1. The third, fourth, and all higher central moments of the normal distribution are either zero or a function of the variance:

$$E\left[\left(\tilde{R}_p - E\left[\tilde{R}_p\right]\right)^n\right] = 0 \text{ for } n \text{ odd},$$

and

$$E\left[\left(\tilde{R}_p - E\left[\tilde{R}_p\right]\right)^n\right] = \frac{n!}{(n/2)!}\left(\frac{1}{2}V\left[\tilde{R}_p\right]\right)^{n/2} \quad \text{for } n \text{ even.}$$

Proof. Moments of the Gaussian Distribution □

$n = 2k + 1$

$$\begin{aligned}
\mathbb{E}\left[\left(\tilde{R} - \mathbb{E}\left[\tilde{R}\right]\right)^n\right] &= \int_{-\infty}^{\infty} \frac{(x-\mu)^{2k+1}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{(x-\mu)^{2k+1}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} d(x-\mu) \\
&= \int_{-\infty}^{0} \frac{(x-\mu)^{2k+1}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} d(x-\mu) \\
&+ \int_{0}^{\infty} \frac{(x-\mu)^{2k+1}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} d(x-\mu) \\
&= \int_{\infty}^{0} \frac{(\mu-x)^{2k+1}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(\mu-x)^2}{2\sigma^2}\right\} d(\mu-x) \\
&+ \int_{0}^{\infty} \frac{(x-\mu)^{2k+1}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} d(x-\mu) \\
&= 0
\end{aligned}$$

$n = 2k$

$$\begin{aligned}
\mathbb{E}\left[\left(\tilde{R} - \mathbb{E}\left[\tilde{R}\right]\right)^n\right] &= \int_{-\infty}^{\infty} \frac{(x-\mu)^{2k}}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{\sigma^{2k}}{\sqrt{2\pi}} \left(\frac{x-\mu}{\sigma}\right)^{2k} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} d\left(\frac{x-\mu}{\sigma}\right) \\
&= \left[-\frac{\sigma^{2k}}{\sqrt{2\pi}} \left(\frac{x-\mu}{\sigma}\right)^{2k-1} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}\right]_{-\infty}^{\infty} \\
&+ (2k-1)\int_{-\infty}^{\infty} \frac{\sigma^{2k}}{\sqrt{2\pi}} \left(\frac{x-\mu}{\sigma}\right)^{2k-2} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} d\left(\frac{x-\mu}{\sigma}\right) \\
&= (2k-1)(2k-3)\cdots 1 \int_{-\infty}^{\infty} \frac{\sigma^{2k}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} d\left(\frac{x-\mu}{\sigma}\right) \\
&= \frac{(2k)!}{2^k k!}\sigma^{2k} \\
&= \frac{n!}{(n/2)!}\left(\frac{1}{2}\mathbb{V}ar\left[\tilde{R}\right]\right)^{n/2}
\end{aligned}$$

2.The Convexity of the Indifference Curves

For $\lambda \in (0,1)$, two distinguished portfolios $\tilde{R}_1$ and $\tilde{R}_2$ with $\mathbb{E}\left[U\left(\tilde{R}_1\right)\right] = \mathbb{E}\left[U\left(\tilde{R}_2\right)\right]$ (implying on the same indifference curve $\bar{U}$), and some third portfolio $\tilde{R}_3$ satisfying $\bar{R}_3 = \lambda\bar{R}_1 + (1-\lambda)\bar{R}_2$ and $\sigma_3 = \lambda\sigma_1 + (1-\lambda)\sigma_2$,

$$
\begin{aligned}
\mathbb{E}\left[U\left(\tilde{R}_3\right)\right] &= \int_{-\infty}^{\infty} U\left(\bar{R}_3 + \sigma_3 x\right) n(x)\, dx \\
&\geq \int_{-\infty}^{\infty} \left[\lambda U\left(\bar{R}_1 + \sigma_1 x\right) + (1-\lambda)\left(\bar{R}_2 + \sigma_2 x\right)\right] n(x)\, dx \\
&= \lambda \int_{-\infty}^{\infty} \left(\bar{R}_1 + \sigma_1 x\right) n(x)\, dx + (1-\lambda)\lambda \int_{-\infty}^{\infty} \left(\bar{R}_2 + \sigma_2 x\right) n(x)\, dx \\
&= \lambda \mathbb{E}\left[U\left(\tilde{R}_1\right)\right] + (1-\lambda)\mathbb{E}\left[U\left(\tilde{R}_2\right)\right] \\
&= \bar{U},
\end{aligned}
$$

so that the portfolio $\tilde{R}_3$ grants a higher level of utility and lies to the northwest of the indifference curve $\bar{U}$. So shows the convexity.

3. **Note that the denominators of λ and γ, given by $\varsigma\delta - \alpha^2$, are guaranteed to be positive when V is of full rank.**

To see this, note that since V is positive definite, so is V^{-1}. Therefore the quadratic form $\left(\alpha\bar{R} - \varsigma e\right)' V^{-1}\left(\alpha\bar{R} - \varsigma e\right) = \alpha^2\varsigma - 2\alpha^2\varsigma + \varsigma^2\delta = \varsigma\left(\varsigma\delta - \alpha^2\right)$ is positive. But since $\varsigma \equiv \bar{R}'V^{-1}\bar{R}$ is a positive quadratic form, then $\left(\varsigma\delta - \alpha^2\right)$ must also be positive.

Remark. **1) V is positive definite, so is V^{-1}. But why?**

We have learned the theorem : A is a symmetric positive definite if and only if all the eigenvalues of A are positive. V is positive definite ,so that $Av = rv$ and $r > 0$ (r is an eigenvalue and v is the corresponding eigenvector).

$$
\begin{aligned}
Av &= rv \\
A^{-1}Av &= A^{-1}rv \\
v &= rA^{-1}v \\
A^{-1}v &= \frac{1}{r}v
\end{aligned}
$$

We can get A^{-1} has eigenvalue $\frac{1}{r}$ which is always positive (as r is the eigenvalue of A). A^{-1} is positive definite from the theorem.

2) V is the covariance martrix of the return on the n assets and V is full rank. Why V is positive definite?

Proof. Denote $\omega = (w_1, w_2, \cdots w_n)^T \in R^n$,and ω is a nonzero vector.

$$
\begin{aligned}
\omega' V \omega &= \sum_{i,j} w_i w_j \sigma_{ij} \\
&= \sum_{i,j} \left(E\left\{(R_i - ER_i)(R_j - ER_j)\right\}\right) w_i w_j \\
&= E \sum_{i,j} \left(w_i w_j \left\{(R_i - ER_i)(R_j - ER_j)\right\}\right) \\
&= E\left(\sum_{i=1}^{n} w_i (R_i - ER_i)\right)^2 \\
&= E\left((w_1R_1 + w_2R_2 + \cdots + w_nR_n) - ER_p\right)^2 \text{ where } R_p = w_1R_1 + w_2R_2 + \cdots + w_nR_n \\
&= Var(R_p) \geq 0
\end{aligned}
$$

$Var(R_p) \neq 0$ as V is full rank . So we have V is positive definite. □

4. To see that the slope of the hyperbola asymptotes to a magnitude of $\sqrt{\varsigma\delta - \alpha^2/\delta}$,use $\sigma_p^2 = \frac{1}{\delta} + \frac{\delta\left(\bar{R}_p - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2}$ to substitute for $\left(\bar{R}_p - \frac{\alpha}{\delta}\right)$ in $\frac{\partial \bar{R}_p}{\partial \sigma_p} = \frac{\varsigma\delta - \alpha^2}{\delta\left(\bar{R}_p - \frac{\alpha}{\delta}\right)}\sigma_p$ to obtain $\partial \bar{R}_p / \partial \sigma_p = \pm\sqrt{(\varsigma\delta - \alpha^2)}/\sqrt{\delta - 1/\sigma_p^2}$. Taking the limit of this expression as $\sigma_p \to \infty$ gives the desired result.

Remark. Some students wonder that why the intercept of hyperbola asymptotes is $R_{\mathbf{mv}} = \frac{\alpha}{\delta}$.

We can imagine that the the hyperbola asymptote intersects the efficient frontier at $\sigma_p = \infty$. Denote the intercept by R_{Int} .

We have:

$$
\begin{aligned}
\bar{R}_p = R_{Int} + \sqrt{\frac{\varsigma\delta - \alpha^2}{\delta}}\sigma_p \quad &\to \quad R_{Int} + \sqrt{\frac{\varsigma\delta - \alpha^2}{\delta}} \bullet \sqrt{\frac{1}{\delta} + \frac{\delta\left(\bar{R}_p - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2}} \text{ as } \bar{R}_P \to \infty \\
\bar{R}_p - R_{Int} \quad &\to \quad \sqrt{\frac{\varsigma\delta - \alpha^2}{\delta^2} + \left(\bar{R}_p - \frac{\alpha}{\delta}\right)^2} \text{ as} \bar{R}_p \to \infty
\end{aligned}
$$

We have $R_{Int} = \frac{\alpha}{\delta}$ which is R_{mv}.