Advanced Microeconomics II

Expected Utility Theory

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A Gamble

Lottery A

Prize	\$2,500,000	\$500,000	\$0
Probability	0	1	0

Lottery B

Prize	\$2,500,000	\$500,000	\$0
Probability	0.1	0.89	0.01

Which lottery would you prefer?

Another Gamble

Lottery C

Prize	\$2,500,000	\$500,000	\$0
Probability	0	0.11	0.89

Lottery D

Prize	\$2,500,000	\$500,000	\$0
Probability	0.1	0	0.90

Which lottery would you prefer?

Choice Under Uncertainty

- Many important economic decisions involve risk.
- How do we model choice when it involves uncertainty?
- Standard choice theory works OK but uncertainty has a structure we can use to restrict preferences a "rational" person might hold.

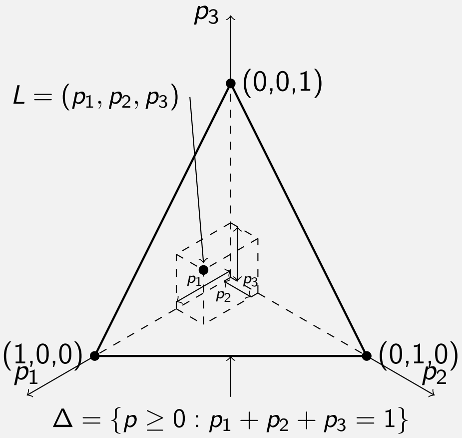
Modelling Risk

- C - set of all possible consequences (assume finite)
  - Index consequences by $n = 1, \dots, N$.
- Probabilities of outcomes are objectively known.

Definition

A simple lottery L is a list $L = (p_1, \dots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

Lottery Simplex

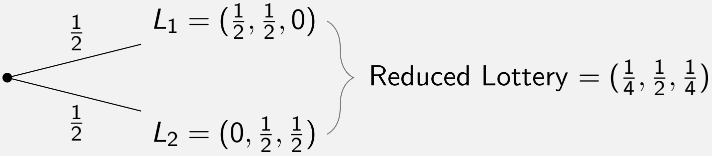


Compund Lotteries

Definition

Given K simple lotteries $L_k = (p_1^k, \ldots, p_N^k, k = 1 \ldots, K)$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \ldots, K$.

<u>EXAMPLE</u> $C = \{1, 2, 3\}$



Preferences Over Lotteries

- Assume consequentialism - only consequences matter.
- $\mathcal{L}$ - the set of all simple lotteries over the set of outcomes C.
- $\succeq$ is a rational preference relation on $\mathcal{L}$.

Definition

The preference relation $\succeq$ on the space of simple lotteries $\mathcal{L}$ is continuous if for any $L, L', L'' \in \mathcal{L}$ the sets

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succeq L''\} \subset [0,1]$$

and

$$\{\alpha \in [0,1] : L'' \succeq \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed.

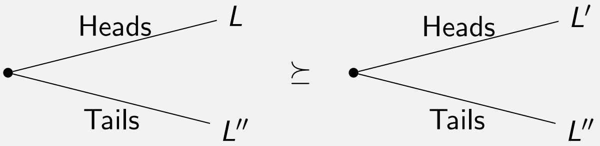
Independence Axiom

Definition

The preference relation $\succeq$ on the space of simple lotteries $\mathcal{L}$ satisfies the independence axiom if for any $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$ we have

$$L \succeq L' \text{ iff } \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''\}.$$

EXAMPLE



if and only if
$L \succeq L'$

Expected Utility Form

Definition

The utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if there is an assignment of numbers $(u_1, \ldots, u_n)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathcal{L}$ we have

$$U(L) = u_1 p_1 + \ldots + u_N p_N$$

- Such a function is called a von Neumann-Morgenstern (v.N-M) expected utility function.
- Linear function in the probabilities.

Linearity of Expected Utility Form

Proposition

A utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if and only if it is linear, that is, if and only if

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathcal{L}, k = 1 \ldots, K$ and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0, \sum_k \alpha_k = 1$.

Proof

- Let L be a simple lottery $L = (p_1, \ldots, p_N)$. L can be written as $\sum_n p_n L^n$, where $L^n = (\ldots, 1, \ldots)$. Let $(u_1, \ldots, u_N)$ be numbers such that $U(L^n) = u_n$.
  Then $U(L) = U(\sum_n p_n L^n) = \sum_n p_n U(L^n) = \sum_n p_n u_n$.
  So $U(\cdot)$ has expected utility form.
- In the other direction, consider a compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $L_k = (p_1^k, \ldots, p_N^k)$.
  Hence $U(\sum_k \alpha_k L_k) = \sum_n u_n(\sum_k \alpha_k p_n^k) = \sum_k \alpha_k(\sum_n u_n p_n^k) = \sum_k \alpha_k U(L_k)$.

Affine Transformations

Proposition

Suppose that $U : \mathcal{L} \to \mathbb{R}$ *is a v.N-M expected utility function for the preference relation* $\succeq$ *on* $\mathcal{L}$*. Then* $\tilde{U} : \mathcal{L} \to \mathbb{R}$ *is another v.N-M utility function for* $\succeq$ *if and only if there are scalars* $\beta > 0$ *and* γ *such that* $\tilde{U}(L) = \beta U(L) + \gamma$ *for every* $L \in \mathcal{L}$*.*

Proof

W.L.O.G assume $\exists\ \overline{L}$ and $\underline{L}$, such that $\overline{L} \succeq L \succeq \underline{L}$ and $\overline{L} \succ \underline{L}$ for all $L \in \mathcal{L}$.

- Let $U(\cdot)$ be a v.N-M expected utility function and $\tilde{U}(L) = \beta U(L) + \gamma$, then

$$\begin{aligned}
\tilde{U}\left(\sum_k \alpha_k L_k\right) &= \beta U\left(\sum_k \alpha_k L_k\right) + \gamma \\
&= \beta\left[\sum_k \alpha_k U\left(L_k\right)\right] + \gamma \\
&= \sum_k \alpha_k \left[\beta U\left(L_k\right) + \gamma\right] \\
&= \sum_k \alpha_k \tilde{U}\left(L_k\right)
\end{aligned}$$

Proof Cont.

- In the reverse direction, assume $\tilde{U}(\cdot)$ and $U(\cdot)$ have the expected utility form. Let $L \in \mathcal{L}$ and define $\lambda_L \in [0,1]$ by $U(L) = \lambda_L U(\overline{L}) + (1-\lambda_L)U(\underline{L})$. Thus $\lambda_L = \frac{U(L)-U(\underline{L})}{U(\overline{L})-U(\underline{L})}$. Since $\lambda_L U(\overline{L}) + (1-\lambda_L)U(\underline{L}) = U(\lambda_L \overline{L} + (1-\lambda_L)\underline{L})$, then $L \sim \lambda_L \overline{L} + (1-\lambda_L)\underline{L}$, so

$$\begin{aligned}\tilde{U}(L) &= \tilde{U}(\lambda_L \overline{L} + (1-\lambda_L)\underline{L}) \\ &= \lambda_L \tilde{U}(\overline{L}) + (1-\lambda_L)\tilde{U}(\underline{L}) \\ &= \lambda_L(\tilde{U}(\overline{L}) - \tilde{U}(\underline{L})) + \tilde{U}(\underline{L})\end{aligned}$$

Thus $\tilde{U}(L) = \beta U(L) + \gamma$, where $\beta = \frac{\tilde{U}(\overline{L})-\tilde{U}(\underline{L})}{U(\overline{L})-U(\underline{L})}$ and $\gamma = \tilde{U}(\underline{L}) - U(\underline{L})\frac{\tilde{U}(\overline{L})-\tilde{U}(\underline{L})}{U(\overline{L})-U(\underline{L})}$

Implications

- Differences in utilities have meaning.
- Identify $\succeq$ up to affine transformations.

<u>Example</u>

There are 4 outcomes. If $u_1 - u_2 > u_3 - u_4$, then we have $\frac{1}{2}(u_1 + u_4) > \frac{1}{2}(u_2 + u_3)$, thus $L = (\frac{1}{2}, 0, 0, \frac{1}{2}) \succ L'(0, \frac{1}{2}, \frac{1}{2}, 0)$

Expected Utility Theorem

Proposition

Suppose that the rational preference relation $\succeq$ on the space of lotteries $\mathcal{L}$ satisfies the continuity and independence axioms. Then $\succeq$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$ we have

$$L \succeq L \textit{ if and only if } \sum_{n=1}^{N} u_n p_n \succeq \sum_{n=1}^{N} u_n p'_n.$$

- Advantages - technically convenient, a useful guide to action
- Disadvantage - Poor descriptive theory.

Expected Utility Theorem Proof

Since the number of outcomes are finite and $\succeq$ satisfies the independence axiom, $\exists \overline{L}$ and $\underline{L}$, such that $\overline{L} \succeq L \succeq \underline{L}$ and $\overline{L} \succ \underline{L}, \forall L \in \mathcal{L}$. W.l.o.g. $\overline{L} \succ \underline{L}$.

- Step 1. If $L \succ L'$ and $\alpha \in (0,1)$, then $L \succ \alpha L + (1-\alpha)L' \succ L'$ (Why?)

Expected Utility Theorem Proof Cont.

- Step 2. Let $\alpha, \beta \in [0,1]$. Then $\beta\overline{L} + (1-\beta)\underline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L}$ if and only if $\beta > \alpha$.
  - ($\Leftarrow$)
    If $\beta > \alpha$, then
    $$\beta\overline{L} + (1-\beta)\underline{L} = \gamma\overline{L} + (1-\gamma)[\alpha\overline{L} + (1-\alpha)\underline{L}],$$
    where $\gamma = \frac{\beta-\alpha}{1-\alpha} \in (0,1]$.
    $$\begin{aligned} &\overline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L} \\ \Rightarrow &\gamma\overline{L} + (1-\gamma)[\alpha\overline{L} + (1-\alpha)\underline{L}] \succ \alpha\overline{L} + (1-\alpha)\underline{L} \text{ (Why?)} \\ \Rightarrow &\beta\overline{L} + (1-\beta)\underline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L}. \end{aligned}$$
  - ($\Rightarrow$)
    If $\beta = \alpha$, then $\beta\overline{L} + (1-\beta)\underline{L} \sim \alpha\overline{L} + (1-\alpha)\underline{L}$.
    If $\alpha > \beta$,then from above $\alpha\overline{L} + (1-\alpha)\underline{L} \succ \beta\overline{L} + (1-\beta)\underline{L}$.

Expected Utility Theorem Proof Cont.

- Step 3. For any $L \in \mathcal{L}, \exists \alpha_L$ such that $\alpha_L \overline{L} + (1-\alpha_L)\underline{L} \sim L$.
  - Define

$$B = \{\alpha \in [0,1] : \alpha \overline{L} + (1-\alpha)\underline{L} \succeq L\}$$
$$W = \{\alpha \in [0,1] : L \succeq \alpha \overline{L} + (1-\alpha)\underline{L}\}$$

  - B and W are non-empty, closed and cover $[0,1]$. (Why?)
  - $\exists \alpha$ belonging to both sets.
  - Uniqueness follows from the result of step 2.

Expected Utility Theorem Proof Cont.

- Step 4. $U : \mathcal{L} \to \mathbb{R}$ that assigns $U(L) = \alpha_L$ for $\forall L \in \mathcal{L}$ represents $\succeq$.
  - By step 3, we have $L \succeq L'$ if and only if
    $$\alpha_L \overline{L} + (1 - \alpha_L)\underline{L} \succeq \alpha'_L \overline{L} + (1 - \alpha'_L)\underline{L}.$$
  - By step 2,
    $$\alpha_L \overline{L} + (1 - \alpha_L)\underline{L} \succeq \alpha'_L \overline{L} + (1 - \alpha'_L)\underline{L}$$
    if and only if $\alpha_L \geq \alpha'_L$.
- What's left?

Expected Utility Theorem Proof Cont.

- Step 5. $U(\cdot)$ that assigns $U(L) = \alpha_L$ for $\forall L \in \mathcal{L}$ has the expected utility form.
  - By definition

$$L \sim U(L)\overline{L} + (1 - U(L))\underline{L}$$
$$L' \sim U(L')\overline{L} + (1 - U(L'))\underline{L}$$

  - So

$$\begin{aligned}
\beta L + (1-\beta)L' &\sim \beta[U(L)\overline{L} + (1 - U(L))\underline{L}] \\
&\quad + (1-\beta)[U(L')\overline{L} + (1 - U(L'))\underline{L}] \\
&\sim [\beta U(L) + (1-\beta)U(L')]\overline{L} \\
&\quad + [1 - \beta U(L) - (1-\beta)U(L')]\underline{L}
\end{aligned}$$

  - By construction of $U(.)$

$$U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta)U(L')$$

Allais Paradox

Lottery A

Prize	\$2,500,000	\$500,000	\$0
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Lottery B

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Lottery C

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Allais Paradox

- If $A \succeq_i B$ then

$$u_{05} > (.10)u_{25} + (.89)u_{05} + (.01)u_0.$$

- Add $(.89)u_0 - (.89)u_{05}$ to both sides.

$$(.11)u_{05} + (.89)u_0 > (.10)u_{25} + (.90)u_0.$$

- Any v.N-M utility function must have $C \succeq_i D$.
- Other paradoxes - St Petersburg, Ellsberg, Machina
- Alternative theories of choice under uncertainty - regret, prospect