# Advanced Macroeconomics II

Linlin Niu

WISE, Xiamen University

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# A Simple RBC Model

## The environment

1) *Preference:*

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}}{1-\eta}\right]$$

2) *Technology:*

$$C_t + K_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1}$$

How to define "$Z_t$": $log\, Z_t = (1-\varphi) log \bar{Z} + \varphi log Z_{t-1} + \epsilon_t$, where $\epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$, $\varphi < 1$. As a parameter, usually $\bar{Z}$ is normalized to 1.

3) *Endowment:*

$$N_t = 1, \quad K_{-1} > 0$$

4) *Information:* decision made based on all information $I_t$ up to time $t$.

# A Simple RBC Model

## The social planner's problem

$$\max_{(C_t, K_t)_{t=0}^{\infty}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}}{1-\eta} \right]$$

$$\begin{aligned}
& s.t. K_{-1},\ Z_0 \\
C_t + K_t &= Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1} \\
\log Z_t &= (1-\varphi) \log \bar{Z} + \varphi \log Z_{t-1} + \epsilon_t \\
& \epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2)
\end{aligned}$$

# A Simple RBC Model

To solve:

The *Lagrangian*:

$$L = \max_{(C_t,K_t)_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \left[\beta^t \left(\frac{C_t^{1-\eta}-1}{1-\eta} - \lambda_t(C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta)K_{t-1})\right)\right]$$

The necessary FOCs:

$$\begin{aligned}
\frac{\partial L}{\partial C_t} &: \quad 0 \overset{!}{=} C_t^{-\eta} - \lambda_t \\
\frac{\partial L}{\partial K_t} &: \quad 0 \overset{!}{=} \beta^t[-\lambda_t] + \beta^{t+1} E_t\left[(-\lambda_{t+1})[-\rho Z_{t+1} K_t^{\rho-1} - (1-\delta)]\right] \\
&\Longrightarrow \quad 0 \overset{!}{=} -\lambda_t + \beta E_t\left[\lambda_{t+1}\left(\rho Z_{t+1} K_t^{\rho-1} + (1-\delta)\right)\right] \\
\frac{\partial L}{\partial \lambda_t} &: \quad 0 \overset{!}{=} C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta)K_{t-1}
\end{aligned}$$

# A Simple RBC Model

## To solve:

*Transversality condition*:
For uniqueness and non-explosive solution

$\Rightarrow$ The expected marginal utility of consuming the left-over capital discounted as of today converges to zero (when $T \to \infty$)

$$0 = \underset{T \to \infty}{E_0} \left[ \beta^T C_T^{-\eta} K_T \right]$$

# A Simple RBC Model

## To solve for state states:

Collect all the necessary coditions and exogenous process:

$$
\begin{aligned}
C_t &= Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t \\
R_t &= \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \\
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta} R_{t+1}\right] \\
logZ_t &= (1-\phi)log\bar{Z} + \phi logZ_{t+1} + \varepsilon_t, \varepsilon_t \sim iid\, N(0,\sigma^2)
\end{aligned}
$$

*Note* : we have defined the gross return of capital; we substitute out $\lambda_t$ with marginal utility of consumption. The number of variables does not change.

Important: #(equation) = #(variable)

# A Simple RBC Model

## To solve for state states:

To find the S.S of economy, drop time indices:

$$
\begin{aligned}
\bar{C} &= \bar{Z}\bar{K}^{\rho} + (1-\delta)\bar{K} - \bar{K} \\
\bar{R} &= \rho\bar{Z}\bar{K}^{\rho-1} + (1-\delta) \\
1 &= \beta \cdot \bar{R}
\end{aligned}
$$

Three variables/equations need to be solved for $(\bar{C},\ \bar{K},\ \bar{R})$ given the parameters $(\bar{Z},\ \beta,\ \rho,\ \delta)$.

The sequence of solutions

$$
\begin{aligned}
\bar{R} &= 1/\beta \\
\bar{K} &= \left(\frac{\rho\bar{Z}}{\bar{R}-1+\delta}\right)^{1/(1-\rho)} \\
hence &: \quad \bar{Y} = \bar{Z}\bar{K}^{\rho} \qquad \bar{C} = \bar{Y} - \delta\bar{K} \\
\bar{C}/\bar{K} &= \bar{Y}/\bar{K} - \delta = \bar{Z}\bar{K}^{\rho-1} - \delta \\
&= \frac{\bar{R}-1+\delta}{\rho} - \delta = \frac{1-\beta+\delta\beta}{\rho\beta} - \delta
\end{aligned}
$$

# A Simple RBC Model

## Same model environment, but competitive equilibrium

$$\left(C_t,\, N_t,\, K_t,\, R_t,\, W_t\right)_{t=0}^{\infty}$$

$\boxed{\text{Household:}}$

Given $K_{-1}^{(s)}$, $W_t$, and $R_t$, decide on providing how much $N_t^{(s)}$, $K_t^{(s)}$, and consuming $C_t$.

$$\max_{\left(C_t, K_t^{(s)}, N_t^{(s)}\right)_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}}{1-\eta}\right]$$

$$\begin{aligned} & s.t. K_{-1}, \\ C_t + K_t^{(s)} &= W_t N_t^{(s)} + R_t K_{t-1}^{(s)} \end{aligned}$$

plus no-Ponzi-game condition (similar to the transversality condition)

$$0 = \lim_{t \to \infty} E_0\left[\left(\Pi_{s=1}^{t} R_s^{-1}\right) K_t\right]$$

to ensure the capital left-over when time approaches infinity discounted as of today has zero present value.

# A Simple RBC Model

## Same model environment, but competitive equilibrium

$\boxed{\text{Firm:}}$

Given $W_t$, and $R_t$, decide on how much capital to rent and labor to hire, $N_t^{(d)}$, $K_t^{(d)}$, hence how much to produce $Y_t(N_t^{(d)}, K_t^{(d)})$.

$$\max_{K_{t-1}^{(d)}, N_t^{(d)}} Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta) K_{t-1}^{(d)} - W_t N_t^{(d)} - R_t K_{t-1}^{(d)}$$

where

$$logZ_t = (1-\phi) log \bar{Z} + \phi logZ_{t+1} + \varepsilon_t, \varepsilon_t \sim iid\, N(0, \sigma^2)$$

# A Simple RBC Model

## Same model environment, but competitive equilibrium

Market Clearing Conditions:

Capital Market:

$$K_{t-1}^{(d)} = K_{t-1}^{(s)}$$

Labor Market:

$$N_t^{(d)} = N_s^{(s)} = 1$$

Goods Market:

$$C_t + K_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} + (1-\delta) K_{t-1}$$

By *Walras' law*, we only need two out of these three conditions.

# A Simple RBC Model

## To solve competitive equilibrium: solve every party's FOCs

Household:

$$L = \max_{(C_t, K_t)_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - \lambda_t (C_t + K_t - R_t K_{t-1} - W_t N_t)\right)\right]$$

$\Rightarrow$ Household FONCs:

$$\begin{aligned}
\frac{\partial L}{\partial C_t} &: \quad 0 \overset{!}{=} C_t^{-\eta} - \lambda_t \\
\frac{\partial L}{\partial K_t} &: \quad 0 \overset{!}{=} -\lambda_t + \beta E_t[\lambda_{t+1} R_{t+1}] \\
\frac{\partial L}{\partial \lambda_t} &: \quad 0 \overset{!}{=} C_t + K_t - W_t - R_t K_{t-1}
\end{aligned}$$

# A Simple RBC Model

## To solve competitive equilibrium: solve every party's FOCs

Firm:

$$\max_{K_{t-1}^{(d)},N_t^{(d)}} Z_t(K_{t-1})^{\rho} N_t^{1-\rho} + (1-\delta)K_{t-1} - W_t N_t - R_t K_{t-1}$$

Firm's FONCs:

$$\begin{aligned} W_t &= (1-\rho) Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{-\rho} \\ R_t &= \rho Z_t \left(K_{t-1}^{(d)}\right)^{\rho-1} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta) \end{aligned}$$

Dropping ${}^{(d)}$ and using

$$Y_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho}$$

The FONCs amount to

$$\begin{aligned} W_t N_t &= (1-\rho) Y_t \\ R_t K_{t-1} &= \rho Y_t + (1-\delta) K_{t-1} \end{aligned}$$

$$\text{or} \quad r_t K_{t-1} - \delta K_{t-1} = \rho Y_t$$

# A Simple RBC Model

## To solve competitive equilibrium: market clearing

Market clearing by dropping $^{(d)}$ and $^{(s)}$, we have done it.
Combining the FONCs and the exogoneous process, we reduce to the same system of equations (after deleting $W_t$, $Y_t$, $\lambda_t$):

$$
\begin{aligned}
C_t &= Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t \\
R_t &= \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \\
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta} R_{t+1}\right] \\
log Z_t &= (1-\phi) log \bar{Z} + \phi log Z_{t+1} + \varepsilon_t, \varepsilon_t \sim iid\, N(0,\sigma^2)
\end{aligned}
$$

# A Simple RBC Model

Both ways work. You can opt to the simpler one.

# A Simple RBC Model

## Log-linearization

$$
\begin{aligned}
C_t &= \bar{C}e^{\hat{c}_t} \\
K_t &= \bar{K}e^{\hat{k}_t}
\end{aligned}
$$

Budget constraint:

$$
\begin{aligned}
\bar{C}e^{\hat{c}_t} &= \bar{Z}\bar{K}^{\rho}e^{\hat{z}_t}e^{\rho\hat{k}_{t-1}} + (1-\delta)\bar{K}e^{\hat{k}_{t-1}} - \bar{K}e^{\hat{k}_t} \\
\bar{C} + \bar{C}\cdot\hat{c}_t &\approx \bar{Z}\bar{K}^{\rho} + \bar{Z}\bar{K}^{\rho}(z_t + \rho\hat{k}_{t-1}) + (1-\delta)\bar{K}\hat{k}_{t-1} - \delta\bar{K} - \bar{K}\hat{k}_t
\end{aligned}
$$

Using the steady state relationship $\bar{C} = \bar{Z}\bar{K}^{\rho} + (1-\delta)\bar{K} - \bar{K}$, $\bar{Y} = \bar{Z}\bar{K}^{\rho}$, and $\bar{C} = \bar{Y} - \delta\bar{K}$

$$
\begin{aligned}
\bar{C}\hat{c}_t &\approx \bar{Z}\bar{K}^{\rho}(\hat{z}_t + \rho\hat{k}_{t-1}) + (1-\delta)\bar{K}\hat{k}_{t-1} - \bar{K}\hat{k}_t \\
\hat{c}_t &\approx \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}\hat{k}_t
\end{aligned}
$$

# A Simple RBC Model

## Log-linearization

Interest rate definition

$$
\begin{aligned}
R_t &= \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \\
\bar{R} e^{\hat{r}_t} &= \rho \bar{Z} \bar{K}^{\rho-1} e^{\hat{z}_t + (\rho-1)\hat{k}_{t-1}} + (1-\delta) \\
\bar{R} + \bar{R}\hat{r}_t &\approx \rho \bar{Z} \bar{K}^{\rho-1} + (1-\delta) + \rho \bar{Z} \bar{K}^{\rho-1} \left[\hat{z}_t + (\rho-1)\hat{k}_{t-1}\right]
\end{aligned}
$$

Using the steady state relationship

$$
\frac{1}{\beta} = \bar{R} = \rho \bar{Z} \bar{K}^{\rho-1} + (1-\delta)
$$

$$
\begin{aligned}
\bar{R}\hat{r}_t &\approx \rho \bar{Z} \bar{K}^{\rho-1} \left[\hat{z}_t + (\rho-1)\hat{k}_{t-1}\right] \\
\hat{r}_t &\approx \beta \left[\frac{1}{\beta} - (1-\delta)\right] \left[\hat{z}_t - (1-\rho)\hat{k}_{t-1}\right] \\
\hat{r}_t &\approx \left[1 - \beta(1-\delta)\right] \left[\hat{z}_t - (1-\rho)\hat{k}_{t-1}\right]
\end{aligned}
$$

# A Simple RBC Model

## Log-linearization

Euler equation

$$
\begin{aligned}
1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\eta} R_{t+1}\right] \\
1 &= E_t\left[\beta\left(\frac{\bar{C}}{\bar{C}} e^{\hat{c}_{t+1}-\hat{c}_t}\right)^{-\eta} \bar{R} e^{\hat{r}_{t+1}}\right] \\
1 &\approx E_t\left[\beta \bar{R} e^{\eta(\hat{c}_t-\hat{c}_{t+1})+\hat{r}_{t+1}}\right]
\end{aligned}
$$

Use the steady state relationship

$$1 = \beta \bar{R}$$

$$0 \approx E_t\left[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\right]$$

# A Simple RBC Model

## Log-linearization

Technology process

$$\begin{aligned} log\left(\bar{Z}e^{\hat{z}_t}\right) &= (1-\psi)log\bar{Z} + \psi log\left(\bar{Z}e^{\hat{z}_{t-1}}\right) + \varepsilon_t \\ \hat{z}_t &= \psi\hat{z}_{t-1} + \varepsilon_t \end{aligned}$$

which holds exactly.

# A Simple RBC Model

## Log-linearization: results

*Collect* all the four L.L. equations, which form a system of linear and homogeneous equations.

1. $\hat{c}_t \approx \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}\hat{k}_t$
2. $\hat{r}_t \approx [1-\beta(1-\delta)]\left[\hat{z}_t - (1-\rho)\hat{k}_{t-1}\right]$
3. $0 \approx E_t\left[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\right]$
4. $\hat{z}_t = \psi\hat{z}_{t-1} + \varepsilon_t$

$\Longrightarrow$ Four equations, four variables! $(\hat{c}_t, \hat{z}_t, \hat{k}_{t-1}, \hat{r}_t)$, but some variables appear in different time periods. for example, equation 1 is a second order difference equation in $\hat{k}_t$.

Our next task is to solve for this linear system in terms of the exogenous variable and predetermined variable $(\hat{z}_t,\ \hat{k}_{t-1})$.

# Summary on the procedure of log-linearization

1. Determine and collect all equilibrium conditions, i.e. FOCs, constraints, exogenous variable processes. #(equation)=#(variable).
2. Determine steady states.
3. Multiplying out variables, i.e. if $X_t(1-Q_t)$ then multiply out to get $X_t - X_t Q_t$, then replace $X_t$ with $\bar{X}e^{\hat{x}_t}$.
4. Collect all exponential terms wherever possible. E.g. $\bar{X}e^{\hat{x}_t}\bar{Y}e^{\hat{y}_t} = \bar{X}\bar{Y}e^{\hat{x}_t+\hat{y}_t}$.
5. Approximate $e^{\hat{x}_t}$ with first order expansion $e^{\hat{x}_t} \approx 1 + \hat{x}_t$ (only when it's necessary).
6. Collect all constant terms and verify that they cancel out by using the steady state relationship. Further, delete any higher order terms such as to impose $\hat{x}_t\hat{y}_t \approx 0$.
7. Collect all variables to form a system of linear and homogeneous equations.