Week 8 – Bond and interest rate options

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Outline

- BSM-style models
  - Shortcomings of Black-Scholes-Merton for bond options
  - Black's model short-dated bond options
- Models with stochastic interest rates
  - Binomial model of short-term yields
    - Estimating and incorporating volatility
    - Inferring long-term yields
  - Pricing in binomial models
    - Valuing bonds
    - Calibrating models with bond market data
    - European bond options
    - Interest rate caps and floors
  - Intro to continuous-time interest rate models

Common options in the fixed income marketplace

- Bonds with embedded options
  - Callable bonds
    - Prepayable mortgages
  - Puttable bonds

- Futures options
  - Treasury notes and bonds
  - Short-term rates (LIBOR, SOFR)

- Options on interest rates
  - Caps, floors and collars
  - Swaptions

Black-Scholes-Merton misprices bond options

- Critical assumptions underlying the traditional Black-Scholes option pricing model don't hold for bond prices:
  - Constant return volatility parameter for the underlying security
  - Log-normally distributed prices, and positive probability of arbitrarily high future price
  - Constant risk-free rate assumed over life of option
  - Many bond options are American and early exercise can be optimal

- The problems with binomial models in *bond prices* are similar

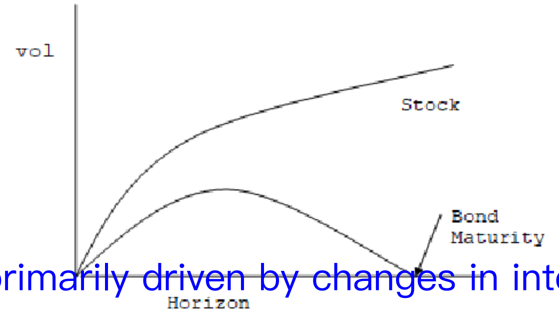
<u>The fix</u>: Price options on fixed income securities using stochastic models of interest rates, not of bond prices

Example: Shortcomings of Black-Scholes-Merton for Pricing Bond Options

- What is the price of a 3-year European call option on a 3-year zero coupon bond with exercise price \$110 (per \$100 face)?
  - The answer is obviously zero. option is worthless because the bond will pay off \$100 with certainty at the end of the three years and the option will always expire out of the money.
  - Under the assumption of r = 10% and 4% annual bond price volatility, the Black-Scholes formula gives a price of 7.78!

Figure: Price volatility vs. horizon -- Stocks vs. Bonds

constant volatility parameter assumption is violated



> *Note:* When stock prices are lognormal, stock price volatility at horizon t is $\sigma\sqrt{t}$, where σ is a constant

The price volatility of bonds is primarily driven by changes in interest rates. Initially, volatility of bond prices also increases with horizon because interest rates become more uncertain looking forward over a longer time period. However, there's a second offsetting effect that isn't present for stocks, that's––that the remaining duration of the bond becomes shorter as time passes. We know that the price sensitivity to interest rate changes is proportional to duration.So as duration shrinks, so does the volatility of the bond price.

Black's model for pricing short-dated bond options

- Despite the problems that arise applying the Black-Scholes-Merton model to bond options, a variant of it, Black's model, is often used when:

  1. the option payoff depends only on the value of a variable at a particular point in time but not on the path it took to get there, and

  2. it is reasonable to assume that the distribution of the variable at that point in time is normally or log-normally distributed (formula below is for log-normal distribution)

- Those assumptions are often realistic for bond options when the life of the option is much shorter than the maturity of the bond

  early in the life of the bond,the pattern of volatility is much the same as it is for a stock

- Black's model is versatile and is used for a variety of applications:
  - We saw earlier that it's used to price options on commodities and futures options
  - It's also used to price and quote interest rate caps and floors, and swaptions.

Black's Model for the Price of a European Option

For a European option on a variable V, define:

T	= maturity date of the option
F	= forward price of V for a contract with maturity T
X	= strike price of option
r	= spot yield for maturity T (continuous basis)
σ	= volatility of F (as fraction of F)
V_T	= value of V at time T
F_T	= value of F at time T
N	= cumulative normal distribution
c	= value of call
p	= value of put

A call option pays $\max(V_T - X, 0)$ at time T. Note that this equals $\max(F_T - X, 0)$, since the spot price converges to the forward price. Then

$$c = e^{-rT}[FN(d_1) - XN(d_2)]$$

$$p = e^{-rT}[XN(-d_2) - FN(-d_1)]$$

where

$$d_1 = \frac{\ln(F/X) + \sigma^2 T/2}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Programmed into spreadsheet “BSM & Black’s Model.xls”

Models with stochastic interest rates

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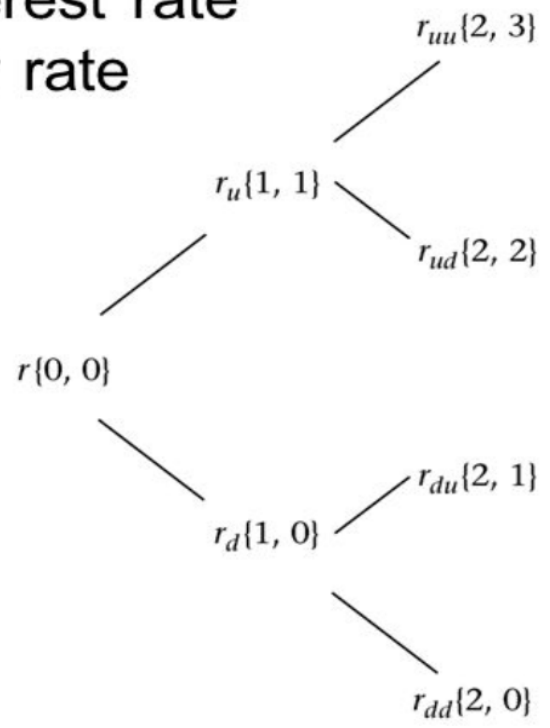
Modeling the evolution of interest rates

- **Payoffs on fixed income derivatives depend on the future yield curve**
  - Pricing derivatives requires a model of the stochastic evolution of the yield curve that answers:
    - how are spot and forward rates expected to move over time?
    - what is the volatility of those movements?
- First we will work with **simple binomial models,** to learn to price interest rate derivatives and understand how to calibrate the models to match market prices.
- Then we will briefly looks at some **continuous time models** of stochastic yield curves and how they can be used for pricing using Monte Carlo or a binomial representation

A generic binomial tree in short rates

Three-period interest rate tree of the 1-year rate



The choice variables for a modeler are:

short term interest rate

(1) the short rate at each node, and (2) the probability that rates go up or down at each node.

For tractability, the evolution of rate changes and probabilities are usually restricted to a simple form.

Different models are distinguished by how those rules are set.

- Probabilities of up or down can be physical or risk neutral
- Probabilities at each node can be constant or can vary
- Infinite choices for how to calibrate; danger of over-fitting
- Different dynamic models restrict the choices in different ways

Volatility

- Key input for all stochastic interest rate models
- Ties down vertical distance between nodes in a tree
- Can estimate from historical data or use implied volatilities

from option prices

Estimating Volatility from Historical Data

Procedure:

a) collect recent sample of yields (e.g., daily data on 1-year rates)

b) calculate sample std. dev. of the yields

- here we calculate proportional changes
- but in some models risk is measured in levels

c) annualize by multiplying by $\sqrt{365}$ (or $\sqrt{250}$)

- or convert to volatility over t-day period by multiplying by $\sqrt{t}$
- note: this assumes rate changes are uncorrelated over time

$$\text{variance} = \sum_{s=1}^{N} \frac{(X_s - E[X])^2}{N-1}$$

X_s = percentage yield change from previous day
N = number of observations
$E(X)$ = average percentage yield change

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<u>Example:</u> Estimating volatility of short-term yields

Date	Observed Yield	Proportional Change
7/16	0.0388	
7/17	0.039	0.00515
7/20	0.0391	0.00256
7/21	0.0393	0.00512
7/22	0.039	-0.00763
7/23	0.0383	-0.01795
7/24	0.0385	0.00522
7/27	0.0385	0.00000
7/28	0.0381	-0.01039
7/29	0.0383	0.00525
7/30	0.0386	0.00783
	Mean	**-0.00048**
	std dev (daily)	**0.00857**
	std dev (annual)	**0.16379**

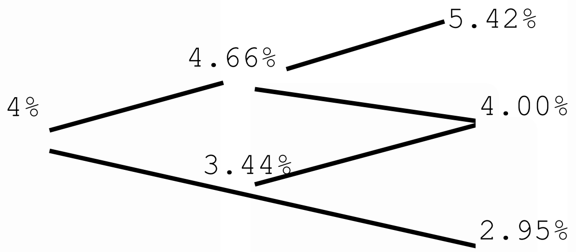
e.g. [(.039-.0388)/.0388] =.00515

(here the annual standard deviation was found by multiplying by $\sqrt{365}$ *.)*

Example: Capturing volatility in multiplicative model without drift

$r_{t+1,H} = r_t(1 + d)$ and $r_{t+1,L} = r_t/(1 + d)$
r_0 = .04 = current 1-year rate
Estimate of **σ =** .164 = annual volatility
Assume probability of rates up or down = .5 risk neutral probability

Then in a tree with each step representing one year, setting d = **σ** makes the model volatility match observed volatility.



<u>Proof that d = σ:</u>

$$\sigma = \text{std dev of } \frac{r_{t+1} - r_t}{r_t} \cong \text{std dev of } [\ln(r_{t+1}) - \ln(r_t)].$$

Since r_{t+1} will equal $(1+d)r_t$ or $r_t/(1+d)$, then $\ln(r_{t+1}) - \ln(r_t)$ =
ln(1+d) or –ln(1+d), which is approximately equal to d or –d, for small d.

Then $E(\ln(r_{t+1}) - \ln(r_t)) = .5(d) + .5(-d) = 0$,

and std. dev. of $[\ln(r_{t+1}) - \ln(r_t)] = [.5(d)^2 + .5(-d)^2]^{1/2} = d$.

Note: The vertical distance between nodes at a point in time in some models is set to be two standard deviations in levels any bond pricing model based on the evolution of short term rates is also a model of the evolution of the entire yield curve

Going from short rates to long rate

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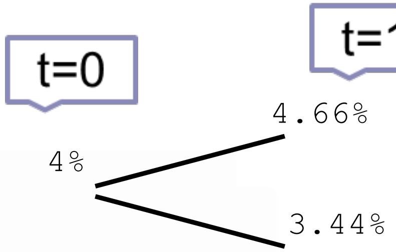
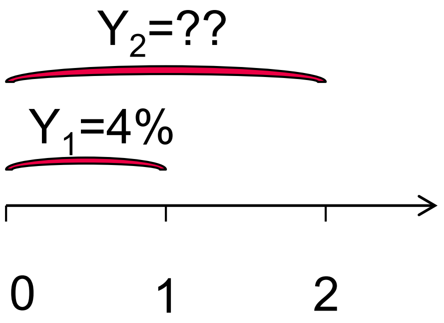
Example: Derive the 2 period yield curve based on the following short rates (assume risk-neutral $pr_{up} = pr_{down} = .5$)



Consider a 2 period, risk-free zero coupon bond that pays \$100 in two periods.



$P_1(4.66\%) = 100/1.0466 = 95.547$
$P_1(3.44\%) = 100/1.0344 = 96.674$

$$P_0(4.00\%) = \frac{.5(95.547)+.5(96.674)}{1.04} = 92.414$$

$92.414 = 100/(1+Y_2)^2 \Rightarrow Y_2 = 4.02\%$

Valuing bonds with embedded options

Value Callable Bond = Value Non-Callable Bond - Value Call Option

call option reduces the bond's value because it's to the advantage of the bond issuer,causing bond buyers to pay less for the bond.

Value Puttable Bond = Value Non-Puttable Bond + Value Put Option

Strategy for pricing callable bond:

- Use interest rate model to price non-callable bond.
- Use same interest rate model to price embedded call option.

Assume the following annual binomial tree is correct for risk-neutral pricing of bonds and bond options: **(pr(up) = pr(down) = .5)**

$r_{HH} = 6.757\%$

$r_H = 4.976\%$

$r_0 = 3.500\%$ $\qquad$ $r_{HL} = 5.532\%$

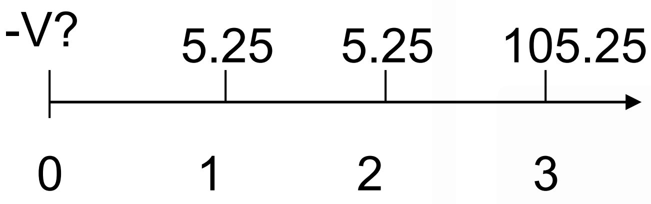
$r_L = 4.074\%$

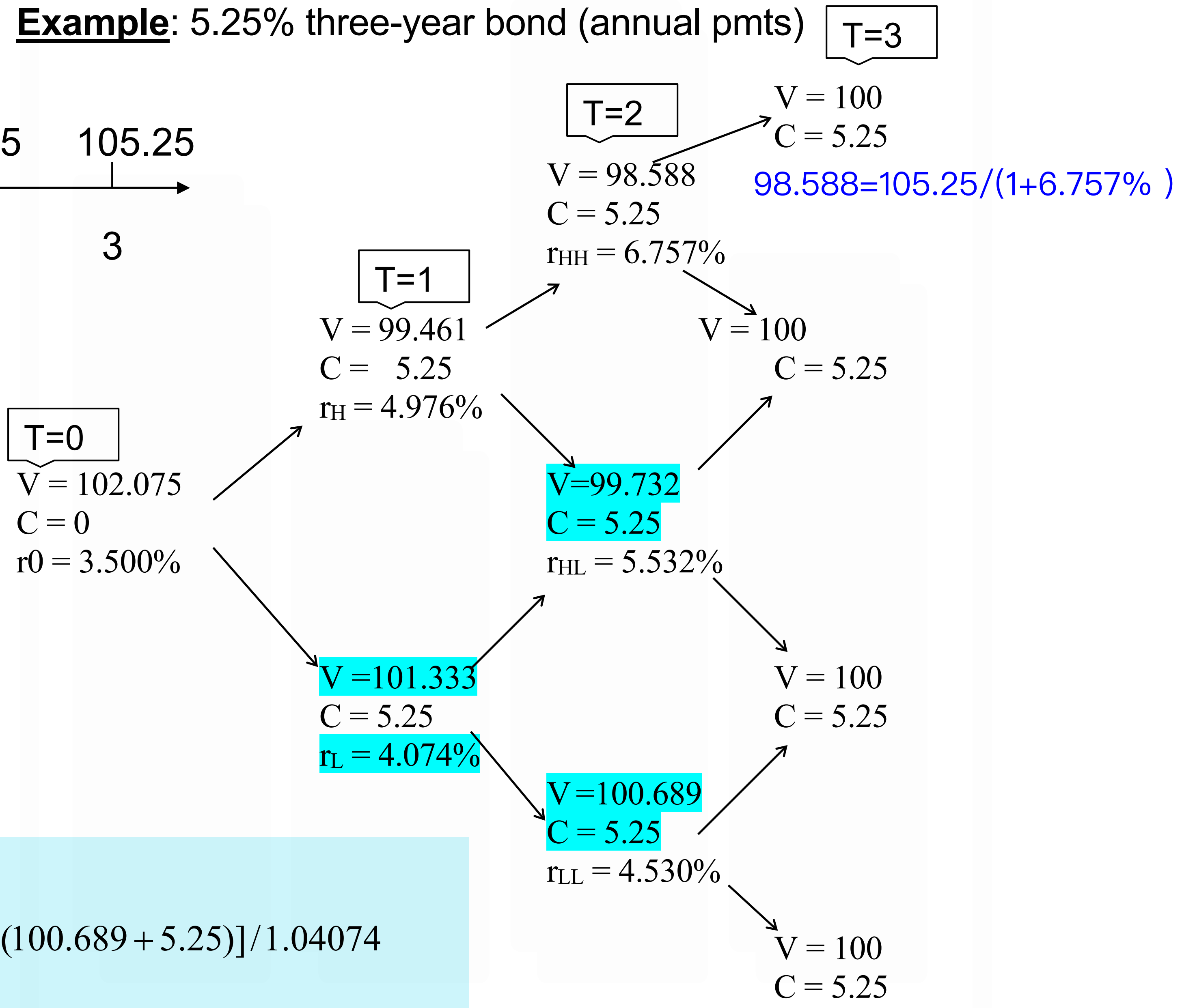
$r_{LL} = 4.530\%$

We will look at where these values came from a little later.

The tree can be used to price option-free bonds:

Example: 5.25% three-year bond (annual pmts)





e.g., $101.333 =$

$$.5[(99.732 + 5.25) + (100.689 + 5.25)]/1.04074$$

Pricing a European Call Option

<u>Example</u>: Assume that the 5.25% bond is callable at the end of two years for $99.50. after the year two coupon has been paid

What is the value of the call option? What is the value of the callable bond?

T=0

V_{call}= 0.383

r0 = 3.500%

T=1

V_{call} = 0.110

r_H = 4.976%

V_{call} =0.683

r_L = 4.074%

V_{call} = 0

r_{HH} = 6.757%

V_{call}=99.732-99.5=.232

r_{HL} = 5.532%

V_{call}=100.689-99.5=1.19

r_{LL} = 4.530%

e.g., .110=.5(.232 + 0)/1.04976

The call is worth $0.383.

The callable bond is worth $102.075 - $.383 = $101.692.

Option Value in Terms of the Spread

The cost of the option can be represented in terms of the change in the quoted yield.

In the last examples, the yield to maturity of the option-free bond solves:

$$102.075 = \frac{5.25}{(1+y)} + \frac{5.25}{(1+y)^2} + \frac{105.25}{(1+y)^3}; \quad y = 4.495\%$$

The yield to maturity of the bond with an European call option solves:

$$101.692 = \frac{5.25}{(1+y^*)} + \frac{5.25}{(1+y^*)^2} + \frac{105.25}{(1+y^*)^3}; \quad y^* = 4.633\%$$

The borrower pays about 14 bps each year for the option.

- The **OAS (options adjusted spread)** is different! *It is defined as the difference between the yield on the bond and the risk-free rate, removing the effect of embedded options.*

Calibrating binomial models

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General iterative procedure:

Guess parameters

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Calculate model-implied values for traded bonds and options

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Compare calculated values with market values; if same stop.

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Adjust parameter values and recalculate

Example: calibrating interest rate tree

Say you observe the current term structure (annual rates):

Year	Spot Rate	Implied One-Year Forward (t-1)
1	3.500%	3.500%
2	4.010%	4.523%
3	4.531%	5.580%

(1+3.5%)*(1+4.523%)=(1+4.01%)^2

Price of a two year, 4% coupon bond (annual payments)

$$\frac{4}{1.035}+\frac{104}{(1.0401)^2}=100$$

Price of a three year, 4.5% coupon bond (annual payments)

$$\frac{4.5}{1.035}+\frac{4.5}{(1.0401)^2}+\frac{104.5}{(1.04531)^3}=100$$

Goal: <u>Construct binomial model for evolution of one year rate that correctly prices these bonds, and that's consistent with volatility assumption</u>.

Some parameters are fixed by assumption (which ones depend on the model chosen):

$p = 1/2$ (equal probability up or down move) risk neutral probability

$\sigma = .1$ (volatility of one-year rate estimated from data)

A standard implementation of volatility is:

$$\begin{tikzcd} & r_H \\ r_0 \arrow[ur] \arrow[dr] & \\ & r_L \end{tikzcd}$$

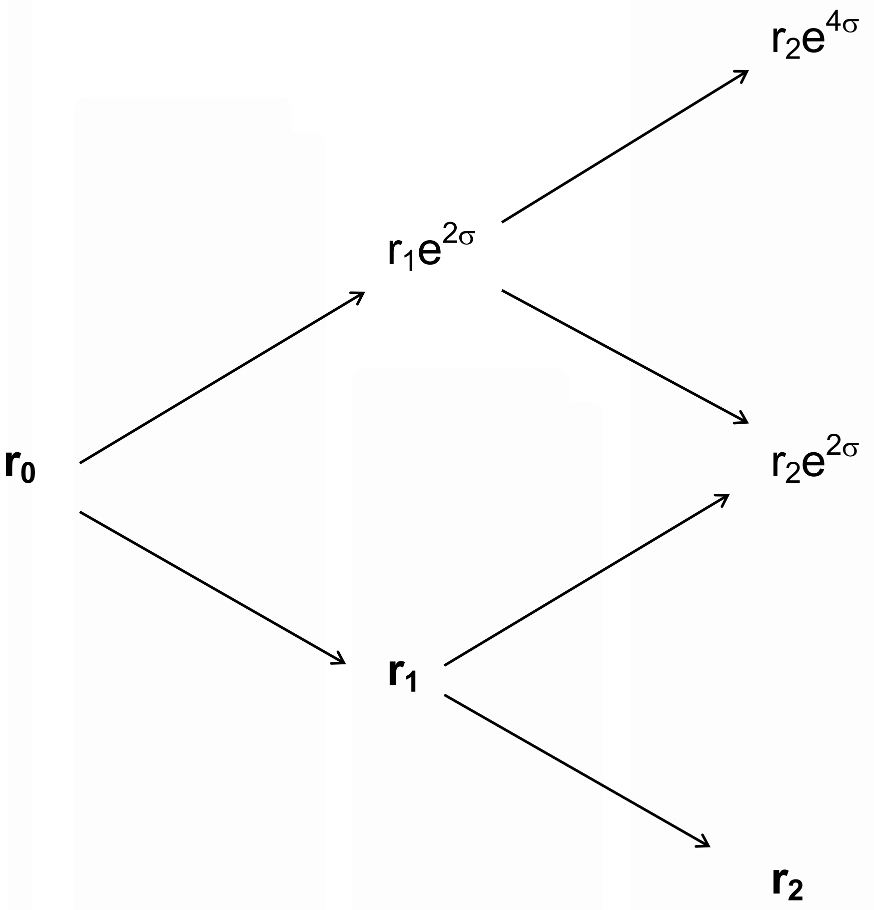
$r_H = r_L\, e^{2\sigma}$ where $e = 2.71828\ldots$

For instance,

$r_L = 4.074\%$ implies $r_H = 4.074\% \times e^{2\times.1} = 4.976\%$.

Notice that $e^{2\sigma} \cong 1 + 2\sigma$ for 2σ small. Then with an equal probability of an up or down, the variance is $.5(r(1+ 2\sigma) - r(1+\sigma))^2 + .5(r - r(1+\sigma))^2 = (r\sigma)^2$.

Binomial Interest Rate Tree



Each period there is one free parameter: the one year rate along the lowest path, r_t.

Let’s take the two period case.

$r_0 = 3.5\%$ (from current term structure)

For now assume $r_1 = 4.5\%$ ($= r_L$).

Then $r_H = 4.5\% \times e^{2 \times .1} = 5.496\%$

Then lattice for evolution of one-year rates is:

$$\begin{tikzcd} & 5.496 \\ 3.5 \arrow[ur] \arrow[dr] & \\ & 4.5 \end{tikzcd}$$

4.5 > 3.5, because yield curve is upward sloping

Pricing two-period coupon bond on tree

Method:

Find bond value at each node, working backwards from final period.

Discount expected one-step-ahead payoff at one-year rate at that node.

V = 100
C = 4.00

V = 98.582
C = 4.00
r_H = 5.496%

V = 99.567
C = 0.00
r_0 = 3.500%

V = 100
C = 4.00

V = 99.522
C = 4.00
r_L = 4.500%

V = 100
C = 4.00

Notes:
98.582 = 104/1.05496

$$99.567 = \frac{1}{2}\left[\frac{98.582+4}{1.035} + \frac{99.522+4}{1.035}\right]$$

What's wrong with this model?

It misprices the two-year bond!

How can we fix It?

The minimum one period rate starting in one year, r_1, was chosen arbitrarily.

Pick a new rate and repeat process.

Should the new rate be higher or lower?

Repeat process of picking rate, filling out lattice, and pricing bond.

Iterate until bond price is correct.

Practice Problem: Verify that the model prices the bond correctly at r_1=4.074

To extend the tree out to three years, price the three year bond, adjusting the guess of the short rate at the bottom of the tree…

choose the lowest node on the tree, R2, to match the known price of a three–year bond. We do this holding R0 and R1 fixed so that the model will continue to correctly price one and two–year bonds.

Also see the spreadsheet **"tree-fitter"** for a more general implementation of this model that you can experiment with:

- It allows the volatility to vary in future periods.
- It also illustrates how a model can be calibrated using implied volatilities.

Practice Problem: Using Tree_Fitter.xls:

Set the model's input parameters to correspond to the example that we have just been working on, and verify that the resulting interest rate tree is the same.

Finding option prices

Inputs:
1. Current bond price
2. Strike price of option
3. Time to expiration
4. Coupon rate
5. **Expected interest rate volatility over life of option**

↓

Option Pricing Model

↓

Output:
Theoretical option price

Finding implied volatility

Inputs:
1. Current bond price
2. Strike price of option
3. Time to expiration
4. Coupon rate
5. **Observed option price**

↓

Option Pricing Model

↓

Output:
Implied interest rate volatility

can also be purchased in the over–the–counter market for an up front fee based on some amount of notional principal but independent of any particular loan.

Valuing caps, floors and collars

- An **interest rate cap** is an option that limits the maximum amount of interest paid on a floating rate liability.

- An **interest rate floor** is an option that limits the minimum amount of interest paid on a floating rate liability.

- An **interest rate collar** is a contract with both a cap and a floor. long cap and a short floor from the perspective of a borrower.

All are valued using a similar approach; we'll focus on caps

- **Common floating rate indices:**
    - LIBOR
    - Treasury

On a floating rate liability, rates are reset at a stated frequency at the stated index rate plus a fixed spread.

Valuing caps as call options on rates

- A cap can be thought of as a strip of call options on the variable interest rate R.
- The separate elements of the cap are sometimes called "caplets." one for each interval between interest rate resets

$$\text{Payoff} = \text{principal x period length x max}[R_t - R_x, 0]$$

R_t = rate at time; Rx = cap rate
index + spread

Example: **An Interest Rate Cap**

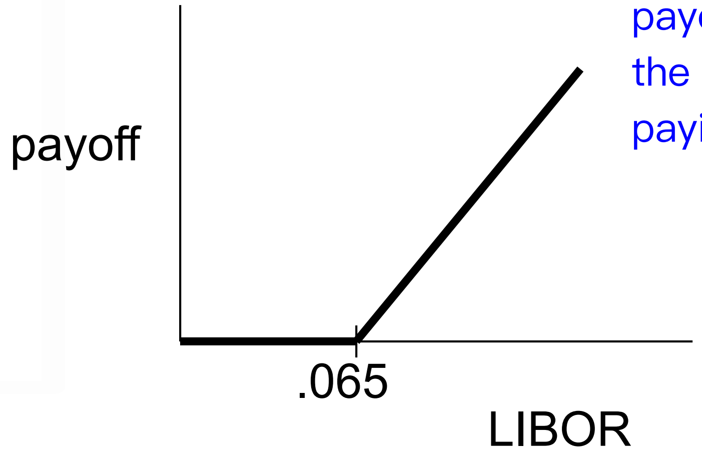
You take out a three year floating rate loan for $100,000, indexed to the six-month LIBOR rate +50 basis points. The rate resets every six months. The rate is capped at 7%.

At the end of each six month period the cap pays:

$$\$100{,}000 \times .5 \times \max[\text{LIBOR}_{6mo.} +.005 - .07, 0]$$

6 months before

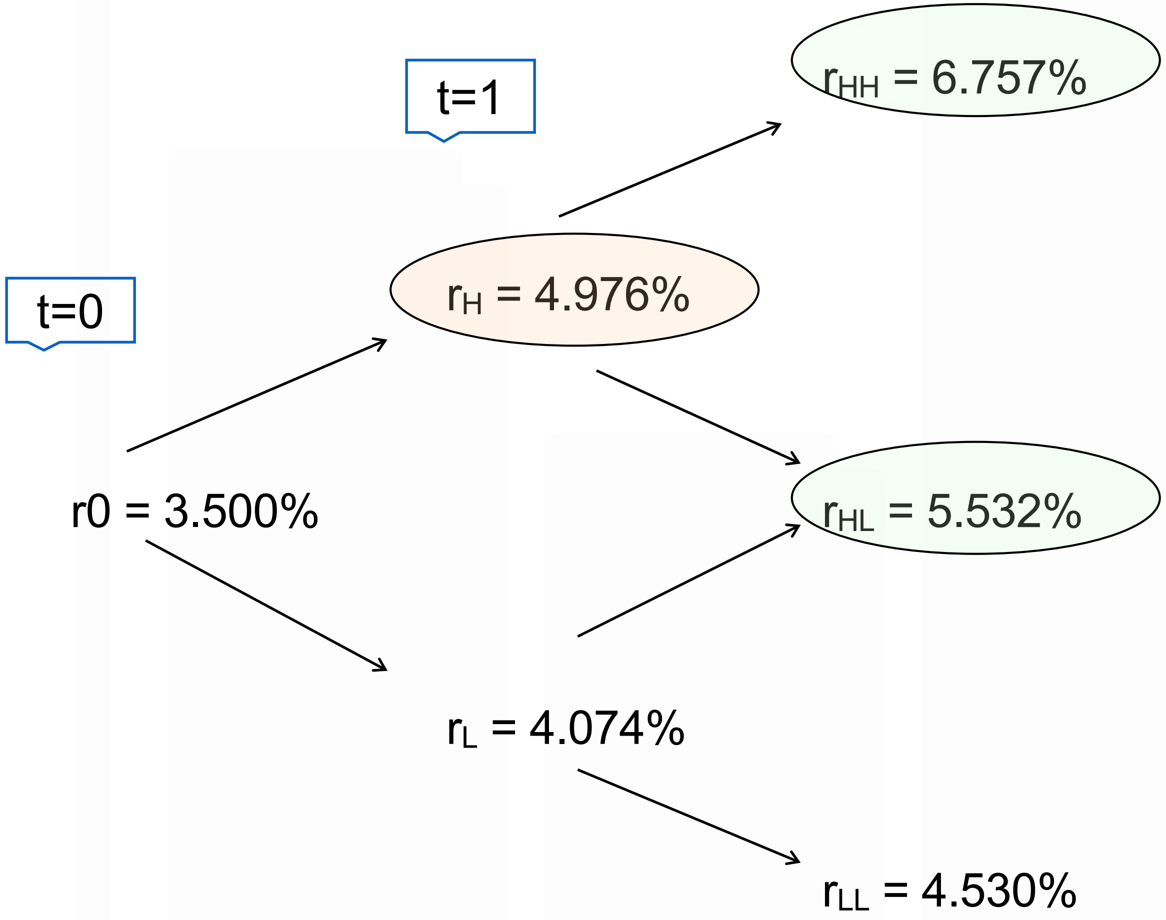
(Relatedly, a floor is a strip of short put options on a variable rate)



payoff in opportunity cost terms as the amount of interest that you avoid paying because of the cap

Example: Pricing an Interest Rate Cap

Assume same process as before for evolution of one year rates:



step length corresponds to the time period between interest rate resets

Consider a three year $100,000 loan, indexed to the one year rate, with an annual reset frequency. ***What is the value of a 4.75% rate cap?***

There are two caplets:

The first pays off (.04976 - .0475)\$100,000 = \$226 **at the end of the second year** if rates go to 4.976%, and zero otherwise.

Discounting back:

$$\text{V(caplet 1)} = \frac{.5(\$226)}{(1.035)(1.04976)} + .5(\$0) = \$104$$

The second pays off at the end of the third year:

payoff = (.06757 - .0475)\$100,000 = \$2,007 @ 6.757%
payoff = (.05532 - .0475)\$100,000 = \$782 @ 5.532%

$$\text{V(caplet 2)} = \frac{.25(\$2007)}{(1.035)(1.04976)(1.06757)} + \frac{.25(\$782)}{(1.035)\boxed{(1.04074)}(1.05532)}$$

$$+ \frac{.25(\$782)}{(1.035)\boxed{(1.04976)}(1.05532)} + .25(\$0) = \$775$$

V(cap) = \$104 + \$775 = \$879

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Introduction to continuous time interest rate model

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One factor models in the short rate

Basic idea

In all such models, the short-term (instantaneous) rate evolves according to:

$$dr = a(r,t)dt + \sigma(r,t)dZ$$

a(r,t) is the drift in short-term rates over time

$\sigma(r,t)$ is the volatility

dZ is an increment to a Brownian motion; *dZ* is distributed N(0,dt)

The resulting rate process is an "Ito Process"

- Notice that both the drift and volatility can be functions of the current level of short rates and time.
- Called "one factor" because formula depends only on one source of volatility
- Various models make different assumptions on drift & volatility functions
- Some implementations are based on physical probabilities, but often they are implemented as risk-neutral models.

Examples of one-factor models

Well-known models include

- Vasicek (1977)
- Cox, Ingersoll, Ross (1985)
- Ho and Lee (1986)
- Hull and White (1990)
- Black, Derman, Toy (1990)
- Black and Karasinski (1991)

In general the goal is to pick a model that generates bond and option prices matching observed prices, with realistic dynamics, and that is not too difficult to compute.

Examples of one-factor models

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Cox-Ingersoll-Ross (CIR)

The short-term interest rate process is assumed to follow:

$$dr = \underbrace{\alpha(b-r)dt}_{\text{mean reverting}} + \sigma\boxed{\sqrt{r}}dZ$$

reduce the variance as the rate gets close to 0. That has the effect of preventing rates from going negative

Parameters

b = long-run target rate

α = speed of adjustment towards target rate

σ = volatility of short-term rate

- Strengths:
  - rates are mean reverting
  - rates can't go negative
  - it can be solved to get a closed form solution for the yield curve

- Weakness:
  - it does not capture yield curve dynamics as accurately as the leading models

Implementing the CIR Model

Link to spreadsheet

European options can be priced using Monte Carlo simulation of the model

American options can be priced using a binomial representation of the implied short rate process

Implementation with Monte Carlo

Step 1: Discretize model and simulate short term rate paths

1. Observe current short rate $r(0)$.
2. Calculate successive short rates on a time path using discrete time step and draws from a standard Normal distribution "$\varepsilon(t)$":

e.g., from the model $dr = a(r,t)dt + \sigma(r,t)dZ$, and given $r(0)$,

$$r(t+\Delta t) = r(t) + a(r,t)\Delta t + \sigma(r,t)\varepsilon(t)\sqrt{\Delta t}$$

The term $\varepsilon(t)\sqrt{\Delta t}$ is the discrete representation of dZ.

Implementation with Monte Carlo

Step 2: Calculate the implied term structure of interest rates

- Find t-period spot yield at time 0 by pricing zero coupon bond with certain payoff of $1 at time t.
  - Discount the $1 to the present along each sample path, using the rates on that path
  - The price "*P*" of the zero coupon bond is the average of the present values across all the sample paths
  - The spot yield to time "t", r(t), on a continuous basis solves $P = \$1e^{-r(t)t}$

- Similarly you can find the implied term structure on future dates by calculating the term structure starting from different starting values for the short-term rate.

Implementation with Monte Carlo

Step 3: Use to price European-style options

1. Calculate $ option payoff at expiration on each path
   - Often need term structure calculations to calculate payoffs
2. Discount the option payoff to the present along each sample path, using the short-term rates on that path
3. The option value is the average of the present values across all the sample paths

Key concept: In general, the price of any contingent claim is the expected future payoff discounted to the present using risk-neutral probabilities.

Implementation with binomial trees

When the optimal exercise policy must be determined to price the option, it is most straightforward to use a lattice model, often a binomial tree.

This requires turning the continuous time model of the short rate into an interest rate tree.

- E.g., $dr = a(r,t)dt + \sigma(r,t)dZ$ becomes a recombining binomial tree with:

$$\begin{aligned} r(up) &= r + a(r,t)\Delta\tau + \sigma(r,t)\sqrt{\Delta\tau} \\ r(down) &= r + a(r,t)\Delta\tau - \sigma(r,t)\sqrt{\Delta\tau} \end{aligned}$$

Notation: "r" is the current node. The values *r(up)* and *r(down)* are the possible short-term rates one time step later.

As with Monte Carlo simulation, once you have the short-rate tree, if necessary you can get the entire yield curve at each node by pricing zero coupon bonds of different maturities, going forward from that node.

Solve for options prices as in the examples using binomial trees in Topic 5 class notes.

Comparing Monte Carlo and binomial approaches

Creates equivalent set of short rates and prices for bonds and options

Each color traces one MC time path

- Red has probability of 3 ups
- Green has probability of 2 ups and a down, etc.

Monte Carlo is easier when it is applicable

Problem is that can't infer optimal strategic choices along a given path

Same paths in binomial tree are ordered so that can optimize working backwards