# Solutions for problem set 07

1.(a)from

$$\begin{aligned} C_t &= \beta_1^0 + \beta_2^0(Y_t - T_t) + \varepsilon_t \\ T_t &= \gamma_1^0 + \gamma_2^0 Y_t + v_t \\ Y_t &= C_t + G_t \end{aligned}$$

we can get

$$Y_t - T_t = \frac{\beta_1^0 - \gamma_1^0 - \gamma_2^0\beta_1^0}{1 - \beta_2^0 + \beta_2^0\gamma_2^0} + \frac{1 - \gamma_2^0}{1 - \beta_2^0 + \beta_2^0\gamma_2^0}(G_t + \varepsilon_t) + \frac{1}{1 - \beta_2^0 + \beta_2^0\gamma_2^0} v_t$$

$$E\left[(Y_t - T_t)\varepsilon_t\right] = \frac{1 - \gamma_2^0}{1 - \beta_2^0 + \beta_2^0\gamma_2^0}\sigma_u^2$$

if $\gamma_2^0 = 1$, OLS estimator of $\beta$ is consistence. otherwise, the OLS estimator of $\beta$ is NOT consistence.

(b)Yes, We can use $G_t$ as a valid instrumental variable and get a consistent estimator $\hat{\beta}$ by 2sls.since $G_t$ is exogeneous variable. $\operatorname{cov}(G_t, \varepsilon_t) = 0$ and $\operatorname{cov}(G_t, Y_t - T_t) = \frac{1-\gamma_2^0}{1-\beta_2^0+\beta_2^0\gamma_2^0}$.
Stage1:

$$Y_t - T_t = \gamma_0 + \gamma_1 G_t + \mu_t$$

$$Y - T = G \cdot \gamma + \mu \quad G = (1, G_t)$$

$$\hat{\gamma}_{ols} = (G'G)^{-1}G'(Y - T)$$

$$(Y \overset{\wedge}{-} T) = G \cdot \hat{\gamma}_{ols} = G(G'G)^{-1}G'(Y - T)$$

Stage2:

$$C_t = \beta_1^0 + \beta_2^0\,(Y_t \overset{\wedge}{-} T_t) + \tilde{\varepsilon}_t$$

$$YI_t = (Y_t \overset{\wedge}{-} T_t) \quad YI = (1, YI_t)$$

$$C = YI \cdot \beta + \tilde{\varepsilon}$$

$$\hat{\beta}_{2sls} = (YI'YI)^{-1}YI'C$$

where

$$YI = (1, G(G'G)^{-1}G'(Y - T))$$

(c)

$$G_t = \frac{\beta_1^0\gamma_2^0 + \gamma_1^0 - \beta_2^0\gamma_1^0}{1 - \beta_2^0 + \beta_2^0\gamma_1^0 - \gamma_2^0} + \frac{\gamma_2^0}{1 - \beta_2^0 + \beta_2^0\gamma_1^0 - \gamma_2^0}\varepsilon_t + \frac{1 - \beta_2^0}{1 - \beta_2^0 + \beta_2^0\gamma_1^0 - \gamma_2^0} v_t + \frac{1 - \beta_2^0 + \beta_2^0\gamma_2^0}{1 - \beta_2^0 + \beta_2^0\gamma_1^0 - \gamma_2^0} w_t$$

$$E(G_t\varepsilon_t) = \frac{\gamma_2^0}{1 - \beta_2^0 + \beta_2^0\gamma_1^0 - \gamma_2^0}\sigma_u^2$$

for $\gamma_2^0 \neq 0$, $E(G_t\varepsilon_t) \neq 0$. Therefore, $G_t$ is not a valid instrumental variable.

2. (a)

$$Y_t = X'_t\beta^0 + \varepsilon_t = \beta_0^0 + X_{1t}\beta_1^0 + \varepsilon_t$$
$$E(X_{1t}\varepsilon_t) = E\left[(v_t + u_t)(w_t + u_t)\right] = E(u_t^2) = \sigma_u^2 = 1$$

therefore, OLS estimator $\hat{\beta}$ is inconsistent for $\beta^0$. (b)

$$Z_{1t} = w_t - \varepsilon_t = -u_t$$

$$E(Z_{1t}\varepsilon_t) = E\left[(-u_t)(w_t + u_t)\right] = -E(u_t^2) = -\sigma_u^2 = -1$$

that means $Z_t$ is NOT a valid instrumental vector.

(c)

$$E(v_t\varepsilon_t) = E\left[v_t(w_t + u_t)\right] = 0$$
$$E(V_{1t}v_t) = E\left[(v_t + u_t)v_t\right] = E(v_t^2) = 1$$

Thus, $v_t$ is a valid instrumental variable, $Z_t$ is a valid instrumental variable.
stage1:

$$X = Z \cdot \gamma + \mu$$
$$\hat{\gamma} = (Z'Z)^{-1}Z'X$$
$$\hat{X} = X(Z'Z)^{-1}Z'X$$

stage2:

$$Y = \hat{X}\beta + \tilde{\varepsilon}$$

$$\begin{aligned}
\hat{\beta}_{2sls} &= (\hat{X}'\hat{X})^{-1}\hat{X}'Y \\
&= \left[X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'Y \\
&= \left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'Y \\
&= \left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'(X\beta + \varepsilon) \\
&= \beta_0 + \left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'\varepsilon
\end{aligned}$$

$$\begin{aligned}
\sqrt{n}(\hat{\beta}_{2sls} - \beta_0) &= \left[\frac{X'Z}{n}\left(\frac{Z'Z}{n}\right)^{-1}\frac{Z'X}{n}\right]^{-1}\frac{X'Z}{n}\left(\frac{Z'Z}{n}\right)^{-1}\frac{Z'\varepsilon}{\sqrt{n}} \\
&= \left[\hat{Q}_{XZ}(\hat{Q}_{ZZ})^{-1}\hat{Q}_{ZX}\right]^{-1}\hat{Q}_{XZ}(\hat{Q}_{ZZ})^{-1}\frac{Z'\varepsilon}{\sqrt{n}}
\end{aligned}$$

By WLLN, $\hat{Q}_{XZ} \xrightarrow{p} Q_{XZ}$, $\hat{Q}_{ZZ} \xrightarrow{p} Q_{ZZ}$, $\hat{Q}_{ZX} \xrightarrow{p} Q_{ZX}$,

$$\left[\hat{Q}_{XZ}\hat{Q}_{ZZ}^{-1}\hat{Q}_{ZX}\right]^{-1}\hat{Q}_{XZ}\hat{Q}_{ZZ}^{-1} \xrightarrow{p} \left[Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right]^{-1}Q_{XZ}Q_{ZZ}^{-1} \triangleq A$$

By CLT,

$$\frac{Z'\varepsilon}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} Z_t\varepsilon_t \xrightarrow{p} N\left(0, V_1\right)$$

$$\begin{aligned} V_1 &= E\left(Z_t{Z'}_t\varepsilon_t^2\right) \\ &= E\left(Z_t{Z'}_tE\left(\varepsilon_t^2\left|Z_t\right.\right)\right) \\ &= 2E\left(Z_t{Z'}_t\right) = 2Q_{ZZ} \end{aligned}$$

$$\sqrt{n}(\hat{\beta}_{2sls} - \beta_0) \rightarrow N\left(0, V\right)$$

$$\begin{aligned} V &= AV_1A = 2\big(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\big)^{-1} \\ &= 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= 2I_2 \end{aligned}$$

$$\sqrt{n}(\hat{\beta}_{2sls} - \beta_0) \rightarrow N\left(0, 2I_2\right)$$

(d)

$$JF = \left(R\hat{\beta} - r\right)\left(S^2R(X'X)^{-1}R'\right)^{-1}\left(R\hat{\beta} - r\right) \sim \chi_J^2$$

where $S^2 = \frac{\hat{e}'\hat{e}}{n-K}$, $\hat{e} = Y - X\hat{\beta}_{2sls}$. for the second stage, $\tilde{e} = Y - \hat{X}\hat{\beta}_{2sls}$, $\tilde{S}^2 = \frac{\tilde{e}'\tilde{e}}{n-K}$.

$$\begin{aligned} \tilde{e} &= Y - \hat{X}\hat{\beta}_{2sls} \\ &= (Y - X\hat{\beta}_{2sls}) + (X - \hat{X})\hat{\beta}_{2sls} \\ &= \hat{e} + \hat{\mu}\hat{\beta}_{2sls} \end{aligned}$$

where $\hat{\mu}$ is the estimate residual from the first stage.

$$\begin{aligned} \tilde{e}'\tilde{e} &= \left(\hat{e} + \hat{\mu}\hat{\beta}_{2sls}\right)'\left(\hat{e} + \hat{\mu}\hat{\beta}_{2sls}\right) \\ &= \hat{e}'\hat{e} + 2\hat{e}'\hat{\mu}\hat{\beta}_{2sls} + \hat{\beta}_{2sls}\hat{\mu}'\hat{\mu}\hat{\beta}_{2sls}' \\ &= \sum\hat{e}_t^2 + 2\hat{\beta}_{2sls}^1\sum\hat{e}_t\hat{\mu}_t + \left(\hat{\beta}_{2sls}^1\right)^2\sum\hat{\mu}_t^2 \end{aligned}$$

$$\frac{\tilde{e}'\tilde{e}}{n-K}=\frac{\sum \hat{e}_t^2}{n-K}+\frac{2}{n-K}\hat{\beta}_{2sls}^1\sum \hat{e}_t\hat{\mu}_t+\frac{1}{n-K}\left(\hat{\beta}_{2sls}^1\right)^2\sum \hat{\mu}_t^2 \xrightarrow{p} \sigma^2+2\beta_0^1\text{cov}(\mu,e)+\left(\beta_0^1\right)^2\sigma_\mu^2$$

thus, $\tilde{S}^2$ is not a consistent estimation for $S^2$. therefore, $J\tilde{F}$ from the second stage does not converge to $\chi_J^2$.

3.(a)because $E(X_t\varepsilon_t)=0$ and $E(X_tX'_t)=Q_{XX}$ is nonsingular. we can regard $X_t$ as an instrumental variable for itself. That is , $Z=X$.

$$\begin{aligned}\hat{\beta}_{2sls}&=\left[X'X(X'X)^{-1}X'X\right]^{-1}X'X(X'X)^{-1}X'Y\\&=(X'X)^{-1}X'Y\\&=\hat{\beta}\end{aligned}$$

$\hat{\beta}$ is a special 2SLS estimator $\hat{\beta}_{2sls}$ with instrumental variable $X$.

(b)

$$\begin{aligned}\hat{\beta}&=(X'X)^{-1}X'Y\\&=(X'X)^{-1}X'(X\beta^0+\varepsilon)\\&=\beta^0+(X'X)^{-1}X'\varepsilon\end{aligned}$$

$$\begin{gathered}\hat{\beta}-\beta^0\xrightarrow{P}Q_{XX}E(X_t\varepsilon_t)=0\\ \sqrt{n}(\hat{\beta}-\beta^0)=\left(\frac{X'X}{n}\right)^{-1}\frac{X'\varepsilon}{\sqrt{n}}\xrightarrow{P}Q_{XX}^{-1}\frac{\sum X_t\varepsilon_t}{\sqrt{n}}\\ \text{var}\left(\sqrt{n}(\hat{\beta}-\beta^0)\right)\xrightarrow{P}Q_{XX}^{-1}E(X_tX'_t\varepsilon_t^2)Q_{XX}^{-1}=\sigma^2Q_{XX}^{-1}\end{gathered}$$

By CLT,

$$\sqrt{n}(\hat{\beta}-\beta^0)\xrightarrow{d}N\left(0,\sigma^2Q_{XX}^{-1}\right)$$

$$\begin{aligned}\hat{\beta}_{2sls}&=\left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'Y\\&=\beta_0+\left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'\varepsilon\\&=\beta_0+\left[\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'X}{n}\right]^{-1}\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'\varepsilon}{n}\\&\xrightarrow{P}0\end{aligned}$$

$$\sqrt{n}(\hat{\beta}_{2sls}-\beta_0)=\left[\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'X}{n}\right]^{-1}\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'\varepsilon}{\sqrt{n}}\xrightarrow{P}\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)Q_{XZ}Q_{ZZ}^{-1}\frac{Z'\varepsilon}{\sqrt{n}}$$

$$\text{var}\left(\frac{Z'\varepsilon}{\sqrt{n}}\right) = E(Z_t {Z'}_t \varepsilon_t^2) = \sigma^2 Q_{ZZ}$$

$$\text{var}\left(\hat{\beta}_{2sls}\right) = \left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1} Q_{XZ}Q_{ZZ}^{-1}Q_{ZZ}Q_{ZZ}^{-1}Q_{XZ}\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}\sigma^2 = \sigma^2\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}$$

By CLT,

$$\sqrt{n}(\hat{\beta}_{2sls} - \beta_0) \xrightarrow{d} N\left(0, \sigma^2\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}\right)$$

$$Q_{XX} - Q_{XZ}Q_{ZZ}^{-1}Q_{ZX} = E(X_t {X'}_t) - E(X_t {Z'}_t)[E(Z_t {Z'}_t)]^{-1} E(Z_t {X'}_t)$$

$$\begin{aligned}
\frac{X'X}{n} - \frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'X}{n} &= \frac{1}{n}X'(I - Z(Z'Z)^{-1}Z')X \\
&= \frac{1}{n}X'M_Z X \\
&= \frac{1}{n}(M_Z X)'M_Z X \\
&= \frac{1}{n}D'D
\end{aligned}$$

So $Q_{XX} - Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}$ is positive semi-definite. $\hat{\beta}$ is more asymptotically efficient than $\hat{\beta}_{2sls}$

(c)

$$\hat{\beta} - \beta^0 = (X'X)^{-1}X'\varepsilon = (\frac{X'X}{n})^{-1}\frac{X'\varepsilon}{n}$$

$$\sqrt{n}(\hat{\beta} - \beta^0) = (\frac{X'X}{n})^{-1}\frac{X'\varepsilon}{\sqrt{n}} = (\frac{\sum {X'}_t X_t}{n})^{-1}\frac{\sum {X'}_t \varepsilon_t}{\sqrt{n}}$$

$$(\frac{\sum {X'}_t X_t}{n})^{-1} \xrightarrow{P} Q_{XX}^{-1}$$

$$\frac{\sum X_t \varepsilon_t}{\sqrt{n}} = E\left(\frac{\sum X_t \varepsilon_t}{\sqrt{n}}\right) + o_p\left(\sqrt{\text{var}\left(\frac{\sum X_t \varepsilon_t}{\sqrt{n}}\right)}\right)$$

since $E\left(\frac{\sum X_t \varepsilon_t}{\sqrt{n}}\right) = 0$ and $\text{var}\left(\frac{\sum X_t \varepsilon_t}{\sqrt{n}}\right) = O_p(1)$,

$$\frac{\sum X_t \varepsilon_t}{\sqrt{n}} = O_p(1)$$

$$\left(\frac{X'X}{n}\right)^{-1}\frac{X'\varepsilon}{\sqrt{n}} - Q_{XX}^{-1}\frac{\sum X_t \varepsilon_t}{\sqrt{n}} \xrightarrow{P} 0$$

$$\sqrt{n}(\hat{\beta} - \beta^0) - Q_{XX}^{-1}\frac{\sum X_t \varepsilon_t}{\sqrt{n}} \xrightarrow{P} 0$$

So,

$$\sqrt{n}(\hat{\beta} - \beta^0) = Q_{XX}^{-1}\frac{\sum X_t \varepsilon_t}{\sqrt{n}} + O_p(1)$$

(d)

$$\sqrt{n}(\hat{\beta}_{2sls}-\beta_0)=\left[\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'X}{n}\right]^{-1}\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'\varepsilon}{\sqrt{n}}$$

by WLLN,

$$\left[\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\frac{Z'X}{n}\right]^{-1}\frac{X'Z}{n}(\frac{Z'Z}{n})^{-1}\xrightarrow{P}\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)Q_{XZ}Q_{ZZ}^{-1}$$

$$\frac{\sum Z_t\varepsilon_t}{\sqrt{n}}=E\left(\frac{\sum Z_t\varepsilon_t}{\sqrt{n}}\right)+O_p\left(\sqrt{\text{var}\left(\frac{\sum Z_t\varepsilon_t}{\sqrt{n}}\right)}\right)$$

$$\text{var}\left(\frac{\sum Z_t\varepsilon_t}{\sqrt{n}}\right)=O_p(1)$$

$$\frac{\sum Z_t\varepsilon_t}{\sqrt{n}}=O_p(1)$$

$$\sqrt{n}(\hat{\beta}_{2sls}-\beta^0)=\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}Q_{XZ}Q_{ZZ}^{-1}\frac{\sum X_t\varepsilon_t}{\sqrt{n}}+o_p(1)$$

(e)By (c) and (d)

$$\begin{aligned}\sqrt{n}(\hat{\beta}-\hat{\beta}_{2sls})&=\sqrt{n}(\hat{\beta}-\beta^0)-\sqrt{n}(\hat{\beta}_{2sls}-\beta^0)\\&=Q_{XX}^{-1}\frac{\sum X_t\varepsilon_t}{\sqrt{n}}+o_p(1)-\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}Q_{XZ}Q_{ZZ}^{-1}\frac{\sum X_t\varepsilon_t}{\sqrt{n}}-o_p(1)\\&=\frac{1}{\sqrt{n}}\sum\left(Q_{XX}^{-1}X_t-\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}Q_{XZ}Q_{ZZ}^{-1}X_t\right)\varepsilon_t+o_p(1)\end{aligned}$$

(f) Denote $\Lambda_{XZ}=\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}Q_{XZ}Q_{ZZ}^{-1}$ and $A_t=Q_{XX}^{-1}X_t-\Lambda_{XZ}X_t$
since the asymptotic distribution of $\sqrt{n}(\hat{\beta}-\hat{\beta}_{2sls})$ is determined by the leading term.

$$\frac{1}{n}\sum\left(Q_{XX}^{-1}X_t-\Lambda_{XZ}X_t\right)\varepsilon_t=Q_{XX}^{-1}\sum\frac{X_t\varepsilon_t}{n}-\Lambda_{XZ}\sum\frac{Z_t\varepsilon_t}{n}\xrightarrow{P}0$$

$$\begin{aligned}&\text{var}\left(\sqrt{n}(\hat{\beta}-\hat{\beta}_{2sls})\,|XZ\right)=\sum_{t=1}^{n}A_t\tfrac{1}{n}\text{var}\left(\varepsilon_t\,|X_tZ_t\right)A'_t\\&=\tfrac{1}{n}\sigma^2\sum_{t=1}^{n}A_tA'_t=\tfrac{1}{n}\sigma^2\sum_{t=1}^{n}\left(Q_{XX}^{-1}X_t-\Lambda_{XZ}Z_t\right)\left(Q_{XX}^{-1}X_t-\Lambda_{XZ}Z_t\right)'\\&=\tfrac{1}{n}\sigma^2\sum_{t=1}^{n}\left(Q_{XX}^{-1}X_tX'_tQ_{XX}^{-1}-\Lambda_{XZ}Z_tX'_tQ_{XX}^{-1}-Q_{XX}^{-1}X_tZ'_t\Lambda'_{XZ}+\Lambda_{XZ}Z_tZ'_t\Lambda'_{XZ}\right)\\&\xrightarrow{P}\sigma^2\left[Q_{XX}^{-1}Q_{XX}Q_{XX}^{-1}-\Lambda_{XZ}Q_{ZX}Q_{XX}^{-1}-Q_{XX}^{-1}Q_{XZ}\Lambda'_{XZ}+\Lambda_{XZ}Q_{ZZ}\Lambda'_{XZ}\right]\\&=\sigma^2\left[\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}-Q_{XX}^{-1}\right]\end{aligned}$$

$$\sqrt{n}(\hat{\beta}-\hat{\beta}_{2sls})\xrightarrow{d}N\left(0,\sigma^2\left[\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}-Q_{XX}^{-1}\right]\right)$$

(g)

$$\frac{\sqrt{n}(\hat{\beta}-\hat{\beta}_{2sls})}{\sqrt{\sigma^2\left(\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}-Q_{XX}^{-1}\right)}}\xrightarrow{d}N\left(0,1\right)$$

$$n(\hat{\beta}-\hat{\beta}_{2sls})\left[\left(Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}\right)^{-1}-Q_{XX}^{-1}\right]^{-1}(\hat{\beta}-\hat{\beta}_{2sls})'\sim\chi_J^2$$

where $J$ is the number of endogeneous variable.

4. (a)

$$e_t=Y_t-\hat{X}_t'\hat{\beta}_{2sls}=Y_t-X'_t\hat{\beta}_{2sls}+(X'_t-\hat{X}_t')\hat{\beta}_{2sls}$$

$$e=Y-X\hat{\beta}_{2sls}+(X-\hat{X})\hat{\beta}_{2sls}=\hat{e}+(X-\hat{X})\hat{\beta}_{2sls}$$

$$S^2=\tfrac{e'e}{n-K}=\tfrac{(\hat{e}+\hat{\mu}\hat{\beta}_{2sls})'(\hat{e}+\hat{\mu}\hat{\beta}_{2sls})}{n-K}$$
$$=\tfrac{1}{n-K}\sum_{t=1}^{n}\left[\hat{e}_t^2+2\hat{\beta}_{2sls}'\hat{\mu}_t\hat{e}_t+\left(\hat{\beta}_{2sls}'\hat{\mu}_t\right)^2\right]$$

$$\hat{e}_t=X'_t(\hat{\beta}-\hat{\beta}_{2sls})+\varepsilon_t$$

$$\hat{\mu}_t=Z'_t(\gamma-\hat{\gamma})+\mu_t$$

$$\frac{1}{n-K}\sum_{t=1}^{n}\hat{e}_t^2\xrightarrow{P}\sigma^2$$

$$\frac{1}{n-K}\sum_{t=1}^{n}\hat{\beta}_{2sls}'\hat{\mu}_t\hat{e}_t\xrightarrow{P}\beta_0E(X_t\varepsilon_t)$$

$$\frac{1}{n-K}\sum_{t=1}^{n}\left(\hat{\beta}_{2sls}'\hat{\mu}_t\right)^2\xrightarrow{P}\beta_0\Sigma_\mu\beta_0$$

$$S^2=\frac{e'e}{n-K}\xrightarrow{P}\sigma^2+\beta_0E(X_t\varepsilon_t)+\beta_0\Sigma_\mu\beta_0\neq\sigma^2$$

Thus, $S^2$ is not consistent for $\sigma^2$.

(b)
since $S^2$ is not consistent for $\sigma^2$, $S^2R\hat{Q}_{\hat{X}\hat{X}}^{-1}R'$ is not a consistent estimator for the variance of $\sqrt{n}(R\hat{\beta}_{2sls}-r)$, so $\chi_J^2$ is not valid under $H_0$ let $\hat{\mu}=X-\hat{X}$, then $e=\hat{e}+\hat{\mu}\hat{\beta}_{2sls}$