Advanced Microeconomics II

Static Games of Imperfect Information

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Games with Incomplete Information

Players don't always have full information about the other player's payoffs.

- Industrial Organization
  - Existing firm's may not know an new entrant's costs.
  - Existing firm's may have better information about market demand.
- Labour
  - Employers do not observe potential employee's ability perfectly.
- Auctions
  - Bidders don't know other bidders value.

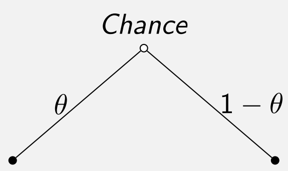
How to model?

Bayesian Games

Simplest version: Players choose actions simultaneously.

- Uncertainty is over other player's characteristics and his action.
- These are games of incomplete information.
- Translate into a game of imperfect information.
  - Introduce the "chance" player.
- Such games are known as *Bayesian* games.

Battle of the Sexes Example



H

W	M	F
M	$2,1$	$0,0$
F	$0,0$	$1,2$

H

W	M	F
M	$2,2$	$0,0$
F	$0,0$	$1,1$

- θ, the probability that W thinks H prefers F to M.
- $1-\theta$, the probability that W thinks H prefers M to F.

Chance

- Chance acts first.
- Chance draws a “type” for each player.
- A player’s type contains all that player’s private information.
- Draws come from a common probability distribution.
- Player’s update information about other’s type using Bayes rule.
  - If each player’s type is independent then this is irrelevant.

Bayesian Game

Definition

A Bayesian game consists of

- a finite set N (the set of players)
- a finite set Ω (the set of states)

and for each player $i \in N$

- a set A_i (the set of actions available to player i)
- a finite set T_i (the set of signals that may be observed by player i) and a function $\tau_i : \Omega \to T_i$ (the signal function of player i)
- a probability measure p_i on Ω (the prior belief of player i) for which $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
- a preference relation $\succeq_i$ on the set of probability measures over $A \times \Omega$ (the preference relation of player i), where $A = \times_{j \in N} A_j$.

Normal Form Representation of a Static Bayesian Game

Definition

The normal-form representation of a Bayesian game is $\{N, (A_i)_{i=1}^n, (T_i)_{i=1}^n, (p_i)_{i=1}^n, (u_i)_{i=1}^n\}$. For each player $i \in N$

- A_i, actions of player i ($A = \times_{j \in N} A_j$);
- T_i, types of player i ($T = \times_{j \in N} T_j$);
- $p_i \in \Delta T$ for which $p_i(t) \geq 0$ for all $t \in T$, prior belief of player i;
- $u_i : A \times T \to R$, utility of player i over outcomes and types.

Standard assumptions

- Lotteries over $A \times T$ are evaluated using expected utility:
$$U_i(\alpha) = \sum_{(a,t) \in A \times T} \alpha(a,t) u_i(a,t) \text{ for each } \alpha \in \Delta(A \times T).$$
- Common prior over T; (Harsanyi doctrine)
$$p_i(t) = p(t) \text{ for each } i \in N.$$

Bayesian Cournot Game

- Firm 1 has cost $C_1(q_1) = c_1 q_1$.
- Firm 2’s cost is unknown by Firm 1. Firm 1 only knows the probability of Firm 2’s cost function.

$$C_2(q_2) = \begin{cases} c_H q_2, & \text{with probability } \theta, \\ c_L q_2, & \text{with probability } 1 - \theta. \end{cases}$$

- Market demand: $P = a - q_1 - q_2$
- Profit: $\pi_i = q_i(a - q_1 - q_2 - c_i), i = 1, 2$

Nash Equilibrium of a Bayesian Game

Definition

A strategy in a Bayesian game for player i is a function $S_i : T_i \to \Delta(A_i)$

Definition

A Nash equilibrium of a Bayesian game $\{N, (A_i)_{i=1}^n, (T_i)_{i=1}^n, p, (u_i)_{i=1}^n\}$ is a vector of strategies, $(s_1^*, \ldots, s_n^*)$, where $\forall i$ and $\forall t_i \in T_i$, $s_i^*(t_i)$ solves

$$\max_{s_i \in \Delta(A_i)} \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), s_i, s_{i+1}^*(t_{i+1}), \ldots, s_n^*(t_n), t)$$

where $p_i(t_{-i}|t_i) = \frac{p(t_i, t_{-i})}{p(t_i)} = \frac{p(t_i, t_{-i})}{\sum_{t_{-i} \in T_{-i}} p(t_i, t_{-i})}$

Bayesian Cournot Game Example

$$c_2 : \max_{q_2^j} (P(q_1, q_2) - c_j) q_2^j \Rightarrow q_2^{j*} = \frac{a - q_1^* - c_j}{2}; j \in \{L, H\}.$$

$$c_1 : \max_{q_1} \theta(a - q_2^H - q_1 - c_1) q_1 + (1 - \theta)(a - q_2^L - q_1 - c_1) q_1$$

$$\Rightarrow q_1^* = \frac{a - \theta q_2^{H*} - (1 - \theta) q_2^{L*} - c_1}{2}.$$

Thus

$$q_2^{H*} = \frac{a - 2c_H + c_1}{3} + \frac{(1 - \theta)(c_H - c_L)}{6},$$

$$q_2^{L*} = \frac{a - 2c_L + c_1}{3} - \frac{\theta(c_H - c_L)}{6},$$

$$q_1^* = \frac{a - 2c_1 + \theta c_H + (1 - \theta) c_L}{3}.$$

With complete information the NE is $(q_1, q_2) = \left(\frac{a + c_2 - 2c_1}{3}, \frac{a + c_1 - 2c_2}{3}\right)$.

Bayesian Battle of the Sexes

W \ H	M	F
M	$2 + t_W, 1$	$0, 0$
F	$0, 0$	$1, 2 + t_H$

- $t_W \sim U[0, x]$
- $t_H \sim U[0, x]$
- $t_W \perp t_H$

Look for a BNE where W plays M iff $t_W \geq w$, P_2 plays F iff $t_H \geq h$.

- For W, M is optimal if $\frac{h}{x}(2 + t_W) \geq \frac{x - h}{x} \times 1 \Rightarrow t_W \geq \frac{x}{h} - 3 = w$
- For H, F is optimal if $\frac{w}{x}(2 + t_H) \geq \frac{x - w}{x} \times 1 \Rightarrow t_H \geq \frac{x}{w} - 3 = h$
- Thus $w = h \Rightarrow w^2 + 3w - x = 0 \Rightarrow w = h = \frac{-3 + \sqrt{9 + 4x}}{2}$
- W plays M with probability $\frac{x - w}{x} = 1 - \frac{-3 + \sqrt{9 + 4x}}{2x}$,
- $\lim_{x \to 0} \frac{x - w}{x} = \frac{2}{3}$

Mixed Strategy Interpretation

- Let $G = \{N, (A_i), (u_i)\}$ be a finite strategic game.
- For each $i \in N$ and $a \in A$ let $\epsilon_i(a)$ be a random variable with range $[-1, 1]$ where $\epsilon_i = (\epsilon_i(a))_{a \in A}$ has a continuously differentiable density function and an absolutely continuous distribution function.
  - Denote f_i as the distribution of ϵ_i.
  - Denote $\epsilon = (\epsilon_i)_{i \in N}$
- Let $G(\epsilon) = \{N, (A_i)_{i=1}^n, (T_i)_{i=1}^n, p, (u_i)_{i=1}^n\}$ be the Bayesian game in which
  - $T_i = [-1, 1]^{|A|}$
  - $p(t) = \times_{i \in N} f_i(t_i)$ ($(\epsilon_i)_{i \in N}$ are independent)
  - $u_i(a, \epsilon) = u_i(a) + \epsilon_i(a)$

Purification

Proposition (Harsanyi, 1973, Theorems 2 and 7)

For almost any game G and any collection ϵ of random variables satisfying the conditions above, almost any mixed strategy Nash equilibrium of G is the mixed strategy profile associated with the limit, as the size γ of the perturbation vanishes, of a sequence of pure strategy equilibria of the Bayesian games $G(\gamma\epsilon)$ in each of which the action chosen by each type is strictly optimal.

Proposition (Harsanyi, 1973, Theorem 5)

The limit, as the size γ of the perturbation vanishes, of any convergent sequence of pure strategy equilibria of the Bayesian games $G(\gamma\epsilon)$ in each of which the action chosen by each type is strictly optimal is associated with a mixed strategy equilibrium of G.

First-Price Sealed-Bid Auction

- Two bidders.
- Bidder i has valuation v_i for the good, values are independent, $v \sim U[0,1]$.
- Each bidders set of actions is the set of possible bids (nonnegative numbers).
- The bidder whose bid is the highest gets the good. If there is a tie, the winner is decided by coin flip.
- Strategy is a function of value, $b_i(v_i)$.

First-Price Sealed-Bid Auction

$b_i(v_i)$ solves

$$\max_{b_i}(v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\text{Prob}\{b_i = b_j(v_j)\}$$

Let's look for a linear equilibrium.

- Assume $b_i(v_i) = a_i + c_i v_i$
- $\text{Prob}\{b_i > a_j + c_j v_j)\} = \text{Prob}\{v_j < \frac{b_i - a_j}{c_j}\} = \frac{b_i - a_j}{c_j}$
- Since $\text{Prob}\{b_i = b_j(v_j)\} = 0$, then $\max_{b_i}(v_i - b_i)\frac{b_i - a_j}{c_j}$
- F.O.C$\Rightarrow b_i = \frac{v_i + a_j}{2} \Rightarrow a_i = \frac{a_j}{2}, c_i = \frac{1}{2}$
- Similarly, we get $a_j = \frac{a_i}{2}$ and $c_j = \frac{1}{2}$
- Hence, $b_i(v_i) = \frac{v_i}{2}$

First-Price Sealed-Bid Auction Cont.

Proposition

If the players' strategies are symmetric, strictly increasing and differentiable, there exists a unique Bayesian Nash equilibrium.

- Players i and j adopt $b(\cdot)$, $b(\cdot)$ is strictly increasing and differentiable
- Given value v_i, player i's optimal bid b_i solves
$$\max_{b_i}(v_i - b_i)\text{Prob}\{b_i > b(v_j)\}$$
- Let $b^{-1}(\cdot)$ denote the inverse function of $b(\cdot)$, $\text{Prob}\{b_i > b(v_j)\} = \text{Prob}\{b^{-1}(b_i) > v_j\} = b^{-1}(b_i)$
$$\frac{\partial u_i(b_i, v_i)}{\partial b_i} = -b^{-1}(b_i) + (v_i - b_i)\frac{\partial}{\partial b_i}b^{-1}(b_i) = 0$$
- Equilibrium requires that $b_i = b(v_i)$:
$$-b^{-1}(b(v_i)) + (v_i - b(v_i))\frac{\partial}{\partial b_i}b^{-1}(b(v_i)) = 0$$

First-Price Sealed-Bid Auction Cont.

$$-b^{-1}(b(v_i)) + (v_i - b(v_i))\frac{\partial}{\partial b_i}b^{-1}(b(v_i)) = 0$$

$$-v_i + (v_i - b(v_i))\frac{1}{b'(v_i)} = 0 \Rightarrow b'(v_i)v_i + b(v_i) = v_i$$

- Integrating both sides of the equation, we get

$$b(v_i)v_i = \frac{1}{2}v_i^2 + k, \text{ where } k \text{ is a constant}$$

- No player bid more than her valuation, $b(v_i) \leq v_i$
- $b(0) \leq 0 \Rightarrow b(0) = 0 = \frac{1}{2}0^2 + k \Rightarrow k = 0$
- $b(v_i) = v_i/2$

A Double Auction

- One good is owned by the seller.
- The buyer's valuation for the good is v_b, the seller's is v_s. These valuations are private information and are drawn from independent uniform distribution on $[0, 1]$.
- The seller names an asking price p_s
- The buyer simultaneously names an offer price p_b
- If $p_b \geq p_s$, then trade occurs at price $p = \frac{p_b+p_s}{2}$; if $p_b < p_s$, then no trade occurs.
- If there is no trade, both players' utilities are 0; if the buyer gets the good for price p, the buyer's utility is $v_b - p$ and the seller's utility is $p - v_s$.

A Double Auction: Equilibrium Definition

A pair of strategies $\{p_b(v_b), p_s(v_s)\}$ is a Bayesian Nash equilibrium if:

- for each $v_b \in [0, 1]$, $p_b(v_b)$ solves

$$\max_{p_b}\left[v_b - \frac{p_b + \mathrm{E}[p_s(v_s)|p_b \geq p_s(v_s)]}{2}\right]\mathrm{Prob}\{p_b \geq p_s(v_s)\}. \quad (1)$$

- for each $v_s \in [0, 1]$, $p_s(v_s)$ solves

$$\max_{p_s}\left[\frac{p_s + \mathrm{E}[p_b(v_b)|p_b(v_b) \geq p_s]}{2} - v_s\right]\mathrm{Prob}\{p_b(v_b) \geq p_s\}. \quad (2)$$

Assume a linear Bayesian Nash equilibrium of the double auction,

$$\begin{aligned} p_s(v_s) &= a_s + c_s v_s \qquad & p_s(v_s) &\sim U[a_s, a_s + c_s] \\ p_b(v_b) &= a_b + c_b v_b \qquad & p_b(v_b) &\sim U[a_b, a_b + c_b] \end{aligned}$$

A Double Auction: Analysis

- Equation (26) becomes

$$\max_{p_b}\left[v_b - \frac{1}{2}\left(p_b + \frac{a_s + p_b}{2}\right)\right]\left(\frac{p_b - a_s}{c_s}\right).$$

- Equation (2) becomes

$$\max_{p_s}\left[\frac{1}{2}\left(p_s + \frac{p_s + a_b + c_b}{2}\right) - v_s\right]\left(\frac{a_b + c_b - p_s}{c_b}\right).$$

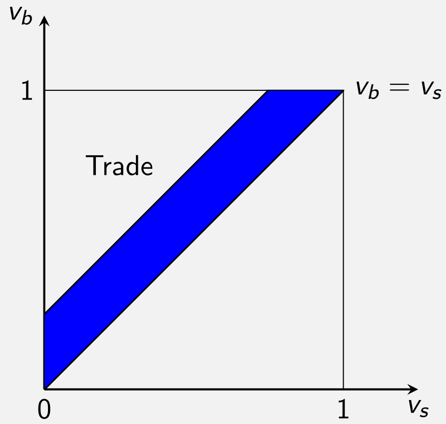
- First-order conditions:

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s; p_s = \frac{2}{3}v_s + \frac{1}{3}(a_b + c_b).$$

- Hence $p_b(v_b) = \frac{2}{3}v_b + \frac{1}{12}$, $p_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}$.

A Double Auction: Equilibrium

Trade occurs if and only if $p_b \geq p_s$. Thus, trade occurs in the linear equilibrium if and only if $v_b \geq v_s + \frac{1}{4}$. The equilibrium misses some valuable trades.



The Revelation Principle

- How can I maximize revenue ?
- What mechanism works best ?
  - Entry fee
  - Reserve price

Use the Revelation Principle to simplify this problem

- Bidders can restrict attention to the following class of games, direct mechanisms
  - The bidders simultaneously make claims (possibly dishonest) about their types (each player's only action).
  - Given the bidders' claims $(\tau_1, \dots, \tau_n)$, bidder i pays $x_i(\tau_1, \dots, \tau_n)$ and receives the good with probability $q_i(\tau_1, \dots, \tau_n) \geq 0$, where $\sum_{i=1}^n q_i(\tau_1, \dots, \tau_n) \leq 1$.

The Revelation Principle

- We can restrict attention to those direct mechanisms in which it is a Bayesian Nash equilibrium for each bidder to tell the truth.
- Find $\{x_1(\tau_1, \dots, \tau_n), \dots, x_n(\tau_1, \dots, \tau_n)\}$ and probability functions $\{q_1(\tau_1, \dots, \tau_n), \dots, q_n(\tau_1, \dots, \tau_n)\}$ for which $\tau_i(t_i) = t_i$ is an equilibrium strategy for each player.
- A direct mechanism in which truth-telling is a Bayesian Nash Equilibrium is called incentive-compatible.

The Revelation Principle

Proposition

The Revelation Principle For any Bayesian Nash equilibrium of any Bayesian game one can construct an incentive-compatible direct mechanism in which player's receive the same equilibrium payoffs.

- When bidders have independent, private values, Myerson (1981) determines which direct mechanisms have a truth-telling equilibrium, and which of these equilibria maximizes revenue.
- The Revelation Principle guarantees that no other mechanism has a Bayesian Nash equilibrium that generates higher revenue.
- Symmetric Bayesian Nash equilibrium we study is equivalent to this payoff-maximizing truth-telling equilibrium.

The Revelation Principle: Proof

- Consider a static Bayesian game $G = \{(A_i)_{i=1}^n, (T_i)_{i=1}^n, (p_i)_{i=1}^n, (u_i)_{i=1}^n\}$.
- Consider a Bayesian Nash equilibrium $s^* = (s_1^*, \dots, s_n^*)$ of this game.
- We will construct a direct mechanism with a truth-telling equilibrium that represents s^*.
- Redefine action spaces, $\tilde{A}_i = T_i$, and payoffs, $\tilde{u}_i(\tau, t) = u_i[s^*(\tau), t]$.
- If other players tell the truth, then player i chooses τ_i such that

$$\max_{\tau_i \in T_i} u_i[s_1^*(t_1), \dots, s_{i-1}^*(t_{i-1}), s_i^*(\tau_i), s_{i+1}^*(t_{i+1}), \dots, s_n^*(t_n), t].$$

- We know that $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} u_i[s_1^*(t_1), \dots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \dots, s_n^*(t_n), t].$$

- Hence, $\tau_i(t_i) = t_i$ (truth-telling) is an equilibrium.

A Double Auction: Revelation Principle

What is the equivalent incentive-compatible direct mechanism to the Double Auction equilibrium we found.

- Players announce types ($\tau_i \in [0, 1]$)

$$u_b(\tau, t) = \begin{cases} v_b - \frac{(\tau_b+\tau_s)}{3} - \frac{1}{6} & \text{if } \tau_b \geq \tau_s + 1/4 \\ 0 & \text{otherwise.} \end{cases}$$

$$u_s(\tau, t) = \begin{cases} \frac{(\tau_b+\tau_s)}{3} + \frac{1}{6} - v_s & \text{if } \tau_b \geq \tau_s + 1/4 \\ 0 & \text{otherwise.} \end{cases}$$

- Is truth-telling an equilibrium?
- For each $v_b \in [0, 1]$, v_b solves

$$\max_{\tau_b} \left[v_b - \frac{\tau_b + \mathrm{E}[v_s | \tau_b \geq v_s + 1/4]}{3} - \frac{1}{6} \right] \Pr\{\tau_b \geq v_s\}.$$

Double Auction Efficiency

- This equilibrium yields higher expected gains for the players than any other Bayesian equilibrium. (Myerson and Sattherthwaite 1983)
- The result is much more general:
  - Add individual rationality
  - Let $v_b \sim F_b[x_b, y_b]$, $v_s \sim F_s[x_s, y_s]$; F_b and F_s are continuous;
  - $y_b > x_s$ (some trades are efficient)
  - $y_s > x_b$ (some trades are inefficient)
  - There is no bargaining game that has a Bayesian Nash equilibrium in which trade occurs if and only if it is efficient.