15.455x Sample Exam Questions

These sample exam problems are each worth 24 points. All sub-parts are weighted equally.

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1. Suppose that an asset price S_t follows a lognormal, continuous-time stochastic process,

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}B,$$

where μ, σ are constants and B is a standard Brownian motion. Use Itô's lemma to find stochastic differential equations expressing $\mathrm{d}V$ in terms of $\mathrm{d}t$ and $\mathrm{d}B$ for the following functions $V(S,t)$. Are they Itô processes?

   (a) $V = \alpha S + \beta$,

   (b) $V = S^\gamma$,

   (c) $V = e^{r(T-t)}S$,

   where α, β, γ, r, and T are constants.

2. Let a stationary discrete-time stochastic process x_t be given by

$$x_t = A + Bx_{t-2} + Cz_t,$$

where $z_t \sim \mathcal{N}(0,1)$ is an IID Gaussian white-noise process, and A, B, C are constants.

(a) What is the unconditional mean of the process x_t?

(b) An analyst decides to construct a forecast f_τ for future values of the process by taking its expected value, conditional on information available up through the time t when the forecast is made. That is,

$$f_\tau \equiv \mathrm{E}_t\big[x_\tau \,|\, x_t, x_{t-1}, \ldots\big], \quad \tau > t.$$

Let $A = 0.1$, $B = 0.2$, $C = 0.3$, and suppose that two recent values $x_1 = 0.4$, $x_2 = 0.5$ have just been observed. What are the one-step-ahead and two-step-ahead forecasts? That is, at time $t = 2$, what are the forecasts f_3 and f_4? What is the variance of the forecasts?

3. (a) Consider the quadratic form defined by

$$Q(x, y) = 2x^2 + 12xy - 7y^2.$$

Using Lagrange multipliers, find the location and value of the extrema of Q subject to the constraint $x + 3y = 5$. Determine whether each solution is a maximum, minimum, or neither.

(b) Two assets have correlation ρ, and their volatilities are 2σ and σ respectively. What are the weights of a minimum-variance, fully-invested portfolio of the two assets, and what is its risk? That is, minimize the portfolio variance $\sigma_p^2 = \mathbf{w}^\top C \mathbf{w}$, where C is the covariance matrix and $\mathbf{w}$ is an asset weight vector whose components satisfy the budget constraint $w_1 + w_2 = 1$.

(c) In the problem above, for what values of ρ and σ will the solution also satisfy an inequality constraint $0 \leq w_i \leq 1$? (That is, the optimal portfolio is also unlevered and long-only.)

4. The returns on a set of N assets are believed to follow the mean-reverting process

$$R_{it} - \mu_i = -\lambda(R_{i(t-1)} - \mu_i) + \sigma_i z_{it},$$

where μ_i, σ_i, λ are constants, $i = 1, \ldots N$; $|\lambda| < 1$; and

$$\mathbb{E}[z_{it}] = 0; \quad \mathbb{E}[z_{it} z_{js}] = \begin{cases} 1 & \text{if } t = s \text{ and } i = j, \\ 0 & \text{if } t \neq s \text{ or } i \neq j; \end{cases}$$

A market-neutral long/short trading strategy attempts to profit by investing capital in weights assigned according to

$$w_{it} = -\frac{1}{N}(R_{it} - \overline{R}_t),$$

where the market average return is defined by

$$\overline{R}_t = \frac{1}{N}\sum_{i=1}^{N} R_{it}.$$

Assume there are no transaction costs and the risk-free rate $R_f = 0$. Find the expected portfolio return

$$\mathbb{E}[R_p] = \mathbb{E}\left[\sum_i w_{i(t-1)} R_{it}\right]$$

in terms of the parameters given. Under what conditions is this expected return positive?

5. Two stocks have prices S_1 and S_2 that follow geometric Brownian motion with the same stochastic process $\mathrm{d}B$:

$$\mathrm{d}S_1 = \mu_1 S_1 \ \mathrm{d}t + \sigma_1 S_1 \ \mathrm{d}B,$$
$$\mathrm{d}S_2 = \mu_2 S_2 \ \mathrm{d}t + \sigma_2 S_2 \ \mathrm{d}B.$$

(a) A contract has value $V = S_1 S_2$. You can show that V also follows geometric Brownian motion. What are its drift and volatility parameters?

(b) What is the process followed by $1/V$?

(c) A call option on V with strike K has value $C(t, V)$ and payoff at expiration $\max(S_1 S_2 - K, 0)$. What PDE does the option satisfy?