Homework 2

1. Suppose the following assumptions hold:
Assumption 1.1: $\{Y_t, X_t'\}'$ is an i.i.d. random sample with

$$Y_t = X'_t\beta^0 + \varepsilon_t,$$

for some unknown parameter β^0 and unobservable random disturbance ε_t.
Assumption 1.2: $E\left(\varepsilon_t \left| X_t \right.\right) = 0$ a.s.
Assumption 1.3:
(i) $W_t = W\left(X_t\right)$ is a positive function of X_t;
(ii) The $K \times K$ matrix $E\left(X_t W X'_t\right) = Q_w$ is finite and nonsingular.
(iii) $E\left(W_t^8\right) \le C < \infty$, $E\left(X_{jt}^8\right) \le C < \infty$ for all $0 \le j \le k$, and $E\left(\varepsilon_t^4\right) \le C$;
Assumption 1.4: $V_w = E\left(X_t W_t^2 X'_t \varepsilon_t^2\right)$ is finite and nonsingular.

We consider the so-called weighted least squares (WLS) estimator for β^0:

$$\hat{\beta}_w = \left(n^{-1}\sum_{t=1}^{n} X_t W X'_t\right)^{-1} n^{-1}\sum_{t=1}^{n} X_t W Y_t.$$

(a) Show that $\hat{\beta}_w$ is the solution to the following problem

$$\min_{\beta}\sum_{t=1}^{n} W_t (Y_t - X'_t\beta)^2.$$

(b) Show that $\hat{\beta}_w$ is consistent for β^0;
(c) Show that $\sqrt{n}\left(\hat{\beta}_w - \beta^0\right) \xrightarrow{d} N\left(0, \Omega_w\right)$ for some $K \times K$ finite and positive definite matrix Ω_w. Obtain the expression of Ω_w under (i) conditional homoskedasticity $E\left(\varepsilon_t^2 \left| X_t \right.\right) = \sigma^2$ a.s. and (ii) conditional heteroskedasticity $E\left(\varepsilon_t^2 \left| X_t \right.\right) \ne \sigma^2$.
(d) Propose an estimator $\hat{\Omega}_w$ for Ω_w, and show that $\hat{\Omega}_w$ is consistent for Ω_w under conditional homoskedasticity and conditional heteroskedasticity respectively.
(e) Construct a test statistic for $H_0: \ R\beta^0 = r$, where r is a $J \times K$ matrix and r is a $J \times 1$ vector under conditional homoskedasticity and under conditional heteroskedasticity respectively. Derive the asymptotic distribution of the test statistic under H_0 in each case.
(f) Suppose $E\left(\varepsilon_t^2 \left| X_t \right.\right) = \sigma^2\left(X_t\right)$ is known, and we set $W_t = \sigma^{-1}\left(X_t\right)$. Construct a test statistic for $H_0: \ R\beta^0 = r$, where r is a $J \times K$ matrix and r is a $J \times 1$ vector. Derive the asymptotic distribution of the test statistic under H_0.

2. Consider the problem of testing conditional homoskedasticity $\left(H_0: \ E\left(\varepsilon_t^2 \left| X_t \right.\right) = \sigma^2 \right)$ for a linear regression model

$$Y_t = X'_t\beta^0 + \varepsilon_t,$$

where X_t is a $K \times 1$ vector consisting of an intercept and explanatory variables. To test conditional homoskedasticity, we consider the auxiliary regression

$$\begin{aligned}\varepsilon_t &= vech(X_t X'_t)'\gamma + v_t \\ &= U'_t\gamma + v_t\end{aligned}$$

Suppose Assumptions 4.1, 4.2, 4.3, 4.4, 4.7 hold, and $E\left(\varepsilon_t^4 \,|X_t\right) \neq \mu_4$. That is , $E\left(\varepsilon_t^4 \,|X_t\right)$ is a function of X_t.
(a) Show $\mathrm{var}\,(v_t \,|X_t\,) \neq \sigma_v^2$ under H_0. That is, the disturbance v_t in the auxiliary regression model displays conditional heteroskedasticity.
(b) Suppose ε_t is directly observable. Construct an asymptotically valid test for the null hypothesis H_0 of conditional homoskedasticity. Justify your reasoning and test statistic.