Contents

Answers to Chapter 1 Exercises

1. Suppose there are two lotteries $P = \{p_1, \ldots, p_n\}$ and $P^* = \{p_1^*, \ldots, p_n^*\}$. Let $V(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i U_i$ be an individual's expected utility function defined over these lotteries. Let $W(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i Q_i$ where $Q_i = a + bU_i$ and a and b are constants. If $P^* \succ P$, so that $V(p_1^*, \ldots, p_n^*) > V(p_1, \ldots, p_n)$, must it be the case that $W(p_1^*, \ldots, p_n^*) > W(p_1, \ldots, p_n)$? In other words, is W also a valid expected utility function for the individual? Are there any restrictions needed on a and b for this to be the case?

   Answer: If $V(p_1^*, \ldots, p_n^*) > V(p_1, \ldots, p_n)$ then this implies $\sum_{i=1}^{n} p_i^* U_i > \sum_{i=1}^{n} p_i U_i$. If b is a positive constant, then we can multiply both sides of the inequality without changing the sign: $\sum_{i=1}^{n} p_i^* bU_i > \sum_{i=1}^{n} p_i bU_i$. Since $\sum_{i=1}^{n} p_i^* = \sum_{i=1}^{n} p_i = 1$, we can then add the constant a to each side of the inequality to obtain $\sum_{i=1}^{n} p_i^* (a + bU_i) > \sum_{i=1}^{n} p_i (a + bU_i)$. But this is simply $W(p_1^*, \ldots, p_n^*) > W(p_1, \ldots, p_n)$. Hence for W to be a valid expected utility function for the individual, a can be a constant of any sign but b must be positive.

2. (Allais paradox) Asset A pays \$1,500 with certainty, while asset B pays \$2,000 with probability 0.8 or \$100 with probability 0.2. If offered the choice between asset A or B, a particular individual would choose asset A. Suppose, instead, that the individual is offered the choice between asset C and asset D. Asset C pays \$1,500 with probability 0.25 or \$100 with probability 0.75, while asset D pays \$2,000 with probability 0.2 or \$100 with probability 0.8. If asset D is chosen, show that the individual's preferences violate the independence axiom.

   Answer: Using the notation in our text, let the set of outcomes be $x = \{100, 1500, 2000\}$. Then $P^A = \{0, 1, 0\}$, $P^B = \{0.2, 0, 0.8\}$, $P^C = \{0.75, 0.25, 0\}$, and $P^D = \{0.80, 0, 0.20\}$. Define asset E as $P^E = \{1, 0, 0\}$. Then if $P^A \succ P^B$, the independence axiom states that

$$\lambda P^A + (1-\lambda) P^E \succ \lambda P^B + (1-\lambda) P^E$$
$$0.25 P^A + 0.75 P^E \succ 0.25 P^B + 0.75 P^E$$
$$P^C \succ P^D$$

   Thus, if the individual chooses asset D over asset C, independence is violated.

3. Verify that the HARA utility function in equation (1.33) becomes the constant absolute-risk-aversion utility function when $\beta = 1$ and $\gamma = -\infty$. *Hint:* recall that $e^a = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

Answer: For $\beta = 1$, $U(W) = \frac{1-\gamma}{\gamma}\left(\frac{\alpha W}{1-\gamma}+1\right)^{\gamma}$. This can be written as $U(W) = \frac{1-\gamma}{\gamma}\left(\frac{\alpha W}{1-\gamma}+1\right)^{\gamma-1}\left(\frac{\alpha W}{1-\gamma}+1\right)$. Letting $x = 1-\gamma$, it can be re-written as

$$U(W) = \frac{x}{1-x}\frac{1}{\left(\frac{\alpha W}{x}+1\right)^{x}}\left(\frac{\alpha W}{x}+1\right)$$

As $\gamma \to -\infty$, $x \to +\infty$. Thus, the limiting value of the first term in the above expression is -1, the limiting value of the second expression is $1/e^{\alpha W}$, while the limiting value of the third expression is 1. Hence, $\lim_{\gamma\to-\infty} U(W) = -e^{-\alpha W}$.

4. Consider the individual's portfolio choice problem given in equation (1.42). Assume $U(W) = \ln(W)$ and the rate of return on the risky asset equals $\tilde{r} = \begin{cases} 4r_f & \text{with probability } \frac{1}{2} \\ -r_f & \text{with probability } \frac{1}{2} \end{cases}$. Solve for the individual's proportion of initial wealth invested in the risky asset, A/W_0.

Answer: From the first order condition

$$E[U'(\tilde{W})(\tilde{r} - r_f)] = 0$$

we have

$$\frac{\frac{1}{2}3r_f}{W_0(1+r_f)+A3r_f} + \frac{-\frac{1}{2}2r_f}{W_0(1+r_f)-A2r_f} = 0.$$

Solving for A/W_0, we obtain $A/W_0 = \frac{1+r_f}{12r_f}$.

5. An expected-utility-maximizing individual has constant relative-risk-aversion utility, $U(W) = W^{\gamma}/\gamma$, with relative risk-aversion coefficient of $\gamma = -1$. The individual currently owns a product that has a probability p of failing, an event that would result in a loss of wealth that has a present value equal to L. With probability $1-p$, the product will not fail and no loss will result. The individual is considering whether to purchase an extended warranty on this product. The warranty costs C and would insure the individual against loss if the product fails. Assuming that the cost of the warranty exceeds the expected loss from the product's failure, determine the individual's level of wealth at which she would be just indifferent between purchasing or not purchasing the warranty.

Answer: The level of wealth for which the individual would be indifferent satisfies

$$pU(W-L) + (1-p)U(W) = U(W-C)$$

or

$$\frac{p}{W-L}-\frac{1-p}{W}=-\frac{1}{W-C}$$

which implies

$$\frac{Wp+(1-p)(W-L)}{(W-L)W}=\frac{1}{W-C}$$

or

$$W^2-LW=W^2+CL(1-p)-W(C+L(1-p))$$

so

$$W^*=\frac{CL(1-p)}{C-Lp}$$

For levels of wealth above W^*, the individual would refuse the warranty (self-insure) while for levels of wealth below W^*, the individual would take the warranty. The intuition is as follows. Recall that the Pratt risk premium is $\pi=\frac{1}{2}\sigma^2 R(W)$. In addition, an individual who has constant relative risk aversion has absolute risk aversion, $R(W)$, that declines with wealth. Hence, the warranty cost that this individual is willing to pay, C, which is analogous to the premium, π, declines with wealth.

6. In the context of the portfolio choice problem in equation (1.42), show that an individual with increasing relative risk aversion invests proportionally less in the risky asset as her initial wealth increases.

Answer: Let r^h denote a realization of $\tilde{r}$ such that it exceeds r_f, and let W^h be the corresponding level of $\tilde{W}$. Then for $A \geq 0$, we have $W^h \geq W_0(1+r_f)$. If relative risk aversion, $R_r(W) \equiv WR(W)$, is increasing in wealth, this implies

$$W^h R(W^h) \geq W_0(1+r_f)R(W_0(1+r_f)) \tag{6.1}$$

Multiplying both sides of (6.1) by $-U'(W^h)(r^h-r_f)$, which is a negative quantity, the inequality sign changes:

$$W^h U''(W^h)(r^h-r_f) \leq -U'(W^h)(r^h-r_f)W_0(1+r_f)R(W_0(1+r_f)) \tag{6.2}$$

Next, let r^l denote a realization of $\tilde{r}$ such that it is lower than r_f, and let W^l be the corresponding level of $\tilde{W}$. Then for $A \geq 0$, we have $W^l \leq W_0(1+r_f)$. If relative risk aversion is increasing in wealth, this implies

$$W^l R(W^l) \leq W_0(1+r_f)R(W_0(1+r_f)) \tag{6.3}$$

Multiplying (6.3) by $-U'(W^l)(r^l - r_f)$, which is positive, so that the sign of (6.3) remains the same, we obtain

$$W^l U''(W^l)(r^l - r_f) \le -U'(W^l)(r^l - r_f) W_0 (1 + r_f) R(W_0(1 + r_f)) \tag{6.4}$$

Notice that inequalities (6.2) and (6.4) are of the same form. The inequality holds whether the realization is $\tilde{r} = r^h$ or $\tilde{r} = r^l$. Therefore, if we take expectations over all realizations, where $\tilde{r}$ can be either higher than or lower than r_f, we obtain

$$E[\tilde{W}U''(\tilde{W})(\tilde{r} - r_f)] \le -E[U'(\tilde{W})(\tilde{r} - r_f)] W_0 (1 + r_f) R(W_0(1 + r_f)) \tag{6.5}$$

Since the first term on the right-hand-side is just the first order condition, inequality (6.5) reduces to

$$E[\tilde{W}U''(\tilde{W})(\tilde{r} - r_f)] \le 0 \tag{6.6}$$

Thus, we see that an individual with increasing relative risk aversion has

$$\eta = 1 + \frac{E[U''(\tilde{W})(\tilde{r} - r_f)\tilde{W}]}{-AE[U''(\tilde{W})(\tilde{r} - r_f)^2]} < 1 \tag{6.7}$$

and invests proportionally less in the risky asset as wealth increases.

7. Consider the following four assets whose payoffs are as follows:

$$\text{Asset A} = \begin{cases} X \text{ with probability } p_x \\ 0 \text{ with probability } 1 - p_x \end{cases} \qquad \text{Asset B} = \begin{cases} Y \text{ with probability } p_y \\ 0 \text{ with probability } 1 - p_y \end{cases}$$

$$\text{Asset C} = \begin{cases} X \text{ with probability } \alpha p_x \\ 0 \text{ with probability } 1 - \alpha p_x \end{cases} \qquad \text{Asset D} = \begin{cases} Y \text{ with probability } \alpha p_y \\ 0 \text{ with probability } 1 - \alpha p_y \end{cases}$$

where $0 < X < Y$, $p_y < p_x$, $p_x X < p_y Y$, and $\alpha \in (0,1)$.

a. When given the choice of asset C versus asset D, an individual chooses asset C. Could this individual's preferences be consistent with von Neumann-Morgenstern expected utility theory? Explain why or why not.

Answer: While the expected value of asset D's payoff is $Y\alpha p_y$ which is greater than the expected value of asset C's payoff, $X\alpha p_x$, a risk averse individual may still prefer asset C which has a higher probability of success. Hence, this individual's preferences could be consistent with expected utility theory.

b. When given the choice of asset A versus asset B, an individual chooses asset A. This same individual, when given the choice between asset C and asset D, chooses asset D. Could this individual's preferences be consistent with von Neumann-Morgenstern expected utility theory? Explain why or why not.

Answer: Such preferences are not consistent with von Neumann-Morgenstern expected utility theory. They violate the independence axiom. Consider the set of payoffs $\{0, X, Y\}$. Asset C can be considered to be a compound lottery that with probability α returns asset A and with probability $(1-\alpha)$ returns 0. Similarly, asset D can be considered to be a compound lottery that with probability α returns asset B and with probability $(1-\alpha)$ returns 0. The independence axiom says that if asset A is preferred to asset B, then asset C must be preferred to asset D.

8. An individual has expected utility of the form

$$E[U(\tilde{W})] = E[-e^{-b\tilde{W}}]$$

where $b > 0$. The individual's wealth is normally distributed as $N(\overline{W}, \sigma_W^2)$. What is this individual's *certainty equivalent* level of wealth?

Answer: Note that from the properties of a normally distributed random variable

$$E\left[-e^{-b\tilde{W}}\right] = -e^{-b\overline{W}+\frac{1}{2}b^2\sigma_W^2} = -e^{-b\left(\overline{W}-\frac{b}{2}\sigma_W^2\right)} = U\left(\overline{W}-\frac{b}{2}\sigma_W^2\right)$$

Thus, the individual's certainty equivalent level of wealth is $\overline{W}-\frac{b}{2}\sigma_W^2$.

Answers to Chapter 2 Exercises

1. Prove that the indifference curves graphed in Figure 2.1 are convex if the utility function is concave. *Hint:* suppose there are two portfolios, portfolios 1 and 2, that lie on the same indifference curve, where this indifference curve has expected utility of $\overline{U}$. Let the mean returns on portfolios 1 and 2 be $\overline{R}_{1p}$ and $\overline{R}_{2p}$, respectively, and let the standard deviations of returns on portfolios 1 and 2 be σ_{1p} and σ_{2p}, respectively. Consider a third portfolio located in $(\overline{R}_p, \sigma_p)$ space that happens to be on a straight line between portfolios 1 and 2, that is, a portfolio having a mean and standard deviation satisfying $\overline{R}_{3p} = x\overline{R}_{1p} + (1-x)\overline{R}_{2p}$ and $\sigma_{3p} = x\sigma_{1p} + (1-x)\sigma_{2p}$ where $0 < x < 1$. Prove that the indifference curve is convex by showing that the expected utility of portfolio 3 exceeds $\overline{U}$. Do this by showing that the utility of portfolio 3 exceeds the convex combination of utilities for portfolios 1 and 2 for each standardized normal realization. Then integrate over all realizations to show this inequality holds for expected utilities.

   Answer: We want to show that

$$E[U(\mathring{R}_{3p})] > wE[U(\mathring{R}_{1p})] + (1-w)E[U(\mathring{R}_{2p})] = w\overline{U} + (1-w)\overline{U} = \overline{U}$$

   or

$$\int_{-\infty}^{\infty} U(\overline{R}_{3p} + x\sigma_{3p})n(x)dx > w\int_{-\infty}^{\infty} U(\overline{R}_{1p} + x\sigma_{1p})n(x)dx + (1-w)\int_{-\infty}^{\infty} U(\overline{R}_{2p} + x\sigma_{2p})n(x)dx \qquad (*)$$

   Note that for a given realization of *x*,

$$\begin{aligned} U(\overline{R}_{3p} + x\sigma_{3p}) &= U(w(\overline{R}_{1p} + x\sigma_{1p}) + (1-w)(\overline{R}_{2p} + x\sigma_{2p})) \\ &> wU(\overline{R}_{1p} + x\sigma_{1p}) + (1-w)U(\overline{R}_{2p} + x\sigma_{2p}) \end{aligned}$$

   because $U(\cdot)$ is a concave function. Thus, mutiplying each side of the inequality by $n(x)$, which is always positive, preserves the direction of the inequality. Integrating over all realizations of *x* gives the desired result in *.

2. Show that the covariance between the return on the minimum variance portfolio and the return on *any* other portfolio equals the variance of the return on the minimum variance portfolio. *Hint:* write down the variance of a portfolio that consists of a proportion x invested in the minimum variance portfolio and a proportion $(1-x)$ invested in any other portfolio. Then minimize the variance of this composite portfolio with respect to x.

Answer: Let σ_a^2 be the variance of the return on an arbitrary portfolio and let $\sigma_{a,mv}$ be the covariance of this portfolio's return with that of the minimum variance portfolio. Then the variance of the composite portfolio consisting of proportions x and $(1-x)$ in the minimum variance and arbitrary portfolios, respectively, is

$$x^2\sigma_{mv}^2 + (1-x)^2\sigma_a^2 + 2x(1-x)\sigma_{a,mv}$$

If we minimize this composite portfolio's variance with respect to x, we obtain the first order condition

$$2x\sigma_{mv}^2 - 2(1-x)\sigma_a^2 + 2(1-2x)\sigma_{a,mv} = 0$$

Now since the minimum variance portfolio has, by definition, the smallest variance of all portfolios, it must be the case that $x = 1$ is the solution to this first order condition. Making this substitution, one obtains

$$\sigma_{mv}^2 = \sigma_{a,mv}$$

3. Show how to derive the solution for the optimal portfolio weights for a frontier portfolio when there exists a riskless asset, that is, equation (2.42) given by $\omega^* = \lambda V^{-1}(\bar{R} - R_f e)$ where $\lambda \equiv \frac{\bar{R}_p - R_f}{(\bar{R}-R_f e)'V^{-1}(\bar{R}-R_f e)} = \frac{\bar{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}$. The derivation is similar to the case with no riskless asset.

Answer: Since the objective function is

$$\min_w \frac{1}{2} w'Vw + \lambda\{\bar{R}_p - [R_f + w'(\bar{R} - R_f e)]\}$$

The first order conditions with respect to w and λ are

$$\frac{\partial L}{\partial w} = Vw - \lambda(\bar{R} - R_f e) \tag{a}$$

$$\frac{\partial L}{\partial \lambda} = \bar{R}_p - [R_f + w'(\bar{R} - R_f e)] \tag{b}$$

Re-arranging (a) gives

$$w^* = \lambda V^{-1}(\bar{R} - R_f e) \tag{c}$$

and re-arranging (b) gives

$$\bar{R}_p - R_f = (\bar{R} - R_f e)'w^* \tag{d}$$

Pre-multiplying equation (c) by $(\bar{R} - R_f e)'$ gives

$$(\bar{R} - R_f e)'w^* = \lambda(\bar{R} - R_f e)'V^{-1}(\bar{R} - R_f e) \tag{e}$$

and equating the left-hand side of (d) to the right-hand side of (e), we have

$$\bar{R}_p - R_f = \lambda(\bar{R} - R_f e)'V^{-1}(\bar{R} - R_f e) = \lambda\left[\bar{R}'V^{-1}\bar{R} - 2R_f \bar{R}'V^{-1}e + R_f^2 e'V^{-1}e\right]$$
$$= \lambda\left[\varsigma - 2\alpha R_f + \delta R_f^2\right]$$

or

$$\lambda = \frac{\bar{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}$$

4. Show that when $R_f = R_{mv}$, the optimal portfolio involves $e'\omega^* = 0$.

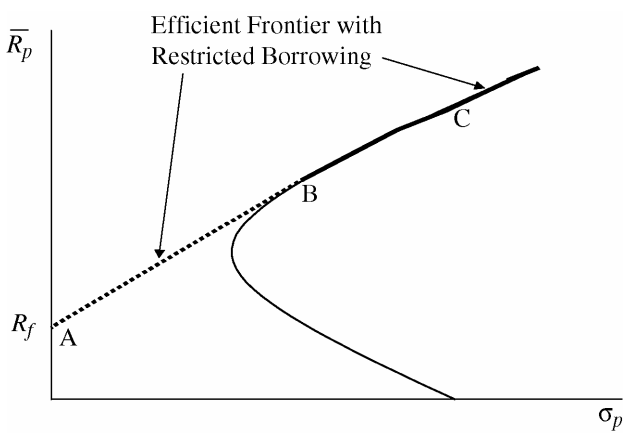
Answer: When $R_f = R_{mv} \equiv \frac{\alpha}{\delta}$, the optimal portfolio weights are $w^* = \lambda V^{-1}(\bar{R} - R_f e) = \lambda V^{-1}(\bar{R} - \frac{\alpha}{\delta} e)$. Pre-multiplying by e', we obtain

$$e'w^* = \lambda\left(e'V^{-1}\bar{R} - e'V^{-1}e\frac{\alpha}{\delta}\right) = \lambda\left(\alpha - \delta\frac{\alpha}{\delta}\right) = 0$$

5. Consider the mean-variance analysis covered in this chapter where there are n risky assets whose returns are jointly normally distributed. Assume that investors differ with regard to their (concave) utility functions and their initial wealths. Also assume that investors can lend at the risk-free rate, $R_f < R_{mv}$, but investors are restricted from risk-free borrowing; that is, no risk-free borrowing is permitted.

   a. Given this risk-free borrowing restriction, graphically show the efficient frontier for these investors in expected portfolio return-standard deviation space $(\bar{R}_p, \sigma_p)$.

   Answer:



If an investor's indifference curve is tangent to the frontier between points A and B, she holds a positive proportion of the risk-free asset and a positive proportion of risky portfolio B. If the investor's indifference curve is tangent to the frontier to the right of B, he holds a combination of any two portfolios on the risky–asset only frontier that would replicate a position to the right of B.

b. Explain why only three portfolios are needed to construct this efficient frontier, and locate these three portfolios on your graph. (Note that these portfolios may not be unique.)

Answer: One portfolio is composed of investing completely in the risk-free asset (point A). The other two portfolios can be any two portfolios of the risky asset only efficient frontier, for example, points B and C. This is because we proved in class that the risky asset only frontier can be constructed from combinations of any two risky asset only efficient portfolios (the separation property).

c. At least one of these portfolios will sometimes need to be sold short to generate the entire efficient frontier. Which portfolio(s) is it (label it on the graph) and in what range(s) of the efficient frontier will it be sold short? Explain.

Answer: For example, if the three portfolios are at A, B, and C, then to the right of C, portfolio B needs to be sold short to obtain a portfolio expected rate of return higher than that of portfolio C. This is because $\overline{R}_p = w\overline{R}_B + (1-w)\overline{R}_C$ requires $w < 0$ for $\overline{R}_p > \overline{R}_C$.

6. Suppose there are n risky assets whose returns are multi-variate normally distributed. Denote their $n \times 1$ vector of expected returns as $\overline{R}$ and their $n \times n$ covariance matrix as V. Let there also be a riskless asset with return R_f. Let portfolio a be on the mean-variance efficient frontier and have an expected return and standard deviation of $\overline{R}_a$ and σ_a, respectively. Let portfolio b be any other (not necessarily efficient) portfolio having expected return and standard deviation $\overline{R}_b$ and σ_b, respectively. Show that the correlation between portfolios a and b equals portfolio b's Sharpe ratio divided by portfolio a's Sharpe ratio, where portfolio i's Sharpe ratio equals $(\overline{R}_i - R_f)/\sigma_i$. (*Hint:* write the correlation as $\text{cov}(R_a, R_b)/(\sigma_a\sigma_b)$, and derive this covariance using the properties of portfolio efficiency.)

Answer: Note that for an efficient portfolio,

$$w_a = \lambda_a V^{-1}(\overline{R} - R_f e) \tag{1}$$

and

$$\lambda_a = \frac{\overline{R}_a - R_f}{(\overline{R} - R_f e)' V^{-1} (\overline{R} - R_f e)} \tag{2}$$

and

$$\begin{aligned} \sigma_a^2 &= w_a' V w_a \\ &= \frac{(\overline{R}_a - R_f)^2}{(\overline{R} - R_f e)' V^{-1} (\overline{R} - R_f e)} \end{aligned} \tag{3}$$

Thus, the correlation between portfolios a and b is

$$\begin{aligned}
\frac{\operatorname{cov}(R_a, R_b)}{\sigma_a \sigma_b} &= \frac{w_a' V w_b}{\sigma_a \sigma_b} \\
&= \lambda_a \frac{(\bar{R} - R_f e)' V^{-1} V w_b}{\sigma_a \sigma_b} \\
&= \lambda_a \frac{(\bar{R} - R_f e)' w_b}{\sigma_a \sigma_b} \\
&= \lambda_a \frac{\bar{R}_b - R_f}{\sigma_a \sigma_b} \\
&= \frac{\lambda_a}{\sigma_a} \frac{\bar{R}_b - R_f}{\sigma_b}
\end{aligned} \tag{4}$$

Now from (2) and (3) we see that

$$\lambda_a = \frac{\sigma_a^2}{\bar{R}_a - R_f} \tag{5}$$

Therefore,

$$\begin{aligned}
\frac{\operatorname{cov}(R_a, R_b)}{\sigma_a \sigma_b} &= \frac{\lambda_a}{\sigma_a} \frac{\bar{R}_b - R_f}{\sigma_b} \\
&= \frac{\sigma_a}{\bar{R}_a - R_f} \frac{\bar{R}_b - R_f}{\sigma_b} \\
&= S_b / S_a < 1
\end{aligned} \tag{6}$$

7. A corn grower has utility of wealth given by $U(W) = -e^{-aW}$ where $a > 0$. This farmer's wealth depends on the total revenue from the sale of corn at harvest time. Total revenue is a random variable $\tilde{s} = \tilde{q}\tilde{p}$, where $\tilde{q}$ is the number of bushels of corn harvested and $\tilde{p}$ is the spot price, net of harvesting costs, of a bushel of corn at harvest time. The farmer can enter into a corn futures contract having a current price of f_0 and a random price at harvest time of $\tilde{f}$. If k is the number of short positions in this futures contract taken by the farmer, then the farmer's wealth at harvest time is given by $\tilde{W} = \tilde{s} - k(\tilde{f} - f_0)$. If $\tilde{s} \sim N(\bar{s}, \sigma_s^2)$, $\tilde{f} \sim N(\bar{f}, \sigma_f^2)$, and $\operatorname{cov}(\tilde{s}, \tilde{f}) = \rho \sigma_s \sigma_f$, then solve for the optimal number of futures contract short positions, k, that the farmer should take.

Answer: We showed that an individual having normally distributed wealth and negative exponential utility maximizes the function

$$\max_k E[\tilde{W}] - \frac{1}{2} a Var[\tilde{W}] = \max_k \bar{s} - k(\bar{f} - f_0) - \frac{1}{2} a \left[\sigma_s^2 + k^2 \sigma_f^2 - 2k\rho\sigma_s\sigma_f \right]$$

This leads to the first order condition

$$-(\bar{f} - f_0) - ak\sigma_f^2 + a\rho\sigma_s\sigma_f = 0$$

or

$$k = \frac{f_0 - \overline{f}}{a\sigma_f^2} + \frac{\rho\sigma_s}{\sigma_f}$$

8. Consider the standard Markowitz mean-variance portfolio choice problem where there are n risky assets and a risk-free asset. The risky assets' $n \times 1$ vector of returns, $\tilde{R}$, has a multivariate normal distribution $N(\overline{R}, V)$, where $\overline{R}$ is the assets' $n \times 1$ vector of expected returns and V is a non-singular $n \times n$ covariance matrix. The risk-free asset's return is given by $R_f > 0$. As usual, assume no labor income so that the individual's end-of-period wealth depends only on her portfolio return; that is, $\tilde{W} = W_0 \tilde{R}_p$, where the portfolio return is $\tilde{R}_p = R_f + w'(\tilde{R} - R_f e)$ where w is an $n \times 1$ vector of portfolio weights for the risky assets and e is an $n \times 1$ vector of 1s. Recall that we solved for the optimal portfolio weights, w^* for the case of an individual with expected utility displaying constant absolute risk aversion, $E[U(\tilde{W})] = E[-e^{-b\tilde{W}}]$. Now, in this problem, consider the different case of an individual with expected utility displaying constant relative risk aversion, $E[U(\tilde{W})] = E[\frac{1}{\gamma}\tilde{W}^{\gamma}]$ where $\gamma < 1$. What is w^* for this constant relative-risk-aversion case? *Hint:* recall the efficient frontier and consider the range of the probability distribution of the tangency portfolio. Also consider what would be the individual's marginal utility should end-of-period wealth be nonpositive. This marginal utility will restrict the individual's optimal portfolio choice.

Answer: Note that constant relative risk-aversion (CRRA) utility, $\frac{1}{\gamma}W^{\gamma}$, $\gamma < 1$, in general is not a defined, real-valued function for $W < 0$. In addition, since marginal utility equals $\partial U(W)/\partial W = W^{\gamma-1} = 1/W^{1-\gamma}$, as wealth approaches zero it becomes infinite:

$$\lim_{W \downarrow 0} \frac{\partial U(W)}{\partial W} = \lim_{W \downarrow 0} \frac{1}{W^{1-\gamma}} = \infty$$

and absolute risk aversion, $R(W) = \frac{(1-\gamma)}{W}$, also becomes infinite. The implication of this is that an investor with CRRA utility would avoid assets that had a positive probability of making total wealth equal 0 (or negative). To see this, note that if we were to solve the investor's maximization problem

$$\max_w E[U(\tilde{W})] = \max_w E[U(W_0[R_f + w'(\tilde{R} - R_f e)])]$$

the first order conditions would be of the form

$$E\left[\frac{\partial U(W)}{\partial W}(R_i - R_f)\right] = 0, \quad i = 1, \ldots, n$$

with R_i being normally distributed, when $w_i \neq 0$, there is a positive probability that $\tilde{R}_p = R_f + w'(\tilde{R} - R_f e) \leq 0$, no matter what the other elements of w. This implies that for those realizations, say where $R_p = W = 0$, then $\frac{\partial U(W)}{\partial W}(R_i - R_f)$ will be positive or negative infinity, and the average of all realizations can never equal 0. Hence, with $\tilde{R}_i$ being normally distributed, the corner solution where the individual puts all wealth in the riskfree asset maximizes expected utility. Note that this is not the case with constant absolute risk aversion, because marginal utility, be^{-bW}, is positive for both positive and negative values of wealth.

Answers to Chapter 3 Exercises

1. Assume that individual investor k chooses between n risky assets in order to maximize the following utility function:

$$\max_{\{\omega_i^k\}} \bar{R}_k - \frac{1}{\theta_k} V_k$$

where the mean and variance of investor k's portfolio are $\bar{R}_k = \sum_{i=1}^{n} \omega_i^k \bar{R}_i$ and $V_k = \sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i^k \omega_j^k \sigma_{ij}$, respectively, and where $\bar{R}_i$ is the expected return on risky asset i, and σ_{ij} is the covariance between the returns on risky asset i and risky asset j. ω_i^k is investor k's portfolio weight invested in risky asset i, so that $\sum_{i=1}^{n} \omega_i^k = 1$. θ_k is a positive constant and equals investor k's *risk tolerance*.

a. Write down the Lagrangian for this problem and show the first-order conditions.

Answer:

$$L = \sum_{i=1}^{n} w_i^k \bar{R}_i - \frac{1}{\theta_k} \sum_{i=1}^{n}\sum_{j=1}^{n} w_i^k w_j^k \sigma_{ij} + \lambda_k \left(1 - \sum_{i=1}^{n} w_i^k\right) \qquad (1)$$

The first order conditions are

$$\bar{R}_i - \frac{2}{\theta_k} \sum_{j=1}^{n} w_j^k \sigma_{ij} = \lambda_k, i = 1, \ldots, n \qquad (2)$$

b. Rewrite the first-order condition to show that the expected return on asset i is a linear function of the covariance between risky asset i's return and the return on investor k's optimal portfolio.

Answer: From (2), note that $\sum_{j=1}^{n} w_j^k \sigma_{ij}$ is the covariance of asset i with investor k's optimal portfolio, that is, $\text{cov}(\tilde{R}_i, \tilde{R}_p^k) = \sum_{j=1}^{n} w_j^k \sigma_{ij}$. Hence, (2) can be re-written as

$$\bar{R}_i = \lambda_k + \frac{2}{\theta_k} \text{cov}(\tilde{R}_i, \tilde{R}_p^k) \qquad (3)$$

c. Assume that investor k has initial wealth equal to W_k and that there are $k = 1, \ldots, M$ total investors, each with different initial wealth and risk tolerance. Show that the equilibrium expected return on asset i is of a similar form to the first-order condition found in part (b), but depends on the *wealth-weighted risk tolerances* of investors and the *covariance of the return on asset i with the market portfolio*. *Hint:* begin by multiplying the first order condition in (b) by investor k's wealth times risk tolerance, and then aggregate over all investors.

Answer: Multiplying (2) by $W_k\theta_k$ gives

$$W_k\theta_k\overline{R}_i - 2W_k\sum_{j=1}^{n} w_j^k\sigma_{ij} = W_k\theta_k\lambda_k \tag{4}$$

Summing over all investors, we obtain

$$\sum_{k=1}^{M} W_k\theta_k\overline{R}_i - 2\sum_{k=1}^{M} W_k\sum_{j=1}^{n} w_j^k\sigma_{ij} = \sum_{k=1}^{M} W_k\theta_k\lambda_k \tag{5}$$

Let $\theta_M \equiv \sum_{k=1}^{M} W_k\theta_k$ be the wealth-weighted risk tolerances of the M investors.
Note also that

$$2\sum_{k=1}^{M} W_k\sum_{j=1}^{n} w_j^k\sigma_{ij} = 2\sum_{j=1}^{n}\sum_{k=1}^{M} w_j^k W_k\sigma_{ij} = 2\text{cov}(\tilde{R}_i, \tilde{R}_M) \tag{6}$$

is equal to two times the covariance between asset i's return and the return on the market portfolio. Thus, (5) can be re-written as

$$\overline{R}_i = \frac{1}{\theta_M}\sum_{k=1}^{M} W_k\theta_k\lambda_k + \frac{2}{\theta_M}\text{cov}(\tilde{R}_i, \tilde{R}_M)$$

2. Let the U.S. dollar ($)/Swiss franc (SF) spot exchange rate be $0.68 per SF and the one-year forward exchange rate be $0.70 per SF. The one-year interest rate for borrowing or lending dollars is 6.00 percent.

   a. What must be the one-year interest rate for borrowing or lending Swiss francs in order for there to be no arbitrage opportunity?

   Answer: The covered interest parity relation for a one-year forward rate states that

$$r_{01}^* = \frac{S_0}{F_{01}}(1+r_{01}) - 1 = \frac{0.68}{0.70}(1.0600) - 1 = 2.97 \text{ percent}$$

   b. If the one-year interest rate for borrowing or lending Swiss francs was less than your answer in part (a), describe the arbitrage opportunity.

   Answer: If $1 + r_{01}^* < \frac{S_0}{F_{01}}(1+r_{01})$, an arbitrage is to

   At date 0:

   (i) borrow $\frac{1}{1+r_{01}^*}$ Swiss francs, agreeing to repay 1 Swiss franc in one year (date 1).

   (ii) Exchange the Swiss francs for $\frac{S_0}{1+r_{01}^*}$ U.S. dollars, and invest these at the interest rate r_{01}.

   (iii) Take a long position in a one-year forward exchange contract on the Swiss franc at rate F_{01}.

At date 1:

(i) Pay $\$F_{01}$ and receive one Swiss franc.

(ii) Use this Swiss franc to repay your borrowing obligation $\frac{1+r_{01}^*}{1+r_{01}^*} = 1$ Swiss franc.

(iii) Realize the U.S. dollar investment of $\frac{S_0}{1+r_{01}^*}(1+r_{01})$.

The net cash flow at date 0 is zero. At date 1 the net cash flow is $\frac{S_0}{1+r_{01}^*}(1+r_{01}) - F_{01}$ U.S. dollars, which is positive based on the assumption that $1+r_{01}^* < \frac{S_0}{F_{01}}(1+r_{01})$.

3. Suppose that the Arbitrage Pricing Theory holds with $k = 2$ risk factors, so that asset returns are given by

$$\tilde{R}_i = a_i + b_{i1}\tilde{f}_1 + b_{i2}\tilde{f}_2 + \tilde{\varepsilon}_i$$

where $a_i \cong \lambda_{f0} + b_{i1}\lambda_{f1} + b_{i2}\lambda_{f2}$. Maintain all of the assumptions made in the notes and, in addition, assume that both λ_{f1} and λ_{f2} are positive. Thus, the positive risk premia imply that both of the two orthogonal risk factors are "priced" sources of risk. Now define two new risk factors from the original risk factors:

$$\tilde{g}_1 = c_1\tilde{f}_1 + c_2\tilde{f}_2$$
$$\tilde{g}_2 = c_3\tilde{f}_1 + c_4\tilde{f}_2$$

Show that there exists a c_1, c_2, c_3, and c_4 such that $\tilde{g}_1$ is orthogonal to $\tilde{g}_2$, they each have unit variance, and $\lambda_{g1} > 0$, but that $\lambda_{g2} = 0$, where λ_{g1} and λ_{g2} are the risk premia associated with $\tilde{g}_1$ and $\tilde{g}_2$, respectively. In other words, show that any economy with two priced sources of risk can also be described by an economy with one priced source of risk.

Answer: Define $C \equiv \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$. Once we find the appropriate risk factor transformation $\begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{bmatrix} = C\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}$, then we can re-define the factor sensitivities

$$\begin{aligned}
\tilde{R}_i &= a_i + [b_{i1} \quad b_{i2}]\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} + \tilde{\varepsilon}_i \\
&= a_i + [b_{i1} \quad b_{i2}]C^{-1}C\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} + \tilde{\varepsilon}_i \\
&= a_i + [b_{i1} \quad b_{i2}]C^{-1}\begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{bmatrix} + \tilde{\varepsilon}_i \\
&= a_i + [\hat{b}_{i1} \quad \hat{b}_{i2}]\begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{bmatrix} + \tilde{\varepsilon}_i
\end{aligned} \tag{1}$$

where $[\hat{b}_{i1} \quad \hat{b}_{i2}] = [b_{i1} \quad b_{i2}]C^{-1}$. This will be true for all assets $i = 1, \ldots, n$. Note, also, that we need each asset's expected rate of return, as predicted by APT, to be the same after the transformation. This will be the case for all assets if

$$\begin{aligned} a_i &= \lambda_{f0} + [b_{i1} \quad b_{i2}]\begin{bmatrix} \lambda_{f1} \\ \lambda_{f2} \end{bmatrix} \\ &= \lambda_{f0} + [b_{i1} \quad b_{i2}]C^{-1}C\begin{bmatrix} \lambda_{f1} \\ \lambda_{f2} \end{bmatrix} \\ &= \lambda_{f0} + [\hat{b}_{i1} \quad \hat{b}_{i2}]\begin{bmatrix} \lambda_{g1} \\ \lambda_{g2} \end{bmatrix} \end{aligned} \tag{2}$$

where $\begin{bmatrix} \lambda_{g1} \\ \lambda_{g2} \end{bmatrix} = C\begin{bmatrix} \lambda_{f1} \\ \lambda_{f2} \end{bmatrix}$. Since we want $\lambda_{g1} > 0$ and $\lambda_{g2} = 0$,

$$C\begin{bmatrix} \lambda_{f1} \\ \lambda_{f2} \end{bmatrix} = \begin{bmatrix} c_1\lambda_{f1} + c_2\lambda_{f2} \\ c_3\lambda_{f1} + c_4\lambda_{f2} \end{bmatrix} = \begin{bmatrix} \lambda_{g1} \\ 0 \end{bmatrix} \tag{3}$$

so that we need $c_3/c_4 = -\lambda_{f2}/\lambda_{f1}$. Note, also, that we require $\tilde{g}_1$ and $\tilde{g}_2$ to be orthogonal factors, so that $E[\tilde{g}_1\tilde{g}_2] = 0$, or

$$\begin{aligned} 0 &= E[(c_1\tilde{f}_1 + c_2\tilde{f}_2)(c_3\tilde{f}_1 + c_2\tilde{f}_2)] \\ &= E[(c_1c_3\tilde{f}_1^2 + (c_2c_3 + c_1c_4)\tilde{f}_1\tilde{f}_2 + c_2c_4\tilde{f}_2^2] \\ &= c_1c_3 + c_2c_4 \end{aligned} \tag{4}$$

which implies

$$c_3/c_4 = -c_2/c_1 \tag{5}$$

Since $c_3/c_4 = -\lambda_{f_2}/\lambda_{f_1}$, this implies

$$c_2/c_1 = \lambda_{f_2}/\lambda_{f_1} \tag{6}$$

or

$$C_2 = \lambda_{f2}\frac{c_1}{\lambda_{f1}} \tag{7}$$

Note that equation (3) also requires

$$\begin{aligned} c_1\lambda_{f1} + c_2\lambda_{f2} &= \lambda_{g1} > 0 \\ c_1\lambda_{f1} + \frac{c_1}{\lambda_{f1}}\lambda_{f2}^2 &= \lambda_{g1} > 0 \\ c_1\left(\lambda_{f1} + \frac{\lambda_{f2}^2}{\lambda_{f1}}\right) &= \lambda_{g1} > 0 \end{aligned} \tag{8}$$

Note that a natural value of c_1 is $c_1 = k\lambda_{f1}$, where k is some positive constant. This would ensure that λ_{g1} is positive since then from (8) we would have $k(\lambda_{f1}^2 + \lambda_{f2}^2) = \lambda_{g1} > 0$. This would imply from (7) that $c_2 = k\lambda_{f2}$. To find the appropriate value of k, note that the last conditions to be satisfied are that the factors have unit variance, $E\left[\tilde{g}_1^2\right] = E\left[\tilde{g}_2^2\right] = 1$, so that

$$c_1^2 + c_2^2 = 1 \tag{9}$$

$$c_3^2 + c_4^2 = 1 \tag{10}$$

Now

$$\begin{aligned} c_1^2 + c_2^2 &= 1 \\ k^2\lambda_{f1}^2 + k^2\lambda_{f2}^2 &= 1 \end{aligned} \tag{11}$$

implies

$$k = \frac{1}{\sqrt{\lambda_{f1}^2 + \lambda_{f2}^2}} \tag{12}$$

Defining $L \equiv \sqrt{\lambda_{f1}^2 + \lambda_{f2}^2}$, we obtain

$$c_1 = \frac{\lambda_{f1}}{L} \tag{13}$$

$$c_2 = \frac{\lambda_{f2}}{L} \tag{14}$$

Finally, note that vaues for c_3 and c_4 that satisfy $c_3/c_4 = -\lambda_{f2}/\lambda_{f1}$, (5), and (10) are

$$c_3 = \frac{-\lambda_{f2}}{L} \tag{15}$$

$$c_4 = \frac{\lambda_{f1}}{L} \tag{16}$$

Alternatively, we could have $c_3 = \lambda_{f2}/L$ and $c_4 = -\lambda_{f1}/L$. Thus, the linear transformation, $\begin{bmatrix}\tilde{g}_1 \\ \tilde{g}_2\end{bmatrix} = C\begin{bmatrix}\tilde{f}_1 \\ \tilde{f}_2\end{bmatrix}$ where C is given in equations (13) to (16) satisfies all of the necessary conditions.

Answers to Chapter 4 Exercises

1. Consider the one-period model of consumption and portfolio choice. Suppose that individuals can invest in a one-period bond that pays a riskless real return of R_{rf} and in a one-period bond that pays a riskless nominal return of R_{nf}. Derive an expression for R_{rf} in terms of R_{nf}, $E[I_{01}]$, and $\text{cov}(M_{01}, I_{01})$.

   Answer: Note that

$$R_{nf} = \frac{1}{E[M_{01}]}$$

and

$$R_{rf} = \frac{1}{E[M_{01} I_{01}]}$$

Therefore the ratio of the two rates are

$$\frac{R_{nf}}{R_{rf}} = \frac{E[M_{01} I_{01}]}{E[M_{01}]} = \frac{E[M_{01}]E[I_{01}] + \text{cov}(M_{01}, I_{01})}{E[M_{01}]} = E[I_{01}] + R_{nf}\text{cov}(M_{01}, I_{01})$$

Re-arranging, we obtain

$$R_{rf} = \frac{R_{nf}}{E[I_{01}] + R_{nf}\text{cov}(M_{01}, I_{01})}$$

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E[mX_a] \end{aligned}$$

   Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, \ldots, k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

Answer: Since

$$p_s = \pi_s m_s 1$$

and

$$m_s = \frac{\delta U'(C_s^*)}{U'(C_0^*)} = \delta \frac{C_0^*}{C_s^*}$$

then

$$p_s = \delta \pi_s \frac{C_0^*}{C_s^*}$$

3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E[m_{01} R_s]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s / C_0} \frac{d(C_s / C_0)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U(C_0) + \delta E[U(\tilde{C}_1)]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma / \gamma$, solve for the elasticity of intertemporal substitution, ε^I.

Answer: Since $m_{01} = \delta U'(C_1)/U'(C_0) = \delta (C_1/C_0)^{\gamma-1}$, for primitive security s, the first order condition is

$$1 = \pi_s \delta (C_s / C_0)^{\gamma-1} R_s$$

Totally differentiating this condition gives

$$0 = \pi_s \delta (\gamma - 1)(C_s / C_0)^{\gamma-2} R_s d(C_s / C_0) + \pi_s \delta (C_s / C_0)^{\gamma-1} dR_s$$

Re-arranging, one obtains

$$\frac{R_s}{C_s/C_0}\frac{d(C_s/C_0)}{dR_s}=\frac{1}{1-\gamma}$$

4. Consider an economy with $k=2$ states of nature, a “good” state and a “bad” state.[1] There are two assets, a risk-free asset with $R_f=1.05$ and a second risky asset that pays cashflows

$$X_2=\begin{bmatrix}10\\5\end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

Answer: Let

$$P=\begin{bmatrix}1/1.05\\6\end{bmatrix}$$

and

$$X=\begin{bmatrix}1 & 10\\1 & 5\end{bmatrix}$$

Then

$$[p_1 \quad p_2]=P'X^{-1}=\begin{bmatrix}\frac{1}{1.05} & 6\end{bmatrix}\begin{bmatrix}-1 & 2\\0.2 & -0.2\end{bmatrix}=[0.2476 \quad 0.7048]$$

Hence, the risk-neutral probabilities are $\hat{\pi}_1\equiv p_1R_f=0.26$ and $\hat{\pi}_2\equiv p_2R_f=0.74$.

b. Suppose that the physical probabilities of the two states are $\pi_1=\pi_2=0.5$. What is the stochastic discount factor for the two states?

Answer: $m_1=p_1/\pi_1=0.495$. $m_2=p_2/\pi_2=1.410$.

5. Consider a one-period economy with two end-of-period states. An option contract pays 3 in state 1 and 0 in state 2 and has a current price of 1. A forward contract pays 3 in state 1 and -2 in state 2. What are the one-period risk-free return and the risk-neutral probabilities of the two states?

Answer: Let the payoff matrix X be

[1]I thank Michael Cliff of Virginia Tech for suggesting this example.

$$X = \begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix}$$

and let the price of the two securities be

$$P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The prices of the elementary securities of the two states are

$$\begin{aligned} p_s &= P'X^{-1}e_s, \quad s = 1,2 \\ &= [1 \quad 0] \begin{bmatrix} 1/3 & 1/2 \\ 0 & -1/2 \end{bmatrix} e_s \\ &= [1/3 \quad 1/2] e_s \end{aligned}$$

Therefore, the risk-free return is

$$\frac{1}{R_f} = p_1 + p_2 = 5/6$$

so $R_f = 6/5 = 1.20.$ The risk-neutral probabilities of the two states are $\hat{\pi}_s = p_s R_f$ or

$$\begin{aligned} \hat{\pi}_1 &= p_1 R_f = \frac{1}{3}\frac{6}{5} = \frac{6}{15} = \frac{2}{5} \\ \hat{\pi}_2 &= p_2 R_f = \frac{1}{2}\frac{6}{5} = \frac{6}{10} = \frac{3}{5} \end{aligned}$$

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (*Hint:* you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

Answer:

$$
\begin{aligned}
0 &= E[m(R_i - R_f)] \\
&= E[(a + bR_m)(R_i - R_f)] \\
&= aE[R_i] - aR_f + bE[R_m R_i] - bR_f E[R_m] \\
&= a(E[R_i] - R_f) + b(E[R_m]E[R_i] + \mathrm{cov}[R_m, R_i] - R_f E[R_m]) \\
&= (E[R_i] - R_f)(a + bE[R_m]) + bCov[R_m, R_i]
\end{aligned}
$$

so

$$
\begin{aligned}
E[R_i] - R_f &= \frac{-b\mathrm{cov}[R_m, R_i]}{a + bE[R_m]} \\
&= -\frac{\mathrm{cov}[R_m, R_i]}{\sigma_m^2} \frac{b\sigma_m^2}{a + bE[R_m]} \\
&= -\beta_i \frac{b\sigma_m^2}{a + bE[R_m]}
\end{aligned}
$$

so that

$$
\gamma = -\frac{b\sigma_m^2}{a + bE[R_m]}
$$

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

Answer: For the risk-free asset, we have

$$
\frac{1}{R_f} = E[a + bR_m]
$$

or

$$
a = \frac{1}{R_f} - bE[R_m]
$$

For the market portfolio, we have

$$
\begin{aligned}
1 = E[(a + bR_m)R_m] &= aE[R_m] + bE\left[R_m^2\right] \\
&= aE[R_m] + b\left(\sigma_m^2 + E[R_m]^2\right)
\end{aligned}
$$

Substituting for *a* from the risk-free asset equation gives

$$
\begin{aligned}
1 &= \left(\frac{1}{R_f} - bE[R_m] \right) E[R_m] + b\left(\sigma_m^2 + E[R_m]^2 \right) \\
&= \frac{E[R_m]}{R_f} + b\sigma_m^2
\end{aligned}
$$

or

$$
b = -\frac{E[R_m] - R_f}{R_f \sigma_m^2}
$$

so

$$a = \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f \sigma_m^2}$$

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

Answer:

$$\begin{aligned} a + bE[R_m] &= \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f) - E[R_m](E[R_m] - R_f)}{R_f \sigma_m^2} \\ &= \frac{1}{R_f} \end{aligned}$$

$$\begin{aligned} \gamma &= -\frac{b\sigma_m^2}{a + bE[R_m]} \\ &= \frac{E[R_m] - R_f}{R_f \sigma_m^2} \sigma_m^2 R_f \\ &= E[R_m] - R_f \end{aligned}$$

7. Consider a two-factor economy with multiple risky assets and a risk-free asset whose return is denoted R_f. The economy's first factor is the return on the market portfolio, R_m, and the second factor is the return on a zero-net-investment portfolio, R_z. In other words, one can interpret the second factor as the return on a portfolio that is long one asset and short another asset, where the long and short positions are equal in magnitude (e.g., $R_z = R_a - R_b$) and where R_a and R_b are the returns on the assets that are long and short, respectively. It is assumed that $\text{cov}(R_m, R_z) = 0$. The expected returns on all assets in the economy satisfy the APT relationship

$$E[R_i] = \lambda_0 + \beta_{im}\lambda_m + \beta_{iz}\lambda_z \tag{*}$$

where R_i is the return on an arbitrary asset i, $\beta_{im} = \text{cov}(R_i, R_m)/\sigma_m^2$, $\beta_{iz} = \text{cov}(R_i, R_z)/\sigma_z^2$, and λ_m and λ_z are the risk premiums for factors 1 and 2, respectively.

Now suppose you are given the stochastic discount factor for this economy, m, measured over the same time period as the above asset returns. It is given by

$$m = a + bR_m + cR_z \tag{**}$$

where a, b, and c are known constants. Given knowledge of this stochastic discount factor in equation (**), show how you can solve for λ_0, λ_m, and λ_z in equation (*) in terms of a, b, c, σ_m, and σ_z. Just write down the conditions that would allow you to solve for the λ_0, λ_m, and λ_z. You need not derive explicit solutions for the λ's since the conditions are nonlinear and may be tedious to manipulate.

Answer: We know from APT that $\lambda_0 = R_f$, $\lambda_m = E[R_m] - R_f$, and $\lambda_z = E[R_z] = E[R_a - R_b]$. Therefore, we can determine these three parameters if we can derive conditions for R_f, $E[R_m]$, and $E[R_z]$ in terms of the other parameters. First, we know that

$$\begin{aligned} \frac{1}{R_f} &= E[m] \\ &= E[a + bR_m + cR_z] \\ &= a + bE[R_m] + cE[R_z] \end{aligned} \tag{1}$$

Second, we can use the condition

$$\begin{aligned} 1 &= E[mR_m] \\ &= E[(a + bR_m + cR_z)R_m] \\ &= aE[R_m] + bE\left[R_m^2\right] + cE[R_z R_m] \\ &= aE[R_m] + b\left(\sigma_m^2 + E[R_m]^2\right) + cE[R_z]E[R_m] \end{aligned} \tag{2}$$

Lastly, we know that for the difference between to risky asset returns, $R_z = R_a - R_b$,

$$\begin{aligned} 0 &= E[mR_z] \\ &= E[(a + bR_m + cR_z)R_z] \\ &= aE[R_z] + bE[R_z R_m] + cE\left[R_z^2\right] \\ &= aE[R_z] + bE[R_z]E[R_m] + c\left(\sigma_z^2 + E[R_z]^2\right) \end{aligned} \tag{3}$$

Equations (1), (2), and (3) are three equations in the three unknowns R_f, $E[R_m]$, and $E[R_z]$. They can be solved in terms of a, b, c, σ_m, and σ_z. Then, according to APT, these solutions R_f^*, $E[R_m]^*$, and $E[R_z]^*$ determine $\lambda_0 = R_f^*$, $\lambda_m = E[R_m]^* - R_f^*$, and $\lambda_z = E[R_z]^*$.

Answers to Chapter 5 Exercises

1. Consider the following consumption and portfolio choice problem. Assume that $U(C_t, t) = \delta^t[aC_t - bC_t^2]$, $B(W_T, T) = 0$, and $y_t \neq 0$, where $\delta = \frac{1}{1+\rho}$ and $\rho \geq 0$ is the individual's subjective rate of time preference. Further, assume that $n = 0$ so that there are no risky assets but there is a single-period riskless asset yielding a return of $R_{ft} = 1/\delta$ that is constant each period (equivalently, the risk-free interest rate $r_f = \rho$). Note that in this problem labor income is stochastic and there is only one (riskless) asset for the individual consumer-investor to hold. Hence, the individual has no portfolio choice decision but must decide only what to consume each period. In solving this problem, assume that the individual's optimal level of consumption remains below the "bliss point" of the quadratic utility function, that is, $C_t^* < \frac{1}{2}a/b, \forall t$.

   a. Write down the individual's wealth accumulation equation from period t to period $t+1$.

   Answer:

$$W_{t+1} = R_{ft}[W_t + y_t - C_t] = \frac{1}{\delta}[W_t + y_t - C_t] \tag{1}$$

   b. Solve for the individual's optimal level of consumption at date $T-1$ and evaluate $J(W_{T-1}, T-1)$. *Hint:* this is trivial.

   Answer: We assume throughout that the individual is below the maximal "bliss point" of the quadratic utility function, that is, $C^* < \frac{1}{2}a/b$. The solution at date $T-1$ is trivial. Since there is no bequest function, in the last period the individual consumes income and remaining wealth, $C^* = W_{T-1} + y_{T-1}$. Since we define

$$\begin{aligned} J(W_t, t) &\equiv \max E_t\left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T)\right] \\ &= \max E_t\left[\sum_{s=t}^{T-1} \delta^s\left[aC_s - bC_s^2\right]\right] \end{aligned} \tag{2}$$

   Thus,

$$\begin{aligned} J(W_{T-1}, T-1) &= \delta^{T-1}\left[aC_{T-1}^* - bC_{T-1}^{*2}\right] \\ &= \delta^{T-1}[a(W_{T-1} + y_{T-1}) - b(W_{T-1} + y_{T-1})^2] \end{aligned} \tag{3}$$

   c. Continue to solve the individual's problem at date $T-2$, $T-3$, and so on—and notice the pattern that emerges. From these results, solve for the individual's optimal level of consumption for any arbitrary date, $T-t$, in terms of the individual's expected future levels of income.

   Answer: Using

$$J(W_t, t) = \max_{C_t}[U(C_t, t) + E_t[J(W_{t+1}, t+1)]] \tag{4}$$

at date $T-2$ we have

$$J(W_{T-2}, T-2) = \max_{C_{T-2}} \delta^{T-2}\left[aC_{T-2} - bC_{T-2}\right]^2 \\ + E_{T-2}[\delta^{T-1}[a(W_{T-1}+y_{T-1}) - b(W_{T-1}+y_{T-1})^2]] \tag{5}$$

Substituting in $W_{T-1} = \frac{1}{\delta}[W_{T-2} + y_{T-2} - C_{T-2}]$ in (5) and taking the first order condition with respect to C_{T-2}, we have

$$\delta^{T-2}[a - 2bC_{T-2}] + \delta^{T-1}\left[-\frac{1}{\delta}a + \frac{2b}{\delta}\left[\frac{1}{\delta}(W_{T-2} + y_{T-2} - C_{T-2}) + E_{T-2}[y_{T-1}]\right]\right] = 0 \tag{6}$$

Thus,

$$C^*_{T-2} = \frac{1}{1+\delta}[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]] \tag{7}$$

and therefore from (5)

$$\begin{aligned} J(W_{T-2}, T-2) &= \delta^{T-2}\left[aC^*_{T-2} - bC^{*\,2}_{T-2}\right] \\ &+ \delta^{T-1}E_{T-2}\left[a\left(\frac{1}{\delta}\left[W_{T-2} + y_{T-2} - C^*_{T-2}\right] + y_{T-1}\right) - b\left(\frac{1}{\delta}\left[W_{T-2} + y_{T-2} - C^*_{T-2}\right] + y_{T-1}\right)^2\right] \\ &= \delta^{T-2}\left\{\frac{a}{1+\delta}\left[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]\right] - \frac{b}{(1+\delta)^2}\left[W_{T-2} + y_{T-2} + \delta E_{T-2}\left[y_{T-1}\right]\right]^2\right\} \\ &+ \delta^{T-1}E_{T-2}\left[\frac{a}{1+\delta}(W_{T-2} + y_{T-2} + \delta y_{T-1}) - \frac{b}{\delta(1+\delta)^2}(W_{T-2} + y_{T-2} + \delta y_{T-1})^2\right] \\ &= \delta^{T-2}\left[a[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]] - \frac{b}{\delta(1+\delta)}\left[W_{T-2} + y_{T-2} + \delta E_{T-2}[y_{T-1}]\right]^2\right] \end{aligned} \tag{8}$$

Next, considering the problem at $T-3$, we have

$$J(W_{T-3}, T-3) = \max_{C_{T-3}} \delta^{T-3}\left[aC_{T-3} - bC_{T-3}\right]^2 + E_{T-3}[J(W_{T-2}, T-2)] \tag{9}$$

Substituting (8) into (9) and then substituting (7) for $W_{T-2} = \frac{1}{\delta}[W_{T-2} + y_{T-3} - C_{T-3}]$, we can then take the first order condition with respect to C_{T-3}. This leads to

$$C^*_{T-3} = \frac{1}{1+\delta+\delta^2}(W_{T-3} + y_{T-3} + E_{T-3}[\delta y_{T-2} + \delta^2 y_{T-1}]) \tag{10}$$

By now, we can see the pattern for the optimal consumption rule:

$$C^*_{T-m} = \frac{1}{1+\delta+\cdots+\delta^{m-1}}\left(W_{T-m} + E_{T-m}\left[\sum_{t=0}^{m-1}\delta^t y_{T-m+t}\right]\right) \tag{11}$$

The rule is to consume a proportion of current "permanent income" where m is the number of years remaining in one's lifetime.

2. Consider the consumption and portfolio choice problem with power utility $U(C_t,t) \equiv \delta^t C_t^\gamma/\gamma$ and a power bequest function $B(W_T,T) \equiv \delta^T W_T^\gamma/\gamma$. Assume there is no wage income ($y_t \equiv 0 \ \forall \ t$) and a constant risk-free return equal to $R_{ft} = R_f$. Also, assume that $n=1$ and the return of the single risky asset, R_{rt}, is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

   a. Derive the first-order conditions for the optimal consumption level and portfolio weight at date $T-1$, C_{T-1}^* and ω_{T-1}^*, and give an explicit expression for C_{T-1}^*.

   Answer: Letting $S_t \equiv W_t - C_t$ and $R_t \equiv R_f + \omega_t^*(R_{rt} - R_f)$

$$\begin{aligned} U_C &= E_{T-1}[B_W R_{T-1}] \\ \delta^{T-1}C_{T-1}^{\gamma-1} &= E_{T-1}\left[\delta^T W_T^{\gamma-1} R_{T-1}\right] = E_{T-1}[\delta^T (S_{T-1}R_{T-1})^{\gamma-1} R_{T-1}] \\ &= \delta^T E_{T-1}\left[R_{T-1}^\gamma\right](W_{T-1} - C_{T-1})^{\gamma-1} \end{aligned} \tag{1}$$

or

$$\begin{aligned} C_{T-1} &= \frac{\left(\delta E_{T-1}\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}}}{1+\left(\delta E_{T-1}\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}}} W_{T-1} \\ &= \frac{a_1}{1+a_1} W_{T-1} = c_1 W_{T-1} \end{aligned} \tag{2}$$

where $c_1 = a_1/(1+a_1)$ and $a_1 \equiv \left(\delta E_{T-1}\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}} = \left(\delta E\left[R_{T-1}^\gamma\right]\right)^{\frac{1}{\gamma-1}}$ where $E[\cdot]$ is the unconditional expectations operator. The unconditional expectation is the same as the conditional one because the distribution of asset returns is assumed to be independent and identically distributed. The first order condition with respect to the portfolio weight implies

$$\begin{aligned} E_{T-1}[B_W R_{r,T-1}] &= R_f E_{T-1}[B_W] \\ \delta^T E_{T-1}\left[\left(S_{T-1}R_{T-1}\right)^{\gamma-1} R_{r,T-1}\right] &= \delta^T R_{f,T-1} E_{T-1}\left[(S_{T-1}R_{T-1})^{\gamma-1}\right] \\ E_{T-1}\left[R_{T-1}^{\gamma-1} R_{r,T-1}\right] &= R_{f,T-1} E_{T-1}\left[R_{T-1}^{\gamma-1}\right] \\ E\left[R_{T-1}^{\gamma-1} R_{r,T-1}\right] &= R_{f,T-1} E\left[R_{T-1}^{\gamma-1}\right] \end{aligned} \tag{3}$$

Equation (3) shows that the optimal portfolio weight is determined independently from the level of wealth or consumption. It depends only on the distribution of returns on the risky asset. Since this distribution is independent and identically distributed, we can again replace the conditional with the unconditional expectations operator. Hence, all individuals with the same constant coefficient of relative risk aversion, γ, chose the same portfolio proportions for the risky and risk-free asset.

b. Solve for the form of $J(W_{T-1},T-1)$.

Answer:

$$\begin{aligned} J(W_{T-1},T-1) &= \delta^{T-1}C_{T-1}^{*\gamma}/\gamma + \delta^T E_{T-1}\left[\left(R_{T-1}^*\left(W_{T-1}-C_{T-1}^*\right)\right)^\gamma/\gamma\right] \\ &= \delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^\gamma W_{T-1}^\gamma/\gamma + \delta^T E_{T-1}\left[R_{T-1}^{*\gamma}\frac{W_{T-1}^\gamma}{\gamma(1+a_1)^\gamma}\right] \\ &= \delta^{T-1}\frac{W_{T-1}^\gamma}{\gamma(1+a_1)^\gamma}\left(a_1^\gamma + \delta E_{T-1}\left[R_{T-1}^{*\gamma}\right]\right) \\ &= \delta^{T-1}b_1 W_{T-1}^\gamma/\gamma \end{aligned} \tag{4}$$

where $b_1 \equiv \left(a_1^\gamma + \delta E\left[R_{T-1}^{*\gamma}\right]\right)/\left(1+a_1\right)^\gamma = \left[a_1/\left(1+a_1\right)\right]^{\gamma-1}$.

c. Derive the first-order conditions for the optimal consumption level and portfolio weight at date $T-2$, C_{T-2}^* and ω_{T-2}^*, and give an explicit expression for C_{T-2}^*.

Answer: The optimality condition for consumption is

$$\begin{aligned} U_C(C_{T-2}^*,T-2) &= E_{T-2}[J_W(W_{T-1},T-1)R_{T-2}] \\ C_{T-2}^{\gamma-1} &= \delta^{T-1}E_{T-2}\left[b_1 W_{T-1}^{\gamma-1}R_{T-2}\right] \\ C_{T-2}^{\gamma-1} &= \delta E_{T-2}\left[b_1\left(S_{T-2}R_{T-2}\right)^{\gamma-1}R_{T-2}\right] \\ &= \delta b_1 E_{T-2}\left[R_{T-2}^\gamma\right]\left(W_{T-2}-C_{T-2}\right)^{\gamma-1} \end{aligned} \tag{5}$$

or

$$\begin{aligned} C_{T-2}^* &= \frac{\left(\delta b_1 E_{T-2}\left[R_{T-2}^\gamma\right]\right)^{\frac{1}{\gamma-1}}}{1+\left(\delta b_1 E_{T-1}\left[R_{T-2}^\gamma\right]\right)^{\frac{1}{\gamma-1}}}W_{T-2} \\ &= \frac{a_2}{1+a_2}W_{T-2} \\ &= c_2 W_{T-2} \end{aligned} \tag{6}$$

where $c_2 = a_2/(1+a_2)$ and $a_2 \equiv \left(b_1\delta E\left[R_{T-2}^\gamma\right]\right)^{\frac{1}{\gamma-1}} = a_1^2/(1+a_1) = a_1c_1$. The optimality condition for the portfolio weight ω_{T-2}^* turns out to be of the same form as at $T-1$:

$$\begin{aligned} E_{T-2}[J_W R_{r,T-2}] &= R_f E_{T-2}[J_W] \\ E_{T-2}[b_1(S_{T-2}R_{T-2})^{\gamma-1}R_{r,T-2}] &= R_f E_{T-1}[b_1(S_{T-2}R_{T-2})^{\gamma-1}] \\ E_{T-2}\left[b_1 R_{T-2}^{\gamma-1}R_{r,T-2}\right] &= R_f E_{T-2}\left[b_1 R_{T-2}^{\gamma-1}\right] \\ E\left[R_{T-2}^{\gamma-1}R_{r,T-2}\right] &= R_f E\left[R_{T-2}^{\gamma-1}\right] \end{aligned} \tag{7}$$

Hence the portfolio weight ω_{T-2}^* is the same as ω_{T-1}^* and does not depend on wealth or consumption, just the distribution of risky asset returns.

d. Solve for the form of $J(W_{T-2}, T-2)$. Based on the pattern for $T-1$ and $T-2$, provide expressions for the optimal consumption and portfolio weight at any date $T-t$, $t = 1, 2, 3, \ldots$.

Answer:

$$\begin{aligned}
J(W_{T-2}, T-2) &= U(C^*_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)] \\
&= \delta^{T-2} C^{*\gamma}_{T-2}/\gamma + E_{T-1}\left[\delta^{T-1} b_1 W^{\gamma}_{T-1}/\gamma\right] \\
&= \delta^{T-2}\left(\frac{a_2}{1+a_2}\right)^{\gamma} W^{\gamma}_{T-2}/\gamma + \delta^{T-1} E_{T-1}\left[b_1\left(R^*_{T-2}\left(W_{T-2} - C^*_{T-2}\right)\right)^{\gamma}/\gamma\right] \\
&= \delta^{T-2}\left(\frac{a_2}{1+a_2}\right)^{\gamma} W^{\gamma}_{T-2}/\gamma + \delta^{T-1} E_{T-1}\left[b_1 R^{*\gamma}_{T-2} \frac{W^{\gamma}_{T-2}}{\gamma(1+a_2)^{\gamma}}\right] \\
&= \delta^{T-2} \frac{W^{\gamma}_{T-2}}{\gamma(1+a_2)^{\gamma}}\left(a_2^{\gamma} + \delta E_{T-2}\left[b_1 R^{*\gamma}_{T-2}\right]\right) \\
&= \delta^{T-1} b_2 W^{\gamma}_{T-2}/\gamma
\end{aligned} \tag{8}$$

where $b_2 \equiv \left(a_2^{\gamma} + \delta E\left[R^{*\gamma}_{T-2}\right]\right)/(1+a_2)^{\gamma}$.

The optimal portfolio weight is the same each period and satisfies

$$E\left[R^{\gamma-1}_{T-2} R_{r,T-2}\right] = R_f E\left[R^{\gamma-1}_{T-2}\right] \tag{9}$$

The level of consumption at $T-1$ is $C^*_{T-2} = c_2 W_{T-2}$ where $c_2 = a_1 c_1/(1+a_1 c_1)$. Hence, $C^*_{T-3} = c_3 W_{T-3}$ where $c_3 = a_1 c_2/(1+a_1 c_2)$. In general, $C^*_{T-t} = c_t W_{T-t}$ where $c_t = a_1 c_{t-1}/(1+a_1 c_{t-1})$.

3. Consider the multiperiod consumption and portfolio choice problem

$$\max_{C_s, \omega_s \forall s} E_t\left[\sum_{s=t}^{T-1} U(C_s, s) + B(W_T, T)\right]$$

Assume negative exponential utility $U(C_s, s) \equiv -\delta^s e^{-bC_s}$ and a bequest function $B(W_T, T) \equiv -\delta^T e^{-bW_T}$ where $\delta = e^{-\rho}$ and $\rho > 0$ is the (continuously compounded) rate of time preference. Assume there is no wage income ($y_s \equiv 0 \ \forall\ s$) and a constant risk-free return equal to $R_{fs} = R_f$. Also, assume that $n = 1$ and the return of the single risky asset, R_{rs}, has an identical and independent normal distribution of $N(\overline{R}, \sigma^2)$ each period. Denote the proportion of wealth invested in the risky asset at date s as ω_s.

a. Derive the optimal portfolio weight at date $T-1$, ω_{T-1}^*. *Hint:* it might be easiest to evaluate expectations in the objective function prior to taking the first-order condition.

Answer: The individual's problem at date $T-1$ is $R_t \equiv R_f + \omega_t^*(R_{rt} - R_f)$

$$
\begin{aligned}
&\max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - E_{T-1}[\delta^T e^{-bW_T}] \\
&= \max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - E_{T-1}\left[\delta^T e^{-b(W_{T-1}-C_{T-1})[R_f+\omega_{T-1}(R_{r,T-1}-R_f)]}\right] \\
&= \max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - \delta^T e^{-b(W_{T-1}-C_{T-1})\left[R_f+\omega_{T-1}\left(\overline{R}-R_f\right)\right]+\frac{1}{2}b^2\left(W_{T-1}-C_{T-1}\right)^2\omega_{T-1}^2\sigma^2}
\end{aligned}
$$

Maximizing this with respect to ω_{T-1} is the same as maximizing

$$\max_{\omega_{T-1}} b(W_{T-1}-C_{T-1})[R_f+\omega_{T-1}(\overline{R}-R_f)] - \frac{1}{2}b^2(W_{T-1}-C_{T-1})^2\omega_{T-1}^2\sigma^2$$

Taking the derivative with respect to ω_{T-1} we have

$$b(W_{T-1}-C_{T-1})(\overline{R}-R_f) - b^2(W_{T-1}-C_{T-1})^2\omega_{T-1}\sigma^2 = 0$$

or

$$\omega_{T-1}^* = \frac{(\overline{R}-R_f)}{(W_{T-1}-C_{T-1})b\sigma^2}$$

b. Solve for the optimal level of consumption at date $T-1$, C_{T-1}^*. C_{T-1}^* will be a function of W_{T-1}, b, ρ, R_f, $\overline{R}$, and σ^2.

Answer:

$$\max_{C_{T-1},\omega_{T-1}} -\delta^{T-1}e^{-bC_{T-1}} - \delta^T e^{-b\left(W_{T-1}-C_{T-1}\right)\left[R_f+\omega_{T-1}\left(\overline{R}-R_f\right)\right]+\frac{1}{2}b^2\left(W_{T-1}-C_{T-1}\right)^2\omega_{T-1}^2\sigma^2}$$

Taking the derivative of this expression with respect to C_{T-1} gives

$$
\begin{aligned}
0 = {}& \delta^{T-1}be^{-bC_{T-1}} - \delta^T\{b[R_f+\omega_{T-1}(\overline{R}-R_f)] - b^2(W_{T-1}-C_{T-1})\omega_{T-1}^2\sigma^2\} \\
&\times e^{-b\left(W_{T-1}-C_{T-1}\right)\left[R_f+\omega_{T-1}\left(\overline{R}-R_f\right)\right]+\frac{1}{2}b^2\left(W_{T-1}-C_{T-1}\right)^2\omega_{T-1}^2\sigma^2}
\end{aligned}
$$

substituting in for ω_{T-1}^* and simplifying leads to

$$
\begin{aligned}
0 = {}& e^{-bC_{T-1}} - \delta\left\{\left[R_f + \frac{(\overline{R}-R_f)^2}{(W_{T-1}-C_{T-1})b\sigma^2}\right] - \frac{(\overline{R}-R_f)^2}{(W_{T-1}-C_{T-1})b\sigma^2}\right\} \\
&\times e^{-b\left(W_{T-1}-C_{T-1}\right)\left[R_f+\frac{\left(\overline{R}-R_f\right)^2}{(W_{T-1}-C_{T-1})b\sigma^2}\right]+\frac{1}{2}\frac{\left(\overline{R}-R_f\right)^2}{\sigma^2}} \\
0 = {}& e^{-bC_{T-1}} - \delta R_f e^{-b\left(W_{T-1}-C_{T-1}\right)R_f-\frac{1}{2}\frac{\left(\overline{R}-R_f\right)^2}{\sigma^2}}
\end{aligned}
$$

This implies

$$e^{-bC_{T-1}} = e^{\ln\delta + \ln R_f - b(W_{T-1}-C_{T-1})R_f - \frac{1}{2}\frac{(\bar{R}-R_f)^2}{\sigma^2}}$$

or

$$-bC_{T-1} = -\rho + \ln R_f - b\left(W_{T-1} - C_{T-1}\right)R_f - \frac{1}{2}\frac{(\bar{R}-R_f)^2}{\sigma^2}$$

Solving for C_{T-1}, we obtain

$$C^*_{T-1} = \frac{R_f}{1+R_f}W_{T-1} + H$$

where $H \equiv \left[\rho - \ln R_f + \frac{1}{2}(\bar{R}-R_f)^2/\sigma^2\right]/[b(1+R_f)]$.

c. Solve for the indirect utility function of wealth at date $T-1$, $J(W_{T-1}, T-1)$.

Answer:

$$\begin{aligned}
J\left(W_{T-1}, T-1\right) &= -\delta^{T-1}e^{-bC^*_{T-1}} - \delta^T e^{-b\left(W_{T-1}-C^*_{T-1}\right)\left[R_f + \omega^*_{T-1}\left(\bar{R}-R_f\right)\right] + \frac{1}{2}b^2\left(W_{T-1}-C^*_{T-1}\right)^2\omega^{*2}_{T-1}\sigma^2} \\
&= -\delta^{T-1}e^{-b\frac{R_f}{1+R_f}W_{T-1} - bH} - \delta^T e^{-b\left[\frac{1}{1+R_f}W_{T-1} - H\right]R_f - \frac{\left(\bar{R}-R_f\right)^2}{\sigma^2} + \frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}} \\
&= -\delta^{T-1}e^{-b\frac{R_f}{1+R_f}W_{T-1} - bH} - \delta^{T-1}e^{-\rho - b\frac{R_f}{1+R_f}W_{T-1} + bR_f H - \frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}} \\
&= -\delta^{T-1}e^{-b\frac{R_f}{1+R_f}W_{T-1}}\left(e^{-bH} + e^{-\rho + bR_f H - \frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}}\right) \\
&= -\delta^{T-1}Ge^{-b\frac{R_f}{1+R_f}W_{T-1}}
\end{aligned}$$

where $G \equiv \left(e^{-bH} + e^{-\rho + bR_f H - \frac{1}{2}\frac{\left(\bar{R}-R_f\right)^2}{\sigma^2}}\right)$ is independent of W_{T-1}.

d. Derive the optimal portfolio weight at date $T-2$, ω^*_{T-2}.

Answer:

$$\begin{aligned}
&\max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}} - E_{T-1}\left[\delta^{T-1}Ge^{-b\frac{R_f}{1+R_f}W_{T-1}}\right] \\
&= \max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}} - E_{T-1}\left[\delta^{T-1}Ge^{-b\frac{R_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f + \omega_{T-2}\left(R_{r,T-2}-R_f\right)\right]}\right] \\
&= \max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}} \\
&\quad -\delta^{T-1}Ge^{-\frac{bR_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f + \omega_{T-2}\left(\bar{R}-R_f\right)\right] + \frac{1}{2}\left(\frac{bR_f}{1+R_f}\right)^2\left(W_{T-2}-C_{T-2}\right)^2\omega^2_{T-2}\sigma^2}
\end{aligned}$$

Maximizing this with respect to ω_{T-1} is the same as maximizing

$$\max_{\omega_{T-1}} \frac{bR_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f+\omega_{T-2}\left(\overline{R}-R_f\right)\right]-\frac{1}{2}\left(\frac{bR_f}{1+R_f}\right)^2\left(W_{T-2}-C_{T-2}\right)^2\omega_{T-2}^2\sigma^2$$

Taking the derivative with respect to ω_{T-2} we have

$$\frac{bR_f}{1+R_f}(W_{T-2}-C_{T-2})(\overline{R}-R_f)-\left(\frac{bR_f}{1+R_f}\right)^2(W_{T-2}-C_{T-2})^2\omega_{T-2}\sigma^2=0$$

or

$$\omega_{T-2}^*=\frac{(\overline{R}-R_f)(1+R_f)}{(W_{T-2}-C_{T-2})bR_f\sigma^2}$$

e. Solve for the optimal level of consumption at date $T-2$, C_{T-2}^*.

Answer:

$$\max_{C_{T-2},\omega_{T-2}} -\delta^{T-2}e^{-bC_{T-2}}-\delta^{T-1}Ge^{-\frac{bR_f}{1+R_f}\left(W_{T-2}-C_{T-2}\right)\left[R_f+\omega_{T-2}\left(\overline{R}-R_f\right)\right]+\frac{1}{2}\left(\frac{bR_f}{1+R_f}\right)^2\left(W_{T-2}-C_{T-2}\right)^2\omega_{T-2}^2\sigma^2}$$

Taking the derivative of this expression with respect to C_{T-2} and simplifying gives

$$e^{-bC_{T-2}}-\delta G\frac{R_f^2}{1+R_f}e^{-\frac{bR_f}{1+R_f}(W_{T-2}-C_{T-2})R_f-\frac{1}{2}\frac{(\overline{R}-R_f)^2}{\sigma^2}}=0$$

This implies

$$e^{-bC_{T-2}}=e^{\ln\frac{\delta GR_f^2}{1+R_f}-\frac{bR_f^2}{1+R_f}(W_{T-2}-C_{T-2})-\frac{1}{2}\frac{\left(\overline{R}-R_f\right)^2}{\sigma^2}}$$

or

$$-bC_{T-2}=\ln\frac{\delta GR_f^2}{1+R_f}-\frac{bR_f^2}{1+R_f}(W_{T-2}-C_{T-2})-\frac{1}{2}\frac{(\overline{R}-R_f)^2}{\sigma^2}$$

Solving for C_{T-2}, we obtain

$$C_{T-2}^*=\frac{R_f^2}{1+R_f+R_f^2}W_{T-2}+K$$

where $K\equiv\left[-\ln\frac{\delta GR_f^2}{1+R_f}+\frac{1}{2}\left(\overline{R}-R_f\right)^2/\sigma^2\right]\left(1+R_f\right)/\left[b\left(1+R_f+R_f^2\right)\right]$.

4. An individual faces the following consumption and portfolio choice problem:

$$\max_{C_t,\omega_t \forall t} E_0\left[\sum_{t=0}^{T-1}\delta^t \ln[C_t] + \delta^T \ln[W_T]\right]$$

where each period the individual can choose between a risk-free asses paying a time-varing return of R_{ft} over the period from t to $t+1$ and a single risky asset. The individual receives no wage income. The risky asset's return over the period from t to $t+1$ is given by

$$R_{rt} = \begin{cases}(1+u_t)R_{ft} \text{ with probability } \frac{1}{2} \\ (1+d_t)R_{ft} \text{ with probability } \frac{1}{2}\end{cases}$$

where $u_t > 0$ and $-1 < d_t < 0$. Let ω_t be the individual's proportion of wealth invested in the risky asset at date t. Solve for the individual's optimal portfolio weight ω_t^* for $t = 0, \ldots, T-1$.

Answer: With log utility, the individual's portfolio choice is myopic. It is the same for a multi-period problem as it would be for a one-period problem. As shown in Chapter 5, the first order conditions for date t are

$$1 = R_{ft} E_t\left[\frac{1}{R_{ft} + \omega_t(R_{rt} - R_{ft})}\right]$$

or

$$\begin{aligned}\frac{2}{R_{ft}} &= \frac{1}{R_{ft} + \omega_t u_t R_{ft}} + \frac{1}{R_{ft} + \omega_t d_t R_{ft}} \\ &= \frac{1}{R_{ft}[1+\omega_t u_t]} + \frac{1}{R_{ft}[1+\omega_t d_t]}\end{aligned}$$

Therefore

$$\begin{aligned}2 &= \frac{2+\omega_t(u_t+d_t)}{[1+\omega_t u_t][1+\omega_t d_t]} \\ 2(1+\omega_t(u_t+d_t)+\omega_t^2 u_t d_t) &= 2+\omega_t(u_t+d_t) \\ 2\omega_t(u_t+d_t)+2\omega_t^2 u_t d_t &= \omega_t(u_t+d_t) \\ 2\omega_t^2 u_t d_t &= -\omega_t(u_t+d_t) \\ \omega_t &= -\frac{u_t+d_t}{2u_t d_t}\end{aligned}$$

Answers to Chapter 6 Exercises

1. Two individuals agree at date 0 to a forward contract that matures at date $2.$ The contract is written on an underlying asset that pays a dividend at date 1 equal to $D_1.$ Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2,$ and let $E_0[\cdot]$ be the expectations operator at date $0.$ What is the value of $E_0[m_{02} f_2]$? Explain your answer.

   Answer: Let S_i be the price of the underlying asset at date i and let D_0 be the date 0 present value of dividends that it pays between dates 0 and $1.$ Pricing using a stochastic discount factor implies $S_0 = E_0[m_{01} D_1] + E_0[m_{02} S_2] \ = D_0 + E_0[m_{02} S_2]$ where D_0 is the date 0 present value of dividends. If we let F_{02} be the forward price, then we know that the payoff to the long party is $f_2 = S_2 - F_{02}.$ This long forward position represents ownership in a share of the underlying asset, a short position (selling) the underlying asset's dividends, and borrowing an amount such that the repayment at date 2 equals $F_{02}.$ Using the stochastic discount factor approach to pricing, we know that $E_0[m_{02} f_2] = E_0[m_{02}(S_2 - F_{02})] = E[m_{02} S_2] - E[m_{02} F_{02}].$ Now note that $S_0 = E_0[m_{01} D_1] + E_0[m_{02} S_2] = D_0 + E_0[m_{02} S_2]$ where D_0 is the date 0 present value of dividends. Also note that $E[m_{02} F_{02}] = E[m_{02}] F_{02} = R_f^{-2} F_{02}.$ Thus we have

$$
\begin{aligned}
E_0[m_{02} f_2] &= E[m_{02} S_2] - E[m_{02} F_{02}] \\
&= S_0 - D_0 - R_f^{-2} F_{02}
\end{aligned}
$$

   However, we know that the absence of arbitrage implies that the forward price satisfies $F_{02} = R_f^2 (S_0 - D_0),$ which implies that $E_0[m_{02} f_2] = 0.$

2. Assume that there is an economy populated by infinitely lived representative individuals who maximize the lifetime utility function

$$
E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-a c_t}\right]
$$

   where c_t is consumption at date t and $a > 0,\ 0 < \delta < 1.$ The economy is a Lucas endowment economy (Lucas 1978) having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

Answer: A result of the Lucas (1978) model is that the price of a risky asset, P_0, satisfies

$$P_0 = E_0\left[\sum_{t=1}^{\infty}\frac{U_c(c_t,t)}{U_c(c_0,0)}d_t\right]$$

In this problem $U(c_t,t) = -\delta^t e^{-ac_t}$, so that $U_c(c_t,t) = a\delta^t e^{-ac_t}$. Also, because this is an endowment economy with one share per individual, we have $c_t = d_t$. Thus,

$$P_0 = E_0\left[\sum_{t=1}^{\infty}\frac{U_c(c_t,t)}{U_c(c_0,0)}d_t\right] = E_0\left[\sum_{t=1}^{\infty}\delta^t e^{-a(d_t-d_0)}d_t\right]$$

3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

Answer:

$$P_t = E_t\left[\sum_{t=1}^{\infty}\delta^j\left(\frac{C_{t+j}^*}{C_t^*}\right)^{\gamma-1}d_{t+j}\right]$$

or

$$\begin{aligned}P_t/d_t &= E_t\left[\sum_{j=1}^{\infty}\delta^j\left(\frac{C_{t+j}^*}{C_t^*}\right)^{\gamma-1}\left(\frac{d_{t+j}}{d_t}\right)\right]\\ &= E_t\left[\sum_{j=1}^{\infty}\delta^j e^{(\gamma-1)\ln(C_{t+j}/C_t)+\ln(d_{t+j}/d_t)}\right]\end{aligned}$$

Now

$$\begin{aligned}\ln\left(C_{t+j}/C_t\right) &= j\cdot\mu_c + \sigma_c\textstyle\sum_{i=1}^{j}\eta_{t+j}\\ \ln\left(D_{t+j}/D_t\right) &= j\cdot\mu_d + \sigma\textstyle\sum_{i=1}^{j}\varepsilon_{t+j}\end{aligned}$$

so that

$$\begin{aligned}P_t/d_t &= E_t\left[\sum_{j=1}^{\infty}\delta^j e^{(\gamma-1)\left(j\cdot\mu_c+\sigma_c\sum_{i=1}^{j}\eta_{t+j}\right)+j\cdot\mu_d+\sigma_d\sum_{i=1}^{j}\varepsilon_{t+j}}\right]\\ &= E_t\left[\sum_{j=1}^{\infty}\delta^j e^{j\left[(\gamma-1)\mu_c+\mu_c\right]+\sum_{i=1}^{j}\left[(\gamma-1)\sigma_c\eta_{t+j}+\sigma_d\varepsilon_{t+j}\right]}\right]\\ &= \sum_{j=1}^{\infty}\delta^j e^{j\left[(\gamma-1)\mu_c+g_d\right]}e^{\frac{j}{2}[(1-\gamma)^2\sigma_c^2+\sigma_d^2-2(1-\gamma)\sigma_c\sigma_d\rho]}\\ &= \sum_{j=1}^{\infty}e^{j[\ln\delta-(1-\gamma)\mu_c+\mu_d+\frac{1}{2}((1-\gamma)^2\sigma_c^2+\sigma_d^2)-(1-\gamma)\sigma_c\sigma_c\rho]}\\ &= \frac{1}{1-\delta e^{-(1-\gamma)\mu_c+\mu_d+\frac{1}{2}\left((1-\gamma)^2\sigma_c^2+\sigma_d^2\right)-(1-\gamma)\sigma_c\sigma_d\rho}}-1\end{aligned}$$

So

$$P_t = d_t \frac{\delta e^{\alpha}}{1-\delta e^{\alpha}}$$

where

$$\alpha \equiv \mu_d - (1-\gamma)\mu_c + \frac{1}{2}[(1-\gamma)^2\sigma_c^2 + \sigma_d^2] - (1-\gamma)\rho\sigma_c\sigma_d$$

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t that is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t\left[d_{t+i}\right]}{R_f^{\,i+1}}1 \tag{1}$$

and

$$b_{t+1} = \begin{cases} \dfrac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1-q_t \end{cases} \tag{2}$$

where $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

a. Show whether or not $p_t = f_t + b_t$, subject to the specifications in (1) and (2), is a valid solution for the price of the risky asset.

Answer: We need to check that (2) satisfies $E_t\left[b_{t+1}\right] = R_f b_t$. While q_t is a random variable as of date $t-1$, it is realized (known) as of date t. Thus,

$$E_t\left[b_{t+1}\right] = \frac{R_f}{q_t} b_t q_t + E_t\left[e_{t+1}\right] q_t + \left(1-q_t\right) E_t[z_{t+1}] = R_f b_t$$

so it is a valid solution.

b. Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{\text{solar}}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

Answer: Since $E_t\left[b_{t+1}\right] = R_f b_t$, we see that

$$\lim_{i\to\infty} E_t[b_{t+i}] = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases} \qquad (*)$$

For limited liability assets, such as oil, we cannot have a bubble path with a price becoming negative, so we need to consider only bubbles with $b_t > 0$. In this case, we see from the above equation (*) that for a bubble solution to exist, the bubble component must be expected to increase infinitely. But this cannot be a rational expectation if there is an upper bound on the price of oil, as would be the case if there was a perfect substitute in perfectly elastic supply. Thus, since p_t cannot rise above p_{solar}, b_t cannot rise above $p_{\text{solar}} - p_t^*$. Thus, a bubble path where b_t must be expected to increase to infinity cannot possibly occur.

c. Suppose p_t is the price of a bond that matures at date $T < \infty$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

Answer: For similar reasons, a rational speculative bubble cannot exist for the price of a bond. Since, at maturity, the bond's price must be $p_T = d_T$ and zero after date T, its price cannot rationally be expected to satisfy equation (*) and increase infinitely. Thus, a bubble path is invalid, and the only rational price is $p_t = p_t^*$.

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{Cs\{\omega_{is}\},\forall s,i} E_t\left[\sum_{s=t}^{T} \delta^s u(C_s)\right]$$

where T < ∞. Explain why a rational speculative asset price bubble could not exist in such an economy.

Answer: With the economy, and therefore assets (or asset markets), having a finite horizon, asset prices could not have the form $p_t = f_t + b_t$ with $b_t \neq 0$ because at date T, $p_T = f_T = d_T$ which is an asset's final dividend payment. Since $b_T = 0$ with certainty, then the bubble process $E_t[b_{t+1}] = \delta^{-1} b_t$ implies $E_{T-1}[b_T] = E_{T-1}[0] = \delta^{-1} b_{T-1}$, or $b_{T-1} = 0$. A similar argument implies $b_t = 0$ for all previous dates, $t < T-1$.

Answers to Chapter 7 Exercises

1. In light of this chapter's discussion of forward contracts on dividend-paying assets, reinterpret Chapter 3's example of a forward contract on a foreign currency. In particular, what are the "dividends" paid by a foreign currency?

   Answer: By purchasing a foreign currency at the initial dollar price S_0, one can invest in a foreign-currency denominated interest-bearing instrument, such as a foreign bond or a foreign-currency denominated bank (certificate of) deposit. The amount of foreign-currency denominated interest per unit of foreign currency that accrues over a time interval of τ can be expressed as $R_f^{*\tau} - 1$ where R_f^* is one plus the per-period foreign currency interest rate. The present value of this interest is $(R_f^{*\tau} - 1)/R_f^{*\tau}$ and its dollar value is $S_0(R_f^{*\tau} - 1)/R_f^{*\tau}$. Thus if we use the formula derived in this chapter, the equilibrium forward price on foreign currency is

$$
\begin{aligned}
F_{0\tau} &= (S_0 - D) = R_f^\tau \\
&= \left(S_0 - S_0 \frac{R_f^{*\tau} - 1}{R_f^{*\tau}} \right) R_f^\tau \\
& S_0 \frac{R_f^\tau}{R_f^{*\tau}}
\end{aligned}
$$

   which is the same formula derived in Chapter 3.

2. What is the lower bound for the price of a three-month European put option on a dividend-paying stock when the stock price is \$58, the strike price is \$65, the annualized, risk-free return is $R_f = e^{0.05}$, and the stock is to pay a \$3 dividend two months from now?

   Answer:

$$
\begin{aligned}
P_0 &\geq \max\left[0, XR_f^{-\tau} + D - S_0\right] \\
&\geq \max[0, 65e^{-\frac{0.05}{4}} + 3e^{-\frac{0.05}{6}} - 58] \\
&\geq \$9.17
\end{aligned}
$$

3. Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices X_1, X_2, and X_3, respectively, where $X_3 > X_2 > X_1$ and $X_3 - X_2 = X_2 - X_1$. All options are written on the same asset and have the same maturity. Show that

$$c_2 \leq \frac{1}{2}(c_1 + c_3)$$

   Hint: consider a portfolio that is long the option having a strike price of X_1, long the option having the strike price of X_3, and short two options having the strike price of X_2.

Answer: The portfolio mentioned in the hint has the following payoffs:

Stock Price	Portfolio Value
$S_\tau \le X_1$	0
$X_1 < S_\tau \le X_2$	$S_\tau - X_1 > 0$
$X_2 < S_\tau \le X_3$	$S_\tau - X_1 - 2(S_\tau - X_2) = X_2 - X_1 - (S_\tau - X_2) \ge 0$
$X_3 < S_\tau$	$S_\tau - X_1 - 2(S_\tau - X_2) + S_\tau - X_3 = X_2 - X_1 - (X_3 - X_2) = 0$

This portfolio has either a zero or positive payoff at expiration. Therefore, it must have a non-negative value, implying $c_1 + c_3 - 2c_2 \ge 0$, or $c_2 \le \frac{1}{2}(c_1 + c_3)$.

4. Consider the binomial (Cox-Ross-Rubinstein) option pricing model. The underlying stock pays no dividends and has the characteristic that $u = 2$ and $d = 1/2$. In other words, if the stock increases (*decreases*) over a period, its value doubles (*halves*). Also, assume that one plus the risk-free interest rate satisfies $R_f = 5/4$. Let there be two periods and three dates: 0, 1, and 2. At the initial date 0, the stock price is $S_0 = 4$. The following option is a type of *Asian* option referred to as an average price call. The option matures at date 2 and has a terminal value equal to

$$c_2 = \max\left[\frac{S_1 + S_2}{2} - 5, 0\right]$$

where S_1 and S_2 are the prices of the stock at dates 1 and 2, respectively. Solve for the no-arbitrage value of this call option at date 0, c_0.

Answer: The stock price tree is

Date 0	Date 1	Date 2		Date 0	Date 1	Date 2
		u^2S				16
	uS Z]				8 Z]	
S Z]		S	=	4 Z]		4
	dS Z]				2 Z]	
		d^2S				1

Since $p = \frac{R_f - d}{u - d} = \frac{1.25 - 0.5}{2 - 0.5} = 0.5$, the formula for the option price is $c = \frac{1}{R_f}[pc_u + (1-p)c_d] = \frac{4}{5}[0.5c_u + 0.5c_d] = \frac{2}{5}[c_u + c_d]$. Using this and the terminal payoff specified in the question, the tree for the call option is

Date 0	Date 1	Date 2
		7
	$\frac{16}{5}$ Z]	
		1
$\frac{32}{25}$ Z]		
		0
	0 Z]	
		0

5. Calculate the price of a three-month American put option on a non-dividend-paying stock when the stock price is \$60, the strike price is \$60, the annualized, risk-free return is $R_f = e^{0.10}$, and the annual standard deviation of the stock's rate of return is $\sigma = .45$, so that $u = 1/d = e^{\sigma\sqrt{\Delta\tau}} = e^{.45\sqrt{\Delta\tau}}$. Use a binomial tree with a time interval of one month.

Answer: $u = e^{.45/\sqrt{12}} = 1.1387$, so that $d = 1/u = 0.8782$. This allows us to calculate the stock price tree

Date 0	Date 1	Date 2	Date 3		Date 0	Date 1	Date 2	Date 3
			u^3S					88.59
		u^2S Z]					77.80 Z]	
	uS Z]		uS			68.33 Z]		68.33
							60 Z]	
S Z]		S Z]		=	60 Z]			
	dS Z]		dS			52.69 Z]		52.69
		d^2S Z]					46.27 Z]	
			d^3S					40.64

Since $R_f^{-\Delta t} = e^{-0.10/12} = 0.9917$ and $p = \dfrac{R_f^{\Delta t} - d}{u - d} = 0.4998$, working backwards from date 3 using $P_t = \max[X - S_t, R_f^{-\Delta t}(pP_{u,t+\Delta t} + (1-p)P_{d,t+\Delta t})]$, one obtains

Date 0	Date 1	Date 2	Date 3		Date 0	Date 1	Date 2	Date 3
			P_{uuu}					0.00
		P_{uu} Z]					0.00 Z]	
	P_u Z]		P_{uud}			1.80 Z]		0.00
P Z]		P_{ud} Z]		=	5.16 Z]		3.63 Z]	
	P_d Z]		P_{udd}			8.61 Z]		7.31
		P_{dd} Z]					13.73* Z]	
			P_{ddd}					19.36

6. Let the current date be t and let $T > t$ be a future date, where $\tau \equiv T - t$ is the number of periods in the interval. Let $A(t)$ and $B(t)$ be the date t prices of single shares of assets A and B, respectively. Asset A pays no dividends but asset B does pay dividends, and the present (date t) value of asset B's known dividends per share paid over the interval from t to T equals D. The per-period risk-free return is assumed to be constant and equal to R_f.

a. Consider a type of forward contract that has the following features. At date t an agreement is made to exchange at date T one share of asset A for F shares of asset B. No payments between the parties are exchanged at date t. Note that F is negotiated at date t and can be considered a forward price. Give an expression for the equilibrium value of this forward price and explain your reasoning.

Answer: The payoff of the contract at date T is $A(T)-FB(T)$. This can be replicated by buying one share of asset B, investing FD dollars at the risk-free rate, and short-selling F shares of asset B. Note that we need to invest FD at the risk-free rate because the dividends on the F shares sold short need to be paid during the time of the short sale. Since this portfolio replicates the forward contract that has zero value, it must be that

$$A(t)+FD-FB(t)=0$$

or

$$F=\frac{A(t)}{B(t)-D}$$

b. Consider a type of European call option that gives the holder the right to buy one share of asset A in exchange for paying X shares of asset B at date T. Give the no-arbitrage lower bound for the date t value of this call option, $c(t)$.

Answer: Consider the value of a long forward contract in part (a) that has a forward price of X shares of asset B. Its date t value is

$$f(t)=A(t)+XD-XB(t)$$

Hence, the call option has a value that is at least as great as this forward contract.

$$c(t)\geq \max[A(t)+XD-XB(t),0]$$

c. Derive a put-call parity relation for European options of the type described in part (b).

Answer: Let portfolio A include one call option, X shares of asset B, and borrowing of XD dollars at the risk-free rate. Its value at date T is

$$\max[A(T)-XB(T),0]+XB(T)+XDR_f^\tau-XDR_f^\tau=\max[A(T),XB(T)]$$

Let portfolio B include one put option and one share of asset A. Its date T value is

$$\max[XB(T)-A(T),0]+A(T)=\max[XB(T),A(T)]$$

Since the portfolios have the same date T value, the absence of arbitrage implies that their date t values must be equal

$$c(t)+XB(t)-XD=p(t)+A(t)$$

Answers to Chapter 8 Exercises

1. A variable, $x(t)$, follows the process

$$dx = \mu dt + \sigma dz$$

where μ and σ are constants. Find the process followed by $y(t) = e^{\alpha x(t) - \beta t}$.

Answer: Using Itô's lemma, we have

$$\begin{aligned} dy &= \frac{\partial y}{\partial x} dx + \frac{\partial y}{\partial t} dt + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}(dx)^2 \\ &= \alpha y dx - \beta y dt + \frac{1}{2}\alpha^2 y \sigma^2 dt \end{aligned}$$

or

$$dy/y = \left(\alpha\mu - \beta + \frac{\alpha^2\sigma^2}{2}\right)dt + \alpha\sigma dz$$

follows geometric Brownian motion.

2. Let *P* be a price index, such as the Consumer Price Index (CPI). Let *M* equal the nominal supply (stock) of money in the economy. For example, *M* might be designated as the amount of bank deposits and currency in circulation. Assume *P* and *M* each follow geometric Brownian motion processes

$$\begin{aligned} \frac{dP}{p} &= \mu_p dt + \sigma_p dz_p \\ \frac{dM}{M} &= \mu_m dt + \sigma_m dz_m \end{aligned}$$

with $dz_p dz_m = \rho dt$. Monetary economists define real money balances, m, to be $m = \frac{M}{P}$. Derive the stochastic process for m.

Answer: Applying Itô's lemma for the case of two state variables:

$$\begin{aligned} dm &= \frac{\partial m}{\partial M} dM + \frac{\partial m}{\partial P} dP + \frac{1}{2}\frac{\partial^2 m}{\partial M^2}(dM)^2 + \frac{1}{2}\frac{\partial^2 m}{\partial P^2}(dP)^2 + \frac{\partial^2 m}{\partial M \partial P}(dM)(dP) \\ &= \frac{1}{P} dM - \frac{M}{P^2} dP + 0 + \frac{M}{P^3}\sigma_p^2 P^2 dt - \frac{1}{P^2}\rho\sigma_p P \sigma_m M dt \\ &= m\frac{dM}{M} - m\frac{dP}{P} + m\sigma_p^2 dt - m\rho\sigma_p\sigma_m dt \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dm}{m} &= \mu_m dt + \sigma_m dz_m - \mu_p dt - \sigma_p dz_p + \left(\sigma_p^2 - \rho\sigma_p\sigma_m\right) dt \\ &= \left(\mu_m - \mu_p + \sigma_p^2 - \rho\sigma_p\sigma_m\right) dt + \sigma_m dz_m - \sigma_p dz_p \end{aligned}$$

which can be written as

$$\frac{dm}{m} = \alpha dt + \sigma dz$$

where $\alpha \equiv \mu_m - \mu_p + \sigma_p^2 - \rho\sigma_p\sigma_m$, $\sigma dz \equiv \sigma_m dz_m - \sigma_p dz_p$, and $\sigma^2 = \sigma_m^2 + \sigma_p^2 - 2\rho\sigma_p\sigma_m$. Thus we see that *m* also follows a geometric Brownian motion process.

3. The value (price) of a portfolio of stocks, $S(t)$, follows a geometric Brownian motion process:

$$dS/S = \alpha_s dt + \sigma_s dz_s$$

while the dividend yield for this portfolio, $y(t)$, follows the process

$$dy = \kappa(\gamma S - y)dt + \sigma_y y^{\frac{1}{2}} dz_y$$

where $dz_s dz_y = \rho dt$ and κ, γ, and σ_y are positive constants. Solve for the process followed by the portfolio's dividends paid per unit time, $D(t) = yS$.

Answer: Applying Itô's lemma

$$\begin{aligned} dD &= \frac{\partial D}{\partial S} dS + \frac{\partial D}{\partial y} dy + \frac{\partial^2 D}{\partial S \partial y}(dS)(dy) \\ &= ydS + Sdy + \sigma_s S \sigma_y y^{\frac{1}{2}} \rho dt \\ &= y\alpha_s S dt + y\sigma_s S dz_s + S\kappa(\gamma S - y)dt + \sigma_y S y^{\frac{1}{2}} dz_y + \sigma_s S \sigma_y y^{\frac{1}{2}} \rho dt \\ &= S[y\alpha_s + \kappa(\gamma S - y) + \sigma_s \sigma_y y^{\frac{1}{2}} \rho]dt + y\sigma_s S dz_s + \sigma_y S y^{\frac{1}{2}} dz_y \end{aligned}$$

4. The Ornstein-Uhlenbeck process can be useful for modeling a time series whose value changes stochastically but which tends to revert to a long-run value (its unconditional or steady state mean). This continuous-time process is given by

$$dy(t) = [\alpha - \beta y(t)]dt + \sigma dz(t)$$

The process is sometimes referred to as an *elastic random walk*. $y(t)$ varies stochastically around its unconditional mean of α/β, and β is a measure of the strength of the variable's reversion to this mean. Find the distribution of $y(t)$ given $y(t_0)$, where $t > t_0$. In particular, find $E[y(t) \mid y(t_0)]$ and $\text{var}[y(t) \mid y(t_0)]$. *Hint:* make the change in variables:

$$x(t) = \left(y(t) - \frac{\alpha}{\beta} \right) e^{\beta(t-t_0)}$$

and apply Itô's lemma to find the stochastic process for $x(t)$. The distribution and first two moments of $x(t)$ should be obvious. From this, derive the distribution and moments of $y(t)$.

Answer: Consider the stochastic process followed by $x(t)$. Applying Itô's lemma, we have

$$\begin{aligned} dx &= e^{\beta(t-t_0)}dy + \beta\left(y - \frac{\alpha}{\beta} \right) e^{\beta(t-t_0)}dt \\ &= e^{\beta(t-t_0)}(\alpha - \beta y)dt + e^{\beta(t-t_0)}\sigma dz + (\beta y - \alpha)e^{\beta(t-t_0)}dt \\ &= e^{\beta(t-t_0)}\sigma dz \end{aligned}$$

Thus we can write

$$x(t) = x(t_0) + \int_{t_0}^{t} \sigma e^{\beta(s-t_0)} dz(s)$$

This implies that $x(t)$ is normally distributed with

$$\begin{aligned} E[x(t) \mid x(t_0)] &= x(t_0) \\ \text{Var}[x(t) \mid x(t_0)] &= \sigma^2 \int_{t_0}^{t} e^{2\beta(s-t_0)} ds = \frac{\sigma^2}{2\beta}\left(e^{2\beta(t-t_0)} - 1 \right) \end{aligned}$$

Converting $x(t)$ back into $y(t)$, we have

$$y(t) = x(t)e^{-\beta(t-t_0)} + \frac{\alpha}{\beta}$$

and

$$y(t_0) = x(t_0) + \frac{\alpha}{\beta}$$

In other words, if we know $x(t_0)$, then we also know $y(t_0)$ and vice-versa. Hence,

$$
\begin{aligned}
E[y(t) \mid y(t_0)] &= E\left[x(t)e^{-\beta(t-t_0)} + \frac{\alpha}{\beta} \mid y(t_0) \right] \\
&= e^{-\beta(t-t_0)} E[x(t) \mid x(t_0)] + \frac{\alpha}{\beta} \\
&= e^{-\beta(t-t_0)} x(t_0) + \frac{\alpha}{\beta} \\
&= e^{-\beta(t-t_0)} \left(y(t_0) - \frac{\alpha}{\beta} \right) + \frac{\alpha}{\beta}
\end{aligned}
$$

Finally, we also know that

$$
\begin{aligned}
\text{var}[y(t) \mid y(t_0)] &= e^{-2\beta(t-t_0)} \ \text{var}[x(t) \mid x(t_0)] \\
&= \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta(t-t_0)}\right)
\end{aligned}
$$

Also note that $y(t)$ is a linear function of $x(t)$. Therefore, since $x(t)$ has a normal distribution, so does $y(t)$.

Answers to Chapter 9 Exercises

1. Suppose that the price of a non-dividend-paying stock follows the process

$$dS = \alpha S dt + \beta S^{\gamma} dz$$

where α, β, and γ are constants. The risk-free interest rate equals a constant, r. Denote $p(S(t),t)$ as the current price of a European put option on this stock having an exercise price of X and a maturity date of T. Derive the equilibrium partial differential equation and boundary condition for the price of this put option using the Black-Scholes hedging argument.

Answer: The derivation is very similar to the standard Black-Scholes derivation:

Itô's lemma says

$$dp = \left[\frac{\partial p}{\partial S}\alpha S + \frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2 p}{\partial S^2}\beta^2 S^{2\gamma}\right]dt + \frac{\partial p}{\partial S}\beta S^{\gamma} dz$$

Now consider forming a portfolio of -1 put option (sell the put option), $\frac{\partial p}{\partial S}$ units of the stock, and invest $B(t) = p(t) - \frac{\partial p}{\partial S}S(t)$ in the risk-free asset. Let $H(t)$ be the value of this portfolio at time *t*, which initially has a zero value. Then the change in value of this portfolio over the next instant is

$$\begin{aligned}
dH(t) &= -dp(t) + \left(\frac{\partial p}{\partial S}\right)dS(t) + \left[p(t) - \frac{\partial p}{\partial S}S(t)\right]rdt \\
&= -\left[\frac{\partial p}{\partial S}\alpha S + \frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2 p}{\partial S^2}\beta^2 S^{2\gamma}\right]dt - \left(\frac{\partial p}{\partial S}\right)\beta S^{\gamma} dz \\
&\quad \left[p(t) - \frac{\partial p}{\partial S}S(t)\right]rdt + \left(\frac{\partial p}{\partial S}\right)\alpha S\, dt + \left(\frac{\partial p}{\partial S}\right)\beta S^{\gamma} dz \\
&= \left[rp(t) - \frac{\partial p}{\partial S}rS(t) - \frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2 p}{\partial S^2}\beta^2 S^{2\gamma}\right]dt.
\end{aligned}$$

Note that the return on this portfolio is instantaneously riskless. Thus, to avoid arbitrage, the rate of return on this "hedge" portfolio must equal the risk-free rate of return. But since the intial value of the portfolio is zero, its future value must also be zero. Therefore, this implies the drift term in the above equation is zero:

$$\frac{\partial p}{\partial t} + \frac{1}{2}\frac{\partial^2 p}{\partial S^2}\beta^2 S^{2\gamma} + rS\frac{\partial p}{\partial S} - rp = 0.$$

which is the equilibrium PDE for the option price. It is solved subject to the boundary condition $p(S(T),T) = \max\left[0, X - S(T)\right]$.

2. Define $P(r(t),\tau)$ as the date t price of a pure discount bond that pays $1 in τ periods. The bond price depends on the instantaneous maturity yield, $r(t)$, which follows the process

$$dr(t) = \alpha[\bar{r} - r(t)]dt + \sigma\sqrt{r}dz$$

where α, γ, and σ are positive constants. If the process followed by the price of a bond having τ periods until maturity is

$$dP(r,\tau)/P(r,\tau) = \mu(r,\tau)dt - \sigma_p(r,\tau)dz$$

and the market price of bond risk is

$$\frac{\mu(r,\tau) - r(t)}{\sigma_p(r,\tau)} = \lambda\sqrt{r}$$

then write down the equilibrium partial differential equation and boundary condition that this bond price satisfies.

Answer: Itô's lemma implies

$$\begin{aligned} dP(r,\tau) &= \frac{\partial P}{\partial r}dr + \frac{\partial P}{\partial t}dt + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr)^2 \\ &= \left[P_r\alpha(\bar{r} - r) + P_t + \frac{1}{2}P_{rr}\sigma^2 r\right]dt + P_r\sigma\sqrt{r}dz \\ &= \mu(r,\tau)P(r,\tau)dt - \sigma_p(r,\tau)P(r,\tau)dz \end{aligned}$$

where $\mu(r,\tau) = \left[P_r\alpha(\bar{r} - r) + P_t + \frac{1}{2}P_{rr}\sigma^2 r\right]/P(r,\tau)$. and $\sigma_p(r,\tau) = -P_r\sigma\sqrt{r}/P(r,\tau)$.

The market price of risk condition can be re-written as

$$\mu(r,\tau) = r(t) + \lambda\sqrt{r}\sigma_p(r,\tau)$$

Subsituting in for $\mu(r,\tau)$ and $\sigma_p(r,\tau)$ and simplifying, we have

$$P_r\alpha(\bar{r} - r) + P_t + \frac{1}{2}P_{rr}\sigma^2 r = rP - \lambda\sigma r P_r$$

This can be re-written as

$$\frac{1}{2}\sigma^2 r P_{rr} + [\alpha\bar{r} + (\lambda\sigma - \alpha)r]P_r - rP - P_\tau = 0$$

and solved subject to the boundary condition $P(r,\tau = 0) = 1$.

3. The date t price of stock A, $A(t)$, follows the process

$$dA/A = \mu_A dt + \sigma_A dz$$

and the date t price of stock B, $B(t)$, follows the process

$$dB/B = \mu_B dt + \sigma_B dq$$

where σ_A and σ_B are constants and dz and dq are Brownian motion processes for which $dzdq = \rho dt$. Let $c(t)$ be the date t price of a European option written on the difference between these two stocks' prices. Specifically, at this option's maturity date, T, the value of the option equals

$$c(T) = \max[0, A(T) - B(T)]$$

a. Using Itô's lemma, derive the process followed by this option.

Answer:
$$dc = \left[\frac{\partial c}{\partial A}\mu_A A + \frac{\partial c}{\partial B}\mu_B B + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial A^2}\sigma_A^2 A^2 + \frac{1}{2}\frac{\partial^2 c}{\partial B^2}\sigma_B^2 B^2 + \frac{\partial^2 c}{\partial A \partial B}\rho\sigma_A A \sigma_B B\right]dt$$
$$+ \frac{\partial c}{\partial A}\sigma_A A dz + \frac{\partial c}{\partial B}\sigma_B B\, dq$$
$$\equiv \mu_c c dt + \frac{\partial c}{\partial A}\sigma_A A dz + \frac{\partial c}{\partial B}\sigma_B B\, dq$$

b. Suppose that you are an option dealer who has just sold (written) one of these options for a customer. You now wish to form a hedge portfolio composed of your unit short position in the option and positions in the two stocks. Let $H(t)$ denote the date t value of this hedge portfolio. Write down an equation for $H(t)$ that indicates the amount of shares of stocks A and B that should be held.

Answer: Let us form a hedge portfolio consisting of -1 units of the option, $w_A(t)$ units of stock A, and $w_B(t)$ units of the stock B, where we also restrict the portfolio to require a zero net investment. The zero net investment restriction implies

$$H(t) = -c(t) + w_A(t)A(t) + w_B(t)B(t) = 0$$

c. Write down the dynamics for $dH(t)$, showing that its return is riskless.

Answer: The hedge portfolio's return can then be written as

$$dH(t) = -dc(t) + w_A(t)dA(t) + w_B(t)dB(t)$$
$$= [-\mu_c c + w_A(t)\mu_A A + w_B(t)\mu_B B]dt$$
$$+\left[-\frac{\partial c}{\partial A}\sigma_A A + w_A(t)\sigma_A A\right]dz + \left[-\frac{\partial c}{\partial B}\sigma_B B + w_B(t)\sigma_B B\right]dq$$

Substituting in $c(t) = w_A(t)A(t) + w_B(t)B(t)$ into the above equation gives

$$dH(t) = [w_A(t)(\mu_A - \mu_c)A + w_B(t)(\mu_B - \mu_c)B]dt$$
$$+\left[-\frac{\partial c}{\partial A}\sigma_A A + w_A(t)\sigma_A A\right]dz + \left[-\frac{\partial c}{\partial B}\sigma_B B + w_B(t)\sigma_B B\right]dq$$

If $w_A(t) = \partial c/\partial A$ and $w_B(t) = \partial c/\partial B,$ which will satisfy the the zero-net investment constraint when the option is homogeneous of degree 1, that is, by Euler's Theorem $c(t) = \frac{\partial c}{\partial A} A(t) + \frac{\partial c}{\partial B} B(t),$ then the portfolio return is

$$dH(t) = \left[\frac{\partial c}{\partial A}(\mu_A - \mu_c)A + \frac{\partial c}{\partial B}(\mu_B - \mu_c)B \right] dt$$

which is riskless.

d. Assuming the absence of arbitrage, derive the equilibrium partial differential equation that this option must satisfy.

Answer: Since the return on the hedge portfolio is riskless, if the value of the portfolio starts out at zero, it must remain zero in the absence of arbitrage. Therefore, the drift of $dH(t)$ must be zero, implying

$$\frac{\partial c}{\partial A}\mu_A A + \frac{\partial c}{\partial B}\mu_B B - \mu_c \left[\frac{\partial c}{\partial A} A + \frac{\partial c}{\partial B} A \right] = 0$$

$$\frac{\partial c}{\partial A}\mu_A A + \frac{\partial c}{\partial B}\mu_B B - \mu_c c = 0$$

$$\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial A^2}\sigma_A^2 A^2 + \frac{1}{2}\frac{\partial^2 c}{\partial B^2}\sigma_B^2 B^2 + \frac{\partial^2 c}{\partial A \partial B}\rho\sigma_A A \sigma_B B = 0$$

which can then be solved subject to the boundary condition $c(A(T), B(T), T) = \max[0, A(T) - B(T)].$

4. Let $S(t)$ be the date t price of an asset that continuously pays a dividend that is a fixed proportion of its price. Specifically, the asset pays a dividend of $\delta S(t)dt$ over the time interval $dt.$ The process followed by this asset's price can be written as

$$ds = (\mu - \delta)Sdt + \sigma Sdz$$

where σ is the standard deviation of the asset's rate of return and μ is the asset's total expected rate of return, which includes its dividend payment and price appreciation. Note that the total rate of return earned by the owner of one share of this asset is $dS/S + \delta dt = \mu dt + \sigma dz.$ Consider a European call option written on this asset that has an exercise price of X and a maturity date of $T > t.$ Assuming a constant interest rate equal to r, use a Black-Scholes hedging argument to derive the equilibrium partial differential equation that this option's price, $c(t),$ must satisfy.

Answer: Itô's lemma implies

$$dc = \left[\frac{\partial c}{\partial S}(\mu - \delta)S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 \right] dt + \frac{\partial c}{\partial S}\sigma Sdz.$$

Now consider forming a portfolio that includes -1 unit of the option and a position in the underlying stock and the risk-free asset. We restrict this portfolio to require zero net investment, that is, after selling one unit of the call option and taking a hedge position in the underlying stock, the remaining surplus or deficit of funds is made up by borrowing or lending at the risk-free rate. Moreover, we require that the portfolio be self-financing in the sense that any surplus or deficit of funds from the option and stock positions are made up by investing or acquiring funds at the risk-free rate. Hence, if we let $w(t)$ be the number of shares invested in the stock, then this zero net investment, self-financing restriction implies that the amount invested in the risk-free asset for all dates t must be $B(t) = c(t) - w(t)S(t)$. Therefore, denoting the value of this hedge portfolio as $H(t)$ implies that its instantaneous return satisfies

$$dH(t) = -dc(t) + w(t)(dS(t) + \delta S(t)dt) + [c(t) - w(t)S(t)]rdt$$

Substituting in for $dc(t)$ and $dS(t)$, we obtain

$$\begin{aligned} dH(t) = &-\left[\frac{\partial c}{\partial S}(\mu - \delta) + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right]dt - \frac{\partial c}{\partial S}\sigma S dz \\ &+ w(t)(\mu S dt + \sigma S dz) + [c(t) - w(t)S(t)]rdt \end{aligned}$$

Now consider selecting the number of shares invested in the stock to equal $w(t) = +\partial c/\partial S$. This implies

$$\begin{aligned} dH(t) &= -\left[\frac{\partial c}{\partial S}(\mu - \delta)S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right]dt - \frac{\partial c}{\partial S}\sigma S dz \\ &\quad + \frac{\partial c}{\partial S}(\mu S dt + \sigma S dz) + \left[c(t) - \frac{\partial c}{\partial S}S(t)\right]rdt \\ &= \left[-\frac{\partial c}{\partial t} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} + rc(t) - (r - \delta)S(t)\frac{\partial c}{\partial S}\right]dt \end{aligned}$$

The return on this portfolio is instantaneously riskless, so to avoid arbitrage it must equal the competitive risk-free rate of return, r. But since we restricted the hedge portfolio to require zero-net investment at the initial date, say $t = 0$, then $H(0) = 0$ and

$$dH(0) = rH(0)dt = r0dt = 0$$

This implies $H(t) = 0 \ \forall t$ so that $dH(t) = 0 \ \forall t$. This no-arbitrage implies

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} + (r - \delta)S\frac{\partial c}{\partial S} - rc = 0$$

Answers to Chapter 10 Exercises

1. In this problem, you are asked to derive the equivalent martingale measure and the pricing kernel for the case to two sources of risk. Let S_1 and S_2 be the values of two risky assets that follow the processes

$$dS_i/S_i = \mu_i dt + \sigma_i dz_i, i = 1,2$$

where both μ_i and σ_i may be functions of S_1, S_2, and t, and dz_1 and dz_2 are two independent Brownian motion processes, implying $dz_1 dz_2 = 0$. Let $f(S_1, S_2, t)$ denote the value of a contingent claim whose payoff depends solely on S_1, S_2, and t. Also let $r(t)$ be the instantaneous, risk-free interest rate. From Itô's lemma, we know that the derivative's value satisfies

$$df = \mu_f f dt + \sigma_{f1} f dz_1 + \sigma_{f2} f dz_2$$

where $\mu_f f = f_3 + \mu_1 S_1 f_1 + \mu_2 S_2 f_2 + \frac{1}{2}\sigma_1^2 S_1^2 f_{11} + \frac{1}{2}\sigma_2^2 S_2^2 f_{22}$, $\sigma_{f1} f = \sigma_1 S_1 f_1$, $\sigma_{f2} f = \sigma_2 S_2 f_2$ and where the subscripts on f denote the partial derivatives with respect to its three arguments, S_1, S_2, and t.

a. By forming a riskless portfolio composed of the contingent claim and the two risky assets, show that in the absence of arbitrage an expression for μ_f can be derived in terms of r, $\theta_1 \equiv \frac{\mu_1 - r}{\sigma_1}$, and $\theta_2 \equiv \frac{\mu_2 - r}{\sigma_2}$.

Answer: Form a portfolio of -1 units of the contingent claim, f_1 units of the risky asset 1, and f_2 units of risky asset 2. Let H be the value of this portfolio. Then

$$H = -f + f_1 S_1 + f_2 S_2 \tag{1}$$

The change in value of this portfolio over the next instant is

$$\begin{aligned} dH &= -df + f_1 dS_1 + f_2 dS_2 \\ &= -\mu_f f dt - \sigma_{f1} f dz_1 - \sigma_{f2} f dz_2 \\ &\quad + f_1 \mu_1 S_1 dt + f_1 \sigma_1 S_1 dz_1 + f_2 \mu_2 S_2 dt + f_2 \sigma_2 S_2 dz_2 \\ &= [f_1 \mu_1 S_1 + f_2 \mu_2 S_2 - \mu_f f] dt \end{aligned} \tag{2}$$

Since the portfolio is riskless, the absence of arbitrage implies that it must earn the risk-free rate. Denoting the (possibly stochastic) instantaneous risk-free rate as $r(t)$, we have

$$dH = [f_1 \mu_1 S_1 + f_2 \mu_2 S_2 - \mu_f f] dt = rH dt = r[-f + f_1 S_1 + f_2 S_2] dt \tag{3}$$

which implies

$$f_1\mu_1S_1 + f_2\mu_2S_2 - \mu_f f = r[-f + f_1S_1 + f_2S_2] \tag{4}$$

If we substitute $f_1 = \frac{\sigma_{f1}f}{\sigma_1 S_1}$ and $f_2 = \frac{\sigma_{f2}f}{\sigma_2 S_2}$ into (4) and re-arrange, one obtains

$$\frac{\sigma_{f1}f}{\sigma_1}\mu_1 + \frac{\sigma_{f2}f}{\sigma_2}\mu_2 - \mu_f f = r\left[-f + \frac{\sigma_{f1}f}{\sigma_1} + \frac{\sigma_{f2}f}{\sigma_2}\right] \tag{5}$$

$$\begin{aligned}\mu_f &= r + \frac{\mu_1 - r}{\sigma_1}\sigma_{f1} + \frac{\mu_2 - r}{\sigma_2}\sigma_{f2} \\ &= r + \theta_1\sigma_{f1} + \theta_2\sigma_{f2}\end{aligned} \tag{6}$$

b. Define the risk-neutral processes $d\hat{z}_1$ and $d\hat{z}_2$ in terms of the original Brownian motion processes, and then give the risk-neutral process for df in terms of $d\hat{z}_1$ and $d\hat{z}_2$.

Answer: Let $d\hat{z}_1 = dz_1 + \theta_1(t)dt$ and $d\hat{z}_2 = dz_2 + \theta_2(t)dt$. Then the process for df is:

$$\begin{aligned}df &= (r + \theta_1\sigma_{f1} + \theta_2\sigma_{f2})fdt + \sigma_{f1}fdz_1 + \sigma_{f2}fdz_2 \\ &= (r + \theta_1\sigma_{f1} + \theta_2\sigma_{f2})fdt + \sigma_{f1}fd\hat{z}_1 + \sigma_{f2}fd\hat{z}_2 - \theta_1\sigma_{f1}fdt - \theta_2\sigma_{f2}fdt \\ &= rfdt + \sigma_{f1}fd\hat{z}_1 + \sigma_{f2}fd\hat{z}_2\end{aligned} \tag{7}$$

c. Let $B(t)$ be the value of a "money market fund" that invests in the instantaneous maturity, risk-free asset. Show that $F(t) \equiv f(t)/B(t)$ is a martingale under the risk-neutral probability measure.

Answer: Applying Itô's lemma, one obtains

$$\begin{aligned}dF &= \frac{1}{B}df - \frac{f}{B^2}dB = rFdt + \sigma_{f1}Fd\hat{z}_1 + \sigma_{f2}Fd\hat{z}_2 - rFdt \\ &= \sigma_{f1}Fd\hat{z}_1 + \sigma_{f2}Fd\hat{z}_2\end{aligned} \tag{8}$$

d. Let $M(t)$ be the state price deflator such that $f(t)M(t)$ is a martingale under the physical probability measure. If

$$dM = \mu_m dt + \sigma_{m1}dz_1 + \sigma_{m2}dz_2$$

what must be the values of μ_m, σ_{m1}, and σ_{m2} that preclude arbitrage? Show how you solve for these values.

Answer: Define $f^m = fM$ and apply Itô's lemma:

$$\begin{aligned}df^m &= fdM + Mdf + (df)(dM) \\ &= [f\mu_m + M\mu_f f + (\sigma_{f1}f\sigma_{m1} + \sigma_{f2}f\sigma_{m2})]dt \\ &\quad + [f\sigma_{m1} + M\sigma_{f1}f]dz_1 + [f\sigma_{m2} + M\sigma_{f2}f]dz_2\end{aligned} \tag{9}$$

If f^m is a martingale, then its drift must be zero, implying

$$\mu_f = -\frac{\mu_m}{M} - \frac{\sigma_{f1}\sigma_{m1} + \sigma_{f2}\sigma_{m2}}{M} \tag{10}$$

Now consider the case in which f is the instantaneously riskless asset, that is, $f(t) = B(t)$ is the money market investment. This implies that $\sigma_{f1} = \sigma_{f2} = 0$ and $\mu_f = r(t)$. Using (10), requires

$$r(t) = -\frac{\mu_m}{M} \tag{11}$$

In other words, the expected rate of change of the pricing kernel must equal minus the instantaneous risk-free interest rate. Next, consider the general case where the asset f is risky, so that $\sigma_{f1} \neq 0$ and $\sigma_{f1} \neq 0$. Using (10) and (11) together, we obtain

$$\begin{aligned}\mu_f &= r(t) - \frac{\sigma_{f1}\sigma_{m1} + \sigma_{f2}\sigma_{m2}}{M} \\ &= r(t) - \frac{\sigma_{m1}}{M}\sigma_{f1} - \frac{\sigma_{m2}}{M}\sigma_{f2}\end{aligned} \tag{12}$$

Comparing (12) to (6), we see that

$$\begin{aligned}-\frac{\sigma_{m1}}{M} &= \theta_1(t) \\ -\frac{\sigma_{m2}}{M} &= \theta_2(t)\end{aligned} \tag{13}$$

2. The Cox, Ingersoll, and Ross (Cox, Ingersoll, and Ross 1985b) model of the term structure of interest rates assumes that the process followed by the instantaneous maturity, risk-free interest rate is

$$dr = \alpha(\gamma - r)dt + \sigma\sqrt{r}dz$$

where α, γ, and σ are constants. Let $P(t,\tau)$ be the date t price of a zero-coupon bond paying \$1 at date $t+\tau$. It is assumed that $r(t)$ is the only source of uncertainty affecting $P(t,\tau)$. Also, let $\mu_p(t,\tau)$ and $\sigma_p(t,\tau)$ be the instantaneous mean and standard deviation of the rate of return on this bond and assume

$$\frac{\mu_p(t,\tau) - r(t)}{\sigma_p(t,\tau)} = \beta\sqrt{r}$$

where β is a constant.

a. Write down the stochastic process followed by the pricing kernel (state price deflator), $M(t)$, for this problem, that is, the process dM/M. Also, apply Itô's lemma to derive the process for $\mathrm{m}(t) \equiv \ln(M)$, that is, the process $d\mathrm{m}$.

Answer: We know that pricing kernels satisfy

$$dM/M = -r(t)dt - \theta(t)dz$$

where $r(t)$ satisfies $dr = \alpha(\gamma - r)dt + \sigma\sqrt{r}dz$ and $\theta(t) = \beta\sqrt{r}$ is the market price of risk given in the question. Using Itô's lemma,

$$\begin{aligned} d\mathrm{m} &= -\left(r + \frac{1}{2}\theta^2\right)dt - \theta(t)dz \\ &= -\left(r + \frac{1}{2}\beta^2 r\right)dt - \beta\sqrt{r}dz \\ &= -r\left(1 + \frac{1}{2}\beta^2\right)dt - \beta\sqrt{r}dz \end{aligned}$$

b. Let the current date be 0 and write down the formula for the bond price, $P(0,\tau)$, in terms of an expectation of $\mathrm{m}_\tau - \mathrm{m}_0$. Show how this can be written in terms of an expectation of functions of integrals of $r(t)$ and β.

Answer: Since the bond pays \$1 at date τ, we have

$$P(0,\tau) = E_0[M_\tau / M_0] = E_0[e^{m_\tau - m_0}]$$

From the answer in (2.a) we see that $d\mathrm{m} = -r\left(1 + \frac{1}{2}\beta^2\right)dt - \beta\sqrt{r}dz$, so that

$$P(0,\tau) = E_0[e^{m_\tau - m_0}] = E_0[e^{-\left(1+\frac{1}{2}\beta^2\right)\int_0^\tau r(t)dt - \beta\int_0^\tau \sqrt{r(t)}dz(t)}]$$

3. If the price of a non-dividend-paying stock follows the process $dS/S = \mu dt + \sigma dz$ where σ is constant, and there is a constant risk-free interest rate equal to r, then the Black-Scholes derivation showed that the no-arbitrage value of a standard call option having τ periods to maturity and an exercise price of X is given by $c = SN(d_1) - Xe^{-r\tau}N(d_2)$ where $d_1 = [\ln(S/X) + (r + \frac{1}{2}\sigma^2)\tau]/(\sigma\sqrt{\tau})$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

A *forward start* call option is similar to this standard option but with the difference that the option exercise price, X, is initially a random variable. The exercise price is set equal to the contemporaneous stock price at a future date prior to the maturity of the option. Specifically, let the current date be 0 and the option maturity date be τ. Then at date t where $0 < t < \tau$, the option's exercise price, X, is set equal to the date t value of the stock, denoted as $S(t)$. Hence, $X = S(t)$ is a random variable as of the current date 0.

For a given date t, derive the date 0 value of this forward start call option. *Hint*: note the value of a standard call option when $S = X$, and then use a simple application of risk-neutral pricing to derive the value of the forward start option.

Answer: At the future date *t* when *X* is set equal to $S(t)$, we have

$$c(t) = S(t)N(\nu_1) - S(t)e^{-r(\tau-t)}N(\nu_2) = S(t)[N(\nu_1) - e^{-r(\tau-t)}N(\nu_2)] \equiv S(t)K$$

where $\nu_1 = \left(r + \tfrac{1}{2}\sigma^2\right)(\tau - t)/\left(\sigma\sqrt{\tau - t}\right)$ and $\nu_2 = \nu_1 - \sigma\sqrt{\tau - t}$ are independent of $S(t)$. Hence, we see that the call option at date t is equal to a proportion, K, times the future stock price. Therefore, using the risk-neutral valuation technique, the current value of the forward start option is given by

$$c_0 = e^{-rt}\hat{E}[S(t)K] = e^{-rt}K\hat{E}[S(t)] = e^{-rt}Ke^{rt}S(0) = KS(0)$$

This result shows that the forward start option is equal to the value of a standard Black-Scholes option where *X* = *S* (0) and the time until maturity is $\tau - t$.

4. If the price of a non-dividend-paying stock follows the process $dS/S = \mu dt + \sigma dz$ where σ is constant, and there is a constant risk-free interest rate equal to *r*, then the Black-Scholes showed that the no-arbitrage value of a standard call option having τ periods to maturity and an exercise price of X is given by $c = SN(d_1) - Xe^{-r\tau}N(d_2)$ where $d_1 = [\ln(S/X) + (r + \tfrac{1}{2}\sigma^2)\tau]/(\sigma\sqrt{\tau})$ and $d_2 = d_1 - \sigma\sqrt{\tau}$. Based on this result and a simple application of risk-neutral pricing, derive the value of the following *binary* options. Continue to assume that the underlying stock price follows the process $dS/S = \mu dt + \sigma dz$, the risk-free interest rate equals *r*, and the option's time until maturity equals τ.

   a. Consider the value of a *cash-or-nothing call*, *cnc*. If $S(T)$ is the stock's price at the option's maturity date of T, the payoff of this option is

$$cnc_T = \begin{cases} F & \text{if} \quad S(T) > X \\ 0 & \text{if} \quad S(T) \leq X \end{cases}$$

   where *F* is a fixed amount. Derive the value of this option when its time until maturity is τ and the current stock price is S. Explain your reasoning.

   Answer: Using risk-neutral valuation, $cnc_t = e^{-r\tau}\hat{E}_t[cnc_T] = e^{-r\tau}F\hat{\pi}(S(T) > X)$, where $\hat{\pi}(S(T) > X)$ is the risk-neutral probability that $S(T) > X$. From the standard Black-Scholes call option price $c = SN(d_1) - Xe^{-r\tau}N(d_2)$, ones sees that $\hat{\pi}(S(T) > X) = N(d_2)$, since only when $S(T) > X$ (the call option is in the money), does one "pay" the exercise price, X. Hence, we have $cnc_t = e^{-r\tau}FN(d_2)$.

   b. Consider the value of an *asset-or-nothing call, anc*. If $S(T)$ is the stock's price at the option's maturity date of T, the payoff of this option is

$$anc_T = \begin{cases} S(T) & \text{if} \quad S(T) > X \\ 0 & \text{if} \quad S(T) \leq X \end{cases}$$

Derive the value of this option when its time until maturity is τ and the current stock price is S. Explain your reasoning.

Answer: Using risk-neutral valuation, $anc_t = e^{-r\tau}\hat{E}[anc_T]$. But this equals a standard call option plus a cash-or–nothing call paying an amount equal to the exercise price $(F = X)$. Hence, the value of an asset-or-nothing call is $SN(d_1) - Xe^{-r\tau}N(d_2) + Xe^{-r\tau}N(d_2) = SN(d_1)$.

5. Outline a derivation of the form of the multivariate state price deflator given in equations (10.33) and (10.34).

Answer: Let $S = (S_1 \ldots S_n)'$ be an $nx1$ vector of asset prices. They are assumed to follow the process

$$dS = \alpha(S)dt + v(S)dZ \tag{1}$$

where $\alpha(S)$ is an $nx1$ vector of expected returns and asset prices and $v(S)$ is an nxn matrix of volatility terms multiplying the $nx1$ vector of Wiener processes. Note that the elements of $\alpha(S)$ and $v(S)$ may be functions of each of the asset prices as well as calendar time, t. $v(S)$ is assumed to be of full rank, which is equivalent to assuming that the returns on any one asset cannot be perfectly hedged by the returns on a portfolio of the other assets. In this sense, none of the assets are assumed to be redundant. Now if the contingent claim is a function of these asset prices and calendar time, that is, $c(S,t)$, then Itô’s lemma gives its process to be of the form:

$$dc = \mu_c c dt + \sum\nolimits_c c dZ \tag{2}$$

where

$$\mu_c c = c_s'\alpha(S) + c_t + \frac{1}{2} tr[vv'c_{ss}] \tag{3}$$

$$\sum\nolimits_c c = c_s' v \tag{4}$$

and where $c_s = (c_{S_1} \ldots c_{S_2})'$ is an $nx1$ vector of first derivatives and c_{ss} is an nxn matrix of second derivatives whose i, j^{th} element is $c_{S_i S_j}$. $tr[X]$ is the trace of matrix X, that is, the sum of its diagonal elements. If a hedge portfolio composed of -1 unit of the contingent claim and c_{S_i} units of asset i, $i = 1, \ldots, n$ is formed, its value is

$$H = -c + c_s' S \tag{5}$$

and the instantaneous change in the value of the portfolio is

$$\begin{aligned} dH &= -dc + c_s' dS \\ &= -\mu_c c dt - c_s' v dZ + c_s'\alpha(S)dt + c_s' v(S) dZ \\ &= [c_s'\alpha(S) - \mu_c c]dt \\ &= r(t)Hdt = r(t)[-c + c_s' S] \end{aligned} \tag{6}$$

This implies

$$c_s'\alpha(S) - \mu_c c = -rc + rc_s'S \tag{7}$$

or

$$\mu_c c - rc = c_s'(\alpha(S) - rS) \tag{8}$$

Now consider forming *n* portfolios from the *n* underlying securities. The goal is to make these *n* portfolios be "primitive" securities that each depend of a single dz_i process and have unit variance. This can be done by using share amounts of each of the original n securities equal to the rows of v^{-1}. Thus, letting $P = (P_1 \ldots P_n)'$ be the $nx1$ vector of prices for the n primitive portfolios, they will equal

$$P = v^{-1}S \tag{9}$$

and the process followed by these primitive security portfolios is

$$\begin{aligned} dP &= v^{-1}dS = v^{-1}\alpha(s)dt + v^{-1}vdZ \\ &= \mu_p dt + IdZ \end{aligned} \tag{10}$$

where $\mu_p = (\mu_{p1} \ldots \mu_{pn})' = v^{-1}\alpha(S)$ and I is the nxn identity matrix. Using (9) and (10), we can re-write (8) as

$$\mu_c c - rc = c_s'v(\mu_p - rP) \tag{11}$$

Now from the definition that $\Theta = (\theta_1 \ldots \theta_n)'$ is the $n \times 1$ vector of market prices of risks associated with each of the Brownian motions, $\Theta = \mu_p - rP$. Using this and equation (4), equation (11) can be re-written as

$$\begin{aligned} \mu_c c - rc &= c_s'v(\mu_p - rP) \\ &= \Sigma_c c\Theta \end{aligned} \tag{12}$$

or

$$\mu_c - r = \Sigma_c\Theta \tag{13}$$

To deduce the form of the pricing kernel, if

$$dM = \mu_m dt + \sigma_m' dZ \tag{14}$$

where $\sigma_m = (\sigma_{m1} \ldots \sigma_{mn})'$ is an $nx1$ vector, then define $c^m = cM$ and apply Itô's lemma:

$$\begin{aligned} dc^m &= cdM + Mdc + (dc)(dM) \\ &= [c\mu_m + M\mu_c c + \Sigma_c c\sigma_m]dt + [c\sigma_m' + M\Sigma_c c]dZ \end{aligned} \tag{15}$$

If $c^m = cM$ satisfies $c_t M_t = E_t[c_T M_T]$, that is, c^m is a martingale, then its drift in (15) must be zero, implying

$$\mu_c = -\frac{\mu_m}{M} - \frac{\Sigma_c \sigma_m}{M} \tag{16}$$

For the case in which c is the instantaneously riskless asset, all of the elements of Σ_c are zero and therefore $\mu_c = r(t)$. This requires

$$r(t) = -\frac{\mu_m}{M} \tag{17}$$

Next, consider the case where asset c is one of the primitive securities, say the i^{th} one. In this case $\mu_c = \mu_{pi}/P_i$ and $\Sigma_c \sigma_m = (1/P_i)\sigma_{mi}$. Hence, using equation (17), equation (16) becomes

$$\frac{\mu_{pi}}{P_i} = r - \frac{\sigma_{mi}}{P_i M} \tag{18}$$

Rearranging (18), we have

$$\begin{aligned} -\frac{\sigma_{mi}}{M} &= \mu_{pi} - rP_i \\ &= \theta_i \end{aligned} \tag{19}$$

From this we deduce that the pricing kernel follows the process $dM/M = -r(t)dt$ $-$ $\Theta(t)'dZ$.

6. Consider a continuous-time version of a Lucas endowment economy (Lucas 1978). It is assumed that there is a single risky asset (e.g., fruit tree) that produces a perishable consumption good that is paid out as a continuous dividend, g_t. This dividend satisfies the process

$$dg_t/g_t = \alpha dt + \sigma dz$$

where α and σ are constants. There is a representative agent who at date 0 maximizes lifetime consumption given by

$$E_0 \int_0^\infty U(C_t, t) dt$$

where $U(C_t, t) = e^{-\phi t} C_t^\gamma / \gamma$, $\gamma < 1$. Under the Lucas endowment economy assumption, we know that in equilibrium $C_t = g_t$.

a. Let $P_t(\tau)$ denote the date t price of a riskless discount (zero-coupon) bond that pays one unit of the consumption good in τ periods. Derive an (Euler equation) expression for $P_t(\tau)$ in terms of an expectation of a function of future dividends.

Answer:

$$P_t(\tau) = E_t\left[\frac{U_c(c_{t+\tau}, t+\tau)}{U_c(c_t, t)}\right] = E_t\left[e^{-\phi\tau}\left(\frac{g_{t+\tau}}{g_t}\right)^{\gamma-1}\right] \tag{3}$$

b. Let $m_{t,t+\tau} \equiv M_{t+\tau}/M_t$ be the stochastic discount factor (pricing kernel) for this economy. Based on your answer in part (a), write down the stochastic process for M_t. *Hint:* find an expression for M_t and then use Itô's lemma.

Answer: Since the pricing kernel satisfies

$$P_t(\tau) = E_t[m_{t,t+\tau}] = E_t\left[\frac{M_{t+\tau}}{M_t}\right] \tag{4}$$

then comparing (3) and (4) we see that $M_t = e^{-\phi t} g_t^{\gamma-1}$. Applying Itô's lemma, we have

$$\begin{aligned} dM &= (\gamma-1)e^{-\phi t} g^{\gamma-2} dg - \phi e^{-\phi t} g^{\gamma-1} dt + \frac{1}{2}(\gamma-1)(\gamma-2)e^{-\phi t} g^{\gamma-3}(dg)^2 \\ &= (\gamma-1)M\left(\frac{dg}{g}\right) - \phi M dt + \frac{1}{2}(\gamma-1)(\gamma-2)M\left(\frac{dg}{g}\right)^2 \end{aligned}$$

Therefore

$$dM/M = \left[(\gamma-1)\alpha - \phi + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma^2\right]dt + (\gamma-1)\sigma dz$$

c. Based on your previous answers, write down the instantaneous, risk-free real interest rate. Is it constant or time varying?

Answer: It is constant. The drift of dM/M equals minus the interest rate. Hence, the interest rate, r, is given by

$$r = (1-\gamma)\alpha + \phi - \frac{\sigma^2}{2}(1-\gamma)(2-\gamma)$$

Answers to Chapter 11 Exercises

1. Verify that (11.11) holds by using Itô's lemma to find the process followed by $\ln(S(t))$.

   Answer: Itô's lemma says that the process followed by $\ln(S(t))$ is

$$\begin{aligned} d\ln S(t) &= \frac{1}{S}[(\mu - \lambda k)Sdt + \sigma S dz] - \frac{1}{2S^2}\sigma^2 S^2 dt + [\ln YS - \ln S]dq \\ &= \left(\mu - \lambda k - \frac{1}{2}\sigma^2\right)dt + \sigma dz + (\ln Y)dq \end{aligned}$$

   This implies

$$\ln[S(t)/S(0)] = \left(\mu - \lambda k - \frac{1}{2}\sigma^2\right)t + \sigma(\tilde{z}_t - z_0) + \sum_{i=1}^{n} \ln \tilde{Y}_i$$

   where $\sum_{i=1}^{n} \ln \tilde{Y}_i$ are the n log jump sizes over the period from 0 to t. Taking the exponential of each side of this equation, and defining $Y_0 = 1$, we obtain

$$\begin{aligned} S(t)/S(0) &= e^{(\mu - \lambda k - \frac{1}{2}\sigma^2)t + \sigma(\tilde{z}_t - z_0) + \sum_{i=1}^{n} \ln \tilde{Y}_i} \\ &= e^{(\mu - \lambda k - \frac{1}{2}\sigma^2)t + \sigma(\tilde{z}_t - z_0)} \prod_{i=0}^{n} \tilde{Y}_i \\ &= e^{(\mu - \lambda k - \frac{1}{2}\sigma^2)t + \sigma(\tilde{z}_t - z_0)} \tilde{Y}(n) \end{aligned}$$

2. Let $S(t)$ be the U.S. dollar price of a stock. It is assumed to follow the process

$$dS/S = [\mu_s - \lambda k]dt + \sigma_s dz_s + \gamma(\tilde{Y})dq \qquad (*)$$

   where dz_s is a standard Wiener process, $q(t)$ is a Poisson counting process, and $\gamma(\tilde{Y}) = (\tilde{Y} - 1)$. The probability that q will jump during the time interval dt is λdt. $k \equiv E[\tilde{Y} - 1]$ is the expected jump size. Let F be the foreign exchange rate between U.S. dollars and Japanese yen, denominated as U.S. dollars per yen. F follows the process

$$dF/F = \mu_f dt + \sigma_f dz_f$$

   where $dz_s dz_f = \rho dt$. Define $x(t)$ as the Japanese yen price of the stock whose U.S. dollar price follows the process in (*). Derive the stochastic process followed by $x(t)$.

Answer: Applying Itô's lemma for the case of jump-diffusion processes,

$$dx = \frac{1}{F}[S(\mu_s - \lambda k)dt + S\sigma_s dz_s] - \frac{S}{F^2}(dF) + \frac{S}{F^3}(dF)^2$$
$$-\frac{1}{F^2}[S(\mu_s - \lambda k)dt + S\sigma_s dz_s](dF) + \left(\frac{SY}{F} - \frac{S}{F}\right)dq$$

Simplifying, we have

$$dx/x = (\mu_s - \lambda k)dt + \sigma_s dz_s - \mu_f dt - \sigma_f dz_f + \sigma_f^2 dt - \sigma_s \sigma_f \rho dt + (Y-1)dq$$
$$= (\mu_s - \lambda k - \mu_f + \sigma_f^2 - \sigma_s \sigma_f \rho)dt + \sigma_s dz_s - \sigma_f dz_f + (Y-1)dq$$

3. Suppose that the instantaneous-maturity, default-free interest rate follows the jump-diffusion process

$$dr(t) = \kappa[\theta - r(t)]dt + \sigma dz + r\gamma(Y)dq$$

where dz is a standard Wiener process $q(t)$ is a Poisson counting process having the arrival rate of λdt. The arrival of jumps is assumed to be independent of the Wiener process, dz. $\gamma(Y) = (Y-1)$ where $Y > 1$ is a known positive constant.

a. Define $P(r,\tau)$ as the price of a default-free discount bond that pays \$1 in τ periods. Using Itô's lemma for the case of jump-diffusion processes, write down the process followed by $dP(r,\tau)$.

Answer: $dP(r,\tau) = \left[\kappa(\theta - r)P_r - P_\tau + \frac{1}{2}\sigma^2 P_{rr}\right]dt + \sigma P_r dz + \left[P(rY,\tau) - P(r,\tau)\right]dq$

b. Assume that the market price of jump risk is zero, but that the market price of Brownian motion (dz) risk is given by ϕ, so that $\phi = [\alpha_p - r(t)]/\sigma_p$, where $\alpha_p(r,\tau)$ is the expected rate of return on the bond and $\sigma_p(\tau)$ is the standard deviation of the bond's rate of return from Brownian motion risk (not including the risk from jumps). Derive the equilibrium partial differential equation that the value $P(r,\tau)$ must satisfy.

Answer: From Itô's lemma, the expected rate of return on the bond is

$$\alpha_p(r,\tau) = \frac{1}{P}\left[\kappa(\theta - r)P_r - P_\tau + \frac{1}{2}\sigma^2 P_{rr}\right] + \frac{\lambda}{P}[P(rY,\tau) - P(r,\tau)]$$

and its standard deviation from Brownian motion risk is

$$\sigma_p(\tau) = -\sigma\frac{P_r}{P}.$$

Thus, if $\phi = [\alpha_p - r(t)]/\sigma_p$ or

$$\alpha_p = r(t) + \phi\sigma_p = r(t) - \phi\sigma\frac{P_r}{P}$$
$$= \frac{1}{P}\left[\kappa(\theta - r)P_r - P_\tau + \frac{1}{2}\sigma^2 P_{rr}\right] + \frac{\lambda}{P}[P(rY,\tau) - P(r,\tau)]$$

then

$$r(t)-\phi\sigma\frac{P_r}{P}=\frac{1}{P}\left[\kappa(\theta-r)P_r-P_\tau+\frac{1}{2}\sigma^2P_{rr}\right]+\frac{\lambda}{P}[P(rY,\tau)-P(r,\tau)]$$

or

$$\frac{1}{2}\sigma^2P_{rr}+[\kappa(\theta-r)+\phi\sigma]P_r-rP-P_\tau+\lambda[P(rY,\tau)-P]=0$$

4. Suppose that a security's price follows a jump-diffusion process and yields a continuous dividend at a constant rate of δdt. For example, its price, $S(t)$, follows the process

$$dS/S=[\mu(S,t)-\lambda k-\delta]dt+\sigma(S,t)dz+\gamma(\tilde{Y})dq$$

where $q(t)$ is a Poisson counting process and $\gamma(\tilde{Y})=(\tilde{Y}-1)$. Also let $k\equiv E[\tilde{Y}-1]$; let the probability of a jump be λdt; and denote $\mu(S,t)$ as the asset's total expected rate of return. Consider a forward contract written on this security that is negotiated at date t and matures at date T where $\tau=T-t>0$. Let $r(t,\tau)$ be the date t continuously compounded, risk-free interest rate for borrowing or lending between dates t and T. Assuming that one can trade continuously in the security, derive the equilibrium date t forward price using an argument that rules out arbitrage. *Hint:* some information in this problem is extraneous. The solution is relatively simple.

Answer: Consider the two portfolios created at date t:

At date t:
Portfolio A: a long forward position in the security that matures at date T and has forward price F.
Portfolio B: a purchase of $e^{-\delta\tau}$ shares of the security, worth $Se^{-\delta\tau}$, and risk-free borrowing of $Fe^{-r(t,\tau)\tau}$.

Inbetween dates t and T:
Portfolio B: as the security pays dividends at rate δdt, reinvest them in the security at the current price.

At date T:
Portfolio A: S_T-F
Portfolio B: one share of the security, worth S_T, and borrowing of F.

Since Portfolios A and B have the same date T value, their values at date t must be the same. Since the long position in the forward contract has a date t value of zero, this implies

$$0=Se^{-\delta\tau}-Fe^{-r(t,\tau)\tau}$$

or

$$F=Se^{[r(t,\tau)-\delta]\tau}$$

Answers to Chapter 12 Exercises

1. Consider the following consumption and portfolio choice problem. An individual must choose between two different assets, a stock and a short (instantaneous) maturity, default-free bond. In addition, the individual faces a stochastic rate of inflation, that is, uncertain changes in the price level (e.g., the Consumer Price Index). The price level (currency price of the consumption good) follows the process

$$dP_t/P_t = \pi dt + \delta d\zeta$$

The nominal (currency value) of the stock is given by S_t. This nominal stock price satisfies

$$dS_t/S_t = \mu dt + \sigma dz$$

The nominal (currency value) of the bond is given by B_t. It pays an instantaneous nominal rate of return equal to i. Hence, its nominal price satisfies

$$dB_t/B_t = i dt$$

Note that $d\zeta$ and dz are standard Wiener processes with $d\zeta \, dz = \rho dt$. Also assume π, δ, μ, σ, and i are all constants.

a. What processes do the real (consumption good value) rates of return on the stock and the bond satisfy?

Answer: Denote the real value of the stock as $s_t = S_t/P_t$ and the real value of the bond as $b_t = B_t/P_t$. Then using Itô's lemma, we have that the real processes for the stock and bond are

$$ds_t/s_t = (\mu - \pi + \delta^2 - \sigma\delta\rho)dt + \sigma dz - \delta d\zeta$$
$$db_t/b_t = (i - \pi + \delta^2)dt - \delta d\zeta$$

b. Let C_t be the individual's date t real rate of consumption and ω be the proportion of real wealth, W_t, that is invested in the stock. Give the process followed by real wealth, W_t.

Answer:

$$\begin{aligned} dW &= ([\omega(\mu - \pi + \delta^2 - \sigma\delta\rho) + (1-\omega)(i - \pi + \delta^2)]W - C)dt \\ &\quad + \omega(\sigma dz - \delta dq)W - (1-\omega)\delta W d\zeta \\ &= [\omega(\mu - \sigma\delta\rho - i)W + (i - \pi + \delta^2)W - C]dt + \omega W\sigma dz - \delta W d\zeta \end{aligned}$$

c. Assume that the individual solves the following problem:

$$\max_{C,\omega} E_0 \int_0^\infty U(C_t, t)dt$$

subject to the real wealth dynamic budget constraint given in part (b). Assuming $U(C_t, t)$ is a concave utility function, solve for the individual's optimal choice of ω in terms of the indirect utility-of-wealth function.

Answer: The Bellman equation is

$$0 = \max_{\{C,\omega\}} u(C(t),t) + J_t + J_W[\omega(\mu - \sigma\delta\rho - i)W + (i - \pi + \delta^2)W - C]$$
$$+\frac{1}{2} J_{WW} W^2[\omega^2\sigma^2 + \delta^2 - 2\omega\rho\sigma\delta]$$

The first order conditions are

$$0 = u_C - J_W$$
$$J_W W(\mu - \sigma\delta\rho - i) + J_{WW} W^2(\omega\sigma^2 - \rho\sigma\delta) = 0$$

or

$$\omega = -\frac{J_W}{WJ_{WW}} \frac{(\mu - \sigma\delta\rho - i)}{\sigma^2} + \rho\frac{\delta}{\sigma}$$

d. How does ω vary with ρ? What is the economic intuition for this comparative static result?

Answer: If the difference between the expected real rates of return of the stock and the bond is constant, that is, $(\mu - \sigma\delta\rho - i)$ is constant, then

$$\frac{\partial\omega}{\partial\rho} = \frac{\delta}{\sigma} > 0$$

and we see that the greater the correlation between the stock's return and the price level, the greater the demand for the stock as a hedge against price level changes. If $(\mu - \sigma\delta\rho - i)$ is not constant when ρ changes, then

$$\frac{\partial\omega}{\partial\rho} = \frac{\delta}{\sigma}\left(1 + \frac{J_W}{WJ_{WW}}\right)$$

and the sign depends on $-J_W/(WJ_{WW}) > 0$, which is the reciprocal of the individual's relative risk aversion. The more risk averse the individual, the smaller is the magnitude of $J_W/(WJ_{WW}) < 0$, so that more risk-averse individuals still have $\partial\omega/\partial\rho > 0$, that is, they demand more of the stock as a hedge against price level changes.

2. Consider the individual's intertemporal consumption and portfolio choice problem for the case of a single risky asset and an instantaneously risk-free asset. The individual maximizes expected lifetime utility of the form

$$E_0\left[\int_0^T e^{-\phi t}u(C_t)dt\right]$$

The price of the risky asset, S, is assumed to follow the geometric Brownian motion process

$$dS/S = \mu dt + \sigma dz$$

where μ and σ are constants. The instantaneously risk-free asset pays an instantaneous rate of return of r_t. Thus, an investment that takes the form of continually reinvesting at this risk-free rate has a value (price), B_t, that follows the process

$$dB/B = r_t dt$$

where τ_t is assumed to change over time, following the Vasicek mean-reverting process (Vasicek 1977)

$$dr_t = a[b - r_t]dt + sd\zeta$$

where $dzd\zeta = \rho dt$.

a. Write down the intertemporal budget constraint for this problem.

Answer: Let ω be the proportion of wealth invested in the risky asset. Then the intertemporal budget constraint is

$$dW = \omega[\mu - r]Wdt + [rW - C]dt + \omega W\sigma dz$$

b. What are the two state variables for this consumption-portfolio choice problem? Write down the stochastic, continuous-time Bellman equation for this problem.

Answer: The two state variables are wealth, W_t, and the instantaneous maturity risk-free interest rate, r_t. Therefore, the Bellman equation is

$$0 = \max_{\{C,\omega\}}\left\{e^{-\phi t}u(C_t) + J_t + [\omega(\mu - r)W + rW - C]J_W + a[b-r]J_r \right.$$
$$\left. + \frac{1}{2}\omega^2W^2\sigma^2J_{WW} + \frac{1}{2}s^2J_{rr} + \omega W\rho\sigma sJ_{Wr}\right\}$$

c. Take the first-order conditions for the optimal choices of consumption and the demand for the risky asset.

Answer:

$$e^{-\phi t}u_C = J_W$$
$$(\mu - r)WJ_W + \omega W^2\sigma^2J_{WW} + W\rho\sigma sJ_{Wr} = 0$$

d. Show how the demand for the risky asset can be written as two terms: one term that would be present even if r were constant and another term that exists due to changes in r (investment opportunities).

Answer: Solving for ω, we have

$$\omega = -\frac{J_W}{J_{WW}W}\frac{\mu - r}{\sigma^2} - \frac{J_{Wr}}{J_{WW}W}\frac{\rho s}{\sigma}$$

The second term reflects the influence of changes in the risk-free interest rate on the optimal proportion of wealth held in the risky asset.

3. Consider the following resource allocation-portfolio choice problem faced by a university. The university obtains “utility” (e.g., an enhanced reputation for its students, faculty, and alumni) from carrying out research and teaching in two different areas: the “arts” and the “sciences.” Let C_a be the number of units of arts activities “consumed” at the university and let C_s be the number of science activities consumed at the university. At date 0, the university is assumed to maximize an expected utility function of the form

$$E_0\left[\int_0^\infty e^{-\phi t}u(C_a(t), C_s(t))dt\right]$$

where $u(C_a, C_s)$ is assumed to be increasing and strictly concave with respect to the consumption levels. It is assumed that the cost (or price) of consuming a unit of arts activity is fixed at one. In other words, in what follows we express all values in terms of units of the arts activity, making units of the arts activity the numeraire. Thus, consuming C_a units of the arts activity always costs C_a. The cost (or price) of consuming one unit of science activity at date t is given by $S(t)$, implying that the university’s expenditure on C_s units of science activities costs SC_s. $S(t)$ is assumed to follow the process

$$dS/S = \alpha_s dt + \sigma_s d\zeta$$

where α_s and σ_s may be functions of S.

The university is assumed to fund its consumption of arts and sciences activities from its endowment. The value of its endowment is denoted W_t. It can be invested in either a risk-free asset or a risky asset. The risk-free asset pays a constant rate of return equal to r. The price of the risky asset is denoted P and is assumed to follow the process

$$dP/P = \mu dt + \sigma dz$$

where μ and σ are constants and $dzd\zeta = \rho dt$. Let ω denote the proportion of the university’s endowment invested in the risky asset, and thus $(1-\omega)$ is the proportion invested in the risk-free asset. The university’s problem is then to maximize its expected utility by optimally selecting C_a, C_s, and ω.

a. Write down the university's intertemporal budget constraint, that is, the dynamics for its endowment, W_t.

Answer:

$$\begin{aligned} dW &= \omega W(dP/P) + (1-\omega)Wrdt - (C_a + SC_s)dt \\ &= [\omega(\mu - r)W + rW - C_a - SC_s]dt + \omega W\sigma dz \end{aligned}$$

b. What are the two state variables for this problem? Define a "derived utility of endowment" (wealth) function and write down the stochastic, continuous-time Bellman equation for this problem.

Answer: The two state variables are W_t and S_t. Note that P_t is not since μ and σ are constants, that is, there is a constant investment opportunity set. The Bellman equation is

$$\begin{aligned} 0 = \max_{\omega, C_a, C_s} \; & e^{-\phi t}u(C_a, C_s) + J_t + [\omega(\mu - r)W + rW - C_a - SC_s]J_W + \alpha_s SJ_S \\ & + \frac{1}{2}\omega^2 W^2\sigma^2 J_{WW} + \frac{1}{2}\sigma_s^2 S^2 J_{SS} + \omega\rho\sigma\sigma_s WSJ_{WS} \end{aligned}$$

c. Write down the first-order conditions for the optimal choices of C_a, C_s, and ω.

Answer:

$$e^{-\phi t}\frac{\partial u}{\partial C_a} - J_W = 0$$

$$e^{-\phi t}\frac{\partial u}{\partial C_s} - SJ_W = 0$$

$$(\mu - r)WJ_W + \omega W^2\sigma^2 J_{WW} + \rho\sigma\sigma_s WSJ_{WS} = 0$$

d. Show how the demand for the risky asset can be written as two terms, a standard (single-period) portfolio demand term and a hedging term.

Answer: Re-writing the first order condition for ω, one obtains

$$\omega = -\frac{J_W}{WJ_{WW}}\frac{\mu - r}{\sigma^2} - \frac{J_{WS}}{WJ_{WW}}\frac{\rho\sigma_s S}{\sigma}$$

e. For the special case in which utility is given by $u(C_a, C_s) = C_a^\theta C_s^\beta$, solve for the university's optimal level of arts activity in terms of the level and price of the science activity.

Answer: Using the first order conditions for C_a and C_s, one can show

$$S\frac{\partial u}{\partial C_a} = \frac{\partial u}{\partial C_s} = S\theta C_a^{\theta-1}C_s^\beta = \beta C_a^\theta C_s^{\beta-1}$$

which implies

$$C_a = \frac{\theta}{\beta}SC_s$$

4. Consider an individual's intertemporal consumption, labor, and portfolio choice problem for the case of a risk-free asset and a single risky asset. The individual maximizes expected lifetime utility of the form

$$E_0\left\{\int_0^T e^{-\phi t}u(C_t,L_t)dt + B(W_T)\right\}$$

where C_t is the individual's consumption at date t and L_t is the amount of labor effort that the individual exerts at date t. $u(C_t,L_t)$ is assumed to be an increasing concave function of C_t but a decreasing concave function of L_t. The risk-free asset pays a constant rate of return equal to r per unit time, and the price of the risky asset, S, satisfies the process

$$dS/S = \mu dt + \sigma dz$$

where μ and σ are constants. For each unit of labor effort exerted at date t, the individual earns an instantaneous flow of labor income of $L_t y_t dt$. The return to effort or wage rate, y_t, is stochastic and follows the process

$$dy = \mu_y(y)dt + \sigma_y(y)d\zeta$$

where $dzd\zeta = \rho dt$.

a. Letting ω be the proportion of wealth invested in the risky asset, write down the intertemporal budget constraint for this problem.

Answer: $dW = \omega(\mu - r)Wdt + (rW - C + Ly)dt + \omega W\sigma dz$

b. What are the state variables for this problem? Write down the stochastic, continuous-time Bellman equation for this problem.

Answer: The two state variables are wealth, W_t, and the return to effort, y_t. Therefore, the Bellman equation is

$$0 = \max_{(C,\omega,L)}\left\{e^{-\phi t}u[C_t,L_t] + J_t + [\omega(\mu - r)W + (rW - C + Ly)]J_W + \mu_y J_y \right.$$
$$\left. + \frac{1}{2}\omega^2 W^2\sigma^2 J_{WW} + \frac{1}{2}\sigma_y^2 J_{yy} + \omega W\rho\sigma\sigma_y J_{Wy}\right\}$$

c. Take the first-order conditions with respect to each of the individual's decision variables.

Answer:

$$e^{-\phi t}u_c = J_W$$
$$e^{-\phi t}u_L = -yJ_W$$
$$(\mu - r)WJ_W + \omega W^2\sigma^2 J_{WW} + W\rho\sigma\sigma_y J_{Wy} = 0$$

d. Show how the demand for the risky asset can be written as two terms: one term that would be present even if y were constant and another term that exists due to changes in y.

Answer: $$\omega = -\frac{J_W}{J_{WW}W}\frac{\mu - r}{\sigma^2} - \frac{J_{Wy}}{J_{WW}W}\rho\frac{\sigma_y}{\sigma}$$

e. If $u(C_t, L_t) = \gamma \ln[C_t] + \beta \ln[L_t]$, solve for the optimal amount of labor effort in terms of the optimal level of consumption.

Answer: Using the first order conditions, we have $u_L = yu_c$. Substituting in for the assumed form of the utility function gives $L = -\frac{\beta C}{\gamma y}$.

5. Consider an individual's intertemporal consumption and portfolio choice problem for the case of two risky assets (with no risk-free asset). The individual maximizes expected lifetime utility of the form

$$E_0\left\{\int_0^\infty e^{-\phi t} u(C_t)dt\right\}$$

where C_t is the individual's consumption at date t. The individual's portfolio can be invested in a stock whose price, S, follows the process

$$dS/S = \mu dt + \sigma dz$$

and a default-risky bond whose price, B, follows the process

$$dB = rBdt - Bdq$$

where dq is a Poisson counting process defined as

$$dq = \begin{cases} 1 & \text{if a default occurs} \\ 0 & \text{otherwise} \end{cases}$$

The probability of a default occurring over time interval dt is λdt. μ, σ, r, and λ are assumed to be constants. Note that the bond earns a rate of return equal to r when it does not default, but when default occurs, the total amount invested in the bond is lost; that is, the bond price goes to zero, $dB = -B$. We also assume that if default occurs, a new default-risky bond, following the same original bond price process given above, becomes available, so that the individual can always allocate her wealth between the stock and a default-risky bond.

a. Letting ω be the proportion of wealth invested in the stock, write down the intertemporal budget constraint for this problem.

Answer: $dW = (\omega(\mu - r)W + rW - c)dt + \omega W\sigma dz - (1-\omega)Wdq$

b. Write down the stochastic, continuous-time Bellman equation for this problem. *Hint:* recall that the Dynkin operator, $L[J]$, reflects the drift terms from applying Itô's lemma to J. In this problem, these terms need to include the expected change in J from jumps in wealth due to bond default.

Answer: Note that when default occurs, W goes to ωW so that $J(W)$ goes to $J(\omega W)$. Therefore, the Bellman equation is

$$0 = \max_{\{C,\omega\}} \left\{ e^{-\phi t} u(C_t) + J_t + \left[\omega(\mu - r)W + rW - C\right] J_W + \frac{1}{2}\omega^2 W^2 \sigma^2 J_{WW} + \lambda[J(\omega W) - J(W)] \right\}$$

c. Take the first-order conditions with respect to each of the individual's decision variables.

Answer:

$$e^{-\phi t} u_c = J_W$$

$$(\mu - r)WJ_W(W) + \omega W^2 \sigma^2 J_{WW}(W) + \lambda W J_W(\omega W) = 0$$

d. Since this problem reflects constant investment opportunities, it can be shown that when $u(C_t) = c^\gamma/\gamma,\ \gamma < 1$, the derived utility-of-wealth function takes the form $J(W,t) = ae^{-\phi t} W^\gamma/\gamma$, where a is a positive constant. For this constant relative-risk-aversion case, derive the conditions for optimal C and ω in terms of current wealth and the parameters of the asset price processes. *Hint:* an explicit formula for ω in terms of all of the other parameters may not be possible because the condition is nonlinear in ω.

Answer: Substituting the given functional forms into the above first-order conditions, we find

$$C^{\gamma-1} = aW^{\gamma-1}$$

or

$$C = a^{\frac{1}{\gamma-1}} W$$

and

$$(\mu - r)W^{1-\gamma} + \omega\sigma^2(1-\gamma)W^{1-\gamma} + \lambda(\omega W)^{1-\gamma} = 0$$

or

$$\omega = \frac{\mu - r}{(1-\gamma)\sigma^2} + \frac{\lambda}{(1-\gamma)\sigma^2}\omega^{\gamma-1}$$

e. Maintaining the constant relative-risk-aversion assumption, what is the optimal ω if $\lambda = 0$? Assuming the parameters are such that $0 < \omega < 1$ for this case, how would a small increase in λ affect ω, the proportion of the portfolio held in the stock?

Answer: From the optimality condition for ω, we see $\omega = \frac{\mu - r}{(1-\gamma)\sigma^2}$ when $\lambda = 0$. From this point, a small increase in λ, taking r as given, increases ω since if we differentiate the optimality condition with respect to ω and λ we have

$$d\omega = -\frac{\lambda}{\sigma^2}\omega^{\gamma-2}d\omega + \frac{1}{(1-\gamma)\sigma^2}\omega^{\gamma-1}d\lambda$$

or

$$d\omega\left[1+\frac{\lambda}{\sigma^2}\omega^{\gamma-2}\right] = \frac{1}{(1-\gamma)\sigma^2}\omega^{\gamma-1}d\lambda$$

Thus,

$$\frac{d\omega}{d\lambda} = \frac{\omega^{\gamma-1}}{(1-\gamma)\sigma^2 + \lambda(1-\gamma)\omega^{\gamma-2}} > 0$$

6. Show that a log utility investor's optimal consumption for the continuous time problem, equation (12.50), is comparable to that of the discrete-time problem, equation (5.33).

Answer: Optimal consumption for the discrete time problem is

$$C_t^* = \frac{1}{1+\delta+\cdots+\delta^{T-t}}W_t = \frac{1-\delta}{1-\delta^{T-t+1}}W_t \tag{5.34}$$

where δ was the individual's subjective discount factor. The analogous continuous time problem analyzed in the current chapter showed that a log utility investor's optimal consumption was of the form

$$C_t^* = \frac{\rho}{1-(1-\rho)e^{-\rho(T-t)}}W_t \tag{12.59}$$

where ρ is the individual's subjective rate of time preference. If we let $\delta = e^{-\rho}$, then substituting this into the discrete time consumption plan one obtains

$$\begin{aligned} C_t^* &= \frac{1-e^{-\rho}}{1-e^{-\rho(T-t+1)}}W_t \\ &= \frac{1-e^{-\rho}}{1-e^{-\rho}e^{-\rho(T-t)}}W_t \end{aligned}$$

Since $e^{-\rho} = 1-\rho+\frac{\rho^2}{2!}-\frac{\rho^3}{3!}+\cdots$, we see that the discrete time and continuous time solutions are approximately equal. The difference is likely due to all consumption occurring at the beginning of each period in the discrete time case whereas consumption is spread evenly over time in the continuous time case.

7. Use the martingale approach to consumption and portfolio choice to solve the following problem. An individual can choose between a risk-free asset paying the interest rate r and a single risky asset whose price satisfies the geometric Brownian motion process

$$\frac{dS}{S} = \mu dt + \sigma dz$$

where r, μ, and σ are constants. This individual's lifetime utility function is time separable, has no bequest function, and displays constant relative risk aversion:

$$E_t\left[\int_t^T e^{-\rho s}\frac{C_s^\gamma}{\gamma}ds\right]$$

a. Assuming an absence of arbitrage, state the form of the market price of risk, θ, in terms of the asset return parameters and write down the process followed by the pricing kernel, dM/M. You need not give any derivations.

Answer: In the absence of arbitrage, the market price of risk is $\theta = (\mu - r)/\sigma$ and the pricing kernel satisfies

$$dM/M = -rdt - \theta dz$$

which is geometric Brownian motion.

b. Write down the individual's consumption choice problem as a static maximization subject to a wealth constraint, where W_t is current wealth and λ is the Lagrange multiplier for the wealth constraint. Derive the first-order conditions for $C_s \forall s \in [t,T]$ and solve for the optimal C_s as a function of λ and M_s.

Answer:

$$\begin{aligned}\max_{C_S \forall s\in[t,T]} & E_t\left[\int_t^T e^{-\rho s}\frac{C_s^\gamma}{\gamma}ds\right] \\ & +\lambda\left(M_t W_t - E_t\left[\int_t^T M_s C_s ds\right]\right)\end{aligned}$$

The first order conditions are

$$e^{-\rho s}C_s^{\gamma-1} = \lambda M_s \quad \forall s\in[t,T]$$

Solving, one obtains

$$C_s^* = e^{-\frac{\rho s}{1-\gamma}}\lambda^{-\frac{1}{1-\gamma}}M_s^{-\frac{1}{1-\gamma}}, \quad \forall s\in[t,T]$$

c. Write down the valuation equation for current wealth, W_t, in terms of λ, M_t, and an integral of expected functions of the future values of the pricing kernel. Given the previous assumptions that the asset price parameters are constants, derive the closed-form solution for this expectation.

Answer:

$$\begin{aligned}W_t &= \lambda^{-\frac{1}{1-\gamma}}E_t\left[\int_t^T \frac{M_s}{M_t}e^{-\frac{\rho s}{1-\gamma}}M_s^{-\frac{1}{1-\gamma}}ds\right] \\ &= \lambda^{-\frac{1}{1-\gamma}}M_t^{-1}\left[\int_t^T e^{-\frac{\rho s}{1-\gamma}}E_t\left[M_s^{-\frac{\gamma}{1-\gamma}}\right]ds\right]\end{aligned}$$

Now note that if $m_t = \ln(M_t)$, then from Itô's lemma we have $dm = -(r + \frac{1}{2}\theta^2)dt - \theta dz$, which follows arithmetic Brownian motion. Thus

$$\ln M_s = \ln M_t - \left(r + \frac{1}{2}\theta^2\right)(s-t) - \theta(z(s) - z(t))$$
$$M_s = M_t e^{-\left(r+\frac{1}{2}\theta^2\right)(s-t)-\theta(z(s)-z(t))}$$
$$M_s^{-\frac{\gamma}{1-\gamma}} = M_t^{-\frac{\gamma}{1-\gamma}} e^{\frac{\gamma}{1-\gamma}\left(r+\frac{1}{2}\theta^2\right)(s-t)+\frac{\gamma}{1-\gamma}\theta(z(s)-z(t))}$$

Therefore,

$$E_t\left[M_s^{-\frac{\gamma}{1-\gamma}}\right] = M_t^{-\frac{\gamma}{1-\gamma}} E_t\left[e^{\frac{\gamma}{1-\gamma}\left(r+\frac{1}{2}\theta^2\right)(s-t)+\frac{\gamma}{1-\gamma}\theta(z(s)-z(t))}\right]$$
$$= M_t^{-\frac{\gamma}{1-\gamma}} e^{\frac{\gamma}{1-\gamma}\left(r+\frac{1}{2}\theta^2\right)(s-t)+\frac{\gamma^2}{2(1-\gamma)^2}\theta^2(s-t)}$$
$$= M_t^{-\frac{\gamma}{1-\gamma}} e^{\frac{\gamma}{1-\gamma}\left[r+\frac{\theta^2}{2(1-\gamma)}\right](s-t)}$$

Making this substitution into the above, we have

$$W_t = \lambda^{-\frac{1}{1-\gamma}} M_t^{-\frac{1}{1-\gamma}}\left[\int_t^T e^{-\frac{\rho s}{1-\gamma}+\frac{\gamma}{1-\gamma}\left[r+\frac{\theta^2}{2(1-\gamma)}\right](s-t)} ds\right]$$
$$= \lambda^{-\frac{1}{1-\gamma}} M_t^{-\frac{1}{1-\gamma}} e^{-\frac{\rho t}{1-\gamma}}\left[\int_t^T e^{\frac{\gamma}{1-\gamma}\left[r-\frac{\rho}{\gamma}+\frac{\theta^2}{2(1-\gamma)}\right](s-t)} ds\right]$$
$$= \lambda^{-\frac{1}{1-\gamma}} M_t^{-\frac{1}{1-\gamma}} e^{-\frac{\rho t}{1-\gamma}}\left[\int_t^T e^{-a(s-t)} ds\right]$$

where $a \equiv \frac{-\gamma}{1-\gamma}\left[r - \frac{\rho}{\gamma} + \frac{\theta^2}{2(1-\gamma)}\right]$. Calculating this integral, one obtains

$$W_t = \lambda^{-\frac{1}{1-\gamma}} M_t^{-\frac{1}{1-\gamma}} e^{-\frac{\rho t}{1-\gamma}}\left[\frac{1-e^{-a(T-t)}}{a}\right]$$

d. From the answer in part (c), show that optimal consumption is of the form

$$C_t^* = \frac{a}{1-e^{-a(T-t)}} W_t$$

where a is a function of r, ρ, γ, and θ.

Answer: Noting from above that $C_s^* = \lambda^{-\frac{1}{1-\gamma}} M_s^{-\frac{1}{1-\gamma}} e^{-\frac{\rho s}{1-\gamma}}$, we have

$$W_t = C_t^*\left[\frac{1-e^{-a(T-t)}}{a}\right]$$

or

$$C_t^* = \frac{a}{1-e^{-a(T-t)}} W_t$$

e. Describe how you next would calculate the optimal portfolio proportion invested in the risky asset, ω, given the results of parts (a)–(d).

Answer: The dynamics of wealth equal

$$dW = \omega(\mu - r)Wdt + (rW - C)dt + W\omega\sigma dz$$

and since $W(M_t, t)$ we have from Itô's lemma that

$$dW = \left(-rMW_M + \frac{\partial W}{\partial t} + \frac{1}{2}\theta^2 M^2 W_{MM}\right)dt - W_M \theta M dz$$

This implies

$$\omega = -\frac{W_M}{W\sigma}\theta M$$

and from the answer in 1.c we see that $W_M = -\frac{W}{(1-\gamma)M}$. Substituting this into the above, we arrive at the result

$$\omega = \frac{\theta}{(1-\gamma)\sigma} = \frac{\mu - r}{(1-\gamma)\sigma^2}$$

Answers to Chapter 13 Exercises

1. Consider a CIR economy similar to the log utility example given in this chapter. However, instead of the productive technologies following the processes of equation (13.46), assume that they satisfy

$$d\eta_i/\eta_i = \hat{\mu}_i x\,dt + \sigma_i dz_i, \quad i = 1, \ldots, n$$

In addition, rather than assume that the state variable follows the process (13.47), suppose that it is given by

$$dx = (a_0 + a_1 x)dt + b_0 d\zeta$$

Where $dz_i d\zeta = \rho_i dt$. It is assumed that $a_0 > 0$ and $a_1 < 0$.

a. Solve for the equilibrium risk-free interest rate, r, and the process it follows, dr. What parametric assumptions are needed for the unconditional mean of r to be positive?

Answer: We can write the technologies' $n \times 1$ vector of expected rates of return as $\mu = \hat{\mu}x$. This implies that the equilibrium interest rate equals

$$r = \frac{e'\Omega^{-1}\hat{\mu}x - 1}{e'\Omega^{-1}e} = \alpha_0 x + \alpha_1$$

where $\alpha_0 \equiv (e'\Omega^{-1}\hat{\mu})/e'\Omega^{-1}e > 0$ and $\alpha_1 \equiv -1/e'\Omega^{-1}e < 0$ are both constants. Since $dr = \alpha_0 dx$ and $x = (r - \alpha_1)/\alpha_0$, the process for the risk free rate can be written as

$$\begin{aligned} dr &= \alpha_0(a_0 + a_1 x)dt + \alpha_0 b_0 d\zeta \\ &= \alpha_0(a_0 + a_1(r - \alpha_1)/\alpha_0)dt + \alpha_0 b_0 d\zeta \\ &= (\alpha_0 a_0 - a_1\alpha_1 + a_1 r)dt + \alpha_0 b_0 d\zeta \\ &= \kappa(\theta - r)dt + \sigma d\zeta \end{aligned}$$

where $\kappa = -a_1 > 0$, $\theta = -(\alpha_0 a_0 - a_1\alpha_1)/a_1 = -\alpha_0 a_0/a_1 + \alpha_1$, and $\sigma = \alpha_0 b_0$. All of these newly defined parameters are constants. θ will be positive when $\alpha_0 a_0 > a_1\alpha_1$, a condition that is necessary for the unconditional mean of the interest rate to be positive.

b. Derive the optimal (market) portfolio weights for this economy, ω^*. How does ω^* vary with r?

Answer: For this problem, we have

$$
\begin{aligned}
\omega^* &= \Omega^{-1}(\boldsymbol{\mu}\alpha - re) \\
&= \Omega^{-1}\left(\boldsymbol{\mu}\frac{r-\alpha_1}{\alpha_0} - re\right) \\
&= \Omega^{-1}\left(r\boldsymbol{\mu}\frac{e'\Omega^{-1}e}{e'\Omega^{-1}\boldsymbol{\mu}} - \boldsymbol{\mu}\frac{\alpha_1}{\alpha_0} - re\right) \\
&= r\left[\frac{\Omega^{-1}\boldsymbol{\mu}e'\Omega^{-1}e - e\Omega^{-1}e'\Omega^{-1}\boldsymbol{\mu}}{e'\Omega^{-1}\boldsymbol{\mu}}\right] - \frac{\Omega^{-1}\boldsymbol{\mu}\alpha_1}{\alpha_0} \\
&= -\frac{\Omega^{-1}\boldsymbol{\mu}\alpha_1 e'\Omega^{-1}e}{e'\Omega^{-1}\boldsymbol{\mu}} = \frac{\Omega^{-1}\boldsymbol{\mu}}{e'\Omega^{-1}\boldsymbol{\mu}}
\end{aligned}
$$

Therefore, we see that the vector of optimal risky-asset portfolio proportions is constant, independent of r.

c. Derive the partial differential equation for $P(r,t,T)$, the date t price of a default-free discount bond that matures at date T. Does this equation look familiar?

Answer: Note that Ω is constant, as is ω^*. Therefore, let $\Upsilon \equiv \omega^{*\prime}\phi_1$ where ϕ_1 is the $n \times 1$ vector whose i^{th} element equals $\sigma\sigma_i\rho_i = \alpha_0 b_0 \sigma_i \rho_i$. Hence, Υ is a scalar constant. Therefore, based on equation (13.42), the partial differential equation (PDE) for $P(r,t,T)$ takes the form

$$0 = \frac{1}{2}P_{rr}\sigma^2 + P_t + P_r[\kappa(\theta - r) - \Upsilon] - rP$$

This equation is the same form as the one derived in Chapter 9, equation (9.26). Thus, we have provided an equilibrium justification for the Vasicek model that we derived earlier based on a no-arbitrage argument. Hence, the solution to this PDE is of the form (9.28).

2. Consider the intertemporal consumption-portfolio choice model and the Intertemporal Capital Asset Pricing Model of Merton and its general equilibrium specification by Cox, Ingersoll, and Ross.

a. What assumptions are needed for the single-period Sharpe-Treyner-Linter-Mossin CAPM results to hold in this multiperiod environment where consumption and portfolio choices are made continuously?

Answer: For the single-period CAPM results to obtain, one needs to assume constant investment opportunities so that risky asset rates of return have constant means, variances, and covariances and the risk-free interest rate is constant. This implies that risky asset prices follow geometric Brownian motion and are lognormally distributed over discrete intervals. Alternatively, if all individuals have log utility, the CAPM results hold instantaneously, though asset prices' means and covariance matrix can be changing over time.

b. Briefly discuss the portfolio choice implications of a situation in which the instantaneous real interest rate, $r(t)$, is stochastic, following a mean-reverting process such as the square root process of Cox, Ingersoll, and Ross or the Ornstein-Uhlenbeck process of Vasicek. Specifically, suppose that individuals can hold the instantaneous-maturity risk-free asset, a long-maturity default-free bond, and equities (stocks) and that a rise in $r(t)$ raises all assets' expected rates of return. How would the results differ from the single-period Markowitz portfolio demands? In explaining your answer, discuss how the results are sensitive to utility displaying greater or lesser risk aversion compared to log utility.

Answer: The instantaneous-maturity interest rate can be interpreted as a state variable that changes investment opportunities. Suppose that risk-aversion is greater than log utility, so that the wealth effect exceeds the substitution effect. In this case a rise in $r(t)$ leads to an increase in optimal consumption $(\partial C/\partial r > 0)$ and a fall in $r(t)$ leads to a decline in optimal consumption $(\partial C/\partial r > 0)$. To hedge against unfavorable shifts in investment opportunities, there will be an additional hedging demand for assets that have a positive return when $r(t)$ declines. An example of such an asset is a long-term bond, so that there should be a demand for long term bonds that exceed what would be predicted by the single-period Markowitz results. Conversely, suppose that risk-aversion is less than log utility, so that the substitution effect is greater than the wealth effect. In this case a rise in $r(t)$ leads to a decline in optimal consumption $(\partial C/\partial r < 0)$ and a fall in $r(t)$ leads to a rise in optimal consumption $(\partial C/\partial r > 0)$. To hedge against unfavorable shifts in investment opportunities, there will be an additional hedging demand for assets that have a negative return when $r(t)$ declines. In this case, individuals might want to short a long-term bond to hedge against a rise in $r(t)$, that is, there will be a hedging demand for long-term borrowing (e.g., via a mortgage) so that there should be a net demand for long term bonds that is less than what would be predicted by the single-period Markowitz results. Instantaneous portfolio demands would be the same as predicted by the single-period Markowitz model for the case of log utility.

3. Consider a continuous-time version of a Lucas endowment economy. Let C_t be the aggregate dividends paid at date t, which equals aggregate consumption at date t. It is assumed to follow the lognormal process

$$dC/C = \mu_c dt + \sigma_c dz_c \tag{1}$$

where μ_c and σ_c are constants. The economy is populated with representative individuals whose lifetime utility is of the form

$$E_t\left[\int_t^\infty e^{-\rho s}\frac{C_s^\gamma}{\gamma}ds\right] \tag{2}$$

a. Solve for the process followed by the continuous-time pricing kernel, M_t. In particular, relate the equilibrium instantaneous risk-free interest rate and the market price of risk to the parameters in equation (1) and utility function (2) above.

Answer: Note that $M_t = \partial U(C_t, t)/\partial C_t = e^{-\rho t} C_t^{\gamma-1}$. Hence, applying Itô's lemma we have that the process for M_t is

$$\begin{aligned} dM &= (\gamma-1)e^{-\rho t}C^{\gamma-2}dC + \frac{1}{2}(\gamma-1)(\gamma-2)e^{-\rho t}C^{\gamma-3}(dC)^2 - \rho e^{-\rho t}C^{\gamma-1}dt \\ &= \left[(\gamma-1)\mu_c + \frac{1}{2}(\gamma-1)(\gamma-2)\sigma_c^2 - \rho\right]e^{-\rho t}C^{\gamma-1}dt - (1-\gamma)\sigma_c e^{-\rho t}C^{\gamma-1}dz_c \end{aligned}$$

or

$$\frac{dM}{M} = -\left[(1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)(2-\gamma)\sigma_c^2 + \rho\right]dt - (1-\gamma)\sigma_c dz_c$$

Hence, we see that $r = \rho + (1-\gamma)\mu_c - \frac{1}{2}(1-\gamma)(2-\gamma)$ and $\theta = (1-\gamma)\sigma_c$.

b. Suppose that a particular risky asset's price follows the process

$$dS/S = \mu_s dt + \sigma_s dz_s$$

where $dz_s dz_c = \rho_{sc} dt$. Derive a value for μ_s using the pricing kernel process.

Answer: Because $S_t M_t = E_t[S_T M_T]$, we know that $d(SM)$ must follow a martingale (zero drift) process. Using Itô's lemma we have

$$\begin{aligned} d(SM) &= MdS + SdM + dSdM \\ &= MS(dS/S) + MS(dM/M) + MS(dSdM) \end{aligned}$$

Thus,

$$\frac{dSM}{SM} = (\mu_s - r - \sigma_s\theta\rho_{sc})dt + \sigma_s dz_s - \theta dz_c$$

Setting the drift equal to zero, we have

$$\frac{\mu_s - r}{\sigma_s} = \theta\rho_{sc}$$

c. From the previous results, show that Merton's Intertemporal Capital Asset Pricing Model (ICAPM) and Breeden's Consumption Capital Asset Pricing Model (CCAPM) hold between this particular risky asset and the market portfolio of all risky assets.

Answer: We note that aggregate consumption is the dividends paid by the market portfolio. Hence, the return on the market and aggregate consumption must be perfectly correlated. This implies

$$\frac{\mu_m - r}{\sigma_m} = \theta = (1-\gamma)\sigma_c$$

Now Breeden's CCAPM says

$$\mu_s - r = \left(\frac{\beta_{sc}}{\beta_{mc}}\right)(\mu_m - r)$$

where $\beta_{ic} = Cov((dS_i/S_i, dC/C))/Var(dC/C)$. Thus,

$$\mu_s - r = \left(\frac{\sigma_s \sigma_c \rho_{sc}}{\sigma_m \sigma_c}\right)(\mu_m - r)$$

or

$$\frac{\mu_s - r}{\sigma_s} = \rho_{sc}\frac{\mu_m - r}{\sigma_m} = \rho_{sc}\theta$$

which is implied by our previous derivation in 3.b. Hence, the CCAPM holds and this result also shows that the ICAPM (and single-period CAPM) relationship also holds.

Answers to Chapter 14 Exercises

1. In the Constantinides habit persistence model, suppose that there are three, rather than two, technologies. Assume that there are the risk-free technology and two risky technologies:

$$dB/B = rdt$$
$$dS_1/S_1 = \mu_1 dt + \sigma_1 dz_1$$
$$dS_2/S_2 = \mu_2 dt + \sigma_2 dz_2$$

where $dz_1 dz_2 = \phi dt$. Also assume that the parameters are such that there is an interior solution for the portfolio weights (all portfolio weights are positive). What would be the optimal consumption and portfolio weights for this case?

Answer: Let ω_1 and ω_2 be the proportions of wealth held in risky technology one and two, respectively. Then the dynamics for wealth are now

$$dW = \{[(\mu_1 - r)\omega_1 + (\mu_2 - r)\omega_2 + r]W - C(t)\}\,dt + \sigma_1\omega_1 W\,dz_1 + \sigma_2\omega_2 W\,dz_2$$

The Bellman equation then becomes

$$\begin{aligned} 0 &= \max_{\{C_t,\omega_1,\omega_2\}} \{U(C_t, x_t, t) + L[e^{-\rho t}J]\} \\ &= \max_{\{C_t,\omega_1,\omega_2\}} \Big\{ e^{-\rho t}\gamma^{-1}(C_t - bx_t)^{\gamma} + e^{-\rho t}J_W[((\mu_1 - r)\omega_1 + (\mu_2 - r)\omega_2 + r)W - C_t] \\ &\quad + \frac{1}{2}e^{-\rho t}J_{WW}\left[\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + 2\phi\sigma_1\sigma_2\omega_1\omega_2\right]W^2 + e^{-\rho t}J_x(C_t - ax_t) - \rho e^{-\rho t}J \Big\}. \end{aligned}$$

Taking the first order conditions with respect to C_t, ω_1, and ω_2, one obtains

$$(C_t - bx)^{\gamma-1} = J_W - J_x, \text{ or}$$
$$C_t^* = bx_t + [J_W - J_x]^{\frac{1}{\gamma-1}},$$

and

$$(\mu_1 - r)WJ_W + \left(\omega_1\sigma_1^2 + \omega_2\phi\sigma_1\sigma_2\right)W^2 J_{WW} = 0$$
$$(\mu_2 - r)WJ_W + \left(\omega_2\sigma_2^2 + \omega_1\phi\sigma_1\sigma_1\right)W^2 J_{WW} = 0$$

Solving the above two linear equations in the two unknowns, ω_1 and ω_2, one obtains

$$\omega_1^* = \frac{J_W}{(-WJ_{WW})}\left[\frac{\mu_1 - r}{\sigma_1^2(1-\phi^2)} - \phi\frac{\mu_2 - r}{\sigma_1\sigma_2(1-\phi^2)}\right]$$

$$\omega_2^* = \frac{J_W}{(-WJ_{WW})}\left[\frac{\mu_2 - r}{\sigma_2^2(1-\phi^2)} - \phi\frac{\mu_1 - r}{\sigma_1\sigma_2(1-\phi^2)}\right]$$

Note that for this constant investment opportunity set case, the ratio ω_1^*/ω_2^* is always constant. Thus, the proportion of each risky asset to the total of the both risky assets, $\delta_1 \equiv \frac{\omega_1^*}{\omega_1^*+\omega_2^*}$ and $\delta_2 \equiv \frac{\omega_2^*}{\omega_1^*+\omega_2^*}$ are constant. Hence, the solution to the consumption-portfolio choice problem is identical to the problem solved by Constantinides for the case of one risky asset where the mean and variance of the single asset's rate of return are

$$\mu = \mu_1\delta_1 + \mu_2\delta_2$$
$$\sigma^2 = \delta_1^2\sigma_1^2 + \delta_2^2\sigma_2^2 + 2\delta_1\delta_2\phi\sigma_1\sigma_2$$

2. Consider an endowment economy where a representative agent maximizes utility of the form

$$\max\sum_{t=0}^{\infty}\delta^t\frac{(C_t - X_t)^\gamma}{\gamma}$$

where X_t is a level of external habit and equals $X_t = \theta\overline{C}_{t-1}$, where $\overline{C}_{t-1}$ is aggregate consumption at date $t-1$.

a. Write down an expression for the one-period, risk-free interest rate at date t, $R_{f,t}$.

Answer: The pricing kernal for this problem is

$$m_{t,t+1} = \frac{\delta^{t+1}(C_{t+1} - X_{t+1})^{\gamma-1}}{\delta^t(C_t - X_t)^{\gamma-1}}$$

which in equalibrium equals

$$m_{t,t+1} = \delta\left(\frac{\overline{C}_{t+1} - \theta\overline{C}_t}{\overline{C}_t - \theta\overline{C}_{t-1}}\right)^{\gamma-1}$$

Therefore, the one-period riskless interest rate is $1/R_{f,t} = E_t[m_{t,t+1}]$ or

$$R_{f,t} = \frac{(\overline{C}_t - \theta\overline{C}_{t-1})^{\gamma-1}}{\delta E_t\left[(\overline{C}_{t+1} - \theta\overline{C}_t)^{\gamma-1}\right]}$$

b. If consumption growth, C_{t+1}/C_t, follows an independent and identical distribution, is the one-period riskless interest rate, $R_{f,t}$, constant over time?

Answer: Note that the risk-free rate can be written as

$$R_{f,t} = \frac{(1-\theta\bar{C}_{t-1}/\bar{C}_t)^{\gamma-1}}{\delta E_t\left[(\bar{C}_{t+1}/\bar{C}_t-\theta)^{\gamma-1}\right]}$$

If C_{t+1}/C_t follows an independent and identical distribution, the expectation in the denominator of the above expression will be constant. However, the numerator in the expression will be changing as the realizations of $\bar{C}_{t-1}$ and $\bar{C}_t$ change. Hence, $R_{f,t}$ will not be constant, but time varying.

3. The following problem is based on the work of Menzly, Santos, and Veronesi (Menzly, Santos, and Veronesi 2001). Consider a continuous-time endowment economy where agents maximize utility that displays external habit persistence. Utility is of the form

$$E_t\left[\int_0^\infty e^{-\rho t}\ln(C_t - X_t)dt\right]$$

and aggregate consumption (dividend output) follows the lognormal process

$$dC_t/C_t = \mu dt + \sigma dz$$

Define Y_t as the inverse surplus consumption ratio, that is, $Y_t \equiv \frac{C_t}{C_t - X_t} = \frac{1}{1-(X_t/C_t)} > 1$. It is assumed to satisfy the mean-reverting process

$$dY_t = k(\bar{Y} - Y_t)dt - \alpha(Y_t - \lambda)dz$$

where $\bar{Y} > \lambda \geq 1$ is the long-run mean of the inverse surplus, $k > 0$ reflects the speed of mean reversion, $\alpha > 0$. The parameter λ sets a lower bound for Y_t, and the positivity of $\alpha(Y_t - \lambda)$ implies that a shock to the aggregate output (dividend-consumption) process decreases the inverse surplus consumption ratio (and increases the surplus consumption ratio). Let P_t be the price of the market portfolio. Derive a closed-form expression for the price-dividend ratio of the market portfolio, P_t/C_t. How does P_t/C_t vary with an increase in the surplus consumption ratio?

Answer: Denote the representative individual's stochastic discount factor between dates t and $\tau > t$ as $m_{t,\tau}$. Then since

$$U_c(C_t, X_t) = e^{-\rho t}\frac{1}{C_t - X_t}$$

we have

$$m_{t,\tau} = \frac{U_c(C_\tau, X_\tau)}{U_c(C_t, X_t)} = e^{-\rho(\tau - t)}\frac{C_t - X_t}{C_\tau - X_\tau}$$

Let P_t be the date t value of the market portfolio of all assets. Specifically, it is the claim to the dividend stream equal to aggregate consumption from dates t to ∞. Thus

$$\begin{aligned}
P_t &= E_t\left[\int_t^\infty m_{t,\tau} C_\tau d\tau\right] \\
&= E_t\left[\int_t^\infty e^{-\rho(\tau-t)} \frac{C_t - X_t}{C_\tau - X_\tau} C_\tau d\tau\right] \\
&= (C_t - X_t) E_t\left[\int_t^\infty e^{-\rho(\tau-t)} \frac{C_\tau}{C_\tau - X_\tau} d\tau\right] \\
&= (C_t - X_t) E_t\left[\int_t^\infty e^{-\rho(\tau-t)} Y_\tau d\tau\right]
\end{aligned}$$

Since $E_t[Y_T] = \overline{Y} + (Y_t - \overline{Y})e^{-k(T-t)}$, then

$$\begin{aligned}
\frac{P_t}{C_t} &= \frac{1}{Y_t} E_t\left[\int_t^\infty e^{-\rho(\tau-t)} Y_\tau d\tau\right] \\
&= \frac{1}{Y_t} \int_t^\infty e^{-\rho(\tau-t)} E_t[Y_\tau] d\tau \\
&= \frac{1}{Y_t} \int_t^\infty e^{-\rho(\tau-t)} [\overline{Y} + (Y_t - \overline{Y}) e^{-k(\tau-t)}] d\tau \\
&= \frac{1}{Y_t} \left(\frac{\overline{Y}}{\rho} + \frac{Y_t - \overline{Y}}{\rho + k}\right) \\
&= \frac{1}{\rho + k} \left(1 + \frac{k}{\rho} \frac{\overline{Y}}{Y_t}\right)
\end{aligned}$$

Therefore P_t/C_t declines with an increase in the inverse surplus consumption ratio, so that it increases with an increase in the surplus consumption ratio.

4. Consider an individual's consumption and portfolio choice problem when her preferences display habit persistence. The individual's lifetime utility satisfies

$$E_t\left[\int_t^T e^{-\rho s} u(C_s, x_s)\, ds\right] \tag{1}$$

where C_s is date s consumption and x_s is the individual's date s level of habit. The individual can choose among a risk-free asset that pays a constant rate of return equal to r and n risky assets. The instantaneous rate of return on risky asset i satisfies

$$dP_i/P_i = \mu_i dt + \sigma_i dz_i, \quad i = 1, \ldots, n \tag{2}$$

where $dz_i dz_j = \sigma_{ij} dt$ and μ_i, σ_i, and σ_{ij} are constants. Thus, the individual's level of wealth, W, follows the process

$$dW = \sum_{i=1}^n \omega_i (\mu_i - r) W\, dt + (rW - C_t)\, dt + \sum_{i=1}^n \omega_i W \sigma_i\, dz_i \tag{3}$$

where ω_i is the proportion of wealth invested in risky asset i. The habit level, x_s, is assumed to follow the process

$$dx = f(\overline{C}_t, x_t)dt \tag{4}$$

where $\overline{C}_t$ is the date t consumption that determines the individual's habit.

a. Let $J(W, x, t)$ be the individual's derived utility-of-wealth function. Write down the continuous-time Bellman equation that $J(W, x, t)$ satisfies.

Answer:

$$0 = \max_{C_t, \{\omega_{i,t}\}} [e^{-\rho s} u(C_s, x_s) + L[J]].$$

$$0 = \max_{C_t, \{\omega_{i,t}\}} \left[e^{-\rho s} u(C_s, x_s) + \frac{\partial J}{\partial t} + \left[\sum_{i=1}^{n} \omega_i (\mu_i - r)W + (rW - C) \right] \frac{\partial J}{\partial W} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \omega_i \omega_j W^2 \frac{\partial^2 J}{\partial W^2} + f(\overline{C}_t, x_t) \frac{\partial J}{\partial x} \right]$$

b. Derive the first-order conditions with respect to the portfolio weights, ω_i. Does the optimal portfolio proportion of risky asset i to risky asset j, ω_i / ω_j, depend on the individual's preferences? Why or why not?

Answer: The first order conditions are

$$0 = J_W (\mu_i - r)W + J_{WW} \sum_{j=1}^{n} \sigma_{ij} \omega_j W^2, i = 1, \ldots, n.$$

which are n linear equations in n unknowns. The solutions are

$$\omega_i^* = -\frac{J_W}{J_{WW} W} \sum_{j=1}^{n} v_{ij} (\mu_j - r), i = 1, \ldots, n.$$

where $[v_{ij}] \equiv \Omega^{-1}$. Thus, the proportion of wealth in risky asset i to risky asset k is a constant, that is,

$$\frac{\omega_i^*}{\omega_k^*} = \frac{\sum_{j=1}^{n} v_{ij} (\mu_j - r)}{\sum_{j=1}^{n} v_{kj} (\mu_j - r)}$$

This means that the individual splits his portfolio between the risk-free asset, paying return r, and a portfolio of the risky assets that holds the n risky assets in constant proportions. This is due to the "constant investment opportunity set" assumption, that is, that μ_i, σ_i, and σ_{ij} are constants. Hence, two "mutual funds," one holding only the risk-free asset and the other holding a risky asset portfolio with the above weights would satisfy all investors. Only the investor's preferences, current level of wealth, W_t, and the investor's time horizon determine how much is put in the first fund and how much is allocated to the second.

c. Assume that the consumption, $\bar{C}_t$, in equation (4) is such that the individual's preferences display an internal habit, similar to the Constantinides model (Constantinides 1990). Derive the first-order condition with respect to the individual's date t optimal consumption, C_t.

Answer: In this case, $\bar{C}_t = C_t$, the individual's consumption. Hence the first order condition is

$$e^{-\rho t} u_C(C_t, x_t) - \frac{\partial J}{\partial W} + \frac{\partial f(C_t, x_t)}{\partial C_t} \frac{\partial J}{\partial x} = 0$$

d. Assume that the consumption, $\bar{C}_t$, in equation (4) is such that the individual's preferences display an external habit, similar to the Campbell-Cochrane model (Campbell and Cochrane 1999). Derive the first-order condition with respect to the individual's date t optimal consumption, C_t.

Answer: In this case, $\bar{C}_t = C_t^A$, where C_t^A is per capita aggregate consumption. The individual takes this as given, so that the first order condition is simply

$$e^{-\rho t} u_C(C_t, x_t) - \frac{\partial J}{\partial W} = 0$$

Answers to Chapter 15 Exercises

1. In the Barberis, Huang, and Santos model, verify that the first-order conditions (15.16) and (15.17) lead to the envelope condition (15.18).

   Answer: The derived utility of wealth function in (15.15) can be written as

$$J(W_t, z_t) = \frac{C_t^{*\gamma}}{\gamma} + E_t\left[b_0 \delta \bar{C}_t^{\gamma-1} w_t^* \hat{v}(R_{t+1}, z_t) + \delta J(W_{t+1}, z_{t+1})\right]$$

   where C_t^* and w_t^* are the optimal consumption and portfolio weights that satisfy the first order conditions (15.16) and (15.17). Totally differentiating with respect to W_t one obtains[1]

$$J_W(W_t, z_t) = C_t^{*\gamma-1}\frac{\partial C_t^*}{\partial W_t} + E_t\left[b_0 \delta \bar{C}_t^{\gamma-1} \hat{v}(R_{t+1}, z_t)\frac{\partial w_t^*}{\partial W_t}\right] + E_t\left[\delta J_W(W_{t+1}, z_{t+1})\left\{R_f\left(1 - \frac{\partial C_t^*}{\partial W_t}\right) + (R_{t+1} - R_f)\frac{\partial w_t^*}{\partial W_t}\right\}\right]$$

   Substituting (15.16) gives

$$J_W(W_t, z_t) = E_t\left[b_0 \delta \bar{C}_t^{\gamma-1} \hat{v}(R_{t+1}, z_t)\frac{\partial w_t^*}{\partial W_t}\right] + E_t\left[\delta J_W(W_{t+1}, z_{t+1})\left\{R_f + (R_{t+1} - R_f)\frac{\partial w_t^*}{\partial W_t}\right\}\right]$$

   and substituting (15.17) gives

$$J_W(W_t, z_t) = E_t\left[\delta J_W(W_{t+1}, z_{t+1})\right]$$

   Using (15.16) once again, we see that the above equation equals $J_W(W_t, z_t) = C_t^{\gamma-1}$.

2. In the Barberis, Huang, and Santos model, solve for the price-dividend ratio, P_t/D_t, for Economy II when utility is standard constant relative risk aversion, that is,

$$E_0\left[\sum_{t=0}^{\infty} \delta^t \frac{C_t^\gamma}{\gamma}\right]$$

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[1] Recall that $W_{t+1} = (W_t + Y_t - C_t)R_f + w_t(R_{t+1} - R_f)$.

Answer: As shown in Chapter 6, the price of the market portfolio equals

$$P_t = E_t\left[\sum_{j=1}^{\infty}\delta^j \frac{C_{t+j}^{\gamma-1}}{C_t^{\gamma-1}} D_{t+j}\right]$$

or

$$\begin{aligned} P_t/D_t &= E_t\left[\sum_{j=1}^{\infty}\delta^j \left(\frac{C_{t+j}}{C_t}\right)^{\gamma-1}\frac{D_{t+j}}{D_t}\right] \\ &= E_t\left[\sum_{j=1}^{\infty}\delta^j e^{(\gamma-1)\ln(C_{t+j}/C_t)+\ln(D_{t+j}/D_t)}\right] \end{aligned}$$

Now

$$\begin{aligned} \ln(C_{t+j}/C_t) &= j\cdot g_C + \sigma_C \textstyle\sum_{i=1}^{j}\eta_{t+j} \\ \ln(D_{t+j}/D_t) &= j\cdot g_D + \sigma_D \textstyle\sum_{i=1}^{j}\varepsilon_{t+j} \end{aligned}$$

so that

$$\begin{aligned} P_t/D_t &= E_t\left[\sum_{j=1}^{\infty}\delta^j e^{(\gamma-1)\left(j.g_C+\sigma_C\sum_{i=1}^{j}n_{t+j}\right)+j.g_D+\sigma_D\sum_{i=1}^{j}\varepsilon_{t+j}}\right] \\ &= Et\left[\sum_{j=1}^{\infty}\delta^j e^{j[(\gamma-1)g_C+g_D]+\sum_{i=1}^{j}[(\gamma-1)\sigma C\eta_{t+j}+\sigma_D\varepsilon_{t+j}]}\right] \\ &= \sum_{j=1}^{\infty}\delta^j e^{j[(\gamma-1)g_C+g_D]}e^{\frac{j}{2}[(1-\gamma)^2\sigma_C^2+\sigma_D^2-2(1-\gamma)\sigma_C\sigma_D\rho]} \\ &= \sum_{j=1}^{\infty}e^{j[\ln\delta-(1-\gamma)g_C+g_D+\frac{1}{2}\left((1-\gamma)^2\sigma_C^2+\sigma_D^2\right)-(1-\gamma)\sigma_C\sigma_D\rho]} \\ &= \frac{1}{1-\delta e^{-(1-\gamma)g_C+g_D+\frac{1}{2}\left((1-\gamma)^2\sigma_C^2+\sigma_D^2\right)-(1-\gamma)\sigma_C\sigma_D\rho}}-1 \end{aligned}$$

3. In the Kogan, Ross, Wang, and Westerfield model, verify that $\lambda = e^{-\gamma\eta\sigma^2 T}$ satisfies the equality $W_{r,0} = W_{n,0}$.

Answer: Equating $W_{r,0}$ to $W_{n,0}$ leads to

$$\frac{E_0\left[\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}D_T^{\gamma}\right]}{E_0\left[\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{1-\gamma}D_T^{\gamma-1}\right]} = \frac{E_0\left[(\lambda\xi_T)^{\frac{1}{1-\gamma}}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}D_T^{\gamma}\right]}{E_0\left[\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{1-\gamma}D_T^{\gamma-1}\right]}$$

or since the denominators are equal

$$E_0\left[D_T^{\gamma}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right] = E_0\left[D_T^{\gamma}(\lambda\xi_T)^{\frac{1}{1-\gamma}}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$

Now if $\lambda = e^{-\gamma\eta\sigma^2 T}$, then

$$\lambda\xi_T = e^{-\gamma\eta\sigma^2 T} e^{-\frac{1}{2}\sigma^2\eta^2 T + \sigma\eta(z_T - z_0)}$$
$$= e^{-\left(\gamma+\frac{1}{2}\eta\right)\eta\sigma^2 T + \sigma\eta(z_T - z_0)}$$

and

$$(\lambda\xi_T)^{\frac{1}{1-\gamma}} = e^{-\frac{1}{1-\gamma}\left(\gamma+\frac{1}{2}\eta\right)\eta\sigma^2 T + \frac{\sigma\eta}{1-\gamma}(z_T - z_0)}$$

Also note that

$$D_T^\gamma = e^{\gamma\left[\mu-\frac{1}{2}\sigma^2\right]T + \gamma\sigma(z_T - z_0)}$$

Thus, substituting these expressions into the above equality of expectations and canceling like deterministic terms we obtain

$$E_0\left[e^{\gamma\sigma(z_T - z_0)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right] = E_0\left[e^{-\frac{1}{1-\gamma}(\gamma+\frac{1}{2}\eta)\eta\sigma^2 T + \left(\frac{\eta}{1-\gamma}+\gamma\right)\sigma(z_T - z_0)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$

Now consider a change in probability measure, call it the W measure, by defining $dw_t = dz_t - \gamma\sigma dt$ with $w_0 = z_0$, so that $w_T - w_0 = z_T - z_0 - \gamma\sigma T$. From Chapter 10, we know that this change in measure implies that for some random variable, $x_T, E_0[x_T] = E_0^W[x_T\varsigma_T]$ where $dP_T/dW_T = \varsigma_T = e^{\frac{1}{2}\gamma^2\sigma^2 T - \gamma\sigma(z_T - z_0)}$ is the Radon-Nikodym derivative.[2] Viewing x_T as the arguments of the above expectations and making the substitution of $w_T - w_0 = z_T - z_o - \gamma\sigma T$, we obtain

$$E_0^W\left[\varsigma_T e^{\gamma\sigma(w_T - w_0 + \gamma\sigma T)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$
$$= E_0^W\left[\varsigma_T e^{-\frac{1}{1-\gamma}\left(\gamma+\frac{1}{2}\eta\right)\eta\sigma^2 T + \left(\frac{\eta}{1-\gamma}+\gamma\right)\sigma(w_T - w_0 + \gamma\sigma T)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$
$$\text{or } E_0^W\left[e^{\frac{1}{2}\gamma^2\sigma^2 T - \gamma\sigma(w_T - w_0 + \gamma\sigma T) + \gamma\sigma(w_T - w_0 + \gamma\sigma T)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$
$$= E_0^W\left[e^{\frac{1}{2}\gamma^2\sigma^2 T - \gamma\sigma(W_T + W_0 + \gamma\sigma T) - \frac{1}{1-\gamma}\left(\gamma+\frac{1}{2}\eta\right)\eta\sigma^2 T + \left(\frac{n}{1-\gamma}+\gamma\right)\sigma(w_T - w_0 + \gamma\sigma T)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$
$$\text{or } E_0^w\left[e^{\frac{1}{2}\gamma^2\sigma^2 T}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$
$$= E_0^w\left[e^{\frac{1}{2}\gamma^2\sigma^2 T - \frac{1}{1-\gamma}(\gamma+\frac{1}{2}\eta)\eta\sigma^2 T + \frac{\eta}{1-\gamma}\sigma(w_T - w_0 + \gamma\sigma T)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$

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[2]Recall that in Chapter 10 we showed that under the Q measure $c(t) = \hat{E}_t\left[e^{-\int_t^T r(s)ds}c(T)\right]$ while under the P measure $c(T) = E_t\left[M(T)/M(t)c(T)\right] = E_t\left[e^{-\int_t^T\left[r(s)-\frac{1}{2}\theta^2(s)\right]ds - \int_t^T\theta(s)dz(s)}c(T)\right]$. Hence, in this case where $dM/M = -rdt - \theta dz$, the Radon-Nikodym derivative is the ratio of the arguments under the expectations operators, that is, $dQ_T/dP_T =$ $(M_T/M_t)/e^{-\int_t^T r(s)ds} = e^{\int_t^T\frac{1}{2}\theta^2(s)ds - \int_t^T\theta(s)dz(s)}$. If r and θ are constants, this implies $dQ_T/dp_T = e^{\frac{1}{2}\theta^2(T-t) - \theta(z_T - z_t)}$.

or

$$E_0^W\left[\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]=E_0^W\left[e^{-\frac{1}{1-\gamma}\eta^2\sigma^2T+\frac{1}{1-\gamma}\eta\sigma(w_T-w_0)}\left[1+(\lambda\xi_T)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$

Now note that

$$\begin{aligned}\lambda\xi_T&=e^{-\left(\gamma+\frac{1}{2}\eta\right)\eta\sigma^2T+\sigma\eta(w_T-w_0+\gamma\sigma T)}\\&=e^{-\frac{1}{2}\eta^2\sigma^2T+\sigma\eta(w_T-w_0)}\\&\equiv\xi_T^W\end{aligned}$$

Thus, the equality of expectations can be written

$$E_0^W\left[\left[1+\left(\xi_T^W\right)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]=E_0^W\left[\left(\xi_T^W\right)^{\frac{1}{1-\gamma}}\left[1+\left(\xi_T^W\right)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]$$

Now this condition can be written as

$$E_0^W\left[\left[1-\left(\xi_T^W\right)^{\frac{1}{1-\gamma}}\right]\left[1+\left(\xi_T^W\right)^{\frac{1}{1-\gamma}}\right]^{-\gamma}\right]=0$$

It can be verified that this is the derivative at $x=0$ of the function

$$F(x)=2E_0^W\left[\left(e^{\frac{1}{2(1-\gamma)}xT}+e^{-\frac{1}{2(1-\gamma)}xT}\left(\xi_T^W\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}\right]$$

that is, it equals $\partial F(x=0)\partial x$. By making a final change in measure to, say, the Q, measure $dQ_T/dW_T=\psi_T=e^{-\frac{1}{8}\eta^2\sigma^2T+\frac{1}{2}\eta\sigma(w_T-w_0)}$, it can be shown that $F(x)=F(-x)$, which implies $\partial F(x=0)\partial x$.

4. In the Kogan, Ross, Wang, and Westerfield model, suppose that both representative individuals are rational but have different levels of risk aversion. The first type of representative individual maximizes utility of the form

$$E_0\left[\frac{C_{r,T}^{\gamma_1}}{\gamma_1}\right]$$

and the second type of representative individual maximizes utility of the form

$$E_0\left[\frac{C_{n,T}^{\gamma_2}}{\gamma_2}\right]$$

Where $1>\gamma_1>\gamma_2$. Assuming $W_{r,0}=W_{n,0}$, solve for the equilibrium price of the risky asset deflated by the discount bond maturing at date T.

Answer: The two different types of rational individuals have optimization problems that lead to

$$C_{r,T}^{\gamma_1-1} = \lambda_r M_T$$
$$C_{n,T}^{\gamma_2-1} = \lambda_n M_T$$

Substituting out for M_T implies

$$C_{r,T}^{\gamma_1-1} = \lambda C_{n,T}^{\gamma_2-1}$$

where $\lambda \equiv \lambda_r / \lambda_n$. This implies

$$C_{n,T} = \lambda^{\frac{1}{1-\gamma_2}} C_{r,T}^{(\gamma_1-1)/(\gamma_2-1)}$$

Because $C_{r,T} + C_{n,T} = D_T$, we have

$$C_{r,T}^{\gamma_1-1} = \lambda (D_T - C_{r,T})^{\gamma_2-1}$$
$$C_{r,T}^{\gamma_2-1} = (1/\lambda)(D_T - C_{r,T})^{\gamma_1-1}$$

Let $C_{r,T}^*(D_T,\lambda)$ and $C_{n,T}^*(D_T,\lambda)$ be the solutions to the above. Also, note that the present value of terminal consumption must equal the individuals' initial wealths (discounted by the value of a *T*-period discount bond), so that

$$\begin{aligned} W_{r,0} &= \frac{E_0[C_{r,T} M_T / M_0]}{E_0[M_T/M_0]} = \frac{E_0[C_{r,T} M_T]}{E_0[M_T]} \\ &= \frac{E_0\left[C_{r,T} C_{r,T}^{\gamma_1-1}/\lambda_r\right]}{E_0[M_T]} = \frac{E_0\left[C_{r,T}^{\gamma_1}/\lambda_r\right]}{E_0[M_T]} \end{aligned} \tag{0.1}$$

$$\begin{aligned} W_{n,0} &= \frac{E_0[C_{n,T} M_T / M_0]}{E_0[M_T/M_0]} = \frac{E_0[C_{n,T} M_T]}{E_0[M_T]} \\ &= \frac{E_0\left[C_{n,T} C_{n,T}^{\gamma_2-1}/\lambda_n\right]}{E_0[M_T]} = \frac{E_0\left[C_{n,T}^{\gamma_2}/\lambda_n\right]}{E_0[M_T]} \end{aligned} \tag{0.2}$$

Because it was assumed that $W_{r,0} = W_{n,0}$, we have

$$\begin{aligned} E_0\left[C_{r,T}^{\gamma_1}\right] &= \lambda E_0\left[C_{n,T}^{\gamma_2}\right] \\ &= \lambda E_0[(D_T - C_{r,T})^{\gamma_2}] \end{aligned}$$

or

$$\lambda = \frac{E_0[C^*_{r,T} = (D_T, \lambda)^{\gamma_1}]}{E_0[(D_T - C^*_{r,T}(D_T, \lambda))^{\gamma_2}]}$$

or

$$E_0[C^*_{r,T}(D_T, \lambda)^{\gamma_1} - \lambda(D_T - C^*_{r,T}(D_T, \lambda))^{\gamma_2}] = 0$$

This condition is only a function of the terminal distribution of D_T. It can be solved numerically to determine the value of λ as a function of the model parameters γ_1, γ_2, μ, and σ. Once we have this value, say λ^*, then we can write $C^*_{r,T}(D_T, \lambda^*) = C^*_{r,T}(D_T)$. Then the price of the risky asset is

$$\begin{aligned} S_t &= \frac{E_t[D_T M_T / M_t]}{E_t[M_T / M_t]} = \frac{E_t[D_T M_T]}{E_t[M_T]} \\ &= \frac{E_t[D_T C^*_{r,T}(D_T)^{\gamma_1 - 1} / \lambda_r]}{E_t[C^*_{r,T}(D_T)^{\gamma_1 - 1} / \lambda_r]} = \frac{E_t[D_T C^*_{r,T}(D_T)^{\gamma_1 - 1}]}{E_t[C^*_{r,T}(D_T)^{\gamma_1 - 1}]} \end{aligned}$$

which can be computed numerically, say by Monte Carlo simulation of the expectations.

Answers to Chapter 16 Exercises

1. Show that the maximization problem in objective function (16.6) is equivalent to the maximization problem in (16.4).

Answer: The expression in equation (16.4) can be re-written as

$$\max_{X_i} -e^{-a_i R_f (W_{0i} - P_0 X_i)} E[e^{-a_i X_i \tilde{P}_1} \mid I_i]$$

The term $E[e^{-a_i X_i \tilde{P}_1} \mid I_i]$ is in the form of the moment generating function for the random variable $\tilde{P}_1$. Since $\tilde{P}_1$ was assumed to be normally distributed, it has a well-known moment generating function. Substituting in this function, one obtains

$$\max_{X_i} -e^{-a_i R_f (W_{0i} - P_0 X_i)} e^{-a_i X_i E\left[\tilde{P}_1 \middle| I_i\right] + \frac{1}{2} a_i^2 X_i^2 \operatorname{Var}\left[\tilde{P}_1 \middle| I_i\right]}$$

Because the exponential function is monotonic, the above maximization problem is equivalent to

$$\max_{X_i} X_i (E[\tilde{P}_1 \mid I_i] - R_f P_0) - \frac{1}{2} a_i X_i^2 \operatorname{var}[\tilde{P}_1 \mid I_i]$$

2. Show that the results in (16.8) can be derived from Bayes rule and the assumption that $\tilde{P}_1$ and $\tilde{y}_i$ are normally distributed.

Answer: By assumption, the marginal distribution for $\tilde{P}_1$ is $N(m, \sigma^2)$. Since $\tilde{y}_i = \tilde{P}_1 + \tilde{\varepsilon}_i$ and $\tilde{\varepsilon}_i$: $N(0, \sigma_i^2)$ and is assumed to be independent of $\tilde{P}_1$, we have that the marginal distribution for $\tilde{y}_i$ is $N(m, \sigma^2 + \sigma_i^2)$. The covariance of $\tilde{P}_1$ and $\tilde{y}_i$ equals $E[(\tilde{P}_1 - m)(\tilde{y}_i - m)] = E[(\tilde{P}_1 - m)(\tilde{P}_1 + \tilde{\varepsilon}_i - m)] = E[(\tilde{P}_1 - m)^2 + (\tilde{P}_1 - m)\tilde{\varepsilon}_i] = E[A(\tilde{P}_1 - m)^2] = \sigma^2$. Thus, the correlation between $\tilde{P}_1$ and $\tilde{y}_i$ equals $\rho_i = \sigma^2 / \left[\sigma\sqrt{\sigma^2 + \sigma_i^2}\right] = \sigma / \sqrt{\sigma^2 + \sigma_i^2}$. Now Bayes Rule relates conditional and marginal probabilities. For a continuous distribution such as the normal, if we have two random variables, say x and z whose joint and marginal probability density functions are $f(x,z)$, $f(x)$, and $f(z)$ respectively, then Bayes Rule says that the conditional probability densities satisfy

$$f(x \mid z) = \frac{f(z \mid x) f(x)}{f(z)} = \frac{f(x,z)}{f(z)}$$

Let the joint bivariate normal density for $\tilde{P}_1$ and $\tilde{y}_i$ be $n(P_1, y_i, \rho_i)$ and denote the marginal normal densities of $\tilde{P}_1$ and $\tilde{y}_i$ by $n(P_1)$ and $n(y_i)$. Give the assumed means variances and covariances, we have

$$n(P_1, y_i, \rho_i) = \frac{1}{2\pi\sigma\sqrt{\sigma^2 + \sigma_i^2}\sqrt{1-\rho_i^2}} \exp\left[-\frac{\frac{(P_1-m)^2}{\sigma^2} - \frac{2\rho_i (P_1-m)(y_i-m)}{\sigma\sqrt{\sigma^2+\sigma_i^2}} + \frac{(y_i-m)^2}{\sigma^2+\sigma_i^2}}{2(1-\rho_i^2)} \right]$$

$$n(P_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(P_1-m)^2}{2\sigma^2} \right]$$

$$n(y_i) = \frac{1}{\sqrt{\sigma^2+\sigma_i^2}\sqrt{2\pi}} \exp\left[-\frac{(y_i-m)^2}{2(\sigma^2+\sigma_i^2)} \right]$$

Now $n(P_1, y_i, \rho_i)$ can be re-arranged to take the form

$$n(P_1, y_i, \rho_i) = n(y_i)\frac{1}{\sigma\sqrt{1-\rho_i^2}\sqrt{2\pi}} \exp\left[-\frac{\left(P_1 - \left[m\left(1-\rho_i^2\right) + y_i\right]\right)^2}{2\sigma^2\left(1-\rho_i^2\right)} \right]$$

Thus, Bayes Rule states that the conditional density of P_1 given y_i is

$$\begin{aligned} n(P_1 \mid y_i) &= \frac{n(P_1, y_i, \rho_i)}{n(y_i)} \\ &= \frac{1}{\sigma\sqrt{1-\rho_i^2}\sqrt{2\pi}} \exp\left[-\frac{(P_1 - [m(1-\rho_i^2) + y_i])^2}{2\sigma^2(1-\rho_i^2)} \right] \end{aligned}$$

Notice that $n(P_1 \mid y_i)$ is of the same form as a univariate normal density function for the variable P_1 having a mean of $[m(1-\rho_i^2) + y_i]$ and a variance of $\sigma^2(1-\rho_i^2)$. Thus, from this observation we can see that $E[P_1 \mid y_i] = m(1-\rho_i^2) + y_i$ and $\mathrm{var}[P_1 \mid y_i] = \sigma^2(1-\rho_i^2)$.

3. Consider a special case of the Grossman model. Traders can choose between holding a risk-free asset, which pays an end-of-period return of R_f, and a risky asset that has a beginning-of-period price of P_0 per share and an end-of-period payoff (price) of $\tilde{P}_1$ per share. The unconditional distribution of $\tilde{P}_1$ is assumed to be $N(m, \sigma^2)$. The risky asset is assumed to be a derivative security, such as a futures contract, so that its *net supply equals zero*.

   There are two different traders who maximize expected utility over end-of-period wealth, $\tilde{W}_{1i}$, $i = 1, 2$. The form of the ith trader's utility function is

$$U_i(\tilde{W}_{1i}) = -e^{-a_i \tilde{W}_{1i}}, \quad a_i > 0$$

At the beginning of the period, the ith trader observes y_i, which is a noisy signal of the end-of-period value of the risky asset

$$y_i = \tilde{P}_1 + \tilde{\varepsilon}_i$$

where $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$ and is independent of $\tilde{P}_1$. Note that the variances of the traders' signals are the same. Also assume $E[\varepsilon_1 \varepsilon_2] = 0$.

a. Suppose each trader does not attempt to infer the other trader's information from the equilibrium price, P_0. Solve for each of the traders' demands for the risky asset and the equilibrium price, P_0.

Answer: Maximizing investor i's utility with respect to the amount invested in the risky asset, X_i, leads to

$$X_i = \frac{E[\tilde{P}_1 | I_i] - R_f P_0}{a_i \operatorname{var}[\tilde{P}_1 | I_i]}$$

where if I_i just equals y_i, then $E[\tilde{P}_1 | I_i] = m + \rho^2 (y_i - m)$ and $\operatorname{var}\left[\tilde{P}_1 | I_i\right] = \sigma^2 (1-\rho^2)$, where $\rho^2 \equiv \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}$. Substituting these conditional moments into the above equation gives

$$X_i = \frac{m + \rho^2 (y_i - m) - R_f P_0}{a_i \sigma^2 (1-\rho^2)}$$

Since the asset is in zero net supply, $X_1 + X_2 = 0$, so that

$$\frac{m + \rho^2 (y_1 - m) - R_f P_0}{a_1 \sigma^2 (1-\rho^2)} + \frac{m + \rho^2 (y_2 - m) - R_f P_0}{a_2 \sigma^2 (1-\rho^2)} = 0$$

Solving for P_0, one obtains

$$P_0 = \frac{1}{R_f}\left[(1-\rho^2) m + \rho^2 \frac{a_1 y_2 + a_2 y_1}{a_1 + a_2}\right]$$

b. Now suppose each trader does attempt to infer the other's signal from the equilibrium price, P_0. What will be the rational expectations equilibrium price in this situation? What will be each of the traders' equilibrium demands for the risky asset?

Answer: The result of the Grossman model is that the price is fully revealing. The equilibrium price is the same as it would be if both signals were public information. This implies that the equilibrium price is

$$P_0 = \frac{m(1-\bar{\rho}^2) + \bar{\rho}^2 (y_1 + y_2)/2}{R_f}$$

where $\overline{\rho}^2 \equiv \frac{\sigma^2}{\sigma^2+\sigma_\varepsilon^2/2}$. Trader i's equilibrium demand, X_i, is given by using $E[\tilde{P}_1 \mid I_i] = m(1-\overline{\rho}^2) + \frac{1}{2}\overline{\rho}^2(y_1+y_2)$ and $\mathrm{var}[\tilde{P}_1 \mid I_i] = \sigma^2(1-\overline{\rho}^2)$, that is,

$$X_i = \frac{m(1-\overline{\rho}^2) + \frac{1}{2}\overline{\rho}^2(y_1+y_2) - R_f P_0}{a_i \sigma^2 (1-\overline{\rho}^2)}$$

Substituting the equilibrium price in the above equation proves that $X_1 + X_2 = 0$. However, we have even a stronger result. Note that since the numerators of X_i, $i = 1, 2$ are the same, it must be the case that $X_1 = X_2 = 0$, so that in this fully revealing case each individual takes a zero position in the risky asset even though their risk aversions differ.

4. In the Kyle model (Kyle 1985), replace the original assumption **Better-Informed Traders** with the following new one:

The single risk-neutral *insider* is assumed to have better information than the other agents. He observes a signal of the asset's end-of-period value equal to

$$s = \tilde{v} + \tilde{\varepsilon}$$

where $\tilde{\varepsilon} \sim N(0, \sigma_s^2)$, $0 < \sigma_s^2 < \sigma_v^2$, and $\tilde{\varepsilon}$ is distributed independently of $\tilde{u}$ and $\tilde{v}$. The insider does not observe $\tilde{u}$ but chooses to submit a market order of size x that maximizes his expected end-of-period profits.

a. Suppose that the market maker's optimal price-setting rule is a linear function of the order flow

$$p = \mu + \lambda(u + x)$$

Write down the expression for the insider's expected profits given this pricing rule.

Answer: The insider's maximization problem is now

$$\begin{aligned}\max_x E[(s - \tilde{\varepsilon} - \mu - \lambda(x + \tilde{u}))x \mid s] &= \max_x E[(s - \mu - \lambda x)x - (\lambda \tilde{u} + \tilde{\varepsilon})x] \\ &= \max_x (s - \mu - \lambda x)x\end{aligned}$$

since $E[\tilde{u}] = E[\tilde{\varepsilon}] = 0$.

b. Take the first-order condition with respect to x and solve for the insider's optimal trading strategy as a function of the signal and the parameters of the market maker's pricing rule.

Answer: The solution to the insider's problem is

$$x = \alpha + \beta s$$

where $\alpha \equiv -\frac{\mu}{2\lambda}$ and $\beta \equiv \frac{1}{2\lambda}$.

5. Consider a variation of the Kyle model (Kyle 1985). Replace the orginal assumption **Liquidity Traders** with the following new one:

   Noise traders have needs to trade that are exogenous to the model. It is assumed that they, as a group, submit a “market” order to buy $\tilde{u}$ shares of the asset, where $\tilde{u} \sim N(0,\sigma_u^2)$. $\tilde{u}$ and $\tilde{v}$ are assumed to be correlated with correlation coefficient ρ.

   Note that the only change is that, instead of the original Kyle model’s assumption that $\tilde{u}$ and $\tilde{v}$ are uncorrelated, they are now assumed to have nonzero correlation coefficient ρ.

   a. Suppose that the market maker’s optimal price-setting rule is a linear function of the order flow

$$p = \mu + \lambda(u + x)$$

   Write down the expression for the insider’s expected profits given this pricing rule. *Hint:* to find the conditional expectation of $\tilde{u}$, it might be helpful to write it as a weighted average of $\tilde{v}$ and another normal random variable uncorrelated with $\tilde{v}$.

   Answer: The insider’s maximization problem is now

$$\max_x E[(v - \mu - \lambda(x + \tilde{u}))x \mid v] = \max_x (v - \mu - \lambda(x + E[\tilde{u} \mid v]))x$$

   Now using the fact that u and v are jointly normal with correlation ρ, we can write

$$\frac{\tilde{u}}{\sigma_u} = \rho \frac{\tilde{v} - p_0}{\sigma_v} + \sqrt{1-\rho^2}\,\tilde{\varepsilon}$$

   where it is assumed that $\tilde{\varepsilon} \sim N(0,1)$ and is uncorrelated with $\tilde{v}$. Note that writing $\tilde{u}$ this way is consistent with $\tilde{u} \sim N(0,1)$ and having $\tilde{u}$ correlation of ρ with $\tilde{v}$. Thus we see that

$$E[\tilde{u} \mid v] = \rho \sigma_u \frac{v - p_0}{\sigma_v}$$

   Substituting this into the above expected profit function gives

$$\max_x \left(v - \mu - \lambda \left(x + \rho \sigma_u \frac{v - p_0}{\sigma_v} \right) \right) x$$

   b. Take the first-order condition with respect to x and solve for the insider’s optimal trading strategy as a function of v and the parameters of the market maker’s pricing rule.

   Answer: The first order condition is

$$v - \mu - \lambda \rho \sigma_u \frac{v - p_0}{\sigma_v} - 2\lambda x = 0$$

or

$$
\begin{aligned}
x &= \rho\sigma_u \frac{p_0}{2\sigma_v} - \frac{\mu}{2\lambda} + v\frac{1-\lambda\rho\sigma_u/\sigma_v}{2\lambda} \\
&= \rho\sigma_u \frac{p_0}{2\sigma_v} - \frac{\mu}{2\lambda} + v\left[\frac{1}{2\lambda} - \frac{\rho\sigma_u}{2\sigma_v}\right] \\
&= -\frac{\mu}{2\lambda} + \frac{1}{2\lambda}v + (p_0 - v)\frac{\rho\sigma_u}{2\sigma_v}
\end{aligned}
$$

c. For a given pricing rule (given μ and λ) and a realization of $v > p_0$, does the insider trade more or less when $\rho > 0$ compared to the case of $\rho = 0$? What is the intuition for this result? How might a positive value for ρ be interpreted as some of the liquidity traders being better-informed traders? What insights might this result have for a market with multiple insiders (informed traders)?

Answer: Note that if $v > p_0$ and $\rho > 0$, the insider trades less. A positive value for ρ can be interpreted as some of the liquidity traders being better-informed traders such that they tend to move their trading in the direction of *v*. The result that the insider trades less when he faces a market with other insiders is a general result that each insider realize that there is less noise trading to camouflage trades. Competition to trade on insider information lowers the profits of each insider.

Answers to Chapter 17 Exercises

1. Consider the following example of a two-factor term structure model (Jegadeesh and Pennacchi 1996; Balduzzi, Das, and Foresi 1998). The instantaneous-maturity interest rate is assumed to follow the physical process

$$dr(t) = \alpha[\gamma(t) - r(t)]dt + \sigma_r dz_r$$

and the physical process for the interest rate's stochastic "central tendency," $\gamma(t)$, satisfies

$$d\gamma(t) = \delta[\bar{\gamma} - \gamma(t)]dt + \sigma_\gamma dz_\gamma$$

where $dz_r dz_\gamma = \rho dt$ and $\alpha > 0$, σ_r, $\delta > 0$, $\bar{\gamma} > 0$, σ_γ, and ρ are constants. In addition, define the constant market prices of risk associated with dz_r and dz_γ to be θ_r and θ_γ. Rewrite this model using the affine model notation used in this chapter and solve for the equilibrium price of a zero-coupon bond, $P(t,T)$.

Answer: Following the notation in the text, let the state variable process be

$$d\mathrm{x} = \mathrm{a}(\mathrm{x})dt + \mathrm{b}d\mathrm{z} \tag{1}$$

where $\mathrm{x}(t) = (r(t)\gamma(t))'$,

$$\mathrm{a}(\mathrm{x}) = \begin{pmatrix} 0 \\ \delta\bar{\gamma} \end{pmatrix} + \begin{pmatrix} -\alpha & \alpha \\ 0 & -\delta \end{pmatrix}\mathrm{x} \tag{2}$$

$$\mathrm{b} = \begin{pmatrix} \sigma_r & 0 \\ \sigma_\gamma \rho & \sigma_\gamma\sqrt{1-\rho^2} \end{pmatrix} \tag{3}$$

and $d\mathrm{z} = (dz_1 \quad dz_2)'$ is a vector of independent Brownian motion processes so that $dz_1 dz_2 = 0$. Note that the covariance matrix in (3) results in the same variances and covariances as in the case of the correlated Brownian motions dz_r and dz_γ. To find $\Theta = (\theta_1 \quad \theta_2)'$, which is the vector of market prices of risk associated with the independent Brownian motions, note from (3) we can take $dz_1 = dz_r$, so that $\theta_1 = \theta_r$. To find θ_2, note from (3) that $dz_\gamma = \rho dz_1 + \sqrt{1-\rho^2}dz_2$. Since $d\hat{z}_\gamma = dz_\gamma + \theta_\gamma dt$, and we want $d\hat{z}_\gamma = \rho d\hat{z}_1 + \sqrt{1-\rho^2}d\hat{z}_2$, this implies that $dz_\gamma + \theta_\gamma dt = \rho(dz_1 + \theta_r dt) + \sqrt{1-\rho^2}(dz_2 + \theta_2 dt)$. By equating the coefficients of the dt terms we have $\theta_\gamma = \rho\theta_r + \sqrt{1-\rho^2}\theta_2$ or $\theta_2 = [\theta_\gamma - \rho\theta_r]/\sqrt{1-\rho^2}$.

Now equation (17.14) in the text becomes

$$\mathrm{a}(\mathrm{x}) - \mathrm{b}\Theta = \kappa \ (\overline{\mathrm{x}} - \mathrm{x})$$
$$\begin{pmatrix} -\sigma_r\theta_r \\ \delta\overline{\gamma} - \sigma_\gamma\theta_\gamma \end{pmatrix} + \begin{pmatrix} -\alpha & \alpha \\ 0 & -\delta \end{pmatrix}\mathrm{x} = \kappa \quad \overline{\mathrm{x}} - \kappa\mathrm{x} \tag{4}$$

Thus from (4) we see that

$$\kappa = \begin{pmatrix} \alpha - \alpha \\ 0 \quad \delta \end{pmatrix} \tag{5}$$

and

$$\begin{pmatrix} -\sigma_r\theta_r \\ \delta\overline{\gamma} - \sigma_\gamma\theta_\gamma \end{pmatrix} = \begin{pmatrix} \alpha\left(\overline{x}_1 - \overline{x}_2\right) \\ \delta\overline{x}_2 \end{pmatrix} \tag{6}$$

so that

$$\overline{x} = \begin{pmatrix} \overline{\gamma} - \sigma_\gamma\theta_\gamma/\delta - \sigma_r\theta_r/\alpha \\ \overline{\gamma} - \sigma_\gamma\theta_\gamma/\delta \end{pmatrix} \tag{7}$$

Further, equation (17.15) in the text becomes

$$\mathrm{b} = \Sigma\sqrt{\mathrm{s}(\mathrm{x})}$$
$$\begin{pmatrix} \sigma_r & 0 \\ \sigma_\gamma\rho & \sigma_\gamma\sqrt{1-\rho^2} \end{pmatrix} = \Sigma\sqrt{\mathrm{s}(\mathrm{x})} \tag{8}$$

which implies that we can take $\Sigma = \mathrm{b}$ and let s be the identity matrix where $s_{oi} = 1$ and $\mathrm{s}_{1i} = 0$. Lastly, from $r(t,\mathrm{x}) = \alpha + \beta'\mathrm{x}$, we have that $\alpha = 0$ and $\beta' = (1 \quad 0)$.

To derive the formula for a zero-coupon bond's price, $P(t,T,\mathrm{x}) = e^{-Y(t,T,\mathrm{x})(T-t)}$, where $Y(t,T,\mathrm{x})\tau = A(\tau) + \mathrm{B}(\tau)'\mathrm{x}$ and $\tau \equiv T - t$, we need to find $A(\tau)$ and $\mathrm{B}(\tau)$ from the ordinary differential equations (17.18) and (17.19) subject to the boundary conditions $A(0) = 0$ and $\mathrm{B}(0)$ is a 2×1 vector of zeros. Starting with (17.19) we have

$$\frac{\partial \mathrm{B}(\tau)}{\partial \tau} = \beta - \kappa'\mathrm{B}(\tau) - \frac{1}{2}\sum_{i=1}^{n}[\Sigma'\mathrm{B}(\tau)]_i^2\mathrm{s}_{1i}$$
$$\begin{pmatrix} \frac{\partial B_1(\tau)}{\partial \tau} \\ \frac{\partial B_2(\tau)}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \alpha B_1 \\ -\alpha B_1 + \delta B_2 \end{pmatrix} \tag{9}$$

A solution to the ordinary differential equation $\partial B_1/\partial\tau + \alpha B_1 = 1$ subject to the boundary condition $B_1(0) = 0$ is

$$B_1(\tau) = (1 - e^{-\alpha\tau})/\alpha \tag{10}$$

Substituting this into the second line of (9), we have the ordinary differential equation $\partial B_2/\partial\tau + \delta B_2 = 1 - e^{-\alpha\tau}$, subject to the boundary condition $B_2(0) = 0$. Its solution is

$$B_2(\tau) = \frac{\alpha}{\alpha-\delta}\left(\frac{1-e^{-\delta\tau}}{\delta} - \frac{1-e^{-\alpha\tau}}{\alpha}\right) \tag{11}$$

Now equation (17.18) is

$$\begin{aligned}\frac{\partial A(\tau)}{\partial\tau} &= \alpha + (\kappa\overline{\mathrm{x}})'\mathrm{B}(\tau) - \frac{1}{2}\sum_{i=1}^{n}[\Sigma'\mathrm{B}(\tau)]_i^2 s_{0i}\\
&= -\frac{\sigma_r\theta_r}{\alpha}(1-e^{-\alpha\tau}) + \frac{\alpha(\delta\overline{\gamma}-\sigma_\gamma\theta_\gamma)}{\alpha-\delta}\left(\frac{1-e^{-\delta\tau}}{\delta} - \frac{1-e^{-\alpha\tau}}{\alpha}\right)\\
&\quad -\frac{1}{2}\left[\frac{\sigma_r}{\alpha}(1-e^{-\alpha\tau}) + \frac{\alpha\sigma_\gamma\rho}{\alpha-\delta}\left(\frac{1-e^{-\delta\tau}}{\delta} - \frac{1-e^{-\alpha\tau}}{\alpha}\right)\right]^2\\
&\quad -\frac{1}{2}\left[\frac{\alpha\sigma_\gamma\sqrt{1-\rho^2}}{\alpha-\delta}\left(\frac{1-e^{-\delta\tau}}{\delta} - \frac{1-e^{-\alpha\tau}}{\alpha}\right)\right]^2\end{aligned} \tag{12}$$

Integrating equation (12) produces the expression

$$\begin{aligned}A(\tau) = \int^{\tau}\Bigg\{&-\frac{\sigma_r\theta_r}{\alpha}(1-e^{-\alpha t}) + \frac{\alpha(\delta\overline{\gamma}-\sigma_\gamma\theta_\gamma)}{\alpha-\delta}\left(\frac{1-e^{-\delta t}}{\delta} - \frac{1-e^{-at}}{\alpha}\right)\\
&-\frac{1}{2}\left[\frac{\sigma_r}{\alpha}(1-e^{-\alpha t}) + \frac{\alpha\sigma_\gamma\rho}{\alpha-\delta}\left(\frac{1-e^{-\delta t}}{\delta} - \frac{1-e^{-at}}{\alpha}\right)\right]^2\\
&-\frac{1}{2}\left[\frac{\alpha\sigma_\gamma\sqrt{1-\rho^2}}{\alpha-\delta}\left(\frac{1-e^{-\delta t}}{\delta} - \frac{1-e^{-at}}{\alpha}\right)\right]^2\Bigg\}dt + c\end{aligned} \tag{13}$$

where c is a constant determined by the boundary condition $A(0) = 0$. Computing this integral of exponential terms and solving for c is straightforward but tedious. The resulting expression for $A(\tau)$ can be written as

$$\begin{aligned}A(\tau) = &-\overline{\gamma}\left(\tau - \frac{1}{\alpha-\delta}\left[\alpha\frac{1-e^{-\delta\tau}}{\delta} - \delta\frac{1-e^{-a\tau}}{\alpha}\right]\right)\\
&-\beta'\kappa^{-1}(\mathrm{G}(\tau) - \mathrm{I}\tau)\varphi + \frac{1}{2}\beta'\mathrm{G}(\tau)\Omega\mathrm{G}(\tau)'\beta\\
&+\frac{1}{2}\beta'[\kappa^{-1}\mathrm{G}(\tau)\Omega + \Omega\mathrm{G}(\tau)'(\kappa^{-1})']\beta\\
&+\frac{1}{2}\beta'\kappa^{-1}\mathrm{C}(\kappa^{-1})'\beta\tau\end{aligned} \tag{14}$$

where I is a 2×2 identity matrix, $\varphi' \equiv (\theta_r \sigma_r \quad \theta_\gamma \sigma_\gamma)$,

$$G(\tau) \equiv \begin{pmatrix} B_1 & B_2 \\ 0 & \frac{1-e^{-\delta\tau}}{\delta} \end{pmatrix}$$

$$C \equiv \begin{pmatrix} \sigma_r^2 & \sigma_r \sigma_\gamma \rho \\ \sigma_r \sigma_\gamma \rho & \sigma_\gamma^2 \end{pmatrix}$$

and Ω is a 2×2 matrix with elements

$$\Omega_{11} = -\frac{\sigma_r^2}{2\alpha} - \frac{\sigma_r \sigma_\gamma \rho}{\alpha+\delta} + \frac{\alpha\delta\sigma_\gamma^2}{2(\alpha-\delta)^2}\left(\alpha - \delta + \frac{4\alpha\delta}{\alpha+\delta}\right)$$

$$\Omega_{12} = \Omega_{21} = -\frac{\sigma_r \sigma_\gamma \rho}{\alpha+\delta} + \frac{\alpha\sigma_\gamma^2}{\alpha-\delta}\left(\frac{1}{\alpha+\delta} - \frac{1}{2\delta}\right)$$

$$\Omega_{22} = -\frac{\sigma_\gamma^2}{2\delta}$$

2. Consider the following one-factor quadratic Gaussian model. The single state variable, $x(t)$, follows the risk-neutral process

$$dx(t) = \kappa[\bar{x} - x(t)]dt + \sigma_x \, d\hat{z}$$

and the instantaneous-maturity interest rate is given by $r(t,x) = \alpha + \beta x(t) + \gamma x(t)^2$. Assume κ, $\bar{x}$, α, and γ are positive constants and that $\alpha - \frac{1}{4}\beta^2/\gamma \geq 0$, where β also is a constant. Solve for the equilibrium price of a zero-coupon bond, $P(t,T)$.

Answer: For this one-factor specification, we have

$$P(t,T,x) = \exp(-A(\tau) - B(\tau)x(t) - C(\tau)x(t)^2) \tag{1}$$

The ordinary differential equations (17.28), (17.29), and (17.30) become

$$\frac{\partial A(\tau)}{\partial \tau} = \alpha + \kappa\bar{x}B(\tau) - \frac{1}{2}B(\tau)^2\sigma_x^2 + C(\tau)\sigma_x^2 \tag{2}$$

$$\frac{\partial B(\tau)}{\partial \tau} = \beta - \kappa B(\tau) - 2C(\tau)B(\tau)\sigma_x^2 + 2C(\tau)\kappa\bar{x} \tag{3}$$

$$\frac{\partial C(\tau)}{\partial \tau} = \gamma - 2\kappa C(\tau) - 2C(\tau)^2\sigma_x^2 \tag{4}$$

subject to the boundary conditions $A(0) = B(0) = C(0) = 0.$ The solutions to these first-order ordinary differential equations can be found in Anh, Dong-Hyun, Robert F. Dittmar, and A. Ronald Gallant 2002 "Quadratic Term Structure Models: Theory and Evidence," Review of Financial Studies 15, 243–288 and Singleton, Kenneth and Qiang Dai 2003 "Fixed Income Pricing," in Handbook of Economics and Finance, Chapter 20, C. Constantinides, M. Harris, and R. Stulz, eds., North Holland. Define $\Gamma \equiv \sqrt{\kappa^2 + 2\gamma\sigma_x^2}$. Then the solution to the non-linear, first-order ordinary differential equation (4) is

$$C(\tau) = \frac{\gamma(e^{2\Gamma\tau} - 1)}{(\kappa + \Gamma)(e^{2\Gamma\tau} - 1) + 2\Gamma} \tag{5}$$

This value for $C(\tau)$ can be substituted into (3) to obtain a first-order, linear differential equation for $B(\tau)$ whose solution is

$$B(\tau) = C(\tau)\left[\frac{2\kappa\left(\overline{x} + \frac{\beta}{2\gamma}\right)}{\Gamma} \frac{e^{\Gamma\tau} - 1}{e^{\Gamma\tau} + 1} + \frac{\beta}{\gamma}\right] \tag{6}$$

Finally, substituting (5) and (6) into (2), the right-hand side of (2) is entirely a function of τ. This can be integrated directly to find the value of $A(\tau)$ with the integrating constant determined by $A(0) = 0.$ This complicated function for $A(\tau)$ is not reproduced here, but Anh, Dittmar, and Gallant (2002) give its formula for the case of $\beta = 0$ and $\gamma = 1$, conditions which they show are needed to empirically identify the other parameters.

3. Show that for the extended Vasicek model when $\overline{r}(t) \equiv \frac{1}{\alpha}\partial f(0,t)/\partial t + f(0,t) + \sigma_r^2(1 - e^{-2\alpha t})/(2\alpha^2)$, then $P(0,T) = \hat{E}[\exp(-\int_0^T r(s)ds)] = \exp(-\int_0^T f(0,s)ds)$.

Answer: Starting from (17.52):

$$dr = \alpha[\overline{r}(t) - r(t)]dt + \sigma_r d\hat{z} \tag{1}$$

and substituting in $\overline{r}(t) \equiv \frac{1}{\alpha}\partial f(0,t)/\partial t + f(0,t) + \sigma_r^2(1 - e^{-2\alpha t})/(2\alpha^2)$, then (17.52) and (17.51) show that the integral equation of (1) can be written as

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \frac{\sigma_r^2}{\alpha}\left(e^{-\alpha(t-u)} - e^{-2\alpha(t-u)}\right)du + \int_0^t \sigma_r e^{-\alpha(t-u)} d\hat{z}(u) \\ &= f(0,t) + \frac{\sigma_r^2}{\alpha^2}\left[\frac{1}{2} - e^{-\alpha t} + \frac{1}{2}e^{-2\alpha t}\right]^2 + \int_0^t \sigma_r e^{-\alpha(t-u)} d\hat{z}(u) \\ &= f(0,t) + \frac{\sigma_r^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \int_0^t \sigma_r e^{-\alpha(t-u)} d\hat{z}(u) \end{aligned} \tag{2}$$

We can use (2) to substitute in for $r(s)$ in the bond price equation to obtain

$$
\begin{aligned}
P(0,T) &= \hat{E}_0\left[\exp\left(-\int_0^T r(s)ds\right)\right] \\
&= \hat{E}_0\left[\exp\left(-\int_0^T\left[f(0,s)+\frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha s})^2+\int_0^s \sigma_r e^{-\alpha(s-u)}d\hat{z}(u)\right]ds\right)\right] \\
&= \exp\left(-\int_0^T f(0,s)+\frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha s})^2 ds\right)\hat{E}_0\left[\exp\left(-\int_0^T\left[\int_0^s \sigma_r e^{-\alpha(s-u)}d\hat{z}(u)\right]ds\right)\right] \\
&= \exp\left(-\int_0^T f(0,s)+\frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha s})^2 ds\right)\hat{E}_0\left[\exp\left(-\int_0^T \tfrac{\sigma_r}{\alpha}(1-e^{-\alpha s})d\hat{z}(s)\right)\right]
\end{aligned} \tag{3}
$$

where the last line of (3) is obtained by switching the order of integration. Now note that the risk-neutral Brownian motion process $d\hat{z}(s)$ are mean zero, normally distributed independent increments, so that exp $\left(-\int_0^T \tfrac{\sigma_r}{\alpha}(1-e^{-\alpha s})d\hat{z}(s)\right)$ is lognormally distributed. Thus, the expectation of this variable equals the exponential of one-half of its variance. This allows us to write

$$
\begin{aligned}
P(0,T) &= \exp\left(-\int_0^T f(0,s)+\frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha s})^2 ds\right)\left[\exp\left(\int_0^T \frac{\sigma_r^2}{2\alpha^2}(1-e^{-\alpha s})^2 ds\right)\right] \\
&= \exp\left(-\int_0^T f(0,s)ds\right)
\end{aligned} \tag{4}
$$

4. Determine the value of an *n*-payment interest rate floor using the LIBOR market model.

Answer: Let $f(t,T)$ be the date *t* price of a floorlet whose date $T+\tau$ payoff equals $\tau\max[X-L(T,T,\tau),0]$. Note that it can be shown to be a call option on a bond since

$$
\begin{aligned}
f(T,T) &= P(T,T+\tau)\max\left[\tau X-\tau L(T,T,\tau),0\right] \\
&= P(T,T+\tau)\max\left[\tau X-\left(\frac{1}{P(T,T+\tau)}-1\right),0\right] \\
&= \max\left[(1+\tau X)P(T,T+\tau)-1,0\right] \\
&= \max\left[\frac{1+\tau X}{1+\tau L(T,T,\tau)}-1,0\right]
\end{aligned} \tag{1}
$$

The last line in (1) shows that the floorlet is a call option on a bond having a payoff of $1+\tau X$ at its maturity date of $T+\tau$. To value this floorlet using the LIBOR market model, we compute expectations under the forward rate measure generated by $d\tilde{z} = d\hat{z}+\sigma_I(t,T+\tau)dt =$ $dz+[\Theta(t)+\sigma_I(t,T+\tau)]dt$, so that since $\tau L(t,T,\tau)=P(t,T)/P(t,T+\tau)-1$, (17.65) holds:

$$
\frac{dL(t,T,\tau)}{L(t,T,\tau)} = \left[\sigma_I(t,T+\tau)-\sigma_I(t,T)\right]' d\hat{z} \tag{2}
$$

Define the deflated floorlet prices as $F(t,T) = f(t,T)/P(t,T+\tau)$. Then similar to (17.68) we have that the deflated floorlet price is a martingale under the forward rate measure

$$\frac{dF(t,T)}{F(t,T)} = [\sigma_c(t) + \sigma_I(t,T+\tau)]' d\hat{z}_t \tag{3}$$

Thus

$$\begin{aligned} F\left(t,T\right) &= \hat{E}_t\left[F(T+\tau,T)\right] \\ &= \hat{E}_t\left[\frac{\tau \max[X - L(T,T,\tau),0]}{P(T+\tau,T+\tau)}\right] \end{aligned} \tag{4}$$

Noting that $F(t,T) = f(t,T)/P(t,T+\tau)$ and realizing that $P(T+\tau,T+\tau)=1,$ we can rewrite this as

$$f(t,T) = P(t,T+\tau)\hat{E}_t[\tau \max[X - L(T,T,\tau),0]] \tag{5}$$

Assuming that $L\left(T,T,\tau\right)$ is lognormally distributed under the date $T+\tau$ forward measure allows us to compute (5) as

$$f(t,T) = P(t,T+\tau)[XN(-d_{2T}) - L(t,T,\tau)N(-d_{1T})] \tag{6}$$

where $d_{1T} = \left[\ln(L(t,T,\tau)/X) + \frac{1}{2}v(t,T)^2\right]/v(t,T), d_{2T} = d_{1T} - v(t,T),$ and

$$v(t,T)^2 = \int_t^T \left|\sigma_I(s,T+\tau) - \sigma_I(s,T)\right|^2 ds \tag{7}$$

Because an *n*-payment interest rate floor is a portfolio of n floorlets, the value of this floor, $F_n(t),$ can be written as

$$F_n(t) = \sum_{j=1}^{n} f(t,T_j) \tag{8}$$

A typical floor may have dates $T_1 = t, T_2 = t+\tau, \ . \ . \ . \ , T_n = t + \tau(n-1).$

Answers to Chapter 18 Exercises

1. Consider the example given in the "structural approach" to modeling default risk. Maintain the assumptions made in the chapter but now suppose that a third party guarantees the firm's debtholders that if the firm defaults, the debtholders will receive their promised payment of B. In other words, this third-party guarantor will make a payment to the debtholders equal to the difference between the promised payment and the firm's assets if default occurs. (Banks often provide such a guarantee in the form of a letter of credit. Insurance companies often provide such a guarantee in the form of bond insurance.)

   What would be the fair value of this bond insurance at the initial date, t? In other words, what is the competitive bond insurance premium charged at date t?

   Answer: Let $I(t)$ be the date t value of the insurance for the bond that matures at date T. Then

$$I(T) = \max[0, B - A(T)]$$

We see that the insurance has a payoff that resembles that of a European put option written on the firm's assets having an exercise price of B. Therefore, the present value of the insurance is

$$I(t) = P(t,T)BN(-h_2) - e^{-\delta\tau} AN(-h_1)$$

where $h_1 = [\ln[e^{-\delta\tau} A/(P(t,T)B)] + \frac{1}{2}v^2]/v,\ h_2 = h_1 - v,$ and $v(\tau)$ is given in (9.61).

2. Consider a Merton-type "structural" model of credit risk (Merton 1974). A firm is assumed to have shareholders' equity and two zero-coupon bonds that both mature at date T. The first bond is "senior" debt and promises to pay B_1 at maturity date T, while the second bond is "junior" (or subordinated) debt and promises to pay B_2 at maturity date T. Let $A(t)$, $D_1(t)$, and $D_2(t)$ be the date t values of the firm's assets, senior debt, and junior debt, respectively. Then the maturity values of the bonds are

$$D_1(T) = \begin{cases} B_1 & \text{if } A(T) \geq B_1 \\ A(T) & \text{otherwise} \end{cases}$$

$$D_2(T) = \begin{cases} B_2 & \text{if } A(T) \geq B_1 + B_2 \\ A(T) - B_1 & \text{if } B_1 + B_2 > A(T) \geq B_1 \\ 0 & \text{otherwise} \end{cases}$$

The firm is assumed to pay no dividends to its shareholders, and the value of shareholders' equity at date *T*, *E(T)*, is assumed to be

$$E(T)=\begin{cases}A(T)-(B_1+B_2) & \text{if } A(T)\geq B_1+B_2\\ 0 & \text{otherwise}\end{cases}$$

Assume that the value of the firm's assets follows the process

$$dA/A=\mu dt+\sigma dz$$

where μ denotes the instantaneous expected rate of return on the firm's assets and σ is the constant standard deviation of return on firm assets. In addition, the continuously compounded, risk-free interest rate is assumed to be the constant r. Let the current date be t, and define the time until the debt matures as $\tau\equiv T-t$.

a. Give a formula for the current, date t, value of shareholders' equity, $E(t)$.

Answer: Shareholders' equity is analogous to a call option written on the firm's assets with exercise price B_1+B_2. Therefore,

$$E(t)=A(t)N(h_1)-(B_1+B_2)P(t,T)N(h_2)$$

where $h_1=\left[\ln A(t)/[P(t,T)(B_1+B_2)]+\tfrac{1}{2}v^2\right]/v$, $h_2=h_1-v$, and $v(\tau)$ is given in (9.61).

b. Give a formula for the current, date t, value of the senior debt, $D_1(t)$.

Answer: The senior debt's payoff is analogous to a risk-free payoff of B_1, less the value of a put option on the firm's assets having exercise price B_1. Hence, we have

$$\begin{aligned}D_1(t)&=P(t,T)B_1-P(t,T)B_1N(-k_2)+A(t)N(-k_1)\\&=P(t,T)B_1N(k_2)+A(t)N(-k_1)\end{aligned}$$

where $k_1=\left[\ln A(t)/[P(t,T)B_1]+\tfrac{1}{2}v^2\right]/v$, $k_2=k_1-v$, and $v(\tau)$ is given in (9.61).

c. Using the results from parts (a) and (b), give a formula for the current, date t, value of the junior debt, $D_2(t)$.

Answer: Since the value of the firm's assets must equal the sum of the values of the claims on those assets, $A(t)=E(t)+D_1(t)+D_2(t)$. Therefore,

$$\begin{aligned}D_2(t)&=A(t)-E(t)-D_1(t)\\&=A(t)-A(t)N(h_1)+(B_1+B_2)P(t,T)N(h_2)\\&\quad-P(t,T)B_1N(k_2)-A(t)N(-k_1)\\&=A(t)[N(-h_1)-N(-k_1)]+P(t,T)[(B_1+B_2)N(h_2)-B_1N(k_2)]\end{aligned}$$

3. Consider a portfolio of m different defaultable bonds (or loans), where the i^{th} bond has a default intensity of $\lambda_i(t, \mathrm{x})$ where x is a vector of state variables that follows the multivariate diffusion process in (18.7). Assume that the only source of correlation between the bonds' defaults is through their default intensities. Suppose that the maturity dates for the bonds all exceed date $T > t$. Write down the expression for the probability that none of the bonds in the portfolio defaults over the period from date t to date T.

Answer: As given in equation (18.5), the physical probability of the i^{th} bond surviving over the interval from the current date t to date T is

$$E_t\left[\exp\left(-\int_t^T \lambda_i(u, \mathrm{x}(u))du\right)\right]$$

The doubly stochastic modeling of default implies that, given none of the bonds has yet defaulted, the default intensity for the *first* default among the *m* bonds is given by the sum of the default intensities $\sum_{i=1}^{m} \lambda(t, \mathrm{x}(t))$. Similarly, the probability of joint survivorship (that none of the bonds will default) before date T is given by

$$E_t\left[\exp\left(-\int_t^T \sum_{i=1}^{m} \lambda(u, \mathrm{x}(u))du\right)\right]$$

4. Consider the standard "plain vanilla" swap contract described in Chapter 17. In equation (17.74) it was shown that under the assumption that each party's payments were default free, the equilibrium swap rate agreed to at the initiation of the contract, date T_0, equals

$$s_{0,n}(T_0) = \frac{1 - P(T_0, T_{n+1})}{\tau \sum_{j=1}^{n+1} P(T_0, T_j)}$$

where for this contract, fixed-interest-rate coupon payments are exchanged for floating-interest-rate coupon payments at the dates T_1, T_2, . . . , T_{n+1}, where $T_{j+1} = T_j + \tau$ and τ is the maturity of the LIBOR of the floating-rate coupon payments. This swap rate formula is valid when neither of the parties have credit risk. Suppose, instead, that they both have the same credit risk, and it is equivalent to the credit risk reflected in LIBOR interest rates. (Recall that LIBOR reflects the level of default risk for a large international bank.) Moreover, assume a reduced-form model of default with recovery proportional to market value, so that the value of a LIBOR discount bond promising \$1 at maturity date T_j is given by (18.22):

$$D(T_0, T_j) = E_{T_0}\left[e^{-\int_{T_0}^{T_j} R(u,\mathrm{x})du}\right]$$

where the default-adjusted instantaneous discount rate $R(t, \mathrm{x}) \equiv r(t, \mathrm{x}) + \hat{\lambda}(t, \mathrm{x})\hat{L}(t, x)$ is assumed to be the same for both parties. Assume that if default occurs at some date $\tau < T_{n+1}$, the counterparty whose position is in the money (whose position has positive value) suffers a proportional loss of $L(\tau, \mathrm{x})$ in that position. Show that under these assumptions, the equilibrium swap rate is

$$s_{0,n}(T_0) = \frac{1 - D(T_0, T_{n+1})}{\tau \sum_{j=1}^{n+1} D(T_0, T_j)}$$

Answer: This problem is based on Section I of Duffie, Darrell, and Kenneth J. Singleton 1997 "An Econometric Model of the Term Structure of Interest-Rate Swap Yields," *Journal of Finance* 52, 1287–1321. See that article for a more complete explanation of the solution.

The swap can be viewed as an agreement to exchange $n+1$ stochastic cashflows. If K is the swap's fixed annualized coupon rate, then conditional on no default by date T_j, the floating-rate payer's net cashflow (fixed-rate payer's net payment) at date T_j is $\tau[K - L_{ibor}(T_{j-1},\tau)]$, where $L_{ibor}(T_{j-1},\tau)$ is the annualized τ-maturity LIBOR at date $T_{j-1} = T_j - \tau$. (Note that the notation used in Chapter 17 for this spot LIBOR was $L(T_{j-1},T_{j-1},\tau)$. Here we use a slightly different notation to avoid confusion with the proportional loss rate $L(t,\mathrm{x})$.)

When recovery is proportional to market value, a similar argument to that made in the text can be used to compute the present value of the payment at date T_j as

$$\tilde{E}_t\left[e^{-\int_t^{T_j} R(u,\mathrm{x})du}\tau\left[K - L_{ibor}(T_{j-1},\tau)\right]\right] \tag{1}$$

where $R(t,\mathrm{x}) \equiv r(t,\mathrm{x}) + \hat{\lambda}(t,\mathrm{x})\hat{L}(t,\mathrm{x})$. Expression (1) is analogous to equation (18.22) for the case of a defaultable bond with a fixed promised payment. Thus, the present value of all of the swap's $n+1$ payments is

$$\sum_{j=1}^{n+1}\tilde{E}_t\left[e^{-\int_t^{T_j} R(u,\mathrm{x})du}\tau\left[K - L_{ibor}(T_{j-1},\tau)\right]\right] \tag{2}$$

At the inception of the swap, which is assumed to be date T_0, the expression (2) must equal zero for it to be fairly priced. Thus, at date T_0 we have $s_{0,n}(T_0) = K$, so that

$$\sum_{j=1}^{n+1}\tilde{E}_{T_0}\left[e^{-\int_{T_0}^{T_j} R(u,\mathrm{x})du}\tau\left[s_{0,n}(T_0) - L_{ibor}(T_{j-1},\tau)\right]\right] = 0 \tag{3}$$

Rearranging this expression, we obtain

$$\tau s_{0,n}(T_0)\sum_{j=1}^{n+1}\tilde{E}_{T_0}\left[e^{-\int_{T_0}^{T_j} R(u,\mathrm{x})du}\right] = \sum_{j=1}^{n+1}\tilde{E}_t\left[e^{-\int_{T_0}^{T_j} R(u,\mathrm{x})du}\tau L_{ibor}(T_{j-1},\tau)\right] \tag{4}$$

Now note that, by the assumption that LIBOR interest rates reflect credit risk equivalent to the parties of the swap, we have

$$1 + \tau L_{ibor}\left(T_{j-1},\tau\right) = \tilde{E}_{T_{j-1}}\left[e^{\int_{T_{j-1}}^{T_j} R(u,\mathrm{x})du}\right] \tag{5}$$

After substituting in (5), the right-hand side of (4) can be written as

$$\begin{aligned}
&\sum_{j=1}^{n+1} \hat{E}_{T_0}\left[e^{-\int_{T_0}^{T_j} R(u,x)du} \left(\hat{E}_{T_{j-1}}\left[e^{\int_{T_{j-1}}^{T_j} R(u,x)du} \right] - 1 \right) \right] \\
&= \sum_{j=1}^{n+1} \hat{E}_{T_0}\left[e^{-\int_{T_0}^{T_{j-1}} R(u,x)du} \right] - \sum_{j=1}^{n+1} \hat{E}_{T_0}\left[e^{-\int_{T_0}^{T_j} R(u,x)du} \right] \\
&= \sum_{j=1}^{n+1} D(T_0, T_{j-1}) - \sum_{j=1}^{n+1} D(T_0, T_j) \\
&= 1 - D(T_0, T_{n+1})
\end{aligned} \tag{6}$$

where in the last line of (6) we make use of the fact that $D(T_0, T_0) = 1$. Substituting (6) for the right-hand side of (4), we obtain

$$\tau s_{0,n}(T_0) \sum_{j=1}^{n+1} D(T_0, T_j) = 1 - D(T_0, T_{n+1}) \tag{7}$$

or

$$s_{0,n}(T_0) = \frac{1 - D(T_0, T_{n+1})}{\tau \sum_{j=1}^{n+1} D(T_0, T_j)} \tag{8}$$