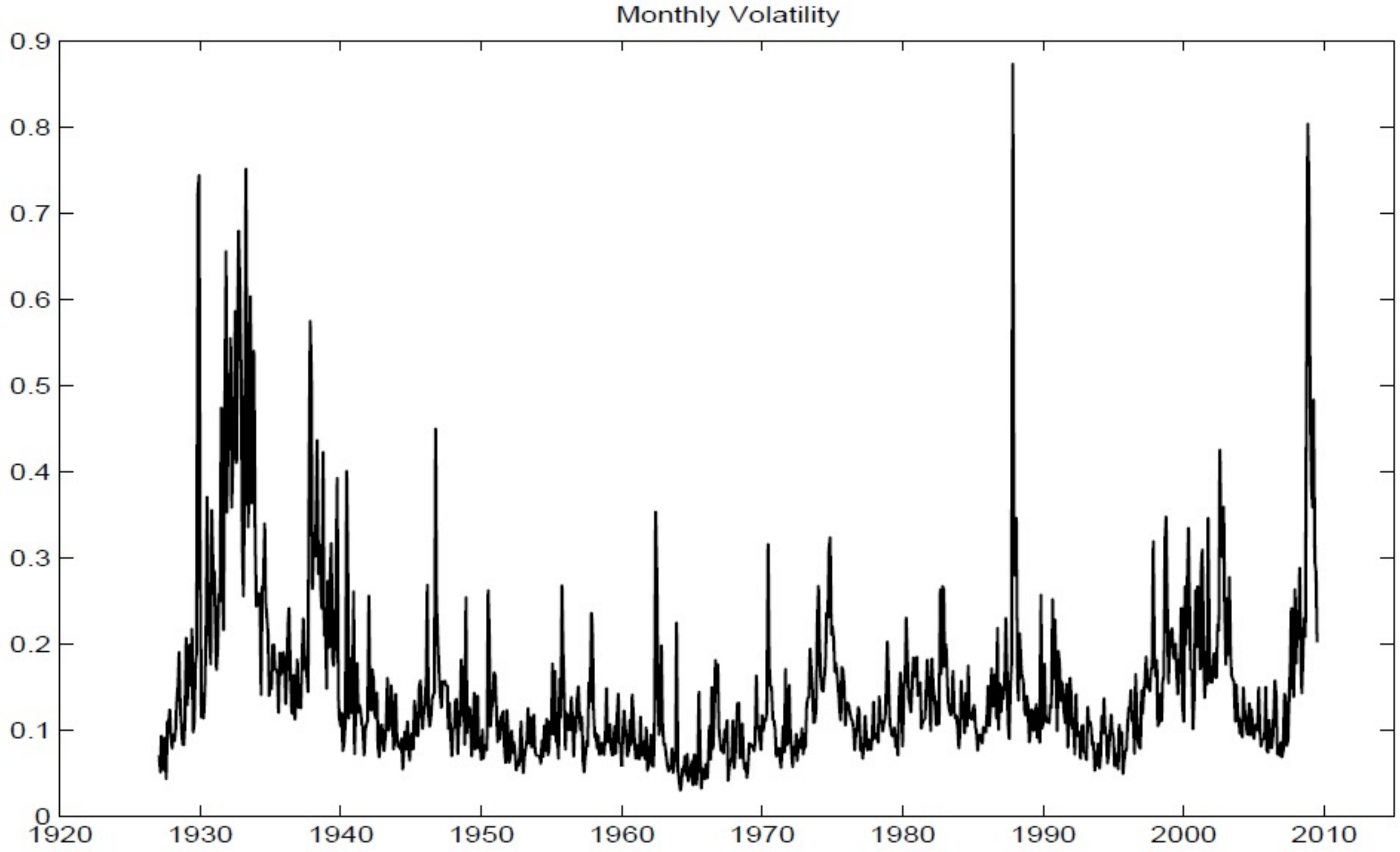
Plus / Minus Sigma Level	Probability of occurring on any given day	How often event is expected to occur	Associated S&P 500 percentage move	Actual S&P 500 occurrences (Jan 1950-2016) vs (expected from normal distribution)
>+-1	31.73%	80 trading days per year	+-0.973%	3534 (expected 5276)
>+-2	4.56%	12 trading days per year	+-1.95%	776 (expected 758)
>+-3	0.27%	1 event every 8 months	+-2.92%	229 (expected 44)
>+-4	6.33×10-3%	Once in 62 years	+-3.89%	98 (expected 1)
>+5	5.73×10-5%	One in 6900 years	+-4.86%	50 (expected 0)
>+-8	1.22×10-13%	Once in 3.2 trillion years	+-7.78%	8 (expected 0)
>+-9	2.25×10-17%	Twice in 20000 trillion years	+-8.76%	7 (expected 0)

Source: Seeking Alpha

https://seekingalpha.com/article/3959933-predicting-stock-market-returns-lose-normal-and-switch-to-laplace

BSM constant volatility assumption is violated in the data

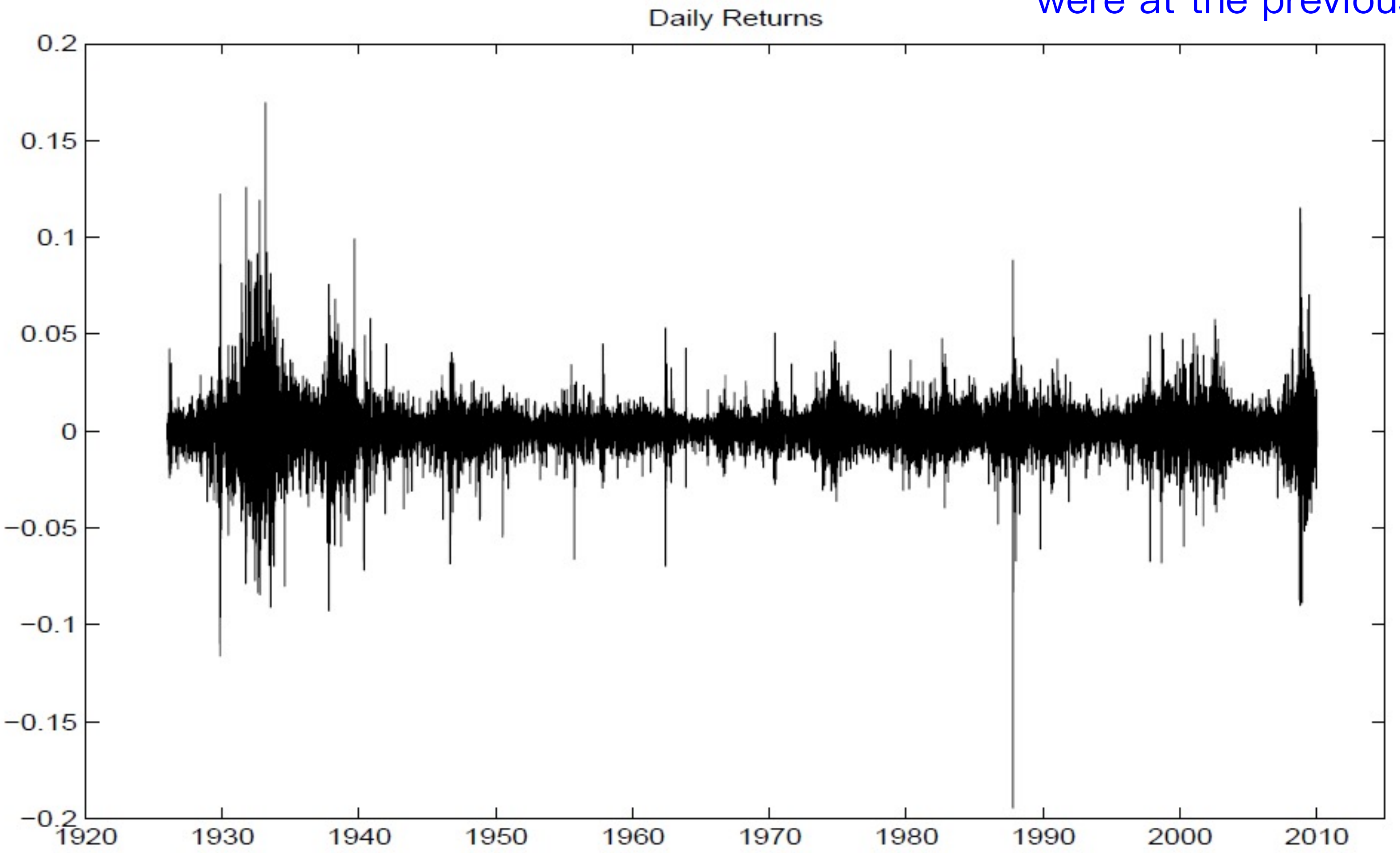
- Black and Scholes model assumes the volatility of stock returns is constant
- But **volatility is in fact stochastic**: it moves randomly over time



BSM assumption of continuous prices is violated in the data

- Black and Scholes model assumes trading is continuous, and prices do not jump
- But **prices sometimes jump discretely** to other levels

reopen prices can be far different than they were at the previous day's close.



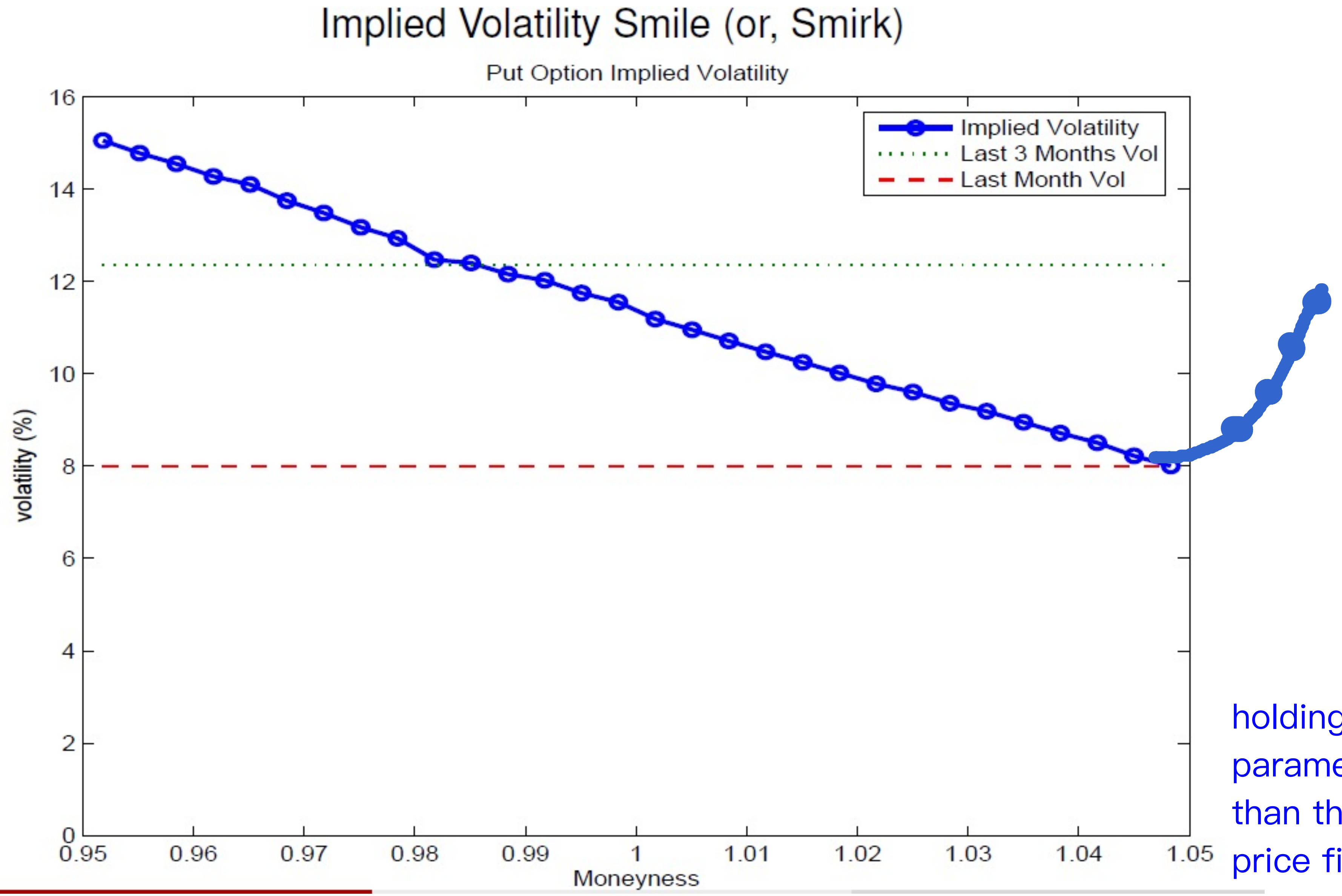
BSM Implied Volatility

- **Implied Volatility:** *The level of volatility σ that once inserted into Black and Scholes formula, it matches the value of a traded option*
- For instance, let $put^{mkt}(K, T)$ be a traded put price at strike price K and time to maturity T
  - E.g. on May 3, $put^{mkt}(1500, .12) = \$20.35$
- Define:

$$\sigma_{\mathbf{Imp}} \text{ is chosen such that } put^{mkt}(K, T) = BSP(S_0, K, T, r, q, \sigma_{\mathbf{Imp}})$$

- Every option has a potentially different implied volatility σ_{Imp}

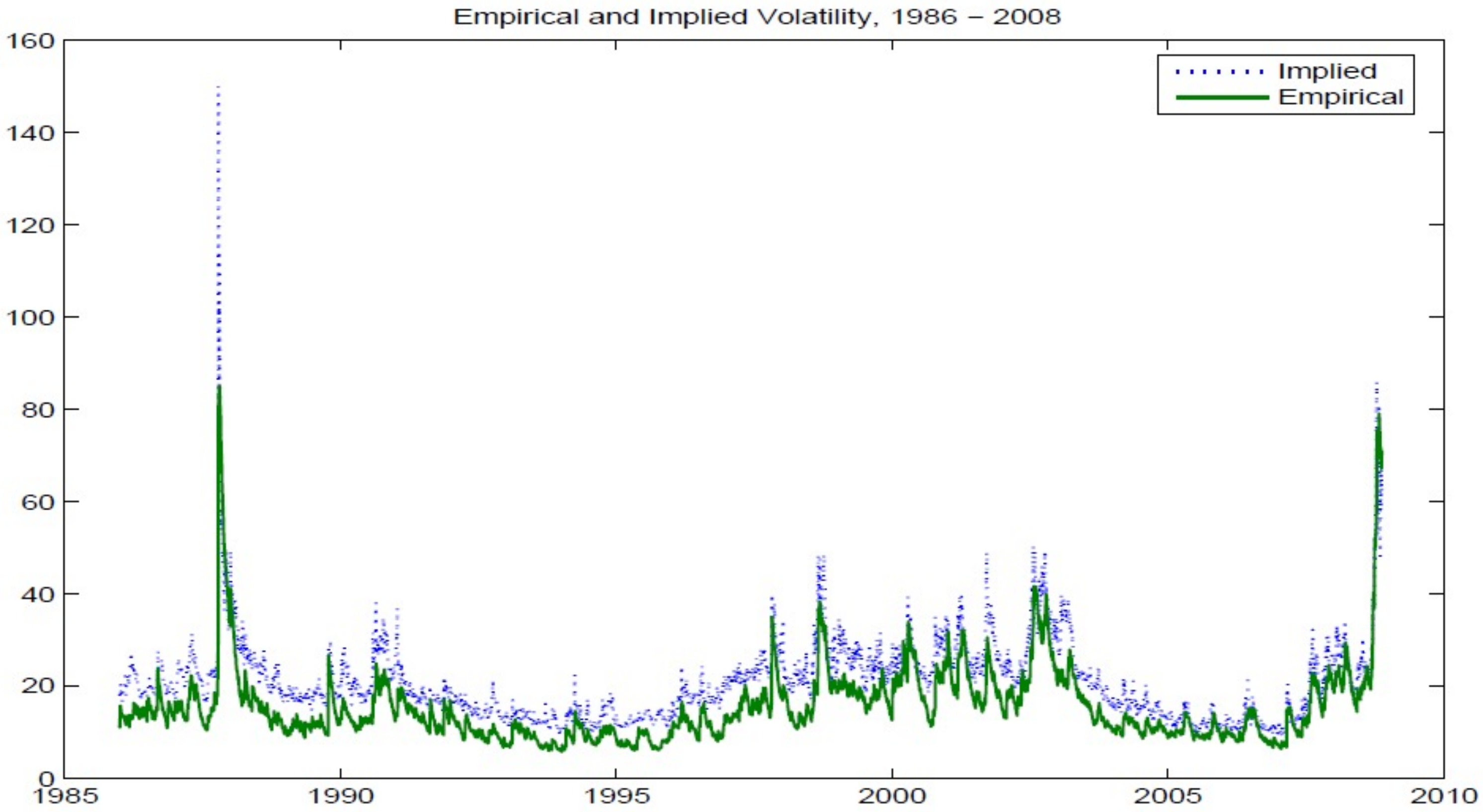
BSM implied volatility as function of moneyness



holding all the parameters other than the strike price fixed

Uses of implied volatility

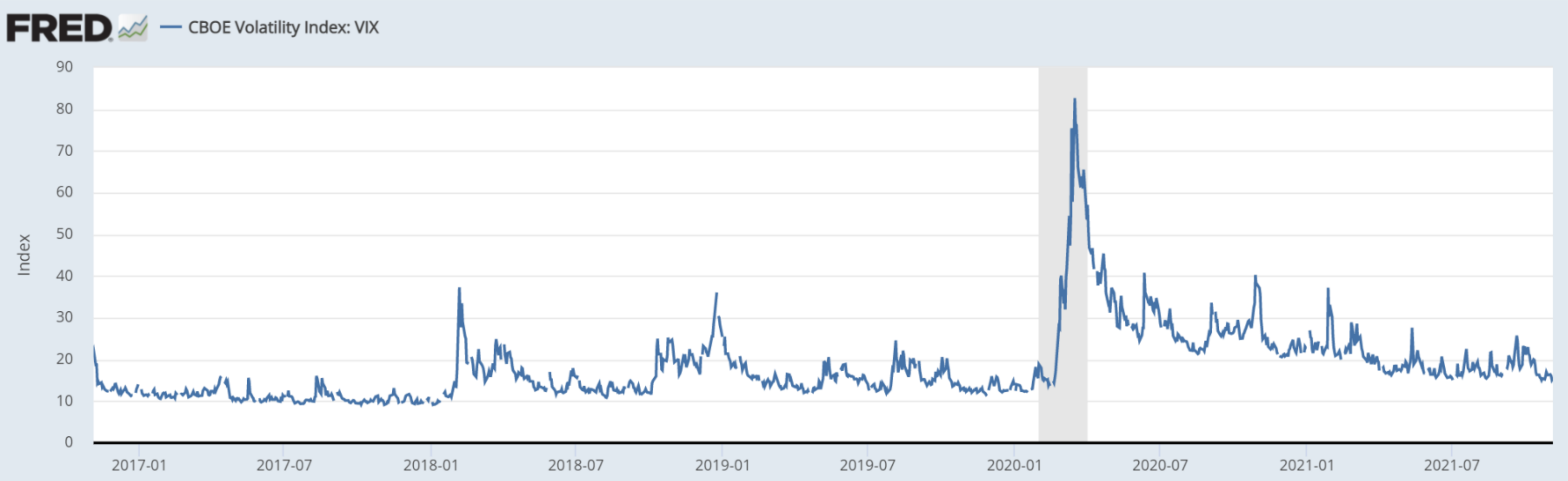
- Gauge the general market uncertainty about future returns
  - Higher uncertainty $\Longrightarrow$ higher option prices $\Longrightarrow$ higher implied volatility
  - It augments return volatility itself, as implied volatility is *Forward Looking* not historical backward looking



implied volatility tends to be higher than historical volatility, and that the difference becomes more pronounced during periods where volatility is unusually high. Those differences are consistent with options prices containing a risk premium that isn't captured by the Black– Scholes– Merton model

Uses of implied volatility

Here is some more recent data on implied volatility for the period 2017 through October 2021



Uses of implied volatility

Gauging the relative price of different options

It is hard to compare the value of options with different strike prices or maturities

- Different strike prices => different **intrinsic values** $S_0 - K$
- Different maturities => different time value of money and uncertainty

Options Prices across Strike Prices and Maturities

		Maturity					
		0.12	0.21	0.39	0.64	0.88	1.13
Strike Prices	1450	8.7	14.65	23.8	34	40.6	47.5
	1475	12.6	19.7	29.1	39.2	47	54.1
	1500	20.35	26.8	36.75	47.3	54.2	61.3
	1525	31.2	36.8	45.6	55.3	62.6	69.5
	1550	47.4	50.2	57	65.4	72.1	78.6
	1575	68.2	67.4	71	77.1	82.8	88.7

On May 3, 2007, $put^{mkt}(1500, 0.12) = \$20.35$, $put^{mkt}(1475, .12) = \$12.60$

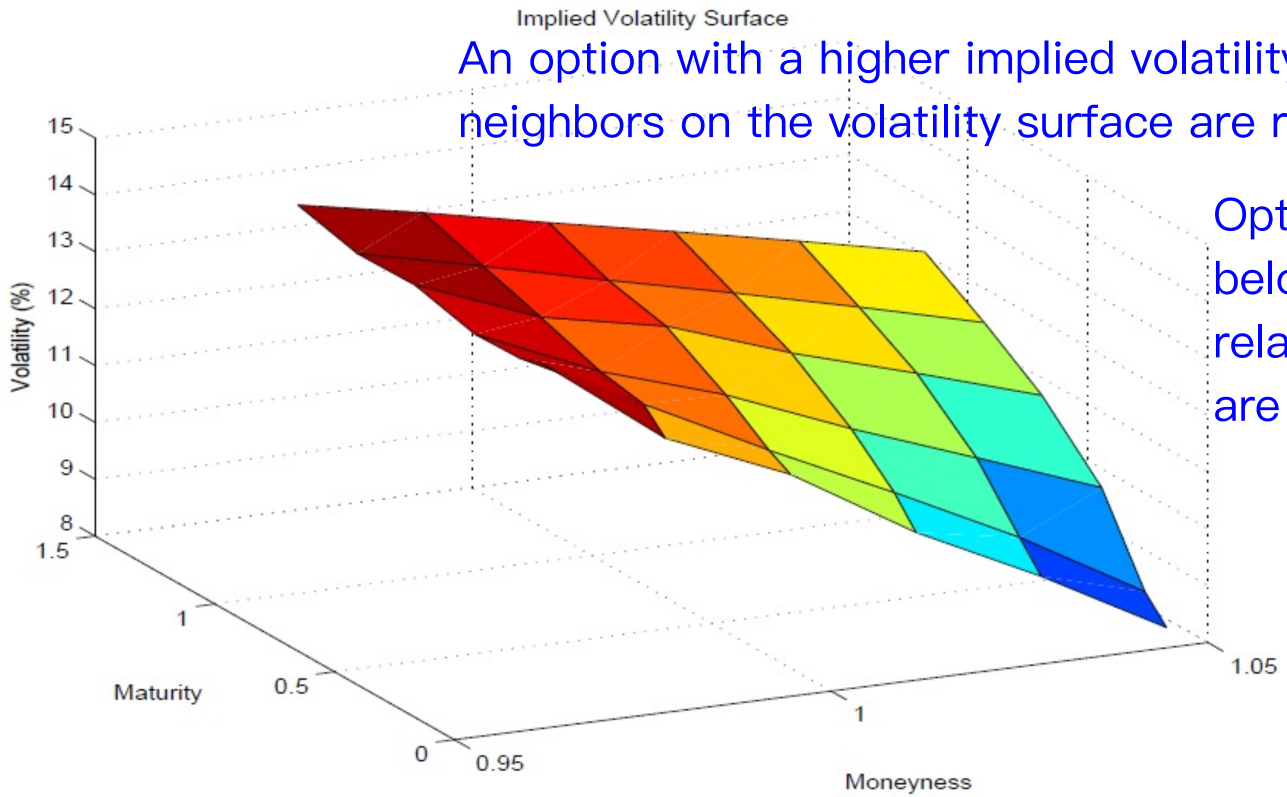
- Is the 7.75 dollar difference only due to the 25 dollar difference in strike prices, or "something else"?

Implied volatility surface

Using implied volatility helps to adjust for differences in intrinsic values and maturities

Implied volatility is a simple measure of how expensive options are relative to each other

Implied Volatility Surface = implied volatilities across strikes and maturities



An option with a higher implied volatility compared to its close neighbors on the volatility surface are relatively expensive

Options that are significantly below the surface, that have a relatively low implied volatility, are cheap.

A trading strategy then is to buy options that seem significantly underpriced and to sell options that appear to be significantly overpriced speculative: directional bet on what will happen to market volatility and avoid a directional bet on whether the market will rise or fall (alternative: straddles and strangles)

Trading implied volatility with the VIX futures contract

- CBOE allows traders to directly take a position in implied volatility by trading the VIX
  - Long side (short side) gains (loses) when volatility rises (falls)
  - Based on average of BSM implied volatility for short-term traded options contracts on S&P500
  - See https://www.cboe.com/products/futures/vx-cboe-volatility-index-vix-futures/contract-specifications

An example of using the VIX to hedge would be for an index fund manager who's compensated based on how closely his portfolio tracks the S&P500 index. Because tracking error tends to rise with volatility,the manager can hedge his compensation by buying a contract whose payoff increases with volatility.

Note that the VIX is based on an average across options with different amounts of moneyness, and that the individual implied volatilities on the underlying contracts may differ significantly from the average that's the VIX

What does this all mean for the BSM model?
What are the alternatives?

- ❑ Despite its inaccuracies BSM serves as a useful benchmark
- ❑ It also works reasonably well to hedge options positions against changes in stock prices using delta or delta-gamma hedging

It remains the basis for widely used implied volatility calculations and for some price quotations

- ❑ **Models have been proposed to correct some of the shortcomings**
  - **Deterministic and Stochastic Volatility Models**
    - Accommodate time-varying volatility
  - **Models with Price Jumps**
    - Accommodate jumps
  - **Implied Tree Models**
    - Produce trees that price options consistently with observed market prices

All of these models are consistent with the idea that OTM puts are expensive relative to BSM prices because investors seeking protection from large losses (e.g., jumps down) must pay a higher (insurance) premium

Deterministic volatility models

- Assume that volatility σ depends on the stock price itself
- **Constant Elasticity of Variance** model: $\sigma(S) = \Sigma \times S^{\alpha}$

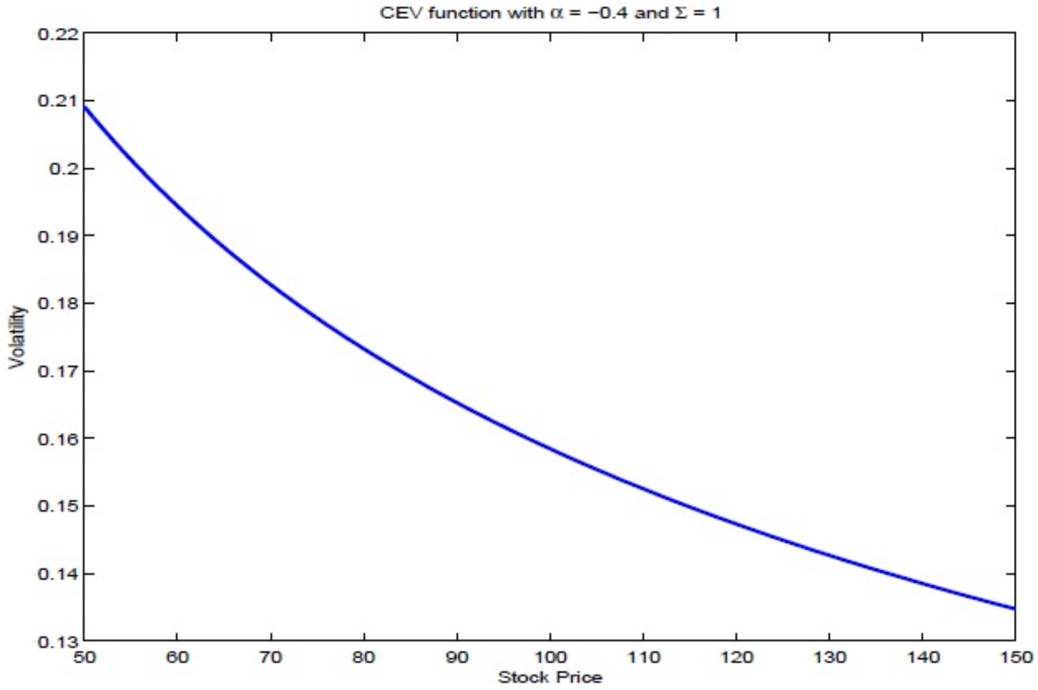
drift   a volatility term

$$\log\left(\frac{S_{t,t+h}}{S_t}\right) = \mu \times h + S_t^{\alpha} \times \Sigma \times \epsilon_t$$

constant which scales volatility

ϵ_t is a normally distributed "shock": $\epsilon_t \sim N(0, h)$

- If $\alpha < 0$, a lower S_t implies a higher volatility $\Longrightarrow$ implied volatility smirk



This model reduces to the Black–Scholes– Merton assumption about stock returns when alpha is set equal to 0

Stochastic volatility models

- Assume that volatility σ_t is moving over time
- The Heston Model: Let $\sigma_t = \sqrt{v_t}$

drift term that represents the expected return over the interval h

volatility term

$$R_{t,t+h} = \mu \times h + \sqrt{v_t} \times \epsilon_{1,t}$$

$$(v_{t+h} - v_t) = \boxed{k \times (\overline{v} - v_t)} \times h + \Sigma \times \sqrt{v_t} \times \epsilon_{2,t}$$

mean reversion: when v is below mean, it increase. when v is above mean reverting level, v tends to decrease.

(It is hard to ensure $v_t > 0$ for every t unless interval size h is very small)

- Result: if $corr(\epsilon_{1,t}, \epsilon_{2,t}) < 0$, i.e. volatility is negatively correlated with stock returns, OTM put options become relatively more expensive $\Longrightarrow$ volatility smirk

correlation is negative

- Intuition: sigma1 is negative → sigma2 is positive
  - A decline in price $\Longrightarrow$ higher volatility $\Longrightarrow$ higher probability of even larger declines $\Longrightarrow$ higher price of insurance against downturns

Jumps in stock prices

We know that periodically there are large jumps in stock prices (e.g., 1987 and 2020)

$$R_{t,t+h} = \mu \times h + \sigma \times \epsilon_t + \omega \times Q_t$$

jump term

volatility during normal times

expected return over an interval of h

- $Q_t = 0$ most of the time, $Q_t = 1$ with small probability
- ω can be a random variable (e.g normal), or a constant parameterizes the size of the jumps.

- Result: If $\omega < 0$ (or $E[\omega] < 0$: ω itself can be random), then OTM put options are relatively more expensive an implied volatility smirk.
  - If $\omega < 0$, it becomes more likely that bad negative outcomes occur
  - Investors willing to pay a higher premium to insure against those bad events

- Pricing with jumps is a bit more complicated
  - The pricing formulas are not as “nice” as the Black and Scholes formula

Implied tree: example

Previously we saw how to find option prices given a stock price tree

With implied trees we start with some observed options prices and calibrate the stock price tree to be consistent with those prices

- We can then use the tree to price other options

For example, consider the binomial tree model

- Given $S_0 = 1502.39$, $\sigma = 12.36\%$, $r = 4.713\%$, $q = 1.91\%$ and $T = .12$, we find $u = exp(\sigma\sqrt{T}) = 1.043746137$ and $d = 1/u = 0.958087378$. Thus, the risk neutral probability

$$q^* = \frac{e^{(r-q)T} - d}{u - d} = 0.528631113$$

- The price of the $K = 1500$ put option come out to be $p_0 = \$28.394$, higher than the traded market price $p^{mkt}(1500, .12) = \$20.35$.

Implied tree: example

$i = 0$ | $i = 1$

$$\begin{array}{l} S_{1,u} = 1568.114 \\ p_{1,u} = 0 \end{array}$$

$$\begin{array}{l} S_0 = 1502.39 \\ q_0^* = 0.528631113 \\ p_0 = e^{-r\times 0.12} \times (1 - q_0^*) \times p_{1,d} = 28.394 \end{array}$$

$$\begin{array}{l} S_{1,d} = 1439.421 \\ p_{1,d} = 60.579 \end{array}$$

- An implied tree has the same logic of implied volatility: Since the model is not working using the original inputs, we look for an alternative specification that makes it work
  - In the above example, we can **choose** $S_{1,u}$ to price the option correctly
  - To avoid too many parameters, define $u = S_{1,u}/S_0$ and define $S_{1,d} = S_0/u$

(Note choosing $S_{1,u}$ really means choosing σ)

Implied tree: example

$i = 0$

$S_0 = 1502.39$
$q_0^* = 0.5446$
$p_0 = e^{-r \times 0.12} \times (1 - q_0^*) \times p_{1,d} = 20.35$

$i = 1$

$S_{1,u} = 1551.26$
$p_{1,u} = 0$

$S_{1,d} = 1455.059$
$p_{1,d} = 44.941$

- What do we use an implied tree for?
  - To price other options
  - For instance, if $K = 1490$, the put price from the binomial tree is \$15.82184291, closer to the market value of $p^{mkt}(1490, .12) = \$17.05$, compared to the original case (which would be \$23.707)

Summary

- BSM formula does not price options accurately.
- Accuracy can be improved by incorporating
  - Stochastic volatility
  - Jumps
  - Fat tails
- Still, BSM is a very useful benchmark
  - Gives decent approximation to prices close to the money, and used for finding hedge ratios
- More complicated models fit the data better and can be used for trading strategies
- BSM has become the industry standard for quoting option prices
  - Quotes are in “implied volatility” terms
  - Implied volatility surfaces provide a simple way to evaluate the relative value of options