Time Series Analysis (I) Lecture 2

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This Lecture

1. Time Series data and examples
2. Lag operators
3. Mean and Autocovariance
4. Stationarity

Time Series data and examples

- $\{y_t\}_{t=1}^{T}$; or $\{y_1, y_2, ..., y_T\}$ is a collection of observations indexed by the date of each observation
  eg: daily stock price sequence, yearly GDP, monthly CPI;
- Some examples of Time series data
  - a time trend : $y_t = t$ or $y_t = a + bt$
  - a constant: $y_t = c$
  - a Gaussian white noise process: $y_t = \varepsilon_t$, where ε_t is $i.i.d$ N(0,1).

Lag Operators

Lag Operator: $L : L(y_t) = y_{t-1}$; and
$L^k(y_t) = L^{k-1}(L(y_t)) = L^{k-1}(y_{t-1}) = ... = y_{t-k}$;

Linear properties:

$$L(ay_t + w_t) = ay_{t-1} + w_{t-1}$$

$$\begin{aligned}(1-\lambda_1 L)(1-\lambda_2 L)y_t &= (1-(\lambda_1+\lambda_2)L+\lambda_1\lambda_2 L^2)y_t \\ &= y_t-(\lambda_1+\lambda_2)y_{t-1}+\lambda_1\lambda_2 y_{t-2}.\end{aligned}$$

First order difference equation:

Consider the following first order difference equation

$$y_t = \phi y_{t-1} + w_t = \phi L y_t + w_t$$
$$(1-\phi L) y_t = w_t$$

Note that when $|\phi| < 1$, the dynamic system is stable ($|\phi| > 1$,, explosive), and we have

$$\frac{1}{1-\phi L} = \lim_{j\to\infty} (1 + \phi L + \phi^2 L^2 +, \ldots, +\phi^j L^j);$$

Thus,

$$\begin{aligned} y_t &= \frac{w_t}{(1-\phi L)} = \lim_{j\to\infty} (1 + \phi L + \phi^2 L^2 +, \ldots, +\phi^j L^j) w_t \\ &= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \ldots . \end{aligned}$$

Some Definitions

- The effect of w_{t-j} on y_t or dynamic multiplier is given by

$$\frac{\partial y_t}{\partial w_{t-j}} = \phi^j.$$

- The impulse-response function is defined as the dynamic multipliers for w_t on y_{t+j} for different j.
- A cumulative effect on y of a transitory change in w_t (just w_t changes, others unchange) is defined as

$$\sum_{j=0}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} = \sum_{j=0}^{\infty} \left(\phi^j\right) = \frac{1}{1-\phi}.$$

- The long-run effect of a permanent (persistent) change in w (all w's change since time t) on y is defined as

$$
\begin{aligned}
& \lim_{j\to\infty}\left(\frac{\partial y_{t+j}}{\partial w_t}+\frac{\partial y_{t+j}}{\partial w_{t+1}}+,\ldots,+\frac{\partial y_{t+j}}{\partial w_{t+j}}\right) \\
= & \lim_{j\to\infty}(\phi^j+\phi^{j-1}+,\ldots,+1)=\frac{1}{1-\phi}.
\end{aligned}
$$

Second order difference equations

Consider a second order difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$

$$(1 - \phi_1 L - \phi_2 L^2) y_t = w_t$$

Let λ_1 and λ_2 (could be complex numbers) satisfy that

$$(1 - \phi_1 L - \phi_2 L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L)$$

We have

$$
\begin{aligned}
y_t &= \frac{w_t}{(1-\phi_1 L-\phi_2 L^2)} = \frac{w_t}{(1-\lambda_1 L)(1-\lambda_2 L)} \\
&= (\lambda_1-\lambda_2)^{-1}\left(\frac{\lambda_1 w_t}{(1-\lambda_1 L)} - \frac{\lambda_2 w_t}{(1-\lambda_2 L)}\right) \\
&= \left(\frac{\lambda_1}{\lambda_1-\lambda_2}(1+\lambda_1 L+\lambda_1^2 L^2+\ldots) - \frac{\lambda_2}{\lambda_1-\lambda_2}(1+\lambda_2 L+\ldots)\right) w_t \\
&= (c_1+c_2)w_t + (c_1\lambda_1+c_2\lambda_2)w_{t-1} + (c_1\lambda_1^2+c_2\lambda_2^2)w_{t-2}\ldots,
\end{aligned}
$$

where

$$
c_1 = \frac{\lambda_1}{\lambda_1-\lambda_2}, c_2 = \frac{-\lambda_2}{\lambda_1-\lambda_2}.
$$

- The effect of w_{t-j} on y_t or dynamic multiplier is given by

$$\frac{\partial y_t}{\partial w_{t-j}} = c_1\lambda_1^j + c_2\lambda_2^j$$

- A cumulative effect on y of a transitory change in w_t is

$$\sum_{j=0}^{\infty} \frac{\partial y_{t+j}}{\partial w_t} = \sum_{j=0}^{\infty} \left(c_1\lambda_1^j + c_2\lambda_2^j \right)$$

- The long-run effect of a permanent change in w on y is defined as

$$\lim_{j\to\infty} \left(\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} +, \dots, \frac{\partial y_{t+j}}{\partial w_{t+j}} \right) = \lim_{j\to\infty} \sum_{i=0}^{j} (c_1\lambda_1^{j-i} + c_2\lambda_2^{j-i})$$

P-th order difference equations

Consider the following P-th order difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} +, ..., + \phi_p y_{t-p} + w_t$$

Factor the polynomial equation as

$$1 - \phi_1 L - \phi_2 L^2 - ..., \phi_p L^p = (1 - \lambda_1 L)(1 - \lambda_2 L)...(1 - \lambda_p L)$$

$\lambda_1, ...\lambda_p$ are the eigenvalues of the following matrix (more detail see Append 1.A page 21)

$$F = \begin{pmatrix} \phi_1, \phi_2, \phi_3, ..., \phi_p \\ 1, 0, 0, ..., 0 \\ 0, 1, 0, ..., 0 \\ 0, 0, 0, ...1, 0 \end{pmatrix}_{p \times p}$$

or the inverse of the roots for the equation

$$1 - \phi_1 z - \phi_2 z^2 -, ..., -\phi_p z^p = 0$$

The stable (boundness) condition for

$$\frac{1}{1-\phi_1 L-\phi_2 L^2-,...,-\phi_p L^p}$$

is the roots for $1-\phi_1 z-\phi_2 z^2-...\phi_p z^p=0$ lie outside of the unit circle or $\lambda_1,...\lambda_p$ lies inside the unit circle.
Once again, one can find

$$\begin{aligned} y_t &= \frac{1}{1-\phi_1 L-\phi_2 L^2-,...,-\phi_p L^p} w_t \\ &= \psi_0 w_t+\psi_1 w_{t-1}+\psi_2 w_{t-2},..., \\ &= \Psi(L) w_t \end{aligned}$$

Ex1: show the long-run effect of w on y given by

$$\lim_{j\to\infty}\left(\frac{\partial y_{t+j}}{\partial w_t}+\frac{\partial y_{t+j}}{\partial w_{t+1}}+,\ldots,\frac{\partial y_{t+j}}{\partial w_{t+j}}\right)=\frac{1}{1-\phi_1-\phi_2-\ldots-\phi_p}$$

Mean and Autocovariance

Suppose that $\{Y_t\}$ is a time series; define the mean function as

$$\mu_t = E(Y_t).$$

Its Autocovariance function is defined as

$$\begin{aligned}\gamma_Y(s,t) &= Cov(Y_s, Y_t) \\ &= E[(Y_s - \mu_s)(Y_t - \mu_t)].\end{aligned}$$

Autocorrelation function (ACF)

$$\rho_Y(s,t) = \frac{Cov(Y_s, Y_t)}{\sqrt{Cov(Y_t, Y_t)Cov(Y_s, Y_s)}}$$

Example: $Y_t = \mu + \varepsilon_t$ where ε_t is a i.i.d $N(0, \sigma^2)$.

$$\begin{aligned}
\mu_t &= E(Y_t) = E(\mu + \varepsilon_t) = \mu \\
\gamma_Y(s,t) &= Cov(Y_s, Y_t) = E[(Y_s - \mu)(Y_t - \mu)] \\
&= E(\varepsilon_s \varepsilon_t) = \left\{ \begin{array}{l} 0, \text{ if } s \neq t \\ \sigma^2, \text{ if } s = t \end{array} \right\}
\end{aligned}$$

Ex2: How about $Y_t = \beta t + \varepsilon_t$?

Stationarity

Covariance stationary or weakly stationary if

$$\begin{aligned} \mu_t &= \mu \text{ for all } t \\ \gamma_Y(t-j,t) &= Cov(Y_{t-j}, Y_t) = \gamma_j \text{ for all } t \text{ and } j. \end{aligned}$$

In other words, μ_t and $\gamma_Y(t-j,t)$ is independent of t.

White Noise: Y_t is defined as a white noise process if

$$E(Y_t) = 0,$$

$$E(Y_t^2) = \sigma^2$$

and

$$\gamma_Y(t-j,t) = \left\{ \begin{array}{l} 0,\ \text{if } j \neq 0 \\ \sigma^2, \text{if } j = 0 \end{array} \right\}.$$

Thus, White Noise process is stationary.

Random Walk: $S_t = \sum_{i=1}^{t} Y_i$, where Y_i is a white noise process $WN(0, \sigma^2)$;

$$\begin{aligned}
E(S_t) &= 0 \\
E(S_t^2) &= t\sigma^2 \\
\gamma_S(t-j, t) &= Cov(S_{t-j}, S_t) \\
&= Cov(S_{t-j}, S_{t-j} + \sum_{i=t-j+1}^{t} Y_i) \\
&= Cov(S_{t-j}, S_{t-j}) \\
&= (t-j)\sigma^2.
\end{aligned}$$

Thus, Random Walk process S_t is not stationary.

Autocorrelation function

Autocorrelation function (ACF) of a stationary process

$$\begin{aligned} \rho_Y(j) &= \frac{Cov(Y_{t+j}, Y_t)}{Cov(Y_t, Y_t)} \\ &= Corr(Y_{t+j}, Y_t) \end{aligned}$$

Note that if a process is stationary then

$$\begin{aligned} \rho_Y(j) &= \rho_Y(-j) \\ \gamma_Y(j) &= \gamma_Y(-j) \end{aligned}$$

Strictly Stationary

Y_t is strictly stationary if for all $k, t_1, ..t_k, y_1, ..., y_k$ and h, the probability

$$\Pr(Y_{t_1} \leq y_1, ..., Y_{t_k} \leq y_k) = \Pr(Y_{t_1+h} \leq y_1, ..., Y_{t_k+h} \leq y_k);$$

in other words, the joint distribution $F(Y_{t_1}, ..., Y_{t_k})$ depends only on the intervals separating the dates $(t_1, ..t_k)$, not on the date itself.

Gaussian process: Y_t is a Gaussian process, if the joint distribution of $(Y_{t_1}, ..., Y_{t_k})$ for any $k, t_1, ..t_k$, is Gaussian distribution.

Strictly stationary does not imply weakly stationary, require finite second moments;

Weakly stationary does not imply strictly stationary,

eg: the third order moment could be time-varying, even the first and second moments are constant.