Advanced Microeconomics II

Bayesian Extensive Games With Observable Actions

Brett Graham

Wang Yanan Institute for Studies in Economics
Xiamen University, China

April 1, 2015

Bayesian Extensive Game With Observable Actions

First let's extend Bayesian games.

Definition

A Bayesian extensive game with observable actions is a tuple $\{\Gamma, (\Theta_i), (p_i), (u_i)\}$ where

- $\Gamma = \{N, H, P\}$ is an extensive game form with perfect information and simultaneous moves

and for each player $i \in N$

- Θ_i is a finite set (the set of possible types of player i); $\Theta = \times_{\{i \in N\}} \Theta_i$
- p_i is a probability measure on Θ_i for which $p_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, and the measures p_i are stochastically independent ($p_i(\theta_i)$ is the probability that player i is selected to be of type θ_i)
- $u_i : \Theta \times Z \to \mathcal{R}$ is a von Neumann-Morgenstern utility function ($u_i(\theta, h)$ is player i's payoff when the profile of types is θ and the terminal history of Γ is h).

Bayesian Extensive Game With Observable Actions

We can associate with any such game an extensive game (with imperfect information and simultaneous moves) in which

- the set of histories is $\{\varnothing\} \cup (\Theta \times H)$ and
- each information set of each player i takes the form

$$I_i(\theta, h) = \{(\theta', h) : \theta' \in \Theta \text{ and } \theta'_i = \theta_i\}$$

  for $i \in P(h)$ (so that the number of histories in $I(\theta, h)$ is the number of members of Θ_{-i}).

Interpretation: Chance first chooses player types. The (otherwise perfect) game is then played.

Example - Tough Chain Store Game

- Chance chooses a Chain Store type: $\Theta_{CS} = \{R(egular), T(ough)\}$.
- The Chain Store is 'Tough' with probability ϵ.
  - A 'Tough' chain store prefers to fight than to cooperate.
- The standard chain-store game is then played.
- The payoff to the potential entrant is

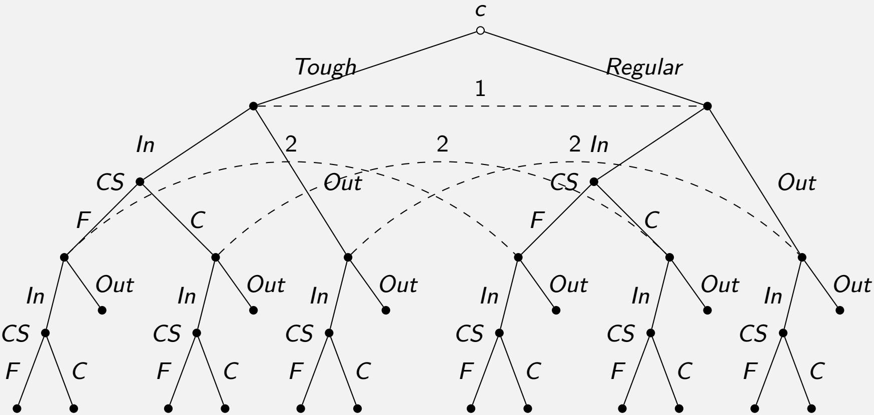
$$u_k(\theta, h) = \begin{cases} b & \text{if } h_k = (In, C) \\ b - 1 & \text{if } h_k = (In, F) \\ 0 & \text{if } h_k = Out, \end{cases}$$

  where $0 < b < 1$.
- The payoff to the chain-store in each market is $u_{CS}(\theta, h) =$

$$\begin{cases} 0 & \text{if } h_k = (In, C) \text{ and } \theta_{CS} = R, \text{ or } h_k = (In, F) \text{ and } \theta_{CS} = T \\ -1 & \text{if } h_k = (In, F) \text{ and } \theta_{CS} = R, \text{ or } h_k = (In, C) \text{ and } \theta_{CS} = T \\ a & \text{if } h_k = Out, \end{cases}$$

  where $a > 1$.

Example - Tough Chain Store Game



Homework: Write down this extensive form game.

Signalling Games

The simplest type of Bayesian extensive game with observable actions.

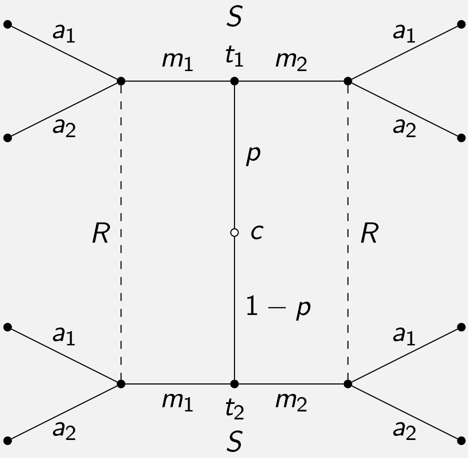
Definition

A signalling game is a Bayesian extensive game with observable actions $\{\Gamma, (\Theta_i), (p_i), (u_i)\}$ in which

- $N = \{S, R\}$.
- $P(\varnothing) = S$ (The 'sender' plays first).
- $P(h) = R$ for $h \in A(\varnothing)$. (The 'receiver' plays second).
- Histories have at most length 2. (The game then ends).
- Θ_R is a singleton. (The 'receiver' has one type).

Interpretation: The sender sends a message about his type. The receiver observes the message and chooses an action. Payoffs are a function of type, message and action.

Extensive Game Form Simple Example



How many pure strategies does each player have?

Strategies

- The first and last strategies of the sender are pooling strategies.
- The second and third strategies of the sender are separating strategies.
- If we consider mixed strategies we can have hybrid strategies.
- If we have more than two types we can have partial pooling/semi-separating strategies.

Equilibrium Requirements - Beliefs

Signalling Requirement 1 After observing any message $m_j \in A(\varnothing)$ the receiver must have a belief about which types could have sent m_j. Denote this belief by the probability distribution $\mu_S(m_j)(t_S)$, where $\mu_S(m_j)(t_S) \geq 0$ for each $t_S \in \Theta_S$, and

$$\sum_{t_S \in \Theta_S} \mu_S(m_j)(t_S) = 1.$$

Equilibrium Requirements - Rationality

Signalling Requirement 2R[*Receiver rationality*] The Receiver's strategy must be optimal. For each message $m_j \in A(\varnothing)$, $s_R^*(m_j)$ solves

$$\max_{a_k \in A(m_j)} \sum_{t_S \in \Theta_S} \mu_S(m_j)(t_S) u_R(t_S, (m_j, a_k)).$$

Signalling Requirement 2S[*Sender rationality*] The Sender's strategy must be optimal. For each type $t_S \in \Theta_S$, $s_S^*(t_S)$ solves

$$\max_{m_j \in A(\varnothing)} u_S(t_S, (m_j, s_R^*(m_j))).$$

Equilibrium Requirements - Bayesian Updating

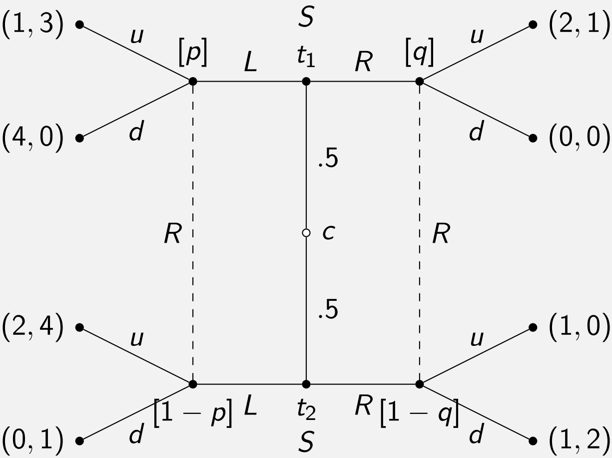
Signalling Requirement 3 For each $m_j \in A(\varnothing)$,

- if there exists $t_S \in \Theta_S$ such that $s_S^*(t_S) = m_j$, then for each $t'_S \in \Theta_S$

$$\mu_S(m_j)(t'_S) = \begin{cases} \frac{p(t'_S)}{\sum\limits_{\{\tilde{t}_S \in \Theta_S | s_S^*(\tilde{t}_S) = m_j\}} p(\tilde{t}_S)} & \text{if } s_S^*(t'_S) = m_j \\ 0 & \text{otherwise.} \end{cases}$$

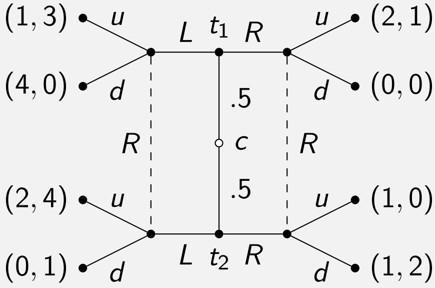
- if there does not exists $t_S \in \Theta_S$ such that $s_S^*(t_S) = m_j$ then what should we do?

Signalling Game - Simple Example



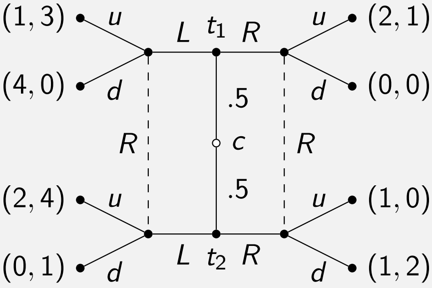
$$p = \mu_S(L)(t_1);\ q = \mu_S(R)(t_1).$$

Receiver Optimal Strategy



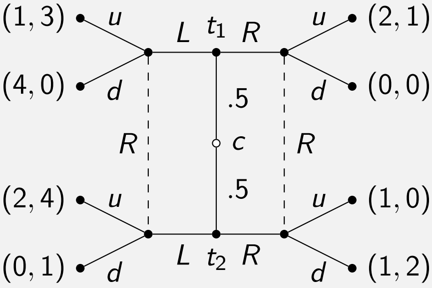
- For all p, $\beta_R(L)(u) = 1$.
- If $q < 2/3$, then $\beta_R(R)(u) = 0$
- If $q = 2/3$, then $\beta_R(R)(u) \in [0, 1]$
- If $q > 2/3$, then $\beta_R(R)(u) = 1$

Type 2 Sender Optimal Strategy



- $\beta_S(t_2)(L) = 1$.
  - Since for all p, $\beta_R(L)(u) = 1$.

Type 1 Sender Optimal Strategy



- If $\beta_R(R)(u) < 1/2$, then $\beta_S(t_1)(L) = 1$.
- If $\beta_R(R)(u) = 1/2$, then $\beta_S(t_1)(L) \in [0, 1]$.
- If $\beta_R(R)(u) > 1/2$, then $\beta_S(t_1)(L) = 0$.

Equilibria

Type 1a:

- $\beta_S(t_1)(L) = 1$, $\beta_S(t_2)(L) = 1$.
- $\beta_R(L)(u) = 1$, $\beta_R(R)(u) = 0$.
- $p = 0.5$, $q < 2/3$.

Type 1b:

- $\beta_S(t_1)(L) = 1$, $\beta_S(t_2)(L) = 1$.
- $\beta_R(L)(u) = 1$, $0 \leq \beta_R(R)(u) \leq 1/2$.
- $p = 0.5$, $q = 2/3$.

Type 2:

- $\beta_S(t_1)(L) = 0$, $\beta_S(t_2)(L) = 1$.
- $\beta_R(L)(u) = 1$, $\beta_R(R)(u) = 1$.
- $p = 0$, $q = 1$.

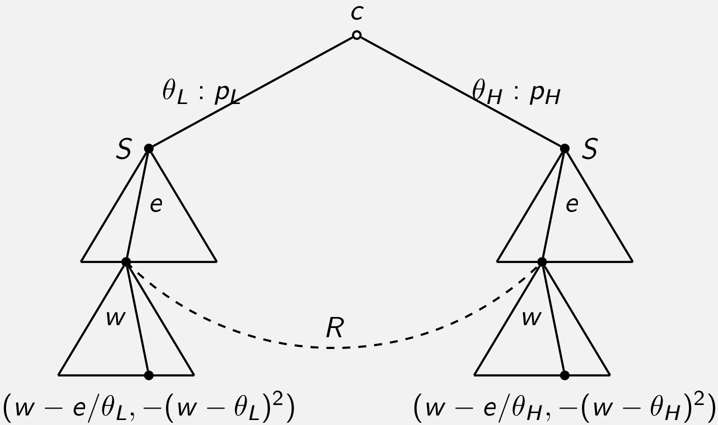
Classify each as a pooling, separating or hybrid equilibrium.

Spence's Model of Education

A worker knows her talent $\theta \in \{\theta_L, \theta_H\}$, while her employer does not. A worker has productivity θ_L with probability p_L and productivity θ_H with probability $p_H = 1 - p_L$. The value of the worker to the employer is θ, but the employer pays the worker a wage w that is equal to the expectation of θ (there is a competitive labour market).

- The worker chooses an amount of education $e \in [0, \infty)$.
- Employer makes an offer $w \in [\theta_L, \theta_H]$ to the worker.
- Payoffs: The worker's payoff is $w - e/\theta$ and the employer's payoff is $-(w - \theta)^2$.

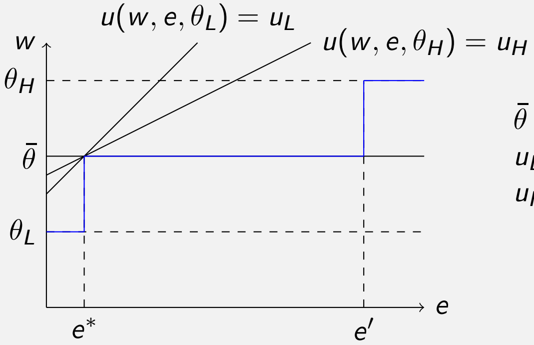
Example - Model of Education Game Tree



Pooling Equilibrium

$$S : e(\theta_H) = e^*;\ e(\theta_L) = e^*.$$

$$R : \mu_S(e)(\theta_H) = \begin{cases} 1 & \text{if } e' \le e \\ p_H & \text{if } e^* \le e < e' \\ 0 & \text{otherwise} \end{cases};\ w(e) = \begin{cases} \theta_H & \text{if } e' \le e \\ \bar{\theta} & \text{if } e^* \le e < e' \\ \theta_L & \text{otherwise.} \end{cases}$$

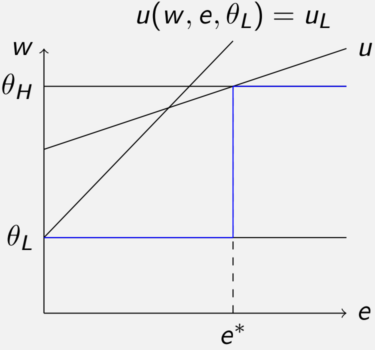


$$\bar{\theta} = p_H\theta_H + p_L\theta_L;$$
$$u_L = u(\bar{\theta}, e^*, \theta_L);$$
$$u_H = u(\bar{\theta}, e^*, \theta_H).$$

Separating Equilibrium

$$S : e(\theta_H) = e^*;\ e(\theta_L) = 0.$$

$$R : \mu_S(e)(\theta_H) = \begin{cases} 1 & \text{if } e^* \leq e \\ 0 & \text{otherwise} \end{cases};\ w(e) = \begin{cases} \theta_H & \text{if } e^* \leq e \\ \theta_L & \text{otherwise.} \end{cases}$$

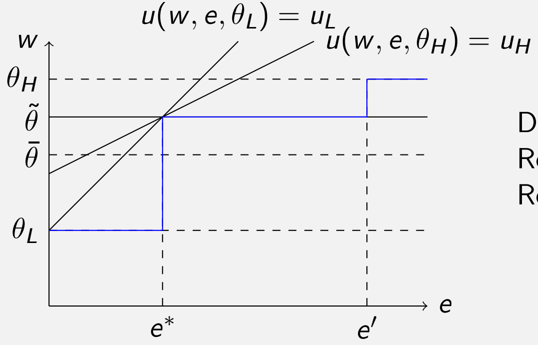


$$u_L = u(\theta_L, 0, \theta_L);$$
$$u_H = u(\theta_H, e^*, \theta_H).$$

Hybrid Equilibrium

$$S: e(\theta_H) = e^*;\ e(\theta_L) = \begin{cases} 0 & \text{with probability } \lambda \\ e^* & \text{with probability } 1-\lambda. \end{cases}$$

$$R: \mu_S(e)(\theta_H) = \begin{cases} 1 & \text{if } e' \leq e \\ \tilde{p} & \text{if } e^* \leq e < e' \\ 0 & \text{otherwise.} \end{cases};\ w(e) = \begin{cases} \theta_H & \text{if } e' \leq e \\ \tilde{\theta} & \text{if } e^* \leq e < e' \\ \theta_L & \text{otherwise.} \end{cases}$$



Define $\lambda, \tilde{p}, \tilde{\theta}$.

Restrictions on e'?

Restrictions on e^*?

Perfect Bayesian Equilibrium

Definition

Let $\{\Gamma, (\Theta_i), (p_i), (u_i)\}$ be a Bayesian extensive game with observable actions, where $\Gamma = \{N, H, P\}$. A perfect Bayesian equilibrium of the game is a pair $((\sigma_i), (\mu_i)) = ((\sigma_i(\theta_i))_{i \in N, \theta_i \in \Theta_i}, (\mu_i(h))_{i \in N, h \in H \setminus Z})$, where $\sigma_i(\theta_i)$ is a behavioral strategy of player i in Γ and $\mu_i(h)$ is a probability measure on θ_i and the following conditions are satisfied.

- *Correct initial beliefs* $\mu_i(\varnothing) = p_i$ for each $i \in N$.
- *Action-determined beliefs* If $i \notin P(h)$ and $a \in A(h)$ then $\mu_i(h, a) = \mu_i(h)$; if $i \in P(h)$, $a \in A(h)$, $a' \in A(h)$, and $a_i = a'_i$ then $\mu_i(h, a) = \mu_i(h, a')$.
- *Sequential rationality*
- *Bayesian updating*

Perfect Bayesian Equilibrium

- *Bayesian updating* If $i \in P(h)$ and a_i is in the support of $\sigma_i(\theta_i)(h)$ for some θ_i in the support of $\mu_i(h)$ then for any $\theta_i' \in \Theta_i$ we have

$$\mu_i(h, a)(\theta_i') = \frac{\sigma_i(\theta_i')(h)(a_i) \cdot \mu_i(h)(\theta_i')}{\sum_{\tilde{\theta}_i \in \Theta_i} \sigma_i(\tilde{\theta}_i)(h)(a_i) \cdot \mu_i(h)(\tilde{\theta}_i)}.$$

- *Sequential rationality* For every nonterminal history $h \in H \backslash Z$, every player $i \in P(h)$, and every $\theta_i \in \Theta_i$

$$O(\sigma_i(\theta_i), \sigma_{-i}, \mu_{-i}|h) \succeq_i O(s_i, \sigma_{-i}, \mu_{-i}|h)$$

for any strategy s_i of player i in Γ.

Chain-Store Equilibrium - Chain-store

- $\mu_{CS}(h)(T)$: the belief by the potential entrants after history h that the chain-store is tough.
- $t(h)$: the number of potential entrants who have moved.

Regular Chain-store strategy

$$\sigma_{CS}(R)(h) = \begin{cases} C & \text{if } t(h) = K \\ F & \text{if } t(h) < K \text{ and } \mu_{CS}(h)(T) \geq b^{K-t(h)} \\ m_{CS}^h & \text{if } t(h) < K \text{ and } \mu_{CS}(h)(T) < b^{K-t(h)} \end{cases}$$

if $P(h) = CS$, where m_{CS}^h is the mixed strategy such that

$$m_{CS}^h(F) = \frac{\left(1 - b^{K-t(h)}\right) \mu_{CS}(h)(T)}{\left(1 - \mu_{CS}(h)(T)\right) b^{K-t(H)}}.$$

Tough Chain-store strategy

$$\sigma_{CS}(T)(h) = F \text{ if } P(h) = CS.$$

Chain-Store Equilibrium - Potential Entrant

Potential entrant k strategy

$$\sigma_k(h) = \begin{cases} Out & \text{if } \mu_{CS}(h)(T) > b^{K-k+1} \\ m_k & \text{if } \mu_{CS}(h)(T) = b^{K-k+1} \\ In & \text{if } \mu_{CS}(h)(T) < b^{K-k+1} \end{cases}$$

if $P(h) = k$, (so that $t(h) = k - 1$), where m_k is the mixed strategy such that

$$m_k(Out) = 1/a.$$

Chain-Store Equilibrium - Beliefs

- Correct initial beliefs: $\mu_{CS}(\varnothing)(T) = \epsilon$.
- For any history h with $P(h) = k$, $\mu_{CS}(h, h^k)(T) =$

$$\begin{cases} \max\{b^{K-k}, \mu_{CS}(h)(T)\} & \text{if } h^k = (In, F) \text{ and } \mu_{CS}(h)(T) > 0 \\ 0 & \text{if } h^k = (In, C) \text{ or } \mu_{CS}(h)(T) = 0 \\ \mu_{CS}(h)(T) & \text{if } h^k = (Out). \end{cases}$$