Homework 3

1. Suppose the following assumptions hold:
Assumption 1.1: $\{Y_t, X_t'\}'$ is an i.i.d. random sample with

$$Y_t = X'_t\beta^0 + \varepsilon_t,$$

for some unknown parameter β^0 and unobservable random disturbance ε_t.
Assumption 1.2: $E\left(\varepsilon_t \left| X_t\right.\right) = 0$ a.s.
Assumption 1.3:
(i) $W_t = W\left(X_t\right)$ is a positive function of X_t;
(ii) The $K \times K$ matrix $E\left(X_t W X'_t\right) = Q_w$ is finite and nonsingular.
(iii) $E\left(W_t^8\right) \leq C < \infty$, $E\left(X_{jt}^8\right) \leq C < \infty$ for all $0 \leq j \leq k$, and $E\left(\varepsilon_t^4\right) \leq C$;
Assumption 1.4: $V_w = E\left(X_t W_t^2 X'_t \varepsilon_t^2\right)$ is finite and nonsingular.

We consider the so-called weighted least squares (WLS) estimator for β^0:

$$\hat{\beta}_w = \left(n^{-1}\sum_{t=1}^{n} X_t W X'_t\right)^{-1} n^{-1}\sum_{t=1}^{n} X_t W Y_t.$$

(a) Show that $\hat{\beta}_w$ is the solution to the following problem

$$\min_{\beta} \sum_{t=1}^{n} W_t (Y_t - X'_t\beta)^2.$$

(b) Show that $\hat{\beta}_w$ is consistent for β^0;
(c) Show that $\sqrt{n}\left(\hat{\beta}_w - \beta^0\right) \xrightarrow{d} N\left(0, \Omega_w\right)$ for some $K \times K$ finite and positive definite matrix Ω_w. Obtain the expression of Ω_w under (i) conditional homoskedasticity $E\left(\varepsilon_t^2 \left| X_t\right.\right) = \sigma^2$ a.s. and (ii) conditional heteroskedasticity $E\left(\varepsilon_t^2 \left| X_t\right.\right) \neq \sigma^2$.
(d) Propose an estimator $\hat{\Omega}_w$ for Ω_w, and show that $\hat{\Omega}_w$ is consistent for Ω_w under conditional homoskedasticity and conditional heteroskedasticity respectively.
(e) Construct a test statistic for $H_0: \quad R\beta^0 = r$, where r is a $J \times K$ matrix and r is a $J \times 1$ vector under conditional homoskedasticity and under conditional heteroskedasticity respectively. Derive the asymptotic distribution of the test statistic under H_0 in each case.
(f) Suppose $E\left(\varepsilon_t^2 \left| X_t\right.\right) = \sigma^2\left(X_t\right)$ is known, and we set $W_t = \sigma^{-1}\left(X_t\right)$. Construct a test statistic for $H_0: \quad R\beta^0 = r$, where r is a $J \times K$ matrix and r is a $J \times 1$ vector. Derive the asymptotic distribution of the test statistic under H_0.

ANSWER:
(a)

$$\min_{\beta} \sum_{t=1}^{n} W_t (Y_t - X'_t\beta)^2$$

From

$$f.o.c: \quad \min_{\beta} \sum_{t=1}^{n} W_t(-2X'_t\beta Y_t + 2X_t X'_t\hat{\beta}) \quad = 0$$

we have

$$\hat{\beta}_w = \left(n^{-1} \sum_{t=1}^{n} X_t W_t X'_t \right)^{-1} n^{-1} \sum_{t=1}^{n} X_t W_t Y_t$$

(b)

$$\begin{aligned}
\hat{\beta}_w &= \left(n^{-1} \sum_{t=1}^{n} X_t W_t X'_t \right)^{-1} \left(n^{-1} \sum_{t=1}^{n} X_t W_t (X'_t \beta + \varepsilon_t) \right) \\
&= \beta^0 + \left(n^{-1} \sum_{t=1}^{n} X_t W_t X'_t \right)^{-1} \left(n^{-1} \sum_{t=1}^{n} X_t W_t \varepsilon_t \right)
\end{aligned}$$

$$E\left|X_{jt} W_t X'_{lt}\right| \leq \left[E\left(X_{jt}^2 W_t^2\right) E\left(X_{lt}^2\right)\right]^{1/2} \leq \left[E\left(X_{jt}^4\right)^{1/2} E\left(W_t^4\right)^{1/2} E\left(X_{lt}^2\right)\right]^{1/2}$$

From assumption

$$E\left(W_t^8\right) \leq C < \infty \quad E\left(X_{jt}^2\right) \quad \leq C < \infty$$

we have

$$E\left|X_{jt} W_t X'_{lt}\right| < \infty \quad 0 \leq j, l \leq k$$

By WLLN,

$$\frac{1}{n} \sum_{t=1}^{n} X_t W_t X'_t \xrightarrow{p} E\left(X_t W_t X'\right) = Q_w$$

and

$$\left(\frac{1}{n} \sum_{t=1}^{n} X_t W_t X'_t \right)^{-1} \xrightarrow{p} Q_w^{-1}$$

$$E\left|X_{jt} W_t \varepsilon_t\right| \leq \left[E\left(X_{jt}^2 W_t^2\right) E\left(\varepsilon_{lt}^2\right)\right]^{1/2} \leq \left[E\left(X_{jt}^4\right)^{1/2} E\left(W_t^4\right)^{1/2} E\left(\varepsilon_{lt}^2\right)\right]^{1/2}$$

From assumption

$$E\left(X_{jt}^2\right) < \infty \quad E\left(W_t^8\right) < \infty \quad E\left(\varepsilon_t^4\right) \quad < C$$

we have

$$E\left|X_{jt} W_t \varepsilon_t\right| < \infty$$

By WLLN,

$$\frac{1}{n} \sum_{t=1}^{n} X_t W_t \varepsilon_t \xrightarrow{p} E\left(X_t W_t \varepsilon_t\right) = 0$$

So,

$$\left(n^{-1} \sum_{t=1}^{n} X_t W_t X'_t \right)^{-1} \left(n^{-1} \sum_{t=1}^{n} X_t W_t \varepsilon_t \right) \xrightarrow{p} Q_w^{-1} \cdot 0 = 0$$

$$\hat{\beta}_w \xrightarrow{p} \beta^0$$

$\hat{\beta}_w$ is consistent for β^0.
(c)

$$\sqrt{n}(\hat{\beta}_w - \beta^0) = \left(\frac{1}{n}\sum_{t=1}^{n} X_t W_t X'_t\right)^{-1} \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n} X_t W_t \varepsilon_t\right)$$

Let $Z_t = X_t W_t \varepsilon_t$, $\bar{Z}_t = \frac{1}{n}\sum\limits_{t=1}^{n} X_t W_t \varepsilon_t$. Then,

$$Var(Z_t) = E\left(X_t W_t^2 X'_t \varepsilon_t^2\right) = V_w$$

$$\begin{aligned} E\left|X_{jt} W_t^2 X_{lt} \varepsilon_t^2\right| &\leq E\big(X_{jt}^2 W_t^4 X_{lt}^2\big)^{1/2} \cdot E\big(\varepsilon_t^4\big)^{1/2} \\ &\leq \left[E\left(X_{jt}^4\right) E\left(W_t^8\right) E\left(X_{lt}^4\right)\right]^{1/4} \cdot E\big(\varepsilon_t^4\big)^{1/2} \end{aligned}$$

since $E\left(W_t^8\right) < \infty$, $E\left(X_{jt}^4\right) < \infty$, $E\left(\varepsilon_t^4\right) < \infty$, we have

$$E\left|X_{jt} W_t^2 X_{lt} \varepsilon_t^2\right| < \infty$$

By CLT,

$$\sqrt{n}\bar{Z} \xrightarrow{d} N\left(0, V_w\right)$$

$$\sqrt{n}(\hat{\beta}_w - \beta^0) = \left(\frac{1}{n}\sum_{t=1}^{n} X_t W_t X'_t\right)^{-1} \sqrt{n}\bar{Z} \xrightarrow{d} N\left(0, \Omega_w\right)$$

where $\Omega_w = Q_w^{-1} V_w Q_w^{-1}$, $Q_w = E\left(X_t W_t X'_t\right)$.
(i) conditional homoskedasticity $E\left(\varepsilon_t^2 \left|X_t\right.\right) = \sigma^2$

$$V_w = E\left(X_t W_t^2 X'_t \varepsilon_t^2\right) = E\left(E\left(X_t W_t^2 X'_t \varepsilon_t^2 \left|X_t\right.\right)\right) = \sigma^2 E\left(X_t W_t^2 X'_t\right) = \sigma^2 Q_{w^2}$$

$$\Omega_w = \sigma^2 Q_w^{-1} Q_{w^2} Q_w^{-1}$$

(ii) conditional heteroskedasticity $E\left(\varepsilon_t^2 \left|X_t\right.\right) \neq \sigma^2$

$$\Omega_w = Q_w^{-1} V_w Q_w^{-1}$$

(d)

(i) under conditional homoskedasticity: $\hat{\Omega}_w = s^2 \hat{Q}_w^{-1} \hat{Q}_{w^2} \hat{Q}_w^{-1}$, where $\hat{Q}_w^{-1} = \left(\frac{1}{n}\sum\limits_{t=1}^{n} X_t W_t X'_t\right)^{-1} \xrightarrow{p} Q_w^{-1}$, $\hat{Q}_{w^2} = \frac{1}{n}\sum\limits_{t=1}^{n} X_t W_t^2 X'_t \xrightarrow{p} Q_{w^2}$

$$\begin{aligned} s^2 &= \frac{1}{n-k} e'e = \frac{1}{n-k}\sum\left(Y_t - X'_t\hat{\beta}\right)^2 = \frac{1}{n-k}\sum\left[\varepsilon_t - X'_t\left(\hat{\beta} - \beta^0\right)\right]^2 \\ &= \frac{n}{n-k}\left(\frac{1}{n}\sum\varepsilon_t^2\right) + \left(\hat{\beta} - \beta^0\right)'\left(\frac{1}{n-k}\sum X_t X'_t\right)\left(\hat{\beta} - \beta^0\right) - 2\left(\hat{\beta} - \beta^0\right)\left(\frac{1}{n-k}\sum X_t \varepsilon_t\right) \\ &\xrightarrow{p} 1 \cdot \sigma^2 + 0 \cdot E\left(X_{jt}^2\right) \cdot 0 + 2 \cdot 0 = \sigma^2 \end{aligned}$$

$$s^2\hat{Q}_w^{-1}\hat{Q}_{w^2}\hat{Q}_w^{-1} \xrightarrow{p} \sigma^2 Q_w^{-1}Q_{w^2}Q_w^{-1}$$

(ii) under conditional heteroskedasticity: $\hat{\Omega}_w = \hat{Q}_w^{-1}\hat{V}_w\hat{Q}_w^{-1}$, where $\hat{Q}_w = \frac{1}{n}\sum\limits_{t=1}^{n} X_t W_t X'_t$, $\hat{V}_w = \frac{1}{n}\sum\limits_{t=1}^{n} X_t W_t^2 X'_t e_t^2$

$$\begin{aligned}
\hat{V}_w &= \frac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t e_t^2 \\
&= \frac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t \Big(\varepsilon_t - (\hat{\beta}-\beta^0)' X_t\Big)^2 \\
&= \frac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t \varepsilon_t^2 + \frac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t \left[(\hat{\beta}-\beta^0)' X_t X'_t (\hat{\beta}-\beta^0)\right] + \frac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t \left[\varepsilon_t X'_t (\hat{\beta}-\beta^0)\right]
\end{aligned}$$

$$\begin{aligned}
E\left|X_{jt}W_t^2 X_{lt}\varepsilon_t^2\right| &\le \left[E\left(X_{jt}^2 W_t^4 X_{lt}^2\right) E\left(\varepsilon_t^4\right)\right]^{1/2} \\
&\le \left[\left(EX_{jt}^4 X_{lt}^4\right)\left(EW_t^8\right)\right]^{1/4}\left(E\varepsilon_t^4\right)^{1/2} \\
&\le \left(EX_{jt}^8\right)^{1/8}\left(EX_{lt}^8\right)^{1/8}\left(EW_t^8\right)^{1/4}\left(E\varepsilon_t^4\right)^{1/2} \\
&\le C < \infty
\end{aligned}$$

It is easy to proof

$$\begin{aligned}
&\tfrac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t \varepsilon_t^2 \to E\left(X_t W_t^2 X'_t \varepsilon_t^2\right) = V_w \\
&\tfrac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t \left[(\hat{\beta}-\beta^0)' X_t X'_t (\hat{\beta}-\beta^0)\right] \to 0 \\
&\tfrac{1}{n}\sum_{t=1}^{n} X_t W_t^2 X'_t \left[\varepsilon_t X'_t (\hat{\beta}-\beta^0)\right] \to 0
\end{aligned}$$

So,

$$\hat{\Omega}_w = \hat{Q}_w^{-1}\hat{V}_w\hat{Q}_w^{-1} \xrightarrow{p} Q_w^{-1}V_wQ_w^{-1} = \Omega_w$$

(e)

(i) under conditional homoskedasticity

$$\sqrt{n}(\hat{\beta}_w - \beta^0) \xrightarrow{d} N\left(0, \sigma^2 Q_w^{-1}Q_{w^2}Q_w^{-1}\right)$$

under $H_0: \quad R\beta^0 = r$,

$$\sqrt{n}R(\hat{\beta}_w - \beta^0) \xrightarrow{d} N\left(0, \sigma^2 R Q_w^{-1}Q_{w^2}Q_w^{-1}R'\right)$$

$$\sqrt{n}(R\hat{\beta}_w - r)\left(\sigma^2 R Q_w^{-1}Q_{w^2}Q_w^{-1}R'\right)^{-1}\sqrt{n}(R\hat{\beta}_w - r) \xrightarrow{d} \chi_J^2$$

by slustky theorem,

$$\sqrt{n}(R\hat{\beta}_w - r)\Big(\sigma^2 R\hat{Q}_w^{-1}\hat{Q}_{w^2}\hat{Q}_w^{-1}R'\Big)^{-1}\sqrt{n}(R\hat{\beta}_w - r) \xrightarrow{d} \chi_J^2$$

(ii) under conditional heteroskedasticity

$$\sqrt{n}(\hat{\beta}_w - \beta^0) \xrightarrow{d} N\left(0, Q_w^{-1} V_w Q_w^{-1}\right)$$

under $H_0: \quad R\beta^0 = r$,

$$\sqrt{n}R(\hat{\beta}_w - \beta^0) \xrightarrow{d} N\left(0, RQ_w^{-1} V_w Q_w^{-1} R'\right)$$

$$\sqrt{n}(R\hat{\beta}_w - r)\left(RQ_w^{-1} V_w Q_w^{-1} R'\right)^{-1} \sqrt{n}(R\hat{\beta}_w - r) \xrightarrow{d} \chi_J^2$$

by slustky theorem,

$$\sqrt{n}(R\hat{\beta}_w - r)\left(R\hat{Q}_w^{-1} \hat{V}_w \hat{Q}_w^{-1} R'\right)^{-1} \sqrt{n}(R\hat{\beta}_w - r) \xrightarrow{d} \chi_J^2$$

(f)

$$\sqrt{n}(\hat{\beta}_w - \beta^0) \xrightarrow{d} N\left(0, Q_w^{-1} V_w Q_w^{-1}\right)$$

$$\begin{aligned}
V_w &= E\left(X_t W_t^2 X'_t \varepsilon_t^2\right) = E\left(X_t W_t^2 X'_t E\left(\varepsilon_t^2\right)\right) = E\left(X_t X'_t\right) \\
Q_w &= E\left(X_t W_t X'_t\right) \\
V_w &= \frac{1}{n}\sum_{t=1}^{n} X_t X'_t \xrightarrow{p} V_w \\
Q_w &= \frac{1}{n}\sum_{t=1}^{n} X_t X'_t W_t \xrightarrow{p} Q_w
\end{aligned}$$

$$\sqrt{n}(R\hat{\beta}_w - r)\left(R\hat{Q}_w^{-1} \hat{V}_w \hat{Q}_w^{-1} R'\right)^{-1} \sqrt{n}(R\hat{\beta}_w - r) \xrightarrow{d} \chi_J^2$$

2. Consider the problem of testing conditional homoskedasticity ($H_0: \quad E\left(\varepsilon_t^2 \left| X_t \right.\right) = \sigma^2$) for a linear regression model

$$Y_t = X'_t \beta^0 + \varepsilon_t,$$

where X_t is a $K \times 1$ vector consisting of an intercept and explanatory variables. To test conditional homoskedasticity, we consider the auxiliary regression

$$\begin{aligned}
\varepsilon_t &= vech(X_t X'_t)' \gamma + v_t \\
&= U'_t \gamma + v_t
\end{aligned}$$

Suppose Assumptions 4.1, 4.2, 4.3, 4.4, 4.7 hold, and $E\left(\varepsilon_t^4 \left| X_t \right.\right) \neq \mu_4$. That is , $E\left(\varepsilon_t^4 \left| X_t \right.\right)$ is a function of X_t.

(a) Show $\operatorname{var}\left(v_t \left| X_t \right.\right) \neq \sigma_v^2$ under H_0. That is, the disturbance v_t in the auxiliary regression model displays conditional heteroskedasticity.

(b) Suppose ε_t is directly observable. Construct an asymptotically valid test for the null hypothesis H_0 of conditional homoskedasticity. Justify your reasoning and test statistic.

ANSWER: Under H_0

$$\begin{aligned}\operatorname{var}(v_t \,|X_t\,) &= \operatorname{var}\left(\varepsilon_t^2 \,|X_t\,\right) \\ &= E\left(\varepsilon_t^4 \,|X_t\,\right) - \sigma^4 \\ &\neq cons\tan t\end{aligned}$$

This means the disturbance v_t in the auxiliary regression model display conditional heteroskedasticity. According to the auxiliary regression model, the null hypothesis is equivalent to

$$H_0 : \quad R\gamma = 0$$

where $R = \begin{pmatrix} 0 & I_J \end{pmatrix}$, I_J is the identity matrix, and $J = \frac{K(K+1)}{2} - 1$. Then,

$$\begin{aligned}\sqrt{n}\hat{\gamma} =& \sqrt{n}(\hat{\gamma} - \gamma_0) + \sqrt{n}(\gamma_0 - 0) \\ =& \sqrt{n}(\hat{\gamma} - \gamma_0) \\ &\xrightarrow{d} N\left(0, Q_u^{-1} V_v Q_u^{-1}\right)\end{aligned}$$

where $V_v = E\left(U_t U'{}_t v_t^2\right)$.
It follows that a robust Wald test statistic

$$W = \sqrt{n}R\hat{\gamma}\left[RQ_u^{-1}V_vQ_u^{-1}R'\right]^{-1}\sqrt{n}R\hat{\gamma} \xrightarrow{d} \chi_J^2$$

Because $\hat{Q}_u \xrightarrow{p} Q_u$and $\hat{V}_u \xrightarrow{p} V_u$, we have the Wald test statistic

$$W = \sqrt{n}R\hat{\gamma}\left[R\hat{Q}_u^{-1}\hat{V}_v\hat{Q}_u^{-1}R'\right]^{-1}\sqrt{n}R\hat{\gamma} \xrightarrow{d} \chi_J^2$$

by the Slutsky theorem.