Solutions: 15.455x Sample Exam Questions

These sample exam problems are each worth 24 points. All sub-parts are weighted equally.

---

1. Suppose that an asset price S_t follows a lognormal, continuous-time stochastic process,

$$\mathrm{d}S = \mu S \, \mathrm{d}t + \sigma S \, \mathrm{d}B,$$

where μ, σ are constants and B is a standard Brownian motion. Use Itô's lemma to find stochastic differential equations expressing $\mathrm{d}V$ in terms of $\mathrm{d}t$ and $\mathrm{d}B$ for the following functions $V(S, t)$. Are they Itô processes?

(a) $V = \alpha S + \beta$,

(b) $V = S^\gamma$,

(c) $V = e^{r(T-t)} S$,

where α, β, γ, r, and T are constants.

Solution: These are all Itô processes.

(a)

$$\mathrm{d}V = \alpha\,\mathrm{d}S = (\alpha\mu S)\,\mathrm{d}t + (\alpha\sigma S)\,\mathrm{d}B_t,$$

(b)

$$\begin{aligned}
\mathrm{d}V = \mathrm{d}S^\gamma &= \frac{(\sigma S)^2}{2}\gamma(\gamma-1)S^{\gamma-2}\,\mathrm{d}t + \gamma S^{\gamma-1}\,\mathrm{d}S \\
&= \gamma S^\gamma\left((\gamma-1)\frac{\sigma^2}{2}\,\mathrm{d}t + \frac{\mathrm{d}S}{S}\right) \\
&= \gamma V\left(\left[\mu + (\gamma-1)\frac{\sigma^2}{2}\right]\,\mathrm{d}t + \sigma\,\mathrm{d}B_t\right), \\
\frac{\mathrm{d}V}{V} &= \gamma\left(\mu + (\gamma-1)\frac{\sigma^2}{2}\right)\,\mathrm{d}t + (\gamma\sigma)\,\mathrm{d}B_t.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{d}V &= -re^{r(T-t)}S\,\mathrm{d}t + e^{r(T-t)}\,\mathrm{d}S \\
&= e^{r(T-t)}S\left(-r\,\mathrm{d}t + \frac{\mathrm{d}S}{S}\right) \\
&= (\mu - r)V\,\mathrm{d}t + \sigma V\,\mathrm{d}B_t, \\
\frac{\mathrm{d}V}{V} &= (\mu - r)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t.
\end{aligned}$$

2. Let a stationary discrete-time stochastic process x_t be given by

$$x_t = A + Bx_{t-2} + Cz_t,$$

where $z_t \sim \mathcal{N}(0,1)$ is an IID Gaussian white-noise process, and A, B, C are constants.

(a) What is the unconditional mean of the process x_t?

(b) An analyst decides to construct a forecast f_τ for future values of the process by taking its expected value, conditional on information available up through the time t when the forecast is made. That is,

$$f_\tau \equiv \mathrm{E}_t\big[x_\tau \,|\, x_t, x_{t-1}, \ldots\big], \quad \tau > t.$$

Let $A = 0.1$, $B = 0.2$, $C = 0.3$, and suppose that two recent values $x_1 = 0.4$, $x_2 = 0.5$ have just been observed. What are the one-step-ahead and two-step-ahead forecasts? That is, at time $t = 2$, what are the forecasts f_3 and f_4? What is the variance of the forecasts?

Solution:

(a) Taking expectations and using stationarity,

$$\mu \equiv \mathrm{E}[x_t] = A + B\mathrm{E}[x_{t-2}] + C\mathrm{E}[z_t] = A + B\mu,$$
$$\mu = \frac{A}{1-B}.$$

(b) At $t = 2$, all returns x_t and noise terms z_t are unknown for $t > 2$ but known for $t \leq 2$. From the return equation for r_t, we have

$$\begin{aligned}
x_3 &= A + Bx_1 + Cz_3, \\
\mathrm{E}[x_3|x_1, x_2] &= A + Bx_1 = 0.1 + (0.2)(0.4) = 0.18, \\
x_4 &= A + Bx_2 + Cz_4, \\
\mathrm{E}[x_4|x_1, x_2] &= A + Bx_2 = 0.1 + (0.2)(0.5) = 0.2, \\
\mathrm{Var}(x_3) &= \mathrm{E}\left[(x_3 - A - Bx_1)^2\right] = \mathrm{E}\left[(Cz_3)^2\right] = C^2 = 0.09, \\
\mathrm{Var}(x_4) &= \mathrm{E}\left[(x_4 - A - Bx_2)^2\right] = \mathrm{E}\left[(Cz_4)^2\right] = C^2 = 0.09.
\end{aligned}$$

where the expectations are taken at $t = 2$.

3. (a) Consider the quadratic form defined by

$$Q(x, y) = 2x^2 + 12xy - 7y^2.$$

Using Lagrange multipliers, find the location and value of the extrema of Q subject to the constraint $x + 3y = 5$. Determine whether each solution is a maximum, minimum, or neither.

(b) Two assets have correlation ρ, and their volatilities are 2σ and σ respectively. What are the weights of a minimum-variance, fully-invested portfolio of the two assets, and what is its risk? That is, minimize the portfolio variance $\sigma_p^2 = \mathbf{w}^\top C\mathbf{w}$, where C is the covariance matrix and $\mathbf{w}$ is an asset weight vector whose components satisfy the budget constraint $w_1 + w_2 = 1$.

(c) In the problem above, for what values of ρ and σ will the solution also satisfy an inequality constraint $0 \leq w_i \leq 1$? (That is, the optimal portfolio is also unlevered and long-only.)

Solution:

(a) To solve along the line $x + 3y = 5$ using Lagrange multipliers, extremize

$$\begin{aligned}\mathcal{L} &= Q(x,y) - \gamma(x + 3y - 5)\\ &= 2x^2 + 12xy - 7y^2 - \gamma(x + 3y - 5).\end{aligned}$$

Then differentiating,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= 4x + 12y - \gamma = 0,\\ \frac{\partial \mathcal{L}}{\partial y} &= 12x - 14y - 3\gamma = 0,\\ \frac{\partial \mathcal{L}}{\partial \gamma} &= x + 3y - 5 = 0.\end{aligned}$$

Eliminating γ by subtracting the second equation from 3 times the first gives

$$y = 0, \quad x = 5$$

with solution

$$Q(5,0) = 50$$

This single critical point is a maximum of Q along the line. Since there is only one critical point, this can be quickly checked by evaluation Q at any other point along the line, such as $Q(-1,2) = -50 < Q_{\max}$

Alternatively, if one substitutes $y = t$, $x = 5 - 3t$ along the line, then $Q = -25t^2 + 50$, an unconstrained function of a single variable whose single maximum is clearly located at $t = 0$.

(b) In components, we have

$$\mathcal{L} = 4\sigma^2 w_1^2 + \sigma^2 w_2^2 + 4\rho\sigma^2 w_1 w_2 - \gamma(w_1 + w_2 - 1)$$

which is extremized for

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial w_1} &= 8\sigma^2 w_1 + 4\rho\sigma^2 w_2 - \gamma = 0,\\ \frac{\partial \mathcal{L}}{\partial w_2} &= 4\rho\sigma^2 w_1 + 2\sigma^2 w_2 - \gamma = 0,\\ \frac{\partial \mathcal{L}}{\partial \gamma} &= w_1 + w_2 - 1 = 0.\end{aligned}$$

Subtracting the first two equations eliminates γ and gives

$$(8 - 4\rho)w_1 + (4\rho - 2)w_2 = 0,$$

so

$$w_2 = \frac{4 - 2\rho}{1 - 2\rho} w_1.$$

Substituting into the constraint finally gives

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{5-4\rho} \begin{pmatrix} 1-2\rho \\ 4-2\rho \end{pmatrix}$$

along with

$$\begin{aligned} \sigma_p^2 &= \frac{\sigma^2}{(5-4\rho)^2} \left[4(1-2\rho)^2 + (4-2\rho)^2 + 4\rho(1-2\rho)(4-2\rho) \right] \\ &= \frac{20 - 16\rho - 20\rho^2 + 16\rho^3}{(5-4\rho)^2} \sigma^2 \\ &= \frac{4-4\rho^2}{5-4\rho} \sigma^2. \end{aligned}$$

(c) The weights w_i are independent of σ. Since the correlation is bounded, $-1 \leq \rho \leq 1$, w_2 is always non-negative. However w_1 changes sign when $\rho \to 1/2$, so the portfolio is unlevered and long-only provided that the correlation $-1 \leq \rho < 1/2$.

4. The returns on a set of N assets are believed to follow the mean-reverting process

$$R_{it} - \mu_i = -\lambda(R_{i(t-1)} - \mu_i) + \sigma_i z_{it},$$

where μ_i, σ_i, λ are constants, $i = 1, \ldots N$; $|\lambda| < 1$; and

$$\mathbb{E}[z_{it}] = 0; \quad \mathbb{E}[z_{it} z_{js}] = \begin{cases} 1 & \text{if } t = s \text{ and } i = j, \\ 0 & \text{if } t \neq s \text{ or } i \neq j; \end{cases}$$

A market-neutral long/short trading strategy attempts to profit by investing capital in weights assigned according to

$$w_{it} = -\frac{1}{N}(R_{it} - \overline{R}_t),$$

where the market average return is defined by

$$\overline{R}_t = \frac{1}{N} \sum_{i=1}^{N} R_{it}.$$

Assume there are no transaction costs and the risk-free rate $R_f = 0$. Find the expected portfolio return

$$\mathbb{E}[R_p] = \mathbb{E}\left[\sum_i w_{i(t-1)} R_{it}\right]$$

in terms of the parameters given. Under what conditions is this expected return positive?

Solution: We consider the more general case of using lag-k returns as the weights. The time series of daily *portfolio returns*, π_t is given by

$$\pi_t(k) = \sum_{i \in U} w_{i(t-k)} R_{it} \ . \tag{1}$$

This depends on the lag parameter k, so the answer should demonstrate the k-dependence of the strategy.

Notice that the weights w depend on the the market average $\overline{R}_t$, which in turn depends on all of the stocks. In the most general case, stocks could be cross-correlated and the result requires a full cross-covariance matrix Γ_k, whose (ij) matrix elements are $\mathrm{Cov}(R_{it}, R_{j,(t-k)})$. Each R_i could be multiplied by every R_j.

In the present case, each stock is correlated only with its own lagged returns, so things are simpler. Γ_k is a diagonal matrix and everything can be written as a sum of independent AR(1) autocovariances $\gamma_k(i)$.

The zeroth-order autocovariance is just the variance itself,

$$\begin{aligned}
\gamma_0 = \mathrm{Var}[R_t] &= \mathrm{E}\left[(R_t - \mu)^2\right] \\
&= \lambda^2 \mathrm{E}\left[(R_{t-1} - \mu)^2\right] + \mathrm{E}\left[(\sigma z_t^2)\right] \\
&= \lambda^2 \gamma_0 + \sigma^2,
\end{aligned}$$

so that

$$\gamma_0 = \frac{\sigma^2}{1 - \lambda^2}.$$

The higher order autocovariances can be obtained by recursion.

$$\begin{aligned}
\gamma_k &= \mathrm{E}\left[(R_t - \mu)(R_{t-k} - \mu)\right] \\
&= -\lambda \mathrm{E}\left[(R_{t-1} - \mu)(R_{t-k} - \mu)\right] \\
&= -\lambda \gamma_{k-1},
\end{aligned}$$

so that

$$\gamma_k = (-\lambda)^k \gamma_0 = \frac{(-\lambda)^k}{1 - \lambda^2} \sigma^2.$$

Now we can use this result to compute the closed-form expectation of the strategy P/L.

$$\begin{aligned}
\mathrm{E}\left[\pi_t(k)\right] &= \frac{1 - N}{N^2} \sum_{i=1}^{N} \gamma_k(i) - \frac{1}{N} \sum_{i=1}^{N} (\mu_i - \bar{\mu})^2 \\
&= -\frac{(-\lambda)^k}{1 - \lambda^2} \left(1 - \frac{1}{N}\right) \left(\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2\right) - \frac{1}{N} \sum_{i=1}^{N} (\mu_i - \bar{\mu})^2.
\end{aligned}$$

The last term, which is always negative, represents dispersion of the means. It is independent of k and vanishes only if all the mean returns are equal. The first term alternates in sign, so that only odd lags contribute positive P/L, and decreases in magnitude with k.

5. Two stocks have prices S_1 and S_2 that follow geometric Brownian motion with the same stochastic process $\mathrm{d}B$:

$$\mathrm{d}S_1 = \mu_1 S_1 \ \mathrm{d}t + \sigma_1 S_1 \ \mathrm{d}B,$$
$$\mathrm{d}S_2 = \mu_2 S_2 \ \mathrm{d}t + \sigma_2 S_2 \ \mathrm{d}B.$$

(a) A contract has value $V = S_1 S_2$. You can show that V also follows geometric Brownian motion. What are its drift and volatility parameters?

(b) What is the process followed by $1/V$?

(c) A call option on V with strike K has value $C(t, V)$ and payoff at expiration $\max(S_1 S_2 - K, 0)$. What PDE does the option satisfy?

Solution: It is convenient to write the series by dividing through such that the right-hand sides are Itô processes with constant coefficients:

$$\begin{aligned}\frac{\mathrm{d}S_1}{S_1} &= \mu_1 \,\mathrm{d}t + \sigma_1 \,\mathrm{d}B,\\ \frac{\mathrm{d}S_2}{S_2} &= \mu_2 \,\mathrm{d}t + \sigma_2 \,\mathrm{d}B.\end{aligned}$$

(a) Applying the product rule to $V = S_1 S_2$,

$$\mathrm{d}V = S_1 \,\mathrm{d}S_2 + S_2 \,\mathrm{d}S_1 + \mathrm{d}S_1 \,\mathrm{d}S_2,$$

so that

$$\begin{aligned}\frac{\mathrm{d}V}{V} &= \frac{\mathrm{d}S_1}{S_1} + \frac{\mathrm{d}S_2}{S_2} + \frac{\mathrm{d}S_1}{S_1}\cdot\frac{\mathrm{d}S_2}{S_2}\\ &= (\mu_1+\mu_2)\,\mathrm{d}t + (\sigma_1+\sigma_2)\,\mathrm{d}B + \sigma_1\sigma_2(\mathrm{d}B)^2\end{aligned}$$

where we can replace $(\mathrm{d}B)^2 \to \mathrm{d}t$ and write

$$\frac{\mathrm{d}V}{V} = (\mu_1+\mu_2+\sigma_1\sigma_2)\,\mathrm{d}t + (\sigma_1+\sigma_2)\,\mathrm{d}B.$$

This standard form for geometric Brownian motion lets us read off that the drift and volatility parameters are

$$\begin{aligned}\mu_V &= \mu_1+\mu_2+\sigma_1\sigma_2,\\ \sigma_V &= \sigma_1+\sigma_2.\end{aligned}$$

(b) For $F(t,V) = 1/V$, apply Itô's formula:

$$\begin{aligned}\mathrm{d}F = \mathrm{d}\left(\frac{1}{V}\right) &= \frac{-1}{V^2}\,\mathrm{d}V + \frac{b^2}{2}\left(\frac{2}{V^3}\right)\mathrm{d}t,\\ \frac{\mathrm{d}F}{F} &= -\frac{\mathrm{d}V}{V} + \frac{(\sigma_1+\sigma_2)^2V^2}{2}\left(\frac{2}{V^2}\right)\mathrm{d}t\\ &= \left[(\sigma_1+\sigma_2)^2 - (\mu_1+\mu_2+\sigma_1\sigma_2)\right]\mathrm{d}t - (\sigma_1+\sigma_2)\,\mathrm{d}B.\end{aligned}$$

This is also in the form of a geometric Brownian motion.

(c) Since V follows a standard geometric Brownian motion, options with V as an underlying have values $C(t,V)$ that satisfy the usual Black-Scholes PDE, with the appropriate volatility parameter for V:

$$\frac{\partial C}{\partial t} + \frac{(\sigma_1+\sigma_2)^2}{2}V^2\frac{\partial^2 C}{\partial V^2} + rV\frac{\partial C}{\partial V} - rC = 0.$$