Advanced Microeconomics II

Extensive Form Perfect Information

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Strategic Entry Example

There are two firms, a potential entrant (E) and an industry incumbent (I) .

- The entrant must decide whether to enter the market (In) or not (Out).
- If the potential entrant stays out then he gains nothing and the incumbent firm gains 2.
- If he enters his payoff depends on the reaction by the incumbent to entry.
    - If the incumbent fights (F), both firms lose 1.
    - If the incumbent cooperates (C), both firms gain 1.

Formulate this as a strategic game and find the Nash equilibria.

Strategic Entry Example

		Incumbent	
		F	C
Entrant	In	$-1,-1$	$1,1$
	Out	$0,2$	$0,2$

- Two pure strategy Nash Equilibria: $(In, C), (Out, F)$.
- What's special about (Out, F)?
- Strategic form does not reflect timing.

Extensive Games

- An extensive game is an explicit description of the sequential structure of strategic interactions.
- Players can condition actions on past history.
- We start with models where when a player makes a choice, he knows perfectly what has happened in the past (perfect information).
- We will study two extensions
  - Bargaining games of alternating offers.
  - Repeated games.
- Nash equilibrium ignores timing of choice so we require a new notion of equilibrium.
  - Sub-game perfect equilibrium

Extensive Games With Perfect Information

Definition

An extensive game with perfect information has

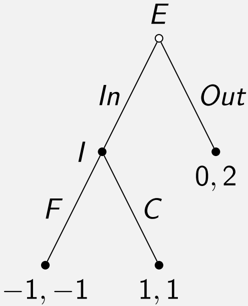
- a set N of players;
- a set H of histories such that
  - the empty sequence $\varnothing$ is a member of H,
  - If $(a^k)_{k=1,\ldots,K} \in H$ where K may be infinite) and $L < K$ then $(a^k)_{k=1,\ldots,L} \in H$, and
  - If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1,\ldots,L} \in H$ for every integer L then $(a^k)_{k=1}^{\infty} \in H$;

  (Each component of a history is an action taken by a player.) A history $(a^k)_{k=1\ldots,K} \in H$ is terminal if it is infinite or there is no a^{K+1} such that $(a^k)_{k=1\ldots,K+1} \in H$. Z is the set of terminal histories;
- A function P that assigns to each member of $H \backslash Z$ a member of N. (P is a player function, $P(h)$ is the player who takes an action after history h.)
- For each player $i \in N$ a preference relation $\succeq_i$ in Z (the preference relation of player i).

Interpretation

- After any nonterminal history h player $P(h)$ chooses an action from $A(h) = \{a : (h, a) \in H\}$.
- $\varnothing$ is the game starting point or initial history.
- $P(\varnothing)$ chooses from $A(\varnothing)$.
- For each choice $a^0 \in A(\varnothing)$, $P(a^0)$ chooses from $A(a^0)$.
- For each choice $a^1 \in A(a^0)$, $P(a^0, a^1)$ chooses from $A(a^0, a^1)$.
- And so on, until we reach a terminal history (no more choices).
- Preferences are generally represented by payoff functions.

Strategic Entry Example

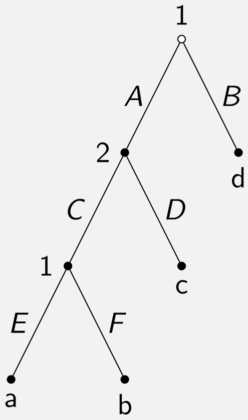


Extensive game representation

- $N = \{I, E\}$.
- $H = \{\varnothing, In, Out, (In, F), (In, C)\}$.
- $P(\varnothing) = E$, and $P(In) = I$.
- $(In, C) \succ_E Out \succ_E (In, F)$ and $Out \succ_I (In, C) \succ_I (In, F)$.

Extensive game form with perfect information is $\{N, H, P\}$.

Example 2

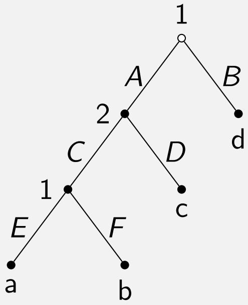


What is the extensive game form?

Player Strategies

Definition

A strategy of player $i \in N$ in an extensive game with perfect information $\{N, H, P, (\succeq_i)\}$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.



How many strategies does each player have?

Nash Equilibrium

Definition

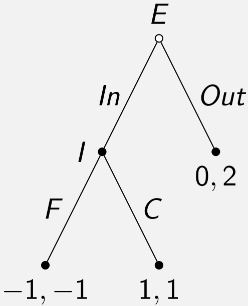
The outcome $O(s) \in Z$ of strategy profile $s = (s_i)_{i \in N}$ is the terminal history such that for $0 \leq k < K$ we have $s_{P(a^1,\ldots,a^k)}(a^1, \ldots, a^k) = a^{k+1}$ where K is the length of $O(s)$.

Definition

A Nash equilibrium of an extensive game with perfect information $\{N, H, P, (\succeq_i)\}$ is a strategy profile s^* such that for every player $i \in N$ we have

$$O(s_i^*, s_{-i}^*) \succeq_i O(s_i, s_{-i}^*) \text{ for every strategy } s_i \text{ of player } i.$$

Strategic Entry Example



What are the set of Nash equilibria?

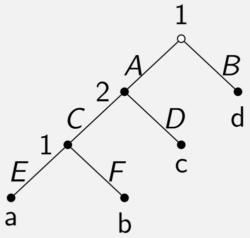
- $(s_E(\varnothing) = In, s_I(In) = C)$
- $(s_E(\varnothing) = Out, s_I(In) = F)$

Strategic Form of Extensive Games

Definition

The strategic form of the extensive game with perfect information $\Gamma = \{N, H, P, (\succeq_i)\}$ is the strategic game $\{N, (S_i), (\succeq'_i)\}$ in which for each player $i \in N$

- S_i is the set of strategies of player $i \in \Gamma$.
- $\succeq'_i$ is defined by $s \succeq'_i s'$ if and only if $O(s) \succeq_i O(s')$ for every $s \in \times_{i \in N} S_i$ and $s' \in \times_{i \in N} S_i$.

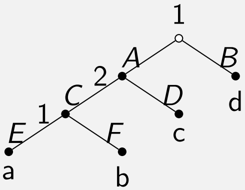


	AC	AD
(A, ACE)	a	c
(A, ACF)	b	c
(B, ACE)	d	d
(B, ACF)	d	d

Reduced Strategic Form of Extensive Games

Definition

Let $\Gamma = \{N, H, P, (\succeq_i)\}$ be an extensive game with perfect information and let $\{N, (S_i), (\succeq'_i)\}$ be its strategic form. For any $i \in N$ define the strategies $s_i \in S_i$ and $s'_i \in S_i$ of player i to be equivalent if for each $s_{-i} \in S_{-i}$ we have $(s_i, s_{-i}) \sim'_j (s'_i, s_{-i})$ for all $j \in N$. The reduced strategic form of Γ is the strategic game $\{N, (S'_i), (\succeq''_i)\}$ in which for each $i \in N$ each set S'_i contains one member of each set of equivalent strategies in S_i and $\succeq''_i$ is the preference ordering over $\times_{j \in N} S'_j$ induced by $\succeq'_i$.



	AC	AD
(A, ACE)	a	c
(A, ACF)	b	c
(B, ACE)	d	d

If $a \neq b$

	AC	AD
(A, ACE)	a	c
(B, ACE)	d	d

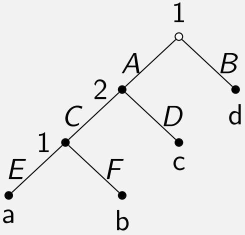
If $a = b$

Subgame

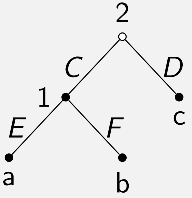
Definition

The subgame of the extensive game with perfect information $\Gamma = \{N, H, P, (\succeq_i)\}$ that follows the history h is the extensive game $\Gamma(h) = \{N, H|_h, P|_h, (\succeq_i |_h)\}$ where

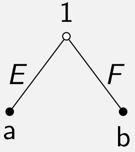
- $H|_h$ is the set of sequences h' of actions for which $(h, h') \in H$,
- $P|_h(h') = P(h, h')$ for each $h' \in H|_h$, and
- $\succeq_i |_h$ is defined by $h' \succeq_i |_h h''$ if and only if $(h, h') \succeq_i (h, h'')$.



$\Gamma(A)$



$\Gamma(AC)$



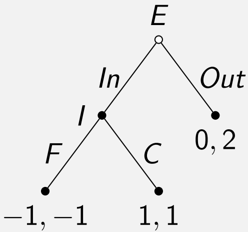
Subgame Perfect Equilibrium

Definition

A subgame perfect equilibrium of an extensive game with perfect information $\Gamma = \{N, H, P, (\succeq_i)\}$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $P(h) = i$ we have

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succeq_i |_h O_h(s_i, s_{-i}^*|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$.



What are the set of subgame perfect equilibria?

- $\{In, (In, C)\}$
- Subgame perfection eliminates Nash equilibria which imply incredible threats.

The one deviation property

Proposition

Let $\Gamma = \{N, H, P, (\succeq_i)\}$ *be a finite horizon extensive game with perfect information. The strategy profile* s^* *is a subgame perfect equilibrium of* Γ *if and only if for every player* $i \in N$ *and every history* $h \in H$ *for which* $P(h) = i$ *we have*

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succeq_i|_h O_h(s_i, s_{-i}^*|_h)$$

for every strategy s_i *of player* i *in the subgame* $\Gamma(h)$ *that differs from* $s_i^*|_h$ *only in the action it prescribes after the initial history of* $\Gamma(h)$*.*

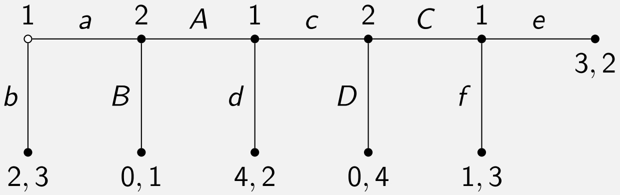
$(\Rightarrow)$ s_i^* is better than any other strategy including any strategy that only deviates after the inital history $\Gamma(h)$.

The one deviation property proof

($\Leftarrow$) If s^* is not a subgame equilibrium then there exists a player i and subgame $\Gamma(h')$ where player i can profitably deviate.

- $l(\Gamma(h'))$ is the length of the longest history in $\Gamma(h')$.
- The number of times player i's profitable deviation differs from s^* is limited by the $l(\Gamma(h'))$ (actually, by the number of times player i plays in $\Gamma(h')$).
- From all profitable deviations of $\Gamma(h')$ choose a strategy s_i' with the least number of deviations.
- h^* is the longest history h (latest profitable deviation) where $s_i'(h) \neq (s_i^*|_{h'})(h)$.
- In the subgame $\Gamma(h', h^*)$, $s_i'|_{h',h^*}$ only differs from $s_i^*|_{h',h^*}$ after history (h', h^*) and is a profitable deviation.

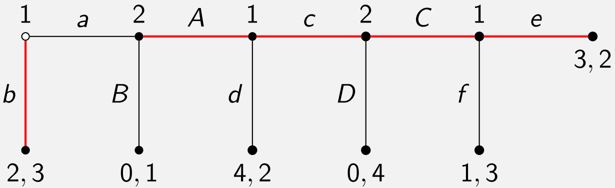
The one deviation property example



What is the SPE of this game?

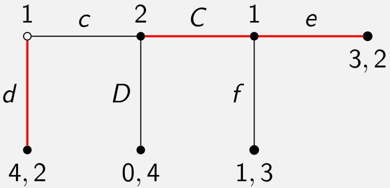
Profitable one-shot deviations

Consider the following strategy profile $\{(b, aAc, aAcCe), (aA, aAcC)\}$.



Construct a profitable one-shot deviation.

Example 1:$\{(b, aAd, aAcCe)\}$ is a profitable one-shot deviation for player 1 in $\Gamma(aA)$.



Kuhn's Theorem

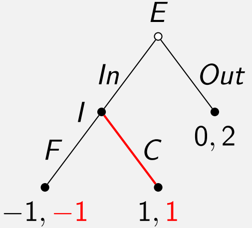
Proposition

Every finite extensive game with perfect information, $\Gamma = \{N, H, P, (\succeq_i)\}$, has a subgame perfect equilibrium.

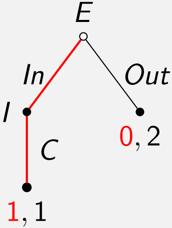
- If $l(\Gamma(h)) = 0$ define $R(h) = h$.
- Let $R(h)$ be defined for all $h \in H$ with $l(\Gamma(h)) \leq k$ for some $k \geq 0$.
- Let h^* be a history for which $l(\Gamma(h^*)) = k + 1$; let $i = P(h^*)$.
- $l(\Gamma(h^*)) = k + 1 \Rightarrow l(\Gamma(h^*, a)) \leq k$ for all $a \in A(h^*)$.
  - Define $s_i(h^*)$ to be a $\succeq_i$-maximizer of $R(h^*, a)$ over $a \in A(h^*)$
  - Define $R(h^*) = R(h^*, s_i(h^*))$.
- This process defines a strategy profile s in Γ; by the one-shot deviation property, s is a subgame perfect equilibrium of Γ.

Backwards Induction

R is referred to as backwards induction. Can be used to find the set of subgame perfect equilibria.



$s_I(In) = C$
$R(In) = (In, C)$

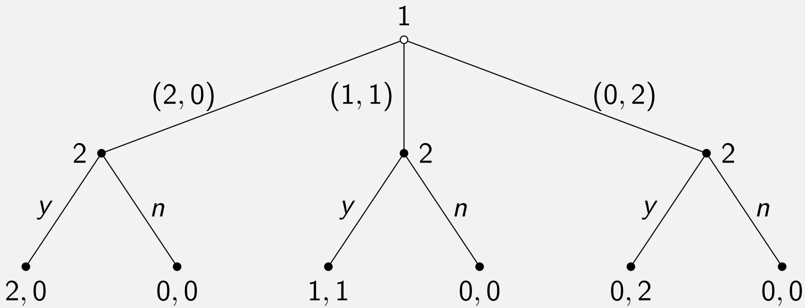


$s_E(\varnothing) = In$
$R(\varnothing) = (In, C)$

The set of subgame perfect equilibria is $\{In, (In, C)\}$.

Backwards Induction - Your Turn

Player 1 proposes an allocation of 2 identical indivisible objects.
Player 2 accepts or rejects the offer.



Find the set of subgame perfect equilibria using backward induction.

Stackleberg Model Of Duopoly

- Firm 1 chooses a quantity $q_1 \geq 0$;
- Firm 2 observes q_1 and chooses a quantity $q_2 \geq 0$;
- The payoff to each firm is given by

$$\pi_i(q_i, q_j) = q_i(1 - q_i - q_j)$$

Extensive Game with Perfect Information and Chance Moves

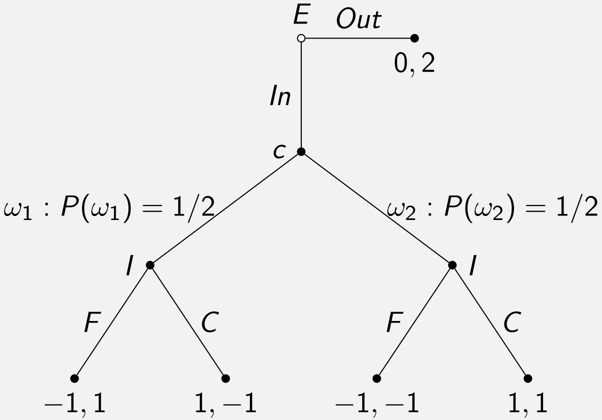
Definition

An extensive game with perfect information and chance moves is a tuple $\{N, H, P, f_c, (\succeq_i)\}$ where N is a finite set of players and H is a set of histories, and

- P is a function from $H \backslash Z$ to $N \cup \{c\}$. c is chance.
- For each $h \in H$ with $P(h) = c, f_c(\cdot|h)$ is a probability measure over $A(h)$; each such measure is independent of every other such measure.
- For each player $i \in N, \succeq_i$ is a preference relation on lotteries over the set of terminal histories.

- Definition of subgame perfect equilibrium is the same as before.
- One deviation property and Kuhn's theorem hold.

Extensive Game with Perfect Information and Chance Moves



Extensive Game with Perfect Information and Simultaneous Moves

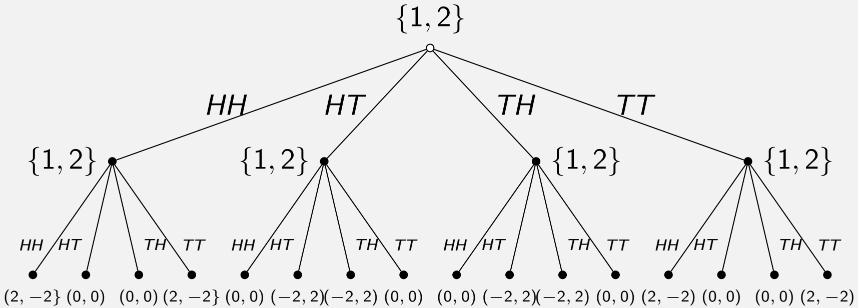
Definition

An extensive game with perfect information and simultaneous moves is a tuple $\{N, H, P, (\succeq_i)\}$ where N is a finite set of players, H is a set of histories, for each $i \in N, \succeq_i$ is player i's preference relation over Z, and

- P is a correspondence from $H \backslash Z$ to N.
- For every $h \in H \backslash Z$ there is a collection $\{A_i(h)\}_{i \in P(h)}$ for which $A(h) = \{a : (h, a) \in H\} = \times_{i \in P(h)} A_i(h)$.

- A strategy of player $i \in N$ is a function that assigns an action in $A_i(h)$ to every nonterminal history h for which $i \in P(h)$.
- Definition of subgame perfect equilibrium is the same as before except that $P(h) = i$ is replaced by $i \in P(h)$.
  - One deviation property holds.
  - Kuhn's theorem does not.

Extensive Game with Perfect Information and Simultaneous Moves



Bank Runs

- Two investors have each deposited D in a bank.
- The bank makes a long-term investment that at maturity will payout $2R$ where $R > D$.
- If the bank if forced to liquidate its investments a total of $2r$ can be recovered, where $D > r > D/2$.
- Investors can withdraw at two dates:
  - Date 1 is before maturity
  - Date 2 is after maturity.
- Assume no discounting.

Bank Runs

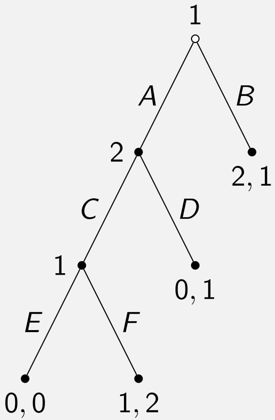
	withdraw	don't
withdraw	r, r	$D, 2r - D$
don't	$2r - D, D$	next stage

	withdraw	don't
withdraw	R, R	$D, 2R - D$
don't	$2R - D, D$	R, R

Interpretation of Strategy

- A strategy is not a plan of action - it requires specification of actions after histories that cannot be reached if a player follows his strategy.
- Alternative interpretation is that the strategy is the belief of the other players.
- Beliefs of others about my action can influence how I rationalize my own action.
  - Players do not choose other players beliefs.
  - Other player's beliefs are required to be the same.
  - Constraints on strategies imply constraints on player beliefs.

Interpretation of Strategy - Example



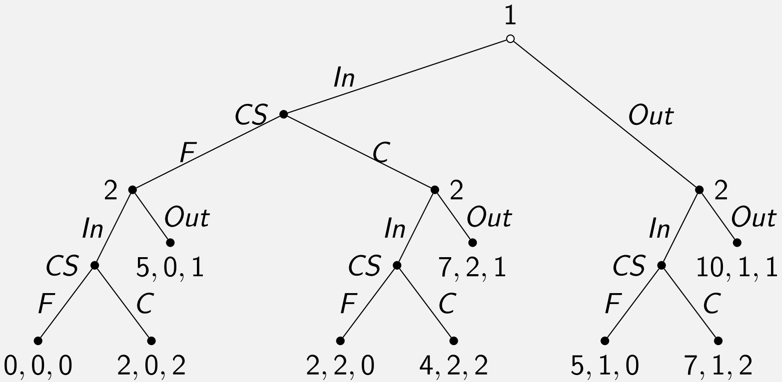
- How can player 2 rationalize A by player 1?
- Contradicts rationality.
- Subgame perfection requires player 2 to maintain rationality assumption even when he sees A.

Chain Store Game

- An incumbent firm faces a sequence of K potential entrants in K different markets. In each market k
  - if entrant stays out, the incumbent gets 5 and the entrant gets 1;
  - if the entrant enters and the incumbent fights, both get 0;
  - if the entrant enters and the incumbent cooperates, both get 2;
  - there are 3 possible outcomes $Q^k = \{Out, (In, C), (In, F)\}$.
- At every point in the game all players observe all previous actions so we have an extensive game of perfect information.
  - $H = \{(\cup_{k=0}^{K} Q^k) \cup (\cup_{k=0}^{K-1}(Q^k \times \{In\}))\}$
  - $P(h) = k + 1$ if $h \in Q^k$ and $P(h) = CS$ if $h \in Q^k \times \{In\}$, for $k = 0, \ldots, K-1$.
  - The payoff of the chain store is the sum of its payoffs in the K markets.

Chain Store Game

Two period Chain Store game



Find the set of subgame perfect equilibria.
Find the set of Nash Equilibria.

Centipede Game

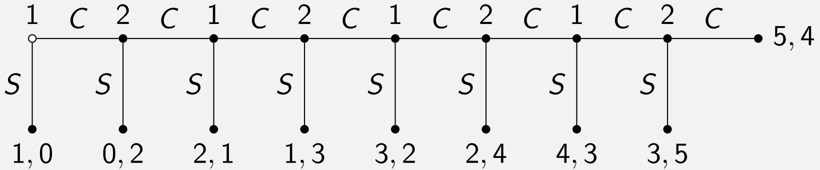
- Two players are involved in a process that they alternately have the opportunity to stop.
- Continuing the game by one period reduces the player's payoff by one but increases the other player's payoff by two.
- The process stops after T periods (T is even).
  - H consists of all sequences $C(t) = (C, \ldots, C)$ of length t for $0 \leq t \leq T$ and all sequences $S(t) = (C, \ldots, C, S)$ consisting of $t - 1$ Cs for $1 \leq t \leq T$.
  - $P(C(t)) = 1$ if t is even and $t < T$, $P(C(t)) = 2$ if t is odd.

$$u_1(S(t)) = \begin{cases} (t+1)/2 & \text{if } t \text{ is odd} \\ t/2 - 1 & \text{if } t \text{ is even} \end{cases} \qquad u_1(C(T)) = T/2 + 1$$

$$u_2(S(t)) = \begin{cases} (t-1)/2 & \text{if } t \text{ is odd} \\ t/2 + 1 & \text{if } t \text{ is even} \end{cases} \qquad u_2(C(T)) = T/2$$

Centipede Game

Eight period Centipede game



Find the set of subgame perfect equilibria.
Find the set of Nash Equilibria.

Predictive Ability