15.415x Foundations of Modern Finance

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Lecture 2: Market Prices and Present Value

Key Concepts

- State-space model for time and risk
- Arbitrage pricing
- Present value and future value
- Nominal vs. real cash flows and returns

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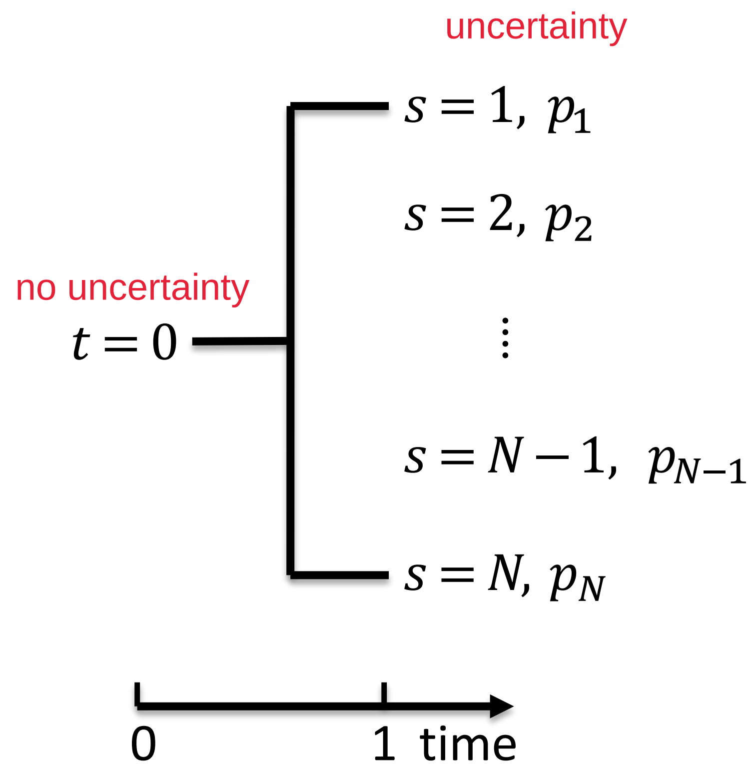
State space model for time and risk

The tree graph with the associated probabilities is called a state space model for the economy.

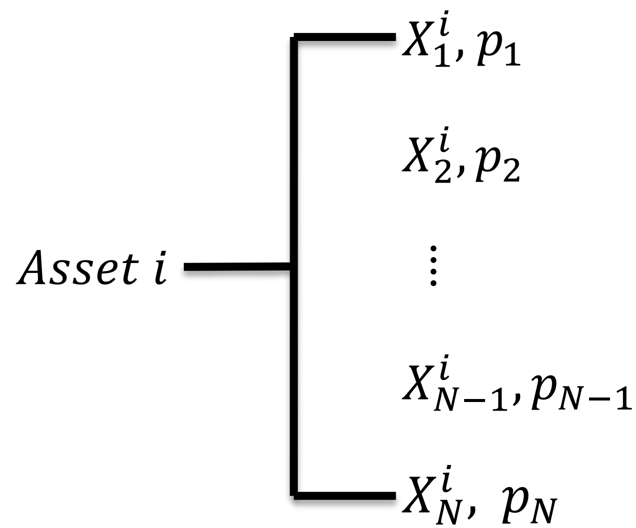
- Consider a simple model to capture the two elements in finance: time and risk.
- There are two dates: $t = 0,\ 1$.
- There are N possible economic states at $t = 1$: $s = 1, \ldots, N,$ with probabilities $p_1, p_2,$ $\ldots,$ p_N.
- States and probabilities are known to all decision makers.



State-space model

- Assume frictionless financial market for simplicity.
- Assets can be traded at time $t = 0$ with payoffs at time $t = 1$.
- The price of an asset is P at $t = 0$ with payoff $X = (X_1, \dots, X_N)$ at $t = 1$.
  - X is a random variable.
- A random payoff is given by the value of its payoff in each state and the corresponding probability:

$$[(X_1, \dots, X_N);\ (p_1, \dots, p_N)]$$

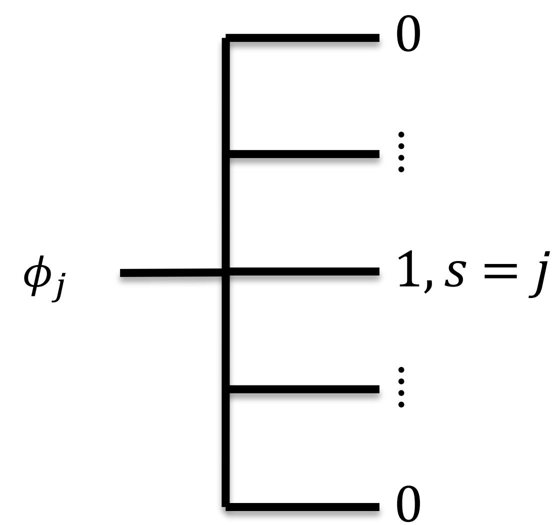


State prices

- Consider primitive state-contingent claims (Arrow-Debreu securities) that pay \$1 in a single state and nothing otherwise.
- Denote the price of the A-D claim on state j by ϕ_j, the state price for state j.
- No arbitrage requires that all state prices must be positive: $\phi_j > 0$ for all j.
- The market is called complete if one can effectively trade A-D securities on each state.

N states, so there are N different Arrow-Debreu securities.
complete market: all the Arrow-Debreu securities are traded in the market.

- Complete market is a useful abstraction.



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Arbitrage pricing

- With the prices of A-D securities, we can price other assets/securities.
- Consider a two-state economy ($N = 2$) with three assets:
  - A-D securities, paying $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$,
  - Asset X paying $\begin{bmatrix}3\\5\end{bmatrix}$.
- How is the price of the third asset related to the prices of the first two?
- Think of the third asset as a portfolio of the first two assets:

$$\begin{bmatrix}3\\5\end{bmatrix} = 3 \times \begin{bmatrix}1\\0\end{bmatrix} + 5 \times \begin{bmatrix}0\\1\end{bmatrix}$$

- Claim: price of asset X is:

$$P = 3 \times \phi_1 + 5 \times \phi_2$$

Arbitrage pricing

- The third asset can be replicated as a portfolio of A-D securities: 3 units of A-D security 1, and 5 units of A-D security 2.
- By no arbitrage, its price must equal the price of the replication portfolio:
- No arbitrage requires:

$$P = 3\,\phi_1 + 5\,\phi_2$$

- If not, agents can generate arbitrage profits. How?
- Law of One Price: Two assets with the same payoff must have the same market price.
- If we have the prices of A-D securities, we can price all other securities: just replicate them as portfolios of A-D securities.
- What if we have prices of a bunch of "composite" securities?

Arbitrage pricing

Example. (Concept check)

Suppose there are two economic states next year.

- Safe government bond pays an interest rate of 5%;
- A stock with price \$100 yields the following payoff next year: (90,120).

What should be a proper set of state prices?

- From the price and payoff of government bond:
$$100 = 105\phi_1 + 105\phi_2$$
- From the price and payoff of the stock:
$$100 = 90\phi_1 + 120\phi_2$$
- Solving for ϕ_1 and ϕ_2 yields:
$$(\phi_1, \phi_2) = \left(\frac{10}{21}, \frac{10}{21}\right)$$

Arbitrage pricing

Example. (Concept check)

- Suppose there are two states next year. The payoff of a share of stock and the probabilities of the states are:

$$[(\$90, \$110); (0.4, 0.6)]$$

- The state prices for the two states are:

$$(\phi_1, \phi_2) = (0.5, 0.4)$$

Questions:

1. What is the stock price today?
2. What is the expected rate of return of the stock?

Arbitrage pricing

Example (cont'd).

- Stock price today:

$$P = \phi_1 X_1 + \phi_2 X_2 = 0.5\,(90) + 0.4\,(110) = 89$$

- Expected rate of return on the stock, $\bar{r}$:

$$\bar{r} = \frac{E[X] - P}{P} = \frac{p_1 X_1 + p_2 X_2}{P} - 1 = \frac{0.4(90) + 0.6(110)}{89} - 1 = \frac{102}{89} - 1 = \frac{13}{89}$$

- Given the expected return on the stock by the market, we have:

$$P\,(1 + \bar{r}) = E[X] \;\;\Rightarrow\;\; P = \frac{E[X]}{1 + \bar{r}} = \frac{102}{1 + 13/89} = 89$$

Arbitrage pricing

- In general, in a complete market we can value any cash flow by the no-arbitrage principle (P1).
- Suppose the firm is considering a project yielding time-1 cash flow:

$$X = (X_1, X_2, \ldots, X_N)$$

- Using prices of A-D securities, we can attach value to this cash flow as

$$P = \phi_1 X_1 + \cdots + \phi_N X_N = PV$$

- This valuation formula encapsulates the arbitrage/relative pricing principle.

present value cash flow

- PV is the present value of the project/asset/CF.
- PV is also given by the expected payoff and the expected rate of return.
- Key idea: Find traded assets with similar cash flows (in timing and risk), use their price/expected return to value the given asset.

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Present value (PV)

Example 1. How much is a sure cash flow of $1,000 in one year worth now?

Market: Safe assets traded in the market offer annual return of 2%.

A potential buyer of the sure CF also expects 2% return. Let the price she is willing to pay be P. Then:

$$P\,(1+0.02) = \$1{,}000$$

Thus,

$$P = \frac{\$1{,}000}{1.02} = \$980$$

which is the CF's present value.

<u>Observation</u>: Present value properly adjusts for time.

Present value (PV)

Example 2. How much is a risky cash flow in one year with a forecasted value of $1,000 worth now?

Market: Traded assets of similar risk offer expected annual return of 20%.

A potential buyer of the risky CF also expects 20% return. Let the price be P. Then:

$$P\,(1 + 0.20) = \$1{,}000$$

Thus, the present value of the risky CF is:

$$P = \frac{\$1{,}000}{1.20} = \$833$$

<u>Observation</u>: Present value properly adjusts for risk.

Present value and discount rate

The current market value of a CF (its PV) is determined by

- its expected payoff;
- discounted at an appropriate discount rate,

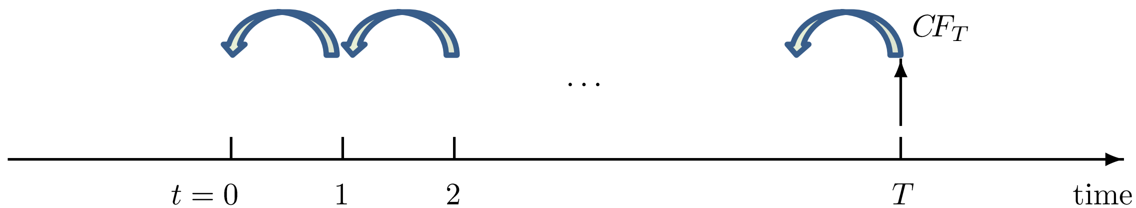
where the discount rate is given by the expected rate of return on traded assets with similar cash flows (in timing and risk):

$$PV = \frac{E[CF]}{1 + \bar{r}}$$

Thus,

- the value of an asset (cash flow) is determined by the financial market (via the discount rate/expected rate of return/required rate of return);
- the discount rate properly adjusts for time and risk;
- the discount rate is also called the opportunity cost of capital (COC) – return offered by similar assets traded in the market.

Present value (PV)



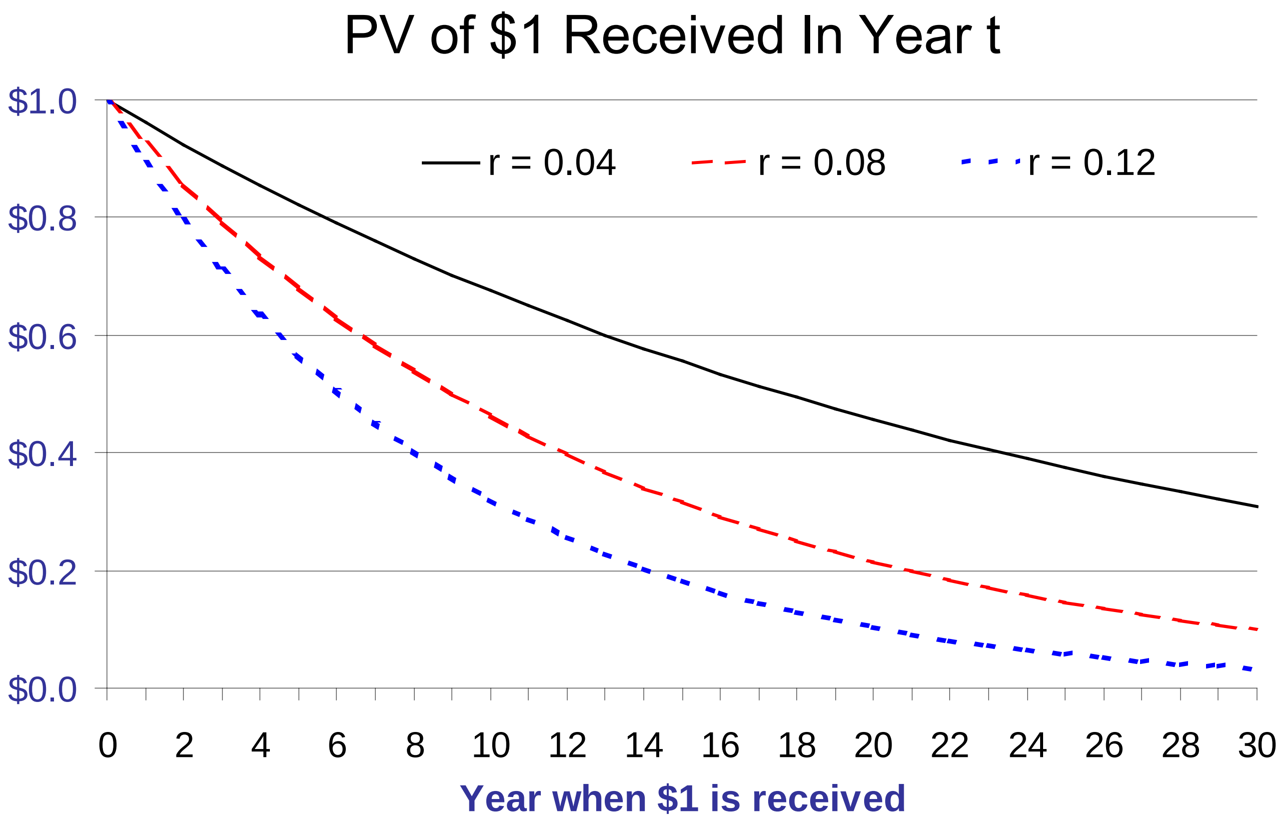
$$PV(CF_T) = \frac{CF_T}{(1+r)^T}$$

From now on, for simplicity, we will use r (instead of $\bar{r}$) to denote discount rates (unless noted otherwise).

Example. (A) \$10M (million) in 5 years or (B) \$15M in 15 years. Which is better if $r = 5\%$?

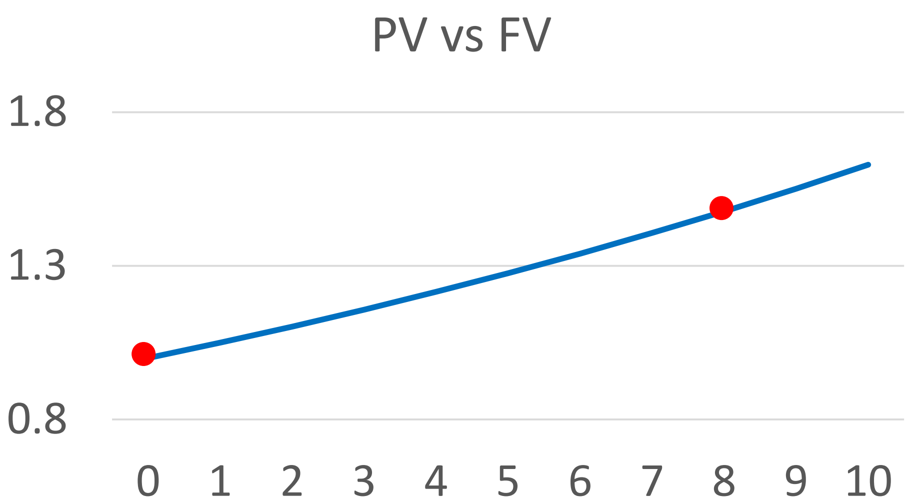
$$PV_A = \frac{10}{1.05^5} = 7.84; \qquad PV_B = \frac{15}{1.05^{15}} = 7.22$$

Present value (PV)



Present vs future value

- We can bring $ back from the future, discounting at the proper discount rate.
- We can also send $ into the future, growing at the proper return rate.



Future value (FV)

- How much will $1 today be worth in one year if the interest rate is 4%?
  - $1 investable at a rate of return $r = 4\%$;
  - FV in 1 year:

$$FV = 1 + r = \$1.04$$

- FV in T years:

$$\begin{aligned} FV &= \$1 \times (1+r) \times \cdots \times (1+r) \\ &= (1+r)^T \end{aligned}$$

Example. Bank pays an annual interest of 4% on 2-year CDs and you deposit $10,000. What is your balance two years later?

$$FV = \$10{,}000 \times (1 + 0.04)^2 = \$10{,}816$$

Present value (PV) and CF

Comparing cash flows:

Example. Drug company has developed a new flu vaccine and needs to choose between two strategies:

- Strategy A: To bring to market in 1 year, invest \$1B (billion) now and returns \$500M (million), \$400M and \$300M in years 1, 2 and 3, respectively.
- Strategy B: To bring to market in 2 years, invest \$200M in years 0 and 1, and returns \$300M in years 2 and 3.

How to value/compare the two strategies (i.e., their CFs)?

Present value (PV) and CF

$$PV(CF_1, CF_2, \ldots, CF_T) = \frac{CF_1}{(1+r)} + \frac{CF_2}{(1+r)^2} + \cdots + \frac{CF_T}{(1+r)^T}$$

Assume that $r = 5\%$.

- Strategy A:

Time	0	1	2	3
Cash Flow	-1,000	500.0	400.0	300.0
Present Value	-1,000	476.2	362.8	259.2
Total PV	98.2			

- Strategy B:

Time	0	1	2	3
Cash Flow	-200	-200.0	300.0	300.0
Present Value	-200	-190.5	272.1	259.2
Total PV	140.8			

Firm should choose strategy B, and its value would increase by \$140.8M (vs. \$98.2M for strategy A).

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Nominal vs real CFs

Example. Inflation is 4% per year. You expect to receive \$1.04 in one year, what is this CF really worth next year?

- The inflation adjusted or real value of \$1.04 in a year is:

$$\text{Real } CF = \frac{\text{Nominal } CF}{1 + \text{ inflation}} = \frac{\$1.04}{1 + 0.04} = \$1.00$$

  - Nominal cash flows $\Rightarrow$ expressed in actual-dollar cash flows.
  - Real cash flows $\Rightarrow$ expressed in constant purchasing power.

- At an annual inflation rate of i, we have:

$$(\text{Real } CF)_t = \frac{(\text{Nominal } CF)_t}{(1 + i)^t}$$

- http://www.tradingeconomics.com/country-list/inflation-rate

Nominal vs real rates

- Nominal rates of return $\Rightarrow$ prevailing market rates.
- Real rates of return $\Rightarrow$ inflation adjusted rates.

Example.

- \$1.00 invested at a 6% interest rate grows to \$1.06 next year.
- If inflation is 4% per year, then its real value is

$$\frac{\$1.06}{1.04} = 1.019$$

- The real rate of return is 1.9%.

$$r_{\text{real}} = \frac{1 + r_{\text{nominal}}}{1 + i} - 1 \approx r_{\text{nominal}} - i$$

Nominal vs real CFs and rates

Example. Sales is \$1M this year and is expected to have a real growth of 2% next year. Inflation is expected to be 4%. The appropriate nominal discount rate is 5%. What is the present value of next year's sales revenue?

- Next year's nominal sales forecast: (\$1M)(1.02)(1.04) = \$1.0608M.

$$PV = \frac{1.0608}{1.05} = 1.0103$$

- Next year's real sales forecast: (\$1M)(1.02) = \$1.02M.

$$r_{\text{real}} = \frac{1 + r_{\text{nominal}}}{1 + i} - 1 = \frac{1.05}{1.04} - 1 = 0.9615\%$$

$$PV = \frac{1.02}{1.009615} = 1.0103$$

- For valuation calculations, treat inflation consistently.
  - Discount nominal cash flows using nominal discount rates.
  - Discount real cash flows using real discount rates.

Summary

- State-space model for time and risk
- Arbitrage/relative pricing
- Present value and future value
- Nominal and real cash flows and returns