Recitation 12

Spring 2021

Question 1

Stock ABC has a current price of \$90. Consider four options maturing in one year:

- One-year put option with strike price \$90
- One-year put option with strike price \$110
- One-year call option with strike price \$90
- One-year call option with strike price \$110

Construct a payoff diagram for the following scenarios:

(a) Buy two put options with strike price \$90 and one bond with face value \$20.

(b) Buy one put option with strike \$90 and one put option with strike price \$110.

(c) Short two puts with strike price \$90 and long one call with strike price \$110.

Solutions:

(a) Buy two put options with strike price \$90 and one bond with face value \$20.

The payoff from a single put option with strike price $K = \$90$ is described by:

$$\text{Payoff} = \begin{cases} 90 - S_1 & \text{if } S_1 \leq 90 \\ 0 & \text{if } S_1 > 90 \end{cases}$$

With two put options, the payoffs are doubled:

$$\text{Payoff} = \begin{cases} 180 - 2S_1 & \text{if } S_1 \leq 90 \\ 0 & \text{if } S_1 > 90 \end{cases}$$

The payoff diagram for two puts is shown below:

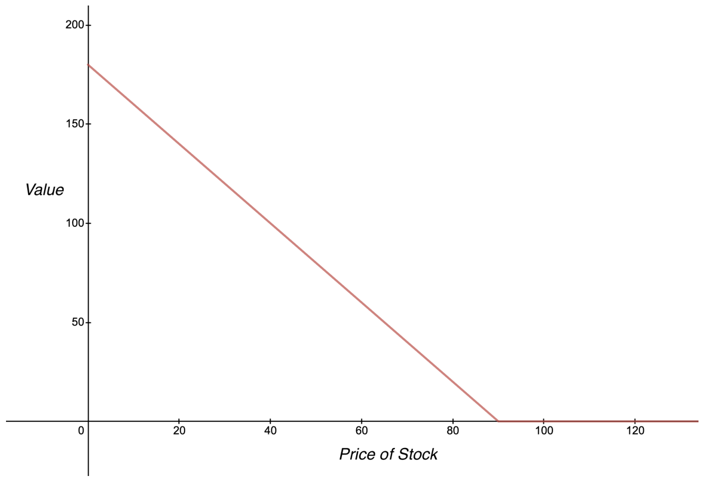


Figure 1: Two put options, $K = 90$

The payoff from the bond is \$20 no matter the stock price S and looks like the following:

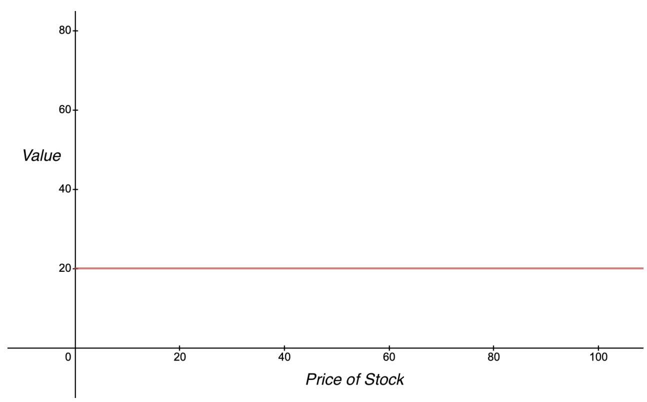


Figure 2: Bond, $F = 20$

Altogether, the payoff structure for the two put options and the bond is:

$$\text{Payoff} = \begin{cases} 200 - 2S_1 & \text{if } S_1 \leq 90 \\ 20 & \text{if } S_1 > 90 \end{cases}$$

The payoff diagram is shown below:

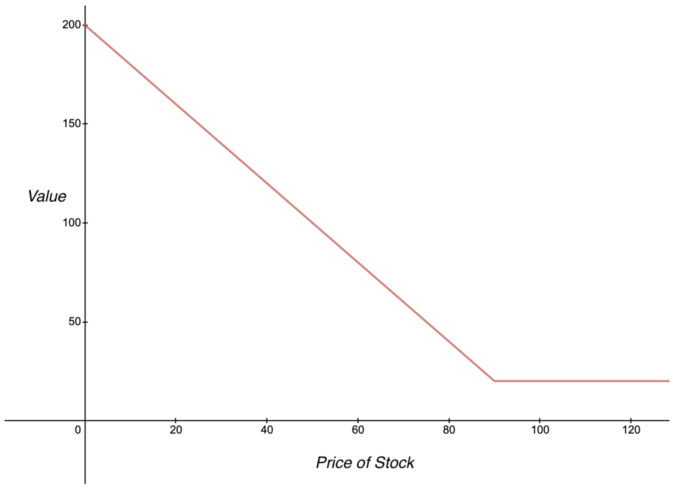


Figure 3: Portfolio

(b) Buy one put option with strike \$90 and one put option with strike price \$110.

The payoffs from the put option with strike price \$90 are described in part a. The payoffs from the put with strike price \$110 are similar and described below:

$$\text{Payoff} = \begin{cases} 110 - S_1 & \text{if } S_1 \leq 110 \\ 0 & \text{if } S_1 > 110 \end{cases}$$

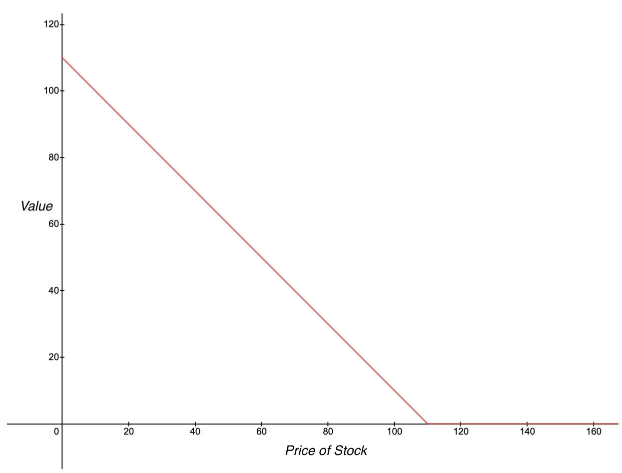


Figure 4: Put option, $K = 110$

Altogether, the portfolio payoffs are described by:

$$\text{Payoff} = \begin{cases} 200 - 2S_1 & \text{if } S_1 \leq 90 \\ 110 - S_1 & \text{if } 90 < S_1 \leq 110 \\ 0 & \text{if } S_1 > 110 \end{cases}$$

The payoff diagram is shown below:

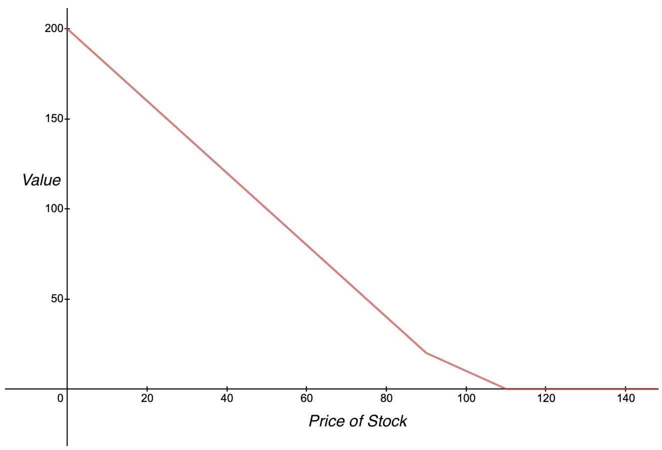


Figure 5: Portfolio

(c) Short two puts with strike price \$90 and long one call with strike price \$110.

If you short a put with strike price \$90, your payoff is $S_1 - 90$ if the put is in-the-money ($S_1 \leq 90$), since the counterparty will exercise the right to sell the stock to you at the price of \$90. In other words, you will pay \$90 for a stock with a market price of S_1. Your payoff is 0 if $S_1 > 90$. By shorting two puts, the payoffs double and are:

$$\text{Payoff} = \begin{cases} -180 + 2S_1 & \text{if } S_1 \leq 90 \\ 0 & \text{if } S_1 > 90 \end{cases}$$

Graphically, this looks like:

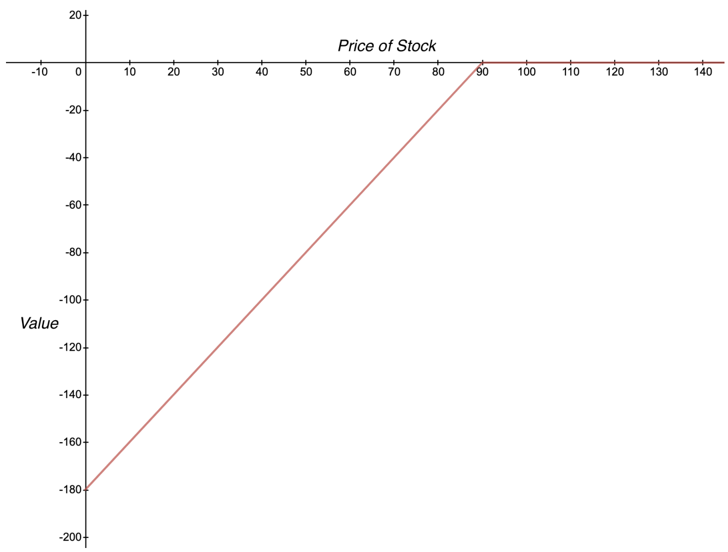


Figure 6: Short two puts, $K = 90$

The payoffs from the long position on a single call option with strike price \$110 are:

$$\text{Payoff} = \begin{cases} 0 & \text{if } S_1 < 110 \\ S_1 - 110 & \text{if } S_1 \geq 110 \end{cases}$$

Graphically, this looks like:

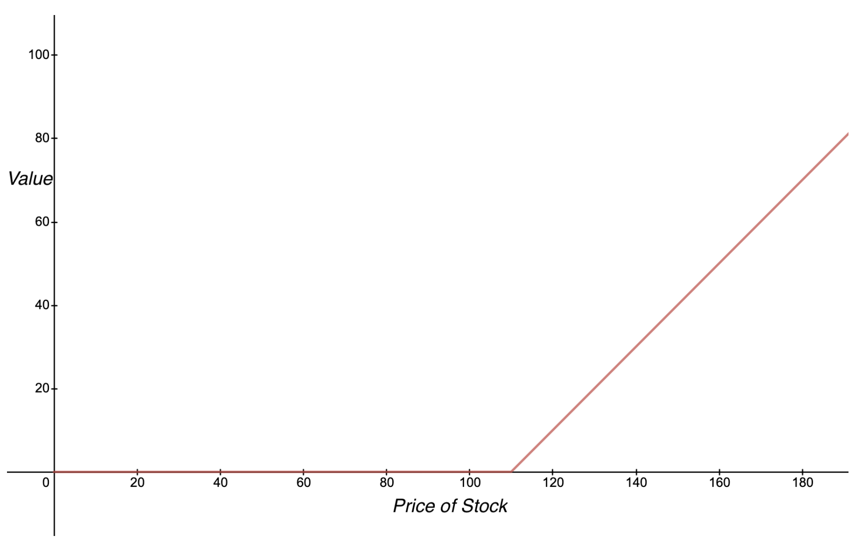


Figure 7: Call option, $K = 110$

Altogether, the payoffs for the portfolio are:

$$\text{Payoff} = \begin{cases} -180 + 2S_1 & \text{if } S_1 \leq 90 \\ 0 & \text{if } 90 < S_1 < 110 \\ S_1 - 110 & \text{if } S_1 \geq 110 \end{cases}$$

The payoff diagram is shown below:

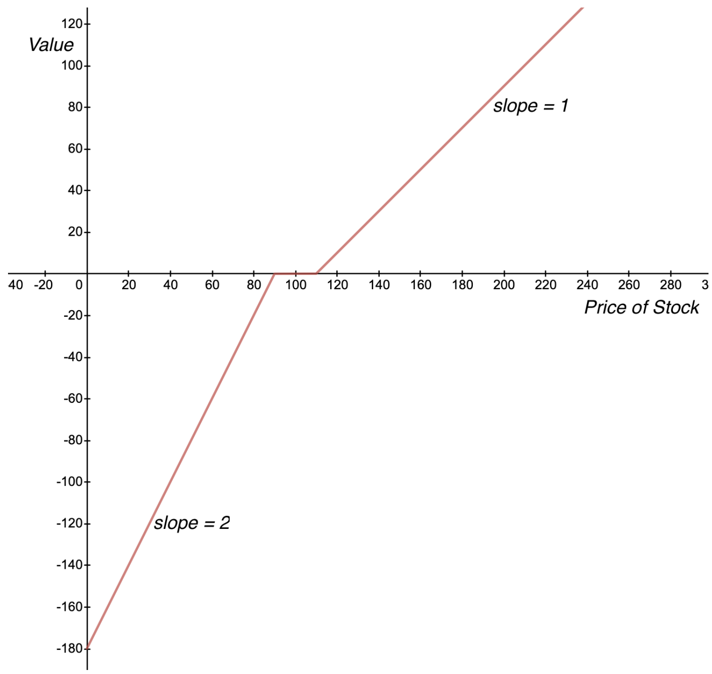


Figure 8: Portfolio

Question 2

Consider the following payoff function $X(S)$:

$$X(S) = \begin{cases} 110 - 6S & S \in [0, 10) \\ 70 - 2S & S \in [10, 20) \\ 90 - 3S & S \in [20, 30) \\ 0 & S \in [30, 40) \\ 2S - 80 & S \geq 40 \end{cases}$$

(a) Plot the payoff diagram for $X(S)$.

(b) Construct a portfolio consisting of only European calls and puts to replicate the payoff function $X(S)$, given below. Be specific about how many options of each type are in your portfolio.

Solutions:

The payoff diagram is shown below:

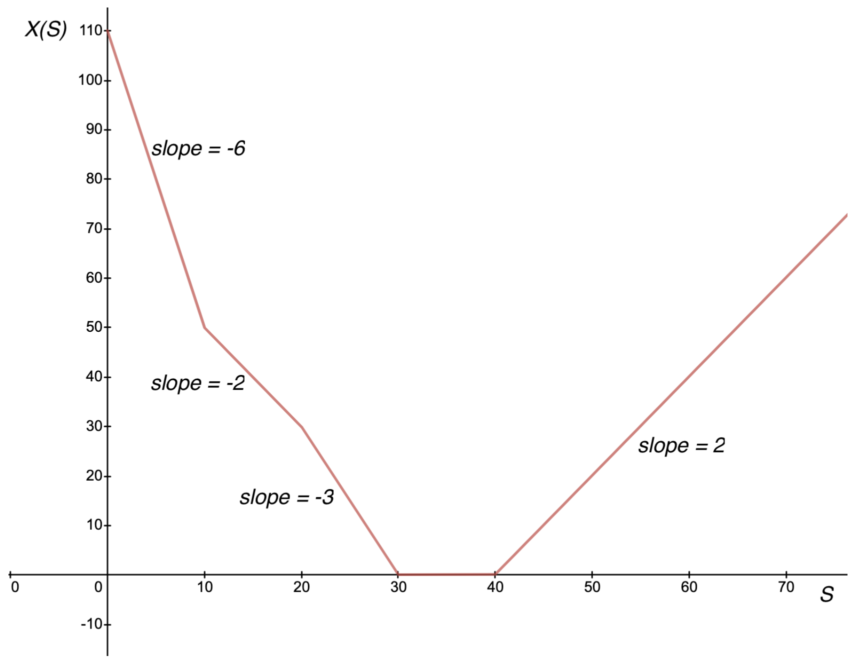


Figure 9: $X(S)$

The solution to this problem is not unique. The easiest way to solve for a portfolio is to start from the right end of the graph and work towards the left.

The right-most segment has a slope of 2 starting at $S = 40$. We can construct this as 2 call options with strike price $K = 40$. The next segment is flat with a payoff of 0, which is accounted for by the call options. The call has a payoff of 0 for $S < 40$ and therefore does not affect the other segments of the graph.

Next is the segment from $S = 20$ to $S = 30$ with a slope of -3. This can be replicated by several put options with a strike price of 30. Since the slope is -3, we want 3 of these put options with a strike price of $K = 30$. The payoff of these puts is 0 for $S > 30$, leaving the segments to the right unaffected. The payoff is $90 - 3S$ for $S < 30$. Our payoffs now look like:

$$\begin{cases} 90 - 3S & S \in [0, 30) \\ 0 & S \in [30, 40) \\ 2S - 80 & S \geq 40 \end{cases}$$

The next segment has a slope of -2, from $S = 10$ to $S = 20$. The 3 put options included in the portfolio with strike price 30 would give this segment a slope of -3, so we need to short a put option with strike price 20 to achieve the slope of -2 in this segment. The payoff of this short is 0 for $S > 20$, leaving the segments to the right unaffected. When $S < 20$, the payoff is now $90 - 3S + (-20 + S) = 70 - 2S$. So our payoffs now look like:

$$\begin{cases} 70 - 2S & S \in [0, 20) \\ 90 - 3S & S \in [20, 30) \\ 0 & S \in [30, 40) \\ 2S - 80 & S \geq 40 \end{cases}$$

Lastly, the segment from $S = 0$ to $S = 10$ has a slope of -6, so we need to buy another 4 put options, each with a strike price of 10. The payoff of these puts is 0 for $S > 10$, leaving the segments to the right unaffected. When $S < 10$, the payoff is now $70 - 2S + 4(10 - S) = 110 - 6S$. So we have the correct payoff structure:

$$\begin{cases} 110 - 6S & S \in [0, 10) \\ 70 - 2S & S \in [10, 20) \\ 90 - 3S & S \in [20, 30) \\ 0 & S \in [30, 40) \\ 2S - 80 & S \geq 40 \end{cases}$$

Altogether, our portfolio consists of:

- Long 2 call options with a strike price of $K = 40$
- Long 3 put options with a strike price of $K = 30$
- Short 1 put option with a strike price of $K = 20$
- Long 4 put options with a strike price of $K = 10$

Question 3

when you want to price an asset, you should find other one or two assets, duplicate the payoff of one asset from others' payoffs, then the present price can also be duplicated from the same portfolio

Suppose ABC is an educational technology startup which has 500,000 shares outstanding. Next year, the value of ABC's assets can either be \$25,000,000 or \$5,000,000, depending on the market adoption of early childhood development applications. ABC has a \$20,000,000 face value bank loan due one year from now.
There are two options on ABC's stock currently traded in the market: a call option with a 1-year maturity and \$7 strike price traded at \$1.50 per option, and a put option with a 1-year maturity and \$7 strike price traded at \$2.52 per option. Suppose that the current one-year risk-free rate is 0%.
What is the current market value of ABC's equity and ABC's debt?

Solutions:

The table below shows the value of assets, debt, and equity in year 1 in each of the two possible states:

debt value is less than asset value

	Asset Value	Debt Value	Equity Value	Share Price
Bad state	\$5mm	\$5mm	\$0mm	\$0
Good state	\$25mm	\$20mm	\$5mm	\$10

The value of equity will be whatever is left after we repay the loan

why does the put-call parity not work in this case?
formula S = C + KB - P assumes that the underlying stock pays no dividends.

We will use the current option prices to impute the current share price. Below are the values of one share and one call option in year 1 in each of the states:

	Share Value	Call Option Value/Payoff
Bad state	\$0	\$0
Good state	\$10	\$3

From this, we can see that, since the share always pays 10/3 as much as the call option, **the price of the share today should be 10/3 the price of the all option today**. So the share price today is $10/3 \times \$1.50 = \$5.00$. the law of one price

Since ABC has 500,000 shares outstanding, the total market value of ABC's equity is $500,000 \times \$5.00 = \$2,500,000$.

Now, we consider the market value of ABC's debt. In the good state, the value of debt is \$20mm, while in the bad state, the value is \$5mm, compared to the face value of \$20mm. We also introduce a default put option for the debtholder to compensate for the value lost in the event of default. This is illustrated below:

face value

	ABC Debt Value	Risk-free Debt	Default Put Option
Bad state	\$5mm	\$20mm	\$15mm
Good state	\$20mm	\$20mm	\$0mm

5 + 15 = 20

We observe that the current value of ABC's debt should equal the current value of \$20mm face value risk-free debt, minus the value of an option that pays \$15mm in the bad state and \$0mm in the good state (the "default put option").

We now need to calculate the value of the default put option based on the given information. We compare the default put option to the put option specified in the problem statement, which pays \$0 in the good state and $\$10 - \$3 = \$7$ in the bad state.

	Default Put Option	Given Put Option
Bad state	\$15mm	\$7
Good state	\$0mm	\$0

We note that the value of the default put option should be \$15mm / 7 times the value of the \$7 strike put option on the stock. So the value of the default put option is $\$15\text{mm}/7 \times \$2.52 = \$5.4\text{mm}$. And, the value of the \$20mm face value risk-free debt is \$20mm since the one-year risk-free rate is 0%.

So the market value of ABC's debt is $\$20\text{mm} - \$5.4\text{mm} = \$14.6\text{mm}$.

The yield to maturity on ABC's debt is $\frac{\$20\text{mm}}{\$14.6\text{mm}} - 1 = 37\%$.

even though the risk-free rate in this economy is 0%, the yield to maturity on this bond is 37%. This high yield to maturity is explained by the fact that, in the event of default, the holder of this loan loses \$15 million out of \$20 million face value. And, therefore, the high YTM captures the risk of this loss.

Question 4

Suppose you observe the following prices of financial instruments in the market:

- The stock price of firm XYZ is \$20 per share
- A European put option on XYZ with 1-year maturity and strike price of \$18 sells for \$3.33
- A European call option on XYZ with 1-year maturity and strike price of \$18 sells for \$7.00

Suppose the continuously compounded risk-free rate is 8%. Is there an arbitrage opportunity? If so, describe a trading strategy designed to profit from this opportunity.

Solutions:

We first examine whether put-call parity holds to determine whether there is an arbitrage opportunity:

$$C + e^{-rT}K \stackrel{?}{=} P + S$$
$$C + e^{-rT}K = \$7 + e^{-8\%} \times 18 = \$23.62$$
$$P + S = \$3.33 + \$20 = \$23.33 \neq \$23.62$$

Since put-call parity does not hold, there is an arbitrage opportunity, as the left-hand side is overpriced relative to the right-hand side. Since the put and stock are underpriced, we can exploit this by:

These transactions generate the immediate profit of $0.29.

1. Sell call at \$7.00
2. Borrow $e^{-8\%} \times 18$ at the risk-free rate of 8%
3. Buy put at \$3.33
4. Buy stock at \$20.00

when it comes to arbitrage, we should buy what is cheap and sell what is expensive

C + KB > P + S

The first two steps raise \$23.62. Steps 3 and 4 cost \$23.33.
What happens a year from now? We examine two cases: one where the stock is above \$18 a share, and one where the stock is below \$18 a share.

Case 1: Stock is above \$18 per share
If the stock is above \$18 per share in year 1, the call expires in-the-money, and the put expires out-of-the-money. You need to deliver $S_T - \$18$ to the counterparty you sold the call to, and repay the loan of \$18.
You sell the share you own for S_T, and pay the $S_T - \$18$ and \$18 to the counterparty and bank, respectively.

- sell the stock to the caller (physical delivery) at $18 and repay the loan
- sell the stock at S_T, pay the caller S_T - 18 (cash delivery), and repay the loan

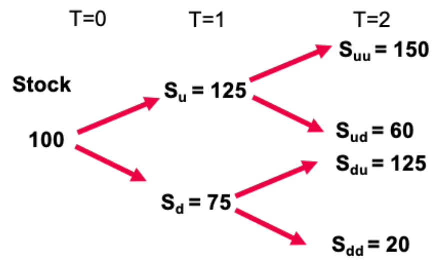
Case 2: Stock is below \$18 per share
If the stock is below \$18 per share in year 1, the call expires out-of-the-money, and the put expires in-the-money. You exercise the put option and receive \$18 for the share you own (or you can think of this as selling at S_T and exercising the put option to receive a payoff of $18 - S_T$). You repay the loan of \$18.

- sell the stock at $18 (physical delivery) and repay the loan
- sell the stock at S_T, get 18 - S_T (cash delivery), and repay the loan

In each case, you have a net payoff of \$0 in year 1, and you receive \$23.62 − \$23.33 = \$0.29 in year 0.

Question 5

Suppose there is a stock with a current stock price of \$100. The graph below shows the evolution of the share price over the next two years:



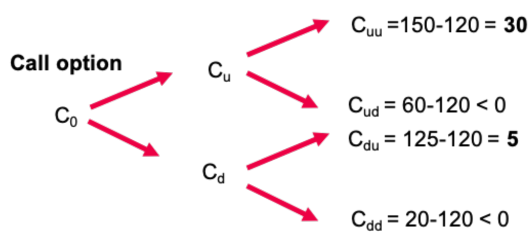
Suppose that the risk-free rate is 3.5% EAR. Assume that there is no arbitrage, and investors can borrow and lend at the risk-free rate.

(a) What is the current (Year 0) price of a European **call** option on this stock with a strike price of \$120, expiring at $T = 2$?

(b) What is the current (Year 0) price of a European **put** option on this stock with a strike price of \$120, expiring at $T = 2$?

Solutions:

(a) We start by creating a tree with the option payoff in different states. If the price at $T = 2$ is over \$120, the payoff is $S_2 - \$120$. Otherwise, the payoff is \$0. The tree is shown below:



First, let's price the option in Year 1 in the "up" state, replicating the payoff with buying a shares and a lending b at the risk-free rate:

$$150a + 1.035b = 30$$
$$60a + 1.035b = 0$$

Solving for a and b gives us $a = 0.333$ and $b = -19.32$. So, the option in year 1 in the "up" state is:

$$C_u = 125 \times 0.333 - 19.32 = 22.34$$

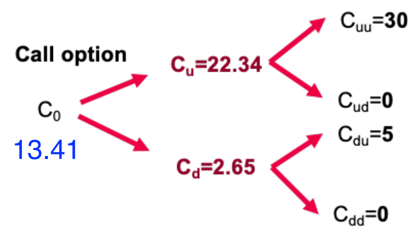
We follow the same steps to price the option in the down state in year 1:

$$125a + 1.035b = 5$$
$$20a + 1.035b = 0$$

Solving yields $a = 0.0476$, $b = -0.9202$. So:

$$C_d = 75 \times 0.0476 - 0.9202 = 2.65$$

The final step is to price the option in year 0:



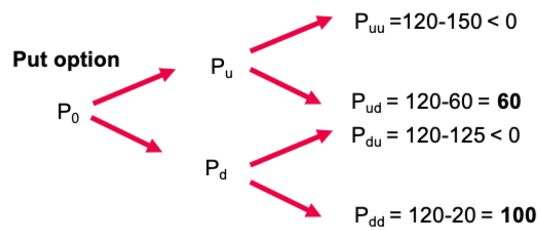
To find C_0, we solve the following system of equations (similar to the year 1 pricing systems of equations):

$$125a + 1.035b = 22.34$$
$$75a + 1.035b = 2.65$$

Solving yields $a = 0.3938$, $b = -25.98$. So, the price of the option in year 0 is:

$$C_0 = 100 \times 0.3938 - 25.98 = \mathbf{13.41}$$

(b) We construct a similar payoff tree, this time for a put. If the price at $T = 2$ is under \$120, the payoff is $120 - S_2$, while the payoff is \$0 otherwise. The tree is shown below:



We'll first find P_u and P_d, then use those to find P_0, similar to the method for pricing the call option.

In the up state, year 1:

$$150a + 1.035b = 0$$
$$60a + 1.035b = 60$$

Solving yields $a = -0.677$, $b = 96.62$. So

$$P_u = 125 \times -0.677 + 96.62 = 13.29$$

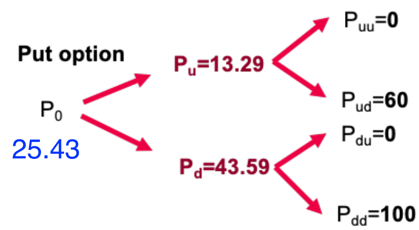
In the down state, year 1:

$$125a + 1.035b = 0$$
$$20a + 1.035b = 100$$

Solving yields $a = -0.952$, $b = 115.02$. So

$$P_u = 75 \times -0.952 + 115.02 = 43.59$$

The final step is to price the put in year 0:



We have the following system of equations in year 0:

$$125a + 1.035b = 13.29$$
$$75a + 1.035b = 43.59$$

Solving yields $a = -0.606$, $b = 86.04$. So, the price of the put in year 0 is:

$$P_0 = 100 \times -0.606 + 86.04 = \mathbf{25.43}$$

Alternatively, we could have priced the put using put-call parity:

$$P_0 + S_0 = \frac{K}{(1+r)^T} + C_0$$

Notice that we're given the effective annual rate and not the continuously compounded APR

$$P_0 = \frac{120}{(1+3.5\%)^2} + 13.41 - 100 = \mathbf{25.43}$$

The put-call parity demonstrates that the prices of the call and the put options we obtained are consistent.