# Quiz 1 Grading Policies & Solutions

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## 1 Solutions

- Prove Jensen's Inequality.

  Proof: Concavity. For $\lambda \in [0,1]$,

$$\lambda U(x_1) + (1-\lambda) U(x_2) \le U(\lambda x_1 + (1-\lambda) x_2).$$

If $U \in C^2$,

$$U'' \le 0.$$

Discrete random variables.

$$\begin{aligned}
\mathbb{E}[U(\tilde{x})] &= \sum_{i=1}^{n} p_i U(x_i) \\
&= \left(1 - \sum_{i=3}^{n} p_i\right) \frac{p_1 U(x_1) + p_2 U(x_2)}{1 - \sum_{i=3}^{n} p_i} + \sum_{i=3}^{n} p_i U(x_i) \\
&\le \left(1 - \sum_{i=3}^{n} p_i\right) U\left(\frac{p_1 x_1 + p_2 x_2}{1 - \sum_{i=3}^{n} p_i}\right) + p_3 U(x_3) + \sum_{i=4}^{n} p_i U(x_i) \\
&= \left(1 - \sum_{i=4}^{n} p_i\right) \left[\frac{1 - \sum_{i=3}^{n} p_i}{1 - \sum_{i=4}^{n} p_i} U\left(\frac{p_1 x_1 + p_2 x_2}{1 - \sum_{i=3}^{n} p_i}\right) + \frac{p_3}{1 - \sum_{i=4}^{n} p_i} U(x_3)\right] + \sum_{i=4}^{n} p_i U(x_i) \\
&\le \left(1 - \sum_{i=4}^{n} p_i\right) U\left(\frac{p_1 x_1 + p_2 x_2 + p_3 x_3}{1 - \sum_{i=4}^{n} p_i}\right) + \sum_{i=4}^{n} p_i U(x_i) \\
&\dots \\
&= (1 - p_n) U\left(\frac{\sum_{i=1}^{n-1} p_i x_i}{1 - p_n}\right) + p_n U(x_n) \\
&\le U\left(\sum_{i=1}^{n} p_i x_i\right) = U(\mathbb{E}[\tilde{x}])
\end{aligned}$$

The procedure above may be converted to an induction.

Continuous random variables. For simplicity, assume $U \in C^2$ [1].

$$U(x(\omega)) \leq U(\mathbb{E}[\tilde{x}]) + U'(\mathbb{E}[\tilde{x}])(x - \mathbb{E}[\tilde{x}])$$

$$\begin{aligned}
\mathbb{E}[U(\tilde{x})] &= \int_{\Omega} U(x(\omega))\, d\mathbb{P}(\omega) \\
&\leq U(\mathbb{E}[\tilde{x}]) + \int_{\Omega} U'(\mathbb{E}[\tilde{x}])(x - \mathbb{E}[\tilde{x}])\, d\mathbb{P}(\omega) \\
&= U(\mathbb{E}[\tilde{x}])
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\mathbb{E}[U(\tilde{x})] &= \mathbb{E}\left[U(\mathbb{E}[\tilde{x}]) + U'(\mathbb{E}[\tilde{x}])(\tilde{x} - \mathbb{E}[\tilde{x}]) + \frac{1}{2}U''(\tilde{x}+\delta)(\tilde{x} - \mathbb{E}[\tilde{x}])^2\right] \\
&= U(\mathbb{E}[\tilde{x}]) + \frac{1}{2}\mathbb{E}[U''(\tilde{x}+\delta)](\tilde{x} - \mathbb{E}[\tilde{x}])^2 \\
&\leq U(\mathbb{E}[\tilde{x}])
\end{aligned}$$

Q.E.D.

Remark: See Stochastic Calculus for Finance Vol.I (Shreve) pp. 30~31 for another cunning proof.

## 2 Grading Policies

- The quiz has 10 marks in total.
- If you have proven the propostion for only one of the above cases you may earn 7 marks for max.
- Listing the concavity condition earns 1 mark. You would not lose anything if your proof is correct but this condition is not shown explicitly.
- Discrete random variables. Proving the bi-point case earns 2 marks, and additional 5 marks for completing the induction.
- Continuous random variables. The notion on the differentiability of the function is compulsory and worth 2 marks. The rigorous usage of the Taylor expansion is required and worth 1 marks.
- If your proof does not help its readers in the sense of comprehensibility you are given no credit.
- THE GRADING POLICIES ARE NOT NEGOTIABLE AND YOUR SCORES ARE FINAL. YOU MAY FILE YOUR QUESTIONS ON THE QUIZ DIRECTLY TO PROF. TSAI.

[1] It may refrain the proof for further generalizations. However the issue is not of our interest. After all this is not a course on theoretical mathematics.