Solutions for problem set 8

1. (a)

$$\begin{aligned}
V_0 &= a\text{var}\left[\sqrt{n}\hat{m}(\beta_0)\right] \\
&= E\left[n\hat{m}(\beta_0)\hat{m}'(\beta_0)\right] - \left\{E\left[\sqrt{n}\hat{m}(\beta_0)\right]\right\}^2 \\
&= E\left[n\hat{m}(\beta_0)\hat{m}'(\beta_0)\right] \\
&= E\left[n\frac{1}{n}\sum_{t=1}^{n} m_t(\beta_0)\frac{1}{n}\sum_{t=1}^{n} m_t(\beta_0)'\right] \\
&= E\left[\frac{1}{n}\sum_{t=1}^{n} m_t(\beta_0)m_t(\beta_0)'\right] \\
&= E\left[m_t(\beta_0)m_t(\beta_0)'\right]
\end{aligned}$$

(b)

$$\begin{aligned}
&\hat{Q}_n(\beta_0) = -\hat{m}(\beta_0)'\hat{W}^{-1}\hat{m}(\beta_0) \\
&\tfrac{d\hat{Q}_n(\beta_0)}{d\beta_0} = -2\tfrac{d\hat{m}_n(\beta_0)}{d\beta_0}\hat{W}^{-1}\hat{m}(\beta_0) = 0 \\
&\tfrac{d\hat{m}_n(\beta_0)}{d\beta_0}\hat{W}^{-1}\hat{m}(\beta_0) = 0
\end{aligned}$$

(c)

$$\begin{aligned}
0 &= \frac{d\hat{m}_n(\hat{\beta})}{d\beta}\hat{W}^{-1}\sqrt{n}\hat{m}(\hat{\beta}) \\
&= \frac{d\hat{m}_n(\hat{\beta})}{d\beta}\hat{W}^{-1}\sqrt{n}\hat{m}(\beta_0) + \frac{d\hat{m}_n(\hat{\beta})}{d\beta}\hat{W}^{-1}\frac{d\hat{m}(\bar{\beta})}{d\beta}\sqrt{n}(\hat{\beta}-\beta_0)
\end{aligned}$$

$$\sqrt{n}(\hat{\beta}-\beta_0) = -\left[\frac{d\hat{m}_n(\hat{\beta})}{d\beta}\hat{W}^{-1}\frac{d\hat{m}(\bar{\beta})}{d\beta}\right]^{-1}\frac{d\hat{m}_n(\hat{\beta})}{d\beta}\hat{W}^{-1}\sqrt{n}\hat{m}(\beta_0)$$

$$\left[\tfrac{d\hat{m}_n(\hat{\beta})}{d\beta}\hat{W}^{-1}\tfrac{d\hat{m}(\bar{\beta})}{d\beta}\right]^{-1}\tfrac{d\hat{m}_n(\hat{\beta})}{d\beta}\hat{W}^{-1} \to A \quad \sqrt{n}\hat{m}(\beta_0) \to N(0,V)$$

$$\sqrt{n}(\hat{\beta}-\beta_0) \to A \cdot N(0,V) \sim N(0,\Omega)$$

where

$$A = (D'_0W^{-1}D_0)^{-1}D'_0W^{-1}$$

$$\frac{d\hat{m}(\hat{\beta})}{d\beta} \to D_0(\beta_0)$$

$$\Omega = (D'_0W^{-1}D_0)^{-1}D'_0W^{-1}VW^{-1}D_0(D'_0W^{-1}D_0)^{-1}$$

(d)The optimal choice of W is $W=V$. Define $\Omega_0=(D'_0V^{-1}D_0)^{-1}$

$$\begin{aligned}
{\Omega^{-1}}_0-\Omega &= D'_0V^{-1}D_0-D'_0W^{-1}D_0(D'_0W^{-1}VW^{-1}D_0)^{-1}D'_0W^{-1}D_0\\
&= D'_0V^{-1/2}\left[I-V^{-1/2}W^{-1}D_0(D'_0W^{-1}VW^{-1}D_0)^{-1}D'_0W^{-1}V^{-1/2}\right]V^{-1/2}D_0\\
&= D'_0V^{-1/2}GV^{-1/2}D_0\\
&= D'_0V^{-1/2}GGV^{-1/2}D_0\\
&= \left[GV^{-1/2}D_0\right]'GV^{-1/2}D_0\\
&= A'A\sim p.s.d
\end{aligned}$$

2.
(a)suppose $m_t(\beta)=Z_t(Y_t-X_t\beta)$, $W=n^{-1}Z_tZ'_t$

$$\beta=\arg\min\left(Z'(Y-X\beta)\right)'(Z_tZ'_t)\left(Z'(Y-X\beta)\right)$$

FOC:

$$(Z'X)'(ZZ)^{-1}\left(Z'(Y-X\hat{\beta})\right)=0$$

$$\hat{\beta}=\left(X'Z(Z'Z)^{-1}Z'X\right)^{-1}X'Z(Z'Z)^{-1}Z'Y=\hat{\beta}_{2sls}$$

(b)for anasymptoticaly optimal GMM, $\Omega_0=\left(D'_0V_0^{-1}D_0\right)^{-1}$
for conditional homoskedasticity, $\hat{\beta}_{2sls}$ and $\hat{\beta}_{GMM}$ have same efficiency.
for conditional heteroskedasticity, from 8.1(d), we know $\hat{\beta}_{GMM}$ is more efficient that $\hat{\beta}_{2sls}$.

3.

$$H_0:\quad g(\beta_0)=R\beta_0-r\ \ =0$$

$$\sqrt{n}\left(\hat{\beta}-\beta_0\right)\to N(0,\Omega_0)$$

$$\sqrt{n}\left(R\hat{\beta}-R\beta_0\right)=\sqrt{n}\left(R\hat{\beta}-r\right)\to N(0,R\Omega_0R')$$

$$n\left(R\hat{\beta}-r\right)'[R\Omega_0R']\left(R\hat{\beta}-r\right)\to\chi^2_J$$

4.

$$\hat{V}=n^{-1}\sum_{t=1}^{n}m_t(\hat{\beta})m_t(\hat{\beta})'$$

by ULLN for $\{m_t(\beta_0)\}$, we have

$$\hat{V}\to E\left[m_t(\beta_0)m_t(\beta_0)'\right]=V_0$$

5.

$$\tilde{V}\to 0\quad \hat{V}\to 0$$

$$\tilde{V}^{-1} - \hat{V}^{-1} \to 0$$

$$\begin{aligned} &nm_t(\hat{\beta})\tilde{V}^{-1}m_t(\hat{\beta}) - nm_t(\hat{\beta})\hat{V}^{-1}m_t(\hat{\beta}) \\ &= nm_t(\hat{\beta})\left(\tilde{V}^{-1} - \hat{V}^{-1}\right)m_t(\hat{\beta}) \\ &\to 0 \end{aligned}$$

6.
Yes. if $\tilde{\beta}$ is consistent, then $\tilde{V}$ is consistent.
So we can replace $\hat{\beta}$ by $\tilde{\beta}$.