Arbitrage

Let x be an arbitrary portfolio of basis assets. Our first concern is with the prices of basis assets being inconsistent, providing possibilities of riskless profit: arbitrage. Mathematically, arbitrage can arise in two different ways: *type-two* involves *redundant basis assets*, whereas type-one arbitrage does not.

Type one arbitrage. There is a portfolio that costs nothing to purchase (for one is paid to hold it) and has non-negative payoff in all states, with a strictly positive payoff in at least one state:

$$S^T x \leqslant 0 \text{ pay nothing, or receive some money today,} \tag{1}$$

$$Ax \geqslant 0 \text{ receive a non-negative amount tomorrow,} \tag{2}$$

$$Ax \neq 0 \text{ this amount is strictly positive in at least one state,} \tag{3}$$

Example. Consider a market with the first two securities. 1st security is a bond with payoff $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$; 2nd security is a stock with pay off $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T$ with probability (1/2, 1/6, 1/3). Suppose that the prices of these securities were $S_1 = 1$ and $S_2 = 1$. Then we could sell one unit of the first security and buy one unit of the second security, which would cost nothing, and obtain a non-negative payoff $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$:

$$\text{Cost: } S^T x = [\ 1 \quad 1\ ] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0,$$

$$\text{Payoff: } Ax = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} > 0.$$

This is very much like receiving a lottery ticket for free. With nothing to pay we have the chance of obtaining 2 in the first state or 1 in the second state without the risk of losing anything in the third state. Another way of looking at the same situation is to realize that the second security pays at least as much as the first security in all states, in mathematical terminology we would say that the payoff of the first security is *stochastically dominated* by the payoff of the second security. Therefore, the second security is unambiguously more valuable than the first security, and it must *command a higher price* than the first security to prevent arbitrage. In our example both securities have the same price—hence the arbitrage opportunity.

Type two arbitrage. The second type of arbitrage is even better. There is a portfolio that has negative price (you are given some money today to hold this portfolio) and pays *identically zero* in *all states* tomorrow:

$$S^T x < 0, \tag{4}$$

$$Ax = 0, \tag{5}$$

Note that the second type of arbitrage *cannot occur* if the basis assets are *linearly independent*, because linear independent implies that $Ax = 0$ only if $x = 0$, in which case trivially $S^T x = 0$. In plain English the second type of arbitrage implies that there is a *mispriced redundant basis asset*. What is meant is the following. The redundant basis asset has a certain price. The redundant asset can also be perfectly replicated from other basis assets. If this replicating portfolio is cheaper or more expensive than the redundant asset itself, the redundant asset is mispriced and there is an easy arbitrage profit from selling the redundant asset and buying the replicating portfolio or vice versa. The absence of the second type of arbitrage means that every marketed payoff has a *unique price* (the so-called *law of one price*) and that prices are linear: the price of a security with payoff Ax must be $S^T x$.

No-Arbitrage Pricing

Having verified that there is no arbitrage among basis assets we can now use the no-arbitrage principle to determine the price of any asset with known payoff. We will refer to the asset whose price we wish to determine as a *focus* asset. This is known as *relative pricing*, since we are trying to calculate the value of the focus asset by taking the prices and payoffs of the basis as given. The two types of arbitrage present themselves differently in pricing, too. If the focus asset is redundant, we can find a replicating portfolio and from the absence of type two arbitrage we conclude that the value of the focus asset is equal to the value of the replicating portfolio.

Suppose now that the focus asset is not redundant. Does this mean we can say nothing about its price? Not quite. We will not be able to pin down the price uniquely, but the absence of type one arbitrage will restrict the price to a range. We can squeeze the payoff of the focus asset between two basis asset portfolios: a super-replicating portfolio that outperforms the focus asset in all scenarios and a sub-replicating portfolio that performs worse than the focus asset in all states. Consequently, the price of the focus asset will have to lie between the value of the most expensive sub-replicating portfolio and the value of the cheapest super-replicating portfolio.

Example. Suppose we have one basis asset with payoff

$$A_{\bullet 1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and price $S_1 = 2$. Let us find the no-arbitrage price bounds for the asset b with payoff:

$$b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Solution. We notice that one unit of the first security performs 'just' better than the second security,

$$b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \leqslant \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \times A_{\bullet 1},$$

the word 'just' referring to the payoff in the first state which is the same for both the focus asset and the super-replicating portfolio. On the other hand, half a unit of the first security performs "just" worse than the focus asset:

$$0.5 \times A_{\bullet 1} = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix} \leqslant \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = b$$

Now the word "just" refers to the payoffs in the second state. All in all, the super-replication price bounds for the focus asset are $0.5 \times S_1 < S_2 < 1 \times S_1$, $1 < S_2 < 2$.

State Prices and the Arbitrage Theorem

The price of an elementary (Arrow-Debreu) security e_j is called a *state price* and is denoted ψ_j. The vector of all state prices is denoted $\psi = \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_m \end{bmatrix}^T$. In a complete market all elementary securities can be *perfectly replicated* and we can therefore find their unique no-arbitrage prices—the state prices—by the perfect replication argument. Note that the elementary securities have non-negative payoff and, therefore, in the absence of type one arbitrage must command positive price. To sum up, no arbitrage in a complete market implies positive state prices. The converse is also true: positive state prices imply no arbitrage.

Example. Suppose that there are three states of the world and we know that the state prices are ψ_1, ψ_2, ψ_3. Find the no-arbitrage price of the security with payoff: $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Solution. We need to write b as a portfolio of elementary securities e_1, e_2, e_3. Trivially,

$$b = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so that the portfolio which combines payoff b from elementary securities e_1, e_2, e_3 is in fact $x = b$. Thus, by the linearity of no-arbitrage pricing, the price of b is $\psi^T x = \psi^T b$. □

Following the previous example, given the state price vector ψ the implied price of securities $A_{\bullet 1}$, $A_{\bullet 2}$,...., $A_{\bullet n}$ is

$$S^T = \begin{bmatrix} S_1 & S_2 & .... & S_n \end{bmatrix} = \begin{bmatrix} \psi^T A_{\bullet 1} & \psi^T A_{\bullet 2} & .... & \psi^T A_{\bullet n} \end{bmatrix} = \psi^T A$$

and after transposition

$$A^T \psi = S. \tag{6}$$

The state prices can be used with great advantage as an indicator of arbitrage.

Theorem. (Arbitrage Theorem). A market with n securities, m states of the world, a security payoff matrix $A \in \mathbb{R}^{m \times n}$ and a security price vector $S \in \mathbb{R}^n$ admits no arbitrage if and only if there is a strictly positive state price vector $\psi \in \mathbb{R}^m$ consistent with the security price vector S, that is

$$S = A^T \psi. \tag{7}$$

□

In a *complete* market the theorem does not tell us anything new. We have already concluded earlier that absence of type-two arbitrage implies unique state prices and that absence of type-one arbitrage forces these state prices to be strictly positive. In an *incomplete* market not all state prices are uniquely determined because not all elementary securities are marketed. Mathematically, the system $S = A^T\psi$ has infinitely many solutions (the situation where there is no solution would imply type two arbitrage). The theorem now has a deeper meaning: it claims that in the absence of arbitrage we can choose these undetermined state prices to be strictly positive. It also claims that if we are unable to do so, then there is arbitrage among the marketd assets.

Example. Suppose we have a market with

$$A = \begin{bmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

and

$$S^T = \begin{bmatrix} 1 & 2 & 0.6 & 1 \end{bmatrix}.$$

Decide whether there are any arbitrage opportunities.

Solution. According to the Arbitrage Theorem there is no arbitrage if and only if there is a vector of strictly positive state prices such that

$$S = A^T\psi, \tag{8}$$

$$\begin{bmatrix} 1 \\ 2 \\ 0.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1.5 & 0.5 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}. \tag{9}$$

Since $r(A) = 3$ the unique candidate for a solution is

$$\psi = (AA^T)^{-1}AS = \begin{bmatrix} 0.2 \\ 0.6 \\ 0.2 \end{bmatrix},$$

and it is easy to verify that this value of ψ solves the state price equation (8) Since all elements of ψ are positive, there is no arbitrage.

State Prices and Asset Returns

When asset prices are not zero, we can rewrite the state price equations $S = A^T\psi$ in terms of returns. Take the original system:

$$\begin{aligned} S_1 &= A_{11}\psi_1 + A_{21}\psi_2 + ... + A_{m1}\psi_m, \\ S_2 &= A_{12}\psi_1 + A_{22}\psi_2 + ... + A_{m2}\psi_m, \\ &\vdots \\ S_n &= A_{1n}\psi_1 + A_{2n}\psi_2 + ... + A_{mn}\psi_m. \end{aligned}$$

Now, divide each equation by its *respective price* to obtain total returns on the right-hand side. Assuming that the first asset is risk-free, the first equation implies

$$1 = R_f(\psi_1 + \psi_2 + ... + \psi_m),$$

whereas for the risky assets we have

$$\begin{aligned} 1 &= \frac{A_{12}}{S_2}\psi_1 + \frac{A_{22}}{S_2}\psi_2 + ... + \frac{A_{m2}}{S_2}\psi_m, \\ &\vdots \\ 1 &= \frac{A_{1n}}{S_n}\psi_1 + \frac{A_{2n}}{S_n}\psi_2 + ... + \frac{A_{mn}}{S_n}\psi_m \end{aligned}$$

From now on we will treat the risky-free asset separately from the risky ones. Let us denote the matrix of *risky* returns $\hat{R}$,

$$\hat{R}^T = \begin{bmatrix} \frac{A_{12}}{S_2} & \frac{A_{22}}{S_2} & \dots & \frac{A_{m2}}{S_2} \\ \vdots & \dots & \dots & \vdots \\ \frac{A_{1n}}{S_n} & \frac{A_{2n}}{S_n} & \dots & \frac{A_{mn}}{S_n} \end{bmatrix}.$$

To sum up, the state price equations can be written as

$$1 = R_f(\psi_1 + \psi_2 + ... + \psi_m), \tag{10}$$

$$1 = \hat{R}^T \psi. \tag{11}$$

From these two identities it is clear that state prices are determined by the basis asset return, independently of the basis asset price.

Risk-Neutral Probabilities

If instead of the state price vector ψ we use a normalized vector q,

$$\begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} = \begin{bmatrix} R_f\psi_1 \\ \vdots \\ R_f\psi_2 \end{bmatrix}, \tag{12}$$

then the bond pricing equation (10) reads

$$q_1 + q_2 + ... + q_m = 1,$$

that is, we can think of q_i as probabilities. It is interesting to see how the pricing formula for risky assets (11) changes in this new light. From (12) we have $\psi = q/R_f$ and substituting this into (11) we obtain

$$1 = \hat{R}^T \frac{q}{R_f} \tag{13}$$

and consequently

$$R_f = \hat{R}^T q. \tag{14}$$

What we have on the left-hand side is the risk-free return, whereas on the right-hand side we obtain the expected return of the risky assets under probabilities q_i. Equivalently, to go back to the prices and payoffs let us multiply each equation in the system (13) by its respective price, whereby we obtain

$$S = \frac{1}{R_f} A^T q. \tag{15}$$