Advanced Macroeconomics II
Log-linearization and Matlab Introduction

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Loglinearization

Growth path: equilibrium level vs. stochastic trajectory

$$A_t^E = A_0 e^{\delta t}$$
$$A_t = A_0 e^{\delta t + \varepsilon_t}$$

$A_t - A_t^E$: the scale depends on equilibrium level.
Consider percentage change

$$\frac{A_t - A_t^E}{A_t^E} = \frac{A_0 e^{\delta t}(e^{\varepsilon_t} - 1)}{A_0 e^{\delta t}} = e^{\varepsilon_t} - 1$$

For $\varepsilon_t \longrightarrow 0$,

$$\begin{aligned} e^{\varepsilon_t} &= e^0 + e^0 \varepsilon_t + \frac{1}{2} e^0 \varepsilon_t^2 + \ldots \\ &\approx 1 + \varepsilon_t \end{aligned}$$

Hence, by first order approximation,

$$\frac{A_t - A_t^E}{A_t^E} = e^{\varepsilon_t} - 1 \approx \varepsilon_t$$

Loglinearization

Example: A Cob-Douglas production function

$$Y_t = A_t \bar{N}^{1-\alpha}$$

where $\bar{N}$ is per capita labor in hours, and assuming capital is constant and normalized to 1.

$$Y_t^E = A_t^E \bar{N}^{1-\alpha}$$

$$\begin{aligned} \frac{Y_t - Y_t^E}{Y_t^E} &= \frac{A_0 e^{\delta t}(e^{\varepsilon_t} - 1)\bar{N}^{1-\alpha}}{A_0 e^{\delta t}\bar{N}^{1-\alpha}} = e^{\varepsilon_t} - 1 \\ &\approx \varepsilon_t \end{aligned}$$

Loglinearization

Example: A Cob-Douglas production function

$$Y_t = A_t \bar{N}^{1-\alpha} K_t^{\alpha}$$

where $\bar{N}$ is per capita labor in hours, and capital is optimally chosen by firm at each period.

Problem

What's the relationship between Y_t and K_t in terms of percentage deviation?

Loglinearization

A simple procedure

Define

$$\hat{x}_t \equiv \log X_t - \log \bar{X}$$

$\bar{X}$: equilibrium, or steady state level of X_t.

$$\begin{aligned}\hat{x}_t &\equiv \log X_t / \bar{X} \\ &= \log(1 + \frac{X_t - \bar{X}}{\bar{X}})\end{aligned}$$

We know that

$$e^{\varepsilon_t} \approx 1 + \varepsilon_t.$$

Equivalently, by taking log,

$$\varepsilon_t \approx \log(1 + \varepsilon_t).$$

Loglinearization

A simple procedure

As a first order approximation when $\frac{X_t - \bar{X}}{\bar{X}}$ is small,

$$\hat{x}_t = \log(1 + \frac{X_t - \bar{X}}{\bar{X}}) \approx \frac{X_t - \bar{X}}{\bar{X}}$$

$\hat{x}_t \times 100\%$ is approximately the percentage change (deviation) with respect the steady state.

To review the definition of $\hat{x}_t$

$$\hat{x}_t = \log X_t / \bar{X}$$

In an alternative form

$$e^{\hat{x}_t} = X_t / \bar{X}$$

$$X_t = \bar{X} e^{\hat{x}_t}$$

Loglinearization

A simple procedure

The most important expression:

$$X_t = \bar{X} e^{\hat{x}_t}$$

Loglinearization based on it:

$$\begin{aligned} X_t &= \bar{X} e^{\hat{x}_t} \approx \bar{X}(1 + \hat{x}_t) \\ &= \bar{X} + \bar{X}\hat{x}_t \end{aligned}$$

X_t is decomposed into:
- the steady state $\bar{X}$
- the deviation term: $\bar{X}\hat{x}_t$ – percentage deviation times the scale of its steady state

Loglinearization

Example 1.

$$\begin{aligned} aX_t &= a\bar{X}e^{\hat{x}_t} \\ &\approx a\bar{X}(1+\hat{x}_t) \\ &= a\bar{X} + a\bar{X}\hat{x}_t \end{aligned}$$

Loglinearization

Example 2.

$$
\begin{aligned}
X_t Y_t &= \bar{X} e^{\hat{x}_t} \bar{Y} e^{\hat{y}_t} \\
&= \bar{X} \bar{Y} e^{\hat{x}_t + \hat{y}_t} \\
&\approx \bar{X} \bar{Y} (1 + \hat{x}_t + \hat{y}_t) \\
&= \bar{X} \bar{Y} + \bar{X} \bar{Y} \hat{x}_t + \bar{X} \bar{Y} \hat{y}_t
\end{aligned}
$$

What if you did not combine $e^{\hat{x}_t}$ and $e^{\hat{y}_t}$ first?

Loglinearization

Example 3. Loglinearizing an equation

$$X_t Y_t = C$$

For any t, the steady state

$$\bar{X}\bar{Y} = C$$

Loglinearize using the previous result:

$$\bar{X}\bar{Y} + \bar{X}\bar{Y}\hat{x}_t + \bar{X}\bar{Y}\hat{y}_t = C$$

Delete the steay state relationship:

$$\begin{aligned} \bar{X}\bar{Y}\hat{x}_t + \bar{X}\bar{Y}\hat{y}_t &= 0 \\ \hat{x}_t + \hat{y}_t &= 0 \end{aligned}$$

A nonlinear relationship becomes linear in its first order approximation!

Loglinearization

Example 4. Loglinearizing an equation

$$\frac{X_t}{Y_t} = C$$

For any t, the steady state

$$\frac{\bar{X}}{\bar{Y}} = C$$

Loglinearize:

$$\begin{aligned}
\frac{\bar{X}e^{\hat{x}_t}}{\bar{Y}e^{\hat{y}_t}} &= C \\
\frac{\bar{X}}{\bar{Y}}e^{\hat{x}_t - \hat{y}_t} &= C \\
\frac{\bar{X}}{\bar{Y}}(1 + \hat{x}_t - \hat{y}_t) &\approx C
\end{aligned}$$

Delete the steay state relationship:

$$\hat{x}_t - \hat{y}_t = 0$$

What if you did not combine $e^{\hat{x}_t}$ and $e^{\hat{y}_t}$ first?

Loglinearization

Example 5. Loglinearizing an equation

$$X_t + Y_t = C$$

For any t, the steady state

$$\bar{X} + \bar{Y} = C$$

Loglinearize:

$$\begin{aligned}
\bar{X}e^{\hat{x}_t} + \bar{Y}e^{\hat{y}_t} &= C \\
\bar{X}(1+\hat{x}_t) + \bar{Y}(1+\hat{y}_t) &\approx C \\
\bar{X} + \bar{X}\hat{x}_t + \bar{Y} + \bar{Y}\hat{y}_t &= C
\end{aligned}$$

Delete the steay state relationship:

$$\bar{X}\hat{x}_t + \bar{Y}\hat{y}_t = 0$$

Loglinearization

Example 6. A simple budget constraint

$$C_t + K_t = Y_t + (1-\delta)K_{t-1}$$

For any t, the steady state

$$\bar{C} + \bar{K} = \bar{Y} + (1-\delta)\bar{K}$$

Loglinearize:

$$\begin{aligned} \bar{C}e^{\hat{c}_t} + \bar{K}e^{\hat{k}_t} &= \bar{Y}e^{\hat{y}_t} + (1-\delta)\bar{K}e^{\hat{k}_{t-1}} \\ \bar{C}(1+\hat{c}_t) + \bar{K}(1+\hat{k}_t) &\approx \bar{Y}(1+\hat{y}_t) + (1-\delta)\bar{K}(1+\hat{k}_{t-1}) \end{aligned}$$

Delete the steay state relationship:

$$\bar{C}\hat{c}_t + \bar{K}\hat{k}_t \approx \bar{Y}\hat{y}_t + (1-\delta)\bar{K}\hat{k}_{t-1}$$

Loglinearization

Example 7. Utility maximization

If a consumer has a per period utility function as

$$U_t = \frac{C_t^{1-\eta}}{1-\eta},$$

then the F.O.C. of utility maximization leads to

$$C_t^{-\eta} = \lambda_t$$

where λ_t is the Lagrangian multiplier.

Loglinearization

Example 7. Utility maximization

The steady state

$$\bar{C}^{-\eta} = \bar{\lambda}$$

Loglinearize:

$$\begin{aligned} (\bar{C}e^{\hat{c}_t})^{-\eta} &= \bar{\lambda}e^{\hat{\lambda}_t} \\ \bar{C}^{-\eta}e^{-\eta\hat{c}_t} &= \bar{\lambda}e^{\hat{\lambda}_t} \\ \bar{C}^{-\eta}(1-\eta\hat{c}_t) &\approx \bar{\lambda}(1+\hat{\lambda}_t) \end{aligned}$$

Delete the steay state relationship:

$$-\eta\hat{c}_t = \hat{\lambda}_t$$

Matlab Introduction