

Prove that the **von Neumann-Morgenstern expected utility function is unique up to a linear monotonic transformation**, which is a cardinal property .

- Suppose that the vNM utility function $U(\cdot)$ represents \succeq .
- Then the vNM utility function, $V(\cdot)$, represents *the same preferences* if and only if for some scalar α and $\beta, \beta > 0$, s.t. $V(p) = \alpha + \beta U(p)$, for all lotteries p .

Proof:

1. Sufficiency

we are to prove if $V(p) = \alpha + \beta U(p)$, for all lotteries p , $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$, then $V(\cdot)$ is a vNM expected utility function that represents the same preference order.

- for $\forall p^1, p^2$ lottery, and $p^1 \succeq p^2$, since $U(\cdot)$ is the vNM expected utility function that represents this preference order, thus

$$p^1 \succeq p^2 \Leftrightarrow U(p^1) \geq U(p^2) \Leftrightarrow \sum_{i=1}^N p_i^1 u_i \geq \sum_{i=1}^N p_i^2 u_i$$

- on the other hand, since $V(p) = \alpha + \beta U(p)$ and $\beta > 0$,

$$U(p^1) \geq U(p^2) \Leftrightarrow \alpha + \beta U(p^1) \geq \alpha + \beta U(p^2) \Leftrightarrow V(p^1) \geq V(p^2)$$

and besides

$$\alpha + \beta \sum_{i=1}^N p_i^1 u_i \geq \alpha + \beta \sum_{i=1}^N p_i^2 u_i \Leftrightarrow \sum_{i=1}^N p_i^1 (\alpha + \beta u_i) \geq \sum_{i=1}^N p_i^2 (\alpha + \beta u_i)$$

- namely, $p^1 \succeq p^2 \Leftrightarrow V(p^1) \geq V(p^2)$, and $V(\cdot)$ is linear in probability, thus $V(\cdot)$ is also a vNM expected utility function represents *the same preferences*.

2. Necessity.

We are to show that if u and v are vNM expected utility functions, then they are linearly related for elementary lottery, and thus for all gambles. As before, we assume that

- All lotteries have **possible payoffs** that are contained in the **set** $X = \{x_1, \dots, x_n\}$ with corresponding probability $P = \{p_1, \dots, p_n\}$.
- define an "**elementary**" or "**primitive**" lottery, e_i , which returns outcome x_i with probability 1 and all other outcomes with probability zero, that is,

$$e_i = \{p_1, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_n\} = \{0, \dots, 0, 1, 0, \dots, 0\}$$

where $p_i = 1$ and $p_j = 0 \forall j \neq i$.

- Without **loss of generality**, suppose that the outcomes are ordered such that

$$e_n \succeq e_{n-1} \succeq \dots \succeq e_1$$

and $e_n \succ e_1$ (otherwise $e_n \sim \dots \sim e_i \sim \dots \sim e_1$, then every utility function is a constant and the result follows immediately).

Because $u(\cdot)$ represents \succeq , we have $u(e_n) \geq \dots \geq u(e_i) \geq \dots \geq u(e_1)$, and $u(e_n) > u(e_1)$. Then for any $i = 1, \dots, n$, $\exists \lambda_i \in [0, 1]$ s.t.

$$u(e_i) = \lambda_i u(e_n) + (1 - \lambda_i) u(e_1) \tag{1}$$

- Now, because $u(\cdot)$ has the expected utility property, (1) implies that if we regard $\lambda_i e_n + (1 - \lambda_i) e_1$ as a compound lottery, we have

$$u(e_i) = \lambda_i u(e_n) + (1 - \lambda_i) u(e_1) = u(\lambda_i e_n + (1 - \lambda_i) e_1)$$

which, because $u(\cdot)$ represents \succeq , means that

$$e_i \sim \lambda_i e_n + (1 - \lambda_i) e_1 \quad (2)$$

- On the other hand, because $v(\cdot)$ also represents \succeq , we must have

$$v(e_i) = v(\lambda_i e_n + (1 - \lambda_i) e_1)$$

since $v(\cdot)$ has the expected utility property,

$$v(e_i) = v(\lambda_i e_n + (1 - \lambda_i) e_1) = \lambda_i v(e_n) + (1 - \lambda_i) v(e_1) \quad (3)$$

- Together, (1) and (3) imply that

$$\frac{u(e_n) - u(e_i)}{u(e_i) - u(e_1)} = \frac{1 - \lambda_i}{\lambda_i} = \frac{v(e_n) - v(e_i)}{v(e_i) - v(e_1)} \quad (4)$$

for any $i = 1, \dots, n$ such that $e_i \succ e_1$ (i.e., such that $\lambda_i > 0$).

- From (4) we may conclude that

$$[u(e_n) - u(e_i)][v(e_i) - v(e_1)] = [v(e_n) - v(e_i)][u(e_i) - u(e_1)] \quad (5)$$

whenever $e_n \succ e_1$.

- Rearranging, (5) can be expressed in the form

$$v(e_i) = \alpha + \beta u(e_i), \text{ for all } i = 1, \dots, n. \quad (6)$$

where

$$\alpha \equiv \frac{u(e_n) v(e_1) - v(e_n) u(e_1)}{u(e_n) - u(e_1)}$$

and

$$\beta \equiv \frac{v(e_n) - v(e_1)}{u(e_n) - u(e_1)}$$

Notice that both α and β are constants (i.e., independent of i), and that β is strictly positive.

- So, for any other arbitrary lottery p , if (p_1, \dots, p_n) is the simple gamble induced by p , then

$$\begin{aligned} v(p) &= \sum_{i=1}^n p_i v(e_i) \\ &= \sum_{i=1}^n p_i (\alpha + \beta u(e_i)) \\ &= \alpha + \beta \sum_{i=1}^n p_i u(e_i) \\ &= \alpha + \beta u(p) \end{aligned}$$

where the first and last equalities follow because $v(\cdot)$ and $u(\cdot)$ have the expected utility property and the second equality follows from (6). ■