Class Note for Advanced Macroeconomics

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Quiz # 1: Log-Linearization

Q1 $\cdots (3')$

$$\boxed{X_t \cdot [1 - Q_t] = \frac{Z_t \cdot [V + X_t]}{V - Y_t}} \tag{1}$$

SOLUTION:
To simplify, we rearrange the equation as product of terms

$$X_t \cdot [1 - Q_t] \cdot [V - Y_t] = Z_t \cdot [V + X_t] \tag{2}$$

$$\Rightarrow \boxed{\text{s.s.}} \quad \bar{X}(1 - \bar{Q})(V - \bar{Y}) = \bar{Z}(V + \bar{X}) \tag{3}$$

Note that V is a parameter, *not* variable. Rewrite Eq.(2) as follows:

$$X_t V - X_t Y_t - X_t Q_t V + X_t Q_t Y_t = Z_t V + Z_t X_t \tag{4}$$

$$\Rightarrow \bar{X} V e^{\hat{X}_t} - \bar{X}\bar{Y} e^{\hat{X}_t + \hat{Y}_t} - \bar{X}\bar{Q} V e^{\hat{X}_t + \hat{Y}_t} + \bar{X}\bar{Q}\bar{Y} e^{\hat{X}_t + \hat{Q}_t + \hat{Y}_t} = \bar{Z} V e^{\hat{Z}_t} + \bar{Z}\bar{X} e^{\hat{Z}_t + \hat{X}_t} \tag{5}$$

Thus, to log-linearize

$$\begin{aligned} \boxed{1.1.} \quad & \bar{X}V(1 + \hat{X}_t) - \bar{X}\bar{Y}(1 + \hat{X}_t + \hat{Y}_t) - \bar{X}\bar{Q}V(1 + \hat{X}_t + \hat{Q}_t) + \bar{X}\bar{Q}\bar{Y}(1 + \hat{X}_t + \hat{Q}_t + \hat{Y}_t) \\ & = \bar{Z}V(1 + \hat{Z}_t) + \bar{Z}\bar{X}(1 + \hat{Z}_t + \hat{X}_t) \end{aligned} \tag{6}$$

$$\begin{aligned} \Rightarrow \quad & \cancel{\bar{X}[(1 - \bar{Q})(V - \bar{Y})]} + \overbrace{\bar{X}[(1 - \bar{Q})(V - \bar{Y}) - \bar{Z}]}^{\bar{Z}V} \cdot \hat{X}_t \\ & - \bar{X}\bar{Y}[1 - \bar{Q}] \cdot \hat{Y}_t - \bar{X}\bar{Q}[V - \bar{Y}] \cdot \hat{Q}_t = \bar{Z}(V + \bar{X}) \cdot \hat{Z}_t + \cancel{\bar{Z}(V + \bar{X})} \end{aligned} \tag{7}$$

That's

$$\boxed{\bar{Z}V \cdot \hat{X}_t - \bar{X}\bar{Y}[1 - \bar{Q}] \cdot \hat{Y}_t - \bar{X}\bar{Q}[V - \bar{Y}] \cdot \hat{Q}_t = \bar{Z}(V + \bar{X}) \cdot \hat{Z}_t} \tag{8}$$

There is alternative approach.

ALTERNATIVE SOLUTION:
We will show how to utilize the Taylor first order expansion to maintain the log-linearization. Recall Eq.(2),

$$X_t \cdot [1 - Q_t] \cdot [V - Y_t] = Z_t \cdot [V + X_t] \tag{9}$$

$$\Rightarrow \boxed{\text{s.s.}} \quad \bar{X}(1 - \bar{Q})(V - \bar{Y}) = \bar{Z}(V + \bar{X}) \tag{10}$$

It's then expanded as

$$\cancel{\bar{X}(1 - \bar{Q})(V - \bar{Y})} + (1 - \bar{Q})(V - \bar{Y})(X_t - \bar{X}) - \bar{X}(V - \bar{Y})(Q_t - \bar{Q})$$

$$-\bar{X}(1-\bar{Q})(Y_t-\bar{Y}) = \cancel{\bar{Z}(V+\bar{X})} + (V+\bar{X})(Z_t-\bar{Z}) + \bar{Z}(X_t-\bar{X}) \tag{11}$$

$$\Rightarrow (1-\bar{Q})(V-\bar{Y})\bar{X}\frac{(X_t-\bar{X})}{\bar{X}} - \bar{X}(V-\bar{Y})\bar{Q}\frac{(Q_t-\bar{Q})}{\bar{Q}} - \bar{X}(1-\bar{Q})\bar{Y}\frac{(Y_t-\bar{Y})}{\bar{Y}}$$
$$= (V+\bar{X})\bar{Z}\frac{(Z_t-\bar{Z})}{\bar{Z}} + \bar{Z}\bar{X}\frac{(X_t-\bar{X})}{\bar{X}} \tag{12}$$

Recall that we define the percentage deviation $\hat{X}_t = \frac{X_t-\bar{X}}{\bar{X}}$, therefore, we are able to obtain

$$\Rightarrow (1-\bar{Q})(V-\bar{Y})\bar{X}\hat{X}_t - \bar{X}(V-\bar{Y})\bar{Q}\hat{Q}_t - \bar{X}(1-\bar{Q})\bar{Y}\hat{Y}_t = (V+\bar{X})\bar{Z}\hat{Z}_t + \bar{Z}\bar{X}\hat{X}_t \tag{13}$$

$$\Rightarrow \overbrace{\left[\bar{X}(1-\bar{Q})(V-\bar{Y}) - \bar{Z}\bar{X}\right]}^{=\bar{Z}(V+\bar{X})-\bar{Z}\bar{X}=\bar{Z}V} \cdot \hat{X}_t - \bar{X}(V-\bar{Y})\bar{Q}\cdot\hat{Q}_t - \bar{X}(1-\bar{Q})\bar{Y}\cdot\hat{Y}_t$$
$$= (V+\bar{X})\bar{Z}\cdot\hat{Z}_t \tag{14}$$

That's

$$\boxed{\bar{Z}V\cdot\hat{X}_t - \bar{X}(V-\bar{Y})\bar{Q}\cdot\hat{Q}_t - \bar{X}(1-\bar{Q})\bar{Y}\cdot\hat{Y}_t = (V+\bar{X})\bar{Z}\cdot\hat{Z}_t} \tag{15}$$

Q2 $\cdots(2')$

$$\boxed{\frac{W_t}{P_t} = \alpha\frac{A_t}{N_t^{1-\alpha}}} \tag{16}$$

ANSWER:

In economic sense, this is the equation for the *real wage*, which equals to the marginal productivity of labor, leaving the capital out (or assumed to be unit). First we need to express the equation in equilibrium, that's

$$\boxed{\text{s.s.}} \qquad \frac{\bar{W}}{\bar{P}} = \alpha\cdot\frac{\bar{A}}{\bar{N}^{1-\alpha}} \tag{17}$$

To log-linearize, we rearrange the equation as follows:

$$\boxed{\text{l.l.}} \qquad \frac{\bar{W}\cdot e^{\hat{W}_t}}{\bar{P}\cdot e^{\hat{P}_t}} = \alpha\cdot\frac{\bar{A}\cdot e^{\hat{A}_t}}{\bar{N}^{1-\alpha}\cdot e^{(1-\alpha)\hat{N}_t}} \tag{18}$$

$$\Rightarrow \frac{\bar{W}}{\bar{P}}\cdot e^{\hat{W}_t-\hat{P}_t} = \alpha\cdot\frac{\bar{A}}{\bar{N}^{1-\alpha}}\cdot e^{\hat{A}_t-(1-\alpha)\hat{N}_t} \tag{19}$$

Further, note that Eq.(17) holds in equilibrium, thus we are able to delimitate this and obtain

$$e^{\hat{W}_t-\hat{P}_t} = e^{\hat{A}_t-(1-\alpha)\hat{N}_t} \tag{20}$$

or equivalently:

$$\hat{\omega}_t \triangleq \hat{W}_t - \hat{P}_t = \hat{A}_t - (1-\alpha)\hat{N}_t \tag{21}$$

ALTERNATIVE SOLUTION:
Or we could first take logarithm on both sides, and then differentiate with respective to time index t, assuming that all the variables to be continuous differentiable.

$$\log \frac{W_t}{P_t} = \log \alpha \frac{A_t}{N_t^{1-\alpha}} \tag{22}$$

$$\Rightarrow \log W_t - \log P_t = \log \alpha + \log A_t - (1-\alpha) \log N_t \tag{23}$$

Differentiating both sides w.r.t t to obtain

$$\frac{dW_t/dt}{W_t} - \frac{dP_t/dt}{P_t} = \frac{dA_t/dt}{A_t} - (1-\alpha)\frac{dN_t/dt}{N_t}, \tag{24}$$

where d denotes the differential operator, i.e., $dX_t = \lim_{\triangle t \to 0} X_{t+\triangle t} - X_t$. The *percentage deviation* from the equilibrium in the continuous sense is defined as

$$\hat{X}_t = \lim_{\triangle t \to 0} \frac{X_{t+\triangle t} - X_t}{X_t} = \frac{\dot{X}_t}{X_t} = \frac{dX_t/dt}{X_t} = \frac{d \log (X_t)}{dt}.$$

Accordingly, Eq.(24) could be expressed as follows

$$\boxed{\hat{W}_t - \hat{P}_t = \hat{A}_t - (1-\alpha)\hat{N}_t}. \tag{25}$$

Q3 $\cdots (2')$

$$\boxed{\lambda_t = \beta \cdot \mathbb{E}\big[\lambda_{t+1}(1+r_t)\big]} \tag{26}$$

$$\text{Hint: define } \hat{r}_t = \log \tfrac{1+r_t}{1+\bar{r}} \tag{27}$$

SOLUTION:
First we need to find out the equilibrium state

$$\bar{\lambda} = \beta\bar{\lambda}(1+\bar{r}) \quad \Rightarrow \quad 1 = \beta(1+\bar{r}). \tag{28}$$

Note that for rational equilibrium, the representative agent's *subjective interpretation of probability* equals to the *objective frequency*. For this reason the expectation operator is leaved out.

To define the percentage deviation of net return as

$$\hat{r}_t = \log \frac{1+r_t}{1+\bar{r}},$$

or, equivalently, we have

$$1 + r_t = (1+\bar{r})e^{\hat{r}_t}.$$

With such transformation, we then log-linearize the equation by following procedure

$$\cancel{\bar{\lambda}} e^{\hat{\lambda}_t} = \mathbb{E}\big[\beta \cancel{\bar{\lambda}} e^{\hat{\lambda}_{t+1}}(1+\bar{r})e^{\hat{r}_t}\big] \tag{29}$$

$$1 + \hat{\lambda}_t = \cancel{\beta(1+\bar{r})}\mathbb{E}\big[1 + \hat{\lambda}_{t+1} + \hat{r}_t\big] \tag{30}$$

That's

$$\boxed{\hat{\lambda}_t = \mathbb{E}\big[\hat{\lambda}_{t+1} + \hat{r}_t\big]} \tag{31}$$

Note that, usually we do *not* have the following property

$$\log\big(\mathbb{E}[X]\big) = \mathbb{E}[\log(X)],$$

which means that you can't simply apply the differential approach by taking logarithm on both sides. I just found some students did mistake over this point.

Q4 $\cdots(3')$

$$\boxed{y_t = A_t\left[\alpha k_{t-1}^{\rho} + (1-\alpha)n_t^{1-\rho}\right]^{\frac{1}{\rho}}} \tag{32}$$

ANSWER:

For this kind of complicated polynomials, I would prefer to firstly rearranging the formula into somehow rather easily-coping one, dismissing the power operator(s). That means, for this equation, to take power of ρ to obtain

$$y_t^{\rho} = A_t^{\rho}\left[\alpha k_{t-1}^{\rho} + (1-\alpha)n_t^{1-\rho}\right] \tag{33}$$

$$\Rightarrow \boxed{\text{s.s}} \quad \bar{y}^{\rho} = \bar{A}^{\rho}\left[\alpha\bar{k}^{\rho} + (1-\alpha)\bar{n}^{1-\rho}\right] \tag{34}$$

For the power function, you should be much careful when express the variables in the stochastic form. For instance, we need to characterize the "power" effect via

$$x_t^{\rho} = (\bar{x}\cdot e^{\hat{x}_t})^{\rho} = \bar{x}^{\rho}\cdot e^{\rho\hat{x}_t}.$$

Thus, applying this technique to log-linearize the above equation

$$\boxed{\text{l.l.}} \quad \bar{y}^{\rho}e^{\rho\hat{y}_t} = \bar{A}^{\rho}e^{\rho\hat{A}_t}\left[\alpha\bar{k}^{\rho}e^{\rho\hat{k}_{t-1}} + (1-\alpha)\bar{n}^{1-\rho}e^{(1-\rho)\hat{n}_t}\right] \tag{35}$$

$$= \bar{A}^{\rho}\alpha\bar{k}^{\rho}e^{\rho\hat{A}_t+\rho\hat{k}_{t-1}} + \bar{A}^{\rho}(1-\alpha)\bar{n}^{1-\rho}e^{\rho\hat{A}_t+(1-\rho)\hat{n}_t} \tag{36}$$

$$\Rightarrow \bar{y}^{\rho}(1+\rho\hat{y}_t) = \bar{A}^{\rho}\alpha\bar{k}^{\rho}\big[1+\rho\hat{A}_t+\rho\hat{k}_{t-1}\big] + \bar{A}^{\rho}(1-\alpha)\bar{n}^{1-\rho}\big[1+\rho\hat{A}_t+(1-\rho)\hat{n}_t\big] \tag{37}$$

$$\Rightarrow \cancel{\bar{y}^{\rho}} + \bar{y}^{\rho}\rho\hat{y}_t = \cancel{\bar{A}^{\rho}\left[\alpha\bar{k}^{\rho} + (1-\alpha)\bar{n}^{1-\rho}\right]} + \bar{A}^{\rho}\alpha\bar{k}^{\rho}\rho\big[\hat{A}_t + \hat{k}_{t-1}\big] \tag{38}$$

$$+ \bar{A}^{\rho}(1-\alpha)\bar{n}^{1-\rho}\rho\hat{A}_t + \bar{A}^{\rho}(1-\alpha)\bar{n}^{1-\rho}(1-\rho)\hat{n}_t. \tag{39}$$

Divide both sides by $\bar{y}^{\rho}\rho$ and use the equilibrium equation to get reduced form

$$\boxed{\hat{y}_t = \hat{A}_t + \frac{\bar{A}^{\rho}\alpha\bar{k}^{\rho}}{\bar{y}^{\rho}}\cdot\hat{k}_{t-1} + \frac{\bar{A}^{\rho}(1-\alpha)(1/\rho-1)\bar{n}^{1-\rho}}{\bar{y}^{\rho}}\cdot\hat{n}_t}. \tag{40}$$