15.455x – Mathematical Methods for Quantitative Finance

Recitation Notes #5

1 Itô's lemma

Let's look further at Itô processes and Itô's lemma.

Exercise: Let

$$F(t, X) = e^{-rt}X^2,$$
$$\mathrm{d}X = (\mu X)\,\mathrm{d}t + (\sigma X)\,\mathrm{d}B.$$

Write $\mathrm{d}F$ in three ways:

- As as function of $\mathrm{d}t$, $\mathrm{d}X$
- As a function of $\mathrm{d}t$, $\mathrm{d}B$,
- As an expression with coefficients involving only t, F and no explicit reference to X.

So in terms of the general form of an Itô process, $a = 0, b = 1$. For pure Brownian motion, therefore,

$$\mathrm{d}F = \frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial B}\,\mathrm{d}B + \frac{1}{2}\frac{\partial^2 F}{\partial B^2}\,\mathrm{d}t$$

Solution: Start with the partial derivatives of F that appear in Itô's formula:

$$\frac{\partial F}{\partial t} = -re^{-rt}X^2 = -rF,$$
$$\frac{\partial F}{\partial X} = e^{-rt}(2X) = 2\frac{F}{X},$$
$$\frac{\partial^2 F}{\partial X^2} = 2e^{-rt} = 2\frac{F}{X^2}.$$

Substituting into the formula,

$$
\begin{aligned}
\mathrm{d}F &= \frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial X}\,\mathrm{d}X + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\,\mathrm{d}t \\
&= \Big[-re^{-rt}X^2 + \frac{(\sigma X)^2}{2}(2e^{-rt})\Big]\,\mathrm{d}t + \Big[e^{-rt}(2X)\Big]\,\mathrm{d}X
\end{aligned}
$$

Now replace $\mathrm{d}X$ with $(\mu X)\,\mathrm{d}t + (\sigma X)\,\mathrm{d}B$ to obtain the form

$$
\begin{aligned}
\mathrm{d}F &= \Big[-re^{-rt}X^2 + \frac{(\sigma X)^2}{2}(2e^{-rt}) + e^{-rt}(\mu X)(2X)\Big]\,\mathrm{d}t + \Big[e^{-rt}(\sigma X)(2X)\Big]\,\mathrm{d}B \\
&= (e^{-rt}X^2)\Big[2\mu + \sigma^2 - r\Big]\,\mathrm{d}t + (e^{-rt}X^2)\Big[2\sigma\Big]\,\mathrm{d}B.
\end{aligned}
$$

Finally, recognizing the pre-factor of both terms as $F = e^{-rt}X^2$, the differential can be written as

$$
\mathrm{d}F = (2\mu + \sigma^2 - r)F\,\mathrm{d}t + (2F\sigma)\,\mathrm{d}B,
$$

which is of the same form as $\mathrm{d}X$, with different coefficients.

To integrate $\mathrm{d}F$, we could therefore follow the same steps we used in lecture for geometric Brownian motion. Or we could appeal directly, using

$$
\begin{aligned}
\mathrm{d}(\log F) &= \mathrm{d}(\log(e^{-rt}X^2)) = -r\,\mathrm{d}t + 2\,\mathrm{d}(\log X) \\
&= (2\mu - \sigma^2 - r)\,\mathrm{d}t + (2\sigma)\,\mathrm{d}B.
\end{aligned}
$$

$$
\begin{aligned}
X &= \mu X dt + \sigma X dB \\
dlogX &= \frac{dX}{X} + \frac{\sigma^2 X^2}{2}\left(-\frac{1}{X^2}\right) \\
&= \mu dt + \sigma dB - \frac{\sigma^2}{2}
\end{aligned}
$$

This can be integrated to give

$$
F_t = F_0 e^{(2\mu - \sigma^2 - r)t + 2\sigma(B_t - B_0)}.
$$

2 Expectations of Brownian processes

We know that $\mathrm{d}B \sim \mathcal{N}(0,\,\mathrm{d}t)$ can be integrated to get

$$
\int_0^t \mathrm{d}B = B_t - B_0 \sim \mathcal{N}(0, t).
$$

For simplicity, we'll set $B_0 = 0$ and then replace the random variable B_t, which has time-dependent variance, with the expression $B_t \to \sqrt{t}Z$, where $Z \sim \mathcal{N}(0,1)$, making the time dependence explicit.

Expectations of functions of B_t can then be evaluated as ordinary Gaussian expectations, whether by using the characteristic function or computing integrals explicitly:

$$\mathrm{E}\left[f(B_t - B_0)\right] = \mathrm{E}\left[f(\sqrt{t}Z)\right] = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} f(\sqrt{t}z)\,\mathrm{d}z.$$

For example if $f(x) = x^4$, then

$$\mathrm{E}\left[B_t^4\right] = \mathrm{E}\left[(\sqrt{t}Z)^4\right] = \frac{t^2}{\sqrt{2\pi}} \int e^{-z^2/2} z^4\,\mathrm{d}z = 3t^2.$$

Exponentials frequently occur inside expectations. Consider this example:

Exercise: Find $\mathrm{E}\left[e^{6X}\right]$ where $\mathrm{d}X = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$.

Solution: Integrate $\mathrm{d}X$ to obtain

$$\begin{aligned} X_t - X_0 &= \mu t + \sigma(B_t - B_0) \\ &= \mu t + \sigma\sqrt{t}Z. \end{aligned}$$

Setting $X_0 = 0$, we need to evaluate

$$\mathrm{E}\left[e^{6(\mu t + \sigma\sqrt{t}Z)}\right] = e^{6\mu t}\,\mathrm{E}\left[e^{6\sigma\sqrt{t}Z}\right].$$

A useful formula:

Similar to the characteristic function and the moment-generating function, it's convenient to compute the expectation for $e^{\alpha Z+\beta}$, where α, β are arbitrary constants.

$$
\begin{aligned}
\mathrm{E}\left[e^{\alpha Z+\beta}\right] &= e^{\beta}\,\mathrm{E}\left[e^{\alpha Z}\right] \\
&= \frac{e^{\beta}}{\sqrt{2\pi}} \int e^{-z^2/2} e^{\alpha z}\,\mathrm{d}z \\
&= \frac{e^{\beta}}{\sqrt{2\pi}} \int e^{-(z^2-2\alpha z)/2}\,\mathrm{d}z \\
&= \frac{e^{\beta}}{\sqrt{2\pi}} \int e^{-(z-\alpha)^2/2} e^{\alpha^2/2}\,\mathrm{d}z \\
&= e^{\alpha^2/2+\beta}.
\end{aligned}
$$

Applying this result, we find

$$
\begin{aligned}
\mathrm{E}\left[e^{6X}\right] &= \mathrm{E}\left[e^{6\mu t+6\sigma\sqrt{t}Z}\right] \\
&= e^{6(\mu+3\sigma^2)t}.
\end{aligned}
$$

3 Solutions of the diffusion equation

First, let's show that

$$
p(z,t) = \int p_0(z-w,t) f(w)\,\mathrm{d}w \tag{1}
$$

is a solution to the diffusion equation, where

$$
p_0(z,t) = \frac{1}{\sqrt{2\pi t}} e^{-z^2/(2t)}. \tag{2}
$$

We can act on both sides of the equation with the differential operator and use linearity to see that

shifted by a constant, the derivatives are exactly the same

$$
\left[\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial z^2}\right] p(z,t) = \int \left[\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial z^2}\right] p_0(z-w,t) f(w)\,\mathrm{d}w = 0
$$

because the only place where t and z appear in the integrand is in the function p_0, which is known the be a solution.

So the integral formula Eq. 1 always gives solutions. (It remains to be shown why they satisfy $p(z,0) = f(z)$.)

Exercise: Find the solution $p(z,t)$ that has initial condition $p(z,0) = z^2$.

Solution:

$$\begin{aligned}
p(z,t) &= \int \frac{1}{\sqrt{2\pi t}} e^{-(z-w)^2/2t} w^2 \, \mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}} \int e^{-u^2/2} (z + u\sqrt{t})^2 \, \mathrm{d}u \\
&= z^2 \Big[\frac{1}{\sqrt{2\pi}} \int e^{-u^2/2} \, \mathrm{d}u \Big] + 2z\sqrt{t} \Big[\frac{1}{\sqrt{2\pi}} \int e^{-u^2/2} u \, \mathrm{d}u \Big] + t \Big[\frac{1}{\sqrt{2\pi}} \int e^{-u^2/2} u^2 \, \mathrm{d}u \Big] \\
&= z^2 + t,
\end{aligned}$$

where we made a change of variable $u = (w - z)/\sqrt{t}$ in the second line.

Exercise: Find the solution $p(z,t)$ that has initial condition

$$p(z,0) = \theta(\kappa - z) = \begin{cases} 1 & z < \kappa \\ 0 & z > \kappa \end{cases}$$

Solution: Since the function $f(w)$ describing the initial conditions is either 0 or 1, the integrand is quite simple:

$$\begin{aligned}
p(z,t) &= \int_{-\infty}^{\kappa} \frac{1}{\sqrt{2\pi t}} e^{-(z-w)^2/2t} \, \mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\kappa - z)/\sqrt{t}} e^{-u^2/2} \, \mathrm{d}u \\
&= \Phi\left(\frac{\kappa - z}{\sqrt{t}} \right),
\end{aligned}$$

where $\Phi(x)$ is the Gaussian cumulative distribution function,

$$\Phi(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, \mathrm{d}z, \tag{3}$$

and the same change of variable, $u = (w - z)/\sqrt{t}$ was used. You can easily check that this satisfies the diffusion equation. Verifying the initial conditions as $t \to 0$ is more delicate due to the factor of $1/\sqrt{t}$.