Solution to P.S. 1

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1. Prove that **an ordinal utility function preserves the preference orderings for any strictly increasing transformation.**

Proof:

Suppose$\succeq$ is a preference relation on the consumption space which is represented by $u(\boldsymbol{x})$. Namely

$$\forall \boldsymbol{x}^1, \boldsymbol{x}^2 \in X,\ \boldsymbol{x}^1 \succeq \boldsymbol{x}^2 \Leftrightarrow u\left(\boldsymbol{x}^1\right) \geq u\left(\boldsymbol{x}^2\right)$$

since f is strictly increasing in the range of u, then

$$u\left(\boldsymbol{x}^1\right) \geq u\left(\boldsymbol{x}^2\right) \Leftrightarrow f\left(u\left(\boldsymbol{x}^1\right)\right) \geq f\left(u\left(\boldsymbol{x}^2\right)\right), \, i.e. \, v\left(\boldsymbol{x}^1\right) \geq v\left(\boldsymbol{x}^2\right)$$

Hence $v(x)$ could fully represent the preference relation$\succeq$.$\blacksquare$

2. Prove that the **von Neumann-Morgenstern expected utility function is unique up to a linear monotonic transformation**, which is a cardinal property .

- Suppose that the vNM utility function $U(\cdot)$ represents $\succeq$.

- Then the vNM utility function, $V(\cdot)$, represents *the same preferences* if and only if for some scalar α and β,$\beta > 0$, s.t. $V(p) = \alpha + \beta U(p)$, for all lotteries p.

Proof:

1. Sufficiency

   we are to prove if $V(p) = \alpha + \beta U(p)$, for all lotteries p, α, $\beta \in \mathbb{R}$ and$\beta > 0$, then $V(\cdot)$ is a vNM expected utility function that represents the same preference order.

   - for$\forall p^1$, p^2lottery, and $p^1 \succeq p^2$, since $U(\cdot)$ is the vNM expected utility function that represents this preference order, thus

   $$p^1 \succeq p^2 \Leftrightarrow U(p^1) \geq U(p^2) \Leftrightarrow \sum_{i=1}^{N} p_i^1 u_i \geq \sum_{i=1}^{N} p_i^2 u_i$$

   - on the other hand, since$V(p) = \alpha + \beta U(p)$ and $\beta > 0$,

   $$U(p^1) \;\; \geq \;\; U(p^2) \Leftrightarrow \alpha + \beta U(p^1) \geq \alpha + \beta U(p^2) \Leftrightarrow V(p^1) \geq V(p^2)$$

   and besides

   $$\alpha + \beta \sum_{i=1}^{N} p_i^1 u_i \;\; \geq \;\; \alpha + \beta \sum_{i=1}^{N} p_i^2 u_i \Leftrightarrow \sum_{i=1}^{N} p_i^1 (\alpha + \beta u_i) \geq \sum_{i=1}^{N} p_i^2 (\alpha + \beta u_i)$$

   - namely, $p^1 \succeq p^2 \Leftrightarrow V(p^1) \geq V(p^2)$, and $V(\cdot)$ is linear in probability, thus $V(\cdot)$ is also a vNM expected utility function.

2. Necessity.

   We are to show that if u and v are vNM expected utility functions, then they are linearly related for elementary lottery, and thus for all gambles. As before, we assume that

- All lotteries have possible payoffs that are contained in the set $X = \{x_1, ..., x_n\}$ with corresponding probability $P = \{p_1, ..., p_n\}$.
- define an "**elementary**" or "**primitive**" lottery, e_i, which returns outcome x_i with probability 1 and all other outcomes with probability zero, that is,

$$e_i = \{p_1, ..., p_{i-1}, p_i, p_{i+1}, ..., p_n\} = \{0, ..., 0, 1, 0, ...0\}$$

  where $p_i = 1$ and $p_j = 0 \; \forall j \neq i$.

- Without loss of generality, suppose that the outcomes are ordered such that

$$e_n \succeq e_{n-1} \succeq ... \succeq e_1$$

  and $e_n \succ e_1$(otherwise $e_n \sim \cdots \sim e_i \sim \cdots \sim e_1$, then every utility function is a constant and the result follows immediately). Because $u(\cdot)$ represents $\succeq$, we have $u(e_n) \geq \ldots \geq u(e_i) \geq \ldots \geq u(e_1)$, and $u(e_n) > u(e_1)$. Then for any $i = 1, \ldots, n$, $\exists \lambda_i \in [0, 1]$ s.t.

$$u(e_i) = \lambda_i u(e_n) + (1 - \lambda_i) u(e_1) \tag{1}$$

- Now, because $u(\cdot)$ has the expected utility property, (1) implies that if we regard $\lambda_i e_n + (1 - \lambda_i) e_1$as a compound lottery, we have

$$u(e_i) = \lambda_i u(e_n) + (1 - \lambda_i) u(e_1) = u(\lambda_i e_n + (1 - \lambda_i) e_1)$$

  which, because $u(\cdot)$ represents $\succeq$, means that

$$e_i \sim \lambda_i e_n + (1 - \lambda_i) e_1 \tag{2}$$

- On the other hand, because $v(\cdot)$ also represents $\succeq$, we must have

$$v(e_i) = v(\lambda_i e_n + (1 - \lambda_i) e_1)$$

since $v(\cdot)$ has the expected utility property,

$$v(e_i) = v(\lambda_i e_n + (1-\lambda_i) e_1) = \lambda_i v(e_n) + (1-\lambda_i) v(e_1) \tag{3}$$

- Together, (1) and (3) imply that

$$\frac{u(e_n) - u(e_i)}{u(e_i) - u(e_1)} = \frac{1-\lambda_i}{\lambda_i} = \frac{v(e_n) - v(e_i)}{v(e_i) - v(e_1)} \tag{4}$$

  for any $i = 1, \ldots, n$ such that $e_i \succ e_1$ (i.e., such that $\lambda_i > 0$).

- From (4) we may conclude that

$$[u(e_n) - u(e_i)][v(e_i) - v(e_1)] = [v(e_n) - v(e_i)][u(e_i) - u(e_1)] \tag{5}$$

whenever $e_n \succ e_1$.

- Rearranging, (5) can be expressed in the form

$$v(e_i) = \alpha + \beta u(e_i), \text{ for all } i = 1, \ldots, n. \tag{6}$$

where

$$\alpha \equiv \frac{u(e_n) v(e_1) - v(e_n) u(e_1)}{u(e_n) - u(e_1)}$$

and

$$\beta \equiv \frac{v(e_n) - v(e_1)}{u(e_n) - u(e_1)}$$

Notice that both α and β are constants (i.e., independent of i), and that β is strictly positive.

- So, for any other arbitrary lottery p, if $(p_1, \ldots, p_n)$ is the simple gamble induced by p, then

$$\begin{aligned} v(p) &= \sum_{i=1}^{n} p_i v(e_i) \\ &= \sum_{i=1}^{n} p_i (\alpha + \beta u(e_i)) \\ &= \alpha + \beta \sum_{i=1}^{n} p_i u(e_i) \\ &= \alpha + \beta u(p) \end{aligned}$$

where the first and last equalities follow because $v(\cdot)$ and $u(\cdot)$ have the expected utility property and the second equality follows from (6). ■