Problem set 5

March 12, 2012

Let $\bar{R} = (\bar{R}_1, \bar{R}_2 ... \bar{R}_n)'$ be an n $\times$ 1 vector of the expected returns of the n assets. Also let V be the n $\times$ n covariance matrix of the returns on the n assets. V is assumed to be of full rank. Next, let $\omega = (\omega_1, \omega_2 ... \omega_n)'$ be an n $\times$ 1 vector of portfolio proportions, such that ω_i is the proportion of total portfolio wealth invested in the i_{th} asset, and e is defined to be an $n \times 1$vector of ones.

1. prove that the variance covariance matrix$V$ is positive definite.

2. prove V^{-1}exists and is positive definite.

3. prove that$\varsigma\delta - \alpha^2 > 0$, where $\alpha = \bar{R}'V^{-1}e = e'V^{-1}\bar{R}$, $\varsigma = \bar{R}'V^{-1}\bar{R}$, $and\, \delta = e'V^{-1}e$ are scalars.

When we derive the portfolio frontier: namely we minimize the portfolio's variance subject to the constraints that the portfolio's expected return equals $\bar{R}_p$ and the portfolio's weights sum to one.

$$\min_{\omega} \frac{1}{2}\omega' V^{-1}\omega + \lambda \left[\bar{R}_p - \omega'\bar{R}\right] + \gamma \left[1 - \omega' e\right] \tag{1}$$

solving this quadratic optimization problem, we get the optimal weights vector

$$\omega^{\star} = a + b\bar{R}_p \tag{2}$$

where$a=\frac{\varsigma V^{-1}e-\alpha V^{-1}\bar{R}}{\varsigma\delta-\alpha^2}$, $b=\frac{\varsigma V^{-1}\bar{R}-\alpha V^{-1}e}{\varsigma\delta-\alpha^2}$. Given equation (2), the variance of the frontier portfolio is

$$\begin{aligned} \sigma_p^2 &= \omega^{\star\prime} V \omega^{\star} \\ &= \frac{\delta \bar{R}_p^2 - 2\alpha \bar{R}_p + \varsigma}{\varsigma\delta - \alpha^2} \\ &= \frac{1}{\delta} + \frac{\delta\left(\bar{R}_p - \frac{\alpha}{\delta}\right)^2}{\varsigma\delta - \alpha^2} \end{aligned} \tag{3}$$

1. prove that equation (2) is a sufficient condition for a frontier portfolio.

2. prove equation (3).

This Problem set is due to Mar, 14.