

Advanced Microeconomics II

Mixed Strategy Nash Equilibrium

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Mixed Strategies

- Existence of NE in strategic games is not very general, e.g. Matching Pennies.
- A possible improvement is to make action sets convex.
- Denote $\Delta(A_i)$ as the set of probability distributions over A_i .
- $\alpha_i \in \Delta(A_i)$ is a **mixed strategy** of player i .
- Let $U_i : \times_{j \in N} \Delta(A_j) \rightarrow \mathcal{R}$ be a von Neumann-Morgenstern utility function for player i that represents preferences over $\times_{j \in N} \Delta(A_j)$. (What assumption is implied?)

Mixed Strategies for Finite Games

For finite A :

- $\alpha_i(a_i)$ is the probability that α_i assigns to a_i .
- If $\alpha_i(a_i) = 1$ then α_i is a **pure strategy**.

$$U_i(\alpha) = \sum_{a \in A} (\prod_{j \in N} \alpha_j(a_j)) u_i(a).$$

Example:

		My wife	
		M	F
Me	M	3, 3	0, 0
	F	0, 0	1, 1

Let α_i^M be the probability that player i plays action M .

$$U_1(\alpha) = \alpha_1^M \alpha_2^M 3 + \alpha_1^M (1 - \alpha_2^M) 0 + (1 - \alpha_1^M) \alpha_2^M 0 + (1 - \alpha_1^M) (1 - \alpha_2^M) 1$$

Mixed Strategies for Infinite Games

For infinite A :

- $\alpha_i(\cdot)$ is the probability distribution function over A_i .

$$U_i(\alpha) = \int_{a \in A} u_i(a) (\prod_{j \in N} \alpha_j(a_j)) da.$$

Example:

$G = \{N, (A_i)_{i=1}^N, (u_i)_{i=1}^N\}$, where
 $N = \{1, 2\}$, $A_1 = A_2 = [0, \infty)$,
 $u_i(a_1, a_2) = \max\{(1 - a_1 - a_2)a_i, 0\}$

Let $\alpha_i(\cdot)$ be the probability distribution over A_i .

$$U_1(\alpha) = \int_0^\infty \int_0^\infty \max\{(1 - a_1 - a_2)a_1, 0\} \alpha_1(a_1) \alpha_2(a_2) da_1 da_2$$

Mixed Strategy Nash Equilibrium

Definition

The **mixed extension** of the strategic game $\{N, (A_i), (u_i)\}$ is the strategic game $\{N, (\Delta(A_i)), (U_i)\}$.

Definition

A **mixed strategy Nash equilibrium of a strategic game** is a Nash equilibrium of its mixed extension. Specifically, for every player $i \in N$

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha_{-i}^*) \text{ for every } \alpha_i \in \Delta(A_i).$$

Existence of Mixed Strategy Nash Equilibrium

Proposition

Every finite strategic game has a mixed strategy Nash equilibrium.

- $\Delta(A_i)$ is a non-empty, compact, convex set.
- U_i is linear in the probabilities over A_i , so it is continuous and quasi-concave in α_i .
- Apply our previous existence theorem.

Mixed Strategy Nash Equilibrium and Best Response

Lemma

Let $G = \{N, (A_i), (u_i)\}$ be a finite game. Then $\alpha^* \in \times_{i \in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G if and only if for every player $i \in N$ every pure strategy in the support of α_i^* is a best response to α_{-i}^* .

- We can write $U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(e(a_i), \alpha_{-i})$, where $e(a_i)$ is the strategy that plays a_i with probability one.
- (\Rightarrow) If a_i in the support of α_i is not a best response, then transfer $\alpha_i(a_i)$ to a best response action.
- (\Leftarrow) If there exists another α_i' that gives a higher payoff then there must be at least one action in the support of α_i' that gives a higher payoff than some action in the support of α_i^* .

Implication: Every action in the support of α_i^* yields the same payoff.

Pure Coordination Game

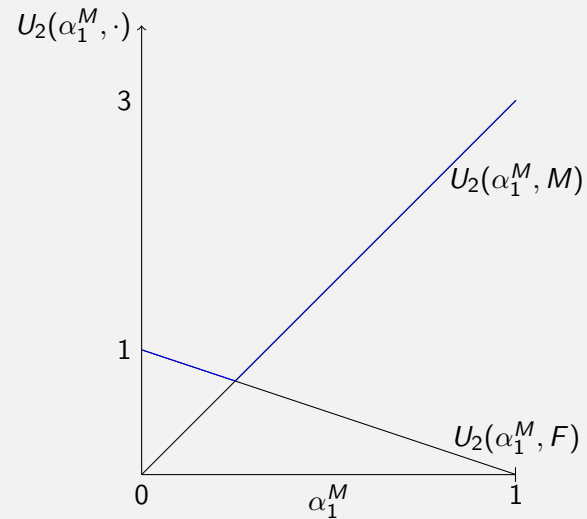
		My wife	
		M	F
Me	M	3, 3	0, 0
	F	0, 0	1, 1

- Two mixed strategy equilibria: $\{M, M\}$ and $\{F, F\}$.
- Other mixed strategy equilibria? What distribution on A_1 makes player 2 indifferent between M and F .
- Let α_1^M be the probability that I watch movies.
 - ▶ So $1 - \alpha_1^M$ is the probability that I watch football.

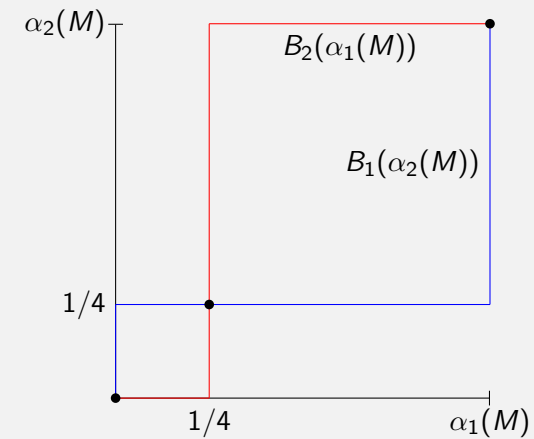
$$\begin{aligned} \alpha_1^M u_2(M, M) + (1 - \alpha_1^M) u_2(F, M) &= \alpha_1^M u_2(M, F) + (1 - \alpha_1^M) u_2(F, F) \\ 3\alpha_1^M &= 1 - \alpha_1^M \\ \alpha_1^M &= 1/4 \end{aligned}$$

- $\{\alpha_1^M = 1/4, \alpha_2^M = 1/4\}$ is a mixed strategy Nash equilibrium.

Deriving Best Response Functions



Best Response Functions



For You

		Player 2	
		L	R
Player 1	U	6, 6	2, 7
	D	7, 2	0, 0

- Are there any pure strategy equilibria?
- Are there any other mixed strategy equilibria?

War of Attrition

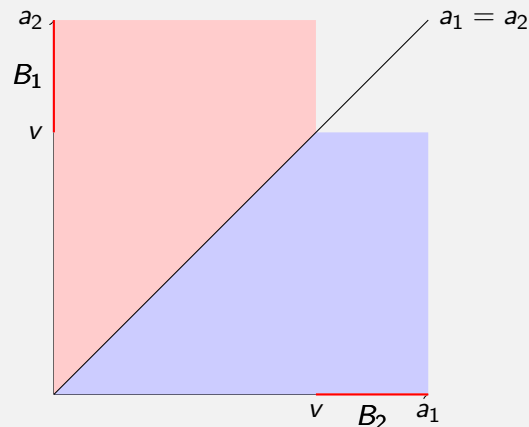
Two players are fighting over an object with value $v > 0$.

- Each player chooses a time to quit.
- The player who quits first loses and pays a cost equal to his quit time.
- The other player pays the same cost but wins the object.
- If they quit at the same time they each receive half the value of the object.

Model this as a strategic game.

Pure Strategy Equilibria

What are the pure strategy equilibria?



Other Mixed Strategy Equilibria

- Assume the equilibrium is symmetric.
- Assume each player's strategy is represented by a continuous distribution $F(a_i)$ with support over $[\underline{a}, \bar{a}]$.
 - ▶ Payoff from each pure strategy is the same, say C .

- Choose an arbitrary pure strategy a_1 . What is the payoff?

$$\int_{\underline{a}}^{a_1} (v - a_2) f(a_2) da_2 + \int_{a_1}^{\bar{a}} -a_1 f(a_2) da_2 \equiv C$$

- Differentiate both sides with respect to a_1 .

$$(v - a_1) f(a_1) + a_1 f(a_1) - \int_{a_1}^{\bar{a}} f(a_2) da_2 = 0$$

- This is a first order differential equation. $vf(a_1) = 1 - F(a_1)$
- So $F(a) = 1 - Ke^{-a/v}$. (What is \underline{a} , \bar{a} and K ?)

Interpretations of Mixed Strategy Equilibria

- As Objects of choice.
 - ▶ Actions are based on guesses and guessing is a psychological operation that is deliberate.
 - ▶ Doesn't model player motivation for randomization but is probably a good description of behavior.
- As a (Stochastic) Steady State
 - ▶ Reinterpret equilibrium as the interaction of n large populations — $\alpha_i(a_i)$ is the steady state frequency of matching with a_i in population i .
- As Pure Strategies in an Extended Game
 - ▶ There exists unmodeled factors that influence behavior.
 - ▶ Hard to accept that deliberate behavior is influenced by factors that do not affect payoffs.
 - ▶ Predicted behavior is fragile, since unobserved changes in external factors can destroy the equilibrium outcome.
 - ▶ In order to interpret an equilibrium in application need to specify these unmodeled effects and how they affect choices.

Interpretations of Mixed Strategy Equilibria

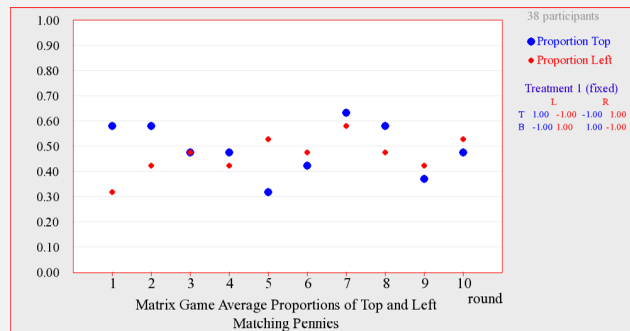
- As Pure Strategies in a Perturbed Game
 - ▶ (Harsanyi 1973) Game is an approximation of Bayesian game where each player's preferences are subject to small shocks.
 - ▶ The mixed strategy profile is the limit of a sequence of pure strategy equilibrium profiles of Bayesian games with successively smaller perturbations on payoffs.
 - ▶ The limit of any sequence of pure strategy equilibrium profiles of Bayesian games with successively smaller perturbations on payoffs is a Nash equilibrium mixed strategy profile of the limit game.
- As Beliefs
 - ▶ A Nash equilibrium is a profile of beliefs β where the other players have a *common* belief β_i about player i 's action and each action in the support of β_i is optimal given β_{-i} .
 - ▶ The equilibrium is a steady state of beliefs rather than actions.

Predictive Ability

Recall the following game from the class experiment.

		Player 2	
		L	R
Player 1	T	1, -1	-1, 1
	B	-1, 1	1, -1

What is the set of Nash equilibria?



Predictive Ability

Total observed choices: 380, proportion of "heads": 199.

Hypothesis test $H_0 : p = 0.5$; $H_1 : p \neq 0.5$.

$z = 0.92$, p -value = 0.36.

Frequency of outcomes

		Player 2	
		L	R
Player 1	T	44	49
	B	44	53

Pearson Chi-squared test = 1.2, p -value = 0.75.

Correlated Equilibrium

Definition

A **correlated equilibrium of a strategic game** $\{N, (A_i), (u_i)\}$ is

- a finite probability space (Ω, π) (Ω is the set of **states** and π is the probability measure on Ω)
- for each player $i \in N$ a partition of \mathcal{P}_i of Ω (player i 's **information partition**)
- for each player $i \in N$ a function $\sigma_i : \Omega \rightarrow A_i$ with $\sigma_i(\omega) = \sigma_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$ (σ_i is player i 's **strategy**)

such that for every $i \in N$ and every function $\tau_i : \Omega \rightarrow A_i$ for which $\tau_i(\omega) = \tau_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\tau_i(\omega), \sigma_{-i}(\omega))$$

Correlated Equilibrium and Nash Equilibrium

Proposition

For every mixed strategy Nash equilibrium α of a finite strategic game $\{N, (A_i), (u_i)\}$ there is a correlated equilibrium $\{(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\}$ in which for each player $i \in N$ the distribution on A_i induced by σ_i is α_i .

- Set $\Omega = A$ and $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$.
- For each $i \in N$ and $b_i \in A_i$ set $P_i(b_i) = \{a \in A : a_i = b_i\}$ and let \mathcal{P}_i consist of the $|A_i|$ sets $P_i(b_i)$.
- Define $\sigma_i(a) = a_i$ for each $a \in A$.
- This is a correlated equilibrium since for each player i , for each $a_i \in A_i$

$$\sum_{\{b \in A : b_i = a_i\}} \pi(b) u_i(a_i, b_{-i}) \geq \sum_{\{b \in A : b_i = a'_i\}} \pi(b) u_i(b'_i, b_{-i}) \text{ for any } b'_i \in A_i.$$

- The distribution on A_i induced by σ_i is α_i .

Correlated Equilibrium and Nash Equilibrium: Application

- Applied to our Pure co-ordination game and the mixed strategy equilibrium $\alpha_1^*(M) = \alpha_2^*(M) = 1/4$.
- Set $\Omega = A$ and $\pi(\{MM\}) = 1/16, \pi(\{MF\}) = \pi(\{FM\}) = 3/16, \pi(\{FF\}) = 9/16$.
- $P_1(M) = \{MM, MF\} = P_1^M, P_1(F) = \{FM, FF\} = P_1^F, \mathcal{P}_1 = \{P_1^M, P_1^F\},$
 $P_2(M) = \{MM, FM\} = P_2^M, P_2(F) = \{MF, FF\} = P_2^F, \mathcal{P}_2 = \{P_2^M, P_2^F\}.$
- $\sigma_1(MM) = \sigma_1(MF) = M, \sigma_1(FM) = \sigma_1(FF) = F,$
 $\sigma_2(MM) = \sigma_2(FM) = M, \sigma_2(MF) = \sigma_2(FF) = F.$

Set of Correlated Equilibrium Payoffs is Convex

Proposition

Let $G = \{N, (A_i), (u_i)\}$ be a strategic game. Any convex combination of correlated equilibrium payoff profiles of G is a correlated equilibrium payoff profile of G .

- Let u^1, \dots, u^K be correlated equilibrium payoff profiles and let $(\lambda^1, \dots, \lambda^K) \in \mathcal{R}^K$ with $\lambda^k \geq 0$ for all k and $\sum_{k=1}^K \lambda^k = 1$.
- For each k , $\{(\Omega^k, \pi^k), (\mathcal{P}_i^k), (\sigma_i^k)\}$ is the correlated equilibrium that generates u^k . (disjoint Ω^k)
- Let $\Omega = \cup_k \Omega^k$ and $\pi(\omega) = \lambda^k \pi^k(\omega)$ where k is such that $\omega \in \Omega^k$.
- For each i let $\mathcal{P}_i = \cup_k \mathcal{P}_i^k$.
- Let $\sigma_i(\omega) = \sigma_i^k(\omega)$ where k is such that $\omega \in \Omega^k$.

Correlated Equilibrium

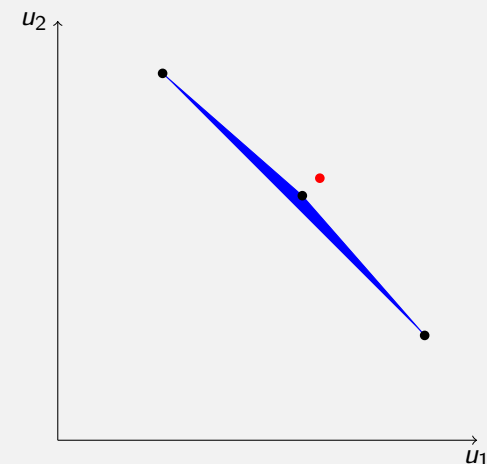
Is the set of correlated equilibrium payoffs the convex hull of mixed strategy Nash equilibrium payoffs?

		Player 2	
		L	R
Player 1	U	6, 6	2, 7
	D	7, 2	0, 0

		Player 2	
		L	R
Player 1	U	y	z
	D	x	-

- Recall the set of Nash equilibrium payoffs are $\{(2, 7), (7, 2), (4\frac{2}{3}, 4\frac{2}{3})\}$.
- Consider $\{(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\}$ where
 - $\Omega = \{x, y, z\}, \pi(x) = \pi(y) = \pi(z) = 1/3.$
 - $\mathcal{P}_1 = \{\{x\}, \{y, z\}\}, \mathcal{P}_2 = \{\{x, y\}, \{z\}\},$
 - $\sigma_1(x) = D, \sigma_1(y) = \sigma_1(z) = U,$
 $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$
- What are the payoffs to each player?
- Is it a correlated equilibrium.

Nash Equilibrium versus Correlated Equilibrium



Simplified State Space

Proposition

Let $G = \{N, (A_i), (u_i)\}$ be a finite strategic game. Every probability distribution over outcomes that can be obtained in a correlated equilibrium of G can be obtained in a correlated equilibrium in which the set of states is A and for each $i \in N$ player i 's information partition consists of all sets of the form $\{a \in A : a_i = b_i\}$ from some action $b_i \in A_i$.

- Let $\{(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\}$ be a correlated equilibrium of G .
- $\{(\Omega', \pi'), (\mathcal{P}'_i), (\sigma'_i)\}$ is also a correlated equilibrium of G .
 - ▶ $\Omega' = A$ and $\pi'(a) = \pi(\{\omega \in \Omega : \sigma(\omega) = a\})$ for each $a \in A$.
 - ▶ \mathcal{P}'_i consists of sets of the type $\{a \in A : a_i = b_i\}$ from some $b_i \in A_i$.
 - ▶ $\sigma'_i(a) = \sigma_i(a)$.