Recitation 11

Spring 2021

Question 1: Forward Interest Rates and Arbitrage

The following table shows current interest rates:

Interest rate	Value
1-yr spot rate	1%
2-yr spot rate	3%
1-yr forward rate in year 1	4%

Construct the following 3 arbitrage strategies:

(a) Pays \$100 today and nothing in the future

(b) Pays \$100 in year 1 and nothing otherwise

why should you go long on 2-yr spot rate?

(c) Pays \$100 in year 2 and nothing otherwise

scenario 1: invest \$100 for 2 years at r2 106.09

scenario 2: invest \$100 for 1 year at r1 and for another year at f1 105.04

Solutions:

The general approach to solving this problem is to:

1. Invest x at spot rate, r_1
2. Invest y at spot rate, r_2
3. Invest z at 1-yr forward rate in year 1, f_1

We will apply this method to construct the 3 arbitrage strategies:

(a) Pays \$100 today and nothing in the future.

The payout in year 0 (today) is $-x - y$, the amount being invested today.

The payout in year 1 is $(1+r_1)x - z$, since we get rate r_1 on the investment of x, and we also invest z in year 1.

The payout in year 2 is $(1+r_2)^2 y + (1+f_1)z$, since we get r_2 compounded over two years on the investment of y in year 0, as well as f_1 on the investment of z in year 1.

Altogether, this yields the following set of equations:

$$\begin{aligned} -x - y &= 100 \\ (1+r_1)x - z &= 0 \\ (1+r_2)^2 y + (1+f_1)z &= 0 \end{aligned}$$

Solving in excel using $r_1 = 1\%$, $r_2 = 3\%$, and $f_1 = 4\%$ yields $x = -10,103.81$, $y = 10,003.81$, and $z = -10,204.85$.

Intuitively, we are going long on the investment with the 2-yr rate since it has a higher return than investing with the 1-yr rate in year 0 and the forward 1-yr rate in year 1.

(b) Pays \$100 in year 1 and nothing otherwise.

We end up with the same set of equations but with a \$0 payout in year 0 and 2, and a \$100 payout in year 1:

$$\begin{aligned} -x - y &= 0 \\ (1+r_1)x - z &= 100 \\ (1+r_2)^2 y + (1+f_1)z &= 0 \end{aligned}$$

Plugging in the interest rates and solving in excel yields $x = -9,904.76$, $y = 9,904.76$, and $z = -10,103.81$.

The intuition for going long on the 2-yr investment and short on the 1-yr and forward 1-yr investments is the same as part a.

(c) Pays \$100 in year 2 and nothing otherwise.

We end up with the same set of equations but with a \$0 payout in year 0 and 1, and a \$100 payout in year 2:

$$\begin{aligned} -x - y &= 0 \\ (1+r_1)x - z &= 0 \\ (1+r_2)^2 y + (1+f_1)z &= 100 \end{aligned}$$

Plugging in the interest rates and solving in excel yields $x = -9,523.81$, $y = 9,523.81$, and $z = -9,619.05$.

The intuition for going long on the 2-yr investment and short on the 1-yr and forward 1-yr investments is the same as parts a and b.

Question 2: Forwards and Arbitrage

Suppose that the current price of a dividend-paying stock equals \$1,000. Suppose that the risk free rate equals 3% and the dividend yield is 2%, both continuously compounded.

(a) You notice that a forward price for delivery of this stock in 2 years is \$1,025. Is there an arbitrage opportunity? If so, how can you exploit it?

(b) Now suppose the forward price for delivery of this stock in 2 years is \$1,015. This forward is underpriced. How can you exploit this?

Solutions:

(a) Given the current stock price, the forward price should be:

$$F_T = e^{(r-y)T} S_0$$

So $F_2 = e^{(0.03-0.02)\times 2} \times 1,000 = \$1,020.20$. This means the forward is overpriced.

We can exploit this by using the following general arbitrage strategy, where the initial cost of the portfolio is zero:

1. Engage in one short forward contract
2. Borrow $e^{-yT} \times S_0$ at the risk-free rate
3. Buy e^{-yT} shares of stock
4. Reinvest continuous stream of dividends by buying additional units of the stock

Then, the number of shares you have in 2 years is:

$$e^{yT} \times \text{number of shares originally bought} = e^{yT} \times e^{-yT} = 1$$

Since you engaged in one short forward contract, you deliver 1 share to the counterparty and collect \$1,025.

You need to repay the loan. The face value is

compounded continuously

$$FV = e^{rT} \times e^{-yT} \times S_0 = S_0 e^{(r-y)T}$$

So $FV = \$1,000e^{(0.03-0.02)\times 2} = \$1,020.20$.

Your profit is $\$1,025 - \$1,020.20 = \$4.80$.

(b) Since the forward contract is underpriced, we use the following arbitrage strategy, where the initial cost of the portfolio is zero:

1. Engage in one long forward contract
2. Short sell e^{-yT} shares of the stock
3. Invest proceeds from the short sale ($e^{-yT} \times S_0$) at the risk free rate

In 2 years, you will need to return $FV = e^{yT} \times \text{number of shares shorted} = e^{yT} \times e^{-yT} = 1$.

Your investment in the risk-free asset is now worth $e^{rT} \times e^{-yT} S_0 = S_0 e^{(r-y)T} = \$1,000 \times e^{(0.03-0.02)\times 2} = \$1,020.20$.

You pay the seller of the forward contract \$1,015, and the seller delivers one share of stock, which you use to cover the short position.

Your profit is $\$1,020.20 - \$1,015 = \$5.20$.

Question 3: Currency Forwards

Suppose that USD/JPY is trading at 105, and the 1-year forward on USD/JPY is trading at 106. The risk-free rate in the US is 1%, and in Japan it is 3%. Construct an arbitrage strategy that gives you \$100 today and nothing in the future.

Solutions:

Suppose we borrow x USD today. We will keep \$100 today as that is our desired payout in year 0. The rest, $x - 100$, will be exchanged to Japanese Yen at the current spot exchange rate (105) and invested into the one-year risk-free Japanese bond that pays 3% interest. That is, we invest $105(x - 100)$ in the risk-free Japanese bond.
In 1 year, we will get $105(x - 100) \times (1 + 0.03)$, since the interest rate is 3%.
In 1 year, our liabilities in the US will be $x \times (1 + .01)$, since the interest rate in the US is 1%. We can convert this to Japanese Yen at the 1-year forward exchange rate to determine the value of liabilities in Japanese Yen in 1 year. This gives us $106x(1 + .01)$.
Since we don't want any payout in year 1, we want the year 1 income to equal the year 1 liability. We have both of these values in JPY and can set them equal:

$$\text{Year 1 income in JPY} = \text{Year 1 liabilities in JPY}$$
$$105 \times 1.03 \times (x - 100) = 106 \times 1.01 \times x$$

What if negative number?
we borrowed in the wrong currency,
you should just reverse the direction

Solving yields $x = \$9,922$ USD. So, we need to borrow \$9,922 USD in year 0, keep \$100 as the year 0 payout, and then convert the remaining \$9,822 to JPY and invest it in 3% interest Japanese risk-free bonds. If we lock in the forward rate, the income in one year is exactly the liabilities owed to the bank in one year.
Note that we could have expressed the year 1 income in USD by dividing by the forward exchange rate. We then could set that equal to the liability in USD to obtain the same answer.
Intuitively, this arbitrage is possible because the forward is mispriced. In other words, the forward exchange rate is too low, or the income in JPY relative to USD is too high. At the given forward exchange rate, you can borrow in the low-yielding currency, convert to the high-yielding currency, and invest in the high-yielding currency to earn a profit.
Note that forwards pricing is done by assuming such a trading strategy can't be profitable (by imposing no-arbitrage).

Question 4: Valuation of Interest Rate Swaps

Suppose ABC Corporation has borrowed **\$3 million** for the next 3 years at a **variable interest rate**. Under the terms of the loan, ABC will pay interest at

the end of Years 1, 2, and 3.
The interest that ABC will need to pay at the end of Year 1 is based on the one-year spot interest rate in Year 0. Likewise, the interest to be paid at the end of Year 2 will be based on the one-year spot interest rate in Year 1, and the interest to be paid at the end of Year 3 will be based on the one-year spot interest rate in Year 2.
(Aside: corporate bank loans typically feature a rate linked to LIBOR)
The current 1-yr, 2-yr, and 3-yr spot rates are given below. These will be used to calculate present values:

Interest rate	Value
1-yr spot rate	3.5%
2-yr spot rate	4.0%
3-yr spot rate	4.2%

(a) The spot interest rates today can be used to calculate the implied one-year spot interest rate that will be in effect during the second year (i.e. the forward rate). Check that $f_1 = 4.502\%$.

(b) Note that, under this loan, the interest rate for year 2 could be higher or lower than f_1, depending on how interest rates evolve. The same logic applies to year 3.

Suppose that ABC is not comfortable with this uncertainty and wants to enter into an interest rate swap, which will fix the interest rates for the 3-year term of this loan. The swap has the following characteristics:

1. Term: 3 years
2. Annual settlement, with settlement dates at the end of Year 1, 2, and 3.
3. ABC will pay a fixed rate r_S during the 3 years of the loan. The counterparty will pay the floating rate.

What is r_S?

Solutions:

(a) Investing a dollar at the one-year spot rates in year 0 and year 1 should yield thee same as a 2-yr investment made in year 0 at the 2-yr spot rate. The implied forward rate f_1 satisfies:

$$(1 + r_1)(1 + f_1) = (1 + r_2)^2$$

where r_1 is the current 1-yr spot rate and r_2 is the current 2-yr spot rate.

Solving yields $f_1 = 4.502\%$.

(b) The interest rate r_S should be fair to both parties, which means that:

PV of interest @ variable rate = PV of interest @ fixed rate

We first need to solve for the expected variable rates to be paid:

- From part a, the expected rate in year 2 is $f_1 = 4.502\%$
- Likewise, the expected rate in year 3 is $f_2 = \frac{(1+r_3)^3}{(1+r_1)(1+f_1)} = 4.601\%$. Note that r_3 is the 3-yr spot rate, r_1 is the 1-yr spot rate, f_1 is the 1-yr forward rate in year 1, and f_2 is the 1-yr forward rate in year 2. fT: from T-1 to T

So, the interest payments each year are: $\mathrm{PV}_0[\tilde{r}_1(T) \text{ at } T+1] = \mathrm{PV}_0[f_{T+1} \text{ at } T+1]$

- Year 1: $\$3\text{mm} \times r_1 = \$3\text{mm} \times 3.5\% = \$105,000.00$ T—>T+1 = T—>T+1
- Year 2: $\$3\text{mm} \times f_1 = \$3\text{mm} \times 4.502\% = \$135,072.46$ f1: from 1 to 2
- Year 3: $\$3\text{mm} \times f_2 = \$3\text{mm} \times 4.601\% = \$138,034.64$

The fixed interest to be paid each year is $\$3\text{mm} \times r_S$.

Setting the PV of variable payments and fixed payments equal yields:

$$\frac{\$3\text{mm} \times 3.5\%}{1+3.5\%} + \frac{\$3\text{mm} \times 4.502\%}{1+4.0\%^{2}} + \frac{\$3\text{mm} \times 4.601\%}{1+4.2\%^{3}} =$$
$$\frac{\$3\text{mm} \times r_S}{1+3.5\%} + \frac{\$3\text{mm} \times r_S}{1+4.0\%^{2}} + \frac{\$3\text{mm} \times r_S}{1+4.2\%^{3}}$$

Solving this yields $r_S = 4.185\%$.

Alternative Method:

Recall that:

$$r_S = \frac{1 - B_T}{\sum_{u=1}^{T} B_u}$$

where B_t is the price of a t-year zero-coupon bond. Applying this to our case where $T = 3$ yields:

$$r_S = \frac{1 - B_3}{B_1 + B_2 + B_3}$$

Given r_1, r_2, and r_3, we have:

$$r_S = \frac{1 - 1/(1+r_3)^3}{1/(1+r_1) + 1/(1+r_2)^2 + 1/(1+r_3)^3}$$

Plugging in the spot rates yields $r_S = 4.185\%$.

the baseline is one-price theory, which means that there is no space for arbitrage but you should be careful about the discount rate, if you are given annual rate 4%, but the time period is 1 month with continuously compounding, then the effective discount rate is exp(4%/12)-1