Quiz 2

Consider the following linear model

$$Y_t = X_t^T \underset{k\times 1}{\beta_0} + \varepsilon_t \tag{1}$$

Assume that $\left\{Y_t, X_t^T\right\}^T$ is jointly stationary and ergodic, $E\left(\varepsilon_t \left| X_t\right.\right) = 0$, $E\left(\varepsilon_t\right) = \sigma^2$, and $\{X_t\varepsilon_t\}$ is an martingale difference sequence. Define $Q = E\left(X_t X'_t\right)$ which is finite and nonsingular, $V = E\left(X_t X'_t \varepsilon_t^2\right)$ is finite and positive definite.

1. Show step by step that the OLS estimator, $\hat{\beta}$, is consistent.
2. Show step by step the asymptotic normality of $\hat{\beta}$.
3. How to test $R\beta_0 = r$.

ANSWER:

1.

$$\begin{aligned}\hat{\beta} &= \left(X'X\right)^{-1} X'Y \\ &= \hat{Q}^{-1} n^{-1} \sum_{t=1}^{n} X_t Y_t\end{aligned}$$

where $\hat{Q}^{-1} = n^{-1} \sum\limits_{t=1}^{n} X_t X'_t$. Substituting $Y_t = X'_t \beta^0 + \varepsilon_t$, we have

$$\hat{\beta} - \beta^0 = \hat{Q}^{-1} n^{-1} \sum_{t=1}^{n} X_t \varepsilon_t$$

Since X_t is ergodic stationary, $\{X_t X'_t\}$ is also ergodic stationary.

$$\hat{Q} \xrightarrow{p} E\left(X_t X'\right) = Q$$

$$\hat{Q}^{-1} \xrightarrow{p} Q^{-1}$$

Since $\{Y_t, X'_t\}$ is ergodic stationary, so $\varepsilon_t = Y_t - X'_t\beta^0$ is ergodic stationary, so is $X_t\varepsilon_t$.

$$n^{-1} \sum_{t=1}^{n} X_t \varepsilon_t \xrightarrow{p} E\left(X_t \varepsilon_t\right) = E\left[E\left(X_t\varepsilon_t \left| X_t\right.\right)\right] = E\left[X_t E\left(\varepsilon_t \left| X_t\right.\right)\right] = E\left(X_t \cdot 0\right) = 0$$

Thus,

$$\hat{\beta} - \beta^0 = \hat{Q}^{-1} n^{-1} \sum_{t=1}^{n} X_t \varepsilon_t = 0$$

This completes the proof.

2. Since $\sqrt{n}\left(\hat{\beta} - \beta^0\right) = \hat{Q}^{-1} n^{-1} \sum\limits_{t=1}^{n} X_t\varepsilon_t$, and $n^{-1} \sum\limits_{t=1}^{n} X_t\varepsilon_t = \sqrt{n}\bar{Z}_n$, $\bar{Z}_n = n^{-1} \sum\limits_{t=1}^{n} Z_t$, $Z_t = X_t\varepsilon_t$.
Because $\{Y_t, X_t\}$ is ergodic stationary, so $X_t\varepsilon_t$ is also stationary ergodic.

$$V = E\left(X_t X'_t \varepsilon_t^2\right) = Var\left(X_t\varepsilon_t\right)$$

$$n^{-1}\sum_{t=1}^{n} X_t\varepsilon_t \xrightarrow{d} N\left(0, V\right)$$

$$\hat{Q}^{-1} \xrightarrow{p} Q^{-1}$$

$$\sqrt{n}\left(\hat{\beta}-\beta^0\right)=\hat{Q}^{-1}n^{-1}\sum_{t=1}^{n} X_t\varepsilon_t \xrightarrow{d} N\left(0, Q^{-1}VQ^{-1}\right)$$

3.

$$\sqrt{n}\left(\hat{\beta}-\beta^0\right)\sim N\left(0, Q^{-1}VQ^{-1}\right)$$

$$\sqrt{n}R\left(\hat{\beta}-\beta^0\right)\sim N\left(0, RQ^{-1}VQ^{-1}R'\right)$$

$$\sqrt{n}\left(R\hat{\beta}-\beta^0\right)\left(RQ^{-1}VQ^{-1}R'\right)^{-1}\sqrt{n}\left(R\hat{\beta}-\beta^0\right)\sim \chi^2_J$$