Exercise 3.5 (Black-Scholes-Merton formula). Let the interest rate r and the volatility $\sigma > 0$ be constant. Let

$$S(t) = S(0)\, e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

be a geometric Brownian motion with mean rate of return r, where the initial stock price $S(0)$ is positive. Let K be a positive constant. Show that, for $T > 0$,

$$\mathbb{E}\left[e^{-rT}\left(S(T) - K\right)^+\right] = S(0)\, N\left(d_+\left(T, S(0)\right)\right) - Ke^{-rT} N\left(d_-\left(T, S(0)\right)\right),$$

where

$$d_\pm\left(T, S(0)\right) = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T\right],$$

and N is the cumulative standard normal distribution function

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{y} e^{-\frac{1}{2}z^2}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{1}{2}z^2}\,dz.$$

Solution:

$$\begin{aligned}
\mathbb{E}\left[e^{-rT}\left(S(T) - K\right)^+\right] &= \mathbb{E}\left[e^{-rT}\left(S(0)\, e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W(T)} - K\right)^+\right] \\
&= \mathbb{E}\left[e^{-rT}\left(\exp\left\{\ln S(0) + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma W(T)\right\} - K\right)^+\right] \\
&= \int_{-\frac{\ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{-rT}\left(\exp\left\{\ln S(0) + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z\right\} - K\right) N'(z)\,dz \\
&= \exp\left\{\ln S(0) - \frac{1}{2}\sigma^2 T\right\}\int_{-\frac{\ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}^{\infty} e^{-\frac{1}{2}z^2 + \sigma\sqrt{T}z}dz - Ke^{-rT}\int_{-d_-(T,S(0))}^{\infty} N'(z)\,dz \\
&= S(0)\int_{-\frac{\ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}}^{\infty} N'(z)\,dz - Ke^{-rT}\int_{-d_-(T,S(0))}^{\infty} N'(z)\,dz \\
&= S(0)\int_{-d_+(T,S(0))}^{\infty} N'(z)\,dz - Ke^{-rT}\int_{-d_-(T,S(0))}^{\infty} N'(z)\,dz \\
&= S(0)\, N\left(d_+\left(T, S(0)\right)\right) - Ke^{-rT} N\left(d_-\left(T, S(0)\right)\right).
\end{aligned}$$

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Exercise 3.8. This problem presents the convergence of the distribution of stock prices in a sequence of binomial models to the distribution of geometric Brownian motion. In contrast to the analysis of Subsection 3.2.7, here we allow the interest rate to be different from zero.

Let $\sigma > 0$ and $r \geq 0$ be given. For each positive integer n, we consider a binomial model taking n steps per unit time. In this model, the interest rate per period is $\frac{r}{n}$, the up factor is $u_n = e^{\sigma/\sqrt{n}}$, and the down factor is $d_n = e^{-\sigma/\sqrt{n}}$. The risk-neutral probabilities are then

$$\tilde{p}_n = \frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}, \tilde{q}_n = \frac{e^{\sigma/\sqrt{n}} - \frac{r}{n} - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}.$$

Let t be an arbitrary positive rational number, and for each positive integer n for which nt is an integer, define

$$M_{nt,n} = \sum_{k=1}^{nt} X_{k,n},$$

where $X_{1,n}, \ldots, X_{n,n}$ are independent, identically distributed random variables with

$$\tilde{\mathbb{P}}\{X_{k,n} = 1\} = \tilde{p}_n, \tilde{\mathbb{P}}\{X_{k,n} = -1\} = \tilde{q}_n, k = 1, \ldots, n.$$

The stock price at time t in this binomial model, which is the result of nt steps from the initial time, is given by (see (3.2.15) for a similar equation)

$$\begin{aligned} S_n(t) &= S(0)\, u_n^{\frac{1}{2}(nt+M_{nt,n})} d_n^{\frac{1}{2}(nt-M_{nt,n})} \\ &= S(0) \exp\left\{\frac{\sigma}{2\sqrt{n}}(nt + M_{nt,n})\right\} \exp\left\{-\frac{\sigma}{2\sqrt{n}}(nt - M_{nt,n})\right\} \\ &= S(0) \exp\left\{\frac{\sigma}{\sqrt{n}} M_{nt,n}\right\}. \end{aligned}$$

This problem shows that as $n \to \infty$, the distribution of the sequence of random variables $\frac{\sigma}{\sqrt{n}} M_{nt,n}$ appearing in the exponent above converges to the normal distribution with mean $\left(r - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$. Therefore, the limiting distribution of $S_n(t)$ is the same as the distribution of the geometric Brownian motion $S(0) \exp\left\{\sigma W(t) + \left(r - \frac{1}{2}\sigma^2\right) t\right\}$ at time t.

(i) Show that the moment-generating function $\varphi_n(u)$ of $\frac{1}{\sqrt{n}} M_{nt,n}$ is given by

$$\varphi_n(u) = \left[e^{\frac{u}{\sqrt{n}}} \left(\frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right) - e^{-\frac{u}{\sqrt{n}}} \left(\frac{e^{\sigma/\sqrt{n}} - \frac{r}{n} - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right)\right]^{nt}.$$

(ii) We want to compute

$$\lim_{n\to\infty} \varphi_n(u) = \lim_{x\downarrow 0} \varphi_{\frac{1}{x^2}}(u),$$

where we have made the change of variable $x = \frac{1}{\sqrt{n}}$. To do this, we will compute $\ln \varphi_{\frac{1}{x^2}}(u)$ and then take the limit as $x \downarrow 0$. Show that

$$\ln \varphi_{\frac{1}{x^2}}(u) = \frac{t}{x^2} \ln\left[\frac{\left(rx^2 + 1\right)\sinh ux + \sinh(\sigma - u)x}{\sinh \sigma x}\right]$$

(the definitions are $\sinh z = \frac{e^z - e^{-z}}{2}$, $\cosh z = \frac{e^z + e^{-z}}{2}$), and use the formula

$$\sinh(A - B) = \sinh A \cosh B - \cosh A \sinh B$$

to rewrite this as

$$\ln \varphi_{\frac{1}{x^2}}(u) = \frac{t}{x^2} \ln\left[\cosh ux + \frac{\left(rx^2 + 1 - \cosh \sigma x\right)\sinh ux}{\sinh \sigma x}\right].$$

(iii) Use the Taylor series expansions

$$\cosh z = 1 + \frac{1}{2}z^2 + O\left(z^4\right), \sinh z = z + O\left(z^3\right),$$

to show that

$$\cosh ux + \frac{\left(rx^2 + 1 - \cosh \sigma x\right) \sinh ux}{\sinh \sigma x} = 1 + \frac{1}{2}u^2x^2 + \frac{rux^2}{\sigma} - \frac{1}{2}ux^2\sigma + O\left(x^4\right). \tag{1}$$

The notation $O\left(x^j\right)$ is used to represent terms of the order x^j.

(iv) Use the Taylor series expansion $\ln(1+x) = x + O\left(x^2\right)$ to compute $\lim_{x\downarrow 0} \varphi_{\frac{1}{x^2}}(u)$. Now explain how you know that the limiting distribution for $\frac{\sigma}{\sqrt{n}} M_{nt,n}$ is normal with mean $\left(r - \frac{1}{2}\sigma^2\right)t$ and variance $\sigma^2 t$.

Solution: (i)

$$\begin{aligned}
\varphi_n(u) &= \mathbb{E}\left[\exp\left\{\frac{u}{\sqrt{n}} M_{nt,n}\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{\frac{u}{\sqrt{n}} \sum_{k=1}^{nt} X_{k,n}\right\}\right] \\
&= \prod_{k=1}^{nt} \mathbb{E}\left[\exp\left\{\frac{u}{\sqrt{n}} X_{k,n}\right\}\right] \\
&= \prod_{k=1}^{nt} \left(\exp\left\{\frac{u}{\sqrt{n}}\right\}\tilde{p}_n + \exp\left\{-\frac{u}{\sqrt{n}}\right\}\tilde{q}_n\right) \\
&= \left[e^{\frac{u}{\sqrt{n}}}\left(\frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right) + e^{-\frac{u}{\sqrt{n}}}\left(\frac{e^{\sigma/\sqrt{n}} - \frac{r}{n} - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right)\right]^{nt}
\end{aligned}$$

(ii)

$$\begin{aligned}
\lim_{n\to\infty} \varphi_n(u) &= \lim_{n\to\infty}\left[e^{\frac{u}{\sqrt{n}}}\left(\frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right) + e^{-\frac{u}{\sqrt{n}}}\left(\frac{e^{\sigma/\sqrt{n}} - \frac{r}{n} - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right)\right]^{nt} \\
&= \lim_{x\downarrow 0} \exp\left\{\frac{\ln\left[e^{ux}\left(\frac{rx^2+1-e^{-\sigma x}}{e^{\sigma x}-e^{-\sigma x}}\right) + e^{-ux}\left(\frac{e^{\sigma x}-rx^2-1}{e^{\sigma x}-e^{-\sigma x}}\right)\right]}{\frac{x^2}{t}}\right\} \\
&= \lim_{x\downarrow 0} \exp\left\{\frac{t}{x^2}\ln\left[e^{ux}\left(\frac{rx^2 + 1 - e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}}\right) + e^{-ux}\left(\frac{e^{\sigma x} - rx^2 - 1}{e^{\sigma x} - e^{-\sigma x}}\right)\right]\right\} \\
&= \lim_{x\downarrow 0} \exp\left\{\frac{t}{x^2}\ln\left[\frac{\left(rx^2+1\right)\left(e^{ux} - e^{-ux}\right) + \left(e^{-(u-\sigma)x} - e^{(u-\sigma)x}\right)}{e^{\sigma x} - e^{-\sigma x}}\right]\right\} \\
&= \lim_{x\downarrow 0} \exp\left\{\frac{t}{x^2}\ln\left[\frac{\left(rx^2+1\right)\sinh ux + \sinh(\sigma - u)x}{\sinh \sigma x}\right]\right\} \\
&= \lim_{x\downarrow 0} \exp\left\{\frac{t}{x^2}\ln\left[\frac{\left(rx^2+1\right)\sinh ux + \sinh \sigma x \cosh ux - \cosh \sigma x \sinh ux}{\sinh \sigma x}\right]\right\} \\
&= \lim_{x\downarrow 0} \exp\left\{\frac{t}{x^2}\ln\left[\cosh ux + \frac{\left(rx^2 + 1 - \cosh \sigma x\right)\sinh ux}{\sinh \sigma x}\right]\right\}
\end{aligned}$$

(iii)

$$\begin{aligned}
\lim_{n\to\infty}\varphi_n(u) &= \lim_{x\downarrow 0}\exp\left\{\frac{t}{x^2}\ln\left[\cosh ux+\frac{\left(rx^2+1-\cosh\sigma x\right)\sinh ux}{\sinh\sigma x}\right]\right\}\\
&= \lim_{x\downarrow 0}\exp\left\{\frac{t}{x^2}\ln\left[1+\frac{1}{2}u^2x^2+O\left(x^4\right)+\frac{\left[rx^2+1-\left(1+\frac{1}{2}\sigma^2x^2+O\left(x^4\right)\right)\right]\left(ux+O\left(x^3\right)\right)}{\sigma x+O\left(x^3\right)}\right]\right\}\\
&= \lim_{x\downarrow 0}\exp\left\{\frac{t}{x^2}\ln\left[1+\frac{1}{2}u^2x^2+O\left(x^4\right)+\frac{\left(r-\frac{1}{2}\sigma^2\right)ux^2}{\sigma}\right]\right\}\\
&= \lim_{x\downarrow 0}\exp\left\{\frac{t}{x^2}\ln\left[1+\frac{1}{2}u^2x^2+\frac{rux^2}{\sigma}-\frac{1}{2}ux^2\sigma+O\left(x^4\right)\right]\right\}
\end{aligned}$$

(iv)

$$\begin{aligned}
\lim_{n\to\infty}\varphi_n(u) &= \lim_{x\downarrow 0}\exp\left\{\frac{t}{x^2}\ln\left[1+\frac{1}{2}u^2x^2+\frac{rux^2}{\sigma}-\frac{1}{2}ux^2\sigma+O\left(x^4\right)\right]\right\}\\
&= \exp\left\{\lim_{x\downarrow 0}\frac{t}{x^2}\ln\left[1+\frac{1}{2}u^2x^2+\frac{rux^2}{\sigma}-\frac{1}{2}ux^2\sigma+O\left(x^4\right)\right]\right\}\\
&= \exp\left\{\lim_{x\downarrow 0}\frac{t}{x^2}\left[\left(\frac{1}{2}u^2+\frac{ru}{\sigma}-\frac{1}{2}u\sigma\right)x^2+O\left(x^4\right)\right]\right\}\\
&= \exp\left\{\frac{1}{2}u^2t-\frac{u}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)t\right\}
\end{aligned}$$

$$\frac{\sigma}{\sqrt{n}}M_{nt,n}\sim\mathcal{N}\left(\left(r-\frac{1}{2}\sigma^2\right)t,\sigma^2t\right)$$

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