Advanced Macroeconomics II

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1. Spectral Analysis and Filtering

Trigonometric Series

Definition

A trigonometric series is a series of the form

$$\frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + \ldots + a_n \cos nx + b_n \sin nx + \ldots \tag{1}$$

where the coefficients a_n and b_n are constants. If these constants satisfy certain conditions specified later, then the series is called a Fourier series.

Each term in (1) has the property of repeating itself in intervals of 2π:

$$\begin{aligned} \cos(x + 2\pi) &= \cos x, \quad \sin(x + 2\pi) = \sin x \\ \cos\left[n(x + 2\pi)\right] &= \cos(nx + 2n\pi) = \cos nx \end{aligned}$$

It follows that if (1) converges for all x, then its sum $f(x)$ must also have this property:

$$f(x+2\pi) = f(x). \tag{2}$$

We say: $f(x)$ has period 2π.
In general, a function $f(x)$ is said to be periodic and have period p, if

$$f(x+p) = f(x), \quad (p > 0) \tag{3}$$

It should be noted that $\cos(2x)$ has in addition the period 2π, the period π, and in general, $\cos(nx)$ and $\sin(nx)$ have the period $2\pi/n$. However, 2π is the smallest period shared by all terms of the series.

If $f(x)$ has the period p, then the substitution

$$\frac{x}{p} = \frac{t}{2\pi} \rightarrow t = \frac{2\pi}{p}x \tag{4}$$

converts $f(x)$ into a function of t having period 2π, for when t increases by 2π, x increases by p.

2 Fourier Series

Let us suppose now that a periodic function $f(x)$ is the sum of a trigonometric series (1), i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{5}$$

What is the relationship between the coefficient a_n and b_n and the function $f(x)$? To answer this, we multiply $f(x)$ by $\cos mx$ and integrate from $-\pi$ to π:

$$\begin{aligned} & \int_{-\pi}^{\pi} f(x) \cos mx dx \\ = & \int_{-\pi}^{\pi} \left[\frac{a_0}{2} \cos mx + \sum_{n=1}^{\infty} (a_n \cos nx \cos mx + b_n \sin nx \cos mx) \right] dx. \end{aligned}$$

Fourier Series

Integrate term by term,

$$\begin{aligned}
& \int_{-\pi}^{\pi} f(x) \cos mx dx \\
= \quad & \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx \\
& + \sum_{n=1}^{\infty} \left\{ a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx \right\} \qquad (6)
\end{aligned}$$

using

$$\begin{aligned}
\cos x \cos y &= \frac{1}{2} \left[\cos(x+y) + \cos(x-y) \right] \\
\sin x \cos y &= \frac{1}{2} \left[\sin(x+y) + \sin(x-y) \right] \qquad (7) \\
\sin x \sin y &= -\frac{1}{2} \left[\cos(x+y) - \cos(x-y) \right] \\
\sin' x &= \cos x, \quad \cos' x = -\sin x \qquad (8)
\end{aligned}$$

Fourier Series

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos nx \cos mx dx \\
= \quad & \frac{1}{2}\int_{-\pi}^{\pi} \cos(nx+mx)dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(nx-mx)dx
\end{aligned}$$

If $n \neq m$,

$$\begin{aligned}
= \quad & \frac{1}{2}\frac{\sin(n+m)x}{n+m}\Big|_{-\pi}^{\pi} + \frac{1}{2}\frac{\sin(n-m)x}{n-m}\Big|_{-\pi}^{\pi} \\
= \quad & 0
\end{aligned}$$

If $n = m \neq 0$,

$$\begin{aligned}
= \quad & \frac{1}{2}\frac{\sin(n+m)x}{n+m}\Big|_{-\pi}^{\pi} + \frac{1}{2}\int_{-\pi}^{\pi} 1dx \\
= \quad & \pi
\end{aligned}$$

Fourier Series

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \sin nx \cos mx dx \\
= \quad & \frac{1}{2}\int_{-\pi}^{\pi} \sin(nx+mx)dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(nx-mx)dx
\end{aligned}
$$

If $n \neq m$,

$$
\begin{aligned}
= \quad & \left[-\frac{1}{2}\frac{\cos(n+m)x}{n+m} - \frac{1}{2}\frac{\cos(n-m)x}{n-m}\right]|_{-\pi}^{\pi} \\
= \quad & 0
\end{aligned}
$$

If $n = m \neq 0$,

$$
\begin{aligned}
= \quad & -\frac{1}{2}\frac{\cos(n+m)x}{n+m}|_{-\pi}^{\pi} + \frac{1}{2}\int_{-\pi}^{\pi} 0dx \\
= \quad & 0
\end{aligned}
$$

Fourier Series

If $m = 0$, then all terms on the RHS of (6) are 0 except the first one, which is

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} f(x) dx = \pi a_0. \tag{9}$$

For any given positive integer m, only the term in a_m gives a result different from 0. Thus

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \pi a_m, \; m = 1, 2, ... \tag{10}$$

Multiplying $f(x)$ by $\sin mx$ and preceeding in the same way, we find

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = \pi b_m, \; m = 1, 2, ... \tag{11}$$

Fourier Series

From the last three formulas, we now conclude that

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx,\ n = 0, 1, 2, ... \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx,\ n = 1, 2, ... \end{aligned} \tag{12}$$

This is the fundamental rule for coefficients in a Fourier series.

Fourier Series

Definition

We define a **Fourier Series** to be any trigonometric series

$$\frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + ... + a_n \cos nx + b_n \sin nx + ... \qquad (13)$$

in which the coefficient a_n and b_n are computed from a function $f(x)$ by equation (12); the series is then called the Fourier Series of $f(x)$.

But $f(x)$ does not necessarily have the representation of equation (13). Concerning $f(x)$ we assume only that the integrals in (12) exists; for this it is sufficient that $f(x)$ be continuous except for a finite number of jumps between $-\pi$ and π.

A magic equation

$$e^{ix} = \cos x + i \sin x.$$

How this comes about?

2.1 An alternative representation

Taylor series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

with $a = 0$, Maclaurin series,

$$\begin{aligned}
f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 ... + \frac{f^{(n)}(0)}{n!}x^n + ... \\
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \forall x \\
e^{ix} &= 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} ... + \frac{i^4x^n}{n!} + ... \\
&= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + ... \qquad (14)
\end{aligned}$$

An alternative representation

Using

$$\sin' x = \cos x, \quad \cos' x = -\sin x$$

We have

$$\begin{aligned}
\sin' x &= \cos x, \;\; \sin'' x = -\sin x, \;\; \sin^{(3)} x = -\cos x, \;\; \sin^{(4)} x = \sin x \\
\sin' 0 &= 1, \;\; \sin'' 0 = 0, \;\; \sin^{(3)} 0 = -1, \;\; \sin^{(4)} 0 = 0 \\
\cos' 0 &= 0, \;\; \cos^{(2)} 0 = -1, \;\; \cos^{(3)} 0 = 0, \;\; \cos^{(4)} 0 = 1
\end{aligned}$$

Taylor expansion:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} + \dots, \;\; \forall x \tag{15}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots, \;\; \forall x \tag{16}$$

An alternative representation

Compare Equations (15) and (16) with (14),

$$e^{ix} = \cos x + i \sin x.$$

Similarly, we can prove

$$e^{-ix} = \cos x - i \sin x.$$

Then,

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} = \frac{i(e^{-ix} - e^{ix})}{2}. \end{aligned}$$

An alternative representation

Review on a complex variable

A complex variable

$$z = x + iy$$

where $\sqrt{-1} = i$, $i^2 = -1$.

$$\begin{aligned}
|z| &= \sqrt{x^2 + y^2} = \sqrt{(x + iy)(x - iy)}. \\
z &= |z| (\cos\omega + i \sin\omega) = |z| e^{i\omega} \\
z^j &= |z|^j e^{i\omega j} = |z|^j (\cos\omega j + i \sin\omega j)
\end{aligned}$$

An alternative representation

A Fourier series:

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{i(e^{-inx} - e^{inx})}{2}\right) \\
&= \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \left(\underbrace{\frac{a_n - ib_n}{2}}_{c_n} e^{inx} + \underbrace{\frac{a_n + ib_n}{2}}_{c_{-n}} \underbrace{e^{-inx}}_{e^{i(-n)x}}\right) \\
&= \sum_{n=-\infty}^{\infty} c_n e^{inx}
\end{aligned}
$$

where $a_n = c_n + c_{-n}$, $(n = 0, 1, 2, \ldots)$, $b_n = i(c_n - c_{-n})$, $(n = 1, 2, \ldots)$.

An alternative representation

The relationship between $f(x)$ and c_n?
The coefficients c_n can be defined directly in terms of $f(x)$:

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,\ n = 0, \pm 1, \pm 2, ... \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx - i\sin nx\right)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx
\end{aligned}
$$

If $n = 0,\ c_n = \frac{1}{2}a_0.$
If $n > 0,\ c_n = \frac{a_n - ib_n}{2}.$
If $n < 0,\ c_n = \frac{a_n + ib_n}{2}.$

An alternative representation

To summarize,

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\ c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \end{aligned}$$

whenever the series converges to $f(x)$.

2.2 Fourier Cosine/Sine Series

Symmetry properties over $[-\pi, \pi]$ and even/odd functions.

Definition

$f(x)$ is called an even function of x, if

$$f(-x) = f(x), \quad x \in [-\pi, \pi].$$

$f(x)$ is called an off function of x, if

$$f(-x) = -f(x), \quad x \in [-\pi, \pi].$$

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f \text{ is odd.} \\ 2\int_0^a f(x)dx, & \text{if } f \text{ is even.} \end{cases}$$

- The product of two even functions or two odd functions is even;
- The product of an odd function and an even function is odd.

Fourier Cosine/Sine Series

Let f be even in the interval $-\pi \leq x \leq \pi$, then

- $f(x) \cos nx$ is even (product of two even functions).
- $f(x) \sin nx$ is odd (product of an odd and an even functions).

Hence,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad (n = 0, 1, 2, ...) \\ &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx dx. \\ b_n &= 0 \qquad\qquad\qquad\qquad (n = 0, 1, 2, ...) \end{aligned}$$

Fourier Cosine/Sine Series

Let f be odd in the interval $-\pi \leq x \leq \pi$, then

- $f(x)\cos nx$ is odd product of an odd and an even functions).
- $f(x)\sin nx$ is even (product of two odd functions).

Hence,

$$\begin{aligned} a_n &= 0 \quad (n = 0, 1, 2, ...) \\ b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx \quad (n = 0, 1, 2, ...) \end{aligned}$$

Fourier Cosine/Sine Series

Thus we have the expansions:

- $f(x)$ is even:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx. \end{aligned}$$

- $f(x)$ is odd:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx \\ b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \end{aligned}$$

3. Spectral Analysis

3.1 Time domain vs. Frequency domain

Definition

For a time series $\{Y_t\}_{t=-\infty}^{\infty}$, the value of a variable Y_t at date t can be described in terms of a sequence of innovations $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ in models of the form

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}.$$

We can analyse the covariance properties between Y_t and Y_τ at distinct dates t and τ. This is known as analysing the properties of $\{Y_t\}_{t=-\infty}^{\infty}$ in the time domain.

3.1.2 Examples of Spectrum for some typical time series

White noise: $Y_t = \varepsilon_t$

$$s_Y(\omega) = \frac{\sigma^2}{2\pi}.$$

MA(1): $Y_t = \varepsilon_t + \theta\varepsilon_{t-1}$

$$s_Y(\omega) = \frac{\sigma^2}{2\pi}\left[1 + \theta^2 + 2\theta\cos(\omega)\right].$$

AR(1): $Y_t = c + \phi Y_{t-1} + \varepsilon_t$, $|\phi| < 1$.

$$s_Y(\omega) = \frac{\sigma^2}{2\pi}\frac{1}{1 + \phi^2 - 2\phi\cos(\omega)}$$

3.1.3 Calculating the autocovariances from the population spectrum

Let $\{\gamma_j\}_{j=-\infty}^{\infty}$ be an absolutely summable sequence of autocovariances, and define $s_Y(\omega)$ as

$$s_Y(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j},$$

then

$$\int_{-\pi}^{\pi} s_Y(\omega) e^{i\omega k} d\omega = \gamma_k \tag{17}$$

$$\Leftrightarrow \int_{-\pi}^{\pi} s_Y(\omega) \cos(\omega k) d\omega = \gamma_k \tag{18}$$

$$\Leftrightarrow 2 \int_{0}^{\pi} s_Y(\omega) \cos(\omega k) d\omega = \gamma_k \tag{19}$$

3.1.4 Interpreting the population spectrum

By setting $k = 0$, from equation (18),

$$\int_{-\pi}^{\pi} s_Y(\omega) d\omega = \gamma_0.$$

The area under the population spectrum between $[-\pi, \pi]$ gives γ_0, the variance of Y_t.

Interpreting the population spectrum

More generally, since $s_Y(\omega)$ is nonnegative, if we were to calculate

$$\int_{-\omega_1}^{\omega_1} s_Y(\omega) d\omega$$

for any ω_1 between 0 and π, the result would be a positive number that we could interpret as the portion of the variance of Y_t that is associated with frequences ω that are less than ω_1 in absolute value. Recalling that $s_Y(\omega)$ is symmetric, the claim is that

$$2\int_{0}^{\omega_1} s_Y(\omega) d\omega.$$

Interpreting the population spectrum

What does it mean to attribute a certain portion of the variance of Y to cycles with frequencies less than or equal to ω_1?
Consider the following special stochastic process. Suppose that the value of Y at date t is determined by

$$Y_t = \sum_{j=1}^{M} \left[\alpha_j \cos\left(\omega_j t\right) + \delta_j \sin\left(\omega_j t\right)\right] \tag{20}$$

where α_j and δ_j are zero-mean random variables, i.e. $E(Y_t) = 0\ \forall t$.

Interpreting the population spectrum

The sequences $\{\alpha_j\}_{j=1}^{M}$ and $\{\delta_j\}_{j=1}^{M}$ are serially uncorrelated and mutually uncorrelated:

$$\begin{aligned}
E(\alpha_j \alpha_k) &= \begin{cases} \sigma_j^2 & \forall\, j = k \\ 0 & \forall\, j \neq k \end{cases} \\
E(\delta_j \delta_k) &= \begin{cases} \sigma_j^2 & \forall\, j = k \\ 0 & \forall\, j \neq k \end{cases} \\
E(\alpha_j \delta_k) &= 0 \quad \forall\, j, k.
\end{aligned}$$

Interpreting the population spectrum

The variance of Y_t is then:

$$\begin{aligned}
E\left(Y_t^2\right) &= \sum_{j=1}^{M}\left[E\left(\alpha_j^2\right)\cos^2\left(\omega_j t\right) + E\left(\delta_j^2\right)\sin^2\left(\omega_j t\right)\right] \\
&= \sum_{j=1}^{M}\sigma_j^2\left[\cos^2\left(\omega_j t\right) + \sin^2\left(\omega_j t\right)\right] \\
&= \sum_{j=1}^{M}\sigma_j^2 \qquad (21)
\end{aligned}$$

Interpreting the population spectrum

Thus, for this process, the portion of the variance of Y that is due to cycles of frequency ω_j is given by σ_j^2. If the frequencies are ordered $0 < \omega_1 < \omega_2 < ... < \omega_M < \pi$, the portion of the variance of Y that is due to cycles of frequency less than or equal to ω_j is given by $\sigma_1^2 + \sigma_2^2 + ... + \sigma_j^2$.

Interpreting the population spectrum

The kth autocovariance of Y is

$$\begin{aligned} E\left(Y_t Y_{t-k}\right) &= \sum_{j=1}^{M}\left\{E\left(\alpha_j^2\right)\cos\left(\omega_j t\right)\cos\left(\omega_j\left(t-k\right)\right)\right. \\ &\quad \left. + E\left(\delta_j^2\right)\sin\left(\omega_j t\right)\sin\left(\omega_j\left(t-k\right)\right)\right\} \\ &= \sum_{j=1}^{M}\sigma_j^2\left\{\cos\left(\omega_j t\right)\cos\left(\omega_j\left(t-k\right)\right)\right. \\ &\quad \left. + \sin\left(\omega_j t\right)\sin\left(\omega_j\left(t-k\right)\right)\right\} \end{aligned} \tag{22}$$

Utilize the trigonometric identity

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B),$$

we obtain

$$E\left(Y_t Y_{t-k}\right) = \sum_{j=1}^{M}\sigma_j^2\cos\left(\omega_j k\right). \tag{23}$$

Interpreting the population spectrum

- This process is covariance stationary, but (23) implies that the sequence of autocovariances $\{\gamma_j\}_{k=0}^{\infty}$ is not absolutely summable.
- The example process of (20) is rather special, that we can attribute a certain proportion of the variance of Y_t to cycles of less than a given frequency.
- However, there is a general result known as the spectral representation theorem which says that any covariance-stationary process Y_t can be expressed in terms of a generalization of (20).

Interpreting the population spectrum

For any fixed frequency ω in $[0, \pi]$, we define random variables $\alpha(\omega)$ and $\delta(\omega)$ and propose to write a stationary process with absolutely summable autocovariances in the form

$$Y_t = \mu + \int_0^{\pi} [\alpha(\omega)\cos(\omega t) + \delta(\omega)\sin(\omega t)]\, d\omega.$$

The random processes represented by $\alpha(\cdot)$ and $\delta(\cdot)$ have the following properties:

- Zero mean;
- For any frequencies $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \pi$, the variables

$$\begin{aligned} E\left[\int_{\omega_1}^{\omega_2} \alpha(\omega)\, d\omega\right]\left[\int_{\omega_3}^{\omega_4} \alpha(\omega)\, d\omega\right] &= 0 \\ E\left[\int_{\omega_1}^{\omega_2} \delta(\omega)\, d\omega\right]\left[\int_{\omega_3}^{\omega_4} \delta(\omega)\, d\omega\right] &= 0 \end{aligned}$$

- For any $0 < \omega_1 < \omega_2 < \pi$ and $0 < \omega_3 < \omega_4 < \pi$,

$$E\left[\int_{\omega_1}^{\omega_2} \alpha(\omega)\, d\omega\right]\left[\int_{\omega_3}^{\omega_4} \delta(\omega)\, d\omega\right] = 0.$$

Interpreting the population spectrum

For such a process, one can calculate the protion of the variance of Y_t that is due to cycles with frequency less than or equal to some specified value ω_1 through a generalization of the procedure used to analyze (20). Moreover, this magnitude turns out to be given by

$$2\int_0^{\omega_1} s_Y(\omega)\, d\omega \tag{24}$$

3.2 The Sample Periodogram

For a covariance-stationary process Y_t with absolutely summable autocovariances, we have defined the value of the population spectrum at frequency ω to be

$$s_Y(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} \tag{25}$$

where

$$\begin{aligned} \gamma_j &\equiv E\left(Y_t - \mu\right)\left(Y_{t-j} - \mu\right) \\ \mu &\equiv E\left(Y_t\right). \end{aligned}$$

Note that the population spectrum is expressed in terms of $\left\{\gamma_j\right\}_{j=-\infty}^{\infty}$, which represents population second moments.

The Sample Periodogram

Given an observed sample of T observations denoted by $y_1, y_2, ..., y_T$, we can calculate up to $T-1$ sample autocovariances from the formulas

$$\hat{\gamma}_j = \begin{cases} T^{-1} \sum_{t=j+1}^{T} (y_t - \bar{y})(y_{t-1} - \bar{y}), & \text{for } j = 0, 1, 2, ..., T-1 \\ \hat{\gamma}_{-j}, & \text{for } j = -1, -2, ..., -T+1 \end{cases} \tag{26}$$

where sample mean:

$$\bar{y} = T^{-1} \sum_{t=1}^{T} y_t. \tag{27}$$

The Sample Periodogram

For any given ω we can then construct the sample analog of (25), which is known as the sample periodogram:

$$\hat{s}_Y(\omega) = \frac{1}{2\pi} \sum_{j=-T+1}^{T-1} \hat{\gamma}_j e^{-i\omega j}. \tag{28}$$

Equivalently:

$$\hat{s}_Y(\omega) = \frac{1}{2\pi} \left[\hat{\gamma}_0 + 2 \sum_{j=1}^{T-1} \hat{\gamma}_j \cos(\omega j) \right]. \tag{29}$$

The Sample Periodogram

It can be shown as in the population spectrum that the area under the periodogram is the sample variance of y:

$$\int_{-\pi}^{\pi} \hat{s}_Y(\omega)\, d\omega = \hat{\gamma}_0.$$

The sample periodogram is also symmetric around $\omega = 0$, so

$$\hat{\gamma}_0 = 2\int_{0}^{\pi} \hat{s}_Y(\omega)\, d\omega.$$

The Sample Periodogram

There also turns out to be a sample analog to the spectral representation theorem,

$$y_t = \hat{\mu} + \sum_{j=1}^{M} \left\{ \hat{\alpha}_j \cos\left[\omega_j (t-1)\right] + \hat{\delta}_j \sin\left[\omega_j (t-1)\right] \right\}. \tag{30}$$

For the case when T is an odd number, $M \equiv \frac{T-1}{2}$, and frequencies are specified as

$$\begin{array}{llll} \omega_1 = \frac{2\pi}{T}, & \omega_2 = \frac{4\pi}{T}, & \ldots & \omega_M = \frac{2M\pi}{T} \leq \pi. \\ p_1 = T, & p_2 = \frac{T}{2}, & \ldots & p_M = \frac{T}{M} \geq 2. \end{array} \tag{31}$$

The Sample Periodogram

Range of frequence $[0, \pi]$.

- If $\omega < 0$, take a special case

$$\begin{aligned} y_t &= \alpha \cos(-\omega t) + \delta \sin(-\omega t) \\ &= \alpha \cos(\omega t) - \delta \sin(\omega t) \end{aligned}$$

  which is indistinguishable between ω and $-\omega$.

- If $\omega > 0$, take an example of $\omega = \frac{3}{2}\pi$,

$$\begin{aligned} y_t &= \alpha \cos(\frac{3}{2}\pi t) + \delta \sin(\frac{3}{2}\pi t) \\ &= \alpha \cos\left[\left(-\frac{\pi}{2}\right) t\right] + \delta \sin\left[\left(-\frac{\pi}{2}\right) t\right] \\ &= \alpha \cos\left(\frac{\pi}{2} t\right) + \delta \sin\left(\frac{\pi}{2} t\right) \end{aligned}$$

  again indistinguishable between $\frac{3}{2}\pi$ and $\frac{\pi}{2}t$.

The Sample Periodogram

Range of frequence $[0, \pi]$.

- The shortest-period cycle one can observe is $\omega = \pi$, $p = \frac{2\pi}{\omega} = 2$. If $\omega = \frac{3}{2}\pi$, the cycle repeats itself every $\frac{4}{3}$ periods. But if the data are observed only at integer dates, the samples data will exhibit cycles that are repeated every four periods, corresponding to the frequency $\omega = \frac{1}{2}\pi$.
- The lowest frequency (longest cycle) in a particular finite sample is $\omega_1 = \frac{2\pi}{T}$, $p = T$. If $p > T$, no inference can be done for it with T observations.

3.3 Uses of Spectral Analysis

3.3.1 Example: US seasonally unadjusted monthly index of manufacturing production.

Sample period: 1947:1 - 1989:11, $T = 513$, $\omega_1 \approx 0.012$, $\omega_M \approx \pi$.
Features of time series:

1. Economic recessions: 1949, 1954, 1958, 1960, 1970, 1974, 1980, 1982. Eight cycles, about five years per cycle. Each cycle has a roughly year-long episodes of falling production.
2. Strong seasonal patterns, e.g., almost always declines in July and recovers in August.
3. Trend.

Features of the sample periodogram:
The contribution to the sample variance of the lowest frequency component ($j \to 0$) is several orders of magnitude larger than the contributions of economic recessions or the seasonal factors.
Transformation: monthly log difference, annual log difference

3.3 Uses of Spectral Analysis

3.3.2 Effects of filtering

Let's start from the following equation

$$X_t = \sum_{j=-\infty}^{\infty} A_j Y_{t-j} = A(L) Y_t.$$

Y_t: the input process; $A(L)$: the linear filter; X_t: the output process. Assume Y_t is $I(0)$, with spectral density $s_Y(\omega)$. Then

$$s_X(\omega) = \left| \sum_{j=-\infty}^{\infty} A_j e^{-ij\omega} \right|^2 s_Y(\omega) = \left| A(e^{-i\omega}) \right|^2 s_Y(\omega)$$

In polar form

$$A(e^{-i\omega}) = \left| A(e^{-i\omega}) \right| e^{i\phi(\omega)}$$

The effects are two folds: amplitude change, and phase shift. A Symmetric filter has no phase shift.

Effects of filtering

Computing the spectral density for an ARMA process is straightforward. Let $X_t \sim ARMA(p,q)$, $\phi(L)X_t = \theta(L)\varepsilon_t$. The spectral density of X_t is

$$s_X(\omega) = \frac{\left|\theta(e^{-i\omega})\right|^2}{\left|\phi(e^{-i\omega})\right|^2}\frac{\sigma^2}{2\pi}.$$

Example 1. AR(1)

$$\begin{aligned} X_t &= \rho X_{t-1} + \varepsilon_t \\ (1-\rho L)X_t &= \varepsilon_t \\ s_X(\omega) &= \left|1-\rho e^{-i\omega}\right|^{-2}\frac{\sigma^2}{2\pi} \end{aligned}$$

e.g. $\rho = 0.8$, $\rho = -0.8$ features different frequencies.

Effects of filtering

Example 2. First difference

$$
\begin{aligned}
X_t &= Y_t - Y_{t-1} \\
&= (1-L)Y_t \\
A(e^{-i\omega}) &= 1 - e^{-i\omega} = 1 - \cos\omega + i\sin\omega \\
\left|A(e^{-i\omega})\right| &= (2 - 2\cos\omega)^{1/2} \\
\left|A(e^{-i\omega})\right|^2 &= 2 - 2\cos\omega
\end{aligned}
$$

There is a phase shift of $\arctan(\frac{\sin\omega}{1-\cos\omega}) > 0$, so that the output is leading the input.

Effects of filtering

Example 3. Moving average

$$X_t = \frac{Y_{t-1} + Y_t + Y_{t+1}}{3} = \frac{L + 1 + L^{-1}}{3} Y_t$$

$$\begin{aligned} A(e^{-i\omega}) &= \frac{e^{-i\omega} + 1 + e^{i\omega}}{3} = \frac{1 + 2\cos\omega}{3} \\ \left|A(e^{-i\omega})\right|^2 &= \left(\frac{1 + 2\cos\omega}{3}\right)^2 \end{aligned}$$

No phase shift.
The filter is 0 at $\omega = \frac{2}{3}\pi$, i.e. it kills the periodicities equal to $p = \frac{2\pi}{\omega} = 3$.

Effects of filtering

Example 4. H-P filter

$$\left[1+\lambda(1-L)^2(1-L^{-1})^2\right]\tau_t = y_t$$

$$\begin{aligned} \tau_t &= F(L)y_t \\ c_t &= y_t - \tau_t = [1-F(L)]\,y_t \end{aligned}$$

For cycle,

$$A(e^{-i\omega}) = \frac{4\lambda(1-\cos\omega)^2}{1+4\lambda(1-\cos\omega)^2}$$

The phase shift is zero, but not for the beginning and end of the sample where the filter is non-symmetric. λ is chosen such that $\left|A(e^{-i\omega})\right|^2 = 0.5$ at a particular frequency ω_0.

Effects of filtering

Example 5. Band Pass filter
A filter designed to shut down all fluctuations outside of a chosen frequency band.