Systems of Equations and Hedging

A system of m equations for n unknowns $x_1, \ldots, x_n$,

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1, \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2, \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m, \end{aligned}$$

can be written in matrix form as

$$\begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{bmatrix} x_1 + \begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$A_{\bullet 1}x_1 + A_{\bullet 2}x_2 + \cdots + A_{\bullet n}x_n = b$$

or

$$Ax = b,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{1}$$

One can think of the columns of A as being n securities in m states, x being a portfolio of the n securities and b another security that we want to hedge. In such a situation the securities in A are called *basis assets* and the security b is called a *focus asset*. We know that Ax gives the pay-off of the portfolio x of basis assets. To solve a system of equations $Ax = b$ therefore means finding a portfolio x of basis assets that replicates (perfectly hedges) the focus asset b.

Typically, the basis assets are *liquid securities* with *known prices*, whereas the focus asset b is an over-the-counter (OTC) security issued by an investment bank. Such securities are issued between two parties and do not have a liquid secondary market. The question is, what is a fair price of the OTC security?

By issuing the focus asset b the bank commits itself to pay different amounts of money in different states of the world and thus it enters into a risky position. Hedging is a simultaneous purchase of another portfolio that *reduces* this risk, and a *perfect hedge* is a portfolio that eliminates the risk completely. Suppose that portfolio x is a perfect hedge to the focus asset b. The bank will issue asset b (promise to pay b_i in state i tomorrow) and simultaneously purchase the replicating portfolio x of basis assets.

How much will the bank charge for issuing the OTC security? To break even, it will charge exactly the cost of the replicating portfolio (plus a fee to cover its overheads). Tomorrow, when the payment of b becomes due it will liquidate the hedging portfolio x. Since x was a perfect hedge, the pay-off of the hedging portfolio Ax will exactly match the liability b in each state of the world. Hence the bank will not have incurred any risk in this operation.

Linear Independence and Redundant Securities

Let the column vectors $A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n} \in \mathbb{R}^m$ represent n securities in m states of the world, in the sense discussed above.

Definition We say that vectors (securities) $A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n}$ are linearly independent if the only solution to

$$A_{\bullet 1}x_1 + A_{\bullet 2} + \cdots + A_{\bullet n}x_n = 0$$

is the trivial portfolio

$$x_1 = 0, x_2 = 0, \cdots, x_n = 0$$

Mathematicians call the sum $A_{\bullet 1}x_1+A_{\bullet 2}x_2+\cdots+A_{\bullet n}x_n$ a linear combination of vectors $A_{\bullet 1}, A_{\bullet 2}, \cdots A_{\bullet n}$ and the numbers $x_1, \cdots, x_n$ are coefficients of the linear combination. To us $x_1, \cdots, x_n$ represent numbers of units of each security in a portfolio and the linear combination represents the *portfolio pay-off.*

The meaning of linear independence is best understood if we look at a situation where $A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n}$ are not linearly independent. From the definition it means that there is a linear combination where at least one of the coefficients $x_1, \cdots, x_n$ is non-zero and

$$A_{\bullet 1}x_1 + A_{\bullet 2} + \cdots + A_{\bullet n}x_n = 0 \tag{2}$$

Without loss of generality we can assume that $x_1 \neq 0$. One can then solve (2) for $A_{\bullet 1}$:

$$A_{\bullet 1} = -\left(A_{\bullet 2}\frac{x_2}{x_1} + \cdots + A_{\bullet n}\frac{x_n}{x_1}\right)$$

The last equality means that $A_{\bullet 1}$ is a linear combination of vectors $A_{\bullet 2}, ..., A_{\bullet n}$ with coefficients $-x_2/x_1$, . . . ,$-x_n/x_1$. In conclusion, if the vectors

$A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n}$ are not linearly independent, then at least one of them can be expressed as a linear combination of the remaining $n-1$ vectors. And vice versa, if vectors $A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n}$ are linearly independent, then none of them can be expressed as a linear combination of the remaining $n-1$ vectors.

Securities that are linear combinations of other securities are called *redundant* and the portfolio which achieves the same pay-off as that of a redundant security is called a *replicating portfolio*. Redundant securities do not add anything new to the market because their pay-off can be synthesized from the pay-off of the remaining securities; instead of trading a redundant security we might equally well trade the replicating portfolio with the same result.

The practical significance of linearly independent securities, on the other hand, is that each additional linearly independent security has a pay-off previously unavailable in the market. The *marketed subspace* is formed by pay-offs of all possible portfolios (linear combinations) of basis assets and is denoted $\mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n})$. As was mentioned above each linearly independent security adds something new to the market—it adds one extra dimension to the marketed subspace. Consequently, the maximum number of linearly independent securities in the marketed subspace is called *the dimension of the marketed subspace*. The definition of dimension is made meaningful by the following theorem.

Theorem (Dimensionality Theorem). Suppose $A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n}$ are n linearly independent vectors. Suppose

$$B_{\bullet 1}, B_{\bullet 2}, ..., B_{\bullet k} \in \mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n})$$

are linearly independent. Then

$$\mathrm{Span}(B_{\bullet 1}, B_{\bullet 2}, ..., B_{\bullet k}) = \mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, ..., A_{\bullet n})$$

if and only if $k = n$. $\square$

We say that the market is *complete* if the market subspace

$$\mathrm{Span}(A_{\bullet 1}, A_{\bullet 2}, \cdots, A_{\bullet n})$$

includes all possible pay-offs over the m states, that is, if it contains *all possible* m-dimensional vectors. A complete market means that whatever distribution of wealth in the m market scenarios one may think of, it can always be achieved as a pay-off from a portfolio of marketed securities. Since the dimension of $\mathbb{R}^m$ is m, another way of saying that the market is complete is to claim that there are m linearly independent basis securities or that the dimension of the marketed subspace is m.