1. convergence of the object functional
   the object functional :
$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \log c_t\right] \tag{1}$$
   for c_t is bounded, $0 \le c_t \le A_t k_t^{1-\alpha}$, series (1) converges
   where
   $\log c_t = \log(\bar{c}_t z_t) = \log \bar{c}_t + \log z_t$
   $\log z_t = \alpha^{-1}\log A_t = \alpha^{-1}[(\gamma t + \sum_{j=1}^{t}\eta_j) + \log A_0]$
$$\Rightarrow (1) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \log \bar{c}_t\right] + \alpha^{-1}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\sum_{j=1}^{t}\beta^t\eta_j\right]$$
$$+ \alpha^{-1}\log A_0 \sum_{t=0}^{\infty}\beta^t + \alpha^{-1}\gamma\sum_{t=0}^{\infty} t\beta^t$$
   and
   $\alpha^{-1}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\sum_{j=1}^{t}\beta^t\eta_j\right] = 0$
   $\alpha^{-1}\log A_0 \sum_{t=0}^{\infty}\beta^t = \frac{\alpha^{-1}\log A_0}{1-\beta}$
   $\alpha^{-1}\gamma\sum_{t=0}^{\infty} t\beta^t = \alpha^{-1}\beta(\sum_{t=0}^{\infty}\beta^t)' = \frac{\alpha^{-1}\beta}{(1-\beta)^2}$
$$\Rightarrow (1) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \log \bar{c}_t\right] + constant \tag{2}$$
   for $\bar{c}_t$ is bounded, $0 \le \bar{c}_t \le \bar{z}_t^{\alpha-1}\bar{k}_t^{1-\alpha}$, series (2) also converges.

2. detrending (I found a more general argument in King, Plosser and Rebelo (2002). Production, Growth and Business Cycles: Technical Appendix.)
   it can be proved that under balanced growth path,
   $\gamma_y := \frac{y_{t+1}}{y_t} = \gamma_k = \gamma_c = e^{\frac{\gamma}{\alpha}}$

   proof :
   Define balanced growth path as c_t, k_t, i_t, y_t grow with constant, but possibly differing rates, therefore:
$$c_t + k_{t+1} = y_t + k_t \Rightarrow \gamma_k = \frac{y_t}{k_t} - \frac{c_t}{k_t} + 1 = constant$$
   $\Rightarrow$ $\frac{y_t}{k_t} = \frac{c_t}{k_t} \Rightarrow y_t = k_t$,
   In this case there is no investment, contradicting to the assumption.
   Or $\frac{y_t}{k_t} = constant \Rightarrow \gamma_y = \gamma_k$
   $\frac{y_t}{c_t} = constant \Rightarrow \gamma_y = \gamma_c$
   which gives $\gamma_y = \gamma_k = \gamma_c$.
   The steady state growth rate :
   $y_t = A_t k_t^{1-\alpha} \Rightarrow \gamma_y = \gamma_A\gamma_k^{1-\alpha} \Rightarrow \gamma_y = \gamma_A^{\frac{1}{\alpha}} = e^{\frac{\gamma}{\alpha}}$

So we need to use $z_t = A_t^{\frac{1}{\alpha}}$ to detrend the variable c_t, k_t, y_t. Since in the steady state, $\bar{k}_{t+1} = \bar{k}_t = k^* \Rightarrow \gamma_{\bar{k}} = \frac{\bar{k}_{t+1}}{\bar{k}_t} = 1$

Therefore, under balanced growth path, $\gamma_k = \gamma_z = \left(\frac{A_{t+1}}{A_t}\right)^{\frac{1}{\alpha}} = e^{\frac{\gamma}{\alpha}}$