Advanced Microeconomics II

Iterated Elimination of Dominated Strategies

Brett Graham

Wang Yanan Institute for Studies in Economics
Xiamen University, China

March 3, 2015

Never-Best Response and Strictly Dominated

Definition

An action $a_i \in A_i$ is a never-best response if it is not a best response to any belief μ_i of player i where $\mu_i \in \Delta(\times A_{-i})$.

Definition

The action $a_i \in A_i$ of player i in the strategic game $\{N, (A_i), (u_i)\}$ is strictly dominated if there is a mixed strategy α_i of player i such that $U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, where $U_i(\alpha_i, a_{-i})$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

Never-Best Response - Strictly Dominated Equivalence

Lemma

An action of a player in a finite strategic game is a never-best response if and only if it is strictly dominated.

Fix a game $G = \{N, (A_i), (u_i)\}$ and a strategy a_i^*.

- Create an auxiliary zero-sum game $G' = \{\{1,2\}, (A_i')_{i=1}^2, (u_i')_{i=1}^2\}$ where
  - $A_1' = A_i \backslash \{a_i^*\}$ and $A_2' = A_{-i}$.
  - $u_1'(a_i, a_{-i}) = u_i(a_i, a_{-i}) - u_i(a_i^*, a_{-i})$

	L	R
T	$3,0$	$0,1$
M	$0,0$	$3,1$
B	$1,1$	$1,0$

G

$\Rightarrow$

	L	R
T	$2,-2$	$-1,1$
M	$-1,1$	$2,-2$

G'

Never-Best Response - Strictly Dominated Equivalence

a_i^* is a never-best response in G if and only if for any mixed strategy of player 2 in G' there is an action of player 1 in G' that yields player 1 a positive payoff.

$$\begin{aligned}
&\forall m_{-i} \in \Delta(A_{-i}), \exists a_i \in A_i : U_i(a_i, m_{-i}) - U_i(a_i^*, m_{-i}) > 0 \\
\Leftrightarrow &\forall m_2 \in \Delta(A_2'), \exists a_1 \in A_1' : U_1'(a_1, m_2) > 0 \\
\Leftrightarrow &\forall m_2 \in \Delta(A_2'), \exists m_1 \in \Delta(A_1') : U_1'(m_1, m_2) > 0 \text{ (Why?)} \\
\Leftrightarrow &\min_{m_2 \in \Delta(A_2')} \max_{m_1 \in \Delta(A_1')} U_1'(m_1, m_2) > 0
\end{aligned}$$

From previous results: G' has a mixed strategy equilibrium $\Rightarrow$

$$\begin{aligned}
&\min_{m_2 \in \Delta(A_2')} \max_{m_1 \in \Delta(A_1')} U_1'(m_1, m_2) > 0 \\
\Leftrightarrow &\max_{m_1 \in \Delta(A_1')} \min_{m_2 \in \Delta(A_2')} U_1'(m_1, m_2) > 0 \\
\Leftrightarrow &\exists m_1^* \in \Delta(A_1') : \forall m_2 \in \Delta(A_2'), U_1'(m_1^*, m_2) > 0 \\
\Leftrightarrow &\exists m_1^* \in \Delta(A_i) : \forall a_{-i} \in A_{-i}, U_i(m_1^*, a_{-i}) - u_i(a_i^*, a_{-i}) > 0
\end{aligned}$$

Iterated Elimination of Strictly Dominated Actions

Definition

The set $X \subset A$ of outcomes of a finite strategic game $\{N, (A_i), (u_i)\}$ survives iterated elimination of strictly dominated strategies if $X = \times_{j\in N} X_j$ and there is a collection $((X_j^t)_{j\in N})_{t=0}^T$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.
- $X_j^{t+1} \subset X_j^t$ for each $t = 0, \dots, T-1$.
- For each $t = 0, \dots, T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is strictly dominated in the game $\{N, (X_i^t), (u_i^t)\}$ where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j\in N} X_j^t$.
- No action in X_j^T is strictly dominated in the game $\{N, (X_i^T), (u_i^T)\}$.

Iterated Elimination - Example

	L	R
T	$3,0$	$0,1$
M	$0,0$	$3,1$
B	$1,1$	$1,0$

X^0

	L	R
T	$3,0$	$0,1$
M	$0,0$	$3,1$

X^1

	R
T	$0,1$
M	$3,1$

X^2

	R
M	$3,1$

X^3

- $X^0 \to X^1$: B is strictly dominated by $\alpha_1(T) = \alpha_1(M) = 1/2$.
- $X^1 \to X^2$: L is strictly dominated by R.
- $X^2 \to X^3$: T is strictly dominated by M.

For You

	b_1	b_2	b_3	b_4
a_1	$0,7$	$2,5$	$7,0$	$0,1$
a_2	$5,2$	$3,3$	$5,2$	$0,1$
a_3	$7,0$	$2,5$	$0,7$	$0,1$
a_4	$0,0$	$0,-2$	$0,0$	$10,-1$

- What are the set of strategies that survive iterated elimination of strictly dominated strategies.

Iterated Elimination and Rationalizable Actions

Proposition

If $X = \times_{j\in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\{N, (A_i), (u_i)\}$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

($\Leftarrow$) First show that if a_i is rationalizable then $a_i \in X_i^T$.

- Let $(Z_j)_{j\in N}$ be the profile of sets that supports a_i.
- For any t, $Z_j \subset X_j^t$ since each action in Z_j is a best response to some belief over Z_{-j}.

($\Rightarrow$) Now show that for any player i any action in X_i^T is rationalizable.

- By definition if $a_i \in X_i^T$ then it is not strictly dominated and is a best response among actions in X_i^T to some belief $\mu_i(a_i)$ over X_{-i}^T.
- It must also be a best response among the actions in A_i.
  - Otherwise $\exists t$, a_i is a best response over X_{-i}^t but not over X_{-i}^{t-1}.
  - $\exists b_i \in X_i^{t-1} \backslash X_i^t$ which is a best response to $\mu_i(a_i)$ over X_{-i}^{t-1}.
  - b_i cannot be strictly dominated in tth round.
- Note that order is not important.

Another Example

	L	R
U	8	0
D	0	0

M_1

	L	R
	4	0
	0	4

M_2

	L	R
	0	0
	0	8

M_3

	L	R
	3	3
	3	3

M_4

- Since order is not important, in each round let's eliminate all strictly dominated strategies in that round.

Iterated Elimination of Weakly Dominated Actions

Definition

The action $a_i \in A_i$ of player i in the strategic game $\{N, (A_i), (u_i)\}$ is weakly dominated if there is a mixed strategy α_i of player i such that $U_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$ and $U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for some $a_{-i} \in A_{-i}$ where $U_i(\alpha_i, a_{-i})$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

- Is a weakly dominated action strictly dominated?
- Is a strictly dominated action weakly dominated?
- A weakly dominated action that is not strictly dominated is a best response to some belief.
- Order matters for iterated elimination of weakly dominated strategies.

Weak Iterated Elimination - Example

	L	R
T	$1,1$	$0,0$
M	$1,1$	$2,1$
B	$0,0$	$2,1$

X^0

	L	R
M	$1,1$	$2,1$
B	$0,0$	$2,1$

X^1

	R
M	$2,1$
B	$2,1$

X^2

- $X^0 \to X^1$: T is weakly dominated by M.
- $X^1 \to X^2$: L is weakly dominated by R.

	L	R
T	$1,1$	$0,0$
M	$1,1$	$2,1$
B	$0,0$	$2,1$

X^0

	L	R
T	$1,1$	$0,0$
M	$1,1$	$2,1$

X^1

	L
T	$1,1$
M	$1,1$

X^2

- $X^0 \to X^1$: B is weakly dominated by M.
- $X^1 \to X^2$: R is weakly dominated by L.

Dominance Solvability

Definition

A strategic game is dominance solvable if all players are indifferent between all outcomes that survive the iterative procedure in which all the weakly dominated actions of each player are eliminated at each stage.

	L	R
T	$1,1$	$0,0$
M	$1,1$	$2,1$
B	$0,0$	$2,1$

X^0

	L	R
M	$1,1$	$2,1$

X^1

- The game is not dominance solvable.

Example

Each of two players announces a non-negative integer equal to at most 100. If $a_1 + a_2 \leq 100$, where a_i is the number announced by player i, then each player i receives payoff of a_i. If $a_1 + a_2 > 100$ and $a_i < a_j$ then player i receives a_i and player j receives $100 - a_i$; if $a_1 + a_2 > 100$ and $a_i = a_j$ then each player receives 50.

Formulate this as a normal form strategic game.