Solution to P.S.5

March 18, 2012

1. solution

   - to prove V is positive definite, we refer to the definition of it. i.e. $\forall x \in \mathbb{R}^n$,

   $$\begin{aligned} x^T V x &= x^T E[(R-\bar{R})(R-\bar{R})^T]x \\ &= E[x^T(R-\bar{R})(R-\bar{R})^T x] \\ &= E[AA^T] \\ &\geq 0 \end{aligned}$$

   where $A = x^T(R-\bar{R})$.

   - besides, since V is full-rank, which means that the eigenvalues of V are non zero.
   - thus, V is positive definite.

2. solution

   - since V is full rank, thus V^{-1} exists(for all the eigenvalues λ_i of V are nonzero), with its eigenvalues are $\frac{1}{\lambda_i}$, where λ_i are the eigenvalues of V.
   - thus when V is a positive definite matrix, it is obvious that $\frac{1}{\lambda_i} > 0$, which means V^{-1} is positive definite as well.

3. solution

- with V^{-1} being positive definite, $(\alpha\bar{R}-\varsigma e)^{'}V^{-1}(\alpha\bar{R}-\varsigma e)>0$.

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\begin{aligned}
(\alpha\bar{R}-\varsigma e)^{'}V^{-1}(\alpha\bar{R}-\varsigma e) &= \alpha^2\bar{R}V^{-1}\bar{R}-\alpha\varsigma\bar{R}V^{-1}e-\alpha\varsigma e^{'}V^{-1}\bar{R}+\varsigma^2 e^{'}V^{-1}e \\
&= \alpha^2\varsigma-\alpha^2\varsigma-\alpha^2\varsigma+\varsigma^2\delta \\
&= \varsigma^2\delta-\alpha^2\varsigma \\
&= \varsigma(\varsigma\delta-\alpha^2)
\end{aligned}
$$

- besides, since $\varsigma=\bar{R}'V^{-1}\bar{R}$ is positive, thus $\varsigma\delta-\alpha^2>0$

4. solution

- the minimization problem is to minimize

$$L=\frac{1}{2}\omega'V^{-1}\omega+\lambda\left[\bar{R}_p-\omega'\bar{R}\right]+\gamma\left[1-\omega'e\right]$$

- $\frac{\partial L}{\partial\omega}=V\omega-\lambda\bar{R}-\gamma e$, and $\frac{\partial^2 L}{\partial\omega^2}=V$.
- when $\omega=\omega^*$, which is derived from the F.O.N.C, we still ensure that $\frac{\partial^2 L}{\partial\omega^2}>0$, indicating that w^*derived from the first order necessary condition is a sufficient condition to ensure the minimum variance is achieved.

5. solution

- refer to the answers provided by the classmates as a bonus question.