Proposition. *Consider the binomial model with N periods. Let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted portfolio process, let X_0 be a real number, and let the wealth process $X_1, \ldots, X_N$ be generated recursively by (. Then the discounted wealth process $\frac{X_n}{(1+r)^n}$, $n = 0, 1, \ldots, N$, is a martingale under the risk-neutral measure; i.e.,*

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right], n = 0, 1, \ldots, N-1.$$

Proof. (7 points)We compute

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \tilde{\mathbb{E}}_n\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \text{(Linearity)} \\
&= \Delta_n \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] + \frac{X_n - \Delta_n S_n}{(1+r)^n} \text{(Taking out what is known)} \\
&= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{X_n}{(1+r)^n}.
\end{aligned}$$

(3 points) We need show that $\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = \frac{S_n}{(1+r)^n}$. □

$$\begin{aligned}
&\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^n}\right](\omega_1 \ldots \omega_n) \\
=&\frac{1}{(1+r)^n} \cdot \frac{1}{1+r}\left[\tilde{p} S_{n+1}(\omega_1 \ldots \omega_n H) + \tilde{q} S_{n-1}(\omega_1 \ldots \omega_n T)\right] \\
=&\frac{1}{(1+r)^n} \cdot \frac{1}{1+r}\left[\tilde{p} u S_n(\omega_1 \ldots \omega_n) + \tilde{q} d S_n(\omega_1 \ldots \omega_n)\right] \\
=&\frac{S_n(\omega_1 \ldots \omega_n)}{(1+r)^n} \cdot \frac{\tilde{p} u + \tilde{q} d}{1+r} \text{ (where } \tilde{p} = \frac{1+r-d}{u-d}, \tilde{q} = \frac{u-1-r}{u-d}.\text{)} \\
=&\frac{S_n(\omega_1 \ldots \omega_n)}{(1+r)^n}.
\end{aligned}$$

□