Homework 3 solution
Econometrics II Spring, 2013

3.8 Answer
Since
$$\hat{\beta}-\tilde{\beta}=(X^{\intercal}X)^{-1}R^{\intercal}[R(X^{\intercal}X)^{-1}R^{\intercal}]^{-1}(R\hat{\beta}-r)$$
(cf. textbook: page 41 in chapter 3), then
$$(\hat{\beta}-\tilde{\beta})^{\intercal}X^{\intercal}X(\hat{\beta}-\tilde{\beta})=(R\hat{\beta}-r)^{\intercal}[R(X^{\intercal}X)^{-1}R^{\intercal}]^{-1}(R\hat{\beta}-r)$$
Hence,
$$\begin{aligned}F &= \frac{(\hat{\beta}-\tilde{\beta})^{\intercal}X^{\intercal}X(\hat{\beta}-\tilde{\beta})/J}{s^2}\\ &= \frac{\sum(\hat{Y}_t-\tilde{Y}_t)^2/J}{s^2}\end{aligned}$$
where we use the fact that $\hat{Y}=X\hat{\beta}$ and $\tilde{Y}=X\tilde{\beta}$.

3.9 cf. textbook: page 43 in chapter 3 and page 27 in chapter 4.

3.11 Answer

Model 1:
$$Y_t=X_t^{\intercal}\beta^0+(D_tX_t)^{\intercal}\alpha^0+\varepsilon_t, t=1,\cdots,n$$
Model 2:
$$Y_t=X_t^{\intercal}\beta^0+\varepsilon_t, t=1,\cdots,n_1$$
Model 3:
$$Y_t=X_t^{\intercal}(\beta^0+\alpha^0)+\varepsilon_t, t=n_1+1,\cdots,n$$
Let $X=\begin{pmatrix} X^1 & \mathbf{0} \\ X^2 & X^2\end{pmatrix}$, where $X^1=(X_t^{\intercal})_{t=1}^{n_1}$, $X^2=(X_t^{\intercal})_{t=n_1+1}^{n}$. In Model 1, the estimators of β^0 and α^0, denoted as $\hat{\beta}$ and $\hat{\alpha}$, satisfy the normal equation which is
$$\begin{aligned}X^{\intercal}X\begin{pmatrix}\hat{\beta}\\ \hat{\alpha}\end{pmatrix} &= \begin{pmatrix} X^1 & \mathbf{0} \\ X^2 & X^2\end{pmatrix}^{\intercal}\begin{pmatrix} X^1 & \mathbf{0} \\ X^2 & X^2\end{pmatrix}\begin{pmatrix}\hat{\beta}\\ \hat{\alpha}\end{pmatrix}\\ &= \begin{pmatrix} X^{1^{\intercal}}X^1+X^{2^{\intercal}}X^2 & X^{2^{\intercal}}X^2 \\ X^{2^{\intercal}}X^2 & X^{2^{\intercal}}X^2\end{pmatrix}\begin{pmatrix}\hat{\beta}\\ \hat{\alpha}\end{pmatrix}\\ &= \begin{pmatrix} X^1 & \mathbf{0} \\ X^2 & X^2\end{pmatrix}^{\intercal}\begin{pmatrix}Y^1\\ Y^2\end{pmatrix}\\ &= \begin{pmatrix} X^{1^{\intercal}}Y^1+X^{2^{\intercal}}Y^2 \\ X^{2^{\intercal}}Y^2\end{pmatrix}\end{aligned}$$

from which we can get

$$(X^{1^\intercal} X^1 + X^{2^\intercal} X^2)\hat{\beta} + X^{2^\intercal} X^2 \hat{\alpha} = X^{1^\intercal} Y^1 + X^{2^\intercal} Y^2 \tag{1}$$

$$X^{2^\intercal} X^2 (\hat{\beta} + \hat{\alpha}) = X^{2^\intercal} Y^2 \tag{2}$$

(1) − (2):

$$X^{1^\intercal} X^1 \hat{\beta} = X^{1^\intercal} Y^1 \tag{3}$$

It is easy to see that (2) and (3) are the normal equations that model 3 and model 2 should satisfy respectively. So the coefficient estimators from these three models are the same. Thus

$$\begin{aligned} SSR_u &= (Y^1 - X^1\hat{\beta})^\intercal (Y^1 - X^1\hat{\beta}) + (Y^2 - X^2(\hat{\beta} + \hat{\alpha}))^\intercal (Y^2 - X^2(\hat{\beta} + \hat{\alpha})) \\ &= SSR_1 + SSR_2 \end{aligned}$$