15.415x Foundations of Modern Finance

Leonid Kogan and Jiang Wang
MIT Sloan School of Management

Lecture 4: Fixed Income Securities

Key Concepts

- Introduction
- Yield Curve
- Discount versus Coupon Bonds
- Relative Bond Valuation
- Yield To Maturity
- Yield Curve Dynamics
- Interest Rate Risk and Bond Duration
- Bond Duration and Convexity
- Inflation Risk

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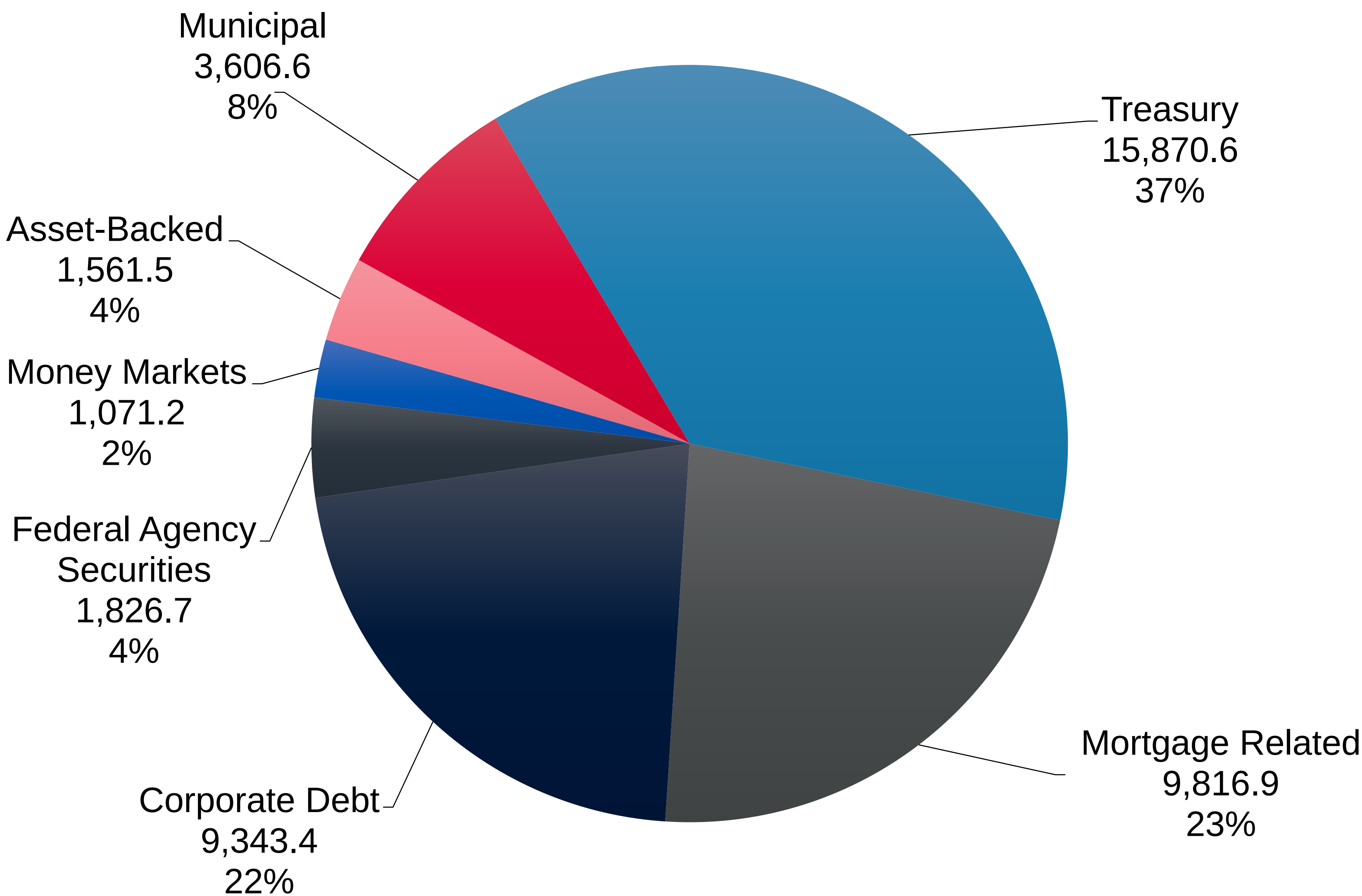
Fixed income securities

- Fixed income securities are financial claims with promised cash flows of fixed amount paid at fixed dates. does not mean that they are risk free
- Major classes of fixed-income securities:
  - Treasury: U.S. Treasuries, Bunds, JGBs, etc.;
  - Federal agency (U.S.): FNMA, FHLMC, etc.; Fannie Mae and Freddie Mac are the agencies that support the functions of the housing market,
  - Municipal securities; exempt from taxes
  - Corporate;
  - Mortgage backed and asset backed.

Outstanding U.S. bond market debt 2019 Q1 ($billions)



Examples of key market participants

Issuers	Intermediaries	Investors
Governments	Dealers: primary and other (secondary)	Pension funds
Municipalities	Investment banks	Insurance companies
Corporations	Credit rating agencies	Mutual funds
		Hedge funds
		Banks
		Individuals

credit rating agencies, such as S&P, serve to provide information about the quality of fixed income

both an investor and an intermediary

Intermediaries facilitate primary transactions, which is when a borrower issues the bond for the first time, as well as secondary market transactions.

Investment banks can act as a dealer holding an inventory of bonds and as a broker making matches between buyers and sellers of fixed income securities. And they can also help corporations or municipalities issue their bonds.

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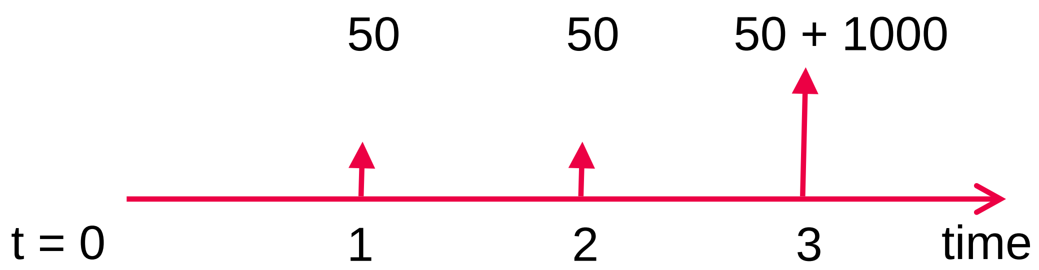
Cash flows of fixed income securities

A 3-year bond with principal of $1,000 and annual coupon payment of 5%



- Cash flow:
  - Maturity;
  - Principal;
  - Coupon.

Valuation of riskless cash flows

- Relative valuation, based on absence of arbitrage.
- Without risk, only time value of money is relevant.
- Prices of traded fixed-income securities provide information needed to value riskless cash flows at hand.

Market information: time value of money

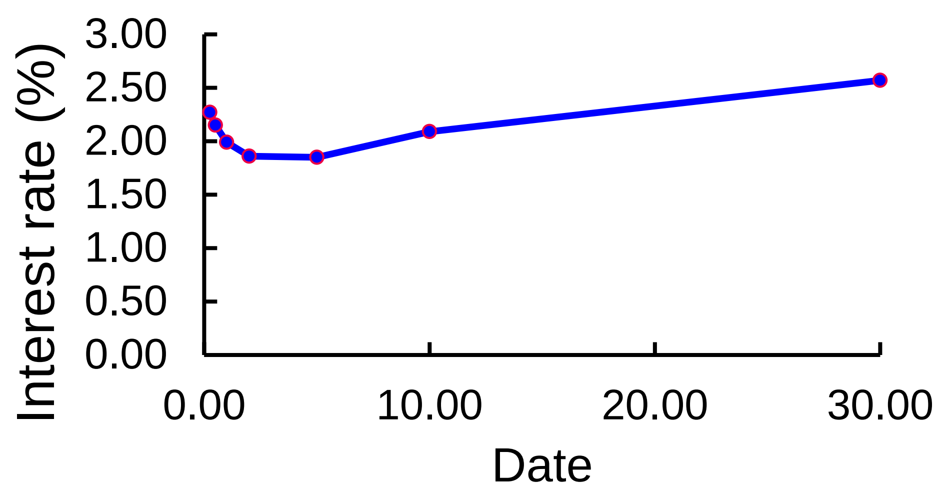
- In the market, the time value of money is captured in many different forms:
- Spot interest rates;
- Prices of discount bonds (zero-coupon bonds);
- Prices of coupon bonds.

Yield curve

- **Spot interest rate** is the current (annualized) interest rate (r_t) for maturity date t: The spot interest rate for maturity t is the rate of return that the fixed-income investor is going to receive by investing today and collecting the payoff t periods from now, t years from now.
  - r_t is for payments only on date t;
  - r_t is different for each date t.
- Spot interest rates on 2019/06/08

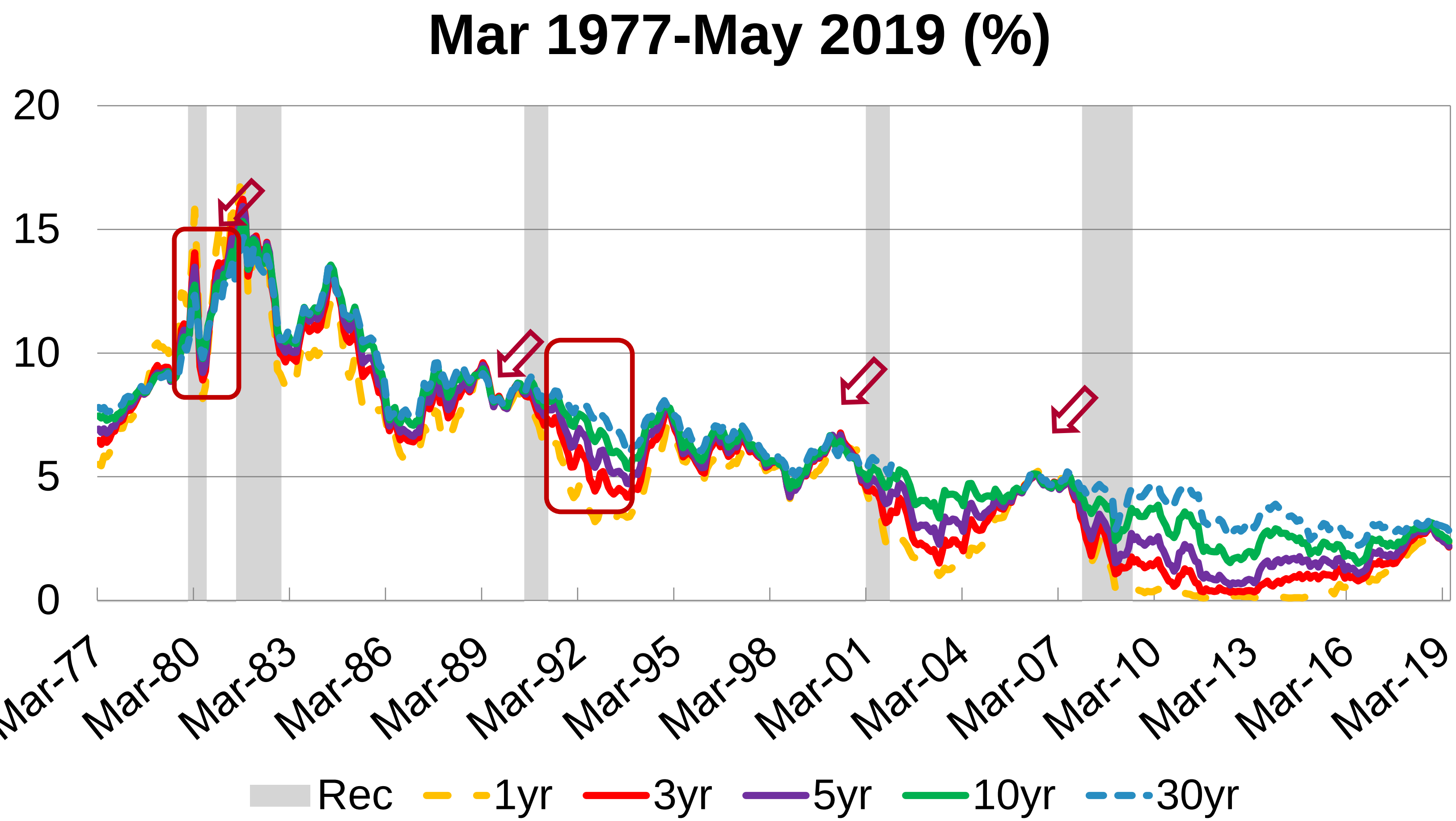
Maturity (year)	0.25	0.5	1	2	5	10	30
Interest Rate (%)	2.27	2.15	1.99	1.86	1.85	2.09	2.57

Yield curve (term structure of interest rates): the set of spot interest rates for different maturities.



Historical U.S. Treasury rates

convergence of interest rates or inversion of the yield curve with the long rates declining relative to short rates tends to happen at the beginning of recessionary periods.



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Bond prices and interest rates

- Let B_t denote the current price (time 0) of a discount bond maturing at t.
- Prices of discount bonds provide information about spot interest rates:

bond price with a face value of $1.00

$$B_t = \frac{1}{(1+r_t)^t} \quad \text{or} \quad r_t = \frac{1}{B_t^{1/t}} - 1$$

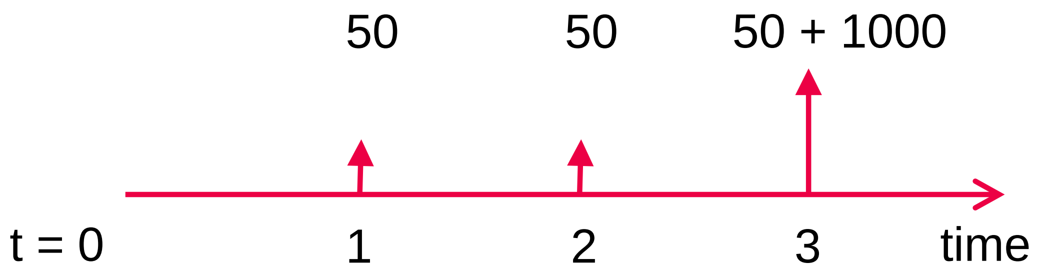
Coupon bonds vs discount bonds

- A coupon bond pays a stream of regular coupon payments and a principal at maturity.
- A coupon bond is a portfolio of discount bonds.
- Relative pricing: establish the value of a coupon bond relative to discount bonds, and vice versa.

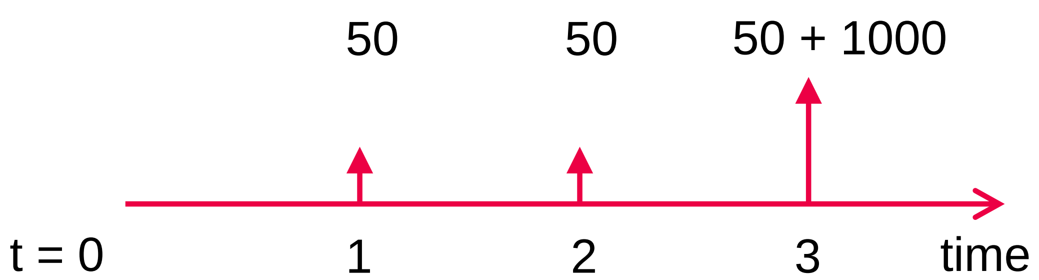
coupon bond as a portfolio of pure discount bonds, where each of the payments delivered by the coupon bond can be visualized as a payoff over a single discount bond with the corresponding maturity.

Coupon bonds vs discount bonds

A 3-year bond of $1,000 par and 5% annual coupon



Portfolio of discount bonds



Pricing a coupon bond

- Law of one price: price of the coupon bond must equal the price of the replicating portfolio of discount bonds:

no arbitrage profits

$$B = \sum_{t=1}^{T} (C_t \times B_t) + (P \times B_T) = \frac{C_1}{1+r_1} + \cdots + \frac{C_{T-1}}{(1+r_{T-1})^{T-1}} + \frac{C_T + P}{(1+r_T)^T}$$

- Suppose that discount bond prices are as follows:

t	1	2	3	4	5
B_t	0.952	0.898	0.863	0.807	0.757

- Coupon bond price is

$$(50)(0.952) + (50)(0.898) + (1{,}050)(0.863) = 998.65$$

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Relative valuation of bonds

- There are 1-year, 2-year, and 3-year bonds (all with face values of $100) traded in the market.

- The coupons are paid in annual installments.

Maturity and type	Coupon rate	Price
A: 1-year discount	--	$96.00
B: 2-year coupon bond	5.0%	$99.30
C: 3-year coupon bond	10.0%	$108.80

- Consider a 3-year discount bond D with the face value of $100, traded at $84.00. Is this bond fairly priced?
  - If not, how can we take advantage of mispricing?

Relative valuation of bonds

- First, visualize bond cash flows as a payoff matrix.

Bond	$t = 1$	$t = 2$	$t = 3$	Price
A	100	0	0	96
B	5	105	0	99.3
C	10	10	110	108.8

- Denote the prices of discount bonds with maturities of 1, 2, and 3 years respectively and face value of $1 as $P_1, P_2,$ and P_3.
- Prices of bonds A, B, and C are related to prices of zero-coupon bonds.
- Consider bond B:

$$5 \times P_1 + 105 \times P_2 + 0 \times P_3 = 99.30$$

Relative valuation of bonds

- Collect all three pricing relations into a system of linear equations:

$$\begin{bmatrix} 100 & 0 & 0 \\ 5 & 105 & 0 \\ 10 & 10 & 110 \end{bmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 96 \\ 99.3 \\ 108.8 \end{pmatrix} \quad \begin{matrix} (A) \\ (B) \\ (C) \end{matrix}$$

- We find a unique solution:

$$P_1 = 0.96, P_2 = 0.90, P_3 = 0.82$$

- The price of bond D is inconsistent with the prices of A, B, and C: the implied price of D is $\$82 < \$84$.

Arbitrage strategy

- We now construct an arbitrage strategy to take advantage of mispricing:
  - This strategy is self-financing: it requires no infusion of capital;
  - It has no risk of losing money;
  - It produces positive profits with a positive probability.
- Construct a portfolio with x_A, x_B, x_C, and x_D shares of bonds A, B, and C, and D respectively.
- We require our portfolio to produce a cash flow of \$1 at time 0, and nothing in periods 1, 2, and 3.
  - This portfolio delivers arbitrage profits.

Arbitrage strategy

- Collect conditions on cash flows at $t = 0, 1, 2, 3$ into a system of linear equations:

negative: cost of buying

positive: payoff

$$\begin{bmatrix} \boxed{\begin{matrix} -96 \\ 100 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} -99.3 \\ 5 \\ 105 \\ 0 \end{matrix}} & \boxed{\begin{matrix} -108.8 \\ 10 \\ 10 \\ 110 \end{matrix}} & \boxed{\begin{matrix} -84 \\ 0 \\ 0 \\ 100 \end{matrix}} \end{bmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \\ x_D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} (t=0) \\ (t=1) \\ (t=2) \\ (t=3) \end{matrix}$$

- Solving the equations, we find

$x_A = -0.0433$ short position

$x_B = -0.0433$ short position

$x_C = 0.4545$ long position

$x_D = -0.5000$ short position

Conclusion

- With a rich collection of fixed-income assets traded in the market, absence of arbitrage imposes strong restrictions on prices of securities relative to each other.
- Prices of coupon bonds, in particular, contain information about the yield curve (interest rates).
- So far we explore implications of arbitrage restrictions in a static framework – trade only at time 0.
  - Arbitrage-based pricing methods can be used in a dynamic setting to describe variation in bond prices over time. trading over multiple periods.

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Yield to maturity

- Yield-to-maturity (YTM) of a bond, denoted by y, solves

$$B = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} + \frac{P}{(1+y)^T}$$

on Bloomberg

- Yield to maturity is a convention for quoting prices: in general, YTM does not represent the expected return on a bond.

- YTM is a function of interest rates of various maturities.

Yield to maturities cannot be used to compare bonds to each other. A bond with a high yield to maturity does not necessarily pay a higher return. return is determined by not only interest rate but also duration. all weight on first year coupon with high spot interest rate vs all weight on last year principle with relatively lower spot interest rate

YTM is some kind of weighted average of the spot rates. Whether higher coupon rate implies higher YTM, depends on the current yield curve.

The par value of bonds definition refers to the principal – the amount of money the bondholder receives when the bond matures. Par value is also called face value or nominal value. It is the amount stipulated in the bond contract. However, par value does not include interest payments.

Yield to maturity: example

- Current spot rates

1 year	2 years
5%	6%

- 2-year Treasury coupon bond with a par value of $100 and a coupon rate of 6%, annual coupon payments:

$$Price = \frac{6}{1+0.05} + \frac{106}{(1+0.06)^2} = 100.0539$$

- Yield to maturity is 5.9706%:

$$100.0539 = \frac{6}{1+0.059706} + \frac{106}{(1+0.059706)^2}$$

YTM vs coupon rate

- Bond price is inversely related to YTM.

bond sells at par: meaning that the price of the bond is equal to the principal payment,

- A bond sells at par only if its coupon rate equals the YTM. Let $P = 1$, coupon rate is c:

principal payment of 1

$$B = \sum_{t=1}^{T} \frac{c}{(1+y)^t} + \frac{1}{(1+y)^T} = \frac{c}{y} + \frac{1}{(1+y)^T}\left(1 - \frac{c}{y}\right)$$

$$c = y \quad \text{iff} \quad B = 1$$

It trades below the par value.

- A bond sells at a discount if its coupon rate is below the YTM, $c < y$.
- A bond sells at a premium if its coupon rate is above the YTM, $c > y$.

the price of the bond will exceed the principal payment

bond price is declining as a function of YTM

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Expectations hypothesis

- Consider two alternative investment strategies:

  1. Invest $1 in a 10-year discount bond;

  2. Invest $1 in a 9-year discount bond + re-invest for 1 more year at the prevailing spot rate.

- Assume that bond returns do not carry a risk premium, the two strategies should produce the same return in expectation.

no risk premium: all of investment strategies
that move money between time 0 and time 10
by investing in different bonds and maybe reinvesting-- they
all earn the same expected rate of return

Expectations hypothesis

- Notation: $r_s(t)$ denotes s-period spot rate at time t. By default, $t = 0$.
- Two strategies should produce the same return in expectation: without risk premium

$$\mathrm{E}_0\left[\left(1 + r_9(0)\right)^9\left(1 + \tilde{r}_1(9)\right)\right] = \left(1 + r_{10}(0)\right)^{10}$$

  - $r_9(0)$ and $r_{10}(0)$ are spot rates, known at time $t = 0$;
  - $\tilde{r}_1(9)$ is random, spot rate at time $t = 9$.
- Without risk premium,

$$\mathrm{E}_0[\tilde{r}_1(9)] = \frac{\left(1+r_{10}(0)\right)^{10}}{\left(1+r_9(0)\right)^9} - 1$$

Expectations hypothesis

- Under the expectations hypothesis, the slope of the yield curve predicts future spot rates.

- Suppose the 10-year spot rate is above the 9-year rate. Then

$$\mathrm{E}_0[\tilde{r}_1(9)] = \frac{\left(1 + r_{10}(0)\right)^9}{\left(1 + r_9(0)\right)^9}\left(1 + r_{10}(0)\right) - 1 > r_{10}(0) > r_9(0)$$

- Suppose the 10-year spot rate is equal to the 9-year rate. Then

$$\mathrm{E}_0[\tilde{r}_1(9)] = \frac{\left(1+r_{10}(0)\right)^9}{\left(1+r_9(0)\right)^9}\left(1 + r_{10}(0)\right) - 1 = r_{10}(0) = r_9(0)$$

Expectations hypothesis

- The slope of the term structure reflects the market's expectations of future short-term interest rates:

$$\mathrm{E}_0[\tilde{r}_1(t)] = \frac{\left(1 + r_{t+1}(0)\right)^{t+1}}{\left(1 + r_t(0)\right)^{t}} - 1$$

Liquidity preference hypothesis

- Investors regard long bonds as riskier than short bonds, earn a premium λ_t -- "risk premium", or "liquidity premium" invest in (t+1)-bond is less liquid and more risky than t-bond plus 1-bond, t+1 should earn more

$$E_0[\tilde{r}_1(t)] + \lambda_t = \left\{ \frac{\left(1 + r_{t+1}(0)\right)^{t+1}}{\left(1 + r_t(0)\right)^t} - 1 \right\}$$

- Implications:
  - Long-term bonds on average receive higher returns than short-term bonds. if lambda is positive
  - Long-term interest rates "over-predict" future short-term rates.
- Term structure reflects expectations of future interest rates + risk (or "liquidity") premium demanded by investors in long bonds.

Hypotheses on interest rates

This empirical fact tells us that the expectations hypothesis does not describe market dynamics very well. Under the expectations hypothesis, average rates of return on long-term bonds and on short-term bonds would have to be the same.

Long-term	Bills
5.9%	3.4%

Average Rates of Return on Treasuries, 1926 - 2018

(Source: Ibbotson Associates, 2019 Yearbook)

- Why long-term bonds may earn a positive premium?
  - Short-term bonds are more money-like: hold value better in the short run; more liquid, extreme case, the rate of return on cash is 0
  - Nominal bonds are exposed to inflation risk – lose value when inflation spikes.

Models of interest rates

- What determines the term structure of interest rates?
  - Expected future spot rates;
  - Risk of long bonds.

- Models of interest rates:
  - Expectations Hypothesis;
  - Liquidity Preference Hypothesis;
  - Dynamic Models: Vasicek, Cox-Ingersoll-Ross, etc.

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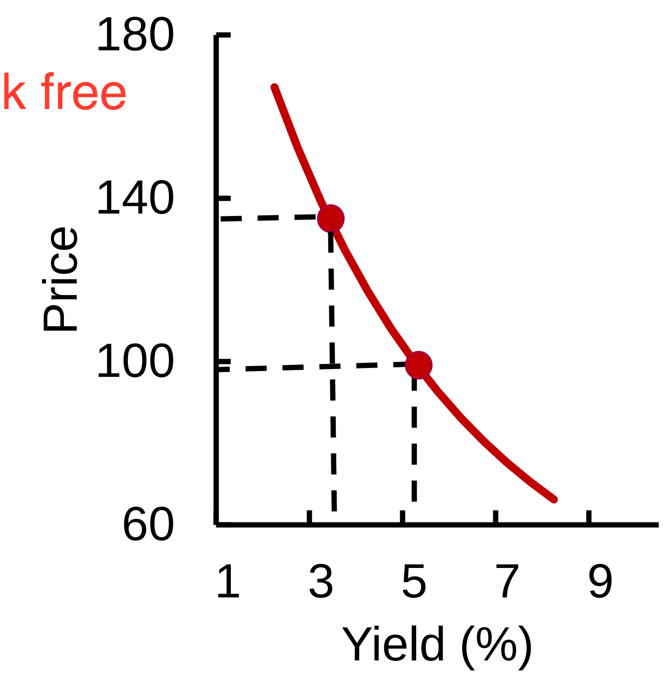
Interest rate risk

- As interest rates change (stochastically) over time, bond prices also change. not risk free
- The value of a bond is subject to interest rate risk.
- Bond prices and bond yields are inversely related:
  - As bond yield rises, bond price falls.



Bond Duration

- Assume a flat term structure at $r_t = y$.
- Measure bond's interest rate risk by its relative price change with respect to a unit change in yield (with a negative sign):
  - Suppose bond price is \$90 at $y = 0.05$.
  - As yield changes to 0.04, bond price rises to \$91.8.
  - Relative price change is $(91.8 - 90)/90 = 0.02$.
  - Normalizing by the change in the yield, risk measure is $\frac{0.02}{0.05-0.04} = 2$.
- We consider infinitesimal changes in bond yield, and use derivatives to define bond risk.

quantitative measure of risk is a measure of sensitivity of the bond price to changes in the yield

Bond Duration

- Measure of bond risk:

inversely related, add '-' to make risk positive

y: yield to maturity

$$-\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}y}$$

special case: discount bond, duration (risk) can be calculated directly

- This is called Modified Duration (MD).

- For a discount bond, $\mathrm{B}_t = (1+y)^{-t}$, hence

$$MD(B_t) = -\frac{1}{B_t}\frac{\mathrm{d}B_t}{\mathrm{d}y} = \frac{t}{1+y}$$

- Modified duration is closely related to physical timing of cash flows:

$$(1+y\ )MD(B_t) = t$$

Macaulay duration

general: coupon bond
how to calculate duration (risk)

- Consider general streams of cash flows, CF_t (e.g., coupons).
- Macaulay duration is the weighted average term to maturity.

$$D = \sum_{t=1}^{T} \left(\frac{PV(CF_t)}{B} \right) t = \frac{1}{B} \sum_{t=1}^{T} \left(\frac{CF_t}{(1+y)^t} \right) t$$

y: yield to maturity

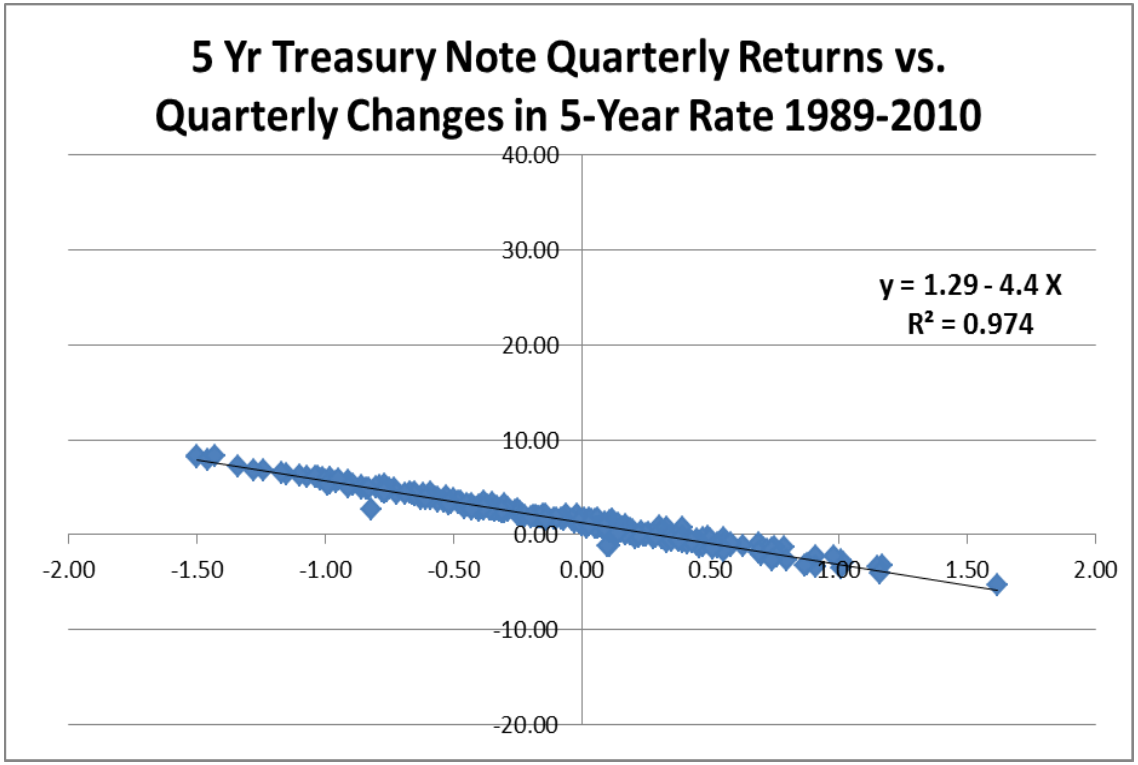
normalize it by the total value of the cash flow B

- Intuitive interpretation – center of gravity of payment tenors.
- Macaulay duration is proportional to Modified duration

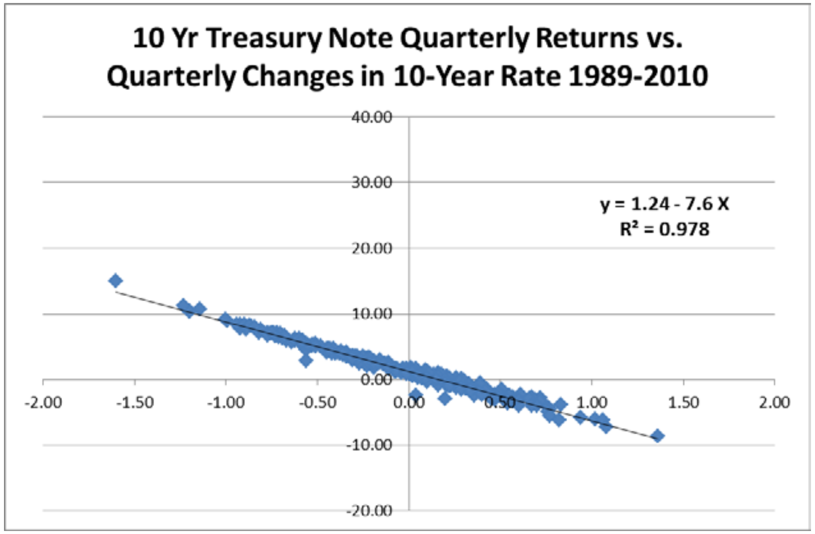
$$MD = \frac{D}{1+y}$$

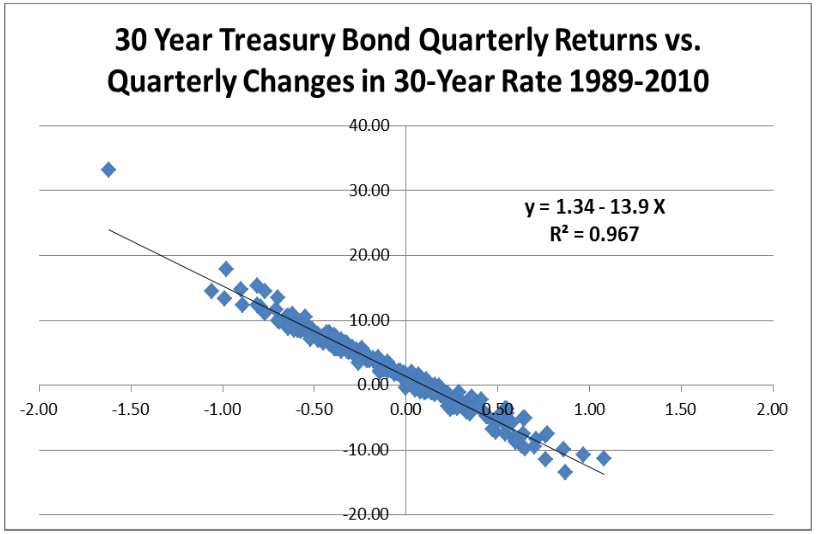
Empirical example

- Regress returns of T-year bonds on changes in T-year interest rate.
- Consider 5, 10, and 30-year Treasury bonds.
- Returns capture price changes in response to interest rate changes: higher slope coefficient indicates longer bond duration.



Empirical example, continued





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Example

- Consider a 4-year T-note: face value \$100 and 7% coupon, selling at \$103.50.

t	CF	PV	$t \times PV(CF)$
1	3.5	3.40	3.40
2	3.5	3.30	6.60
3	3.5	3.20	9.60
4	3.5	3.11	12.44
5	3.5	3.02	15.10
6	3.5	2.93	17.60
7	3.5	2.85	19.95
8	103.5	81.70	653.60
		103.50	738.28

- For T-notes, coupons are paid semi-annually.
- Use 1/2 year (6 months) as time unit.

Example, continued

YTM

- The bond yield (semi-annual) is 3% -- recall that 1 period corresponds to 6 months.
- Macaulay duration: $D = 738.28/103.50 = 7.13$ in $1/2$ year units.

1+YTM

- Modified duration: $MD = D/(1 + y) = 7.13/1.03 = 6.92$.
- If the semi-annual yield moves up by 0.1%, the bond price decreases roughly by 0.692%.
- What if interest rates move by a larger amount – would duration still capture changes in the bond price accurately?

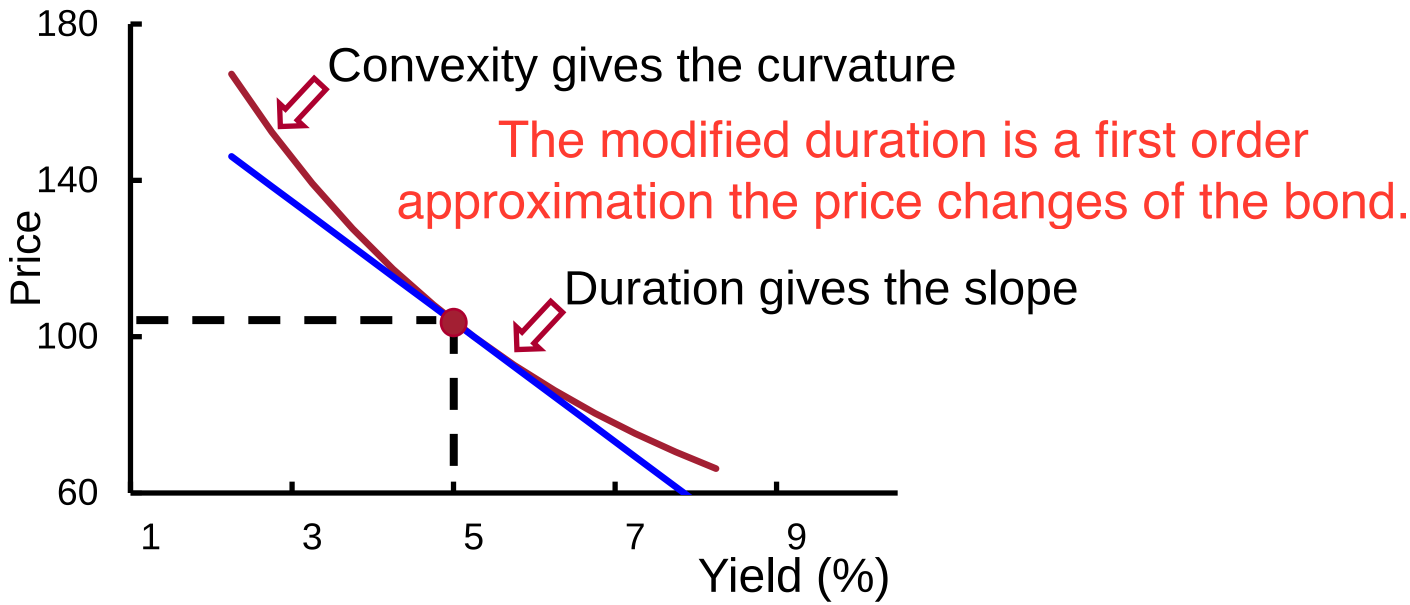
Example, continued

- For small yield changes, pricing by MD is accurate.
- For large yield changes, pricing by MD is inaccurate.

Yield per 6-month period

Yield	Price	Using MD	Difference
0.040	96.63	96.35	0.29
0.035	100.00	99.93	0.07
0.031	102.80	102.79	0.00
0.030	103.50	–	–
0.029	104.23	104.23	0.00
0.025	107.17	107.09	0.08
0.020	110.99	110.67	0.32

Duration and convexity



Convexity, definition

- Bond price is not a linear function of the yield. For larger yield changes, the effect of curvature (i.e., nonlinearity) becomes important.

$$
\begin{aligned}
(\Delta B) &= \frac{\mathrm{d}B}{\mathrm{d}y}(\Delta y) + \frac{1}{2}\frac{\mathrm{d}^2 B}{\mathrm{d}y^2}(\Delta y)^2 + \cdots \\
&\approx [-MD \times (\Delta y) + CX \times (\Delta y)^2] \times B
\end{aligned}
$$

- Convexity (CX) measures the curvature of the bond price as a function of the yield:

$$
CX = \frac{1}{2}\frac{1}{B}\frac{\mathrm{d}^2 B}{\mathrm{d}y^2}
$$

Example

- 10-year bond with 10% coupon and 10% flat yield curve (semi-annual periods: coupon of 5% and yield of 5% per 6-month period).
- Modified Duration: $MD = 6.23$.

6-month yield change is 1/2 of annual yield change

	Annual yield changes (bps)								
	-400	-300	-200	-100	0	100	200	300	400
Price changes	6-month yield change is 150 bps								
Total	29.8	21.3	13.6	6.5	0.0	-6.0	-11.5	-16.5	-21.2
Due to duration	24.9	18.7	12.5	6.2	0.0	-6.2	-12.5	-18.7	-24.9
Due to convexity	4.2	2.4	1.1	0.3	0.0	0.3	1.1	2.4	4.2
Residual	0.6	0.2	0.1	0.0	0.0	-0.0	-0.1	-0.2	-0.5

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Inflation risk

- Most bonds deliver nominal payoffs.
- In the presence of inflation risk, real payoffs are risky even when nominal payoffs are safe.
  - Inflation is the rate of change of the nominal price level.
  - As the price level rises, the real value of a dollar falls – purchasing power declines.
  - Real returns of bonds, all else equal, are inversely related to changes in the price level.

Example

- Suppose that inflation next year is uncertain ex ante, with equally possible rate of 10%, 8% and 6%.
- The real interest rate is 2%.

while nominal securities, like Treasury bonds, may produce safe payoffs in nominal terms, the payoff is risky in real terms.

- The 1-year nominal interest rate will be approximately 10%.
- Real return from investing in a 1-year Treasury security depends on realized inflation.

Year 0 value	Inflation rate (%)	Year 1 nominal payoff	Year 1 real payoff
1,000	0.10	1,100	1,000
1,000	0.08	1,100	1,019 =1100/1.08
1,000	0.06	1,100	1,038 =1100/1.06

Valuation of riskless cash flows

- Relative valuation, based on absence of arbitrage.
- Without risk, only time value of money is relevant.
- Prices of traded fixed-income securities provide information needed to value riskless cash flows at hand.

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