Class Notes for Advanced Macroeconomics

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Solution to Mid-term Exam

1 Explaining Short Terms ($5' \times 4 = 20'$)

Instruments: In this part, you're required to understand the following terms *in economic sense*, and provide reasonable explanations for their *implications* in macroeconomics.

Taylor Principle In a paper on the econometric evaluation of monetary/fiscal policy, Taylor [1993] has proposed that the U.S. monetary policy in recent year could be described by an interest-rate feedback rule of the form $\boxed{2'}$

$$i_t = 4\% + 1.5 \cdot (\pi_t - 2\%) + 0.5 \cdot (y_t - y^*), \tag{1}$$

where i_t denotes the Fed's operating target for the federal funds rate, π_t is the inflation rate $\boxed{1'}$ (measured by the GDP deflator), y_t is the log of real GDP, and y^* is the log of "potential output" (identified empirically with a linear model), that's, the term $(y_t - y^*)$ measures the output gap from the balanced path. The **Taylor Principle** states that *the (eventual) response of the nominal interest rate to a change in the inflation rate must be more than one for one.* $\boxed{2'}$

Friedman Rule In his classic work on "*The Optimum Quantity of Money*", Friedman [1969] proposed that *the aggregate consumer's surplus could be maximized only by setting the nominal interest rates to zero.* The rationale for this policy rule is quite intuitive: While the social cost of producing $\boxed{2'}$ real balances is zero, the private opportunity cost is given by the nominal interest rate. As a result, only when the nominal interest rate is zero are the private and social costs of holding money equated. $\boxed{2'}$

Mathmetically, $i_t = r_t + \pi_t = 0$ by *Fisher equation*, implying that $\pi_t = -r_t$. Start from the consumption Euler equation

$$\begin{aligned}
\mathbb{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)} \cdot R_{t+1}\right] &= 1 \\
\Rightarrow \quad \mathbb{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)} \cdot (1+r_t)\right] &= 1 \\
\Rightarrow \quad r_t &= \frac{1}{\beta} - 1 \\
\Rightarrow \quad \pi_t = -r_t &= 1 - \frac{1}{\beta} < 0, \quad \text{for } \beta \in (0,1), \quad \forall t \geq 0.
\end{aligned} \tag{2}$$

That's, under the Friedman rule, the economy will experience a (moderate) deflation in the long $\boxed{1'}$ run. Is this optimal in the real world? A lot of people argue that monetary policy is a very effective tool in fighting against recessions. If nominal interest rates are zero, they can't be lowered anymore. On the other hand, Friedman's[1969] analysis is correct if we concentrate on the welfare loss associated with holding money.

Super-neutrality of Money **Neutrality** of money is the idea that a change in the stock of money affects only nominal variables in the economy such as prices, wages and exchange rates, with *no* effect on real (inflation-adjusted) variables, like employment, real GDP, and real consumption.

Whereas, **Super-neutrality of Money** is a strong property: As we see, within the classical monetary framework, the Sidrauski's[1967] MIU model exhibits a property called the *Superneutrality of Money*: the steady state values of the capital stock, consumption, and the output are *all* independent of *the growth rate of the nominal money stock*. That's, not only is money neutral, so that proportional changes in the *level* of nominal money balances and prices have no real effects, but changes in the rate of *growth* of nominal money stock also have no effect on the steady state capital stock or, therefore, on output or per capita consumption. In particular, a permanent change in money growth leads to an proportional change in the long-run inflaction rate but leaves the long-run real interest rate, capital stock, and real output level unchanged. $\boxed{2'}$ $\boxed{2'}$ $\boxed{1'}$

Liquidity Effect In macroeconomics, the term "*liquidity effect*" is referred to the observed negative comovement between money supply and nominal interest rate. Most economists, and certainly monetary policymakers, believe that central banks can reduce short-term nominal interest rate by employing policies that lead to faster growth in the money supply. That's, a faster money growth will initially cause nominal interest rates to fall. $\boxed{2'}$ $\boxed{2'}$

Note that an increase in the money supply can have two effects:

(i) it can reduce the real interest rate (the "liquidity effect", more money, i.e. more liquidity, tends to lower the price of money which is equivalent to lowering the interest rate);

(ii) it also forecasts higher future inflation (called the expected inflation or Fisher effect).

Therefore to generate a falling nominal interest rate in response to a positive money supply shock we require the liquidity effect to outweigh the Fisher effect. $\boxed{1'}$

2 Solving a Model of Decentralized Economy with Taxation $(80')$

Consider the following model, where a representative household solves

$$\max_{\{c_t,k_t,n_t\}_{t=0}^{\infty}} E_0\left\{\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\eta}-1}{1-\eta}-An_t\right)\right\} \tag{1}$$

s.t.

$$c_t+k_t=(1-\tau_k)(d_t+1-\delta)k_{t-1}+(1-\tau_n)w_tn_t+g_t \tag{2}$$

where $\beta(0<\beta<1)$, η, A and δ are parameters. The household rents out k_{t-1} units of capital to the representative firm and earns the dividend d_t on each unit of capital. She has to pay a tax τ_k on her total capital income, and τ_n for real wage, w_t, for supplying n_t units of labor. She receives a transfer, g_t, from the government which she takes as given.

The representative firm

The representative firm produces output y_t using a Cobb-Douglas production function with capital and labor as factor inputs. z_t is an exogenous stochastic process with $E_{t-1}[\epsilon_t]=0$ and $Var[\epsilon_t]=\sigma^2$. The firm pays to the representative agent the dividend d_t on rented capital and the real wage w_t on n_t units of labor. $\bar{\gamma}$ is a parameter of steady state productivity.

$$\max_{k_{t-1},n_t} \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} - d_t k_{t-1} - w_t n_t \tag{3}$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

where $\rho < 1$.

The government

The government collects capital income taxes and labor income taxes and transfer g_t units to the agent.

$$g_t = \tau_k (d_t + 1 - \delta) k_{t-1} + \tau_n w_t n_t \tag{4}$$

It might be useful to define expressions for output and the return on capital:

$$y_t = \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} \tag{5}$$

$$R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta \tag{6}$$

where $0 < \theta < 1$.

Since this is a decentralized competitive economy consisting of three sector: the household sector, the firm, and the government. There is distortion tax imposed on the capital stock as well as the consumption. It's not equivalent any more to consider the social planner problem. We need to solve for the competitive equilibrium.

2.1 The Environment

The economy consists of three sectors. The environment is specified as follows: $1'$

— ***Preferences*** — The representative household seeks to maximize the expected value of the discounted sum of the life-time utility of

$$\mathbf{U} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\eta} - 1}{1-\eta} - A n_t \right) \right\}.$$

The "instantaneous utility" is defined as $u(c_t, n_t) = \frac{c_t^{1-\eta}-1}{1-\eta} - A n_t$, which is concave and increasing in consumption, linear and decreasing in labor participation. The parameter η measures the *coefficient of relative risk aversion* (CRRA) as

$$-\frac{u_{c,c}}{u_c} \cdot c = -\frac{-\eta c^{-\eta-1}}{c^{-\eta}} \cdot c = \eta,$$

where $u_c = \partial u(c,n)/\partial c$, and $u_{c,c} = \partial^2 u(c,n)/\partial c^2$. The household is facing her flow budget constraint as

$$c_t + k_t = (1-\tau_k)(d_t + 1 - \delta) k_{t-1} + (1-\tau_n) w_t n_t + g_t,$$

where the government collects capital income taxes and labor income taxes and transfer g_t units to the agent.

$$g_t = \tau_k (d_t + 1 - \delta) k_{t-1} + \tau_n w_t n_t.$$

— ***Technology*** — The representative firm utilizes the Cobb-Douglas production function $1'$

$$y_t = \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta}, \tag{5'}$$

where $\bar{\gamma} > 0$ denotes the initial level of the *total factor productivity* (TFP), θ the share of capital in output. The technology shock, z_t, is driven by the following *exogenous stochastic process*:

$$\begin{aligned} z_t &= \rho z_{t-1} + \varepsilon_t \\ \varepsilon_t &\sim \mathcal{N}(0, \sigma_\varepsilon^2), \quad \text{and } \ 0 \leq \rho < 1. \end{aligned}$$

Capital is the only good that can be accumulated over time. It evolves according to the following accumulation equation:

$$k_t = i_t + (1-\delta) k_{t-1}, \tag{7}$$

where the capital depreciates at the rate of $0 \leq \delta < 1$. The representative firm is set to maximize its profits by borrowing capital from the household sector, and utilizing the labor supply in the competitive market, assuming the general level of price to be normalized. That's

$$\max_{k_{t-1}, n_t} \quad \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} + (1-\delta) k_{t-1} - w_t n_t - r_t k_{t-1},$$

where w_t, and r_t are respectively wage in the competitive labor market, and the capital return. $1'$

— ***Endowment*** — The representative household is endowed with time normalized to be unit, i.e., $l_t + n_t = 1$, where l_t is leisure, capital k_{-1}, and an initial level of technological progress $\bar{\gamma} > 0$, and initial productivity shock z_0 at $t = 0$.

— ***Information*** — All the variables, $\{c_t, i_t, k_t, n_t\}_{t=0}^{\infty}$ are chosen up to the information set $\mathcal{I}_t$.

2.2 Elasticity of Labor Supply

The parameter of A measures the *marginal dis-utility of labor* in the utility function, which could be viewed as the additional utility of one more leisure. Usually, we observe that most people either enter $1'$ or leave employment but do not continuously adjust the number of hours worked. Hansen [1985] argues that this is consistent with a feature of U.S. post-war data: labor volatility is essentially explained by that of the number employed. Consider the following decomposition:

$$var(\ln(H_t)) = var(\ln(\bar{H}_t) + var(\ln(N_t)) + 2cov(\ln(\bar{H}_t, \ln(N_t)),$$

where H_t represents the *total hours worked*, $\bar{H}_t$ the *average hours worked*, N_t the *number of individuals* at work, with all variables beging deviations from trend. Then

- 55% of the variance of the total worked hours H_t comes from the number employed N_t.
- Only 20% arises from variations in average worker hours, $\bar{H}_t$.
- the residual is due to the covariance term.

Accordingly, the *inter-temporal marginal rate of substitute* between the consumption and labor participation is given by

$$MRS = \frac{-u_{n,t}}{u_{c,t}} = \frac{A}{c^{-\eta}} = A \cdot c^{\eta}. \tag{8}$$

In order to simultaneously achieve *high elasticity of aggregate labor supply* and *individually low elasticity of labor supply*: $2'$

- the parameter A should be expected to be *relatively large*;
- the labor is supposed to be *indivisible*.

2.3 Classification of Variables

— ***Endogenous Variables*** — Control variables in the model are the those individual utilize to solve the optimization problem. Here, collecting the control variables and other endogenous variables determined in this model, we have: $2'$

- Consumption: c_t;
- Labor supply: n_t;
- Output: y_t;
- Real Wage: w_t;
- Capital Return: R_t;
- Dividend: d_t;
- Government Transfer: g_t;

— ***State Variables*** — State variables describe the dynamics of the transitionary process of an economy. They could be divided into two sub-categories: $1'$

Endogenous State Variables Capital stock: k_{t-1} (or k_t);

Exogenous State Variables Technology Shocks: z_t.

2.4 The FONCs

— ***The Representative Household*** — The representative household seeks to maximize the expected value of the discounted sum of the life-time utility function subject to his flow budget constraint. That's, she is assumed to operate the optimal problem

$$L : \sum_{t=0}^{\infty} \beta^t \left\{ \left(\frac{c_t^{1-\eta} - 1}{1-\eta} - An_t \right) + \lambda_t \Big[(1-\tau_k)(d_t + 1 - \delta)k_{t-1} + (1-\tau_n)w_t n_t + g_t - c_t - k_t \Big] \right\}$$

Taking derivatives to obtain the FONCs for the representative household: $4'$

$$\begin{aligned}
\frac{\partial L}{\partial c_t} :& \quad c_t^{-\eta} &=& \quad \lambda_t & \text{(9a)}\\
\frac{\partial L}{\partial k_t} :& \quad \lambda_t &=& \quad E_t\big[\beta\lambda_{t+1} \cdot (1-\tau_k)(d_{t+1} + 1 - \delta)\big] & \text{(9b)}\\
\frac{\partial L}{\partial n_t} :& \quad A &=& \quad \lambda_t(1-\tau_n)w_t & \text{(9c)}\\
\frac{\partial L}{\partial \lambda_t} :& \quad c_t + k_t &=& \quad (1-\tau_k)(d_t + 1 - \delta)k_{t-1} + (1-\tau_n)w_t n_t + g_t & \text{(9d)}
\end{aligned}$$

— ***The Firm*** — The representative firm is set to maximize its profits by borrowing capital from the household sector, and utilizing the labor supply in the competitive market, assuming the general level of price to be normalized. That's

$$\max_{k_{t-1}, n_t} \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} - d_t k_{t-1} - w_t n_t$$

This yield the FONCs for the typical firm 2′

$$
\begin{aligned}
\frac{\partial .}{\partial k_{t-1}}: \quad d_t &= \theta \frac{y_t}{k_{t-1}} && \text{(10a)}\\
\frac{\partial .}{\partial n_t}: \quad w_t &= (1-\theta)\frac{y_t}{n_t} && \text{(10b)}
\end{aligned}
$$

where

$$y_t = \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta}.$$

Furthermore, we have the following relationship

$$R_t = \theta \frac{y_t}{k_{t-1}} + 1 - \delta = d_t + 1 - \delta \tag{11}$$

2.5 The Equilibrium

Collecting all the equilibrium conditions including the FONCs for household and firm respectively as well as the budget constrains to get the equations describing the equilibrium of this economy:

$$
\begin{aligned}
\text{Production Function:} \quad y_t &= \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} && \text{(12a)}\\
\text{Wage Equation:} \quad w_t &= (1-\theta)\frac{y_t}{n_t} && \text{(12b)}\\
\text{Dividend:} \quad d_t &= \theta \frac{y_t}{k_{t-1}} && \text{(12c)}\\
\text{Gross Capital Return:} \quad R_t &= d_t + (1-\delta) && \text{(12d)}\\
\text{MRS Between Consumption and Labor:} \quad A &= c_t^{-\eta}(1-\tau_n) w_t && \text{(12e)}\\
\text{Euler Equation:} \quad 1 &= \mathbb{E}_t\big[\beta(1-\tau_k)\cdot\big(\frac{c_{t+1}}{c_t}\big)^{-\eta}\cdot R_{t+1}\big] && \text{(12f)}\\
\text{Government Lump-sum Transfer:} \quad g_t &= \tau_k(d_t+1-\delta)k_{t-1} + \tau_n w_t n_t && \text{(12g)}\\
\text{Household's Budget Constraint:} \quad c_t + k_t &= (1-\tau_k)(d_t+1-\delta)k_{t-1} + (1-\tau_n) w_t n_t + g_t && \text{(12h)}\\
\text{Technology Shocks:} \quad z_t &= \rho z_{t-1} + \varepsilon_t && \text{(12i)}
\end{aligned}
$$

First note that the government's flow budget constraint is always holding under the assumption of *Ricardian Regime*

$$g_t = \tau_k(d_t+1-\delta)k_{t-1} + \tau_n w_t n_t, \tag{12g′}$$

alongside with the identity of output distributed between the capital share and the labor share as:

$$
\begin{aligned}
y_t &= \theta \frac{y_t}{k_{t-1}} \cdot k_{t-1} + (1-\theta)\frac{y_t}{n_t}\cdot n_t\\
&= d_t \cdot k_{t-1} + w_t \cdot n_t. && \text{(13)}
\end{aligned}
$$

Thus, the household's budget constraint could be rewritten as

$$
\begin{aligned}
c_t + k_t &= (1-\tau_k)(d_t+1-\delta)k_{t-1} + (1-\tau_n) w_t n_t + g_t && \text{(12h)}\\
&= (d_t+1-\delta)k_{t-1} + w_t n_t + \underbrace{\big[g_t - (\tau_k(d_t+1-\delta)k_{t-1} + \tau_n w_t n_t)\big]}_{0}\\
&= \underbrace{d_t \cdot k_{t-1} + w_t \cdot n_t}_{y_t} + (1-\delta)k_{t-1}
\end{aligned}
$$

$$\Rightarrow \quad \boxed{c_t + k_t = y_t + (1-\delta)k_{t-1}} \tag{14}$$

And to get rid of w_t by substituting Eq.(12b) in Eq.(12e)

$$\begin{aligned} A = &\quad c_t^{-\eta}(1-\tau_n)w_t && (12e) \\ \Rightarrow &\quad \boxed{A = c_t^{-\eta}(1-\tau_n)(1-\theta)\frac{y_t}{n_t}} && (15) \end{aligned}$$

TO summarize, we have following *six* equations with six unknowns (c_t, R_t, y_t, n_t, k_t, z_t): $\boxed{3'}$

$$\begin{aligned} y_t &= \bar{\gamma} e^{z_t} k_{t-1}^{\theta} n_t^{1-\theta} && (12a') \\ R_t &= \theta \frac{y_t}{k_{t-1}} + (1-\delta) && (12d') \\ 1 &= \mathbb{E}_t\big[\beta(1-\tau_k)\cdot\big(\frac{c_{t+1}}{c_t}\big)^{-\eta}\cdot R_{t+1}\big] && (12f') \\ c_t + k_t &= y_t + (1-\delta)k_{t-1} && (14') \\ A &= c_t^{-\eta}(1-\tau_n)(1-\theta)\frac{y_t}{n_t} && (15') \\ z_t &= \rho z_{t-1} + \varepsilon_t && (12i') \end{aligned}$$

2.6 Hansen's[1985] Indivisible Labor Model

Recall that in the Hansen's[1985] Indivisible Labor Model, there is assumed to be no distortional taxation imposed on the capital and/or consumption. Then the optimal decision FONCs for consumption summarized in the Euler equation, and the labor supply function summary by the marginal rate of substitution between the consumption and labor are respectively:

$$\begin{aligned} 1 &= \mathbb{E}_t\big[\beta\frac{u_{c,t+1}}{u_{c,t}}\cdot R_{t+1}\big] \\ A &= u_{c,t}\cdot w_t \end{aligned}$$

Comparing these two equations with those of our model: The differences are the wedges due to taxation $(1-\tau_k)$ in the Euler equation and $(1-\tau_n)$ in the labor supply FONC equation. Individuals tend to have less incentive to work and production is below the social optimal. $\boxed{2'}$

2.7 The Steady State

Solve for the steady state, i.e. provide formulas for $\bar{k}$, $\bar{y}$, $\bar{R}$, $\bar{c}$ and A given $\boxed{\bar{n}}$ and all other parameters.

— ***Technology Shock*** — For the economy being steady state, we assume no technology shocks, that's, $\varepsilon_t = 0$ for given t, which yields

$$\begin{aligned} &\bar{z} = \rho\bar{z} + 0 \\ \Rightarrow \quad &\boxed{\bar{z} = 0} && (16) \end{aligned}$$

for $\rho < 1$.

— ***Capital Return*** — Solving for the steady state value of the gross capital return in the Euler equation of (12f):

$$\boxed{\bar{R} = \frac{1}{\beta(1-\tau_k)}} \tag{17}$$

— ***Output per capital*** — Notice in the Eq.(12d) we can express the output *per capita* as $1'$

$$\begin{aligned} \bar{R} &= \theta(\frac{\bar{y}}{\bar{k}}) + (1-\delta) \\ \Rightarrow \quad & \boxed{\frac{\bar{y}}{\bar{k}} = \frac{\bar{R}-(1-\delta)}{\theta}} \end{aligned} \tag{18}$$

$1'$

— ***Capital Stock*** — By the production function

$$\begin{aligned} \bar{y} &= \bar{\gamma}\bar{k}^{\theta}\bar{n}^{1-\theta} \\ \Rightarrow \quad \bar{k}^{1-\theta} &= \bar{\gamma}\bar{n}^{1-\theta}\left(\frac{\bar{k}}{\bar{y}}\right) \\ \Rightarrow \quad \bar{k} &= \bar{n}\left(\bar{\gamma}\frac{\bar{k}}{\bar{y}}\right)^{\frac{1}{1-\theta}} = \bar{n}\left(\bar{\gamma}\frac{\bar{y}}{\bar{k}}\right)^{\frac{1}{\theta-1}} \\ \Rightarrow \quad & \boxed{\bar{k} = \bar{n}\left(\frac{\bar{\gamma}\theta}{\bar{R}-(1-\delta)}\right)^{\frac{1}{1-\theta}}} \end{aligned} \tag{19}$$

$1'$

— ***Aggregate Output*** — Thus, by Eq.(19) we obtain the steady state value of the aggregate output as

$$\boxed{\bar{y} = \frac{\bar{R}-(1-\delta)}{\theta}\bar{k}} \tag{20}$$

$1'$

— ***Consumption*** — The steady state value of consumption is obtained via Eq.(14)

$$\boxed{\bar{c} = \bar{y} - \delta\bar{k}} \tag{21}$$

$1'$

— ***Margin Disutility of Labor*** — For A

$$\boxed{A = \bar{c}^{-\eta}(1-\tau_n)(1-\theta)\frac{\bar{y}}{\bar{n}}} \tag{22}$$

2.8 Log-Linearizations

Log-linearize the equations. Define log-deviation of variable x_t as $\hat{x}_t = \log(x_t/\bar{x})$. Write down all necessary log-linearized equations characterizing the equilibrium in the notation of log-deviation term $\hat{x}_t$. (Note: z_t is already in the deviation term for technology factor in the sense of $z_t = \log[(\bar{\gamma}e^{z_t})/(\bar{\gamma}e^0)]$.) $4'$

— ***Production Function*** —

$$\begin{aligned} \bar{y}e^{\hat{y}_t} &= \bar{\gamma}\bar{k}^{\theta}\bar{n}^{1-\theta}e^{z_t+\theta\hat{k}_{t-1}+(1-\theta)\hat{n}_t} \\ \Rightarrow & \boxed{\hat{y}_t = z_t + \theta\hat{k}_{t-1} + (1-\theta)\hat{n}_t} \end{aligned} \tag{23}$$

$\boxed{4'}$

— *Real Interest Rate* —

$$
\begin{aligned}
\bar{R}e^{\hat{r}_t} &= \theta\frac{\bar{y}}{\bar{k}} \cdot e^{\hat{y}_t - \hat{k}_{t-1}} + (1-\delta) \\
\Rightarrow \quad \bar{R}(1+\hat{r}_t) &= \theta\frac{\bar{y}}{\bar{k}} \cdot (1 + \hat{y}_t - \hat{k}_{t-1}) + (1-\delta) \\
\Rightarrow \quad \cancel{\bar{R}} + \bar{R}\hat{r}_t &= \cancel{\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)} + \theta\frac{\bar{y}}{\bar{k}}(\hat{y}_t - \hat{k}_{t-1}) \\
\Rightarrow \quad \bar{R}\hat{r}_t &= \theta\frac{\bar{y}}{\bar{k}}(\hat{y}_t - \hat{k}_{t-1}) \\
\Rightarrow \quad & \boxed{\hat{r}_t = \theta\frac{\bar{y}}{\bar{R}\bar{k}}(\hat{y}_t - \hat{k}_{t-1})}
\end{aligned} \tag{24}
$$

$\boxed{4'}$

— *Euler Equation* —

$$
\begin{aligned}
1 &= \mathbb{E}_t\big[\beta(1-\tau_k) \cdot \big(\frac{c_{t+1}}{c_t}\big)^{-\eta} \cdot R_{t+1}\big] \\
\Rightarrow \quad & \cancel{1 = \mathbb{E}_t\big[\beta(1-\tau_k) \cdot \big(\frac{\bar{c}}{\bar{c}}\big)^{-\eta} \cdot \bar{R}} \cdot e^{-\eta\hat{c}_{t+1} + \eta\hat{c}_t + r_{t+1}}\big] \\
\Rightarrow \quad & \boxed{0 = \mathbb{E}_t\big[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\big]}
\end{aligned} \tag{25}
$$

$\boxed{6'}$

— *Goods Market Clearing* —

$$
\begin{aligned}
\bar{c}e^{\hat{c}_t} + \bar{k}e^{\hat{k}_t} &= \bar{y}e^{\hat{y}_t} + (1-\delta)\bar{k}e^{\hat{k}_{t-1}} \\
\Rightarrow \quad \bar{c}(1+\hat{c}_t) + \bar{k}(1+\hat{k}_t) &= \bar{y}(1+\hat{y}_t) + (1-\delta)\bar{k}(1+\hat{k}_{t-1}) \\
\Rightarrow \quad \cancel{\bar{c} + \bar{k}} + \bar{c}\hat{c}_t + \bar{k}\hat{k}_t &= \cancel{\bar{y} + (1-\delta)\bar{k}} + \bar{y}\hat{y}_t + (1-\delta)\bar{k}\hat{k}_{t-1} \\
\bar{c}\hat{c}_t + \bar{k}\hat{k}_t &= \bar{y}\hat{y}_t + (1-\delta)\bar{k}\hat{k}_{t-1} \\
\Rightarrow \quad \bar{c}\hat{c}_t + \bar{k}\hat{k}_t &= \bar{y}\Big[z_t + \theta\hat{k}_{t-1} + (1-\theta)\hat{n}_t\Big] + (1-\delta)\bar{k}\hat{k}_{t-1} \qquad \text{(By Eq.23)} \\
\Rightarrow \quad & \boxed{\bar{c}\hat{c}_t + \bar{k}\hat{k}_t = \big[\bar{y}\theta + (1-\delta)\bar{k}\big] \cdot \hat{k}_{t-1} + \bar{y}(1-\theta) \cdot \hat{n}_t + \bar{y}z_t}
\end{aligned} \tag{26}
$$

$\boxed{4'}$

— *Labor Supply* —

$$
A = c_t^{-\eta}(1-\tau_n)(1-\theta)\frac{y_t}{n_t} \qquad \text{(By Eq.15)}
$$

$$
\begin{aligned}
\Rightarrow \quad & \cancel{A = \bar{c}^{-\eta}(1-\tau_n)(1-\theta)\frac{\bar{y}}{\bar{n}}} \cdot e^{-\eta\hat{c}_t + \hat{y}_t - \hat{n}_t} \\
\Rightarrow \quad & \boxed{0 = -\eta\hat{c}_t + \hat{y}_t - \hat{n}_t}
\end{aligned} \tag{27}
$$

— *Technology Shock* — Note that z_t is already in the deviation term for technology factor in the sense of $z_t = \log[(\bar{\gamma}e^{z_t})/(\bar{\gamma}e^0)]$.

The technology shock, z_t, is driven by the following *exogenous stochastic process*. $\boxed{2'}$

$$
z_t = \rho z_{t-1} + \varepsilon_t \tag{28}
$$

$$
\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2), \quad \text{and } \ 0 \le \rho < 1.
$$

2.9 Solving for Law of Motion

This means we need to delete $\hat{y}_t$, $\hat{r}_t$, and $\hat{n}_t$ from the budget constraint and the Euler equation. by examine the three other equations.

$$\hat{y}_t = z_t + \theta\hat{k}_{t-1} + (1-\theta)\hat{n}_t \tag{23'}$$

$$\hat{r}_t = \theta\frac{\bar{y}}{\bar{R}\bar{k}}(\hat{y}_t - \hat{k}_{t-1}) \tag{24'}$$

$$0 = -\eta\hat{c}_t + \hat{y}_t - \hat{n}_t \tag{27'}$$

Combining Eq.(23) with Eq.(27) to obtain

$$\cancel{\hat{y}_t} = z_t + \theta\hat{k}_{t-1} + (1-\theta)\hat{n}_t + -\eta\hat{c}_t + \cancel{\hat{y}_t} - \hat{n}_t$$

$$\Rightarrow \boxed{\hat{n}_t = -\frac{\eta}{\theta}\hat{c}_t + \frac{1}{\theta}z_t + \hat{k}_{t-1}}$$

Insert it back to the log-linearized production function,

$$\begin{aligned}
\hat{y}_t &= z_t + \theta\hat{k}_{t-1} + (1-\theta)\hat{n}_t \\
&= z_t + \theta\hat{k}_{t-1} + (1-\theta)\left[-\frac{\eta}{\theta}\hat{c}_t + \frac{1}{\theta}z_t + \hat{k}_{t-1}\right] \\
&= z_t + \theta\hat{k}_{t-1} - \frac{1-\theta}{\theta}\eta\hat{c}_t + \frac{1-\theta}{\theta}z_t + (1-\theta)\hat{k}_{t-1} \\
\Rightarrow &\boxed{\hat{y}_t = \frac{1}{\theta}z_t + \hat{k}_{t-1} - \frac{1-\theta}{\theta}\eta\hat{c}_t}
\end{aligned}$$

Now the interest rate is

$$\begin{aligned}
\hat{r}_t &= \theta\frac{\bar{y}}{\bar{R}\bar{k}}\hat{y}_t - \theta\frac{\bar{y}}{\bar{R}\bar{k}}\hat{k}_{t-1} \\
&= \theta\frac{\bar{y}}{\bar{R}\bar{k}}\left(\frac{1}{\theta}z_t + \hat{k}_{t-1} - \frac{1-\theta}{\theta}\eta\hat{c}_t\right) - \theta\frac{\bar{y}}{\bar{R}\bar{k}}\hat{k}_{t-1} \\
\Rightarrow &\boxed{\hat{r}_t = \frac{\bar{y}}{\bar{R}\bar{k}}z_t - \frac{\bar{y}}{\bar{R}\bar{k}}(1-\theta)\eta\hat{c}_t}
\end{aligned} \tag{29}$$

Insert the result into the budget constraint of Eq.(26):

$$\begin{aligned}
\bar{c}\hat{c}_t + \bar{k}\hat{k}_t &= \bar{y}\hat{y}_t + (1-\delta)\bar{k}\hat{k}_{t-1} \\
\bar{c}\hat{c}_t + \bar{k}\hat{k}_t &= \bar{y}\left[\frac{1}{\theta}z_t + \hat{k}_{t-1} - \frac{1-\theta}{\theta}\eta\hat{c}_t\right] + (1-\delta)\bar{k}\hat{k}_{t-1} \\
\bar{c}\hat{c}_t + \bar{k}\hat{k}_t &= \bar{y}\frac{1}{\theta}z_t + \bar{y}\hat{k}_{t-1} - \frac{1-\theta}{\theta}\eta\bar{y}\hat{c}_t + (1-\delta)\bar{k}\hat{k}_{t-1} \\
(\bar{c} + \frac{1-\theta}{\theta}\eta\bar{y})\hat{c}_t + \bar{k}\hat{k}_t &= \bar{y}\frac{1}{\theta}z_t + \left[\bar{y} + (1-\delta)\bar{k}\right]\hat{k}_{t-1} \\
0 &= -\bar{k}\hat{k}_t + \left(\bar{c} + \bar{k}\right)\hat{k}_{t-1} - (\bar{c} + \frac{1-\theta}{\theta}\eta\bar{y})\hat{c}_t + \bar{y}\frac{1}{\theta}z_t \\
\Rightarrow &\boxed{0 = -\hat{k}_t + \underbrace{(\frac{\bar{c}}{\bar{k}} + 1)}_{a_1}\cdot\hat{k}_{t-1} + \left[\underbrace{-\left(\frac{\bar{c}}{\bar{k}} + \frac{1-\theta}{\theta}\eta\frac{\bar{y}}{\bar{k}}\right)}_{a_2}\right]\cdot\hat{c}_t + \underbrace{\frac{1}{\theta}\frac{\bar{y}}{\bar{k}}}_{a_3}\cdot z_t}
\end{aligned} \tag{30}$$

Or $a_1 = \frac{\bar{c}}{\bar{k}} + 1 = \frac{\bar{c}+\bar{k}}{\bar{k}} = \frac{\bar{y}}{\bar{k}} + 1 - \delta > 1$. Insert the interest rate Eq.(29) into the Euler equation (25)

$$0 = \mathbb{E}_t\big[\eta(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\big]$$
$$0 = \mathbb{E}_t\left[\eta(\hat{c}_t - \hat{c}_{t+1}) + \frac{\bar{y}}{\bar{R}\bar{k}} \cdot z_{t+1} - \frac{\bar{y}}{\bar{R}\bar{k}}(1-\theta)\eta \cdot \hat{c}_{t+1}\right]$$

Dividing both sides by $-\eta$ and rearrange it in compact form:

$$\boxed{0 = \mathbb{E}_t\left[-\hat{c}_t + \underbrace{\left[1 + (1-\theta)\frac{\bar{y}}{\bar{R}\bar{k}}\right]}_{a_5} \cdot \hat{c}_{t+1} + \underbrace{\left[-\frac{\bar{y}}{\eta\bar{R}\bar{k}}\right]}_{a_6} \cdot z_{t+1}\right]} \tag{31}$$

where $a_4 = 0$.

Collecting the equations

$$0 = -\hat{k}_t + a_1\hat{k}_{t-1} + a_2\hat{c}_t + a_3\hat{z}_t$$
$$0 = \mathbb{E}_t\left[-\hat{c}_t + a_4\hat{k}_t + a_5\hat{c}_{t+1} + a_6\hat{z}_{t+1}\right],$$

where $1' \times 6 = 6'$

$$a_1 = \frac{\bar{c}}{\bar{k}} + 1 \qquad a_2 = -\left(\frac{\bar{c}}{\bar{k}} + \frac{1-\theta}{\theta}\eta\frac{\bar{y}}{\bar{k}}\right)$$
$$a_3 = \frac{1}{\theta}\frac{\bar{y}}{\bar{k}} \qquad a_4 = 0$$
$$a_5 = 1 + (1-\theta)\frac{\bar{y}}{\bar{R}\bar{k}} \qquad a_6 = -\frac{\bar{y}}{\eta\bar{R}\bar{k}}$$

2.10 Guessing the Recursive Law of Motion

Assume the following recursive law of motion for the two model variables. $2'$

$$\hat{c}_t = v_{ck} \cdot \hat{k}_{t-1} + v_{cz} \cdot z_t$$
$$\hat{k}_t = v_{kk} \cdot \hat{k}_{t-1} + v_{kz} \cdot z_t$$

2.11 Recursive Equilibrium Law of Motion

For the first equation

$$\begin{aligned} 0 &= -\hat{k}_t + a_1\hat{k}_{t-1} + a_2\hat{c}_t + a_3\hat{z}_t \\ &= -(v_{kk}\hat{k}_{t-1} + v_{kz}z_t) + a_1\hat{k}_{t-1} + a_2\left(v_{ck}\hat{k}_{t-1} + v_{cz}z_t\right) + a_3\hat{z}_t \\ &= -v_{kk}\hat{k}_{t-1} - v_{kz}z_t + a_1\hat{k}_{t-1} + a_2v_{ck}\hat{k}_{t-1} + a_2v_{cz}z_t + a_3\hat{z}_t \end{aligned}$$
$$\Rightarrow \quad \boxed{0 = (-v_{kk} + a_1 + a_2v_{ck}) \cdot \hat{k}_{t-1} + (-v_{kz} + a_2v_{cz} + a_3) \cdot \hat{z}_t} \tag{32}$$

For the second equation

$$\begin{aligned} 0 &= \mathbb{E}_t\left[-\hat{c}_t + a_5\hat{c}_{t+1} + a_6\hat{z}_{t+1}\right] \\ &= \mathbb{E}_t\left[-\left(v_{ck}\hat{k}_{t-1} + v_{cz}z_t\right) + a_5\left(v_{ck}\hat{k}_t + v_{cz}z_{t+1}\right) + a_6\hat{z}_{t+1}\right] \end{aligned}$$

$$= \mathbb{E}_t \Big[- \Big(v_{ck}\hat{k}_{t-1} + v_{cz}z_t \Big) + a_5 v_{ck} \Big(v_{kk}\hat{k}_{t-1} + v_{kz}z_t \Big) + (a_5 v_{cz} + a_6) z_{t+1} \Big]$$

$$\Rightarrow \quad \boxed{0 = (-v_{ck} + a_5 v_{ck} v_{kk}) \cdot \hat{k}_{t-1} + \big[- v_{cz} + a_5 v_{ck} v_{kz} + (a_5 v_{cz} + a_6)\rho \big] \cdot z_t} \tag{33}$$

suing $\mathbb{E}_t\big[z_{t+1}\big] = \mathbb{E}_t\big[\rho z_t + \varepsilon_{t+1}\big] = \rho z_t$.

Note that all these should be holding for any level of the model variables. As a result, we are able to employ the **Method of Undetermined Coefficients**.

For $\boxed{k_{t-1}}$

$$- v_{kk} + a_1 + a_2 v_{ck} \quad = 0 \tag{34a}$$

$$-v_{ck} + a_5 v_{ck} v_{kk} \quad = 0 \tag{34b}$$

Note that Eq.(34b) could be rewritten in following way

$$v_{ck} \cdot (-1 + a_5 v_{kk}) = 0, \tag{34b}$$

by which we conjuncture either v_{ck} or the term $(-1 + a_5 v_{kk})$ is equal to zero. However, the former case should not be correct.

Suppose that v_{ck} were equal to 0, then by the Eq.(34a), we would get

$$-v_{kk} + a_1 + a_2 v_{ck} = 0 \tag{34a$'$}$$

$$\Rightarrow \quad v_{kk} = a_1 = \frac{\bar{c}}{\bar{k}} + 1 > 0,$$

which is *NOT* a stable solution. Thus, the null hypothesis that v_{ck} is zero should be rejected in this case.

Consequently, we have

$$(-1 + a_5 v_{kk}) = 0$$

$$\Rightarrow \quad \boxed{v_{kk} = \frac{1}{a_5} = \frac{1}{1 + (1-\theta)\frac{\bar{y}}{R\bar{k}}} < 1} \tag{35}$$

which is validly a stable solution for this model. Substitute in Eq.(34a) to obtain

$$\boxed{v_{ck} = \frac{v_{kk} - a_1}{a_2} = \frac{1 - a_1 a_5}{a_2 a_5}} \tag{36}$$

Alternatively, you may insert the first equation into the second to get

$$\begin{aligned} a_2 v_{ck} &= v_{kk} - a_1 \\ v_{ck} &= \frac{1}{a_2} v_{kk} - \frac{a_1}{a_2} \end{aligned}$$

Substituting back in Eq.(34b)

$$\frac{a_1}{a_2} - \frac{1}{a_2} v_{kk} + a_5 v_{kk} \left(\frac{1}{a_2} v_{kk} - \frac{a_1}{a_2}\right) \quad = \quad 0$$

$$
\begin{aligned}
\frac{a_5}{a_2}v_{kk}^2 - (\frac{1}{a_2} + \frac{a_1 a_5}{a_2})v_{kk} + \frac{a_1}{a_2} &= 0 \\
v_{kk}^2 - (\frac{1}{a_2}\frac{a_2}{a_5} + \frac{a_1 a_5}{a_2}\frac{a_2}{a_5})v_{kk} + \frac{a_1}{a_2}\frac{a_2}{a_5} &= 0 \\
v_{kk}^2 - (\frac{1}{a_5} + a_1)v_{kk} + \frac{a_1}{a_5} &= 0 \\
(v_{kk} - \frac{1}{a_5})(v_{kk} - a_1) &= 0
\end{aligned}
$$

One should choose the root smaller than 1, which is the first one. Again,

$$
v_{kk} = \frac{1}{a_5}, \qquad v_{ck} = \frac{1}{a_2 a_5} - \frac{a_1}{a_2} = \frac{1 - a_1 a_5}{a_2 a_5}
$$

By comparing coefficients on $\boxed{z_t}$

$$
\begin{aligned}
-v_{kz} + a_2 v_{cz} + a_3 &= 0 \qquad (37a) \\
-v_{cz} + a_5 v_{ck} v_{kz} + (a_5 v_{cz} + a_6)\rho &= 0 \qquad (37b)
\end{aligned}
$$

By Eq.(37a) we have

$$
v_{kz} = a_2 v_{cz} + a_3.
$$

Plug into Eq.(37b) to obtain

$$
\begin{aligned}
& -v_{cz} + a_5 v_{ck}(a_2 v_{cz} + a_3) + (a_5 v_{cz} + a_6)\rho = 0 \qquad (38) \\
& (-1 + a_2 a_5 v_{ck} + a_5\rho)v_{cz} + a_3 a_5 v_{ck} + a_6\rho = 0 \qquad (39) \\
& \Rightarrow \boxed{v_{cz} = \frac{a_3 a_5 v_{ck} + a_6\rho}{1 - a_2 a_5 v_{ck} - a_5\rho}} \qquad (40)
\end{aligned}
$$

and

$$
v_{kz} = a_2 v_{cz} + a_3
$$

TO sum up, we maintain the recursive equilibrium law of motion

$$
\begin{aligned}
\hat{c}_t &= v_{ck} \cdot \hat{k}_{t-1} + v_{cz} \cdot z_t \\
\hat{k}_t &= v_{kk} \cdot \hat{k}_{t-1} + v_{kz} \cdot z_t,
\end{aligned}
$$

where $\boxed{7'}$

$$
\begin{aligned}
v_{kk} &= \frac{1}{a_5} & v_{ck} &= \frac{1 - a_1 a_5}{a_2 a_5} \\
v_{cz} &= \frac{a_3 a_5 v_{ck} + a_6\rho}{1 - a_2 a_5 v_{ck} - a_5\rho} & v_{kz} &= a_2 v_{cz} + a_3
\end{aligned}
$$

2.12 Calibration

$.5' \times 16 = 8'$

Table 1: The Calibration Results

	Set #1	Set #2		Set #1	Set #2
U	−146.6625	−94.6120	$\bar{R}$	1.2626	1.0101
$\bar{k}$	0.4733	12.6631	$\bar{y}$	0.3782	1.2347
$\bar{c}$	0.3664	0.9181	$\bar{A}$	1.3874	2.5820
a_1	1.7740	1.0725	a_2	−2.1943	−0.2458
a_3	2.2193	0.2708	a_4	0	0
a_5	1.4050	1.0618	a_6	−0.6328	−0.0965
v_{ck}	0.4841	0.5316	v_{cz}	0.7846	0.4703
v_{kk}	0.7118	0.9418	v_{kz}	0.4978	0.1552

Note that The discounted sum of the steady state utility is calculated via

$$\mathbf{U} = \sum_{t=0}^{\infty} \beta^t u(\bar{c}, \bar{n}) = u(\bar{c}, \bar{n}) \cdot \frac{1}{1-\beta}.$$

Note that when $\eta = 1$, then the utility function takes the form: $u(c, n) = \ln(c) - A \cdot n$. Therefore

$$\mathbf{U} = \left[\ln(\bar{c}) - A \cdot \bar{n}\right] \cdot \frac{1}{1-\beta}.$$

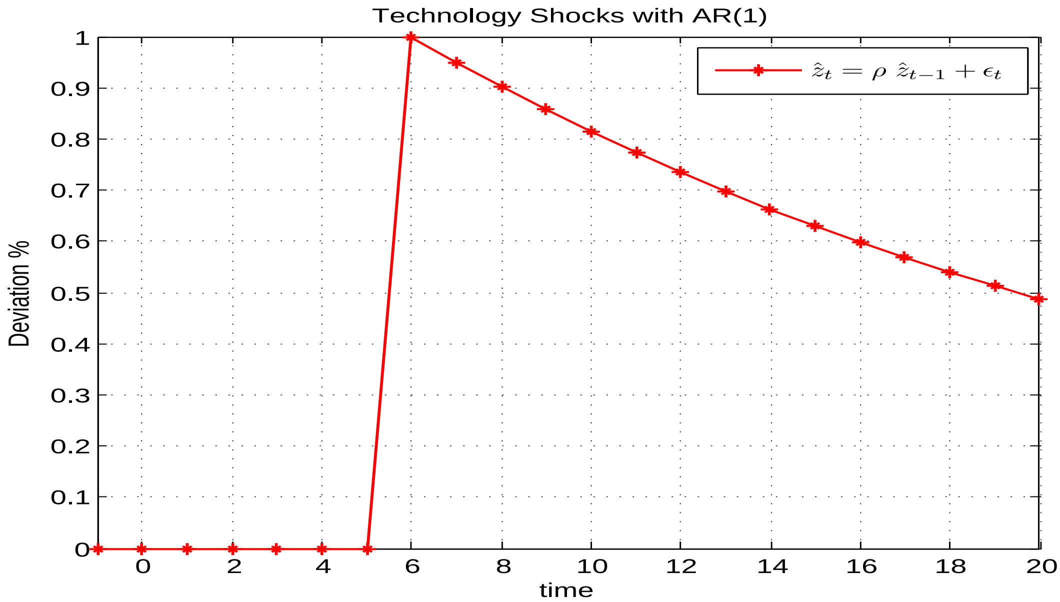
2.13 Impulse Response Analysis

$4'$

Suppose that there occurs a positive technology shock to the production function at period $t = 6$. By the recursive equilibrium law of motion, we could plot the *impulse responses* for the real variables to one unit of technology shocks. Comparing this decentralized model with taxation with Hansen's[1985] Model with No Taxation, it could be summarized that

$4'$

- Generally, these two model share the similar characteristics: *in the short run*, a positive technology shock to the economy would generate higher product, and hence higher capital stock, higher consumption by the household, requiring increase in the real interest rate, as well as greater real wage. This is due to the increase in the *total factor productivity* (TFP).
- However, these two distinct model behavior slightly different in the long run. While our decentralized model with taxation on capital and consumption would eventually converge to the steady state at a relatively high speed, Hansen's[1985] Model presents much stronger persistence in capital stock, consumption, and real wage. This difference confirms the fact that the introduction of distortional tax would result in sub-optimality of the economy. That's, there exists a deadweight social lost associated with the government tax.



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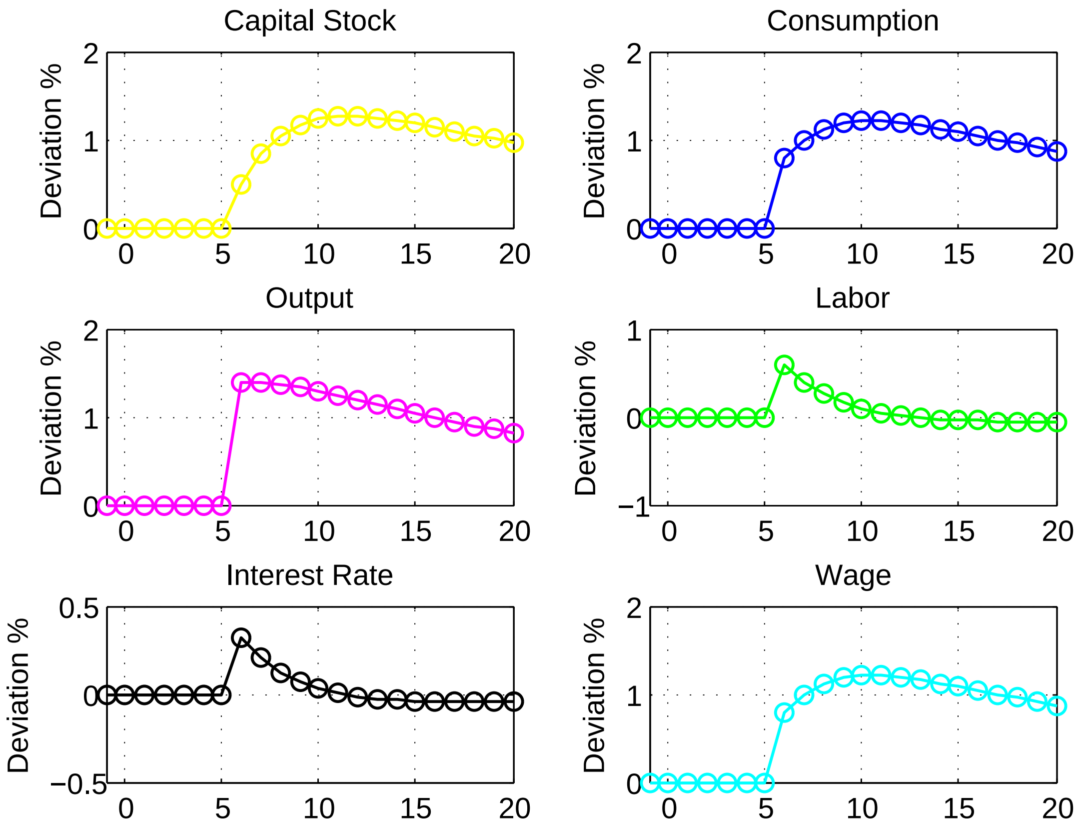

Figure 1: Model with Taxation

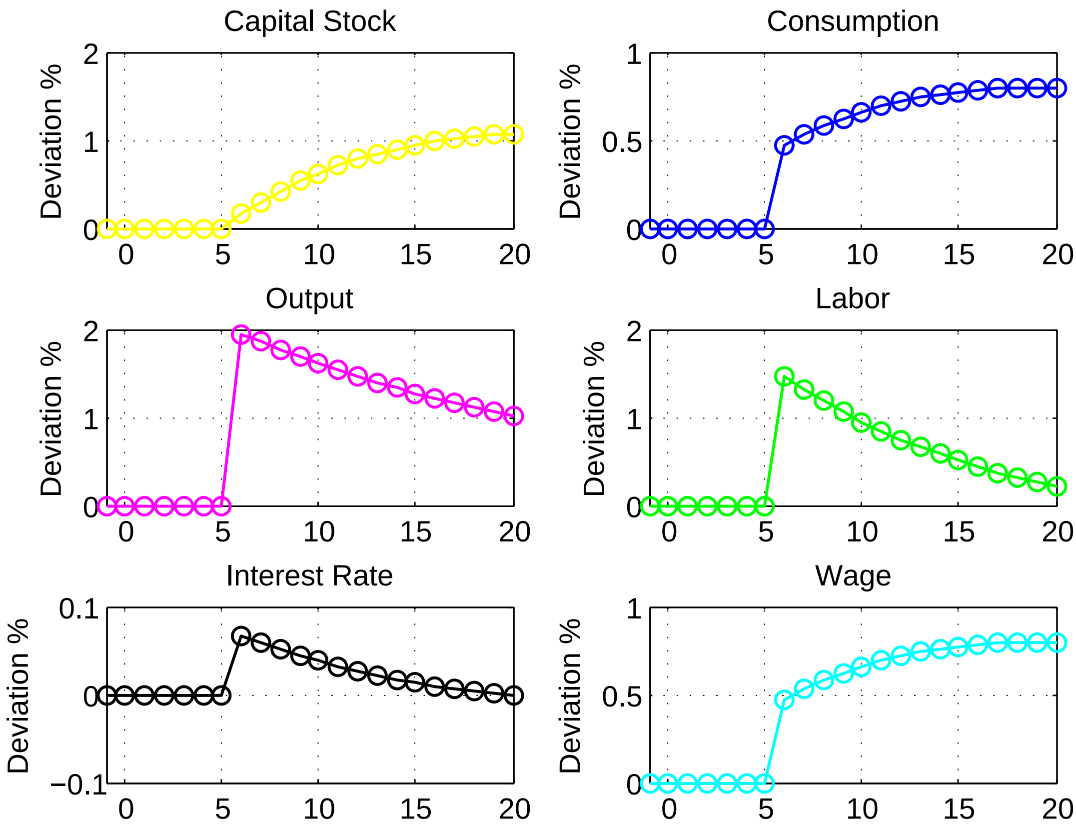

Figure 2: Hansen's[1985] Model with No Taxation