Advanced Microeconomics II

Rationalizability

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Rationalizability

- NE has strong informational requirements.
- Strong assumptions about beliefs.
- What if we rely only on rationality.
- A strategy is permissible if it is a best response to some 'rational' belief about what the other players might do.
- Beliefs about the actions of player j must be rational in that they must also be a best response to some rational belief by j.
- And so on....

Rationalizable Strategies - Pearce

Definition

An action $a_i \in A_i$ is rationalizable in the strategic game $\{N, (A_i), (u_i)\}$ if there exists

- a collection $((X_j^t)_{j \in N})_{t=1}^{\infty}$ of sets with $X_j^t \subset A_j$ for all j and t,
- a belief μ_i^1 of player i whose support is a subset of X_{-i}^1, and
- for each $j \in N$, each $t \geq 1$, and each $a_j \in X_j^t$ a belief $\mu_j^{t+1}(a_j)$ of player j whose support is a subset of X_{-j}^{t+1}

such that

- a_i is a best response to the belief μ_i^1 of player i
- $X_i^1 = \emptyset$ and for each $j \in N \backslash \{i\}$ the set X_j^1 is the set of all $a_j' \in A_j$ such that there is some a_{-i} in the support of μ_i^1 for which $a_j = a_j'$
- for every player $j \in N$ and every $t \geq 1$ every action $a_j \in X_j^t$ is a best response to the belief $\mu_j^{t+1}(a_j)$ for player j
- for each $t \geq 2$ and each $j \in N$ the set X_j^t is the set of all $a_j' \in A_j$ such that there is some player $k \in N \backslash \{j\}$, some action $a_k \in X_k^{t-1}$, and some a_{-k} in the support of $\mu_k^t(a_k)$ for which $a_j' = a_j$.

Rationalizable Strategies - Matching Pennies

How do we rationalize H for player 1?

t	$\mu_1^t(a_1)$	$\mu_2^t(a_2)$	X_1^t	X_2^t
1	$\mu_1^1 : \Pr(a_2 = H) = 1$		$\emptyset$	$\{H\}$
2		$\mu_2^2(H) : \Pr(a_1 = T) = 1$	$\{T\}$	$\emptyset$
3	$\mu_1^3(T) : \Pr(a_2 = T) = 1$		$\emptyset$	$\{T\}$
4		$\mu_2^4(T) : \Pr(a_1 = H) = 1$	$\{H\}$	$\emptyset$
5	$\mu_1^5(H) : \Pr(a_2 = H) = 1$		$\emptyset$	$\{H\}$

There are many possible ways to rationalize H for player 1.

t	$\mu_1^t(a_2)$	$\mu_2^t(a_2)$	X_1^t	X_2^t
1	$\mu_1^1 : \Pr(a_2 = H) = 3/4$		$\emptyset$	$\{H, T\}$
2		$\mu_2^2(H) : \Pr(a_1 = T) = 1$ $\mu_2^2(T) : \Pr(a_1 = H) = 1$	$\{H, T\}$	$\emptyset$
3	$\mu_1^3(H) : \Pr(a_2 = H) = 1$ $\mu_1^3(T) : \Pr(a_2 = T) = 1$		$\emptyset$	$\{H, T\}$

Rationalizable Strategies - Bernheim

Definition

An action $a_i \in A_i$ is rationalizable in the strategic game $\{N, (A_i), (u_i)\}$ if for each $j \in N$ there is a set $Z_j \subset A_j$ such that

- $a_i \in Z_i$
- every action $a_j \in Z_j$ is a best response to a belief $u_j(a_j)$ of player j whose support is a subset of Z_{-j}.

Matching pennies example

- How to rationalize H for player 1.
- Set $Z_1 = \{H, T\}, Z_2 = \{H, T\}$.

Rationalizable Strategies - Equivalence

Lemma

The two definitions of rationalizable are equivalent.

- ($\Rightarrow$) Set $Z_i = \{a_i\} \cup (\cup_{t=1}^{\infty} X_i^t)$ and $Z_j = (\cup_{t=1}^{\infty} X_j^t)$ for each $j \in N \backslash \{i\}$.
- ($\Leftarrow$) Define $\mu_i^1 = \mu_i(a_i)$ and $\mu_j^t(a_j) = \mu_j(a_j)$ for each $j \in N$ and each integer $t \geq 2$.
  - Let $X_i^1 = \emptyset$ and for each $j \in N \backslash \{i\}$ let X_j^1 be the set of all $a_j' \in A_j$ such that there is some a_{-i} in the support of μ_i^1 for which $a_j = a_j'$
  - for each $t \geq 2$ and each $j \in N$ let X_j^t be the set of all $a_j' \in A_j$ such that there is some player $k \in N \backslash \{j\}$, some action $a_k \in X_k^{t-1}$, and some a_{-k} in the support of $\mu_k^t(a_k)$ for which $a_j' = a_j$.

Simple Example

	b_1	b_2	b_3	b_4
a_1	$0, 7$	$2, 5$	$7, 0$	$0, 1$
a_2	$5, 2$	$3, 3$	$5, 2$	$0, 1$
a_3	$7, 0$	$2, 5$	$0, 7$	$0, 1$
a_4	$0, 0$	$0, -2$	$0, 0$	$10, -1$

- What are the set of mixed strategy Nash equilibria?
- What are the set of rationalizable strategies?
- b_4 is not rationalizable.
  - If $\mu_2 : \Pr(a_4) > 1/2$ then b_3 does better than b_4.
  - If $\mu_2 : \Pr(a_4) \leq 1/2$ then b_2 does better than b_4.
- a_4 is not rationalizable.
  - $b_4 \notin Z_2$ - for any belief over $\{b_1, b_2, b_3\}$ a_2 does better than a_4.

Cournot Rationalizable

- $G = \{\{1,2\},(A_i),(u_i)\}$ where $A_i = [0,\infty)$ and $u_i(a_1,a_2) = a_i(1-a_1-a_2)$.
- What are the set of Nash equilibria?
- What are the set of rationalizable strategies?
- The game is symmetric so $Z_1 = Z_2 = Z$.
- $m = \inf Z, M = \sup Z$ so if a_i is rationalizable then $m \leq a_i \leq M$.
- Given a belief $\mu_i(a_j)$ over Z, $B_i(a_j) = (1 - \mathrm{E}(a_j))/2$, so $(1-M)/2 \leq a_i \leq (1-m)/2$.
- The set of best responses to possible beliefs over Z is larger than Z (Why?)
- Thus $\{m \geq (1-M)/2, M \leq (1-m)/2\} \Rightarrow M = m = 1/3$.
- The set of rationalizable strategies is equal to the set of Nash equilibria.

Rationalizable Strategies and Correlated Equilibrium

Lemma

Every action used with positive probability by some player in a correlated equilibrium of a finite strategic game is rationalizable.

- For each player i let Z_i be the set of actions that player i uses with positive probability in the correlated equilibrium.
- $a_i \in Z_i$ is a best response to what belief over Z_{-i}?
  - The same probability distribution as that generated by the strategies of other players, conditional on player i choosing to play a_i.
  - Are these strategies a subset of Z_{-i}? Yes.

Rationalizable Strategies and Independent Probability Distributions

- We could restrict rationalising beliefs to be a product of independent probability distribution over A_{-i}.
- If so, correlated equilibrium strategies are not a subset of rationalizable strategies.

M_1:

	L	R
U	8	0
D	0	0

M_2:

L	R
4	0
0	4

M_3:

L	R
0	0
0	8

M_4:

L	R
3	3
3	3

- M_2 is rationalizable if we allow correlation - $Z_1 = \{U, D\}, Z_2 = \{L, R\}, Z_3 = \{M_2\}$.
- Not so if we restrict beliefs to be the product of independent probability distribution over A_{-i}. Requires $4pq + 4(1-p)(1-q) \geq \max\{8pq, 8(1-p)(1-q), 3\}$.