Put-call parity holds only for European options.

$$c_0 + R_f^{-\tau} X + D = p_0 + S_0 \tag{1}$$

However, it is possible to derive some results for American option prices.

♣ **Put-call parity — American option with no dividends**

It can be shown that, when there are no dividends,

$$S_0 - X \leq C_0 - P_0 \leq S_0 - R_f^{\tau} X$$

Proof :

As in the text we use c_0 and p_0 to denote the European call and put option price, and C_0 and P_0 to denote the American call and put option prices. Because $P_0 \geq p_0$, it follows from the put-call parity that

$$P_0 \geq p_0 = c_0 + R_f^{-\tau} X - S_0$$

and since $c_0 = C_0$, we can get

$$P_0 \geq C_0 + R_f^{-\tau} X - S_0 \text{ or } C_0 - P_0 \leq S_0 - R_f^{-\tau} X.$$

For a further relationship between C_0 and P_0, consider

- Portfolio A: One European call option plus an amount of cash equal to X.
- Portfolio B: One American put option plus one share.

Both options have the same exercise price and expiration date. Assume that the cash in portfolio A is invested at the risk-free rate R_f .

- If the put option is not exercised early, portfolio B is worth $\max[S_\tau, X]$ at τ. Portfolio A is worth

  $$\max[S_\tau - X, 0] + R_f^{\tau} X = \max[S_\tau, X] + R_f^{\tau} X - X$$

  Portfolio A is therefore worth more than portfolio B.

- Suppose next that the put option in portfolio B is exercised early, say, at time t. This means that portfolio B is worth X at time t. However, even if the call option were worthless, portfolio A would be worth $R_f^t X$ at time t.

It follows that portfolio A is worth at least as much as portfolio B in all circumstances. Hence

$$c_0 + X \geq P_0 + S_0$$

Since $c_0 = C_0$,

$$C_0 + X \geq P_0 + S_0$$

or

$$C_0 - P_0 \geq S_0 - X$$

Combining this with the other inequality derived above for $C_0 - P_0$, we obtain

$$S_0 - X \leq C_0 - P_0 \leq S_0 - R_f^{-\tau} X$$

♣ **Put-call parity — American option with dividends**

If dividends are paid, the eqution can be modified to

$$S_0 - D - X \leq C_0 - P_0 \leq S_0 - R_f^{\tau} X$$

Proof :

As in the text we use c_0 and p_0 to denote the European call and put option price, and C_0 and P_0 to denote the American call and put option prices. The present value of the dividends will be denoted by D.

As shown in above, when there are no dividends,

$$C_0 - P_0 \leq S_0 - R_f^{\tau} X$$

Dividends reduce C_0 and increase P_0. Hence, this relationship must also be true when there are dividends.

For a further relationship between C_0 and P_0, consider

- Portfolio A: One European call option plus an amount of cash equal to $D + X$.

- Portfolio B: One American put option plus one share.

Both options have the same exercise price and expiration date. Assume that the cash in portfolio A is invested at the risk-free rate R_f .

- If the put option is not exercised early, portfolio B is worth $\max[S_\tau, X] +$

$R_f^\tau D$ at τ. Portfolio A is worth

$$\max\left[S_\tau - X, 0\right] + R_f^\tau \left(X + D\right) = \max\left[S_\tau, X\right] + R_f^\tau \left(X + D\right) - X.$$

Portfolio A is therefore worth more than portfolio B.

- Suppose next that the put option in portfolio B is exercised early, say, at time t. This means that portfolio B is worth at most $X + R_f^t D$ at time t. However, even if the call option were worthless, portfolio A would be worth $R_f^t \left(X + D\right)$ at time t.

It follows that portfolio A is worth at least as much as portfolio B in all circumstances. Hence

$$c_0 + D + X \geq P_0 + S_0$$

Since $C_0 \geq c_0$,

$$C_0 + D + X \geq P_0 + S_0$$

or

$$C_0 - P_0 \geq S_0 - D - X$$

Combining this with the other inequality derived above for $C_0 - P_0$, we obtain

$$S_0 - D - X \leq C_0 - P_0 \leq S_0 - R_f^{-\tau} X$$