Homework 1 solution
Econometrics II Spring, 2013

1. Show that $T_n = 2[Ln(\hat{\theta}_n) - Ln(\hat{\theta}_n^*)] \to \chi_r^2$,

where $\hat{\theta}_n^* = \hat{\theta}_n - I(\hat{\theta}_n)^{-1}\frac{\partial g^T(\hat{\theta}_n)}{\partial\theta}[\frac{\partial g(\hat{\theta}_n)}{\partial\theta^T}I(\hat{\theta}_n)^{-1}\frac{\partial g^T(\hat{\theta}_n)}{\partial\theta}]^{-1}g(\hat{\theta}_n)$

Proof:

$$\begin{aligned}
T_n &= 2[Ln(\hat{\theta}_n) - Ln(\hat{\theta}_n^*)] \\
&\approx 2[Ln(\hat{\theta}_n) - Ln(\hat{\theta}_n) - \frac{\partial Ln(\hat{\theta}_n)}{\partial\theta^T}(\hat{\theta}_n^* - \hat{\theta}_n) - \frac{1}{2}(\hat{\theta}_n^* - \hat{\theta}_n)^T\frac{\partial^2 Ln(\hat{\theta}_n)}{\partial\theta\partial\theta^T}(\hat{\theta}_n^* - \hat{\theta}_n)] \\
&\approx n(\hat{\theta}_n^* - \hat{\theta}_n)^T I(\theta_0)(\hat{\theta}_n^* - \hat{\theta}_n) \\
&\approx ng(\hat{\theta}_n)^T[\frac{\partial g(\hat{\theta}_n)}{\partial\theta^T}I(\hat{\theta}_n)^{-1}\frac{\partial g^T(\hat{\theta}_n)}{\partial\theta}]^{-1}g(\hat{\theta}_n)
\end{aligned}$$

Under H_0,

$$\sqrt{n}g(\hat{\theta}_n) \xrightarrow{d} N(0, \frac{\partial g(\theta_0)}{\partial\theta^T}I(\theta_0)^{-1}\frac{\partial g^T(\theta_0)}{\partial\theta})$$

and note that

$$\frac{\partial g(\hat{\theta}_n)}{\partial\theta^T}I(\hat{\theta}_n)^{-1}\frac{\partial g^T(\hat{\theta}_n)}{\partial\theta} \xrightarrow{p} \frac{\partial g(\theta_0)}{\partial\theta^T}I(\theta_0)^{-1}\frac{\partial g^T(\theta_0)}{\partial\theta}$$

Therefore

$$T_n \approx ng(\hat{\theta}_n)^T[\frac{\partial g(\hat{\theta}_n)}{\partial\theta^T}I(\hat{\theta}_n)^{-1}\frac{\partial g^T(\hat{\theta}_n)}{\partial\theta}]^{-1}g(\hat{\theta}_n) \xrightarrow{p} ng(\hat{\theta}_n)^T[\frac{\partial g(\theta_0)}{\partial\theta^T}I(\theta_0)^{-1}\frac{\partial g^T(\theta_0)}{\partial\theta}]^{-1}g(\hat{\theta}_n) \xrightarrow{d} \chi_r^2$$

2. Show that

$$I^{12} = -I_{11}^{-1}I_{12}I^{22} \tag{1}$$

$$I^{21} = -I_{22}^{-1}I_{21}I^{11} \tag{2}$$

$$I_{11}^{-1} + I_{11}^{-1}I_{12}EI_{21}I_{11}^{-1} = (I_{11} - I_{12}I_{22}^{-1}I_{21})^{-1} \tag{3}$$

Proof: Let $I = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$, then

$$I^{-1} = \begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix} = \begin{pmatrix} I_{11}^{-1} + I_{11}^{-1}I_{12}EI_{21}I_{11}^{-1} & -I_{11}^{-1}I_{12}E \\ -EI_{21}I_{11}^{-1} & E \end{pmatrix}$$

where

$$I^{22} = E = (I_{22} - I_{21}I_{11}^{-1}I_{12})^{-1}$$

(1)

$$I^{12} = -I_{11}^{-1} I_{12} E = -I_{11}^{-1} I_{12} I^{22}$$

(3)

$$\begin{aligned}
(I_{11} - I_{12} I_{22}^{-1} I_{21})^{-1} &= I_{11}^{-1} + I_{11}^{-1} I_{12} [I_{22} - I_{21} I_{11}^{-1} I_{12}]^{-1} I_{21} I_{11}^{-1} \\
&= I_{11}^{-1} + I_{11}^{-1} I_{12} E I_{21} I_{11}^{-1}
\end{aligned}$$

(2)

$$\begin{aligned}
-I_{22}^{-1} I_{21} I^{11} &= -I_{22}^{-1} I_{21} [I_{11}^{-1} + I_{11}^{-1} I_{12} E I_{21} I_{11}^{-1}] \\
&= -I_{22}^{-1} I_{21} I_{11}^{-1} - I_{22}^{-1} I_{21} I_{11}^{-1} I_{12} E I_{21} I_{11}^{-1} \\
&= -I_{22}^{-1} [I_{id} + I_{21} I_{11}^{-1} I_{12} E] I_{21} I_{11}^{-1} \\
&= -I_{22}^{-1} [E^{-1} E + I_{21} I_{11}^{-1} I_{12} E] I_{21} I_{11}^{-1} \\
&= -I_{22}^{-1} [(E^{-1} + I_{21} I_{11}^{-1} I_{12}) E] I_{21} I_{11}^{-1} \\
&= -I_{22}^{-1} I_{22} E I_{21} I_{11}^{-1} \\
&= -E I_{21} I_{11}^{-1} \\
&= I^{21}
\end{aligned}$$