15.415x Foundations of Modern Finance

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Lecture 14: Portfolio Theory

Key concepts

- Introduction: portfolio choice with mean-variance preferences
- Portfolios with two assets
- Portfolio frontier with multiple risky assets
- Portfolio choice with a safe asset
- Analytics of the portfolio frontier
- Properties of the tangency portfolio
- Non-mean-variance objective

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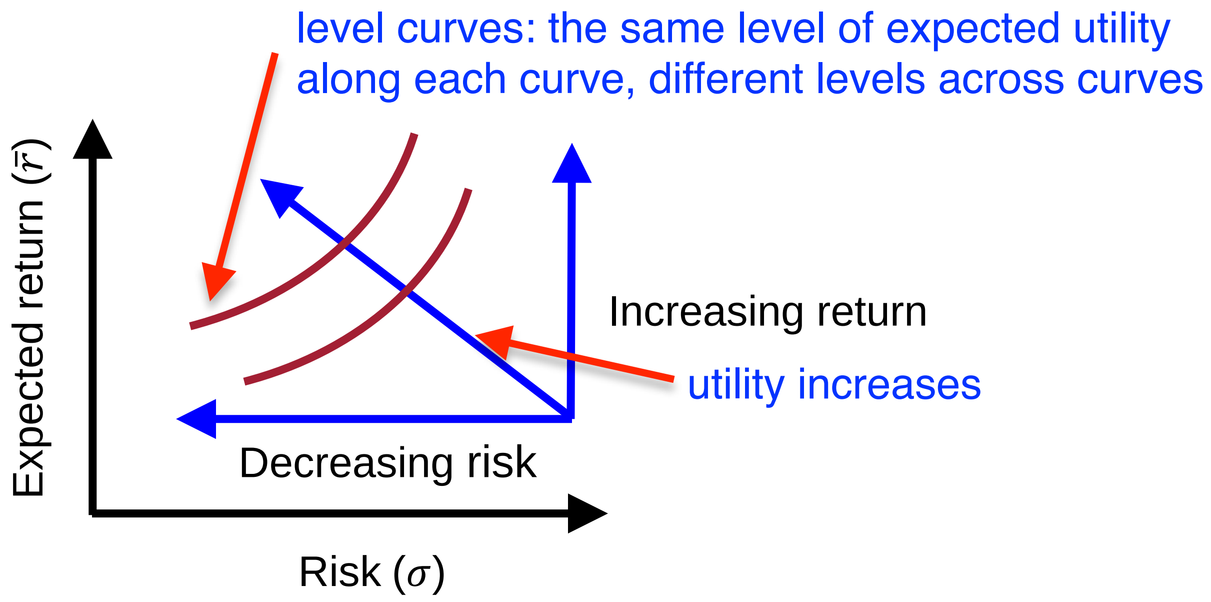
Mean-variance preferences

- How to choose a portfolio: Maximize expected utility.
- Special case – assume investors care only about the first two moments: average return and return variance/volatility (risk).
  - Minimize risk for a given expected return, Or:
  - Maximize expected return for a given risk.



Investor's problem

- Among all the portfolios with a target level of expected return, find the one with the lowest variance.

- Formally, we need to solve the following problem:

$$(P): \qquad \underset{\{w_1,\ldots,w_n\}}{\text{Minimize}} \quad \sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

$$\text{subject to:} \quad (1) \quad \textstyle\sum_{i=1}^{N} w_i = 1$$

$$(2) \quad \textstyle\sum_{i=1}^{N} w_i \, \bar{r}_i = \bar{r}_p$$

- Of all the portfolios [constraint (1)] with an expected return of $\bar{r}_p$ [constraint (2)], find the one that has the lowest variance.

Empirical examples

- We use ETFs to construct portfolios.
- Each ETF in our examples represents an exposure to a distinct asset class.

ETF	Description	Inception date
SPY	US Equity: S&P 500	Jan 22, 1993
AGG	Aggregate bonds	Sep 22, 2003
HYG	High yield bonds	Apr 04, 2007
IAU	Gold	Jan 21, 2005
IYR	US Equity: Real estate sector	Jun 12, 2000

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Two assets, long only

Suppose assets 1 and 2 are perfectly negatively correlated (\rho_{12} = -1, \sigma_1 = \sigma_2). In this case, an equally-weighted portfolio with these two assets has zero variance.

$$\bar{r}_p = w\bar{r}_1 + (1-w)\,\bar{r}_2$$

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}$$

- SPY and AGG (2008/01-2018/12).

Mean Returns	
SPY	AGG
8.70%	3.26%

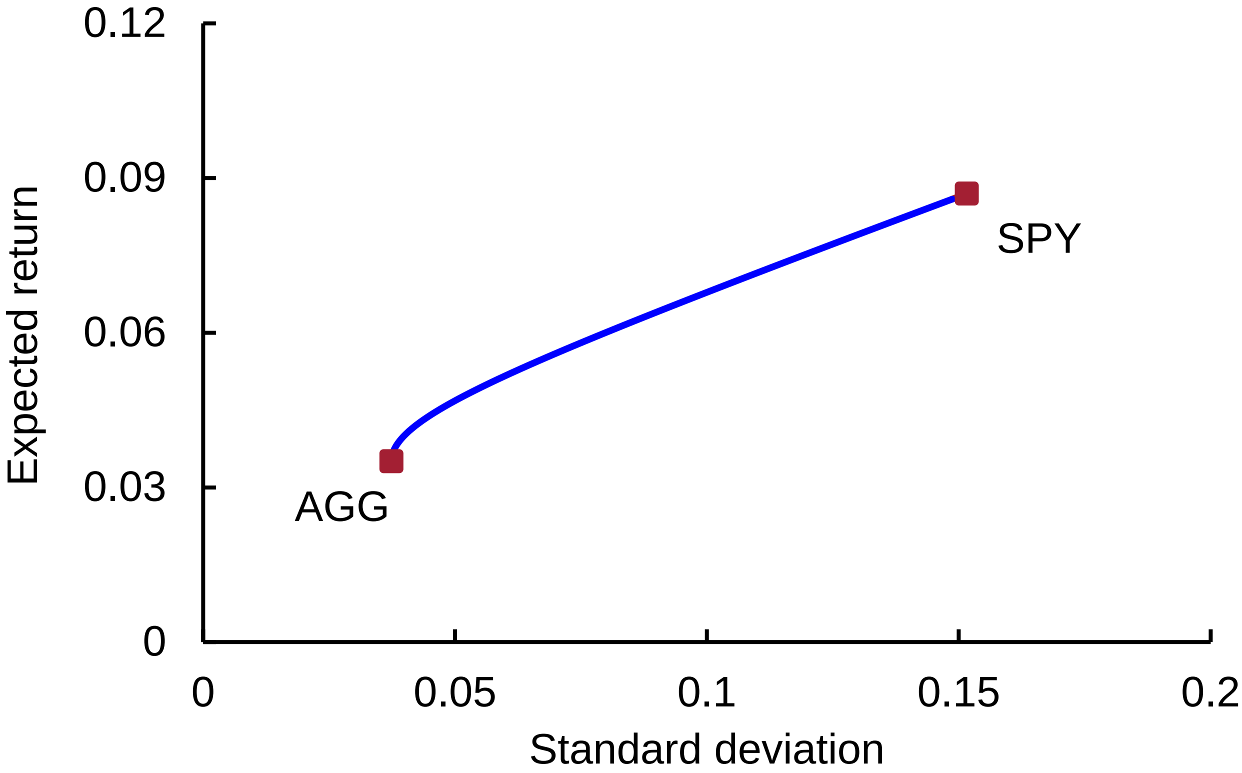
Covariances		
	SPY	AGG
SPY	0.0230	0.0004
AGG	0.0004	0.0014

- Portfolio return and risk (unique solution):

Weight in SPY	0.0%	10.0%	20.0%	30.0%	40.0%	50.0%	60.0%	70.0%	80.0%	90.0%	100.0%
Mean	3.26%	3.80%	4.35%	4.89%	5.44%	5.98%	6.52%	7.07%	7.61%	8.16%	8.70%
St Dev	3.80%	3.83%	4.43%	5.41%	6.61%	7.93%	9.32%	10.75%	12.20%	13.68%	15.16%

Standard deviation^2 = Variance

Two assets, long only



Two assets, with short sales allowed

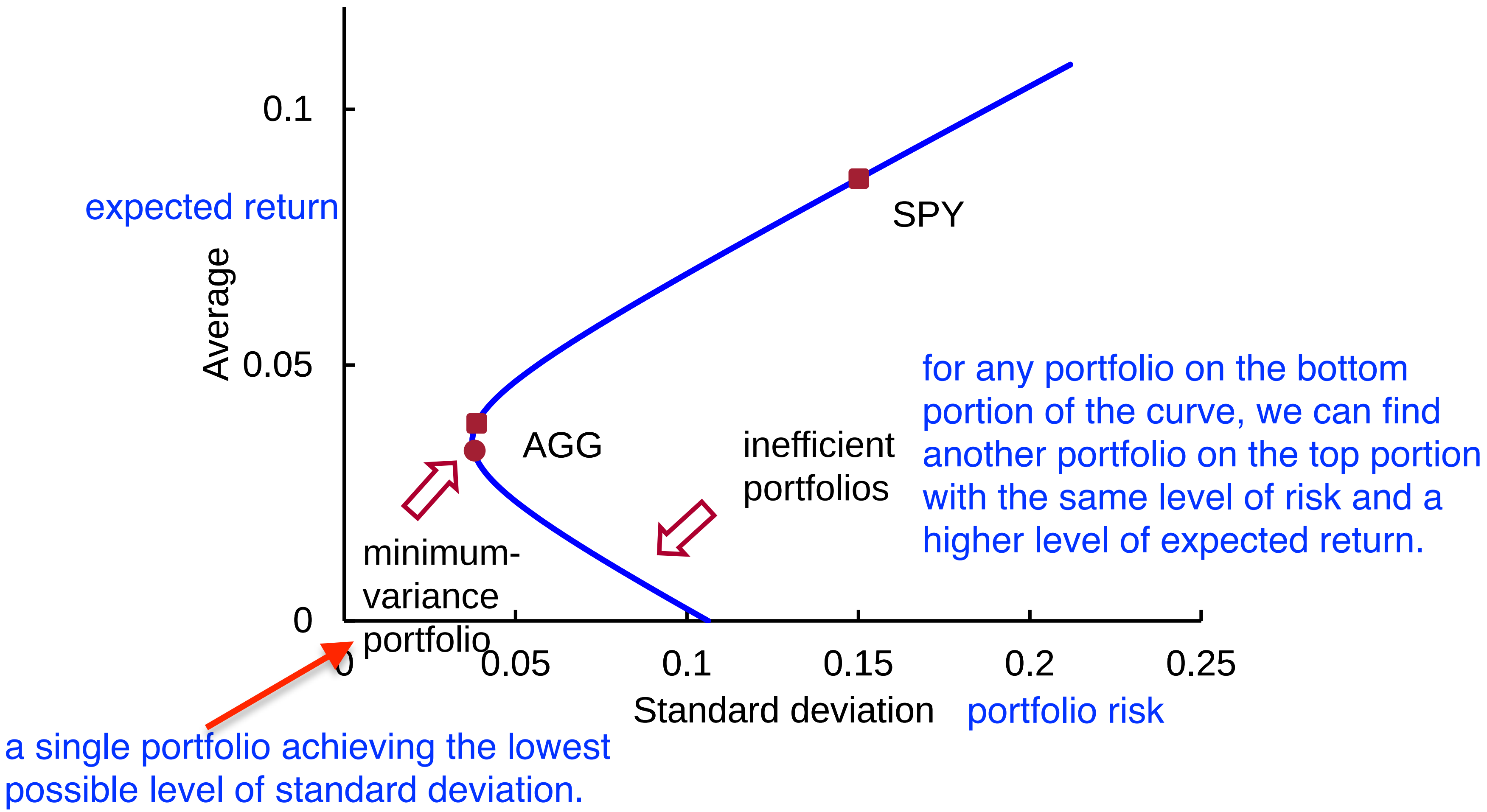
- When short sales are allowed, portfolio weights are unrestricted.

Mean Returns	
SPY	AGG
8.70%	3.26%

Covariances		
	SPY	AGG
SPY	0.0230	0.0004
AGG	0.0004	0.0014

Weight in SPY	-60.0%	-40.0%	-20.0%	0.0%	20.0%	40.0%	60.0%	80.0%	100.0%	120.0%	140.0%
Mean	-0.00%	1.09%	2.17%	3.26%	4.35%	5.44%	6.52%	7.61%	8.70%	9.79%	10.87%
St Dev	10.62%	7.81%	5.32%	3.80%	4.43%	6.61%	9.32%	12.20%	15.16%	18.16%	21.19%

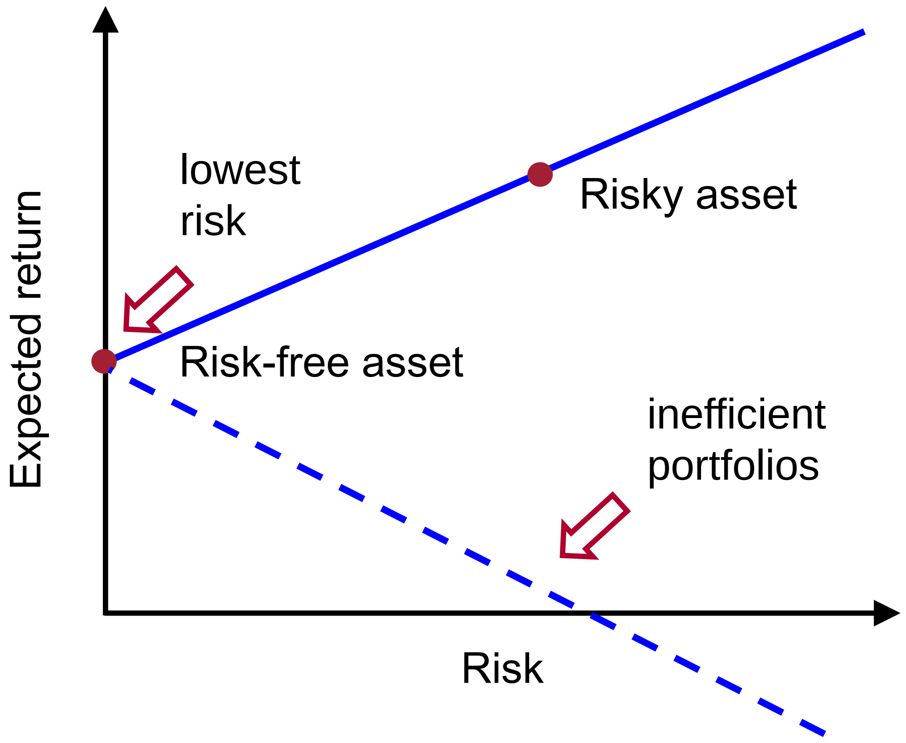
Two assets, with short sales allowed



Two assets: safe and risky

$$\bar{r}_p = w\bar{r}_1 + (1 - w)r_F$$

$$\sigma_p = w\sigma_1$$



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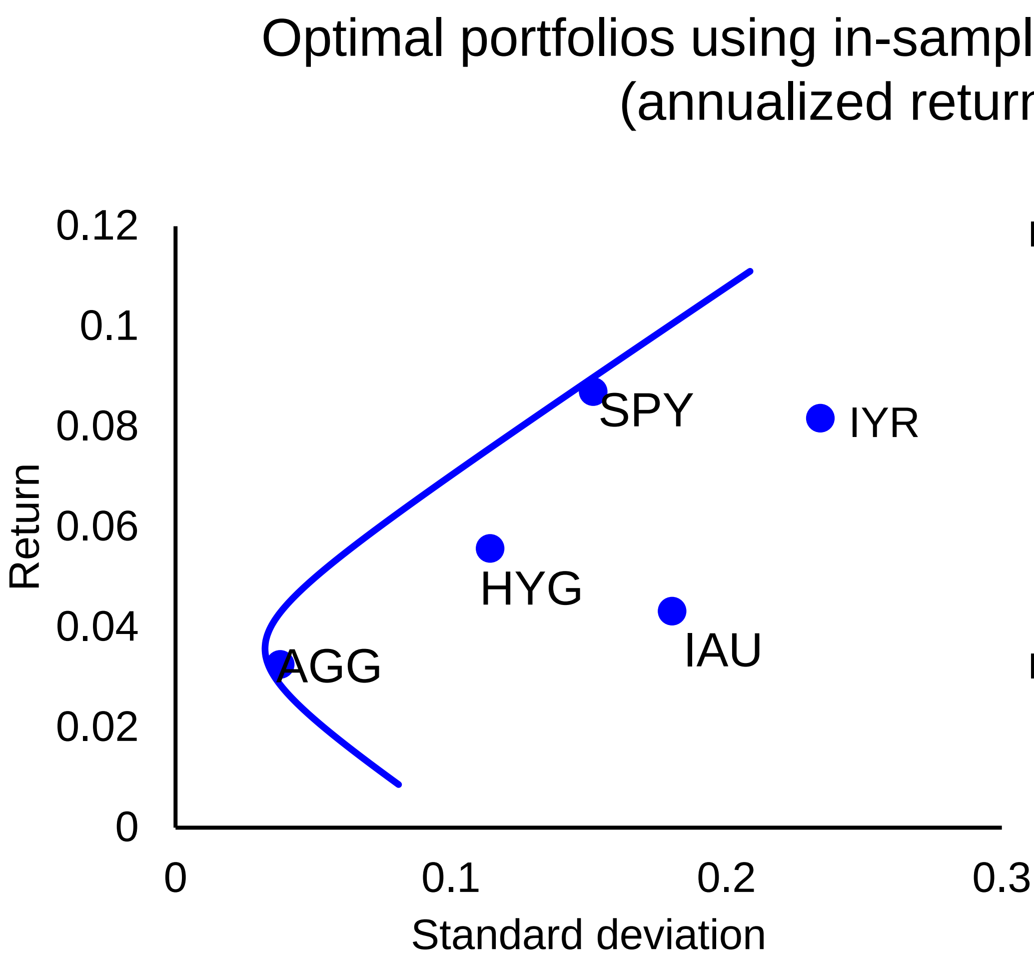
Multiple assets



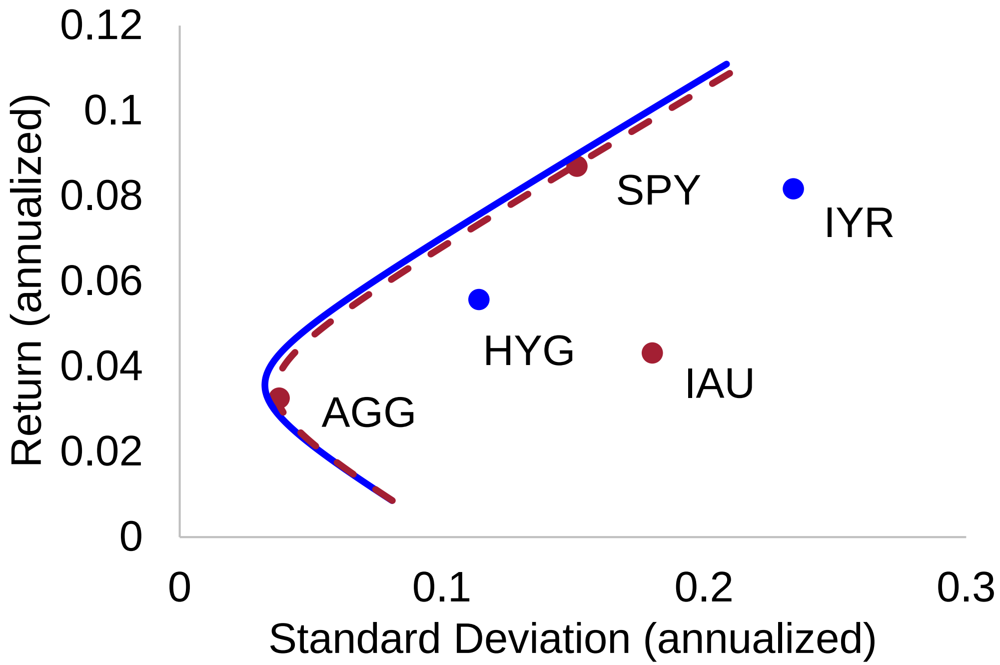
frontier portfolios cannot be dominated

reduce the set of all possible portfolios to a much smaller subset, the efficient frontier

- Given an expected return, the portfolio that minimizes risk (measured by SD or variance) is a mean-variance optimal portfolio. frontier portfolio
- The locus of all frontier portfolios in the mean-SD plane is called portfolio frontier.
- The upper part of the portfolio frontier gives the efficient frontier portfolios.

Multiple assets

Add 3-asset frontier (dashed line): AGG, SPY, and IAU



- When more assets are included, the portfolio frontier improves, i.e., moves toward upper-left: higher mean returns and lower risk.
- Intuition: Since one can always choose to ignore the new assets, including them cannot make one worse off.

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Portfolio frontier with a safe asset

- Observation: A portfolio of risk-free and risky assets can be viewed as a portfolio of two portfolios:
  - the risk-free asset, and
  - a portfolio of only risky assets.
- Example: consider a portfolio with $40 invested in the risk-free asset and $30 each in the Equity Index and LT Bonds:
  - $w_0 = 40\%$ in the risk-free asset,
  - $w_1 = 30\%$ in Equities,
  - $w_2 = 30\%$ in LT Bonds.

Portfolio frontier with a safe asset

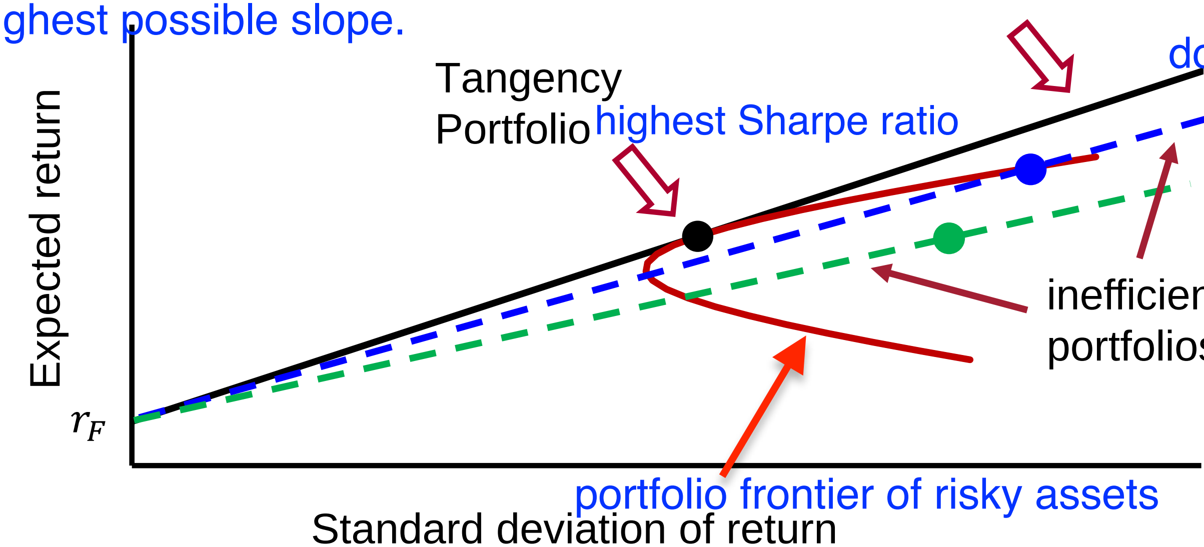
- We can also view the portfolio as follows:
  - $1 - x = 40\%$ in the risk-free asset,
  - $x = 60\%$ in a portfolio of only risky assets, which has:
    - o 50% in Equities,
    - o 50% in LT Bonds.
- Consider a portfolio p with:
  - x invested in a risky portfolio q, and
  - $1 - x$ invested in the risk-free asset.

$$\bar{r}_p = (1 - x) r_F + x \bar{r}_q$$

$$\sigma_p^2 = x^2 \sigma_q^2 \ (\text{use } \mathrm{Var}(r_F) = 0)$$

Portfolio frontier with a safe asset

The efficient frontier obtained by combining the tangency portfolio with the risk-free asset has the highest possible slope.



- With a risk-free asset, frontier portfolios are combinations of:
  - the risk-free asset,
  - the tangency portfolio (consisting of only risky assets).
- The frontier is also called the Capital Market Line (CML).

efficient frontier in the presence of the risk-free asset

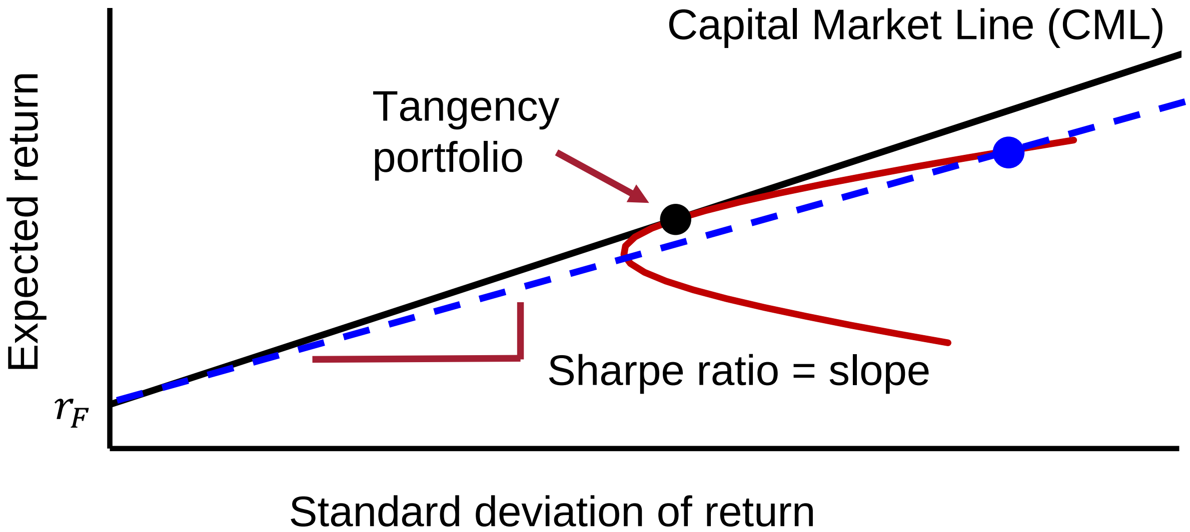
Sharpe ratio

If we think of standard deviation as the quantitative measure of risk, the Sharpe ratio tells us how much compensation in the form of the risk premium a given portfolio provides per unit of risk.

- Sharpe ratio is a measure of a portfolio's risk-return trade-off, equal to the portfolio's risk premium divided by its volatility:

$$\text{Sharpe Ratio} \equiv \frac{\bar{r}_p - r_F}{\sigma_p} \quad \text{(higher is better !)}$$

- The tangency portfolio has the highest possible Sharpe ratio of all portfolios. So do all the portfolios on the CML.



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The tangency portfolio

- Use the optimality condition to characterize the Tangency portfolio explicitly.
  - N risky assets, $i = 1, 2, \ldots, N$,
  - Expected return vector $\bar{r}$,
  - Expected excess return vector $\bar{x}$,
  - Covariance matrix Σ (positive definite),
  - $(N \times 1)$ vector of 1's ι,
  - $(N \times 1)$ vector of portfolio weights w of risky assets.

Then,

  - $\bar{x} = \bar{r} - r_F \, \iota$,
  - The weight in the risk-free asset is $1 - w'\iota$,
  - The expected excess return on portfolio w is $w'\bar{x}$.

The tangency portfolio

Eigenvalue and Generalized Eigenvalue Problems: Tutorial
Boyd, Stephen and Vandenberghe, Lieven. Convex optimization

- Solve for the efficient portfolio:

$$\boxed{w'\Sigma\, w = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \Sigma_{ij}}$$

$$\boxed{w'\bar{x} = \sum_{i=1}^{N} w_i \bar{x}_i}$$

$$\min \quad w'\Sigma\, w \qquad \text{(portfolio variance)}$$

$$\text{s.t.} \quad w'\bar{x} = m \qquad \text{(portfolio expected excess return)}$$

We solve this problem for an arbitrary level of the expected return target.

- Solve the optimization portfolio using the method of Lagrange multipliers:

$$L = w'\Sigma w + 2\lambda\,(m - w'\bar{x}\,)$$

- The first order condition (FOC) $\partial L/\partial w = 0$:

$$2\Sigma w \; - 2\lambda\bar{x} = 0 \;\Rightarrow\; \text{Solution: } w_T = \lambda\,\Sigma^{-1}\,\bar{x}$$

- Tangency portfolio weights on risky assets sum up to one: $w_T'\,\iota = 1$. Find

max sharpe slope
= min reverse

$$\min_w \frac{\sqrt{w'\Sigma w}}{w'\bar{x}} = \min_w \frac{w'\Sigma w}{(w'\bar{x})^2}$$

normalized

$$\min_w w'\Sigma w \quad s.t. \;\; (w'\bar{x})^2 = 1$$

$$\lambda = \frac{1}{\bar{x}'\Sigma^{-1}\iota}$$

$$w_T = \frac{1}{\bar{x}'\Sigma^{-1}\iota}\,\Sigma^{-1}\bar{x}$$

convex opt, KKT, dual = origin

$$L(w,\lambda) = w'\Sigma w + \lambda(w'\bar{x} - 1)$$
$$\nabla_w L = 2\Sigma w + \lambda\bar{x} = 0$$
$$w = -\frac{\lambda}{2}\Sigma^{-1}\bar{x}$$

Example: optimal portfolio

- Consider an asset allocation problem with three asset classes: T-Bills, Stock Market Index, and a Hedge Fund Index.
- Returns on the Stock Market Index (SMI) and Hedge Fund Index (HFI) are uncorrelated.
- What fraction of the Tangency portfolio should be allocated to the Hedge Fund Index?
- What are the Sharpe ratios for the SMI, HFI, and the Tangency portfolio?

	Expected return	Standard deviation
T-Bills	3%	0%
Stock Market Index	9%	20%
Hedge Fund Index	5%	10%

Example: optimal portfolio

- Use the general portfolio solution

$$w_T = \lambda\, \Sigma^{-1} \bar{x}$$

- The covariance matrix of returns is diagonal, easy to invert:

$$w_{SMI} = \lambda \frac{(9\% - 3\%)}{0.20^2} = \lambda \times 1.5$$

$$w_{HFI} = \lambda \frac{(5\% - 3\%)}{0.10^2} = \lambda \times 2.0$$

$$w_{SMI} = \frac{1.5}{1.5 + 2.0} = 43\%$$

$$w_{HFI} = \frac{2.0}{1.5 + 2.0} = 57\%$$

  - Over 50% allocation into the HFI. How does HFI compare to SMI on a stand-alone basis?

Example: optimal portfolio

- Recall that the Sharpe ratio equals expected excess return over volatility:

$$SR_{SMI} = \frac{9\% - 3\%}{20\%} = 30\% \quad vs. \quad SR_{HFI} = \frac{5\% - 3\%}{10\%} = 20\%$$

- HFI has a lower Sharpe ratio yet receives a higher allocation.
  - HFI is an excellent diversifier: it is uncorrelated with SMI.
  - HFI has much lower volatility than SMI.
- Sharpe ratio of the Tangency portfolio:

vs. $SR_{SMI} = 30\%$
and $SR_{HFI} = 20\%$

$$SR_T = \frac{0.43 \times 6\% + 0.57 \times 2\%}{\sqrt{0.43^2 \times 0.2^2 + 0.57^2 \times 0.10^2}} = 36\%$$

- Achieve higher Sharpe ratio thanks to diversification.

thought experiment: if the hedge fund index earns zero risk premium, then the tangency portfolio would allocate 100% to the stock market index. to raise the Sharpe ratio by diversification, low correlation is not sufficient. We need to be able to earn risk premia by taking on uncorrelated risks.

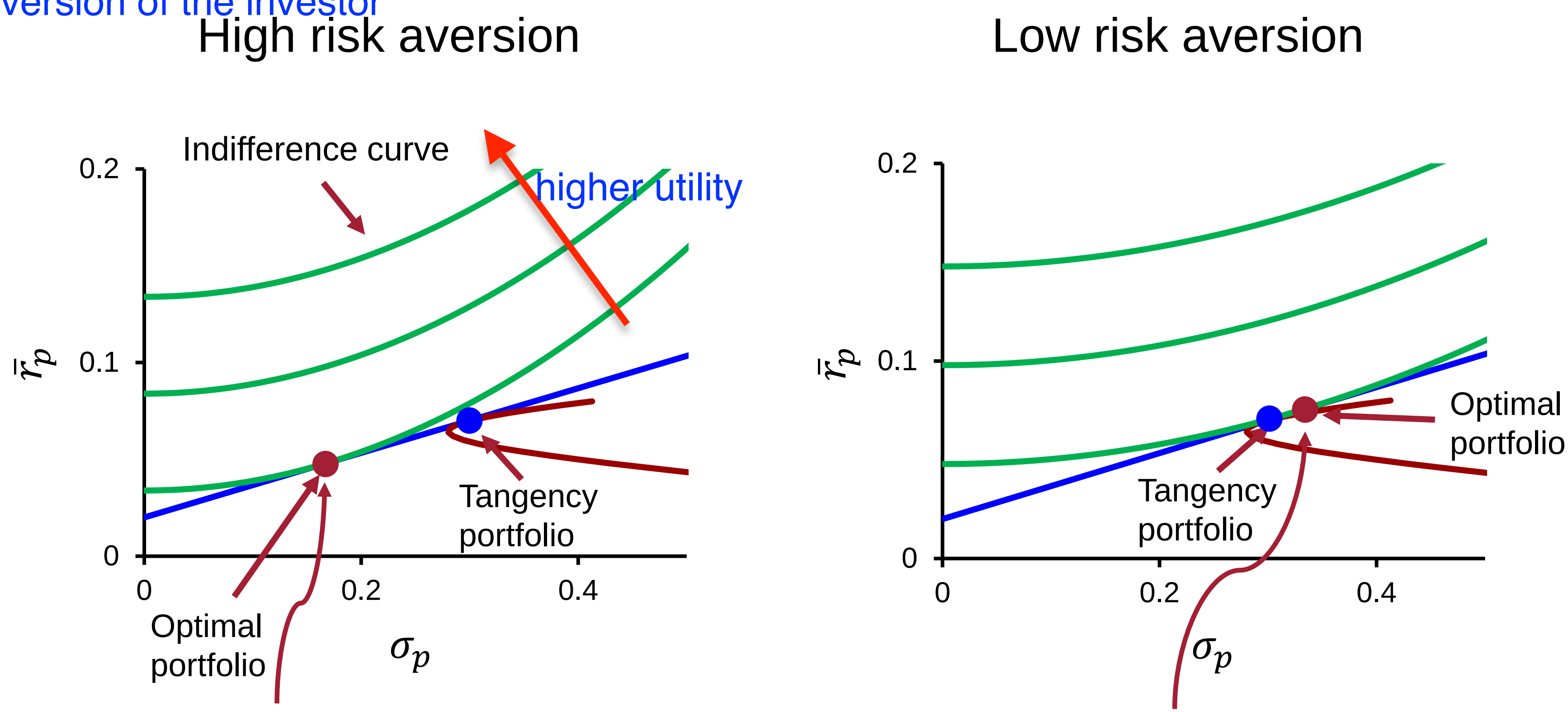
Risk aversion and portfolio choice

compute the tangency portfolio independently of the investor's preferences

- All mean-variance investors choose the tangency portfolio of stocks.

- Individual risk aversion determines allocation to the risky assets.

decide what fraction of the portfolio to allocate to the tangency portfolio, depending on the risk aversion of the investor



invest in the risk-free asset and the tangency portfolio

borrow at the risk-free rate and buy the tangency portfolio

higher Sharpe ratio (good) due to lower risk-free rate (bad), final effect depends on risk aversion. risk-free rate is the cost of borrowing for low risk aversion

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Portfolio and individual assets

- Contribution of an asset to a portfolio:
    - In the presence of a risk-free asset, portfolio's return is:

$$\tilde{r}_p = \left(1 - \sum_{i=1}^{N} w_i\right) r_F + \sum_{i=1}^{N} w_i \tilde{r}_i = r_F + \sum_{i=1}^{N} w_i (\tilde{r}_i - r_F)$$

- Each individual asset contributes to the portfolio on two dimensions:
    - Expected return;
    - Risk measured by return volatility (SD).
- We consider these two aspects separately.

Individual contribution to expected return

- Expected portfolio return is:

$$\bar{r}_p = r_F + \sum_{i=1}^{N} w_i \, (\bar{r}_i - r_F)$$

- Describe the marginal contribution of risky asset i to the expected portfolio return.

  - "Marginal contribution of x to A" means the incremental change of A when x changes by a small amount.

- Marginal contribution is the partial derivative: change in the portfolio properties per unit change in the weight on asset i, holding all other risky asset weights fixed (if we change w_i by δ, the weight on the risk-free asset must change by $-\delta$):

$$\frac{\partial \bar{r}_p}{\partial w_i} = \bar{r}_i - r_F \quad \text{(risk premium of asset } i\text{)}$$

Individual contribution to return volatility

- The variance of portfolio return is given by

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} = w_i^2 \sigma_i^2 + 2\sum_{j \neq i}^{N} w_i w_j \sigma_{ij} + [\text{Terms not including } w_i]$$

- The marginal contribution of asset i to portfolio variance is $2\mathrm{Cov}\left(\tilde{r}_i, \tilde{r}_p\right)$:

$$\frac{\partial\, \sigma_p^2}{\partial w_i} = 2 w_i \sigma_i^2 + 2\sum_{j \neq i}^{N} w_j \sigma_{ij} = 2\sum_{j=1}^{N} w_j \sigma_{ij} = 2\mathrm{Cov}\left(\tilde{r}_i, \sum_{j=1}^{N} w_j \tilde{r}_j\right) = 2\mathrm{Cov}\left(\tilde{r}_i, \tilde{r}_p\right)$$

- The marginal contribution of asset i to portfolio standard deviation is

$$\frac{\partial \sigma_p}{\partial w_i} = \frac{\partial \left(\sigma_p^2\right)^{\frac{1}{2}}}{\partial w_i} = \frac{\mathrm{Cov}\left(\tilde{r}_i, \tilde{r}_p\right)}{\sigma_p} = \frac{\sigma_{ip}}{\sigma_p}$$

Optimality of the tangency portfolio

the margin line can not across CML, or else there is more optimal portfolio above the CML, which contradict optimality of tangency portfolio

- Summarize the marginal contribution of risky asset i to portfolio p by its marginal return-to-risk ratio (RRR):

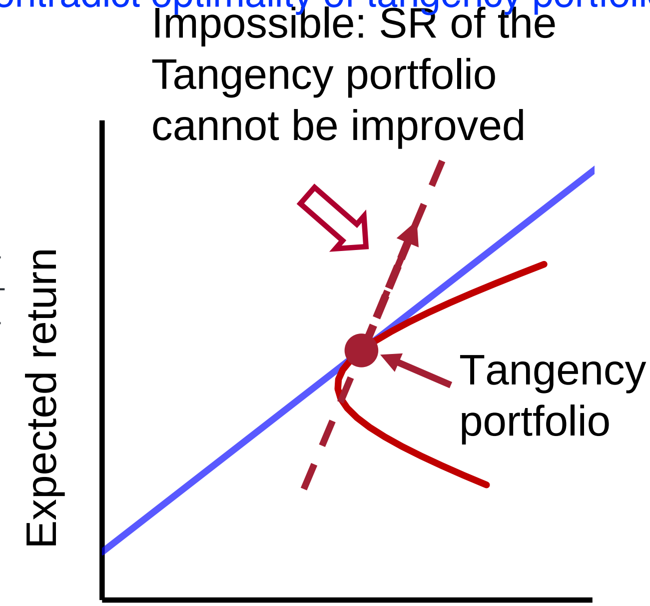
$$RRR_{ip} = \frac{\Delta r_p}{\Delta \sigma_p} = \frac{\frac{\partial r_p}{\partial w_i}\delta w_i}{\frac{\partial \sigma_p}{\partial w_i}\delta w_i} = \frac{\frac{\partial r_p}{\partial w_i}}{\frac{\partial \sigma_p}{\partial w_i}}$$

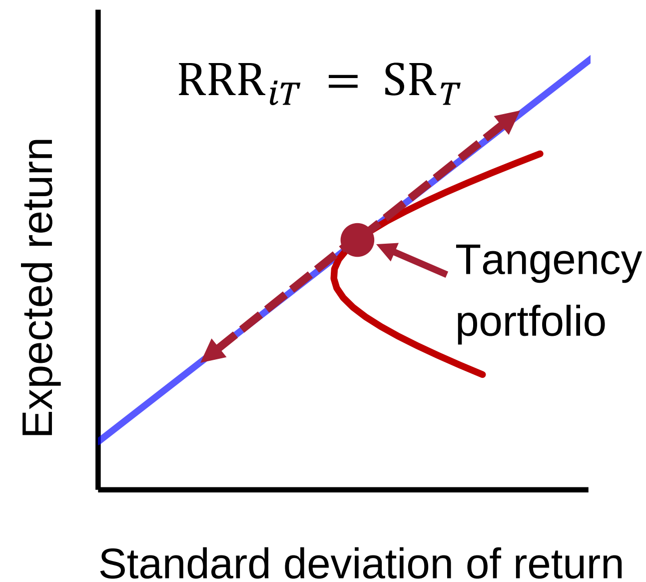
$$\mathrm{RRR}_{ip} = \frac{\left(\frac{\partial \bar{r}_p}{\partial w_i}\right)}{\left(\frac{\partial \sigma_p}{\partial w_i}\right)} = \frac{\bar{r}_i - r_F}{(\sigma_{ip}/\sigma_p)}$$

- Claim: For the tangency portfolio (T), which optimal, the return-to-risk ratio of all risl assets must be the same, and equal to tl Sharpe ratio of the tangency portfolio:

$$\mathrm{RRR}_{iT} = \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T} = \mathrm{SR}_T$$

- Intuition: The SR of a frontier portfolio cann be improved.





Optimality of the tangency portfolio: a supplement

- Claim: For the tangency portfolio (T),

$$\bar{r}_i - r_F = \frac{\sigma_{iT}}{\sigma_T^2}(\bar{r}_T - r_F)$$

$$\text{RRR}_{iT} = \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T} = \text{SR}_T$$

- Derive this result algebraically.

- Tangency portfolio composition is given by $w_T = \lambda\, \Sigma^{-1}\, \bar{x}$.

- Let e_i be the i'th basis vector, $e_i = (0, \ldots, \underbrace{1}_{i}, \ldots 0)'$. Then,

$$\sigma_{iT} = e_i' \Sigma w_T = e_i' \Sigma(\lambda\, \Sigma^{-1}\, \bar{x}) = \lambda e_i' \bar{x} = \lambda \bar{x}_i$$

$$\sigma_T^2 = w_T{}' \Sigma w_T = w_T{}' \Sigma(\lambda\, \Sigma^{-1}\, \bar{x}) = \lambda w_T{}' \bar{x} = \lambda \bar{x}_T$$

And therefore

$$\frac{\bar{x}_i}{\sigma_{iT}} = \frac{\bar{x}_T}{\sigma_T^2} \Rightarrow \text{this is the same as } \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T}$$

Regression interpretation

- Optimality of the tangency portfolio is mathematically equivalent to the intercept α_i in the following regression being equal to zero:

$$\tilde{r}_i - r_F = \alpha_i + \beta_i(\tilde{r}_T - r_F) + \tilde{\varepsilon}_i$$

$$\beta_i = \frac{\sigma_{iT}}{\sigma_T^2}$$

$$\mathrm{RRR}_{iT} = \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T} = \mathrm{SR}_T \quad RRR_{iT} = \frac{\bar{r}_i - r_F}{\beta_i \sigma_T} = \frac{\bar{r}_T - r_F}{\sigma_T} \Rightarrow \alpha_i = 0$$

- If α_i is non-zero, can improve on the Sharpe ratio of the tangency portfolio – the portfolio is not optimal.

- Can test for optimality in the data using regression methods.
  by testing the null hypothesis that the intercept coefficient in this regression is zero.

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Beyond the static mean-variance theory

static: construct the portfolio at time zero and evaluate the outcome next period. do not rebalance the portfolio over time. do not evaluate the time path of the portfolio value.

- In practice, portfolio choice problems are often dynamic:

dynamic tools: dynamic programming, an area of optimization dealing with multi-period decision problems.

  - Household managing investments to support several consumption objectives over the life cycle.

- Even in static problems, the objective may not be captured by the first two moments.

  - Maximize expected return with a penalty for portfolio value falling below a given threshold.

- Can capture various objectives within the expected utility framework, e.g.

$$\max \mathrm{E}\left[U\left(V_p\right)\right], \qquad U(v) = \begin{cases} v, & v \geq \underline{v} \text{ threshold} \\ v + a(v - \underline{v}), & v < \underline{v} \end{cases}, \qquad a > 0$$

Marginal contribution of an asset to portfolio

- Consider the objective

portfolio init value: 1

$$\max \mathrm{E}\left[U\left(1+\tilde{r}_p\right)\right], \ \tilde{r}_p = r_F + \sum_{i=1}^{N} w_i(\tilde{r}_i - r_F)$$

- The first-order optimality condition is

derivative with respect to each weight should be zero

the deduction is true by using definition of Cov and boring calculation

$$\mathrm{E}\left[U'\left(1+\tilde{r}_p\right) \times (\tilde{r}_i - r_F)\right] = 0$$

$$\mathrm{E}\left[U'\left(1+\tilde{r}_p\right)\right] \times (\bar{r}_i - r_F) + \mathrm{Cov}\left(U'\left(1+\tilde{r}_p\right), \tilde{r}_i\right) = 0$$

- Marginal contribution to portfolio risk is captured by $\mathrm{Cov}\left(U'\left(1+\tilde{r}_p\right), \tilde{r}_i\right)$: this places emphasis on nonlinear measures of return co-movement, e.g., $\mathrm{Cov}\left(\tilde{r}_p^2, \tilde{r}_i\right)$.

- Different investors no longer hold the same risky portfolio, the shape of the objective affects the optimal portfolio composition.

individual utility function

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