# Solutions and Hints: Chapter 3 Brownian Motion

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# Outline

# Exercise 3.1

- According to Definition 3.3.3(iii), for $0 \leq t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the $\sigma$-algebra $\mathscr{F}(t)$. Use this property and property (i) of that definition to show that, for $0 \leq t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathscr{F}(t)$.
- Solution:
  $W(u_2) - W(u_1)$ is independent of the $\sigma$-algebra $\mathscr{F}(u_1)$ with $\mathscr{F}(u_1) \supset \mathscr{F}(t)$.
  $\blacksquare$

# Exercise 3.1

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# Exercise 3.2

- Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathscr{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale. (Hint: For $0 \leq s \leq t$, write $W^2(t)$ as $(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s)$.)

- Solution:

$$
\begin{aligned}
&\mathbb{E}_s\left[W^2(t) - t\right] \\
=&\mathbb{E}_s\left[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t\right] \\
=&\mathbb{E}_s\left[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W^2(s) - t\right] \\
=&t - s + W^2(s) - t \\
=&W^2(s) - s.
\end{aligned}
$$

$\blacksquare$

# Exercise 3.2

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- Solution:

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\begin{aligned}
& \mathbb{E}_s\left[W^2(t) - t\right] \\
= & \mathbb{E}_s\left[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t\right] \\
= & \mathbb{E}_s\left[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W^2(s) - t\right] \\
= & t - s + W^2(s) - t \\
= & W^2(s) - s.
\end{aligned}
$$

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# Exercise 3.3

- The kurtosis of a random variable is defined to be the ratio of its fourth central moment to the square of its variance.
- Let $X$ be a normal random variable with mean $\mu$, so that $X-\mu$ has mean zero. Let the variance of $X$, which is also the variance of $X-\mu$, be $\sigma^2$.
- ... the moment-generating function of $X-\mu$ to be $\varphi(u)=\mathbb{E}\left[e^{u(X-\mu)}\right]=e^{\frac{1}{2}\sigma^2u^2}$, where $u$ is a real variable.

# Exercise 3.3

- Solution:

$$\varphi'''(u) = \mathbb{E}\left[(X-\mu)^3 e^{u(X-\mu)}\right] = 3\sigma^4 u e^{\frac{1}{2}\sigma^2 u^2}$$

$$\varphi^{(4)}(u) = \mathbb{E}\left[(X-\mu)^4 e^{u(X-\mu)}\right] = \left(3\sigma^4 + 3\sigma^6 u^2\right) e^{\frac{1}{2}\sigma^2 u^2}$$

$$\varphi^{(4)}(0) = \mathbb{E}\left[(X-\mu)^4\right] = 3\sigma^4$$

■

# Exercise 3.4

- (Other variations of Brownian motion). Theorem 3.4.3 asserts that if $T$ is a positive number and we choose a partition $\Pi$ with points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, then as the number $n$ of partition points approaches infinity and the length of the longest subinterval $\|\Pi\|$ approaches zero, the sample quadratic variation $\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2$ approaches $T$ for almost every path of the Brownian motion $W$. In Remark 3.4.5, we further showed that $\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))(t_{j+1} - t_j)$ and $\sum_{j=0}^{n-1} (t_{j+1} - t_j)^2$ have limit zero. We summarize these facts by the multiplication rules

$$dW(t)\,dW(t) = dt, dW(t)\,dt = 0, dtdt = 0. \tag{1}$$

# Exercise 3.4

- (i) Show that as the number $n$ of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample first variation

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

approaches $\infty$ for almost every path of the Brownian motion $W$.

- Hint:

$$\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \le \max_{0\le k\le n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|.$$

# Exercise 3.4

- Solution: (i) Supposelim$_{n\to\infty}\sum_{j=0}^{n-1}|W(t_{j+1})-W(t_j)|<\infty$,

$$\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2 \le \max_{0\le k\le n-1}|W(t_{k+1})-W(t_k)| \cdot \sum_{j=0}^{n-1}|W(t_{j+1})-W(t_j)|.$$

where

$$\lim_{n\to\infty}\max_{0\le k\le n-1}|W(t_{k+1})-W(t_k)|=0,$$

then $\lim_{n\to\infty}\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2=0$. But $\lim_{n\to\infty}\sum_{j=0}^{n-1}(W(t_{j+1})-W(t_j))^2=T$. A contradiction.
■

# Exercise 3.4

- (ii) Show that as the number $n$ of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample cubic variation

$$\sum_{j=0}^{n-1} \left| W(t_{j+1}) - W(t_j) \right|^3$$

approaches zero for almost every path of the Brownian motion $W$.

# Exercise 3.4

- Solution: (ii)

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \leq$$

$$\max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2$$

$$\to 0 \cdot T = 0 \text{ as } n \to \infty.$$

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# Exercise 3.5

- Black-Scholes-Merton formula.

# Exercise 3.6

- Let $W(t)$ be a Brownian motion and let $\mathscr{F}(t)$, $t \geq 0$, be an associated filtration.

# Exercise 3.6

- (i) For $\mu \in \mathbb{R}$, consider the Brownian motion with drift $\mu$:

$$X(t) = \mu t + W(t).$$

Show that for any Borel-measurable function $f(y)$, and for any $0 \le s < t$, the function

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{ -\frac{(y-x-\mu(t-s))^2}{2(t-s)} \right\} dy$$

satisfies $\mathbb{E}[f(X(t)) | \mathscr{F}(s)] = g(X(s))$, and hence $X$ has the Markov property. We may rewrite $g(x)$ as $g(x) = \int_{-\infty}^{\infty} f(y) p(\tau, x, y) dy$, where $\tau = t - s$ and

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left( -\frac{(y-x-\mu\tau)^2}{2\tau} \right)$$

is the transition density for Brownian motion with drift $\mu$.

# Exercise 3.6

- Solution: (i)

$$
\begin{aligned}
X(t) &= \mu t + W(t) \\
&= \mu(t-s) + W(t) - W(s) + \mu s + W(s) \\
&\sim \mathcal{N}\left(X(s) + \mu(t-s), t-s\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[f(X(t)) \,|\, \mathscr{F}(s)\right] &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{(y-x-\mu(t-s))}{2(t-s)}\right. \\
&= g(x)
\end{aligned}
$$

$\blacksquare$

# Exercise 3.6

- (ii) For $\nu \in \mathbb{R}$ and $\sigma > 0$, consider the geometric Brownian motion
$$S(t) = S(0)\, e^{\sigma W(t) + \nu t}.$$
Set $\tau = t - s$ and
$$p(\tau, x, y) = \frac{1}{\sigma y \sqrt{2\pi\tau}} \exp\left(-\frac{\left(\ln \frac{y}{x} - \nu\tau\right)^2}{2\sigma^2\tau}\right).$$
Show that for any Borel-measurable function $f(y)$ and for any $0 \leq s < t$ the function $g(x) = \int_0^\infty h(y)\, p(\tau, x, y)\, dy$ satisfies $\mathbb{E}[f(S(t)) | \mathscr{F}(s)] = g(S(s))$ and hence $S$ has the Markov property and $p(\tau, x, y)$ is its transition density.

# Exercise 3.6

- Solution: (ii)

$$\begin{aligned} S(t) &= S(s)\, e^{\sigma[W(t)-W(s)]+\nu(t-s)} \\ \mathbb{E}[f(S(t))\,|\mathscr{F}(s)] &= \mathbb{E}\left[f\left(S(s)\, e^{\sigma[W(t)-W(s)]+\nu(t-s)}\right)|\mathscr{F}(s)\right] \\ &= \int_{-\infty}^{\infty} f\left(e^{\ln x+\sigma z+\nu\tau}\right)\frac{1}{\sqrt{2\pi\tau}} e^{-\frac{z^2}{2\tau}}\, dz \\ &= \int_0^{\infty} f(y)\,\frac{1}{\sigma y\sqrt{2\pi\tau}} \exp\left(-\frac{\left(\ln\frac{y}{x}-\nu\tau\right)^2}{2\sigma^2\tau}\right) dy \\ &= \int_0^{\infty} h(y)\, p(\tau,x,y)\, dy = g(S(s)) \end{aligned}$$

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# Exercise 3.7

- Theorem 3.6.2 provides the Laplace transform of the density of the first passage time for Brownian motion. This problem derives the analogous formula for Brownian motions with drift. Let $W$ be a Brownian motion. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \leq t < \infty$, define

$$X(t) = \mu t + W(t),$$

$$\tau_m = \min\{t > 0; X(t) = m\}.$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level $m$. Let $\sigma$ be a positive number and set

$$Z(t) = \exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right) t\right\}.$$

# Exercise 3.7

- (i) Show that $Z(t)$, $t \geq 0$, is a martingale.
- Solution: (i)

$$\begin{aligned}
\mathbb{E}_s[Z(t)] &= \mathbb{E}_s\left[\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}\right] \\
&= Z(s)\,\mathbb{E}_s\left[\exp\left\{\sigma(X(t) - X(s)) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)s\right\}\right] \\
&= Z(s)\int_{-\infty}^{\infty} e^{\sigma y - \frac{1}{2}\sigma^2 s}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{y^2}{2\tau}}\,dy \\
&= Z(s)
\end{aligned}$$

■

# Exercise 3.7

- (i) Show that $Z(t)$, $t \geq 0$, is a martingale.
- Solution: (i)

$$
\begin{aligned}
\mathbb{E}_s[Z(t)] &= \mathbb{E}_s\left[\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}\right] \\
&= Z(s)\,\mathbb{E}_s\left[\exp\left\{\sigma(X(t) - X(s)) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)s\right\}\right] \\
&= Z(s)\int_{-\infty}^{\infty} e^{\sigma y - \frac{1}{2}\sigma^2 s}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{y^2}{2\tau}}\,dy \\
&= Z(s)
\end{aligned}
$$

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# Exercise 3.8

- This problem presents the convergence of the distribution of stock prices in a sequence of binomial models to the distribution of geometric Brownian motion. In contrast to the analysis of Subsection 3.2.7, here we allow the interest rate to be different from zero.
- Let $\sigma > 0$ and $r \geq 0$ be given. For each positive integer $n$, we consider a binomial model taking $n$ steps per unit time. In this model, the interest rate per period is $\frac{r}{n}$, the up factor is $u_n = e^{\sigma/\sqrt{n}}$, and the down factor is $d_n = e^{-\sigma/\sqrt{n}}$. The risk-neutral probabilities are then

$$\tilde{p}_n = \frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}, \tilde{q}_n = \frac{e^{\sigma/\sqrt{n}} - \frac{r}{n} - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}.$$

# Exercise 3.8

- Let $t$ be an arbitrary positive rational number, and for each positive integer $n$ for which $nt$ is an integer, define

$$M_{nt,n} = \sum_{k=1}^{nt} X_{k,n},$$

where $X_{1,n}, \ldots, X_{n,n}$ are independent, identically distributed random variables with

$$\tilde{\mathbb{P}}\left\{X_{k,n} = 1\right\} = \tilde{p}_n, \tilde{\mathbb{P}}\left\{X_{k,n} = -1\right\} = \tilde{q}_n, k = 1, \ldots, n.$$

# Exercise 3.8

- The stock price at time $t$ in this binomial model, which is the result of $nt$ steps from the initial time, is given by (see (3.2.15) for a similar equation)

$$\begin{aligned} S_n(t) &= S(0)\, u_n^{\frac{1}{2}(nt+M_{nt,n})} d_n^{\frac{1}{2}(nt-M_{nt,n})} \\ &= S(0) \exp\left\{ \frac{\sigma}{2\sqrt{n}} (nt + M_{nt,n}) \right\} \exp\left\{ -\frac{\sigma}{2\sqrt{n}} (nt - M_{nt,n}) \right\} \\ &= S(0) \exp\left\{ \frac{\sigma}{\sqrt{n}} M_{nt,n} \right\}. \end{aligned}$$

# Exercise 3.8

- This problem shows that as $n \to \infty$, the distribution of the sequence of random variables $\frac{\sigma}{\sqrt{n}} M_{nt,n}$ appearing in the exponent above converges to the normal distribution with mean $\left(r - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$. Therefore, the limiting distribution of $S_n(t)$ is the same as the distribution of the geometric Brownian motion $S(0) \exp\left\{\sigma W(t) + \left(r - \frac{1}{2}\sigma^2\right) t\right\}$ at time $t$.

# Exercise 3.8

- (i) Show that the moment-generating function $\varphi_n(u)$ of $\frac{1}{\sqrt{n}}M_{nt,n}$ is given by

$$\varphi_n(u) = \left[ e^{\frac{u}{\sqrt{n}}} \left( \frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) - e^{-\frac{u}{\sqrt{n}}} \left( \frac{e^{\sigma/\sqrt{n}} - \frac{r}{n} - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) \right]^{nt}.$$

# Exercise 3.8

- (ii) We want to compute

$$\lim_{n\to\infty} \varphi_n(u) = \lim_{x\downarrow 0} \varphi_{\frac{1}{x^2}}(u),$$

where we have made the change of variable $x = \frac{1}{\sqrt{n}}$. To do this, we will compute $\ln \varphi_{\frac{1}{x^2}}(u)$ and then take the limit as $x \downarrow 0$. Show that

$$\ln \varphi_{\frac{1}{x^2}}(u) = \frac{t}{x^2} \ln \left[ \frac{\left(rx^2+1\right)\sinh ux + \sinh(\sigma - u)x}{\sinh \sigma x} \right]$$

and ... rewrite this as

$$\ln \varphi_{\frac{1}{x^2}}(u) = \frac{t}{x^2} \ln \left[ \cosh ux + \frac{\left(rx^2+1-\cosh \sigma x\right)\sinh ux}{\sinh \sigma x} \right].$$

# Exercise 3.8

- (iii) Use the Taylor series expansions

$$\cosh z = 1 + \frac{1}{2} z^2 + O\left(z^4\right), \sinh z = z + O\left(z^3\right),$$

to show that

$$\cosh ux + \frac{\left(rx^2 + 1 - \cosh \sigma x\right) \sinh ux}{\sinh \sigma x} \\ = 1 + \frac{1}{2} u^2 x^2 + \frac{rux^2}{\sigma} - \frac{1}{2} ux^2 \sigma + O\left(x^4\right). \quad (2)$$

The notation $O\left(x^j\right)$ is used to represent terms of the order $x^j$.

# Exercise 3.8

- (iv) Use the Taylor series expansion $\ln(1+x) = x + O\left(x^2\right)$ to compute $\lim_{x \downarrow 0} \varphi_{\frac{1}{x^2}}(u)$. Now explain how you know that the limiting distribution for $\frac{\sigma}{\sqrt{n}} M_{nt,n}$ is normal with mean $\left(r - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$.

# Exercise 3.9

- (Laplace transform of first passage density). The solution to this problem is long and technical. It is included for the sake of completeness, but the reader may safely skip it.