Advanced Microeconomics II

Bayesian Extensive Games With Observable Actions

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Bayesian Extensive Game With Observable Actions

First let's extend Bayesian games.

Definition

A Bayesian extensive game with observable actions is a tuple $\{\Gamma, (\Theta_i), (p_i), (u_i)\}$ where

- $\Gamma = \{N, H, P\}$ is an extensive game form with perfect information and simultaneous moves

and for each player $i \in N$

- Θ_i is a finite set (the set of possible types of player i); $\Theta = \times_{\{i \in N\}} \Theta_i$
- p_i is a probability measure on Θ_i for which $p_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, and the measures p_i are stochastically independent ($p_i(\theta_i)$ is the probability that player i is selected to be of type θ_i)
- $u_i : \Theta \times Z \to \mathcal{R}$ is a von Neumann-Morgenstern utility function ($u_i(\theta, h)$ is player i's payoff when the profile of types is θ and the terminal history of Γ is h).

Bayesian Extensive Game With Observable Actions

We can associate with any such game an extensive game (with imperfect information and simultaneous moves) in which

- the set of histories is $\{\varnothing\} \cup (\Theta \times H)$ and
- each information set of each player i takes the form

$$I_i(\theta, h) = \{(\theta', h) : \theta' \in \Theta \text{ and } \theta'_i = \theta_i\}$$

  for $i \in P(h)$ (so that the number of histories in $I(\theta, h)$ is the number of members of Θ_{-i}).

Interpretation: Chance first chooses player types. The (otherwise perfect) game is then played.

Example - Tough Chain Store Game

- Chance chooses a Chain Store type: $\Theta_{CS} = \{R(egular), T(ough)\}$.
- The Chain Store is 'Tough' with probability ϵ.
  - A 'Tough' chain store prefers to fight than to cooperate.
- The standard chain-store game is then played.
- The payoff to the potential entrant is

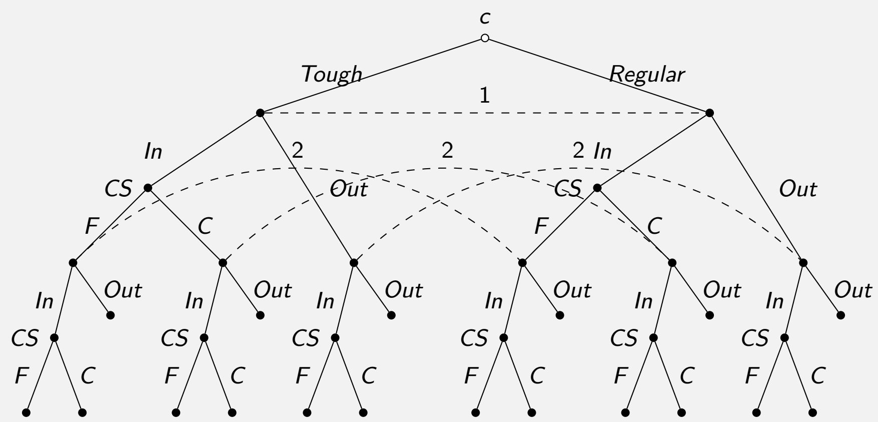
$$u_k(\theta, h) = \begin{cases} b & \text{if } h_k = (In, C) \\ b-1 & \text{if } h_k = (In, F) \\ 0 & \text{if } h_k = Out, \end{cases}$$

  where $0 < b < 1$.
- The payoff to the chain-store in each market is $u_{CS}(\theta, h) =$

$$\begin{cases} 0 & \text{if } h_k = (In, C) \text{ and } \theta_{CS} = R, \text{ or } h_k = (In, F) \text{ and } \theta_{CS} = T \\ -1 & \text{if } h_k = (In, F) \text{ and } \theta_{CS} = R, \text{ or } h_k = (In, C) \text{ and } \theta_{CS} = T \\ a & \text{if } h_k = Out, \end{cases}$$

  where $a > 1$.

Example - Tough Chain Store Game



Homework: Write down this extensive form game.

Signalling Games

The simplest type of Bayesian extensive game with observable actions.

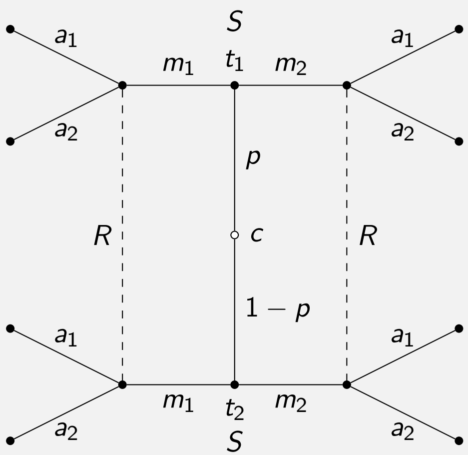
Definition

A signalling game is a Bayesian extensive game with observable actions $\{\Gamma, (\Theta_i), (p_i), (u_i)\}$ in which

- $N = \{S, R\}$.
- $P(\varnothing) = S$ (The 'sender' plays first).
- $P(h) = R$ for $h \in A(\varnothing)$. (The 'receiver' plays second).
- Histories have at most length 2. (The game then ends).
- Θ_R is a singleton. (The 'receiver' has one type).

Interpretation: The sender sends a message about his type. The receiver observes the message and chooses an action. Payoffs are a function of type, message and action.

Extensive Game Form Simple Example



How many pure strategies does each player have?

Strategies

- The first and last strategies of the sender are pooling strategies.
- The second and third strategies of the sender are separating strategies.
- If we consider mixed strategies we can have hybrid strategies.
- If we have more than two types we can have partial pooling/semi-separating strategies.

Equilibrium Requirements - Beliefs

Signalling Requirement 1 After observing any message $m_j \in A(\varnothing)$ the receiver must have a belief about which types could have sent m_j. Denote this belief by the probability distribution $\mu_S(m_j)(t_S)$, where $\mu_S(m_j)(t_S) \geq 0$ for each $t_S \in \Theta_S$, and

$$\sum_{t_S \in \Theta_S} \mu_S(m_j)(t_S) = 1.$$

Equilibrium Requirements - Rationality

Signalling Requirement 2R[*Receiver rationality*] The Receiver's strategy must be optimal. For each message $m_j \in A(\varnothing)$, $s_R^*(m_j)$ solves

$$\max_{a_k \in A(m_j)} \sum_{t_S \in \Theta_S} \mu_S(m_j)(t_S) u_R(t_S, (m_j, a_k)).$$

Signalling Requirement 2S[*Sender rationality*] The Sender's strategy must be optimal. For each type $t_S \in \Theta_S$, $s_S^*(t_S)$ solves

$$\max_{m_j \in A(\varnothing)} u_S(t_S, (m_j, s_R^*(m_j))).$$

Equilibrium Requirements - Bayesian Updating

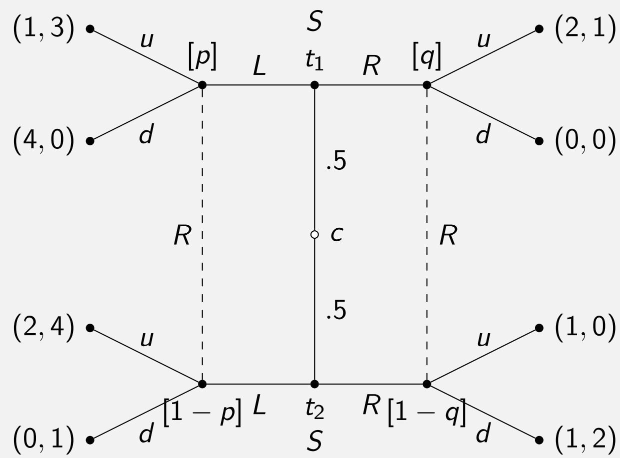
Signalling Requirement 3 For each $m_j \in A(\varnothing)$,

- if there exists $t_S \in \Theta_S$ such that $s_S^*(t_S) = m_j$, then for each $t'_S \in \Theta_S$

$$\mu_S(m_j)(t'_S) = \begin{cases} \dfrac{p(t'_S)}{\sum_{\{\tilde{t}_S \in \Theta_S | s_S^*(\tilde{t}_S) = m_j\}} p(\tilde{t}_S)} & \text{if } s_S^*(t'_S) = m_j \\ 0 & \text{otherwise.} \end{cases}$$

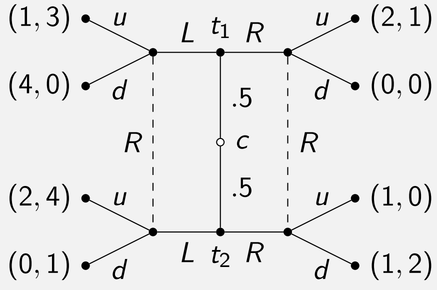
- if there does not exists $t_S \in \Theta_S$ such that $s_S^*(t_S) = m_j$ then what should we do?

Signalling Game - Simple Example



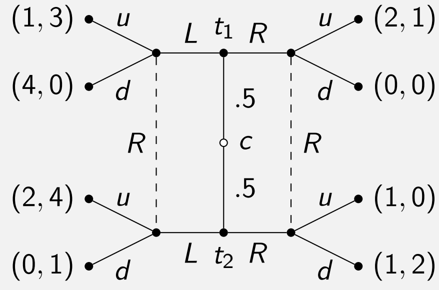
$p = \mu_S(L)(t_1)$; $q = \mu_S(R)(t_1)$.

Receiver Optimal Strategy



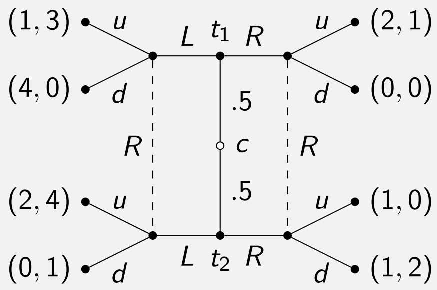
- For all p, $\beta_R(L)(u) = 1$.
- If $q < 2/3$, then $\beta_R(R)(u) = 0$
- If $q = 2/3$, then $\beta_R(R)(u) \in [0, 1]$
- If $q > 2/3$, then $\beta_R(R)(u) = 1$

Type 2 Sender Optimal Strategy



- $\beta_S(t_2)(L) = 1$.
  - Since for all p, $\beta_R(L)(u) = 1$.

Type 1 Sender Optimal Strategy



- If $\beta_R(R)(u) < 1/2$, then $\beta_S(t_1)(L) = 1$.
- If $\beta_R(R)(u) = 1/2$, then $\beta_S(t_1)(L) \in [0, 1]$.
- If $\beta_R(R)(u) > 1/2$, then $\beta_S(t_1)(L) = 0$.

Equilibria

Type 1a:

- $\beta_S(t_1)(L) = 1$, $\beta_S(t_2)(L) = 1$.
- $\beta_R(L)(u) = 1$, $\beta_R(R)(u) = 0$.
- $p = 0.5$, $q < 2/3$.

Type 1b:

- $\beta_S(t_1)(L) = 1$, $\beta_S(t_2)(L) = 1$.
- $\beta_R(L)(u) = 1$, $0 \leq \beta_R(R)(u) \leq 1/2$.
- $p = 0.5$, $q = 2/3$.

Type 2:

- $\beta_S(t_1)(L) = 0$, $\beta_S(t_2)(L) = 1$.
- $\beta_R(L)(u) = 1$, $\beta_R(R)(u) = 1$.
- $p = 0$, $q = 1$.

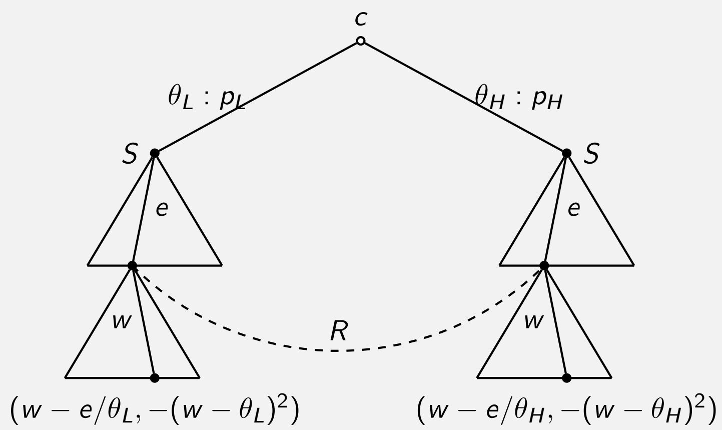
Classify each as a pooling, separating or hybrid equilibrium.

Spence's Model of Education

A worker knows her talent $\theta \in \{\theta_L, \theta_H\}$, while her employer does not. A worker has productivity θ_L with probability p_L and productivity θ_H with probability $p_H = 1 - p_L$. The value of the worker to the employer is θ, but the employer pays the worker a wage w that is equal to the expectation of θ (there is a competitive labour market).

- The worker chooses an amount of education $e \in [0, \infty)$.
- Employer makes an offer $w \in [\theta_L, \theta_H]$ to the worker.
- Payoffs: The worker's payoff is $w - e/\theta$ and the employer's payoff is $-(w - \theta)^2$.

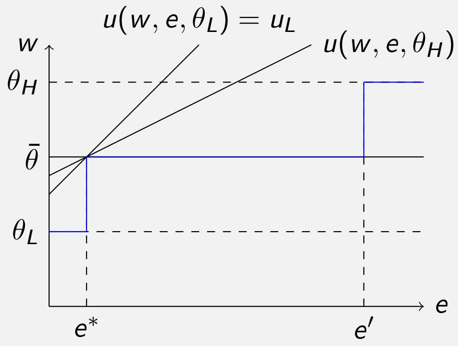
Example - Model of Education Game Tree



Pooling Equilibrium

$$S : e(\theta_H) = e^*;\ e(\theta_L) = e^*.$$

$$R : \mu_S(e)(\theta_H) = \begin{cases} 1 & \text{if } e' \le e \\ p_H & \text{if } e^* \le e < e' \\ 0 & \text{otherwise} \end{cases};\ w(e) = \begin{cases} \theta_H & \text{if } e' \le e \\ \bar{\theta} & \text{if } e^* \le e < e' \\ \theta_L & \text{otherwise.} \end{cases}$$

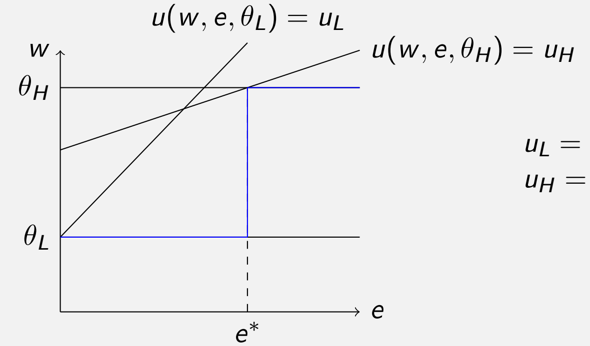


$\bar{\theta} = p_H\theta_H + p_L\theta_L;$
$u_L = u(\bar{\theta}, e^*, \theta_L);$
$u_H = u(\bar{\theta}, e^*, \theta_H).$

Separating Equilibrium

$$S : e(\theta_H) = e^*;\ e(\theta_L) = 0.$$

$$R : \mu_S(e)(\theta_H) = \begin{cases} 1 & \text{if } e^* \le e \\ 0 & \text{otherwise} \end{cases};\ w(e) = \begin{cases} \theta_H & \text{if } e^* \le e \\ \theta_L & \text{otherwise.} \end{cases}$$

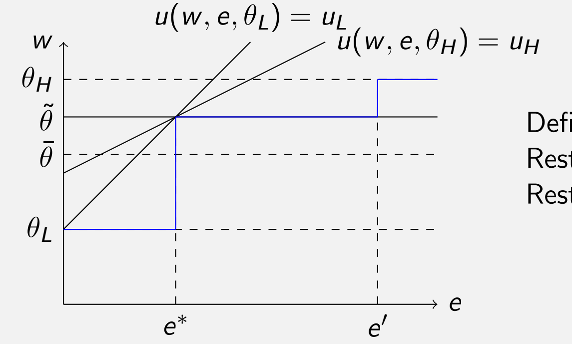


$u_L = u(\theta_L, 0, \theta_L);$
$u_H = u(\theta_H, e^*, \theta_H).$

Hybrid Equilibrium

$$S : e(\theta_H) = e^*;\ e(\theta_L) = \begin{cases} 0 & \text{with probability } \lambda \\ e^* & \text{with probability } 1-\lambda. \end{cases}$$

$$R : \mu_S(e)(\theta_H) = \begin{cases} 1 & \text{if } e' \le e \\ \tilde{p} & \text{if } e^* \le e < e' \\ 0 & \text{otherwise.} \end{cases};\ w(e) = \begin{cases} \theta_H & \text{if } e' \le e \\ \tilde{\theta} & \text{if } e^* \le e < e' \\ \theta_L & \text{otherwise.} \end{cases}$$



Define $\lambda, \tilde{p}, \tilde{\theta}$.
Restrictions on e'?
Restrictions on e^*?

Perfect Bayesian Equilibrium

Definition

Let $\{\Gamma, (\Theta_i), (p_i), (u_i)\}$ be a Bayesian extensive game with observable actions, where $\Gamma = \{N, H, P\}$. A perfect Bayesian equilibrium of the game is a pair $((\sigma_i), (\mu_i)) = ((\sigma_i(\theta_i))_{i\in N, \theta_i \in \Theta_i}, (\mu_i(h))_{i \in N, h \in H\setminus Z})$, where $\sigma_i(\theta_i)$ is a behavioral strategy of player i in Γ and $\mu_i(h)$ is a probability measure on θ_i and the following conditions are satisfied.

- *Correct initial beliefs* $\mu_i(\varnothing) = p_i$ for each $i \in N$.
- *Action-determined beliefs* If $i \notin P(h)$ and $a \in A(h)$ then $\mu_i(h,a) = \mu_i(h)$; if $i \in P(h)$, $a \in A(h)$, $a' \in A(h)$, and $a_i = a'_i$ then $\mu_i(h,a) = \mu_i(h,a')$.
- *Sequential rationality*
- *Bayesian updating*

Perfect Bayesian Equilibrium

- *Bayesian updating* If $i \in P(h)$ and a_i is in the support of $\sigma_i(\theta_i)(h)$ for some θ_i in the support of $\mu_i(h)$ then for any $\theta'_i \in \Theta_i$ we have

$$\mu_i(h,a)(\theta'_i) = \frac{\sigma_i(\theta'_i)(h)(a_i) \cdot \mu_i(h)(\theta'_i)}{\sum_{\tilde{\theta}_i \in \Theta_i} \sigma_i(\tilde{\theta}_i)(h)(a_i) \cdot \mu_i(h)(\tilde{\theta}_i)}.$$

- *Sequential rationality* For every nonterminal history $h \in H\setminus Z$, every player $i \in P(h)$, and every $\theta_i \in \Theta_i$

$$O(\sigma_i(\theta_i), \sigma_{-i}, \mu_{-i}|h) \succeq_i O(s_i, \sigma_{-i}, \mu_{-i}|h)$$

for any strategy s_i of player i in Γ.

Chain-Store Equilibrium - Chain-store

- $\mu_{CS}(h)(T)$: the belief by the potential entrants after history h that the chain-store is tough.
- $t(h)$: the number of potential entrants who have moved.

Regular Chain-store strategy

$$\sigma_{CS}(R)(h) = \begin{cases} C & \text{if } t(h) = K \\ F & \text{if } t(h) < K \text{ and } \mu_{CS}(h)(T) \ge b^{K-t(h)} \\ m^h_{CS} & \text{if } t(h) < K \text{ and } \mu_{CS}(h)(T) < b^{K-t(h)} \end{cases}$$

if $P(h) = CS$, where m^h_{CS} is the mixed strategy such that

$$m^h_{CS}(F) = \frac{\left(1 - b^{K-t(h)}\right)\mu_{CS}(h)(T)}{\left(1-\mu_{CS}(h)(T)\right) b^{K-t(H)}}.$$

Tough Chain-store strategy

$$\sigma_{CS}(T)(h) = F \text{ if } P(h) = CS.$$

Chain-Store Equilibrium - Potential Entrant

Potential entrant k strategy

$$\sigma_k(h) = \begin{cases} Out & \text{if } \mu_{CS}(h)(T) > b^{K-k+1} \\ m_k & \text{if } \mu_{CS}(h)(T) = b^{K-k+1} \\ In & \text{if } \mu_{CS}(h)(T) < b^{K-k+1} \end{cases}$$

if $P(h) = k$, (so that $t(h) = k - 1$), where m_k is the mixed strategy such that

$$m_k(Out) = 1/a.$$

Chain-Store Equilibrium - Beliefs

- Correct initial beliefs: $\mu_{CS}(\varnothing)(T) = \epsilon$.
- For any history h with $P(h) = k$, $\mu_{CS}(h, h^k)(T) =$

$$\begin{cases} \max\{b^{K-k}, \mu_{CS}(h)(T)\} & \text{if } h^k = (In, F) \text{ and } \mu_{CS}(h)(T) > 0 \\ 0 & \text{if } h^k = (In, C) \text{ or } \mu_{CS}(h)(T) = 0 \\ \mu_{CS}(h)(T) & \text{if } h^k = (Out). \end{cases}$$