Solution to P.S.4

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1. solution

   - Given that the third portfolio is a linear combination of portfolio 1 and portfolio 2, it is equivalent to prove that

   $$U(\tilde{R}_3) > \omega\overline{U} + (1-\omega)\overline{U} = \overline{U} \tag{1}$$

   - defining $x = \frac{\tilde{R}_p - \bar{R}_p}{\sigma_p}$, then we have $\tilde{R}_p = \bar{R}_p + x\sigma_p$, where $p = 1, 2, 3$. Then suppose the individual is a vNM expected utility maximizer, then

   $$\int_{\infty}^{\infty} U(\bar{R}_3 + x\sigma_3) f(x)dx > \omega \int_{\infty}^{\infty} U(\bar{R}_1 + x\sigma_1) f(x)dx + (1-\omega) \int_{\infty}^{\infty} U(\bar{R}_2 + x\sigma_2) f(x)dx \tag{2}$$

   where $f(x)$ is the density function.

   - Note that for a given realization of x,

   $$\begin{aligned} U(\bar{R}_3 + x\sigma_3) &= U(\omega(\bar{R}_1 + x\sigma_1) + (1-\omega)(\bar{R}_2 + x\sigma_2)) \qquad (3) \\ &> \omega U(\bar{R}_1 + x\sigma_1) + (1-\omega)U(\bar{R}_2 + x\sigma_2) \end{aligned}$$

   because $U(\cdot)$ is a concave function. Thus, multiplying each side of the inequalities by $f(x)$, which is always positive, preserves the direction of the inequality. Integrating over all the realizations of x gives the desired result in equation (2).

2. solution

- The logic here is extremely simple, we just need to prove when $\omega = 0$ or 1 the efficient frontier is,

$$\bar{R}_p = \frac{\sigma_B \bar{R}_A - \sigma_A \bar{R}_B}{\sigma_B - \sigma_A} + \frac{\bar{R}_B - \bar{R}_A}{\sigma_B - \sigma_A}\sigma_p \tag{4}$$

because in this case $\sigma_p = \mid \omega\sigma_A + (1-\omega)\sigma_B \mid = \omega\sigma_A + (1-\omega)\sigma_B$.

- Then to prove it goes through the the pairs $(\sigma_A, \bar{R}_A)$ and $(\sigma_B, \bar{R}_B)$, we could just substitute σ_A and σ_B into (4), and the conclusion could be proved using simple algebra.