Advanced Microeconomics II
Problem Set 2

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1. (OR 48.1) Consider the three-player game with the payoffs given in the following table. (Player 1 chooses one of the two rows, player 2 chooses one of the two columns, and player 3 chooses one of the three tables.)

A

	L	R
U	$0,0,3$	$0,0,0$
D	$1,0,0$	$0,0,0$

B

	L	R
U	$2,2,2$	$0,0,0$
D	$0,0,0$	$2,2,2$

C

	L	R
U	$0,0,0$	$0,0,0$
D	$0,1,0$	$0,0,3$

(a) Show that the pure strategy equilibrium payoffs are $(1,0,0)$, $(0,1,0)$, and $(0,0,0)$.

Solution: The pure strategy equilibria are (D,L,A), (U,R,A), (D,L,C) and (U,L,C).

(b) Show that there is a correlated equilibrium in which player 3 chooses B and players 1 and 2 play (T,L) and (B,R) with equal probabilities.

Solution: It's easy to show that one of the correlated equilibria with the outcome described above is given by:

- $\Omega = \{x,y\}, \pi(x) = \pi(y) = \frac{1}{2}$;
- $\mathcal{P}_1 = \mathcal{P}_2 = \{\{x\},\{y\}\}, \mathcal{P}_3 = \Omega$;
- $\sigma_1(\{x\}) = U, \sigma_1(\{y\}) = D; \sigma_2(\{x\}) = L, \sigma_2(\{y\}) = R; \sigma_3(\Omega) = B$.

Note that player 3 knows that (U,L) and (D,R) will occur with equal probabilities, so that if he deviates to A or C he will obtain $\frac{3}{2} < 2$.

(c) Explain the sense in which player 3 prefers not to have the information that players 1 and 2 use to coordinate their actions.

Solution: If player 3 were to have the same information as players 1 and 2, which means $\mathcal{P}_3 = \{\{x\},\{y\}\}$, then the above outcome is no longer a correlated equilibrium. Player 3 would prefer to deviate to A on signal x and C on signal y. If each player has perfect information about the other player's choices, the outcome in each state would need to be a Nash equilibrium. In each of the four equilibria identified above the Nash equilibrium payoff for player 3 is zero.

2. Consider the strategic game described in the following table.

	L	R
U	$3,3$	$1,4$
D	$4,1$	$0,0$

(a) (5 points) What are the set of mixed strategy Nash equilibria for this game.

Solution: Note that the set of pure strategy Nash equilibria are a subset of the mixed strategy Nash equilibria. It is obvious that there are 2 pure strategy Nash equilibria in this game, one is (U, R) and the other is (D, L). In order to find the other mixed strategy Nash equilibrium, assume (α, β) is the mixed strategy Nash equilibrium. Consider player 1, to be indifferent between U and D, we must have

$$3\alpha + 1 \times (1-\alpha) = 4\alpha + 0 \times (1-\alpha) \Rightarrow \alpha = \frac{1}{2}$$

Since the game is symmetric, we have $\beta = \frac{1}{2}$, thus $(\frac{1}{2}, \frac{1}{2})$ is a mixed strategy Nash equilibrium. Thus the full set of mixed strategy Nash equilibria is

$$\{\{(1/2, 1/2), (1/2, 1/2)\}, \{(0, 1), (1, 0)\}, \{(1, 0), (0, 1)\}\}.$$

(b) (5 points) Construct a correlated equilibrium for this game with payoffs that are equal to the payoffs in one of the Nash equilibria you constructed in (a).

Solution: Three possible solutions:

1. Let $\Omega = \{x\}$, $\pi(x) = 1$, $P_1 = P_2 = \{x\}$, $\sigma_1(x) = U$ and $\sigma_2(x) = R$. This correlated equilibrium yields the payoff profile $(1, 4)$.
2. Let $\Omega = \{x\}$, $\pi(x) = 1$, $P_1 = P_2 = \{x\}$, $\sigma_1(x) = D$ and $\sigma_2(x) = L$. This correlated equilibrium yields the payoff profile $(4, 1)$.
3. Let $\Omega = \{w, x, y, z\}$, $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1/4$, $P_1 = \{\{w, x\}, \{y, z\}\}$, $P_2 = \{\{w, y\}, \{x, z\}\}$, $\sigma_1(w) = \sigma_1(x) = U$, $\sigma_1(y) = \sigma_1(z) = D$, $\sigma_2(w) = \sigma_2(y) = L$ and $\sigma_2(x) = \sigma_2(z) = R$. This correlated equilibrium yields the payoff profile $(2, 2)$.

(c) (10 points) Construct a correlated equilibrium where payoffs for each player are equal to 2.5.

Solution: Solution 1: Note that the payoff 2.5 is a convex combination of two Nash equilibrium profiles. Let $\Omega = \{x, y\}$, $\pi(x) = \pi(y) = 1/2$, $P_1 = P_2 = \{\{x\}, \{y\}\}$, $\sigma_1(x) = D$, $\sigma_1(y) = U$, $\sigma_2(x) = L$ and $\sigma_2(y) = R$.

Solution 2: First find the correlated equilibrium with the highest joint payoff. Let $\Omega = \{x, y, z\}$, $\pi(x) = \alpha = \pi(z)$, $\pi_(y) = \beta$, $P_1 = \{\{x\}, \{y, z\}\}$, $P_2 = \{\{x, y\}, \{z\}\}$, $\sigma_1(x) = D$, $\sigma_1(y) = \sigma_1(z) = U$, $\sigma_2(x) = \sigma_2(y) = L$ and $\sigma_2(z) = R$. The correlated equilibrium that maximizes joint payoffs solve

$$\max_{\alpha,\beta} 5\alpha + 3\beta \text{ s.t. } 2\alpha + \beta = 1 \text{ and } \alpha \geq \beta.$$

Hence $\alpha = \beta = 1/3$. This correlated equilibrium generates payoffs of $8/3, 8/3$.

To construct a correlated equilibrium with payoffs of $2, 2$, find the convex combination of the correlated equilibrium above and the mixed strategy Nash equilibrium that generate the appropriate payoffs. We need to find λ that solves

$$\lambda 2 + (1 - \lambda)8/3 = 5/2.$$

Hence, $\lambda = 1/4$. So the correlated equilibrium constructed in this way is as follows. Let $\Omega = \{w, x, y, z\}$, $\pi(w) = \pi(x) = \pi(y) = 5/16, \pi(z) = 1/16$, $P_1 = \{\{w, x\}, \{y, z\}\}$, $P_2 = \{\{w, y\}, \{x, z\}\}$, $\sigma_1(w) = \sigma_1(x) = U$, $\sigma_1(y) = \sigma_1(z) = D$, $\sigma_2(w) = \sigma_2(y) = L$ and $\sigma_2(x) = \sigma_2(z) = R$.

3. (OR 63.1) (A location game) Two people chooses whether or not to become a political candidate, and if so which position to take. There is a continuum of citizens, each of whom has a favorite position; the distribution of favorite positions is uniformly distributed between 0 and m. A candidate attracts the votes of those citizens whose favorite positions are closer to his position than to his opponent; if the candidates choose the same position then each receives half of the votes. The winner of the competition is the candidate who receives the most votes. Each person prefers to be the unique winning candidate than to tie for first place, prefers to tie for first place than to stay out of the competition, and prefers to stay out of the competition than to enter and lose.

   (a) Formulate this situation as a strategic game

   Solution: The strategic game consists of:

   - $N = \{1, 2\}$
   - For each player $i \in N, A_i = \{Out\} \cup [0, m]$
   - Let $V_i(a)$ represents the numbers of votes player i receives.That is:

   $$V_i(a) = \begin{cases} \frac{a_i + a_j}{2} & a_i < a_j \\ \frac{m}{2} & a_i = a_j \\ m - \frac{a_i + a_j}{2} & a_i > a_j \end{cases}$$

Then, for each player $i \in N$, $j \in N$ and $i \neq j$, the preference relation for player i, $\succsim_i$, on A is defined as:

$$\{a : V_i > V_j\} \succsim_i \{a : V_i = V_j\} \succsim_i \{Out\} \succsim_i \{a : V_i < V_j\}$$

(b) Find the set of strategies that survive repeated elimination of strictly dominated strategies. If you eliminate all strictly dominated strategies in each round of elimination how many rounds will there be until the set of strategies that survive repeated elimination of strictly dominated strategies is reached.

Solution: We will iteratively eliminate strictly dominated strategies. For each $i \in N, j \in N$ and $i \neq j$:

- $X_i^0 = \{Out\} \cup [0, m]$
- $X_i^1 = [0, m]$
  - For $a_i = m/2$, the outcome is, at worst, $V_i = V_j$. Hence, $a_i = m/2$ strictly dominates $a_i = Out$.
  - For any $a_i \in [0, m]$, all generate the same most preferred outcome (the unique winner) when $a_j = Out$. Hence, none is strictly dominated by any other strategy.
- $X_i^2 = (0, m)$
  - For $a_i = 0$, the outcome is the worst possible for all $a_j \in (0, m)$ and is the second best (tie) for all $a_j \in \{0, m\}$. For $a_i = m/2$, the outcome is the best possible for all $a_j \in [0, m] \backslash \{m/2\}$ and is the second best for $a_j = m/2$. Hence, 0 is strictly dominated by $a_i = m/2$.
  - For any $a_i \in (0, m/2]$, a_i is a best response to $a_i/2$, since the outcome is optimal. For any $a_i \in [m/2, m)$, a_i is a best response to $(m + a_i)/2$, since the outcome is optimal. So no $a_i \in (0, m)$ is strictly dominated.
- $X_i = X_i^2$

The set of strategies that survive iterated elimination of strictly dominated strategies is $X = (0, m) \times (0, m)$ and we need two rounds to reach it.

(c) Find the set of strategies that survive repeated elimination of weakly dominated strategies. If you eliminate all weakly dominated strategies in each round of elimination how many rounds will there be until the set of strategies that survive repeated elimination of weakly dominated strategies is reached? Is order important? Is the game dominance solvable?

Solution: We will iteratively eliminate weakly dominated strategies. For each $i \in$

$N, j \in N$ and $i \neq j$: For each $i \in N, j \in N$ and $i \neq j$:

- $X_i^0 = \{out\} \cup [0, m]$
- $X_i^1 = \left\{\frac{m}{2}\right\}$
  - For any $a_j \in X_j^0 \backslash \{m/2\}$, $a_i = m/2$ generates the optimal outcome, for $a_j = m/2$, $a_i = m/2$ generates the second best otucome. For $a_j = m/2$, $a_i \neq m/2$ generates the worst outcome. Hence, $a_i = m/2$ weakly dominates all other actions.
- $X_i = X_i^1$

The set of strategies that survive iterated elimination of weakly dominated strategies is $X = \left\{\frac{m}{2}\right\} \times \left\{\frac{m}{2}\right\}$ and we need only one round to reach it. In this example, order doesn't matter and the game is dominance solvable.

4. Consider the following two definitions

Definition 1. *An action $a_i \in A_i$ is a* ***never-best response*** *if it is not a best response to any belief of player i over A_{-i}.*

Definition 2. *An action $a_i \in A_i$ is a* ***constrained never-best response*** *if it is not a best response to any belief of player i over A_{-i} where the belief is constrained to be a product of independent probability distributions over each A_k where $k \neq i$.*

Furthermore, consider the following three-player game, where each player receives the same payoff from a given outcome:

M_1:

	L	R
U	9	0
D	0	0

M_2:

	L	R
U	0	9
D	9	0

M_3:

	L	R
U	0	0
D	0	9

M_4:

	L	R
U	6	0
D	0	6

(a) Prove or disprove that M_4 is a never-best response.

Solution: M_4 is not a never-best response. Let μ_i be the belief of player i on the set A_{-i}, then M_4 is a best response to belief:

$$\mu_3(M_4) = \left\{Pr(U, L) = Pr(D, R) = \frac{1}{2}\right\}.$$

(b) Prove or disprove that M_4 is a constrained never-best response.

Solution: M_4 is a constrained never-best response. Assume that the belief over player 1's action set $\{U, D\}$ is $(p, 1-p)$, where p represents the probability that player 1 will play U and the belief over player 2's action set $\{L, R\}$ is $(q, 1-q)$, where q is the probability that player 2 will play L. To ensure that Players 3's expected utility for each action is:

$$\begin{aligned}
u_3(M_1) &= 9pq \\
u_3(M_2) &= 9(p(1-q)+q(1-p)) \\
u_3(M_3) &= 9(1-p)(1-q) \\
u_3(M_4) &= 6pq + 6(1-p)(1-q))
\end{aligned}$$

We need to show that $u_3(M_4) < \max\{u_3(M_1), u_3(M_2), u_3(M_3)\}$ for all $(p, q) \in [0,1] \times [0,1]$. It is sufficent to show that the solution to the following maximization problem is strictly negative:

$$\begin{aligned}
&\max_{p,q} pq + (1-p)(1-q) - 3/5 && (u_3(M_4) - u_3(M_2)) \\
&\text{subject to } p + q \leq (2 + pq)/2 && (u_3(M_4) \geq u_2(M_1)) \\
&\qquad\qquad\quad p + q \geq 1 - pq && (u_3(M_4) \geq u_2(M_3))
\end{aligned}$$

We assume that the first constraint binds while the second does not (these assumptions, along with the second-order conditions, can be shown to be satisfied). The first order conditions are

$$\begin{aligned}
-1 + 2q + \lambda(q/2 - 1) &= 0 \\
-1 + 2p + \lambda(p/2 - 1) &= 0 \\
p + q &= 1 - pq.
\end{aligned}$$

The first-order conditions imply that $p = q = \sqrt{2} - 1$. The value of the objective function is $9 - 6\sqrt{2} - 3/5 \approx -0.085$. Thus, we have proven that M_4 is a never-best response.

(c) Prove or disprove that M_4 is a strictly dominated strategy.

Solution: M_4 is not a strictly dominated strategy. You can use Lemma 60.1 that was proven in class. However, it may be instructive to prove it directly. For M_4 to be strictly dominated by the strategy α_3, we can, without loss of generality assume that $\alpha_3(M_4) = 0$. (Why?). Furthermore, strict domination requires $9\alpha_3(M_1) > 6$, $9\alpha_3(M_2) > 0$ and $9\alpha_3(M_3) > 6$. But, this contradicts $\alpha_3(M_1)+\alpha_3(M_2)+\alpha_3(M_3) = 1$.

(d) Comment on the equivalence of the set of rationalizable strategies and the set of strategies that survive iterated elimination of strictly dominated strategies.

Solution: The set of rationalizable strategies and the set of strategies that survive iterated elimination of strictly dominated strategies are equivalent if we do not require beliefs to be the product of independent proability distributions over the other player's actions. However, if we do require beliefs to be the product of independent proability distributions over the other player's actions then actions that are not strictly dominated (M_4) may not be rationalizable.

5. Recall the definition of a strictly dominated action. The action $a_i \in A_i$ of player i in the strategic game $\{N, (A_i), (u_i)\}$ is *strictly dominated* if there is a mixed strategy α_i of player i such that $U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, where $U_i(\alpha_i, a_{-i})$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

   Two alternate definitions for a mixed strategy α_i follow. The mixed strategy $\alpha_i \in \Delta(A_i)$ of player i in the strategic game $\{N, (A_i), (u_i)\}$ is *strictly dominated* if there is a mixed strategy α'_i of player i such that

   (i) $U_i(\alpha'_i, a_{-i}) > U_i(\alpha_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, where $U_i(\alpha_i, a_{-i})$ is the payoff of player i if he uses the mixed strategy α_i and the other player's vector of actions is a_{-i}; or,

   (ii) $U_i(\alpha'_i, \alpha_{-i}) > U_i(\alpha_i, \alpha_{-i})$ for all $\alpha_{-i} \in \Delta(A_{-i})$, where $U_i(\alpha_i, \alpha_{-i})$ is the payoff of player i if he uses the mixed strategy α_i and the probability distribution over other players' vector of actions is α_{-i}.

   (a) Show that the two definitions above are equivalent.

Solution: $\Rightarrow$ since $U_i(\alpha'_i, a_{-i}) > U_i(\alpha_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, we have

$$U_i(\alpha'_i, \alpha_{-i}) = \sum_{j \neq i} (\prod_{j \in N/i} \alpha_j(a_j)) U_i(\alpha'_i, a_{-i}) > \sum_{j \neq i} (\prod_{j \in N/i} \alpha_j(a_j)) U_i(\alpha_i, a_{-i}) = U_i(\alpha_i, \alpha_{-i})$$

thus $U_i(\alpha'_i, \alpha_{-i}) > U_i(\alpha_i, \alpha_{-i})$ for all $\alpha_{-i} \in \Delta(A_{-i})$.
$\Leftarrow$ since $U_i(\alpha'_i, \alpha_{-i}) > U_i(\alpha_i, \alpha_{-i})$ for all $\alpha_{-i} \in \Delta(A_{-i})$, let α_{-i} assign probability one to a_{-i}, then we have $U_i(\alpha'_i, a_{-i}) > U_i(\alpha_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.
From above we know that the two definitions are equivalent.

   (b) Give an example of a game where there are strictly dominated mixed strategies whose support only contains actions that are not weakly dominated.

Solution:

	L	R
U	$\frac{7}{4}, 1$	$\frac{7}{4}, 1$
M	$2, 1$	$1, 1$
D	$1, 1$	$2, 1$

The mixed strategy in which M and D are each used with probability $\frac{1}{2}$ is strictly dominated by the action U, but M and D are not weakly dominated.