Solutions for Quiz 1

Consider the following classical regression model

$$Y_t = X_t'\beta_0 + \varepsilon_t \ = \beta_0^0 + \sum_{j=1}^{k} \beta_0^0 X_{jt} + \varepsilon_t \quad t = 1, \cdots, n \tag{1}$$

where β_0^0 is an intercept and β_j^0 is a coefficient of $X_j t$.
Suppose we want to test

$$H_0: \quad \beta_1^0 = \beta_2^0 = \cdots = \beta_k^0 = 0$$

Then the F-statistic can be written as $F = \frac{(\tilde{e}'\tilde{e} - e'e)/k}{e'e/(n-k-1)}$ where $e'e$ is the sum of squared residuals from the unrestricted model (1), and $\tilde{e}'\tilde{e}$ is the sum of squared residuals from the following restricted model

$$Y_t = \beta_0^0 + \varepsilon_t \tag{2}$$

(a) show that $F = \frac{R^2/k}{(1-R^2)/(n-k-1)}$ where R^2 is the coefficient of determination of the unrestricted model (1)
(b) show that under H_0,

$$(n-k-1)R^2 \xrightarrow{d} \chi_k^2$$

ANSWER:
(a) It is easy to proof that for model (2) $\hat{\beta}_0^0 = \bar{Y}$, then $\tilde{e}'\tilde{e} = (Y - \bar{Y})'(Y - \bar{Y})$.

$$\begin{aligned} F &= \frac{(\tilde{e}'\tilde{e} - e'e)/k}{e'e/(n-k-1)} \\ &= \frac{(1 - \frac{e'e}{\tilde{e}'\tilde{e}})/k}{\frac{e'e}{\tilde{e}'\tilde{e}}/(n-k-1)} \\ &= \frac{(1 - \frac{e'e}{\tilde{e}'\tilde{e}})/k}{(1 - 1 + \frac{e'e}{\tilde{e}'\tilde{e}})/(n-k-1)} \\ &= \frac{R^2/k}{(1-R^2)/(n-k-1)} \end{aligned}$$

(b) First, note that

$$F = \frac{R^2/k}{(1-R^2)/(n-k-1)}$$

we have

$$kF = (n-k-1)R^2/(1-R^2) \xrightarrow{d} \chi_k^2$$

under H_0. This implies that kF is bounded in probability: that is

$$(n-k-1)R^2/(1-R^2) = O(1)$$

And consequently, given that k is a fixed integer,

$$R^2/(1-R^2) = O(n^{-1}) = o(1)$$

or

$$R^2 \xrightarrow{p} 0$$

Therefore, $1 - R^2 \xrightarrow{p} 1$. By the Slutsky theorem, we then have

$$\begin{gathered} kF \xrightarrow{d} \chi_k^2 \\ \tfrac{(n-k-1)R^2}{(1-R^2)} \xrightarrow{d} \chi_k^2 \\ (n-k-1)R^2 \xrightarrow{d} \chi_k^2 \end{gathered}$$

This completes the proof.