Advanced Macroeconomics II
RBC and Numerical Method

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A Simple RBC Model

The environment

1) *Preference:*

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}}{1-\eta}\right]$$

2) *Technology:*

$$C_t + K_t = Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1}$$

How to define "Z_t": $log\, Z_t = (1-\varphi) log \bar{Z} + \varphi log Z_{t-1} + \epsilon_t$, where $\epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$, $\varphi < 1$. As a parameter, usually $\bar{Z}$ is normalized to 1.

3) *Endowment:*

$$N_t = 1, \quad K_{-1} > 0$$

4) *Information:* decision made based on all information I_t up to time t.

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The social planner's problem

$$\max_{(C_t, K_t)_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}}{1-\eta} \right]$$

$$\begin{aligned}
& s.t. K_{-1},\ Z_0 \\
C_t + K_t &= Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1} \\
\log Z_t &= (1-\varphi) \log \bar{Z} + \varphi \log Z_{t-1} + \epsilon_t \\
& \epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)
\end{aligned}$$

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To solve:

The *Lagrangian*: $L =$

$$\max_{(C_t,K_t)_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \left[\beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - \lambda_t (C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta) K_{t-1}) \right) \right]$$

The necessary FOCs:

$$\begin{aligned}
\frac{\partial L}{\partial C_t} &: \quad 0 \stackrel{!}{=} C_t^{-\eta} - \lambda_t \\
\frac{\partial L}{\partial K_t} &: \quad 0 \stackrel{!}{=} \beta^t [-\lambda_t] + \beta^{t+1} E_t \left[(-\lambda_{t+1}) [-\rho Z_{t+1} K_t^{\rho-1} - (1-\delta)] \right] \\
&\Longrightarrow \quad 0 \stackrel{!}{=} -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\rho Z_{t+1} K_t^{\rho-1} + (1-\delta) \right) \right] \\
\frac{\partial L}{\partial \lambda_t} &: \quad 0 \stackrel{!}{=} C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta) K_{t-1}
\end{aligned}$$

A Simple RBC Model

To solve:

Transversality condition:
For uniqueness and non-explosive solution

$\Rightarrow$ The expected marginal utility of consuming the left-over capital discounted as of today converges to zero (when $T \longrightarrow \infty$).

$$0 = \underset{T\to\infty}{E_0} \left[\beta^T C_T^{-\eta} K_T \right]$$

A Simple RBC Model

To solve for state states:

Collect all the necessary coditions and exogenous process:

$$
\begin{aligned}
C_t &= Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t \\
R_t &= \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \\
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta} R_{t+1}\right] \\
logZ_t &= (1-\phi)log\bar{Z} + \phi logZ_{t+1} + \varepsilon_t, \varepsilon_t \sim iid\, N(0,\sigma^2)
\end{aligned}
$$

Note : we have defined the gross return of capital; we substitute out λ_t with marginal utility of consumption. The number of variables does not change.

Important: #(equation) = #(variable)

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To solve for state states:

To find the S.S of the economy, drop time indices:

$$\begin{aligned}
\bar{C} &= \bar{Z}\bar{K}^{\rho} + (1-\delta)\bar{K} - \bar{K} \\
\bar{R} &= \rho\bar{Z}\bar{K}^{\rho-1} + (1-\delta) \\
1 &= \beta \cdot \bar{R}
\end{aligned}$$

Three variables/equations need to be solved for $(\bar{C}, \bar{K}, \bar{R})$ given the parameters $(\bar{Z}, \beta, \rho, \delta)$.

The sequence of solutions

$$\begin{aligned}
\bar{R} &= 1/\beta \\
\bar{K} &= \left(\frac{\rho\bar{Z}}{\bar{R}-1+\delta}\right)^{1/(1-\rho)} \\
hence &: \quad \bar{Y} = \bar{Z}\bar{K}^{\rho} \qquad \bar{C} = \bar{Y} - \delta\bar{K} \\
\bar{C}/\bar{K} &= \bar{Y}/\bar{K} - \delta = \bar{Z}\bar{K}^{\rho-1} - \delta \\
&= \frac{\bar{R}-1+\delta}{\rho} - \delta = \frac{1-\beta+\delta\beta}{\rho\beta} - \delta
\end{aligned}$$

A Simple RBC Model

Same model environment, but competitive equilibrium

$$\left(C_t,\ N_t,\ K_t,\ R_t,\ W_t\right)_{t=0}^{\infty}$$

$\boxed{\text{Household:}}$ given $K_{-1}^{(s)}$, W_t, and R_t, decide on providing how much $N_t^{(s)}$, $K_t^{(s)}$, and consuming C_t.

$$\max_{(C_t, K_t^{(s)}, N_t^{(s)})_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}}{1-\eta}\right]$$

$$s.t. K_{-1},$$

$$C_t + K_t^{(s)} = W_t N_t^{(s)} + R_t K_{t-1}^{(s)}$$

plus no-Ponzi-game condition (similar to the transversality condition),

$$0 = \lim_{t\to\infty} E_0\left[\left(\Pi_{s=1}^{t} R_s^{-1}\right) K_t\right],$$

to ensure the discounted present value of capital left-over, when time approaches infinity, is zero.

A Simple RBC Model

Same model environment, but competitive equilibrium

$\boxed{\text{Firm:}}$ given W_t, and R_t, decide on how much capital to rent and labor to hire, $N_t^{(d)}$, $K_t^{(d)}$, hence how much to produce $Y_t(N_t^{(d)}, K_t^{(d)})$.

$$\max_{K_{t-1}^{(d)}, N_t^{(d)}} Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta)K_{t-1}^{(d)} - W_t N_t^{(d)} - R_t K_{t-1}^{(d)}$$

where

$$logZ_t = (1-\phi)log\bar{Z} + \phi logZ_{t+1} + \varepsilon_t, \varepsilon_t \sim iid\ N(0, \sigma^2)$$

A Simple RBC Model

Same model environment, but competitive equilibrium

Market Clearing Conditions:

Capital Market:

$$K_{t-1}^{(d)} = K_{t-1}^{(s)}$$

Labor Market:

$$N_t^{(d)} = N_s^{(s)} = 1$$

Goods Market:

$$C_t + K_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} + (1-\delta) K_{t-1}$$

By *Walras' law*, we only need two out of these three conditions.

A Simple RBC Model

To solve competitive equilibrium: solve every party's FOCs

Household:

$$L = \max_{(C_t, K_t)_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - \lambda_t (C_t + K_t - R_t K_{t-1} - W_t N_t)\right)\right]$$

$\Rightarrow$ Household FONCs:

$$\begin{aligned}
\frac{\partial L}{\partial C_t} &: \quad 0 \stackrel{!}{=} C_t^{-\eta} - \lambda_t \\
\frac{\partial L}{\partial K_t} &: \quad 0 \stackrel{!}{=} -\lambda_t + \beta E_t[\lambda_{t+1} R_{t+1}] \\
\frac{\partial L}{\partial \lambda_t} &: \quad 0 \stackrel{!}{=} C_t + K_t - W_t - R_t K_{t-1}
\end{aligned}$$

A Simple RBC Model

To solve competitive equilibrium: solve every party's FOCs

Firm:

$$\max_{K_{t-1}^{(d)}, N_t^{(d)}} Z_t (K_{t-1})^{\rho} N_t^{1-\rho} + (1-\delta) K_{t-1} - W_t N_t - R_t K_{t-1}$$

Firm′s FONCs:

$$\begin{aligned} W_t &= (1-\rho) Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{-\rho} \\ R_t &= \rho Z_t \left(K_{t-1}^{(d)}\right)^{\rho-1} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta) \end{aligned}$$

Dropping $^{(d)}$ and using

$$Y_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho}$$

The FONCs amount to

$$\begin{aligned} W_t N_t &= (1-\rho) Y_t \\ R_t K_{t-1} &= \rho Y_t + (1-\delta) K_{t-1} \end{aligned}$$

$$\text{or} \quad r_t K_{t-1} - \delta K_{t-1} = \rho Y_t$$

A Simple RBC Model

To solve competitive equilibrium: market clearing

Market clearing by dropping $^{(d)}$ and $^{(s)}$, as we have done.
Combining the FONCs and the exogoneous process, we reduce to the same system of equations (after deleting W_t, Y_t, λ_t):

$$
\begin{aligned}
C_t &= Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1} - K_t \\
R_t &= \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \\
1 &= E_t \left[\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\eta} R_{t+1} \right] \\
log Z_t &= (1-\phi) log \bar{Z} + \phi log Z_{t+1} + \varepsilon_t, \varepsilon_t \sim iid\, N(0, \sigma^2)
\end{aligned}
$$

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Both ways work. You can opt to the simpler one.

A Simple RBC Model

Log-linearization

$$
\begin{aligned}
C_t &= \bar{C} e^{\hat{c}_t} \\
K_t &= \bar{K} e^{\hat{k}_t}
\end{aligned}
$$

Budget constraint:

$$
\begin{aligned}
\bar{C} e^{\hat{c}_t} &= \bar{Z} \bar{K}^{\rho} e^{\hat{z}_t} e^{\rho \hat{k}_{t-1}} + (1-\delta) \bar{K} e^{\hat{k}_{t-1}} - \bar{K} e^{\hat{k}_t} \\
\bar{C} + \bar{C} \cdot \hat{c}_t &\approx \bar{Z} \bar{K}^{\rho} + \bar{Z} \bar{K}^{\rho} (z_t + \rho \hat{k}_{t-1}) + (1-\delta) \bar{K} \hat{k}_{t-1} - \delta \bar{K} - \bar{K} \hat{k}_t
\end{aligned}
$$

Using the steady state relationship $\bar{C} = \bar{Z} \bar{K}^{\rho} + (1-\delta) \bar{K} - \bar{K}$, $\bar{Y} = \bar{Z} \bar{K}^{\rho}$, and $\bar{C} = \bar{Y} - \delta \bar{K}$

$$
\begin{aligned}
\bar{C} \hat{c}_t &\approx \bar{Z} \bar{K}^{\rho} (\hat{z}_t + \rho \hat{k}_{t-1}) + (1-\delta) \bar{K} \hat{k}_{t-1} - \bar{K} \hat{k}_t \\
\hat{c}_t &\approx \frac{\bar{Y}}{\bar{C}} \hat{z}_t + \frac{\bar{K}}{\bar{C}} \bar{R} \hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}} \hat{k}_t
\end{aligned}
$$

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Log-linearization

Interest rate definition

$$
\begin{aligned}
R_t &= \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \\
\bar{R} e^{\hat{r}_t} &= \rho \bar{Z} \bar{K}^{\rho-1} e^{\hat{z}_t + (\rho-1)\hat{k}_{t-1}} + (1-\delta) \\
\bar{R} + \bar{R}\hat{r}_t &\approx \rho \bar{Z} \bar{K}^{\rho-1} + (1-\delta) + \rho \bar{Z} \bar{K}^{\rho-1} \left[\hat{z}_t + (\rho-1)\hat{k}_{t-1}\right]
\end{aligned}
$$

Using the steady state relationship

$$
\frac{1}{\beta} = \bar{R} = \rho \bar{Z} \bar{K}^{\rho-1} + (1-\delta)
$$

$$
\begin{aligned}
\bar{R}\hat{r}_t &\approx \rho \bar{Z} \bar{K}^{\rho-1} \left[\hat{z}_t + (\rho-1)\hat{k}_{t-1}\right] \\
\hat{r}_t &\approx \beta \left[\frac{1}{\beta} - (1-\delta)\right] \left[\hat{z}_t - (1-\rho)\hat{k}_{t-1}\right] \\
\hat{r}_t &\approx \left[1 - \beta(1-\delta)\right] \left[\hat{z}_t - (1-\rho)\hat{k}_{t-1}\right]
\end{aligned}
$$

A Simple RBC Model

Log-linearization

Euler equation

$$
\begin{aligned}
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta}R_{t+1}\right] \\
1 &= E_t\left[\beta\left(\frac{\bar{C}}{\bar{C}}e^{\hat{c}_{t+1}-\hat{c}_t}\right)^{-\eta}\bar{R}e^{\hat{r}_{t+1}}\right] \\
1 &\approx E_t\left[\beta\bar{R}e^{\eta(\hat{c}_t-\hat{c}_{t+1})+\hat{r}_{t+1}}\right]
\end{aligned}
$$

Use the steady state relationship

$$1 = \beta\bar{R}$$

$$0 \approx E_t\left[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\right]$$

A Simple RBC Model

Log-linearization

Technology process

$$
\begin{aligned}
log\left(\bar{Z}e^{\hat{z}_t}\right) &= (1-\psi)log\,\bar{Z} + \psi log\left(\bar{Z}e^{\hat{z}_{t-1}}\right) + \varepsilon_t \\
\hat{z}_t &= \psi\hat{z}_{t-1} + \varepsilon_t
\end{aligned}
$$

which holds exactly.

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Log-linearization: results

Collect all the four L.L. equations, which form a system of linear and homogeneous equations.

1. $\hat{c}_t \approx \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}\hat{k}_t$
2. $\hat{r}_t \approx [1-\beta(1-\delta)]\left[\hat{z}_t - (1-\rho)\hat{k}_{t-1}\right]$
3. $0 \approx E_t\left[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\right]$
4. $\hat{z}_t = \psi\hat{z}_{t-1} + \varepsilon_t$

$\Longrightarrow$ Four equations, four variables! $(\hat{c}_t, \hat{z}_t, \hat{k}_{t-1}, \hat{r}_t)$, but some variables appear in different time periods. for example, equation 1 is a second order difference equation in $\hat{k}_t$.

Our next task is to solve for this linear system in terms of the exogenous variable and predetermined variable $(\hat{z}_t,\ \hat{k}_{t-1})$.

Summary on the procedure of log-linearization

1. Determine and collect all equilibrium conditions, i.e. FOCs, constraints, exogenous variable processes. #(equation)=#(variable).
2. Determine steady states.
3. Multiplying out variables, i.e. if $X_t(1 - Q_t)$ then multiply out to get $X_t - X_t Q_t$, then replace X_t with $\bar{X}e^{\hat{x}_t}$.
4. Collect all exponential terms wherever possible. E.g. $\bar{X}e^{\hat{x}_t}\bar{Y}e^{\hat{y}_t} = \bar{X}\bar{Y}e^{\hat{x}_t+\hat{y}_t}$.
5. Approximate $e^{\hat{x}_t}$ with first order expansion $e^{\hat{x}_t} \approx 1 + \hat{x}_t$ (only when it's necessary).
6. Collect all constant terms and verify that they cancel out by using the steady state relationship. Further, delete any higher order terms such as to impose $\hat{x}_t\hat{y}_t \approx 0$.
7. Collect all variables to form a system of linear and homogeneous equations.

Solve for the recursive law of motion with method of undetermined coefficients.

State variables: $\hat{k}_{t-1}$, $\hat{z}_t$

The dynamics of the model should be described by **recursive laws of motion** in terms of the state variables,

$$\begin{aligned}
\hat{k}_t &= v_{kk}\hat{k}_{t-1} + v_{kz}\hat{z}_t \\
\hat{r}_t &= v_{rk}\hat{k}_{t-1} + v_{rz}\hat{z}_t \text{ (It's done!)} \\
\hat{c}_t &= v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t
\end{aligned}$$

We need to solve for v_{kk}, v_{kz}, v_{ck} and v_{cz} the "undetermined" coefficients.

Coefficient interpretation: *elasticities*.

Recursivity

- Substitute the postulated linear recursive law of motion into the I.I. equations until only $\hat{k}_{t-1}$ and $\hat{z}_t$ remain. E.g.

$$E_t \hat{z}_{t+1} = \psi \hat{z}_t$$

$$\begin{aligned} E_t \hat{c}_{t+1} &= E_t \left(v_{ck} \hat{k}_t + v_{cz} \hat{z}_{t+1} \right) \\ &= v_{ck} \left(v_{kk} \hat{k}_{t-1} + v_{kz} \hat{z}_t \right) + v_{cz} \psi \hat{z}_t \\ &= v_{ck} v_{kk} \hat{k}_{t-1} + \left(v_{ck} v_{kz} + v_{cz} \psi \right) \hat{z}_t \end{aligned}$$

Recursivity

- Substitute the postulated linear recursive law of motion into the I.I. equations until only $\hat{k}_{t-1}$ and $\hat{z}_t$ remain. E.g.

$$E_t \hat{z}_{t+1} = \psi \hat{z}_t$$

$$\begin{aligned} E_t \hat{c}_{t+1} &= E_t \left(v_{ck} \hat{k}_t + v_{cz} \hat{z}_{t+1} \right) \\ &= v_{ck} \left(v_{kk} \hat{k}_{t-1} + v_{kz} \hat{z}_t \right) + v_{cz} \psi \hat{z}_t \\ &= v_{ck} v_{kk} \hat{k}_{t-1} + \left(v_{ck} v_{kz} + v_{cz} \psi \right) \hat{z}_t \end{aligned}$$

  - Compare coefficients.

Recursivity

For the first equation (budget constraint)

$$
\begin{aligned}
\hat{c}_t &= \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}\hat{k}_t \\
v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t &= \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}(v_{kk}\hat{k}_{t-1} + v_{kz}\hat{z}_t) \\
v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t &= \frac{\bar{K}}{\bar{C}}(\frac{1}{\beta} - v_{kk})\hat{k}_{t-1} + (\frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}v_{kz})\hat{z}_t
\end{aligned}
$$

Comparing coefficients: since the equation has to be satisfied for any value for $\hat{k}_{t-1}$ and $\hat{z}_t$, we have

$$
\begin{aligned}
v_{ck} &= \frac{\bar{K}}{\bar{C}}(\frac{1}{\beta} - v_{kk}) \\
v_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}v_{kz}
\end{aligned}
$$

Recursivity

For the second equation (return)

$$\begin{aligned} \hat{r}_t &= [1-\beta(1-\delta)]\left[\hat{z}_t-(1-\rho)\hat{k}_{t-1}\right] \\ v_{rk}\hat{k}_{t-1}+v_{rz}\hat{z}_t &= -[1-\beta(1-\delta)](1-\rho)\hat{k}_{t-1}+[1-\beta(1-\delta)]\hat{z}_t \end{aligned}$$

Comparing coefficients, we have

$$\begin{aligned} v_{rk} &= -[1-\beta(1-\delta)](1-\rho) \\ v_{rz} &= [1-\beta(1-\delta)] \end{aligned}$$

Recursivity

For the third equation (Euler equation/asset pricing)

$$
\begin{aligned}
0 &= E_t\left[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\right] \\
0 &= E_t\left[\eta(v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t - v_{ck}\hat{k}_t - v_{cz}\hat{z}_{t+1}) + v_{rk}\hat{k}_t + v_{rz}\hat{z}_{t+1}\right] \\
0 &= E_t\left\{\eta\left[v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t - v_{ck}(v_{kk}\hat{k}_{t-1} + v_{kz}\hat{z}_t)\right]\right. \\
&\quad \left. + v_{rk}(v_{kk}\hat{k}_{t-1} + v_{kz}\hat{z}_t) + (v_{rz} - \eta v_{cz})\hat{z}_{t+1}\right\} \\
0 &= (\eta v_{ck} - \eta v_{ck} v_{kk} + v_{rk} v_{kk})\,\hat{k}_{t-1} + \\
&\quad [\eta v_{cz} - \eta v_{ck} v_{kz} + v_{rk} v_{kz} + (v_{rz} - \eta v_{cz})\psi]\,\hat{z}_t
\end{aligned}
$$

Comparing coefficients, we have

$$
\begin{aligned}
0 &= \eta v_{ck} - \eta v_{ck} v_{kk} + v_{rk} v_{kk} \\
0 &= \eta v_{cz} - \eta v_{ck} v_{kz} + v_{rk} v_{kz} + (v_{rz} - \eta v_{cz})\psi
\end{aligned}
$$

Comparing coefficients...

Collecting the results, and comparing coefficients on $\hat{k}_{t-1}$,

$$\begin{aligned} v_{ck} &= \frac{\bar{K}}{\bar{C}}\left(\frac{1}{\beta} - v_{kk}\right) \\ v_{rk} &= -\left[1-\beta(1-\delta)\right](1-\rho) \\ 0 &= \eta v_{ck} - \eta v_{ck} v_{kk} + v_{rk} v_{kk} \end{aligned}$$

and comparing coefficients on $\hat{z}_t$

$$\begin{aligned} v_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}} v_{kz} \\ v_{rz} &= \left[1-\beta(1-\delta)\right] \\ 0 &= \eta v_{cz} - \eta v_{ck} v_{kz} + v_{rk} v_{kz} + (v_{rz} - \eta v_{cz})\psi \end{aligned}$$

Comparing coefficients on $\hat{k}_{t-1}$

$$
\begin{aligned}
v_{ck} &= \frac{\bar{K}}{\bar{C}}\left(\frac{1}{\beta} - v_{kk}\right) && (1)\\
v_{rk} &= -\left[1-\beta(1-\delta)\right](1-\rho) && (2)\\
0 &= \eta v_{ck} - \eta v_{ck} v_{kk} + v_{rk} v_{kk} && (3)
\end{aligned}
$$

To solve for v_{kk}, plug v_{ck} into equation (3). We get a **characteristic quadratic equation**

$$
\begin{aligned}
0 &= \eta\frac{\bar{K}}{\bar{C}}\left(\frac{1}{\beta} - v_{kk}\right) - \eta\frac{\bar{K}}{\bar{C}}\left(\frac{1}{\beta} - v_{kk}\right)v_{kk} - \left[1-\beta(1-\delta)\right](1-\rho)v_{kk}\\
0 &= \eta\frac{\bar{K}}{\bar{C}}\frac{1}{\beta} - \left\{\eta\frac{\bar{K}}{\bar{C}} + \eta\frac{\bar{K}}{\bar{C}}\frac{1}{\beta} + \left[1-\beta(1-\delta)\right](1-\rho)\right\}v_{kk} + \eta\frac{\bar{K}}{\bar{C}}v_{kk}^2
\end{aligned}
$$

To simplify, divide both sides by $\eta\bar{K}/\bar{C}$

$$
0 = v_{kk}^2 - \left\{1 + \frac{1}{\beta} + \left[1-\beta(1-\delta)\right](1-\rho)\frac{\bar{C}}{\eta\bar{K}}\right\}v_{kk} + \frac{1}{\beta}
$$

Solving the quadratic equation

$$0 = v_{kk}^2 - \gamma v_{kk} + \frac{1}{\beta}$$

where

$$\gamma = 1 + \frac{1}{\beta} + [1 - \beta(1-\delta)]\,(1-\rho)\frac{\bar{C}}{\eta \bar{K}}$$

Using the s.s. solutions to get,

$$\begin{aligned} \bar{C}/\bar{K} &= \frac{1 - \beta + \delta\beta}{\rho\beta} - \delta \\ &= \frac{1 - \beta + \delta\beta(1-\rho)}{\rho\beta} \end{aligned}$$

then

$$\gamma = 1 + \frac{1}{\beta} + \frac{[1 - \beta(1-\delta)]\,(1-\rho)\,[1 - \beta + \delta\beta(1-\rho)]}{\eta\rho\beta}$$

Solving the quadratic equation

The solution is a high school math problem:

$$v_{kk} = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}}$$

Why we delete the root

$$\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}} \ ?$$

Solving the quadratic equation

$$\begin{aligned} (v_{kk} - \lambda_1)(v_{kk} - \lambda_2) &= 0 \\ v_{kk}^2 - (\lambda_1 + \lambda_2)v_{kk} + \lambda_1\lambda_1 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_1\lambda_1 &= \frac{1}{\beta} > 1 \\ \lambda_1 + \lambda_2 &= \gamma > 0 \end{aligned}$$

so λ_1 and λ_2 both positive, one >1, and one <1.
The other root must be bigger than 1. We need to delete the explosive solution!

Solving the quadratic equation

Once v_{kk} is solved, the others can be solved easily.
Plugging it into equation (1),

$$v_{ck} = \frac{\bar{K}}{\bar{C}}\left(\frac{1}{\beta} - v_{kk}\right)$$

we get v_{ck}.

Solving the quadratic equation

Then for coefficients on $\hat{z}_t$

$$\begin{aligned} v_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}} v_{kz} && (4) \\ v_{rz} &= [1-\beta(1-\delta)] && (5) \\ 0 &= \eta v_{cz} - \eta v_{ck} v_{kz} + v_{rk} v_{kz} + (v_{rz} - \eta v_{cz})\psi && (6) \end{aligned}$$

Insert equation (4) and (5) into (6), we can solve for v_{kz}

$$v_{kz} = \frac{v_{rz}\psi + \eta(1-\psi)\frac{\bar{Y}}{\bar{C}}}{-v_{rk} + \eta v_{ck} + \eta(1-\psi)\frac{\bar{K}}{\bar{C}}}$$

Plug it back to equation (4), we solve for v_{cz}.

Now we are done!

Calibration and simulation

Assuming quarterly data with

$$\begin{array}{ll} \beta = 0.99 & \rho = 0.36 \\ \eta = 1.0 & \delta = 0.025 \\ \bar{Z} = 1 & \psi = 0.95 \end{array}$$

Then we get...

Calibration and simulation

Baseline model vs.

$$\eta = 100$$

Does not change steady states.
1) $v_{kk} \uparrow$ due to risk aversion, consumption smoothing, lower intertemporal elasticity of substitution
2) $v_{kz} \downarrow$ less sensitive to technology shock to better smooth consumption

Calibration and simulation

Baseline model vs.

$$\delta = 0.1$$

Reduce steady state size of the economy dramatically due to higher depreciation rate.

	$\bar{K}$	$\bar{Y}$	$\bar{C}$
$\delta = 0.025$	38	3.7	2.75
$\delta = 0.1$	6.4	1.95	1.3

1) $v_{kk} \downarrow$ due to higher depreciation rate
2) $v_{kz} \uparrow$ due to less stock of capital and higher MPK. So return and output both respond more proportionally.

The relationship to a state-space approach

Eliminate the interest rate equation, and set $\hat{z}_t \equiv 0$, we get

$$\begin{aligned} \hat{c}_t &= \frac{\bar{K}}{\beta \bar{C}} \hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}} \hat{k}_t \\ 0 &= \eta(\hat{c}_t - \hat{c}_{t+1}) - [1 - \beta(1-\delta)](1-\rho)\hat{k}_t \end{aligned}$$

Rearrange, we get

$$\begin{aligned} \hat{k}_t - \hat{k}_{t-1} &= (\frac{1}{\beta} - 1)\hat{k}_{t-1} - \frac{\bar{C}}{\bar{K}} \hat{c}_t \\ \hat{c}_{t+1} - \hat{c}_t &= \frac{1}{\eta}[1 - \beta(1-\delta)](1-\rho)(\frac{\bar{C}}{\bar{K}} \hat{c}_t - \frac{1}{\beta}\hat{k}_{t-1}) \end{aligned}$$

The relationship to a state-space approach

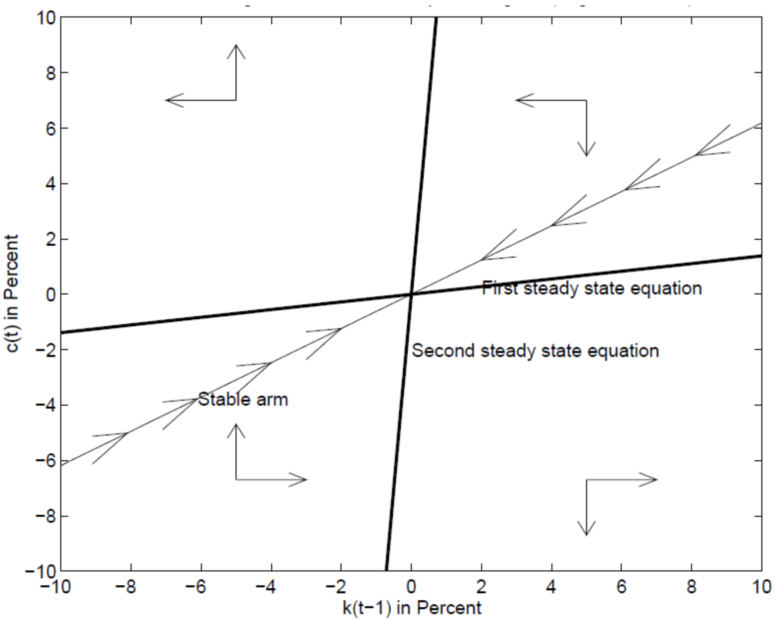
Steady state for each variable:

$\hat{k}_t - \hat{k}_{t-1} = 0$:

$$\hat{c}_t = (\frac{1}{\beta} - 1)\frac{\bar{K}}{\bar{C}}\hat{k}_{t-1}$$

$\hat{c}_{t+1} - \hat{c}_t = 0$:

$$\hat{c}_t = \frac{\bar{K}}{\beta\bar{C}}\hat{k}_{t-1}$$


c(t) in Percent
k(t−1) in Percent
First steady state equation
Second steady state equation
Stable arm
10
8
6
4
2
0
−2
−4
−6
−8
−10

The relationship to a state-space approach

The stable arm is the function:

$$\hat{c}_t = v_{ck}\hat{k}_{t-1}$$

The speed of convergence:

$$\begin{aligned}\hat{k}_t &= v_{kk}\hat{k}_{t-1} \\ \hat{k}_{t+h} &= v_{kk}^h\hat{k}_t\end{aligned}$$

Question: how long does it take to close half of the gap in $\hat{k}_t$ (for the effect of the shock to die out 50%)?

The relationship to a state-space approach

Original nonlinear system

Set $Z_t \equiv \bar{Z}$, and suppress expectation by assuming certainty

$$
\begin{aligned}
C_t &= \bar{Z} K_{t-1}^{\rho} + (1-\delta) K_{t-1} - K_t \\
1 &= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\eta} \left(\rho \bar{Z} K_{t-1}^{\rho-1} + 1 - \delta\right)
\end{aligned}
$$

Rearrange, we get

$$
\begin{aligned}
K_t - K_{t-1} &= \bar{Z} K_{t-1}^{\rho} - \delta K_{t-1} - C_t \\
\frac{C_{t+1}}{C_t} &= \left\{ \beta \left[\rho \bar{Z} \left(\bar{Z} K_{t-1}^{\rho} + (1-\delta) K_{t-1} - C_t \right)^{\rho-1} + 1 - \delta \right] \right\}^{\frac{1}{\eta}}
\end{aligned}
$$

The relationship to a state-space approach

Original nonlinear system

When $\Delta K_t = 0$

$$C_t = \bar{Z} K_{t-1}^{\rho} - \delta K_{t-1}$$

If $C_t > \bar{Z} K_{t-1}^{\rho} - \delta K_{t-1}$, $K_t - K_{t-1} < 0$;
If $C_t < \bar{Z} K_{t-1}^{\rho} - \delta K_{t-1}$, $K_t - K_{t-1} > 0$.

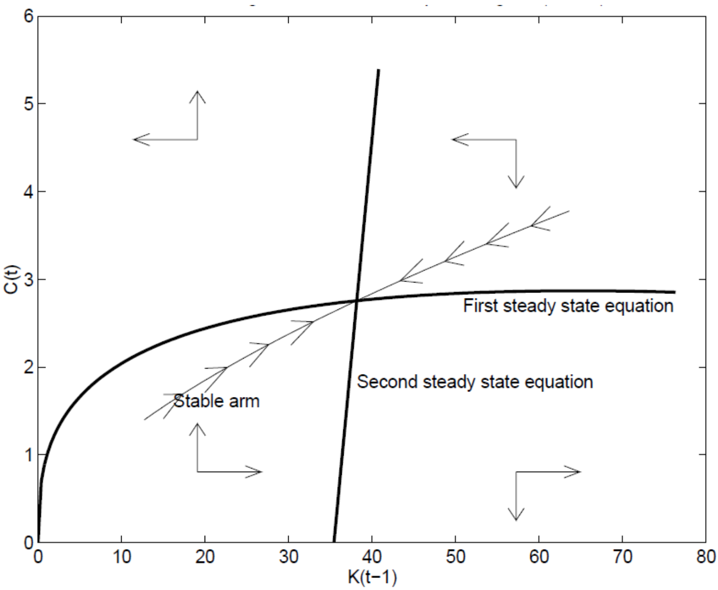
The relationship to a state-space approach

Original nonlinear system

When $\Delta C_{t+1} = 0$, $\frac{C_{t+1}}{C_t} = 1$,

$$
\begin{aligned}
1 &= \left\{\beta\left[\rho\bar{Z}\left(\bar{Z}K_{t-1}^{\rho} + (1-\delta)K_{t-1} - C_t\right)^{\rho-1} + 1 - \delta\right]\right\}^{\frac{1}{\eta}} \\
\frac{\rho\bar{Z}}{1/\beta - 1 + \delta} &= \left(\bar{Z}K_{t-1}^{\rho} + (1-\delta)K_{t-1} - C_t\right)^{1-\rho} \\
C_t &= -\left(\frac{\rho\bar{Z}}{1/\beta - 1 + \delta}\right)^{\frac{1}{1-\rho}} + \bar{Z}K_{t-1}^{\rho} + (1-\delta)K_{t-1} \\
&= \bar{Z}K_{t-1}^{\rho} + (1-\delta)K_{t-1} - \bar{K}
\end{aligned}
$$

If $C_t > -\bar{K} + \bar{Z}K_{t-1}^{\rho} + (1-\delta)K_{t-1}$, $\frac{C_{t+1}}{C_t} > 1$;
If $C_t < -\bar{K} + \bar{Z}K_{t-1}^{\rho} + (1-\delta)K_{t-1}$, $\frac{C_{t+1}}{C_t} < 1$.



C(t)
K(t−1)
0
1
2
3
4
5
6
0
10
20
30
40
50
60
70
80
First steady state equation
Second steady state equation
Stable arm

Solve with Toolkit 4.1

The structure of the problem.

There is an $m \times 1$ endogenous state vector x_t, an $n \times 1$ vector of other endogenous variables y_t, and a $k \times 1$ vector of exogenous stochastic processes z_t. The equilibrium relationships between these variables are fully characterized by the list of equations we just collected after log-linearization. We can cast these equations into three blocks:

$$\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t \\
0 &= E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \\
z_t &= Nz_{t-1} + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0
\end{aligned}$$

where C is of size $l \times n$, $l \geq n$ and of rank n, F is of size $(m + n - l) \times n$, and N has only stable eigenvalues. In total there are $m + n + k$ equations.

Solve with Toolkit 4.1

Recursive law of motion

$$
\begin{aligned}
x_t &= P x_{t-1} + Q z_t \\
y_t &= R x_{t-1} + S z_t \\
z_t &= N z_{t-1} + \epsilon_t
\end{aligned}
$$

Solutions: matrix quadratic equation.

Execute with *Toolkit 4.1*.

Solve with Toolkit 4.1

Cast the log-linearized equations into the system.

An $m \times 1$ endogenous state vector x_t, $\left\{ \hat{k}_t \right\}$.
An $n \times 1$ vector of other endogenous variables y_t, $\left\{ \hat{c}_t, \hat{r}_t, \hat{y}_t \right\}$.
A $k \times 1$ vector of exogenous stochastic processes z_t: $\left\{ \hat{z}_t \right\}$.

1. $\hat{r}_t = [1 - \beta(1 - \delta)] \left[\hat{z}_t - (1 - \rho) \hat{k}_{t-1} \right]$
2. $\hat{c}_t = \frac{\bar{Y}}{\bar{C}} \hat{z}_t + \frac{\bar{K}}{\bar{C}} \bar{R} \hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}} \hat{k}_t$
3. $\hat{y}_t = \hat{z}_t + \rho \hat{k}_{t-1}$
4. $0 = E_t \left[\eta \left(\hat{c}_t - \hat{c}_{t+1} \right) + \hat{r}_{t+1} \right]$
5. $\hat{z}_t = \psi \hat{z}_{t-1} + \varepsilon_t$

Solve with Toolkit 4.1

Cast these equations into three blocks

1) The first block $0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t$

$$
\begin{aligned}
0 &= [1-\beta(1-\delta)]\,\hat{z}_t - [1-\beta(1-\delta)]\,(1-\rho)\hat{k}_{t-1} - \hat{r}_t \\
0 &= \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}\hat{k}_t - \hat{c}_t \\
0 &= \hat{z}_t + \rho\hat{k}_{t-1} - \hat{y}_t
\end{aligned}
$$

2) The second block
$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t]$

$$0 = E_t\left[\eta\left(\hat{c}_t - \hat{c}_{t+1}\right) + \hat{r}_{t+1}\right]$$

3) The third block $z_t = Nz_{t-1} + \epsilon_{t+1};\;\; E_t[\epsilon_{t+1}] = 0$

$$\hat{z}_t = \psi\hat{z}_{t-1} + \varepsilon_t$$

An RBC Model with Indivisible Labor (Hansen, 1985)

Hansen, G. D., 1985, Indivisible labor and the business cycle. *Journal of Monetary Economics* 16, 309-327.

$$U_t = \ln c_t - An_t$$

A: marginal disutility of labor.

An RBC Model with Indivisible Labor (Hansen, 1985)

Motivation

Important labor market phenomena

- Existence of unemployed workers.
- Fluctuations in the rate of unemployment.
- Fluctuations in hours worked are large relative to productivity flucturations.
- Micro studies do not support the strong willingness of individuals to substitute leisure across time.

An RBC Model with Indivisible Labor (Hansen, 1985)

The Motivation

Problem

How to reconcile the micro and macro evidence, such that large aggregate fluctuations in hours worked are consistent with individuals who have little intertemporal substitution of labor.

Solution

Indivisible labor supply $\tilde{n}_t$.

Individual utility function

$$U(c_t) + v(\tilde{n}_t)$$

where labor is inelastic.

An RBC Model with Indivisible Labor (Hansen, 1985)

The Mechanism

- Labor is indivisible: agents either have a job or not, $\tilde{n}_t = 0$ or $\tilde{n}_t = n^*$.
- Agents are assigned to jobs according to a lottery with probability π_t.
- Unemployment insurance is complete, so that individual consumption c_t is independent of job status.
- Total labor supply

$$\begin{aligned} \text{Total} &: \quad H_t = \bar{L}\pi_t n^* \\ \text{Average} &: \quad n_t = \frac{H_t}{\bar{L}} = \pi_t n^* \end{aligned}$$

H_t: total hours worked; $\bar{L}$: total labor force.

An RBC Model with Indivisible Labor (Hansen, 1985)

The Mechanism

- Normalization:
  - $v(0) = 0$, if labor supply is 0, such that the (dis)utility from work at this point is also 0.
  - $\frac{v(n^*)}{n^*} = -A < 0$.
- Expected utility from a representative agent

$$
\begin{aligned}
E\left[U(c_t) + v(\tilde{n}_t)\right] &= U(c_t) + \pi_t v(n_t^*) + (1 - \pi_t)v(0) \\
&= U(c_t) + \pi_t(-An^*) \\
&= U(c_t) - An_t
\end{aligned}
$$

An RBC Model with Indivisible Labor (Hansen, 1985)

Consistency with the empirical facts

$$
\begin{aligned}
n_t &= \frac{H_t}{\bar{L}} \\
&= \frac{H_t}{L_t} \cdot \frac{L_t}{\bar{L}} \\
&\approx n^* \cdot \underbrace{(1 - u_t)}_{\pi_t}
\end{aligned}
$$

u_t, unemployment rate.

$\frac{H_t}{L_t} = \bar{n}_t \approx n^*$, empirically the ratio is very stable.

An RBC Model with Indivisible Labor (Hansen, 1985)

A recent use of the model

Shu-Hua Chen and Jang-Ting Guo:
Progressive Taxation and Macroeconomic——(In)stability with Productive Government Spending.

Explain the cause of "Great Moderation" with policy changes in taxation.

$$Y_t = AK_t^{\alpha} H_t^{1-\alpha} G_t^{\chi}$$

$$\tau_t = 1 - \eta \left(\frac{Y^*}{Y_t}\right)^{\phi}$$