

Advanced Microeconomics II

Extensive Form Perfect Information

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Strategic Entry Example

There are two firms, a potential entrant (E) and an industry incumbent (I).

- The entrant must decide whether to enter the market (In) or not (Out).
- If the potential entrant stays out then he gains nothing and the incumbent firm gains 2.
- If he enters his payoff depends on the reaction by the incumbent to entry.
 - ▶ If the incumbent fights (F), both firms lose 1.
 - ▶ If the incumbent cooperates (C), both firms gain 1.

Formulate this as a strategic game and find the Nash equilibria.

Strategic Entry Example

| | | Incumbent | |
|---------|-------|-----------|--------|
| | | F | C |
| Entrant | In | $-1, -1$ | $1, 1$ |
| | Out | $0, 2$ | $0, 2$ |

- Two pure strategy Nash Equilibria: $(In, C), (Out, F)$.
- What's special about (Out, F) ?
- Strategic form does not reflect timing.

Extensive Games

- An extensive game is an explicit description of the sequential structure of strategic interactions.
- Players can condition actions on past history.
- We start with models where when a player makes a choice, he knows perfectly what has happened in the past (perfect information).
- We will study two extensions
 - ▶ Bargaining games of alternating offers.
 - ▶ Repeated games.
- Nash equilibrium ignores timing of choice so we require a new notion of equilibrium.
 - ▶ Sub-game perfect equilibrium

Extensive Games With Perfect Information

Definition

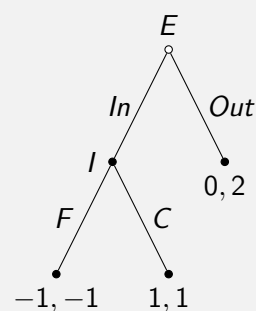
An **extensive game with perfect information** has

- a set N of **players**;
 - a set H of **histories** such that
 - ▶ the empty sequence \emptyset is a member of H ,
 - ▶ If $(a^k)_{k=1,\dots,K} \in H$ where K may be infinite) and $L < K$ then $(a^k)_{k=1,\dots,L} \in H$, and
 - ▶ If an infinite sequence $(a^k)_{k=1}^\infty$ satisfies $(a^k)_{k=1,\dots,L} \in H$ for every integer L then $(a^k)_{k=1}^\infty \in H$;
- (Each component of a history is an **action** taken by a player.) A history $(a^k)_{k=1,\dots,K} \in H$ is **terminal** if it is infinite or there is no a^{K+1} such that $(a^k)_{k=1,\dots,K+1} \in H$. Z is the set of terminal histories;
- A function P that assigns to each member of $H \setminus Z$ a member of N . (P is a **player function**, $P(h)$ is the player who takes an action after history h .)
 - For each player $i \in N$ a preference relation \succeq_i in Z (the **preference relation** of player i).

Interpretation

- After any nonterminal history h player $P(h)$ chooses an action from $A(h) = \{a : (h, a) \in H\}$.
- \emptyset is the game starting point or **initial history**.
- $P(\emptyset)$ chooses from $A(\emptyset)$.
- For each choice $a^0 \in A(\emptyset)$, $P(a^0)$ chooses from $A(a^0)$.
- For each choice $a^1 \in A(a^0)$, $P(a^0, a^1)$ chooses from $A(a^0, a^1)$.
- And so on, until we reach a terminal history (no more choices).
- Preferences are generally represented by payoff functions.

Strategic Entry Example

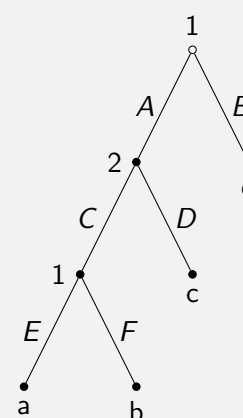


Extensive game representation

- $N = \{I, E\}$.
- $H = \{\emptyset, In, Out, (In, F), (In, C)\}$.
- $P(\emptyset) = E$, and $P(In) = I$.
- $(In, C) \succ_E Out \succ_E (In, F)$ and $Out \succ_I (In, C) \succ_I (In, F)$.

Extensive game form with perfect information is $\{N, H, P\}$.

Example 2

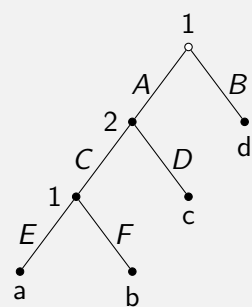


What is the extensive game form?

Player Strategies

Definition

A **strategy of player $i \in N$** in an extensive game with perfect information $\{N, H, P, (\succeq_i)\}$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \setminus Z$ for which $P(h) = i$.



How many strategies does each player have?

Nash Equilibrium

Definition

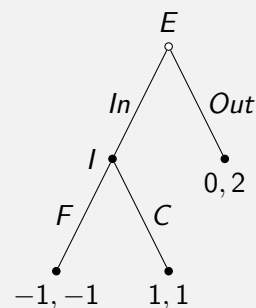
The **outcome** $O(s) \in Z$ of strategy profile $s = (s_i)_{i \in N}$ is the terminal history such that for $0 \leq k < K$ we have $s_{P(a^1, \dots, a^k)}(a^1, \dots, a^k) = a^{k+1}$ where K is the length of $O(s)$.

Definition

A **Nash equilibrium of an extensive game with perfect information** $\{N, H, P, (\succeq_i)\}$ is a strategy profile s^* such that for every player $i \in N$ we have

$$O(s_i^*, s_{-i}^*) \succeq_i O(s_i, s_{-i}^*) \text{ for every strategy } s_i \text{ of player } i.$$

Strategic Entry Example



What are the set of Nash equilibria?

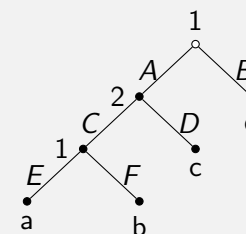
- $(s_E(\emptyset) = In, s_I(In) = C)$
- $(s_E(\emptyset) = Out, s_I(In) = F)$

Strategic Form of Extensive Games

Definition

The **strategic form of the extensive game with perfect information** $\Gamma = \{N, H, P, (\succeq_i)\}$ is the strategic game $\{N, (S_i), (\succeq'_i)\}$ in which for each player $i \in N$

- S_i is the set of strategies of player $i \in \Gamma$.
- \succeq'_i is defined by $s \succeq'_i s'$ if and only if $O(s) \succeq_i O(s')$ for every $s \in \times_{i \in N} S_i$ and $s' \in \times_{i \in N} S_i$.

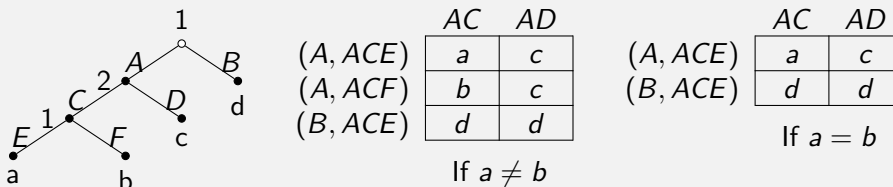


| | AC | AD |
|----------|----|----|
| (A, ACE) | a | c |
| (A, ACF) | b | c |
| (B, ACE) | d | d |
| (B, ACF) | d | d |

Reduced Strategic Form of Extensive Games

Definition

Let $\Gamma = \{N, H, P, (\succeq_i)\}$ be an extensive game with perfect information and let $\{N, (S_i), (\succeq'_i)\}$ be its strategic form. For any $i \in N$ define the strategies $s_i \in S_i$ and $s'_i \in S'_i$ of player i to be **equivalent** if for each $s_{-i} \in S_{-i}$ we have $(s_i, s_{-i}) \sim'_j (s'_i, s_{-i})$ for all $j \in N$. The **reduced strategic form of Γ** is the strategic game $\{N, (S'_i), (\succeq''_i)\}$ in which for each $i \in N$ each set S'_i contains one member of each set of equivalent strategies in S_i and \succeq''_i is the preference ordering over $\times_{j \in N} S'_j$ induced by \succeq'_i .

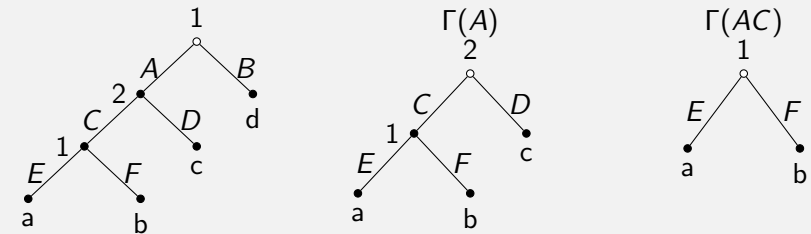


Subgame

Definition

The **subgame of the extensive game with perfect information $\Gamma = \{N, H, P, (\succeq_i)\}$ that follows the history h** is the extensive game $\Gamma(h) = \{N, H|_h, P|_h, (\succeq_i|_h)\}$ where

- $H|_h$ is the set of sequences h' of actions for which $(h, h') \in H$,
- $P|_h(h') = P(h, h')$ for each $h' \in H|_h$, and
- $\succeq_i|_h$ is defined by $h' \succeq_i|_h h''$ if and only if $(h, h') \succeq_i (h, h'')$.



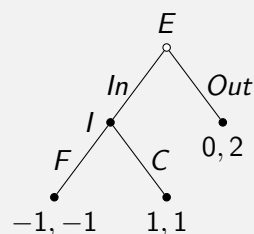
Subgame Perfect Equilibrium

Definition

A **subgame perfect equilibrium of an extensive game with perfect information $\Gamma = \{N, H, P, (\succeq_i)\}$** is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ for which $P(h) = i$ we have

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succeq_i|_h O_h(s_i, s_{-i}^*|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$.



What are the set of subgame perfect equilibria?

- $\{In, (In, C)\}$
- Subgame perfection eliminates Nash equilibria which imply incredible threats.

The one deviation property

Proposition

Let $\Gamma = \{N, H, P, (\succeq_i)\}$ be a finite horizon extensive game with perfect information. The strategy profile s^* is a subgame perfect equilibrium of Γ if and only if for every player $i \in N$ and every history $h \in H$ for which $P(h) = i$ we have

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succeq_i|_h O_h(s_i, s_{-i}^*|_h)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$ that differs from $s_i^*|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

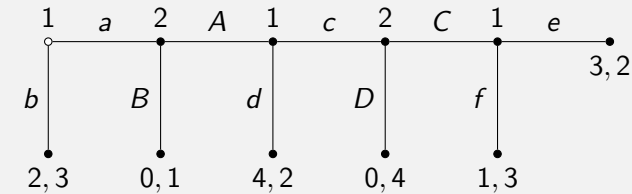
(\Rightarrow) s_i^* is better than any other strategy including any strategy that only deviates after the initial history $\Gamma(h)$.

The one deviation property proof

(\Leftarrow) If s^* is not a subgame equilibrium then there exists a player i and subgame $\Gamma(h')$ where player i can profitably deviate.

- $l(\Gamma(h'))$ is the **length** of the longest history in $\Gamma(h')$.
- The number of times player i 's profitable deviation differs from s^* is limited by the $l(\Gamma(h'))$ (actually, by the number of times player i plays in $\Gamma(h')$).
- From all profitable deviations of $\Gamma(h')$ choose a strategy s'_i with the least number of deviations.
- h^* is the longest history h (latest profitable deviation) where $s'_i(h) \neq (s^*|_{h'})(h)$.
- In the subgame $\Gamma(h', h^*)$, $s'_i|_{h', h^*}$ only differs from $s^*|_{h', h^*}$ after history (h', h^*) and is a profitable deviation.

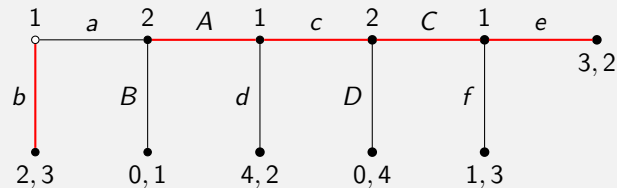
The one deviation property example



What is the SPE of this game?

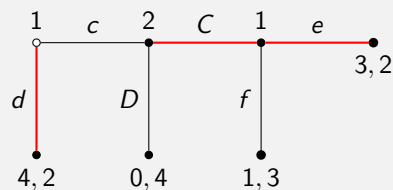
Profitable one-shot deviations

Consider the following strategy profile $\{(b, aAc, aAcCe), (aA, aAcC)\}$.



Construct a profitable one-shot deviation.

Example 1: $\{(b, aAd, aAcCe)\}$ is a profitable one-shot deviation for player 1 in $\Gamma(aA)$.



Kuhn's Theorem

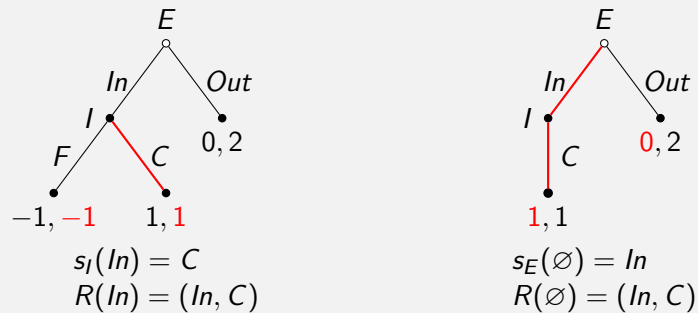
Proposition

Every finite extensive game with perfect information, $\Gamma = \{N, H, P, (\succeq_i)\}$, has a subgame perfect equilibrium.

- If $l(\Gamma(h)) = 0$ define $R(h) = h$.
- Let $R(h)$ be defined for all $h \in H$ with $l(\Gamma(h)) \leq k$ for some $k \geq 0$.
- Let h^* be a history for which $l(\Gamma(h^*)) = k + 1$; let $i = P(h^*)$.
- $l(\Gamma(h^*)) = k + 1 \Rightarrow l(\Gamma(h^*, a)) \leq k$ for all $a \in A(h^*)$.
 - ▶ Define $s_i(h^*)$ to be a \succeq_i -maximizer of $R(h^*, a)$ over $a \in A(h^*)$
 - ▶ Define $R(h^*) = R(h^*, s_i(h^*))$.
- This process defines a strategy profile s in Γ ; by the one-shot deviation property, s is a subgame perfect equilibrium of Γ .

Backwards Induction

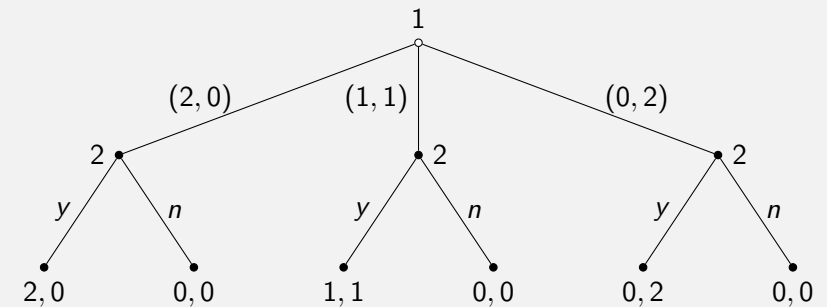
R is referred to as **backwards induction**. Can be used to find the set of subgame perfect equilibria.



The set of subgame perfect equilibria is $\{In, (In, C)\}$.

Backwards Induction - Your Turn

Player 1 proposes an allocation of 2 identical indivisible objects.
 Player 2 accepts or rejects the offer.



Find the set of subgame perfect equilibria using backward induction.

Stackleberg Model Of Duopoly

- Firm 1 chooses a quantity $q_1 \geq 0$;
- Firm 2 observes q_1 and chooses a quantity $q_2 \geq 0$;
- The payoff to each firm is given by

$$\pi_i(q_i, q_j) = q_i(1 - q_i - q_j)$$

Extensive Game with Perfect Information and Chance Moves

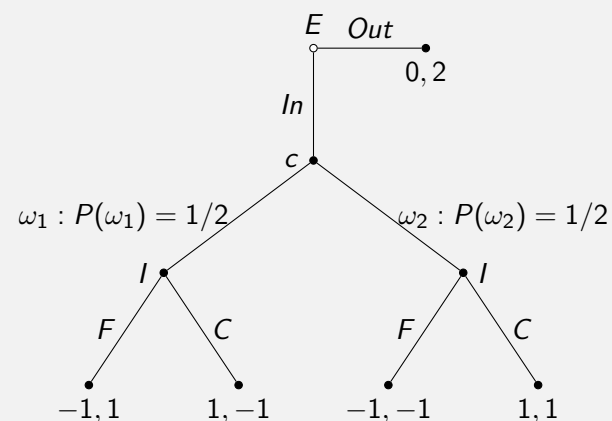
Definition

An **extensive game with perfect information and chance moves** is a tuple $\{N, H, P, f_c, (\succeq_i)\}$ where N is a finite set of players and H is a set of histories, and

- P is a function from $H \setminus Z$ to $N \cup \{c\}$. c is **chance**.
- For each $h \in H$ with $P(h) = c$, $f_c(\cdot|h)$ is a probability measure over $A(h)$; each such measure is independent of every other such measure.
- For each player $i \in N$, \succeq_i is a preference relation on lotteries over the set of terminal histories.

- Definition of subgame perfect equilibrium is the same as before.
- One deviation property and Kuhn's theorem hold.

Extensive Game with Perfect Information and Chance Moves



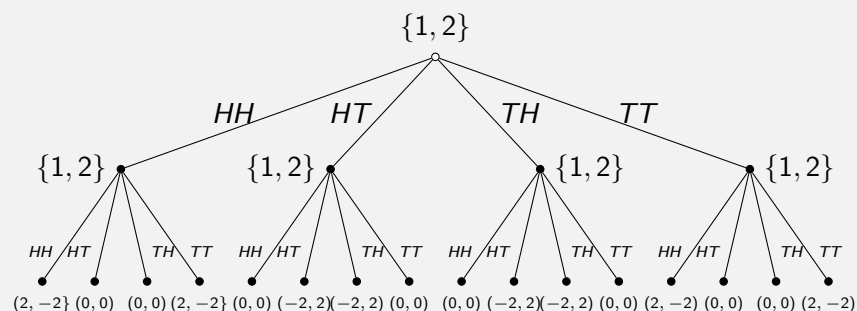
Extensive Game with Perfect Information and Simultaneous Moves

Definition

An **extensive game with perfect information and simultaneous moves** is a tuple $\{N, H, P, (\succeq_i)\}$ where N is a finite set of players, H is a set of histories, for each $i \in N$, \succeq_i is player i 's preference relation over Z , and

- P is a correspondence from $H \setminus Z$ to N .
- For every $h \in H \setminus Z$ there is a collection $\{A_i(h)\}_{i \in P(h)}$ for which $A(h) = \{a : (h, a) \in H\} = \times_{i \in P(h)} A_i(h)$.
- A **strategy** of player $i \in N$ is a function that assigns an action in $A_i(h)$ to every nonterminal history h for which $i \in P(h)$.
- Definition of subgame perfect equilibrium is the same as before except that $P(h) = i$ is replaced by $i \in P(h)$.
 - ▶ One deviation property holds.
 - ▶ Kuhn's theorem does not.

Extensive Game with Perfect Information and Simultaneous Moves



Bank Runs

- Two investors have each deposited D in a bank.
- The bank makes a long-term investment that at maturity will payout $2R$ where $R > D$.
- If the bank is forced to liquidate its investments a total of $2r$ can be recovered, where $D > r > D/2$.
- Investors can withdraw at two dates:
 - ▶ Date 1 is before maturity
 - ▶ Date 2 is after maturity.
- Assume no discounting.

Bank Runs

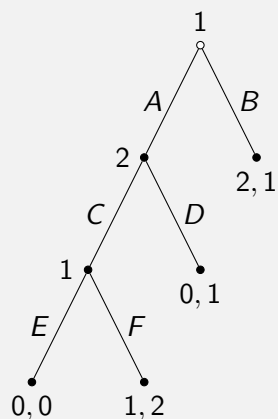
| | | |
|----------|-------------|-------------|
| | withdraw | don't |
| withdraw | r, r | $D, 2r - D$ |
| don't | $2r - D, D$ | next stage |

| | | |
|----------|-------------|-------------|
| | withdraw | don't |
| withdraw | R, R | $D, 2R - D$ |
| don't | $2R - D, D$ | R, R |

Interpretation of Strategy

- A strategy is not a plan of action - it requires specification of actions after histories that cannot be reached if a player follows his strategy.
- Alternative interpretation is that the strategy is the belief of the other players.
- Beliefs of others about my action can influence how I rationalize my own action.
 - ▶ Players do not choose other players beliefs.
 - ▶ Other player's beliefs are required to be the same.
 - ▶ Constraints on strategies imply constraints on player beliefs.

Interpretation of Strategy - Example



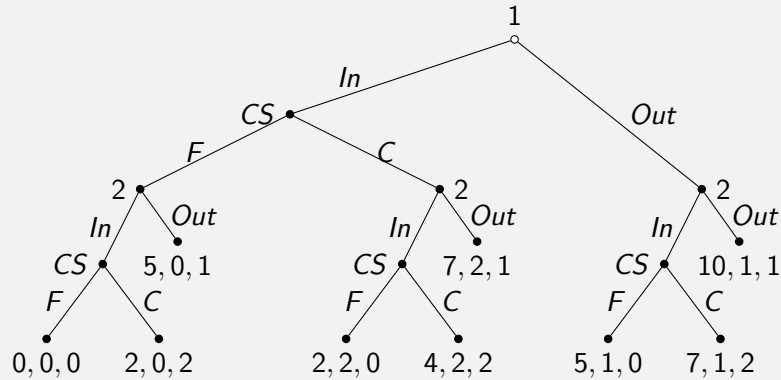
- How can player 2 rationalize A by player 1?
- Contradicts rationality.
- Subgame perfection requires player 2 to maintain rationality assumption even when he sees A.

Chain Store Game

- An incumbent firm faces a **sequence** of K potential entrants in K different markets. In each market k
 - ▶ if entrant stays out, the incumbent gets 5 and the entrant gets 1;
 - ▶ if the entrant enters and the incumbent fights, both get 0;
 - ▶ if the entrant enters and the incumbent cooperates, both get 2;
 - ▶ there are 3 possible outcomes $Q^k = \{Out, (In, C), (In, F)\}$.
- At every point in the game all players observe all previous actions so we have an extensive game of perfect information.
 - ▶ $H = \{(\cup_{k=0}^K Q^k) \cup (\cup_{k=0}^{K-1} (Q^k \times \{In\}))\}$
 - ▶ $P(h) = k + 1$ if $h \in Q^k$ and $P(h) = CS$ if $h \in Q^k \times \{In\}$, for $k = 0, \dots, K - 1$.
 - ▶ The payoff of the chain store is the sum of its payoffs in the K markets.

Chain Store Game

Two period Chain Store game



Find the set of subgame perfect equilibria.
Find the set of Nash Equilibria.

Centipede Game

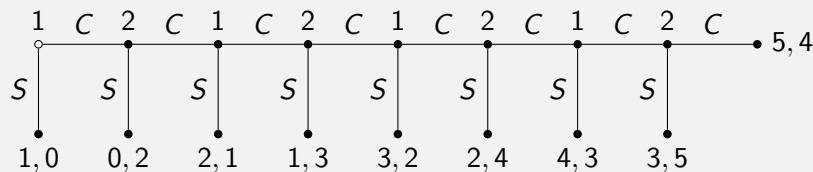
- Two players are involved in a process that they alternately have the opportunity to stop.
- Continuing the game by one period reduces the player's payoff by one but increases the other player's payoff by two.
- The process stops after T periods (T is even).
 - H consists of all sequences $C(t) = (C, \dots, C)$ of length t for $0 \leq t \leq T$ and all sequences $S(t) = (C, \dots, C, S)$ consisting of $t - 1$ Cs for $1 \leq t \leq T$.
 - $P(C(t)) = 1$ if t is even and $t < T$, $P(C(t)) = 2$ if t is odd.

$$u_1(S(t)) = \begin{cases} (t+1)/2 & \text{if } t \text{ is odd} \\ t/2 - 1 & \text{if } t \text{ is even} \end{cases} \quad u_1(C(T)) = T/2 + 1$$

$$u_2(S(t)) = \begin{cases} (t-1)/2 & \text{if } t \text{ is odd} \\ t/2 + 1 & \text{if } t \text{ is even} \end{cases} \quad u_2(C(T)) = T/2$$

Centipede Game

Eight period Centipede game



Find the set of subgame perfect equilibria.
Find the set of Nash Equilibria.

Predictive Ability

