Advanced Microeconomics II

Mixed Strategy Nash Equilibrium

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April 1, 2015

Mixed Strategies

- Existence of NE in strategic games is not very general, e.g. Matching Pennies.
- A possible improvement is to make action sets convex.
- Denote $\Delta(A_i)$ as the set of probability distributions over A_i.
- $\alpha_i \in \Delta(A_i)$ is a mixed strategy of player i.
- Let $U_i : \times_{j \in N} \Delta(A_j) \rightarrow \mathcal{R}$ be a von Neumann-Morgenstern utility function for player i that represents preferences over $\times_{j \in N} \Delta(A_j)$. (What assumption is implied?)

Mixed Strategies for Finite Games

For finite A:

- $\alpha_i(a_i)$ is the probability that α_i assigns to a_i.
- If $\alpha_i(a_i) = 1$ then α_i is a pure strategy.

$$U_i(\alpha) = \sum_{a \in A} (\Pi_{j \in N} \alpha_j(a_j)) u_i(a).$$

Example:

$$\begin{array}{cc|c|c|} & \multicolumn{3}{c}{\text{My wife}} \\ & & M & F \\ \cline{3-4} \text{Me} & M & 3,3 & 0,0 \\ \cline{3-4} & F & 0,0 & 1,1 \\ \cline{3-4} \end{array}$$

Let α_i^M be the probability that player i plays action M.

$$U_1(\alpha) = \alpha_1^M \alpha_2^M 3 + \alpha_1^M (1 - \alpha_2^M) 0 + (1 - \alpha_1^M) \alpha_2^M 0 + (1 - \alpha_1^M)(1 - \alpha_2^M) 1$$

Mixed Strategies for Infinite Games

For infinite A:

- $\alpha_i(.)$ is the probability distribution function over A_i.

$$U_i(\alpha) = \int_{a \in A} u_i(a)(\Pi_{j \in N} \alpha_j(a_j))da.$$

Example:
$G = \{N, (A_i)_{i=1}^N, (u_i)_{i=1}^N\}$, where
$N = \{1, 2\}$, $A_1 = A_2 = [0, \infty)$,
$u_i(a_1, a_2) = \max\{(1 - a_1 - a_2)a_i, 0\}$

Let $\alpha_i(.)$ be the probability distribution over A_i.

$$U_1(\alpha) = \int_0^\infty \int_0^\infty \max\{(1 - a_1 - a_2)a_1, 0\}\alpha_1(a_1)\alpha_2(a_2)da_1da_2$$

Mixed Strategy Nash Equilibrium

Definition

The mixed extension of the strategic game $\{N, (A_i), (u_i)\}$ is the strategic game $\{N, (\Delta(A_i)), (U_i)\}$.

Definition

A mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension. Specifically, for every player $i \in N$

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}) \text{ for every } \alpha_i \in \Delta(A_i).$$

Existence of Mixed Strategy Nash Equilibrium

Proposition

Every finite strategic game has a mixed strategy Nash equilibrium.

- $\Delta(A_i)$ is a non-empty, compact, convex set.
- U_i is linear in the probabilities over A_i, so it is continuous and quasi-concave in α_i.
- Apply our previous existence theorem.

Mixed Strategy Nash Equilibrium and Best Response

Lemma

Let $G = \{N, (A_i), (u_i)\}$ be a finite game. Then $\alpha^* \in \times_{i \in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G if and only if for every player $i \in N$ every pure strategy in the support of α_i^* is a best response to α_{-i}^*.

- We can write $U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(e(a_i), \alpha_{-i})$, where $e(a_i)$ is the strategy that plays a_i with probability one.
- $(\Rightarrow)$ If a_i in the support of α_i is not a best response, then transfer $\alpha_i(a_i)$ to a best response action.
- $(\Leftarrow)$ If there exists another α_i' that gives a higher payoff then there must be at least one action in the support of α_i' that gives a higher payoff than some action in the support of α_i^*.

Implication: Every action in the support of α_i^* yields the same payoff.

Pure Coordination Game

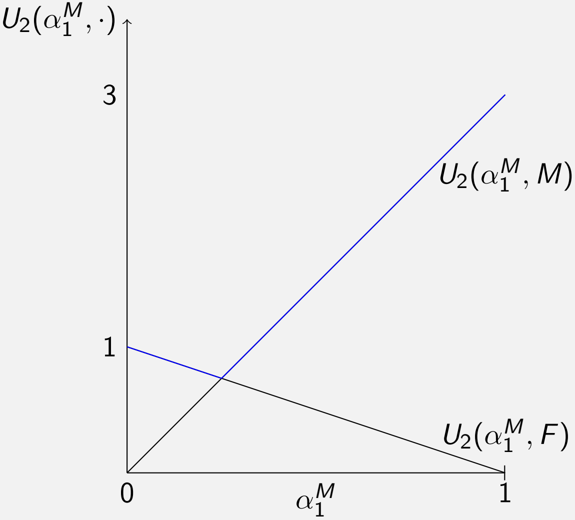
		My wife M	My wife F
Me	M	$3,3$	$0,0$
Me	F	$0,0$	$1,1$

- Two mixed strategy equilibria: $\{M, M\}$ and $\{F, F\}$.
- Other mixed strategy equilibria? What distribution on A_1 makes player 2 indifferent between M and F.
- Let α_1^M be the probability that I watch movies.
  - So $1 - \alpha_1^M$ is the probability that I watch football.

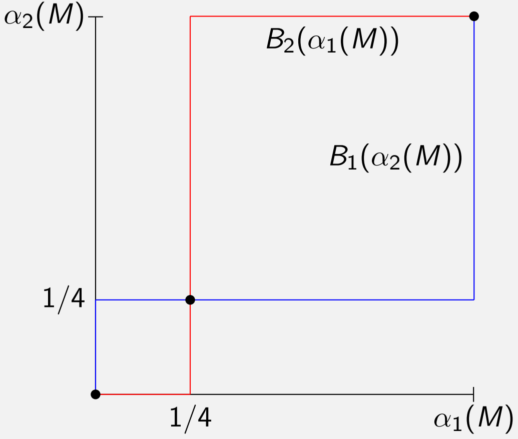
$$\alpha_1^M u_2(M, M) + (1 - \alpha_1^M) u_2(F, M) = \alpha_1^M u_2(M, F) + (1 - \alpha_1^M) u_2(F, F)$$
$$3\alpha_1^M = 1 - \alpha_1^M$$
$$\alpha_1^M = 1/4$$

- $\{\alpha_1^M = 1/4, \alpha_2^M = 1/4\}$ is a mixed strategy Nash equilibrium.

Deriving Best Response Functions



Best Response Functions



For You

		Player 2	
		L	R
Player 1	U	$6, 6$	$2, 7$
	D	$7, 2$	$0, 0$

- Are there any pure strategy equilibria?
- Are there any other mixed strategy equilibria?

War of Attrition

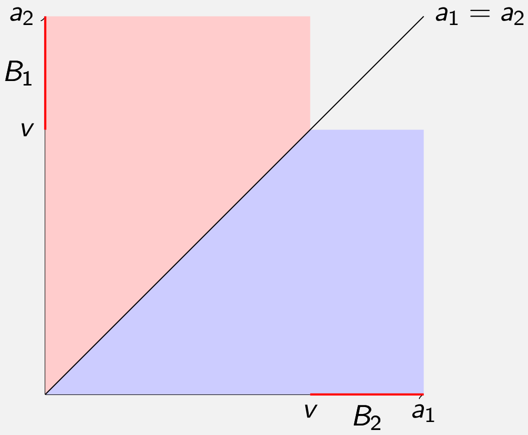
Two players are fighting over an object with value $v > 0$.

- Each player chooses a time to quit.
- The player who quits first loses and pays a cost equal to his quit time.
- The other player pays the same cost but wins the object.
- If they quit at the same time they each receive half the value of the object.

Model this as a strategic game.

Pure Strategy Equilibria

What are the pure strategy equilibria?



Other Mixed Strategy Equilibria

- Assume the equilibrium is symmetric.
- Assume each player's strategy is represented by a continuous distribution $F(a_i)$ with support over $[\underline{a}, \bar{a}]$.
  - Payoff from each pure strategy is the same, say C.
- Choose an arbitrary pure strategy a_1. What is the payoff?

$$\int_{\underline{a}}^{a_1} (v - a_2) f(a_2) da_2 + \int_{a_1}^{\bar{a}} -a_1 f(a_2) da_2 \equiv C$$

- Differentiate both sides with respect to a_1.

$$(v - a_1) f(a_1) + a_1 f(a_1) - \int_{a_1}^{\bar{a}} f(a_2) da_2 = 0$$

- This is a first order differential equation. $vf(a_1) = 1 - F(a_1)$
- So $F(a) = 1 - Ke^{-a/v}$. (What is $\underline{a}$, $\bar{a}$ and K?)

Interpretations of Mixed Strategy Equilibria

- As Objects of choice.
  - Actions are based on guesses and guessing is a psychological operation that is deliberate.
  - Doesn't model player motivation for randomization but is probably a good description of behavior.
- As a (Stochastic) Steady State
  - Reinterpret equilibrium as the interaction of n large populations — $\alpha_i(a_i)$ is the steady state frequency of matching with a_i in population i.
- As Pure Strategies in an Extended Game
  - There exists unmodeled factors that influence behavior.
  - Hard to accept that deliberate behavior is influenced by factors that do not affect payoffs.
  - Predicted behavior is fragile, since unobserved changes in external factors can destroy the equilibrium outcome.
  - In order to interpret an equilibrium in application need to specify these unmodeled effects and how they affect choices.

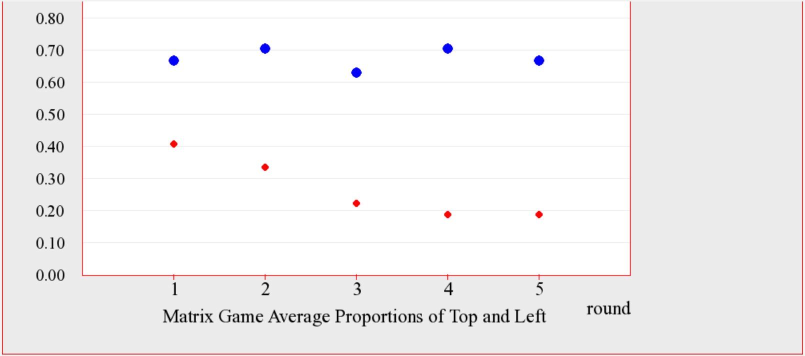
Interpretations of Mixed Strategy Equilibria

- As Pure Strategies in a Perturbed Game
  - (Harsanyi 1973) Game is an approximation of Bayesian game where each player's preferences are subject to small shocks.
  - The mixed strategy profile is the limit of a sequence of pure strategy equilibrium profiles of Bayesian games with successively smaller perturbations on payoffs.
  - The limit of any sequence of pure strategy equilibrium profiles of Bayesian games with successively smaller perturbations on payoffs is a Nash equilibrium mixed strategy profile of the limit game.
- As Beliefs
  - A Nash equilibrium is a profile of beliefs β where the other players have a *common* belief β_i about player *i*'s action and each action in the support of β_i is optimal given β_{-i}.
  - The equilibrium is a steady state of beliefs rather than actions.

Predictive Ability

Recall the following game from the class experiment.

		Player 2	
		L	R
Player 1	T	4, 2	0,0
	B	0, 0	2,4



Predictive Ability

- Total observed outcomes: 135,
- Proportion of T: 0.674 (0.66).
- Proportion of L: 0.267 (0.33).

Frequency of outcomes

		Player 2	
		L	R
Player 1	T	0.18 (0.22)	0.50 (0.44)
	B	0.09 (0.11)	0.24 (0.22)

Pearson Chi-squared test $= 2.75$, p-value $= 0.43$.

Correlated Equilibrium

Definition

A correlated equilibrium of a strategic game $\{N, (A_i), (u_i)\}$ is

- a finite probability space (Ω, π) (Ω is the set of states and π is the probability measure on Ω)
- for each player $i \in N$ a partition of $\mathcal{P}_i$ of Ω (player i's information partition)
- for each player $i \in N$ a function $\sigma_i : \Omega \to A_i$ with $\sigma_i(\omega) = \sigma_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$ (σ_i is player i's strategy)

such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ for which $\tau_i(\omega) = \tau_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\tau_i(\omega), \sigma_{-i}(\omega))$$

Correlated Equilibrium and Nash Equilibrium

Proposition

For every mixed strategy Nash equilibrium α of a finite strategic game $\{N, (A_i), (u_i)\}$ there is a correlated equilibrium $\{(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\}$ in which for each player $i \in N$ the distribution on A_i induced by σ_i is α_i.

- Set $\Omega = A$ and $\pi(a) = \Pi_{j \in N} \alpha_j(a_j)$.
- For each $i \in N$ and $b_i \in A_i$ set $P_i(b_i) = \{a \in A : a_i = b_i\}$ and let $\mathcal{P}_i$ consist of the $|A_i|$ sets $P_i(b_i)$.
- Define $\sigma_i(a) = a_i$ for each $a \in A$.
- This is a correlated equilibrium since for each player i, for each $a_i \in A_i$

$$\sum_{\{b \in A : b_i = a_i\}} \pi(b) u_i(a_i, b_{-i}) \geq \sum_{\{b \in A : b_i = a_i\}} \pi(b) u_i(b_i', b_{-i}) \text{ for any } b_i' \in A_i.$$

- The distribution on A_i induced by σ_i is α_i.

Correlated Equilibrium and Nash Equilibrium: Application

- Applied to our Pure co-ordination game and the mixed strategy equilibrium $\alpha_1^*(M) = \alpha_2^*(M) = 1/4$.
- Set $\Omega = A$ and
  $\pi(\{MM\}) = 1/16, \pi(\{MF\}) = \pi(\{FM\}) = 3/16, \pi(\{FF\}) = 9/16$.
- $P_1(M) = \{MM, MF\} = P_1^M$, $P_1(F) = \{FM, FF\} = P_1^F$,
  $\mathcal{P}_1 = \{P_1^M, P_1^F\}$,
  $P_2(M) = \{MM, FM\} = P_2^M$, $P_2(F) = \{MF, FF\} = P_2^F$,
  $\mathcal{P}_2 = \{P_2^M, P_2^F\}$.
- $\sigma_1(MM) = \sigma_1(MF) = M$, $\sigma_1(FM) = \sigma_1(FF) = F$,
  $\sigma_2(MM) = \sigma_2(FM) = M$, $\sigma_2(MF) = \sigma_2(FF) = F$.

Set of Correlated Equilibrium Payoffs is Convex

Proposition

Let $G = \{N, (A_i), (u_i)\}$ be a strategic game. Any convex combination of correlated equilibrium payoff profiles of G is a correlated equilibrium payoff profile of G.

- Let $u^1, \ldots, u^K$ be correlated equilibrium payoff profiles and let $(\lambda^1, \ldots, \lambda^K) \in \mathcal{R}^K$ with $\lambda^k \geq 0$ for all k and $\sum_{k=1}^K \lambda^k = 1$.
- For each k, $\{(\Omega^k, \pi^k), (\mathcal{P}_i^k), (\sigma_i^k)\}$ is the correlated equilibrium that generates u^k. (disjoint Ω^k)
- Let $\Omega = \cup_k \Omega^k$ and $\pi(\omega) = \lambda^k \pi^k(\omega)$ where k is such that $\omega \in \Omega^k$.
- For each i let $\mathcal{P}_i = \cup_k \mathcal{P}_i^k$.
- Let $\sigma_i(\omega) = \sigma_i^k(\omega)$ where k is such that $\omega \in \Omega^k$.

Correlated Equilibrium

Is the set of correlated equilibrium payoffs the convex hull of mixed strategy Nash equilibrium payoffs?

Player 2

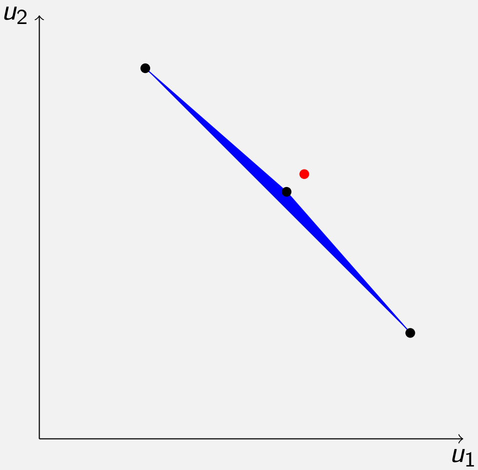
Player 1		L	R
	U	$6,6$	$2,7$
	D	$7,2$	$0,0$

Player 2

Player 1		L	R
	U	y	z
	D	x	$-$

- Recall the set of Nash equilibrium payoffs are $\{(2,7),(7,2),(4\frac{2}{3},4\frac{2}{3})\}$.
- Consider $\{(\Omega,\pi),(\mathcal{P}_i),(\sigma_i)\}$ where
  - $\Omega=\{x,y,z\},\pi(x)=\pi(y)=\pi(z)=1/3$.
  - $\mathcal{P}_1=\{\{x\},\{y,z\}\},\mathcal{P}_2=\{\{x,y\},\{z\}\}$,
  - $\sigma_1(x)=D,\sigma_1(y)=\sigma_1(z)=U$,
    $\sigma_2(x)=\sigma_2(y)=L,\sigma_2(z)=R$
- What are the payoffs to each player?
- Is it a correlated equilibrium.

Nash Equilibrium versus Correlated Equilibrium



Simplified State Space

Proposition

Let $G = \{N, (A_i), (u_i)\}$ be a finite strategic game. Every probability distribution over outcomes that can be obtained in a correlated equilibrium of G can be obtained in a correlated equilibrium in which the set of states is A and for each $i \in N$ player i's information partition consists of all sets of the form $\{a \in A : a_i = b_i\}$ from some action $b_i \in A_i$.

- Let $\{(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\}$ be a correlated equilibrium of G.
- $\{(\Omega', \pi'), (\mathcal{P}'_i), (\sigma'_i)\}$ is also a correlated equilibrium of G.
  - $\Omega' = A$ and $\pi'(a) = \pi(\{\omega \in \Omega : \sigma(\omega) = a\})$ for each $a \in A$.
  - $\mathcal{P}'_i$ consists of sets of the type $\{a \in A : a_i = b_i\}$ from some $b_i \in A_i$.
  - $\sigma'_i(a) = a_i$.