Time series Analysis (I) Lecture 3

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This lecture

1. Moving average process
2. Autoregressive Process
3. Autoregressive Moving average process
4. Yule-Walker Equations
5. Autocovariance generating function
6. Linear filter
7. Invertibility

Moving Average Processes with Order one MA(1)

MA(1) process:

$$Y_t = c + \varepsilon_t + \theta\varepsilon_{t-1}, \text{ where } \varepsilon_t \text{ is } WN(0, \sigma^2).$$

Mean and autocovariance functions:

$$\begin{aligned}
\mu &= E(Y_t) = E(c + \varepsilon_t + \theta\varepsilon_{t-1}) = c; \\
\gamma_0 &= E[(Y_t - \mu)(Y_t - \mu)] = E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_t + \theta\varepsilon_{t-1})] \\
&= E(\varepsilon_t^2 + \theta^2\varepsilon_{t-1}^2) = (1 + \theta^2)\sigma^2; \\
\gamma_1 &= E[(Y_t - \mu)(Y_{t-1} - \mu)] = E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})] \\
&= E(\theta\varepsilon_{t-1}^2) = \theta\sigma^2; \\
\gamma_j &= E[(Y_t - \mu)(Y_{t-j} - \mu)] = 0 \text{ for } j > 1;
\end{aligned}$$

The ACF:

$$\rho_j = \left\{ \begin{array}{c} \frac{\theta}{(1+\theta^2)}, \text{if } j = 1 \\ 0, \ \text{for } j > 1 \end{array} \right\}.$$

Moving Average Processes with q-th order MA(q)

MA(q) process:

$Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + \theta_q \varepsilon_{t-q}$, where ε_t is $WN(0, \sigma^2)$.

Mean and autocovariance functions:

$$\begin{aligned}
\mu &= E(Y_t) = E(c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + \theta_q \varepsilon_{t-q}) = c; \\
\gamma_0 &= E[(Y_t - \mu)(Y_t - \mu)] \\
&= E(\varepsilon_t^2 + \theta_1^2 \varepsilon_{t-1}^2 + ... + \theta_q^2 \varepsilon_{t-q}^2) = (1 + \theta_1^2 +, ... + \theta_q^2)\sigma^2; \\
\gamma_j &= E[(Y_t - \mu)(Y_{t-j} - \mu)] \\
&= (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + ... + \theta_q \theta_{q-j})\sigma^2, \text{for } j = 1, 2, ..., q; \\
\gamma_j &= 0 \text{ for } j > q.
\end{aligned}$$

Moving Average Processes with infinite order

MA(∞) process:

$$Y_t = \mu + \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots, \text{ where } \varepsilon_t \text{ is } WN(0, \sigma^2).$$

Y_t is weakly stationary if $\{\psi_j\}_{j=0}^{\infty}$ is square summable:

$$\sum_{j=0}^{\infty} \psi_j^2 < \infty;$$

a slightly stronger condition is absolutely summable

$$\sum_{j=0}^{\infty} |\psi_j| < \infty.$$

R Demonstration for MA processes

```
See Rcode_lec3
#simulations for MA processes
arima.sim()
help(arima.sim)
require(graphics)
y=arima.sim(n = 200, list(ma = c(0.5)),sd = 1)
#for different value of phi;
plot(y,type='l')
var_y=cov(y,y)
var_y
print("truevalue:")
(1+0.5^2)
help(acf)
covy=acf(y, type = "covariance")
corry=acf(y, type = "correlation")
```

Autoregressive Process with order one: AR(1):

AR(1) process:

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim WN(0, \sigma^2).$$

Y_t is stationary if $|\phi| < 1$(will show later);
Take expectation on both sides, we have

$$\begin{aligned} E(Y_t) &= c + \phi E(Y_{t-1}) \\ \mu &= E(Y_t) = \frac{c}{1-\phi}; \end{aligned}$$

Replace c by μ and do some adjustment, we have

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \varepsilon_t,$$

Time $Y_t - \mu$ on both sides and take expectation,

$$\begin{aligned} E[(Y_t - \mu)^2] &= E[(Y_t - \mu)\phi(Y_{t-1} - \mu)] + E[(Y_t - \mu)\varepsilon_t] \\ &= \phi^2 E[(Y_{t-1} - \mu)^2] + E(\varepsilon_t^2). \end{aligned}$$

Using the definition of stationarity, we have $E[(Y_t - \mu)^2] = E[(Y_{t-1} - \mu)^2] = \gamma_0$. One can solve out

$$\gamma_0 = \frac{\sigma^2}{1 - \phi^2}.$$

Time $Y_{t-j} - \mu$ on both sides and take expectation,

$$E[(Y_{t-j} - \mu)(Y_t - \mu)] = E[(Y_{t-j} - \mu)\phi_1(Y_{t-1} - \mu)] + E[(Y_{t-j} - \mu)\varepsilon_t].$$

One have

$$\gamma_j = \phi\gamma_{j-1}.$$

Using the recursive method, one can solve out

$$\gamma_j = \frac{\phi^j \sigma^2}{1 - \phi^2}, \text{ for all } j \geq 1.$$

Proof of Stationarity:

Given $|\phi| < 1$ and let $\widetilde{\varepsilon}_t = c + \varepsilon_t$ we have

$$\begin{aligned} Y_t &= \frac{1}{1-\phi L}\widetilde{\varepsilon}_t = \widetilde{\varepsilon}_t + \phi\widetilde{\varepsilon}_{t-1} + ...\phi^j\widetilde{\varepsilon}_{t-j} + .. \\ &= c + \varepsilon_t + \phi\left(c + \varepsilon_{t-1}\right) + \phi^2\left(c + \varepsilon_{t-2}\right) + ... \\ &= \frac{c}{1-\phi} + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2_{t-2}\varepsilon_{t-2} + ... \end{aligned}$$

Thus,

$$\mu = E(Y_t) = \frac{c}{1-\phi};$$

$$\begin{aligned} \gamma_0 &= E(Y_t - \mu)^2 = E(\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + ...)^2 \\ &= (1 + \phi^2 + \phi^4 +, ...,)\sigma^2 = \frac{\sigma^2}{1-\phi^2}. \end{aligned}$$

One can solve out γ_j using the basic definition:

$$\begin{aligned}
\gamma_j &= Cov(Y_{t-j}, Y_t) = E[(Y_{t-j} - \mu)(Y_t - \mu)] \\
&= E[(\varepsilon_{t-j} + \phi\varepsilon_{t-j-1} + \phi^2\varepsilon_{t-j-2} + \dots)(\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \dots)] \\
&= (\phi^j + \phi^j\phi^2 + \phi^j\phi^4 +, \dots,)\sigma^2 \\
&= \frac{\phi^j\sigma^2}{1-\phi^2}.
\end{aligned}$$

The autocorrelation is given by

$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi^j.$$

R Demonstration for AR processes

```
require(graphics)
y=arima.sim(n = 200, list(ar = c(0.5)),sd = 1)
plot(y,type='l')
var_y=cov(y,y)
var_y
print("truevalue:")
1/(1-0.5^2)
covy=acf(y, type = "covariance")
corry=acf(y, type = "correlation")
```

Autoregressive Process with Order Two: AR(2)

AR(2) process:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t, \text{ where } \varepsilon_t \sim WN(0, \sigma^2).$$

Using Lag operator,

$$(1 - \phi_1 L - \phi_2 L^2) Y_t = c + \varepsilon_t$$

stationary condition:the roots of $1 - \phi_1 z - \phi_2 z^2 = 0$ are outside of the unit circle.

$$E(Y_t) = c + \phi_1 E(Y_{t-1}) + \phi_2 E(Y_{t-2}) + E(\varepsilon_t)$$

according to stationarity,

$$E(Y_t) = E(Y_{t-1}) = E(Y_{t-2}) = \mu$$

$$\mu = \frac{c}{1 - \phi_1 - \phi_2}$$

AR(2) Cont.

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \varepsilon_t, \ .$$

Thus, by multiplying $(Y_{t-j} - \mu)$ on both sides, and then take expectation

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2}, \text{ for all } j = 1, 2, 3, \ldots.$$

When $j = 1$,one have

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_{-1} = \phi_1 \gamma_0 + \phi_2 \gamma_1.$$

Thus,

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1}{1 - \phi_2};$$

for $j = 2$, we have

$$\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0;$$

dividing by γ_0, we get

$$\rho_2 = \phi_1 \rho_1 + \phi_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2.$$

AR(2) Cont.

$$
\begin{aligned}
\gamma_0 &= E(Y_t-\mu)^2 \\
&= \phi_1 E[(Y_{t-1}-\mu)(Y_t-\mu)] + \phi_2 E[(Y_{t-2}-\mu)(Y_t-\mu)] \\
&\quad + E(\varepsilon_t(Y_t-\mu)) \\
&= \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2.
\end{aligned}
$$

Note the last equation uses the following result:

$$
\begin{aligned}
E(\varepsilon_t(Y_t-\mu)) &= E(\varepsilon_t^2 + \phi_1\varepsilon_t(Y_{t-1}-\mu) + \phi_2\varepsilon_t(Y_{t-2}-\mu)) \\
&= E(\varepsilon_t^2) = \sigma^2.
\end{aligned}
$$

We have

$$\gamma_0 = \phi_1\rho_1\gamma_0 + \phi_2\rho_2\gamma_0 + \sigma^2;$$

then

$$\gamma_0 = \frac{(1-\phi_2)\sigma^2}{(1+\phi_2)((1-\phi_2)^2-\phi_1^2)}.$$

Autoregressive Process with Order p

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t, \text{ where } \varepsilon_t \sim WN(0, \sigma^2).$$

Provided that the roots of $1 - \phi_1 z - \phi_2 z^2 - ..., \phi_p z^p = 0$ are outside of the unit circle, then Y_t is stationary.

$$\mu = \frac{c}{1 - \phi_1 - \phi_2 - ... - \phi_p}.$$

After some adjustment:

$$Y_t - \mu = \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + ... + \phi_p (Y_{t-p} - \mu) + \varepsilon_t.$$

Multiple $Y_{t-j} - \mu$ on both sides and take expectation, we have

$$\gamma_j = \left\{ \begin{array}{c} \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} + ... + \phi_p \gamma_{j-p}, \text{ for all } j = 1, 2, 3, ... \\ \phi_1 \gamma_1 + \phi_2 \gamma_2 + ... + \phi_p \gamma_p + \sigma^2, \text{ for } j = 0. \end{array} \right\}.$$

Yule-Walker equations

Dividing by γ_0, one can get the so called Yule-Walker equations:

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2} + ... + \phi_p \rho_{j-p}, \text{ for all } j = 1, 2, 3, ...$$

One can solve out $\{\rho_1, ...\rho_p\}$ using $j = 1, 2, .., p$.
Ex: Yule-Walker equations for AR(3)

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_3 Y_{t-3} + \varepsilon_t, \text{ where } \varepsilon_t \sim WN(0, \sigma^2).$$

Let

$$\mu = \frac{c}{1 - \phi_1 - \phi_2 - \phi_3}$$

and again

$$Y_t - \mu = \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + \phi_3 (Y_{t-3} - \mu) + \varepsilon_t.$$

Example for Yule-Walker equations for AR(3)

Multiplying both sides by $Y_t - \mu$, $Y_{t-1} - \mu$, ...and taking expectations, and dividing by γ_0, we have

$$
\begin{aligned}
1 &= \phi_1\rho_1 + \phi_2\rho_2 + \phi_3\rho_3 + \frac{\sigma^2}{\gamma_0} \\
\rho_1 &= \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 \\
\rho_3 &= \phi_1\rho_2 + \phi_2\rho_1 + \phi_3 \\
\rho_k &= \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \phi_3\rho_{k-3}, \text{ for all } k > 3
\end{aligned}
$$

One can use equations 1-3 to solve out (ρ_1, ρ_2, ρ_3), and use equation 1 to solve out

$$\gamma_0 = \frac{\sigma^2}{1 - (\phi_1\rho_1 + \phi_2\rho_2 + \phi_3\rho_3)}$$

ARMA(p,q) model

$$\begin{aligned} Y_t &= c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} +, ... \phi_p Y_{t-p} \\ &+ \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \end{aligned}$$

Using Lag operations, we have

$$(1 - \phi_1 L - \phi_2 L^2 ... - \phi_p L^p) Y_t = c + (1 + \theta_1 L + \theta_2 L^2 + ... + \theta_q L^q) \varepsilon_t$$

$$Y_t = \frac{c + (1 + \theta_1 L + \theta_2 L^2 + ..., \theta_q L^q) \varepsilon_t}{1 - \phi_1 L - \phi_2 L^2 ... - \phi_p L^p} = \mu + \psi(L) \varepsilon_t$$

where

$$\psi(L) = \frac{1 + \theta_1 L + \theta_2 L^2 + ..., \theta_q L^q}{1 - \phi_1 L - \phi_2 L^2 ... - \phi_p L^p}.$$

Stationary Condition: the roots of $1 - \phi_1 z - \phi_2 z^2 - ..., \phi_p z^p = 0$ are outside of the unit circle.

Yule-Walker Equations for ARMA(p,q)

Ex: Derive the Yule-Walker equations for ARMA(1,1)

$$Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Autocovariance generating function

From weakly stationary time series: $\{Y_t\}$, its autocovariance function is given as $\{\gamma_j\}_{j=-\infty}^{\infty}$ and

$$\sum_{j=-\infty}^{\infty} |\gamma_j| < \infty$$

Then, Autocovariance generating function (AGF) can be defined as

$$g_Y(z) = \sum_{j=-\infty}^{\infty} \gamma_j z^j.$$

The population spectrum of Y is a special case of AGF:

$$S_Y(w) = \frac{1}{2\pi} g_Y(e^{-iw}) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-ijw}.$$

Autocovariance generating function for MA process

Consider MA process

$$Y_t = \mu + \Psi(L)\varepsilon_t$$

with

$$\Psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \ldots$$

and

$$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

Autocovariance generating function of Y is given by

$$g_Y(z) = \sigma^2 \Psi(z)\Psi(z^{-1}).$$

Two Examples

Ex1: if Y_t is a White noise process, then

$$g_Y(z) = \sigma^2$$

as $\gamma_0 = \sigma^2$ and all other $\gamma_j = 0$ for $j = 1, 2, \ldots$.
Ex 2: if Y_t is a MA(1) process,

$$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

then

$$g_Y(z) = (1 + \theta^2)\sigma^2 + \theta\sigma^2 z$$

as $\gamma_0 = (1 + \theta^2)\sigma^2$, $\gamma_1 = \theta\sigma^2$ and others are zero for $j = 2, \ldots$.

Filter

Original Data series: $\{Y_t\}$, and the data are filtered according to

$$X_t = h(L)Y_t$$

with a linear filter defined as

$$h(L) = \sum_{j=-\infty}^{\infty} h_j L^j$$

and $\sum_{j=-\infty}^{\infty} |h_j| < \infty$.
Autocovariance generating function of X can be expressed as

$$g_X(z) = h(z)h(z^{-1})g_Y(z).$$

Eg: For MA(∞) process

$$Y_t = \mu + \Psi(L)\varepsilon_t$$

with

$$\Psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + ...$$

and $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

Then, we know $g_Y(z) = \sigma^2 \Psi(z)\Psi(z^{-1})$. Given $X_t = h(L)Y_t$, we have

$$g_X(z) = h(z)h(z^{-1})g_Y(z) = \sigma^2 h(z)h(z^{-1})\Psi(z)\Psi(z^{-1}).$$

Invertibility

A linear process $\{Y_t\}$ is invertible (strictly, an invertible function of $\{W_t\}$) if there exists a filter

$$\Psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \ldots$$

with

$$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

such that

$$W_t = \Psi(L) Y_t.$$

MA process and invertibility

Consider the MA(1) process (moving average representation)

$$Y_t = c + \varepsilon_t + \theta\varepsilon_{t-1}, \text{ where } \varepsilon_t \text{ is } WN(0, \sigma^2).$$

Y_t is invertible iff $|\theta| < 1$ and
autoregressive representation:

$$\begin{aligned} \varepsilon_t &= \frac{1}{(1-(-\theta L)}(Y_t - c) \\ &= (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + ...)(Y_t - c). \end{aligned}$$

Remark:

- If invertible, ε_t can be expressed as the function of current and past values of Y_t; Otherwise, need to use the future values of Y_t.
- ε_t associated with the invertible representation is called the *fundamental innovations* for Y_t.

Invertibility for MA(q)

Consider the MA(q) process

$Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + \theta_q \varepsilon_{t-q}$, where ε_t is $WN(0, \sigma^2)$.

Y_t is invertible **iff** the roots of

$$1 + \theta_1 z + \theta_2 z^2 + ...\theta_q z^q = 0$$

lies outside of the unit circle.
It is easy to show

$$\frac{(Y_t - c)}{1 + \theta_1 L + \theta_2 L^2 + ...\theta_q L^q} = \varepsilon_t.$$

Again: ε_t associated with the invertible representation is called the Fundamental innovations for Y_t.

Summary about Invertibility

1. Any AR(p) process is invertible to its innovation process;
2. A MA process is invertible to its innovation means it has an AR representation.