Advanced Microeconomics II
Problem Set 2

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Due 10:00 Apr 9, 2011

1. (Gibbons 1.2) Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, s_1 and s_2, where $0 \le s_i \le 1, i = 1, 2$. If $s_1 + s_2 \le 1$, then the players receive the shares they named; if $s_1 + s_2 > 1$, then both players receive zero.
   (a) Formulate this as a strategic game.
   (b) What are the pure-strategy Nash equilibria of this game?

Solution: From a, we can get the best response function of each player.

$$B_i(s_j) = \begin{cases} 1 - s_j, & \text{if } 0 \le s_j < 1, \\ [0, 1], & \text{if } s_j = 1. \end{cases}$$

Thus we know the pure-strategy Nash equilibria of this game are $(s, 1 - s)$, where $s \in [0, 1]$ and $(1, 1)$.

   (c) Derive a mixed strategy equilibrium where each player randomly choose one of two numbers.
   (d) Consider a symmetric mixed strategy equilibrium $F(.)$, where the support of the mixed strategy is an interval $[a, b]$.
      i. Show that to be a mixed strategy equilibrium, a and b should satisfy $a + b \le 1$.
      ii. Show that to be a mixed strategy equilibrium, a and b should satisfy $a + b \ge 1$.
      iii. Show that to be a mixed strategy equilibrium the probability that $s_i = a$ is not atomless, i.e. $F(a) > 0$.
      iv. Derive such a mixed strategy equilibrium.

2. (Gibbons 1.5) Consider the following two finite versions of the Cournot duopoly model. $P(Q) = a - Q$ is the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. The total cost to firm i of producing quantity q_i is cq_i, where $c < a$.
   (a) First, suppose each firm must choose either half the monopoly quantity, $q_m/2 = (a-c)/4$, or the Cournot equilibrium quantity, $q_c = (a - c)/3$. No other quantities are feasible. Show that this two-action game is equivalent to the Prisoner's Dilemma: each firm has a strictly dominated strategy, and both are worse off in equilibrium than they would be if they cooperated.
   (b) Second, suppose each firm can choose either $q_m/2$, or q_c, or a third quantity, q'. Find a value for q' such that the game is equivalent to the Cournot model presented in class, in the sense that (q_c, q_c) is a unique Nash equilibrium and both firms are worse off

in equilibrium than they could be if they cooperated, but neither firm has a strictly dominated strategy.

3. Consider the strategic game $G = \{N, (A_i), (u_i)\}$. For each $i \in N$, let A_i be a nonempty compact convex subset of Euclidean space and the utility function u_i be continuous and quasi-concave on A_i.
   (a) Prove that $B(a) = \times_{i \in N} B_i(a_{-i})$ is convex, where $B_i(a_{-i})$ is the best response function of player i, i.e. show that if $b \in B(a)$ and $c \in B(a)$ then for any $\lambda \in [0,1]$, $\lambda b + (1-\lambda)c \in B(a)$.
   (b) Let A_i be finite for each $i \in N$. Prove that for each player i, the U_i associated with the mixed extension of G is quasi-concave over $\times_{j \in N} \Delta(A_j)$.

4. Consider the following 2 player game.

	L	R
U	$6,6$	$2,7$
D	$7,2$	$0,0$

   (a) Find the correlated equilibrium that maximizes the sum of the two players payoffs.
   (b) Construct a correlated equilibrium that generates a payoff for both players of $(19/4, 19/4)$.

5. Consider the following three-player game where $A_1 = \{U, D\}$, $A_2 = \{L, R\}$ and $A_3 = \{M_1, M_2, M_3, M_4\}$. The number in each box represents the (equal) payoff to each player.

M_1

	L	R
U	3	0
D	0	0

M_2

	L	R
U	0	3
D	3	0

M_3

	L	R
U	0	0
D	0	3

M_4

	L	R
U	2	0
D	0	2

   (a) What are the set of rationalizable strategies if players beliefs allow correlation between opponents strategies?
   (b) What are the set of rationalizable strategies if players beliefs are restricted to be products of independent probability distributions over opponents strategies?

6. Show that if there is a unique profile of actions that survives iterated elimination of strictly dominated actions, this profile is a Nash equilibrium.

Solution:

Proof 2: Let a^* be the unique outcome that survives iterated elimination of strictly dominated strategies. Since X^T is a singleton it is clear that a^* is a Nash equilibrium of $\{N, (X_i^T), (u_i^T)\}$. Now assume that a^* is a Nash equilibrium of $\{N, (X_i^{t+1}), (u_i^{t+1})\}$ where $0 \leq t < T$, i.e. $u_i^{t+1}(a_i^*, a_{-i}^*) \geq u_i^{t+1}(a_i, a_{-i}^*)$ for all $a_i \in X_i^{t+1}$. Let $a_i' \in X_i \setminus X_i^{t+1}$ for some $i \in N$. Since $a_i' \in X_i \setminus X_i^{t+1}$, a_i' is strictly dominated in $\{N, (X_i^t), (u_i^t)\}$. Hence, a_i' is a never-best response in $\{N, (X_i^t), (u_i^t)\}$. Hence, $u_i^t(a_i', a_{-i}^*) < u_i^t(a^*)$. Since a_i' was chosen arbitrarily it must be that $u_i^t(a_i^*, a_{-i}^*) \geq u_i^t(a_i, a_{-i}^*)$ for all $a_i \in X_i^t$, i.e. a_i^* is a best response to a_{-i}^* in $\{N, (X_i^t), (u_i^t)\}$. Hence, a^* is a Nash equilibrium for $\{N, (X_i^0), (u_i^0)\}=\{N, (A_i), (u_i)\}$.

Proof 3: Let a^* be the unique outcome that survives iterated elimination of strictly dominated strategies. Since a^* survives iterated elimination of strictly dominated strategies a_i^* is rationalizable for each player. Since a^* is unique, the belief that supports a_i^* as a rationalizable action puts probability one on a_{-i}^*. Hence $a_i^* \in B(a_{-i}^*)$ for all $i \in N$, thus a^* is a Nash equilibrium.