Advanced Microeconomics II

Strictly Competitive Games

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March 3, 2015

Strictly Competitive Games

Definition

A strategic game $\{\{1,2\},(A_i),(\succeq_i)\}$ is strictly competitive if for any $a \in A$ and $b \in A$ we have $a \succeq_1 b$ if and only if $b \succeq_2 a$.

Example:

Player 1 \ Player 2	L	R
U	3,4	6,1
D	5,2	4,3

Maxminimizing Strategies

Definition

Let $\{\{1,2\},(A_i),(\succeq_i)\}$ be a strictly competitive strategic game. The action $x^* \in A_1$ is a maxminimizer for player 1 if

$$\min_{y\in A_2} u_1(x^*,y) \geq \min_{y\in A_2} u_1(x,y) \text{ for all } x \in A_1.$$

The action $y^* \in A_2$ is a maxminimizer for player 2 if

$$\min_{x\in A_1} u_2(x,y^*) \geq \min_{x\in A_1} u_2(x,y) \text{ for all } y \in A_2.$$

Example:

Player 1 \ Player 2	L	R
U	3,4	6,1
D	5,2	4,3

Zero-sum Games

Definition

A strategic game $\{\{1,2\},(A_i),(u_i)\}$ is zero-sum if for any $a \in A$ we have $u_2(a) = -u_1(a)$.

Example:

Player 1 \ Player 2	L	R
U	3,−3	6,−6
D	5,−5	4,−4

Zero-sum Games and Strictly Competitive Games

Lemma

Any strategic game can be translated into a zero-sum game that preserves player ordering over preference.

Example:

		Player 2 L	Player 2 R
Player 1	U	3,4	6,1
	D	5,2	4,3

$\Rightarrow$

		Player 2 L	Player 2 R
Player 1	U	3,−3	6,−6
	D	5,−5	4,−4

Zero-sum Games and Nash Equilibrium

Lemma

Let $\{\{1,2\},(A_i),(u_i)\}$ be a zero-sum strategic game. Then $\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$. Further, $y\in A_2$ solves the problem $\max_{y\in A_2}\min_{x\in A_1} u_2(x,y)$ if and only if it solves the problem $\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$.

- For any function f, $-\min_z f(z) = \max_z -f(z)$ and $\arg\min_z f(z) = \arg\max_z -f(z)$.
- $-\min_{x\in A_1} u_2(x,y) = \max_{x\in A_1} -u_2(x,y) = \max_{x\in A_1} u_1(x,y)$.
- $\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$.

Zero-sum Games and Nash Equilibrium

Proposition

Let $G=\{\{1,2\},(A_i),(u_i)\}$ be a zero-sum strategic game.

1. *If (x^*,y^*) is a Nash equilibrium of G then x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.*
2. *If (x^*,y^*) is a Nash equilibrium of G then $\max_x\min_y u_1(x,y) = \min_y\max_x u_1(x,y)$, and thus all Nash equilibria of G yield the same payoffs.*
3. *If $\max_x\min_y u_1(x,y) = \min_y\max_x u_1(x,y)$, x^* is a maxminimizer for player 1, and y^* is a maxminimizer for player 2, then (x^*,y^*) is a Nash equilibrium of G.*

Zero-sum Games and Nash Equilibrium

- (x^*,y^*) is a NE $\Rightarrow u_2(x^*,y^*)\geq u_2(x^*,y)$ for all $y\in A_2$
  $\Rightarrow u_1(x^*,y^*)\leq u_1(x^*,y)$ for all $y\in A_2$
  $\Rightarrow u_1(x^*,y^*) = \min_y u_1(x^*,y)\leq \max_x\min_y u_1(x,y)$.
- (x^*,y^*) is a NE $\Rightarrow u_1(x^*,y^*)\geq u_1(x,y^*)$ for all $x\in A_1$
  $\Rightarrow u_1(x^*,y^*)\geq \min_y u_1(x,y)$ for all $x\in A_1$
  $\Rightarrow u_1(x^*,y^*)\geq \max_x\min_y u_1(x,y)$.
- Repeat for player 2.
- What have we proven?

Zero-sum Games and Nash Equilibrium

- Let $v^* = \max_x \min_y u_1(x, y) = \min_y \max_x u_1(x, y)$.
- Since x^* is a maxminimizer for player 1, $u_1(x^*, y) \geq v^*$ for all $y \in A_2$.
  In particular, $u_1(x^*, y^*) \geq v^*$.
- From the lemma, $\max_y \min_x u_2(x, y) = -v^*$.
  y^* is a maxminimizer for player 2 $\Rightarrow u_2(x, y^*) \geq -v^*$ for all $x \in A_1$
  $\Rightarrow u_1(x, y^*) \leq v^*$ for all $x \in A_1$.
  In particular, $u_2(x^*, y^*) \geq -v^* \Rightarrow u_1(x^*, y^*) \leq v^*$.
- $v^* = u_1(x^*, y^*) \geq u_1(x, y^*)$ for all $x \in A_1$.
- Repeat for player 2.

Notes:

- A way to find Nash equilibria in strictly competitive games.
- Equilibria are interchangeable.
- $u_1(x^*, y^*)$ is the value of the game for player 1.
- $u_1(x^*, y^*)$ is the value of the game for player 1.