Advanced Microeconomics II

Knowledge

Brett Graham

Wang Yanan Institute for Studies in Economics
Xiamen University, China

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Knowledge

- Nash equilibrium assumes that you “know” the other player’s equilibrium strategy.
- How do we model what a person knows?
- How do we model common knowledge?
- Can people agree to disagree?
- Can we express formally the assumptions about knowledge that lie behind the concept of Nash equilibrium?

Information Function

The core of our model of knowledge is Ω, the set of states. Interpretation:

- set of contingencies that are relevant to decision problem, or
- a full description of the world.

Definition

An information function for the set Ω of states is a function P that associates with every state $\omega \in \Omega$ a nonempty subset $P(\omega)$ of Ω.

Definition

An information function P on the set Ω of states is rational if it satisfies the following two conditions.

- (P1) $\omega \in P(\omega)$ for every $\omega \in \Omega$.
- (P2) If $\omega' \in P(\omega)$ the $P(\omega') = P(\omega)$.

Partitional Information Function

Definition

An information function P on the set Ω of states is partitional if there is a partition of Ω such that for any $\omega \in \Omega$ the set $P(\omega)$ is the element of the partition that contains ω.

Lemma

An information function is partitional if and only if it is rational.

- ($\Rightarrow$) Direct from the definitions.
- ($\Leftarrow$) Suppose that P is rational.
  - If $P(\omega)$ and $P(\omega')$ intersect and $\omega'' \in P(\omega) \cap P(\omega')$ then $P(\omega) = P(\omega') = P(\omega'')$ (by (P2))
  - $\cup_{\omega \in \Omega} P(\omega) = \Omega$ (by (P1)).

Knowledge Function

Definition

An event E is any subset of Ω.

Definition

Let E be an event. A knowledge function K is a mapping of events into events, i.e. subsets of Ω into subsets of Ω such that

$$K(E) = \{\omega \in \Omega : P(\omega) \subset E\}.$$

Example: Six Sided Die

- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- $\mathcal{P}_1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}, \mathcal{P}_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.
- $E_1 = \{1\}, E_2 = \{1, 2\}, E_3 = \{1, 2, 3\}, E_4 = \{1, 2, 3, 4\}$.

For each partition and event, find $K(E)$.

Knowledge Function Properties

A knowledge function has the following three properties.

- (K1) $K(\Omega) = \Omega$.
- (K2) If $E \subset F$ then $K(E) \subset K(F)$.
- (K3) $K(E) \cap K(F) = K(E \cap F)$.

Knowledge Function Properties

Lemma

If P satisfies (P1) and (P2) then the associated knowledge function K satisfies the following three properties.

- *(K4 - Axiom of Knowledge)* $K(E) \subset E$.
- *(K5 - Axiom of Transparency)* $K(E) = K(K(E))$.
- *(K6 - Axiom of Wisdom)* $\Omega \backslash K(E) = K(\Omega \backslash K(E))$.

- (K4) If $\omega \in K(E)$ then $P(\omega) \in E$. By (P1) $\omega \in E$.
- (K5) If F is a union of members of a partition then $K(F) = F$. If P satisfies (P1) and (P2) then $K(E)$ is indeed such a union.
- (K6) Since $K(E)$ is a union of members of a partition, $\Omega \backslash K(E)$ is also a partition.

The Puzzle of the Hats

- n rational people are seated around a table, each wearing a hat that is either black or white.
- Each individual observes everybody's hat except his own.
  - $\Omega = \{c \in \{B, W\}^n\}$.
  - $P_i^0(c) = \{(W, c_{-i}), (B, c_{-i})\}$.
- A person must leave the table one minute after they know the color of their own hat.
  - $E_i = \{c : P_i(c) \subset \{c : c_i = B\}$ or $P_i(c) \subset \{c : c_i = W\}$.
- An observer announces that at least one hat is white.
  - Let $F^k = \{c : |\{i : c_i = W\}| = k\}$ (set of states where there are k white hats).
  - $P_i^1(c) = P_i^0(c) \backslash F^0$.
  - If $c \in F^1$ then there exists i such that $P_i^1(c) = \{(W, c_{-i})\}$, $P_i^1(c) \subset E_i$, $K_i(E_i) = \{(W, c_{-i})\}$ and i leaves after one minute.

The Puzzle of the Hats

- If nobody leaves after one minute then $P_i^2(c) = P_i^1(c) \backslash F^1$.
  - If $c \in F^2$ then there exists two i such that $P_i^2(c) = \{(W, c_{-i})\}$, $K_i(E_i) = \{(W, c_{-i})\}$ and both players leave after two minutes.
- If nobody leaves after k minutes then $P_i^{k+1}(c) = P_i^k(c) \backslash F^k$.
  - If $c \in F^{k+1}$ then there exists $k+1$ i such that $P_i^{k+1}(c) = \{(W, c_{-i})\}$, $K_i(E_i) = \{(W, c_{-i})\}$ and $k+1$ players leave after $k+1$ minutes.
- One minute after somebody leaves, everybody else leaves.

Common Knowledge

Definition (1)

Let K_1 and K_2 be the knowledge function of individuals 1 and 2 for the set Ω of states. An event $E \subset \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if ω is a member of every set in the infinite sequence $K_1(E), K_2(E), K_1(K_2(E)), K_2(K_1(E)), \ldots$

Example: Six-sided die. $\Omega = \{1, 2, 3, 4, 5, 6\}$. The partitions induced by knowledge functions K_1 and K_2 are

$$\mathcal{P}_1 = \{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$$
$$\mathcal{P}_2 = \{\{1\}, \{2, 3, 4\}, \{5\}, \{6\}\}$$

- In what states is $E_1 = \{3, 4, 5, 6\}$ common knowledge?
- In what states is $E_2 = \{1, 2, 3, 4\}$ common knowledge?

Common Knowledge

Definition (2)

Let P_1 and P_2 be the information functions of individuals 1 and 2 for the set Ω of states. An event $F \subset \Omega$ is self-evident between 1 and 2 if for all $\omega \in F$ we have $P_i(\omega) \subset F$ for $i = 1, 2$. An event $E \subset \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if there is a self-evident event F for which $\omega \in F \subset E$.

Example: Six-sided die. $\Omega = \{1, 2, 3, 4, 5, 6\}$. The partitions induced by knowledge functions K_1 and K_2 are

$$\mathcal{P}_1 = \{\{1, 2\}, \{3, 4, 5\}, \{6\}\}$$
$$\mathcal{P}_2 = \{\{1\}, \{2, 3, 4\}, \{5\}, \{6\}\}$$

- In what states is $E_1 = \{3, 4, 5, 6\}$ common knowledge?
- In what states is $E_2 = \{1, 2, 3, 4\}$ common knowledge?

Properties of Common Knowledge Events

Lemma

Let P_1 and P_2 be the partitional information functions of individuals 1 and 2 for the set Ω of states, let K_1 and K_2 be the associated knowledge functions, and let E be an event. Then the following three conditions are equivalent.

1. $K_i(E) = E$ *for* $i = 1, 2$.
2. *E is self-evident between 1 and 2.*
3. *E is a union of members of the partition induced by P_i for $i = 1, 2$.*

- (3) directly implies (1).
- (1) implies (2) since given (1) for every $\omega \in E$ we have $P_i(\omega) \subset E$ for $i = 1, 2$.
- (2) implies (3) since given (2) $E = \cup_{\omega \in E} P_i(\omega)$.

Common Knowledge Definition Equivalence

Proposition

Let Ω be a finite set of states, let P_1 and P_2 be the partitional information functions of individuals 1 and 2, and let K_1 and K_2 be the associated knowledge functions. Then an event $E \subset \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ according to Definition 1 if and only if it is common knowledge between 1 and 2 in the state ω according to definition 2.

- ($\Rightarrow$) E is common knowledge between 1 and 2 in ω according to definition 1.
  - $\exists F = K_i(K_j(K_i \cdots K_i(F) \cdots))$ for which $\omega \in F$.
  - Since Ω is finite we must have $F = K_j(F) = K_i(K_j(F)) = K_i(F)$.
  - F is self-evident so E is common knowledge.

Common Knowledge Definition Equivalence

Proposition

Let Ω be a finite set of states, let P_1 and P_2 be the partitional information functions of individuals 1 and 2, and let K_1 and K_2 be the associated knowledge functions. Then an event $E \subset \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ according to Definition 1 if and only if it is common knowledge between 1 and 2 in the state ω according to Definition 2.

- ($\Leftarrow$) E is common knowledge between 1 and 2 in ω according to definition 2.
  - $\exists$ a self-evident event F such that $\omega \in F \subset E$.
  - From previous lemma $F = K_i(F) = K_j(F)$ so $F = K_i(K_j(K_i \cdots K_i(F) \cdots))$.
  - By (K2) $\omega \in F \subset K_i(K_j(K_i \cdots K_i(E) \cdots))$ so E is common knowledge.

Can People Agree to Disagree?

- Can it be common knowledge that two individuals with the same prior belief assign different probabilities to the same event.
- If we want to model a situation where people have different posterior beliefs what assumptions do we need to make about prior beliefs.
- ρ is the common prior probability distribution on the set Ω of states.
- $\rho(E|P_i(\omega))$ is the probability player i attaches to event E in state ω.
- The event "individual i assigns probability η_i to the event E" is

$$\{\omega \in \Omega : \rho(E|P_i(\omega)) = \eta_i\}$$

People With Common Priors Cannot Agree to Disagree

Proposition

Suppose that the set Ω of states is finite and individuals 1 and 2 have the same prior belief. If each individual's information function is partitional and it is common knowledge between 1 and 2 in some state $\omega^* \in \Omega$ that individual 1 assigns probability η_1 to event E and individual 2 assigns probability η_2 to E then $\eta_1 = \eta_2$.

- There is a self-evident event $F \ni \omega^*$ and

$$F \subset \{\omega \in \Omega : \rho(E|P_1(\omega)) = \eta_1\} \cap \{\omega \in \Omega : \rho(E|P_2(\omega)) = \eta_2\}$$

- For each individual $F = \cup_{\omega \in F} P_1(\omega) = \cup_{\omega \in F} P_2(\omega)$ which is finite. (Why?)
- For each $P_i(\omega) \subset F, \rho(E|P_i(\omega)) = \eta_i$.
- Since the information function is partitional, $\rho(E|F) = \eta_i$.

Knowledge and Solution Concepts

- We can now formally model the assumptions implied by different solution concepts applied to $\{N, (A_i), (u_i)\}$.
- Ω is the set of states, each of which describes the environment relevant to the game. Formally, each $\omega \in \Omega$ consists of
  - $P_i(\omega)$ where P_i is partitional.
  - $a_i(\omega) \in A_i$
  - $\mu_i(\omega)$, a probability measure on $A_{-i} = \times_{j \in N \setminus \{i\}} A_j$
- We assume G is common knowledge.

Knowledge and Nash Equilibrium

Proposition

Suppose that in the state $\omega \in \Omega$ each player $i \in N$

1. *knows the other players' actions:* $P_i(\omega) \subset \{\omega' \in \Omega : a_{-i}(\omega') = a_{-i}(\omega)\}$;
2. *has a belief that is consistent with his knowledge: the support of* $\mu_i(\omega)$ *is a subset of* $\{a_{-i}(\omega') \in A_{-i} : \omega' \in P_i(\omega)\}$;
3. *is rational:* $a_i(\omega)$ *is a best response of player* i *to* $\mu_i(\omega)$.

Then $(a_i(\omega))_{i \in N}$ *is a Nash equilibrium of* G.

- From 3, player i's action is a best response to his belief.
- From 2, player i's belief puts probability one on action profile a_i of the other players.
- From 1, player i's belief is correct.

Knowledge and Mixed Strategy Nash Equilibrium

Proposition

Suppose that $|N| = 2$ *and that in the state* $\omega \in \Omega$ *each player* $i \in N$

1. *knows the other player's beliefs:* $P_i(\omega) \subset \{\omega' \in \Omega : \mu_j(\omega') = \mu_j(\omega)\}$ *for* $j \neq i$*;*
2. *has a belief that is consistent with his knowledge: the support of* $\mu_i(\omega)$ *is a subset of* $\{a_{-i}(\omega') \in A_{-i} : \omega' \in P_i(\omega)\}$*;*
3. *knows that the other is rational: for any* $\omega' \in P_i(\omega)$ *the action* $a_j(\omega')$ *is a best response of player* j *to* $\mu_j(\omega)$*.*

Then the mixed strategy profile $(\alpha_1, \alpha_2) = (\mu_2(\omega), \mu_1(\omega))$ *is a mixed strategy Nash equilibrium of* G*.*

Knowledge and Nash Equilibrium

Choose an action a_i^* in the support of $\mu_j(\omega)$.

- From 2, there is some state ω' consistent with player j's information in which player i plays a_i^*.
- From 3, a_i^* is a best response for player i to his belief in state ω'.
- From 1, player i's belief in the state ω' equals his belief in state ω.

Requirement of 2 Players

	L	R
U	$2, 3, 0$	$2, 0, 0$
D	$0, 3, 0$	$0, 0, 0$

A

	L	R
U	$0, 0, 0$	$0, 2, 0$
D	$3, 0, 0$	$3, 2, 0$

B

State	α	β	γ	δ	ϵ	η
Probability$\times$63	32	16	8	4	2	1
1's action	U	D	D	D	D	D
2's action	L	L	L	L	L	L
3's action	A	B	A	B	A	B
1's partition	$\{\alpha\}$	$\{\beta$	$\gamma\}$	$\{\delta$	$\epsilon\}$	$\{\eta\}$
2's partition	$\{\alpha$	$\beta\}$	$\{\gamma$	$\delta\}$	$\{\epsilon$	$\eta\}$
3's partition	$\{\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{\delta\}$	$\{\epsilon\}$	$\{\eta\}$

Knowledge and Mixed Strategy Nash Equilibrium

Proposition

If all players have a common prior and in some state

- *rationality is mutual knowledge and*
- *the player's beliefs are common knowledge*

then the beliefs in that state form a mixed strategy Nash equilibrium

See Aumann and Brandenburger (1995).

Knowledge and Rationalizability

Proposition

Suppose that $|N| = 2$ and that in the state $\omega \in \Omega$ it is common knowledge between the players that each player's belief is consistent with his knowledge and that each player is rational. That is, suppose that there is a self-evident event $F \ni \omega$ such that for every $\omega' \in F$ and each $i \in N$ Suppose that there is a self-evident event F such that $\omega \in F$ such that for every $\omega' \in F$ and each $i \in N$

1. *the support of $\mu_i(\omega')$ is a subset of $\{a_j(\omega'') \in A_j : \omega'' \in P_i(\omega')\}$ for $j \neq i$;*
2. *the action $a_i(\omega')$ is a best response of player i to $\mu_i(\omega')$.*

Then for each $i \in N$ the action $a_i(\omega)$ is rationalizable in G.

- Let $Z_i = \{a_i(\omega') \in A_i : \omega' \in F\}$.
- By 2 for any $\omega \in F, a_i(\omega')$ is a best response to $\mu_i(\omega')$.
- By 1 the support of $\mu_i(\omega')$ is a subset of $\{a_j(\omega'') \in A_j : \omega'' \in P(\omega')\}$.
- Since F is self-evident we have $P_i(\omega') \subset F$ so that

E-mail Game - No Communication

	A	B
A	M, M	$1, -L$
B	$-L, 1$	$0, 0$

G_a (probability $1 - p$)

	A	B
A	$0, 0$	$1, -L$
B	$-L, 1$	M, M

G_b (probability p)

$$L > M > 1; p < 1/2$$

- $N = \{1, 2\}, \Omega = \{(a, b\}, A_i = \{A, B\}$ for each $i \in N$.
- $p_1(b) = p_2(b) = 1 - p_1(a) = 1 - p_2(a) = p$.
- $T_1 = \{a, b\}$, $T_2 = \{ab\}$.
- $\tau_1(a) = a, \tau_1(b) = b, \tau_2(a) = \tau_2(b) = ab$.

- What are the Nash Equilibria?

- (A, A) in each state.

E-mail Game - Perfect Communication

	A	B
A	M, M	$1, -L$
B	$-L, 1$	$0, 0$

G_a (probability $1 - p$)

	A	B
A	$0, 0$	$1, -L$
B	$-L, 1$	M, M

G_b (probability p)

$$L > M > 1; p < 1/2$$

- $N = \{1, 2\}, \Omega = \{a, b\}, A_i = \{A, B\}$ for each $i \in N$
- $p_1(b) = p_2(b) = 1 - p_1(a) = 1 - p_2(a) = p$.
- $T_1 = T_2 = \{a, b\}$.
- $\tau_1(a) = \tau_2(a) = a, \tau_1(b) = \tau_2(b) = b$.

- What are the Nash Equilibria?

- (A, A) in state a, (B, B) in state b (Pareto optimal).
- (A, A) in state a, (A, A) in state b.
- (A, A) in state a, (α, α) in state b where $\alpha(A) = \frac{M-1}{L+M-1}$.

E-mail Game - Imperfect Communication

	A	B
A	M, M	$1, -L$
B	$-L, 1$	$0, 0$

G_a (probability $1 - p$)

	A	B
A	$0, 0$	$1, -L$
B	$-L, 1$	M, M

G_b (probability p)

$$L > M > 1; p < 1/2$$

- $N = \{1, 2\}, A_i = \{A, B\}, T_1 = T_2 = \{0, 1, 2, \ldots\}$,
- $\Omega = \{(Q_1, Q_2) : Q_1 = Q_2 \text{ or } Q_1 = Q_2 + 1\}$.
- For each i, $\tau_i(Q_1, Q_2) = Q_i$.
- If $\omega = (0, 0)$ then payoffs are derived from G_a, otherwise, payoffs are determined by G_b.

E-mail Game - Imperfect Communication Beliefs

Initial beliefs: $p_i(0,0) = 1 - p$.

q	0	1	2
$(q+1,q)$	$p\epsilon$	$p(1-\epsilon)^2\epsilon$	$p(1-\epsilon)^4\epsilon$
$(q+1,q+1)$	$p(1-\epsilon)\epsilon$	$p(1-\epsilon)^3\epsilon$	$p(1-\epsilon)^5\epsilon$

$p_i(q+1,q) = p\epsilon(1-\epsilon)^{2q}$ and $p_i(q+1,q+1) = p\epsilon(1-\epsilon)^{2q+1}$

Updated beliefs $\tilde{p}_i(\omega|\tau_i(\omega))$: $\tilde{p}_1((0,0)|0) = (1-p)/(1-p) = 1$

- $\tilde{p}_2((0,0)|0) = \frac{1-p}{1-p+p\epsilon} > \frac{1}{2}$ since $p < \frac{1}{2}$.
- $\tilde{p}_2((1,0)|0) = \frac{p\epsilon}{1-p+p\epsilon} < \frac{1}{2}$ since $p < \frac{1}{2}$.
- $\tilde{p}_1((q+1,q)|q+1) = \frac{p\epsilon(1-\epsilon)^{2q}}{p\epsilon(1-\epsilon)^{2q}+p\epsilon(1-\epsilon)^{2q+1}} = \frac{\epsilon}{\epsilon+\epsilon(1-\epsilon)} > \frac{1}{2}$.
- $\tilde{p}_1((q+1,q+1)|q+1) = \frac{p\epsilon(1-\epsilon)^{2q+1}}{p\epsilon(1-\epsilon)^{2q}+p\epsilon(1-\epsilon)^{2q+1}} = \frac{\epsilon(1-\epsilon)}{\epsilon+\epsilon(1-\epsilon)} < \frac{1}{2}$.
- $\tilde{p}_2((q,q)|q) = \frac{p\epsilon(1-\epsilon)^{2q-1}}{p\epsilon(1-\epsilon)^{2q-1}+p\epsilon(1-\epsilon)^{2q}} = \frac{\epsilon}{\epsilon+\epsilon(1-\epsilon)} > \frac{1}{2}$.
- $\tilde{p}_2((q+1,q)|q) = \frac{p\epsilon(1-\epsilon)^{2q}}{p\epsilon(1-\epsilon)^{2q-1}+p\epsilon(1-\epsilon)^{2q}} = \frac{\epsilon(1-\epsilon)}{\epsilon+\epsilon(1-\epsilon)} < \frac{1}{2}$.

E-mail Game - Imperfect Communication Equilibrium

Proposition

The electronic mail game has a unique Nash equilibrium, in which both players always choose A.

- Show when player 1 sees 0, player 1 should play A, i.e., $(Q_1, Q_2) = (0, 0)$.
- Show when player 2 sees 0, player 2 should play A, i.e., $(Q_1, Q_2) \in \{(0, 0), (1, 0)\}$.
- Assume that when player 2 sees q he plays A, i.e., $(Q_1, Q_2) \in \{(q, q), (q + 1, q)\}$, e.g. $\{(0, 0), (1, 0)\}$..
  - Show that when player 1 sees $q + 1$, she should play A, i.e., $(Q_1, Q_2) \in \{(q + 1, q), (q + 1, q + 1)\}$, e.g. $\{(1, 0), (1, 1)\}$.
- Assume that when player 1 sees $q + 1$ she plays A, i.e., $(Q_1, Q_2) \in \{(q + 1, q), (q + 1, q + 1)\}$, e.g. $\{(1, 0), (1, 1)\}$.
  - Show that when player 2 sees $q + 1$, he should play A, i.e., $(Q_1, Q_2) \in \{(q + 1, q + 1), (q + 2, q + 1)\}$, e.g. $\{(1, 1), (2, 1)\}$.