Advanced Microeconomics II
Problem Set 3

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1. Armies 1 and 2 are fighting over an island initially held by a battalion of army 2. Army 1 has K battalions and army 2 has L. Whenever the island is occupied by one army the opposing army can launch an attack. The outcome of the attack is that the occupying battalion and one of the attacking battalions are destroyed; the attacking army wins and, so long as it has battalions left, occupies the island with one battalion. The commander of each army is interested in maximizing the number of surviving battalions but also regards the occupation of the island as worth more than one battalion but less than two. (If after an attack, neither army has any battalions left, then the payoff to each commander is 0.) Analyze this situation as an extensive game and, using the notion of subgame perfect equilibrium, predict the winner as a function of K and L.

Solution: We model the situation as an extensive game in which at each history at which player i occupies the island and player j has at least two battalions left, player j has two choices: conquer the island or terminate the game. The first player to move is player 1.

We show that in every subgame in which army i is left with y_i battalions ($i = 1, 2$) and army j occupies the island, army i attacks if and only if either $y_i > y_j$, or $y_i = y_j$ and y_i is even.

The proof is by induction on $\min\{y_1, y_2\}$.

- If $\min\{y_1, y_2\} \leq 1$, then, since the island is initially held by a battalion of army j, $y_j = 1$. If $y_i = 1$ and army j attacks then army j receives a payoff of zero which is worse than not attacking and receiving a payoff of 1.
- Now assume that we have proved the claim whenever $\min\{y_1, y_2\} \leq m$ for some $m \geq 1$.
- Suppose that $\min\{y_1, y_2\} = m + 1$. There are two cases.
  - either $y_i > y_j$, or $y_i = y_j$ and y_i is even: If army i attacks then it occupies the island and is left with $y_i - 1$ battalions. By the induction hypothesis army j does not launch a counterattack, in any subgame perfect equilibrium, so that the attack is worthwhile.
  - either $y_i < y_j$, or $y_i = y_j$ and y_i is odd: If army i attacks then it occupies the island and is left with $y_i - 1$ battalions; army j is left with $y_j - 1$ battalions. Since either $y_i - 1 < y_j - 1$ or $y_i - 1 = y_j - 1$ and is even, it follows from the inductive hypothesis that in all subgame perfect equilibria there is a counterattack. Since the commander of army i is interested in maximizing the

number of surviving battalions but also regards the occupation of the island as worth more than one battalion but less than two, thus army i is better off not attacking.

According to the above analysis, the winner is army i, where

$$i = \begin{cases} 1 & \text{if } K > L \text{ or } K = L \text{ and } K \text{ is even} \\ 2 & \text{if } K < L \text{ or } K = L \text{ and } K \text{ is odd.} \end{cases}$$

2. How many Nash equilibria are there in the 2 period Chain Store game?

Solution: According to the 2 period Chain-Store game in the lecture, player 1 has 2 strategies (there is one history where he has 2 choices), player 2 has 8 strategies (there are 3 histories where he has 2 choices) and the chain store has 16 strategies (there are 4 histories where he has 2 choices). The strategic game is shown below.

	IFI,ICI,OI	IFI,ICI,OO	IFI,ICO,OI	IFI,ICO,OO	IFO,ICI,OI	IFO,ICI,OO	IFO,ICO,OI	IFO,ICO,OO
IF,IFIF,ICIF,OIF	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IF,IFIF,ICIF,OIC	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IF,IFIF,ICIC,OIF	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IF,IFIF,ICIC,OIC	0, 0, 0	0, 0, 0	0, 0, 0	0, 0, 0	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IF,IFIC,ICIF,OIF	2, 0, 2	2, 0, 2	2, 0, 2	2, 0, 2	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IF,IFIC,ICIF,OIC	2, 0, 2	2, 0, 2	2, 0, 2	2, 0, 2	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IF,IFIC,ICIC,OIF	2, 0, 2	2, 0, 2	2, 0, 2	2, 0, 2	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IF,IFIC,ICIC,OIC	2, 0, 2	2, 0, 2	2, 0, 2	2, 0, 2	5, 0, 1	5, 0, 1	5, 0, 1	5, 0, 1
IC,IFIF,ICIF,OIF	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1
IC,IFIF,ICIF,OIC	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1
IC,IFIC,ICIF,OIF	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1
IC,IFIC,ICIF,OIC	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1	2, 2, 0	2, 2, 0	7, 2, 1	7, 2, 1
IC,IFIF,ICIC,OIF	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1
IC,IFIF,ICIC,OIC	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1
IC,IFIC,ICIC,OIF	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1
IC,IFIC,ICIC,OIC	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1	4, 2, 2	4, 2, 2	7, 2, 1	7, 2, 1

I

	IFI,ICI,OI	IFI,ICI,OO	IFI,ICO,OI	IFI,ICO,OO	IFO,ICI,OI	IFO,ICI,OO	IFO,ICO,OI	IFO,ICO,OO
IF,IFIF,ICIF,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IF,IFIF,ICIF,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$
IF,IFIF,ICIC,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IF,IFIF,ICIC,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$
IF,IFIC,ICIF,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IF,IFIC,ICIF,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$
IF,IFIC,ICIC,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IF,IFIC,ICIC,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$
IC,IFIF,ICIF,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IC,IFIF,ICIF,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$
IC,IFIC,ICIF,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IC,IFIC,ICIF,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$
IC,IFIF,ICIC,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IC,IFIF,ICIC,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$
IC,IFIC,ICIC,OIF	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$	$5,1,0$	$10,1,1$
IC,IFIC,ICIC,OIC	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$	$7,1,2$	$10,1,1$

O

If you solve for all of the equilibria using the payoff matrix, you will find 56 Nash equilibria. There are 4 possible Nash equilibrium payoffs: $(10, 1, 1)$, $(7, 1, 2)$, $(7, 2, 1)$ and $(4, 2, 2)$, each one associated with one of the four possible industry configurations.

Note that for payoffs $(10, 1, 1)$ and $(7, 1, 2)$ a change in strategy of CS along the equilibrium path cannot influence player 2's action since player 1 chooses O. Futhermore, for the payoff $(7, 2, 1)$ a change in strategy of CS along the equilibrium path can influence player 2's action but cannot improve the outcome for CS since along the equilibrium path player 2 chooses O. As a result, in each of the Nash equilibria associated with the first three payoffs, the strategies for each player are only restricted along the equilibrium path (once for player 1 and player 2, twice for the chain store firm); for example, if $s_1(\varnothing) = O$ then $s_{CS}(I) = F$, but, after a history (I, F) or (I, C) strategies are unrestricted. Hence, there is one strategy for firm 1, 4 strategies for firm 2 and 4 strategies for the chain store firm consistent with each of the first three Nash equilibrium payoffs.

For the last payoff, play after a history of O is unrestricted for both player 2 and the chain store firm. However, since the chain store can influence player 2's action by his choice $s_{CS}(I)$, and thus, influnce his payoff in the second market, it may be profitable for the chain store to fight an entry of player 1. The direct effect is a loss of one in the first market but if $s_2(IF) = I$ the chain store firm gains three. The net result is a profitable deviation. Hence, this represents an additional restriction on strategies when the equilibrium outcome is $(4, 2, 2)$. Hence, there is one strategy for firm 1, 2 strategies for firm 2 and 4 strategies for the chain store firm consistent with the Nash equilibrium payoff $(4, 2, 2)$. Thus, the total number of Nash equilibria is 56.

3. Consider the split the pie game with T periods.
   (a) Find the SPE when $T = 2$.

Solution: In period 2, player 1 offers $(1, 0)$ to player 2 after any history, player 2 accepts any offer.

In period 1, player 2 offers $(\delta_1, 1-\delta_1)$ to player 1 after any history, player 1 accepts any offer x if $x_1 \geq \delta_1$.
In period 0, player 1 offers $(1-\delta_2(1-\delta_1), \delta_2(1-\delta_1))$ to player 2, player 2 accepts any offer x if $x_2 \geq \delta_2(1-\delta_1)$.

(b) Find the SPE when $T = 3$.

Solution: In period 3, player 2 offers $(0, 1)$ to player 1 after any history, player 1 accepts any offer.
In period 2, player 1 offers $(1-\delta_2, \delta_2)$ to player 2 after any history, player 2 accepts any offer x if $x_2 \geq \delta_2$.
In period 1, player 2 offers $(\delta_1(1-\delta_2), 1-\delta_1(1-\delta_2))$ to player 1, player 1 accepts any offer x if $x_1 \geq \delta_1(1-\delta_2)$. In period 0, player 1 offers $(1-\delta_2(1-\delta_1(1-\delta_2)), \delta_2(1-\delta_1(1-\delta_2)))$ to player 2, player 2 accepts any offer x if $x_2 \geq \delta_2(1-\delta_1(1-\delta_2))$.

(c) Find the SPE when T is even.

Solution: Denote $f_1(t) = \sum_{i=1}^{\frac{T-t}{2}} \delta_1^{i-1}\delta_2^i(1-\delta_1)$ and $g_1(t) = \sum_{i=0}^{\frac{T-1-t}{2}} \delta_1^i\delta_2^i(1-\delta_1)$.
Then in the SPE, in period T player 1 offers $(1, 0)$ and player 2 accepts any offer. In an even period $t < T$ player 1 offers $(1-f_1(t), f_1(t))$ and player 2 accepts any offer x if $x_2 \geq f_1(t)$. In an odd period t player 2 offers $(1-g_1(t), g_1(t))$ and player 1 accepts any offer x if $x_1 \geq 1-g_1(t)$.

(d) Find the SPE when T is odd.

Solution: Denote $f_2(t) = \sum_{i=1}^{\frac{T-t}{2}} \delta_2^{i-1}\delta_1^i(1-\delta_2)$ and $g_2(t) = \sum_{i=0}^{\frac{T-1-t}{2}} \delta_2^i\delta_1^i(1-\delta_2)$. In the SPE, in period T player 2 offers $(0, 1)$ and player 1 accepts any offer. In an odd period $t < T$ player 2 offers $(f_2(t), 1-f_2(t))$ and player 1 accepts any offer x if $x_1 \geq f_2(t)$. In an even period t player 1 offers $(g_2(t), 1-g_2(t))$ and player 2 accepts any offer x if $x_2 \geq 1-g_2(t)$.

(e) What are the limit payoffs when $T \to \infty$?

Solution: The limit outcome as $T \to \infty$ is $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2-\delta_1\delta_2}{1-\delta_1\delta_2}\right)$, so the limit of the equilibrium of the T-period alternating offer bargaining game is equal to the equilibria of the infinite horizon alternating offer bargaining game.

4. Consider the extension of Rubinstein's infinite horizon alternating offer bargaining model from two players to three players. Every player has the same discount factor δ.

   In period 1, player 1 makes an offer which consists of (p_1, p_2, p_3) which represents the share of each player. Then player 2 decides to accept or to reject the offer within the same period. If player 2 accepts the offer, then player 3 is asked to accept or reject the offer within the same period. If both player 2 and player 3 accept the offer, then the bargaining is over and each player takes p_i. If either player 2 or player 3 rejects, the first person who rejects the present offer will initiate the next round.

In period t, player i initiates the offer, which specifies the shares of every player, Then player $i+1(\text{mod } 3)$ either accepts or rejects the offer. If player $i+1(\text{mod } 3)$ accepts the offer, then player $i+2(\text{mod } 3)$ either accepts or rejects the offer. If both players accept the offer, then the bargaining is over and player i receives a payoff of $\delta^{t-1}p_i$. If $j \neq i$ is the first player who rejects the offer, then the next round starts with player j's offer.

(a) (10 points) Calculate a stationary subgame perfect equilibrium.

Solution: Let $x = (x_1, x_2, x_3)$ denote the offer made by player 1 where x_i denotes player i's share of the pie. Similarly, let $y = (y_1, y_2, y_3)$ denote the offer made by player 2 and let $z = (z_1, z_2, z_3)$ denote the offer made by player 3. Obviously,

$$x_1 + x_2 + x_3 = 1; \quad y_1 + y_2 + y_3 = 1; \quad z_1 + z_2 + z_3 = 1.$$

If player 2 is the only one to reject the offer then he makes the offer next period. If player 3 is the only one to reject the offer then he makes the offer next period. Hence player 1 will make player 2 and player 3 an offer that makes them indifferent between rejecting and accepting. Similarly, for player 2 and player 3 when it is there turn to make an offer. Hence,

$$x_2^* = \delta y_2^*; \quad y_3^* = \delta z_3^*; \quad z_1^* = \delta x_1^*; \quad x_3^* = \delta z_3^*; \quad y_1^* = \delta x_1^*; \quad z_2^* = \delta y_2^*.$$

So, we can use these nine equations to solve for the nine unknowns which gives

$$x_1^* = y_2^* = z_3^* = \frac{1}{1+2\delta} \text{ and } x_2^* = x_3^* = y_1^* = y_3^* = z_1^* = z_2^* = \frac{\delta}{1+2\delta}.$$

Player 1 offers (x_1^*, x_2^*, x_3^*) if it is his turn to offer, accepts any offer from player 2 if $y_1 \geq y_1^*$, accepts any offer from player 3 if $z_1 \geq z_1^*$ and rejects offers otherwise. Player 2 offers (y_1^*, y_2^*, y_3^*) if it is his turn to offer, accepts any offer from player 1 if $x_2 \geq x_2^*$, accepts any offer from player 3 if $z_2 \geq z_2^*$ and rejects offers otherwise. Player 3 offers (z_1^*, z_2^*, z_3^*) if it is his turn to offer, accepts any offer from player 1 if $x_3 \geq x_3^*$, accepts any offer from player 2 if $y_3 \geq y_3^*$ and rejects offers otherwise.

(b) (10 points) Can you sustain $(0.5, 0.5, 0)$ which is agreed upon in the initial round as an outcome of a subgame perfect equilibrium if δ is sufficiently close to 1. Explain your answer.

Solution: Yes. The equilibrium may be described by four commonly-held "states", which denote the share of the pie recieved by each player in the sub-game perfect equilibrium outcome associated with that state. The four states are $x = (0.5, 0.5, 0)$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. In state y each player i makes the proposal y and accepts the proposal z if and only if $z_i \geq \delta y_i$. The initial state is x.

Transitions between states occur only after a proposal has been made, before the response. If, in any state y, player i proposes z with $z_i > y_i$ then the state becomes e_j , where $j \neq i$ is the player with the lowest index for whom $z_j < 1/2$. Such a player j exists, and the requirement that $\delta \geq 1/2$ guarantees that it is optimal for him to reject player i's proposal. I leave it to you to check that this is indeed a subgame perfect equilibrium.

5. Prove the following proposition.

Proposition 1. *Let w be a strictly enforceable feasible payoff profile of $G = \{N, (A_i), (u_i)\}$. For all $\epsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|w' - w| < \epsilon$.*

Solution: Let $w = \sum_{a \in A}(\beta_a/\gamma)u(a)$ be a strictly enforceable feasible payoff profile, where β_a for each $a \in A$ is an integer and $\gamma = \sum_{a \in A}(\beta_a)$, and let $(a^k)_{k=1}^{\gamma}$ be the cycling sequence of action profiles for which the cycle of length γ contains β_a repetitions of a for each $a \in A$. Let s_i be the strategy of player i in the repeated game that chooses a_i^t in each period t unless there was a previous period t' in which a single player other than i deviated from $a^{t'}$, in which case it chooses $(p_{-j})_i$, where j is the deviant in the first such period t' and (p_{-j}) is the profile of strategies that generate player j's minimax payoff v_j.

Let $\epsilon > 0$. Let $w'(\delta)$ be the payoff profile in the δ-discounted infinitely repeated game associated with the sequence $(a^k)_{k=1}^{\infty}$. Then,

$$w'(\delta) = (1 - \delta) \sum_{i=0}^{\infty} \delta^{i\gamma} \sum_{k=1}^{\gamma} \delta^k u(a^k) = \frac{(1 - \delta) \sum_{k=1}^{\gamma} \delta^k u(a^k)}{1 - \delta^{\gamma}}.$$

Using L'Hôpital's rule shows that

$$\lim_{\delta \to 1} w'(\delta) = \frac{\sum_{k=1}^{\gamma} u(a^k)}{\gamma} = w.$$

Hence, there exists $\tilde{\delta}$ such that for all $\delta > \tilde{\delta}$, $\left|w'(\delta) - w\right| < \epsilon$. It remains to show that the strategies are equilibrium strategies.

Let M_i and m_i be the maximum and minimum stage game payoff for player i. Let δ_i solve

$$(1 - \delta)M_i + \delta v_i \leq (1 - \delta)m_i + \delta w_i.$$

Such a δ exists since $w_i > v_i$ for each player i. For all $\delta > \max\{\delta_1, \ldots, \delta_n\}$, $s = (s_i)_{i \in N}$ is a Nash equilibrium of the δ-discounted infinitely repeated game.

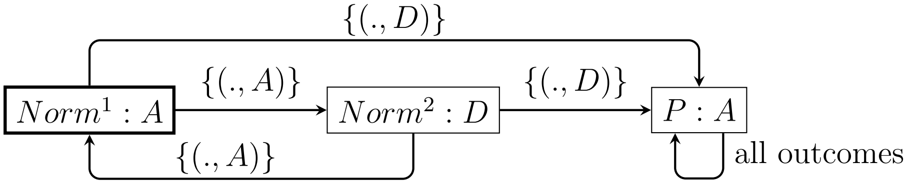
So for $\delta > \underline{\delta} = \max\{\tilde{\delta}, \delta_1, \ldots, \delta_n\}$, the strategy profile s is a Nash equilibrium of the repeated game with a payoff profile w' within ϵ of w.

6. Consider the following stage game.

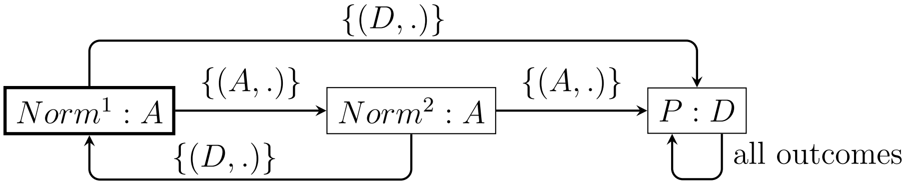
	A	D
A	$2,2$	$0,1$
D	$5,4$	$1,0$

(a) Construct a pair of strategies that generate the average per-period payoffs of $(3.5, 3)$, and are a Nash equilibrium but are not a subgame perfect equilibrium when players use the limit of means criterion to evaluate payoffs.

Solution: The sequence of outcomes that generate the average per-period payoffs of $(3.5, 3)$ are $(AA, AD, AA, AD, \ldots)$. A machine M_1 that supports this sequence of outcomes as a Nash equilibrium is

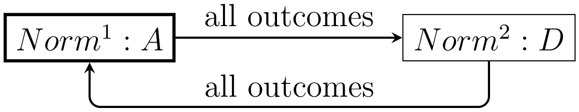


A machine M_2 that supports this sequence of outcomes as a Nash equilibrium is

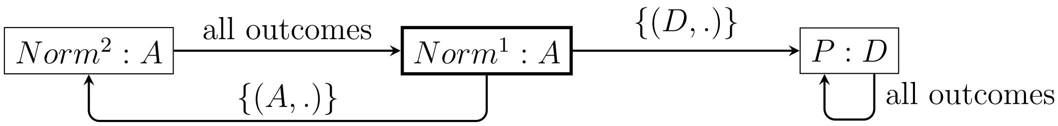


Since A is a dominant strategy for player 2, and player 2 always chooses A along the equilibrium path, there is never any incentive for player 2 to deviate and thus no need for punishment of player 2. We also know D is a dominant strategy for player 1. Thus, we do not require punishment to induce player 1 to play D when required. Thus, we can simplify the machines.

M_1 becomes



M_2 becomes

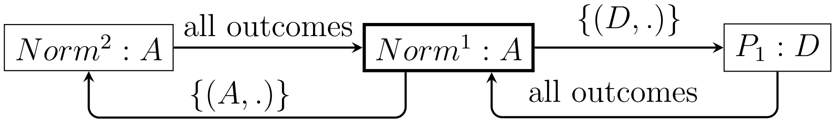


Note that these strategies are not subgame perfect equilibria, because it is not in player 2's interest to punish player 1 forever. Hence, in the subgame after a history in which player 1 deviates, the strategies do not form a Nash equilibrium.

(b) Construct a pair of strategies that generate average per-period payoffs of $(3.5, 3)$, and are a subgame perfect equilibrium when players use the limit of means criterion to evaluate payoffs but not a subgame perfect equilibrium when players use the overtaking criterion to evaluate payoffs.

Solution: To ensure that player 2 is willing to enforce the punishment for player 1, the punishment should be for a finite period of time. The player 1 stage game profit from deviating when the outcome should be (AA) equals 3. Player 1 needs to be punished for one period to wipe out the gains since $(2, 5, 2, 5, \ldots) \succ_1 (5, 1, 5, 1, 5, 1, \ldots)$. Recall that only infinite deviations can generate a outcome that each player is not indifferent to.

The machine M_1 that represents these strategies is the same as in part (a). The machine M_2 that represents these strategies is



Note that player 2 has no incentive not to follow the punishment strategy since they revert to equilibrium payoffs once the punishment phase is over, so, in the limit, player 2 gets the same payoff whether or not he punishes player 1. However, this is not a sub-game perfect equilibrium if players use the overtaking criterion since player 2 could deviate when called to punish and generate the sequence of payoffs $(4, 2, 4, 2, \ldots)$ which is strictly preferred by player 2 to $(0, 2, 4, 2, \ldots)$.

(c) Construct a pair of strategies that generate the average per-period payoffs of $(3.5, 3)$, and are a subgame perfect equilibrium when players use overtaking criterion to evaluate payoffs.

Solution: Punishing player 1 for 2 periods as before will ensure player 1 has no profitable deviation along the equilibrium path. However, players must also have the incentive to apply punishments. If punishments are min-max punishments, which then transition back to the norm once finished, then no machine with finite states can describe such strategies that are SPE. Punishments are required to be of ever increasing (but still finite) length in order to be sub-game perfect.

The machines M_1 and M_2 each have the same states and transition functions: only their output functions differ.

- States:
  - $Norm^1$: Initial State
  - $Norm^2$: second period of $(a^k)_{k=1}^2$ cycle.
  - $P(j, t)$: Punishment phase of player j after a player j deviation with t periods remaining, where t is a positive integer, $j \in \{1, 2\}$.
  - $P(j, t)$: Punishment phase of player 2 after a player 2 deviation with s periods remaining, where s is a positive integer.
- Output function for player 1:
  - In $Norm^1$: choose A.

- In $Norm^2$: choose D.
- In $P(1,t)$: choose D.
- In $P(2,t)$: choose A.

- Output function for player 2:
  - In $Norm^1$: choose A.
  - In $Norm^2$: choose A.
  - In $P(1,t)$: choose D.
  - In $P(2,t)$: choose A.

- Transition function:
  - $\tau_i(Norm^1, a) = \begin{cases} P(1,1) & \text{if } a = (DA) \\ Norm^2 & \text{otherwise.} \end{cases}$
  - $\tau_i(Norm^2, a) = Norm^1$ for all $a \in A$.
  - $\tau_i(P(1,t), a) = \begin{cases} P(1,t-1) & \text{if } 1 < t \text{ and } a \neq (DA) \\ Norm^1 & \text{if } t = 1 \text{ and } a \neq (DA) \\ P(2, 3t+1\}) & \text{if } a = (DA), \end{cases}$
  - $\tau_i(P(2,t), a) = \begin{cases} P(2,t-1) & \text{if } 1 < t \text{ and } a \neq (DA) \\ Norm^1 & \text{if } t = 1 \text{ and } a \neq (DA) \\ P(1, T(2,t)) & \text{if } a = (DA), \end{cases}$

    where $T(2,t)$ is the largest even number greater than $\max\{3, \frac{3(t+1)}{5}\}$ if t is odd and the largest odd number greater than $\max\{3, \frac{3(t+1)}{5}\}$ if t is even.

I leave it as an exercise for you to check that punishments are credible.

7. Consider a game in which the following strategic game is repeated twice:

Player 2

		b_1	b_2	b_3
	a_1	$10, 10$	$2, 12$	$0, 13$
Player 1	a_2	$12, 2$	$5, 5$	$0, 0$
	a_3	$13, 0$	$0, 0$	$1, 1$

The players observe the actions chosen in the first play of the game prior to the second play.

(a) What are the pure strategy sub-game perfect Nash equilibrium payoffs of this game?

Solution: The two pure strategy Nash equilibria of this simultaneous-move game are (a_2, b_2) and (a_3, b_3). Thus, $(a_2b_2, a_2b_2), (a_3b_3, a_3b_3), (a_2b_2, a_3b_3)$ and (a_3b_3, a_2b_2) are four sub-game perfect Nash equilibria outcomes of this game.

In addition, the payoff of each player in the stage game Nash equilibrium outcome of (a_2b_2) is 5 and the payoff of each player in the stage game Nash equilibrium outcome of (a_3b_3) is 1. The difference between the stage game Nash equilibrium payoffs for each player is 4. Hence, any stage game outcome where each player's most profitable deviation is no more than 4 is sustainable in the first period. The outcome is supported by punishing deviations with the lower Nash equilibrium in the second period. Hence, (a_1b_1, a_2b_2), (a_1b_2, a_2b_2), (a_1b_3, a_2b_2), (a_2b_1, a_2b_2) and (a_3b_1, a_2b_2) are additional sub-game equilibrium outcomes.

Thus the pure strategy sub-game perfect Nash equilibrium payoffs of this game are (2,2), (10,10), (6,6), (15,15), (7,17), (17,7), (5,18) and (18,5).

(b) What are the pure strategy subgame perfect Nash equilibria of this game? In particular, how many pure strategy sub-game perfect Nash equilibria are there?

Solution: The number of strategies that support each individual outcome is large. We shall examine them one-by-one. First consider the second-stage outcome (a_2b_2).

- If the first-stage outcome is a Nash equilibrium outcome then strategies off the equilibrium path are unrestricted (players cannot do better in either stage no matter what the actions chosen off the equilibrium path). Hence, there are 2^8 subgame perfect equilibrium strategies associated with (a_2b_2, a_2b_2) and 2^8 subgame perfect equilibrium strategies associated with (a_3b_3, a_2b_2).
- If the first-stage equilibrium outcome is not a Nash equilibrium, only first-stage single-player profitable deviations need to be punished, i.e., for other outcomes the subsequent second-stage Nash equilibrium outcome is unrestricted.
  - For the outcome (a_1b_1, a_2b_2) the first-stage single-player profitable deviations are $(a_1b_2), (a_1b_3), (a_2b_1)$ and (a_3b_1). Hence, there are four outcomes $(a_2b_2), (a_2b_3), (a_3b_2)$ and (a_3b_3) where second-stage outcomes are unrestricted and there are 2^4 sub-game perfect equilibria associated with the outcome (a_1b_1, a_2b_2).
  - For the outcome (a_1b_2, a_2b_2) the first-stage single-player profitable deviations are (a_1b_3) and (a_2b_2). Hence, there are 2^6 sub-game perfect equilibria associated with the outcome (a_1b_2, a_2b_2).
  - For the outcome (a_2b_1, a_2b_2) an argument symmetric to that for the outcome (a_1b_2, a_2b_2) holds. Hence, there are 2^6 sub-game perfect equilibria associated with the outcome (a_2b_1, a_2b_2).
  - For the outcome (a_1b_3, a_2b_2) the first-stage single-player profitable deviation is (a_3b_3). Hence, there are 2^7 sub-game perfect equilibria associated with the outcome (a_1b_2, a_2b_2).
  - For the outcome (a_3b_1, a_2b_2) an argument symmetric to that for the outcome (a_1b_3, a_2b_2) holds. Hence, there are 2^7 sub-game perfect equilibria associated with the outcome (a_3b_1, a_2b_2).

Now consider the second-stage outcome (a_3b_3). Since players are getting the worst stage game payoff in period two, the outcome in period one must be a Nash equilibrium of the stage game. We need to ensure that any first-stage deviation (that necessarily hurts the player in stage 1) does not lead to a reward in stage two that more than compensates for this first-stage loss, i.e. we cannot reward players for first-stage deviations that cost less than 4.

- For the outcome (a_2b_2, a_3b_3) the first-stage single-player deviations that incur a loss less than four are (a_2b_1) and (a_1b_2). Hence, there are 2^6 sub-game perfect equilibria associated with the outcome (a_2b_2, a_2b_2).
- For the outcome (a_3b_3, a_3b_3) the first-stage single-player deviations that incur a loss less than four are $(a_3b_1), (a_3b_2), (a_2b_3)$ and (a_1b_3). Hence, there are 2^4 sub-game perfect equilibria associated with the outcome (a_3b_3, a_3b_3).

Hence, the total number of pure strategy sub-game perfect Nash equilibria is $2^9 + 2^8 + 2^7 + 2^6 + 2^5 = 992$.