Advanced Microeconomics II

Finitely Repeated Games

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Repeated Games

Modeling repeated games examines the potential implications of long-term interactions.

- current actions influence future behaviour.
- allows for cooperation, revenge, threats.
- to sustain cooperation player's need to be
  - rewarded for cooperation;
  - punished for cheating (defecting);
- If we use SPE, punishments must be credible. Players must be sufficiently compensated for punishing cheaters.

Finitely Repeated Games

Definition

For any positive integer T, a T-period finitely repeated game of the strategic game $\{N, (A_i), (u_i)\}$ is an extensive game with perfect information and simultaneous moves $\{N, H, P, (\succeq_i^*)\}$ in which

- $H = \{\varnothing\} \cup (\cup_{t=1}^{T} A^t)$ where A^t is the set of possible sequences of outcomes in A of length t.
- $P(h) = N$ for each nonterminal history $h \in H$.
- the preference relation $\succeq_i^*$ of player i on each terminal history $h \in Z$ is represented by the function $\sum_{t=1}^{T} u_i(a^t)/T$.

Prisoner's Dilemma Example

	C	D
C	$3,3$	$1,4$
D	$4,1$	$2,2$

- Players play the Prisoner's dilemma for T periods.
- For each $h \in Z$, $u_i(h) = \sum_{t=1}^{T} u_i(a^t)/T$.

Consider the following symmetric strategies.

- Always cooperate: after any history play C.
- Never cooperate: after any history play D.
- Tit-for- tat: start with C, then play whatever my opponent played last period.
- Grim trigger: start with C, play C in period t if h is (C, C) in every previous period, otherwise play D.

- Which of these are Nash equilibrium strategies?
- Which of these are SPE strategies?

Enforceable Outcomes

For every $a \in A$ denote by $u(a)$ the vector $(u_i(a))_{i \in N}$.

Definition

Player i's minmax payoff in G (denoted v_i) is

$$v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

Definition

A payoff profile w is enforceable if $w_i \geq v_i$ for all $i \in N$. A payoff profile w is strictly enforceable if $w_i > v_i$ for all $i \in N$. An outcome $a \in A$ is a (strictly) enforceable outcome of G if $u(a)$ is (strictly) enforceable.

Denote by $p_{-i}(i)$ the solution to player i's minmax problem.

Enforceable Outcomes - Examples

	C	D
C	3, 3	1, 4
D	4, 1	2, 2

	C	D	E
C	3, 3	1, 4	0, 0
D	4, 1	2, 2	0.5, 0
E	0, 0	0, 0.5	0, 0

	C	D	E
C	3, 3	1, 4	0, 0
D	4, 1	2, 2	0, 0
E	0, 0	0, 0	0.5, 0.5

In each game

- what is the set of pure strategy Nash equilibria?
- what is player 1's minmax payoff?
- what is player 2's minmax payoff?
- what are the set of enforceable outcomes?

Nash Equilibria

Proposition

If the payoff profile in every Nash equilibrium of the strategic game G is the profile (v_i) of minmax payoffs in G then for any value of T the outcome $(a^1, \ldots, a^T)$ of every Nash equilibrium of the T-period repeated game of G has the property that a^t is a Nash equilibrium of G for all $t = 1 \ldots, T$.

- Suppose t is the latest period for which a^t is not a Nash equilibrium of G.
- There exists some player who can do better in period t. Thus, he has a profitable deviation.
  - Play his best strategy in period t.
  - After t play a strategy that gives him at least his minmax payoff (depends on $s_{-i}(h)$).

Examples

	C	D
C	3, 3	1, 4
D	4, 1	2, 2

	C	D	E
C	3, 3	1, 4	0, 0
D	4, 1	2, 2	0.5, 0
E	0, 0	0, 0.5	0, 0

	C	D	E
C	3, 3	1, 4	0, 0
D	4, 1	2, 2	0, 0
E	0, 0	0, 0	0.5, 0.5

For which of these examples do the conditions of the proposition apply?

Examples

	C	D	E
C	$3,3$	$1,4$	$0,0$
D	$4,1$	$2,2$	$0.5,0$
E	$0,0$	$0,0.5$	$0,0$

Are there other Nash equilibria?

- Can we sustain an average payoff for player 2 above 2?
  - $s_i(\varnothing) = C$
  - For $t = 1, \ldots, T-1$, $s_i(h^{t-1}) = C$ if h contains only (C, C), otherwise play E.
  - For $t = T$, $s_i(h^{t-1}) = D$ if h contains only (C, C), otherwise play E.
  - What is the average payoff as T gets large.

Examples

	C	D	E
C	$3,3$	$1,4$	$0,0$
D	$4,1$	$2,2$	$0.5,0$
E	$0,0$	$0,0.5$	$0,0$

Are there other Nash equilibria?

- Can we sustain an average payoff for player 2 below 2?
  - $s_1(\varnothing) = D$, $s_2(\varnothing) = C$
  - For $t = 1, \ldots, T-1$, $s_1(h^{t-1}) = D$ if h contains only (D, C), otherwise play E.
  - For $t = 1, \ldots, T-1$, $s_2(h^{t-1}) = C$ if h contains only (D, C), otherwise play E.
  - For $t = T$, $s_1(h^{t-1}) = D$ if h contains only (D, C), otherwise play E.
  - For $t = T$, $s_2(h^{t-1}) = D$ if h contains only (D, C), otherwise play E.
  - What is the average payoff as T gets large.

Nash Folk Theorem for Finitely Repeated Games

Proposition

If $G = \{N, (A_i), (u_i)\}$ has a Nash equilibrium $\hat{a}$ in which the payoff of every player i exceeds his minmax payoff v_i then for any strictly enforceable outcome a^* of G and any $\epsilon > 0$ there exists an integer T^* such that if $T > T^*$ the T-period repeated game of G has a Nash equilibrium in which the payoff of each player i is within ϵ of $u_i(a^*)$.

Denote $p(j)$ as the profile of strategies that gives player j his minmax payoff.

- Each player starts by playing a_i^*.
- At time $t \leq T - L$ play a_i^* if nobody has deviated. If one player (say j) deviated at $t-1$ play $p(j)_i$ forever after.
- From $T - L < t \leq T$ if nobody has deviated for each $t < T - L$ play $\hat{a}_i$.

Nash Folk Theorem for Finitely Repeated Games

Need to ensure no profitable deviation. Requires that L is large enough so that

$$\max_{a_i \in A_i} u_i(a_i, a_{-i}^*) - u_i(a^*) \leq L(u_i(\hat{a}) - v_i) \text{ for all } i \in N.$$

Need payoffs to be within ϵ of $u_i(a^*)$. Choose T^* such that

$$\left| \frac{(T^* - L)u_i(a^*) + Lu_i(\hat{a})}{T^*} - u_i(a^*) \right| < \epsilon \text{ for all } i \in N.$$

Are these SPE strategies?

Examples

	C	D	E
C	$3, 3$	$1, 4$	$0, 0$
D	$4, 1$	$2, 2$	$0.5, 0$
E	$0, 0$	$0, 0.5$	$0, 0$

- For the outcome (C, C), what is L, what is T^*?
- For the outcome (D, C), what is L, what is T^*?

Subgame Perfect Equilibrium

Proposition

If the strategic game G has a unique Nash equilibrium payoff profile then for any value of T the action profile chosen after any history in any subgame perfect equilibrium of the T-period repeated game of G is a Nash equilibrium of G.

- In any subgame that starts in period T the outcome must be a Nash equilibrium of G
- Since player payoffs in period T are independent of history, the outcome in $T - 1$ must be a Nash equilibrium.
- And so on...

Examples

	C	D	E
C	$3, 3$	$1, 4$	$0, 0$
D	$4, 1$	$2, 2$	$0, 0$
E	$0, 0$	$0, 0$	$0.5, 0.5$

What if there are multiple equilibria?

- Can we sustain average payoffs above $(2, 2)$?
  - $s_i(\varnothing) = C$
  - For $t = 1, \dots, T - 1$, $s_i(h^{t-1}) = C$ if h contains only (C, C), otherwise play E.
  - For $t = T$, $s_i(h^{t-1}) = D$ if h contains only (C, C), otherwise play E.
  - What is the average payoff as T gets large.
- Is this an SPE?