Advanced Microeconomics II

Extensive Form Games of Perfect Information

Brett Graham

Wang Yanan Institute for Studies in Economics
Xiamen University, China

April 1, 2015

Extensive Games With Imperfect Information

In previous models, players were not perfectly informed

- In strategic games there is uncertainty over simultaneous actions.
- In Bayesian games, there is uncertainty over simultaneous actions and other player's private information.
- In extensive form perfect information games players do not know other player's future actions.

In extensive games with imperfect information, there is additional uncertainty about past moves.

Extensive Games

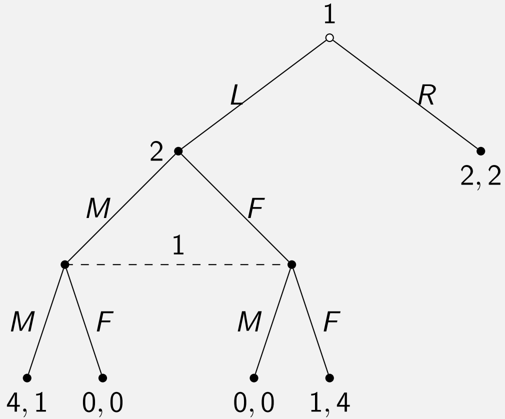
Definition

An extensive game is the same as an extensive game with perfect information and chance moves except we add

- For each player $i \in N$ a partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$ with the property that $A(h) = A(h')$ whenever h and h' are in the same member of the partition. For $I_i \in \mathcal{I}_i$ we denote by $A(I_i)$ the set $A(h)$ and by $P(I_i)$ the player $P(h)$ for any $h \in I_i$. ($\mathcal{I}_i$ is the information partition of player i; a set $I_i \in \mathcal{I}_i$ is an information set of player i.)

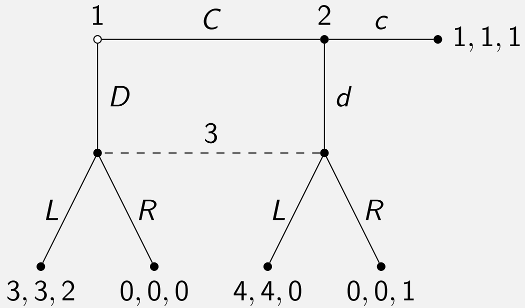
Refer to $\{N, H, P, f_c, (\mathcal{I}_i)_{i \in N}\}$ as an extensive game form.

Example - Battle of the Sexes With an Outside Option



- $N = \{1, 2\}$, $H = \{\varnothing, L, R, LM, LF, LMM, LMF, LFM, LFF\}$
- $P(\varnothing) = P(LM) = P(LF) = 1$, $P(L) = 2$
- $\mathcal{I}_1 = \{\{\varnothing\}, \{LM, LF\}\}$, $\mathcal{I}_2 = \{\{L\}\}$
- f_c is not required since $c \notin N$.

Example - Selten's Horse



- $N = \{1, 2, 3\}$, $H = \{\varnothing, C, D, Cc, Cd, DL, DR, CdL, CdR\}$,
- $P(\varnothing) = 1$, $P(C) = 2$, $P(D) = P(Cd) = 3$,
- $\mathcal{I}_1 = \{\{\varnothing\}\}, \mathcal{I}_2 = \{\{C\}\}, \mathcal{I}_3 = \{\{D, Cd\}\}$
- f_c is not required since $c \notin N$.

Spence's Model of Education

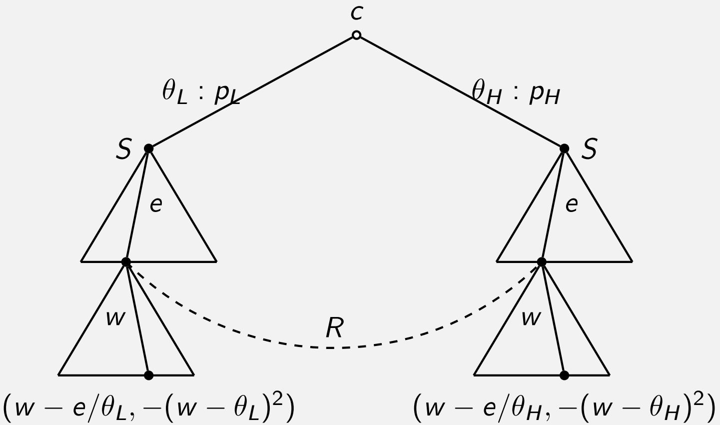
A worker knows her talent $\theta \in \{\theta_L, \theta_H\}$, while her employer does not. A worker has productivity θ_L with probability p_L and productivity θ_H with probability $p_H = 1 - p_L$. The value of the worker to the employer is θ, but the employer pays the worker a wage w that is equal to the expectation of θ (there is a competitive labour market).

- The worker chooses an amount of education $e \in [0, \infty)$.
- Employer makes an offer $w \in [\theta_L, \theta_H]$ to the worker.
- Payoffs: The worker's payoff is $w - e/\theta$ and the employer's payoff is $-(w - \theta)^2$.

Example - Spence's Model of Education

- $T = \{\theta_L, \theta_H\}, E = [0, \theta_H^2], W = [\theta_L, \theta_H]$.
- $N = \{c, 1, 2\}$,
- $H = \{\varnothing\} \cup T \cup T \times E \cup T \times E \times W$,
- $P(\varnothing) = c$, $P(\theta) = 1$ for all $\theta \in T$, $P(\theta, e) = 2$ for all $(\theta, e) \in T \times E$
- $\mathcal{I}_1 = \{\{\theta_L\}, \{\theta_H\}\}$ $\mathcal{I}_2 = \cup_{e \in E}\{\{(\theta_L, e), (\theta_H, e)\}\}$.
- $f_c(\theta_L|\varnothing) = p_L, f_c(\theta_H|\varnothing) = p_H$

Example - Model of Education Game Tree



Pure Strategies

Definition

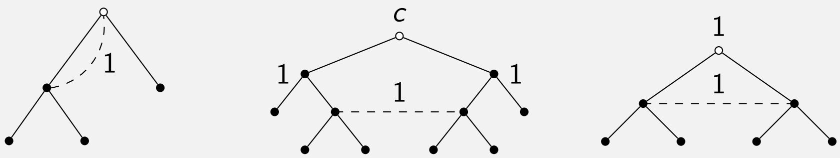
A pure strategy of player $i \in N$ in an extensive game $\{N, H, P, f_c, (\mathcal{I}_i), (\succeq_i)\}$ is a function that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$.

Perfect Recall

Definition

An extensive game form has perfect recall if for each player i, for each information set $I_i \in \mathcal{I}_i$ and for each $h, h' \in I_i$

- there does not exist $\tilde{h} \neq \varnothing$ such that $h = (h', \tilde{h})$ or $h' = (h, \tilde{h})$ and
- if there exists $I_i' \in \mathcal{I}_i$ such that there exists $\tilde{h} \in I_i', \hat{h} \neq \varnothing$ such that $h = (\tilde{h}, \hat{h})$ then there exists $\tilde{h}' \in I_i', \hat{h}' \neq \varnothing$ such that $h' = (\tilde{h}', \hat{h}')$ and the action taken at I_i' is the same for both h and h'.



Mixed and Behavioural Strategies

Definition

A mixed strategy of player $i \in N$ in an extensive game is a probability measure over the set of player i's pure strategies.

Definition

A behavioural strategy of player $i \in N$ in an extensive game is a collection $\beta_i(I_i)_{I_i \in \mathcal{I}_i}$ of independent probability measures, where $\beta_i(I_i)$ is a probability measure over $A(I_i)$.

Mixed and Behavioural Strategies Equivalence

Definition

An outcome $O(\sigma)$ of σ, where $\sigma = (\sigma_i)_{i \in N}$ is the probability distribution over terminal histories that results when each player i follows the precepts of σ_i.

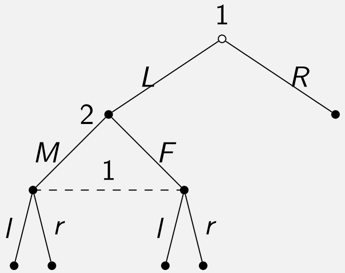
Definition

Two strategies of a player are outcome equivalent if for every collection of pure strategies of the other players the two strategies induce the same outcome.

Proposition

For any mixed strategy of a player in a finite extensive form game with perfect recall there is an outcome-equivalent behavioural strategy.

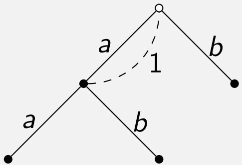
Example - Battle of the Sexes With an Outside Option



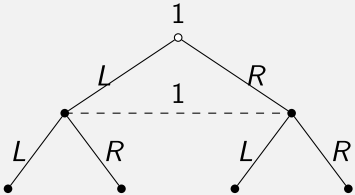
- Player 1 has 4 possible pure strategies: $\{(Ll), (Lr), (Rl), (Rr)\}$.
- Player 1 has 2 information sets $I_1 = \{\varnothing\}, I_2 = \{LM, LF\}$

- What is a mixed strategy equivalent to the behavioural strategy $\beta_1(I_1)(L) = 3/4, \beta_1(I_2)(l) = 1/4$?
- What is a behavioural strategy equivalent to the mixed strategy $\alpha(Ll) = 1/4, \alpha(Lr) = 1/8, \alpha(Rl) = 1/8, \alpha(Rr) = 1/2$?
- The mixed strategy can be derived as a product of the behavioural strategy probabilities.
- The behavioural strategy is derived from the mixed strategy probabilities using Bayes rules where possible.

Non-equivalence for Games with Imperfect Recall



- Player 1 has one information set.
- Let $\beta_1(I_1)(a) = p$.
- No outcome-equivalent mixed strategy exists.



- Player 1 has four pure strategies: $\{(LL), (LR), (RL), (RR)\}$.
- Let $\alpha_1(LL) = \alpha_1(RR) = 1/2$.
- No outcome-equivalent behavioural strategy exists.

Nash Equilibrium

Definition

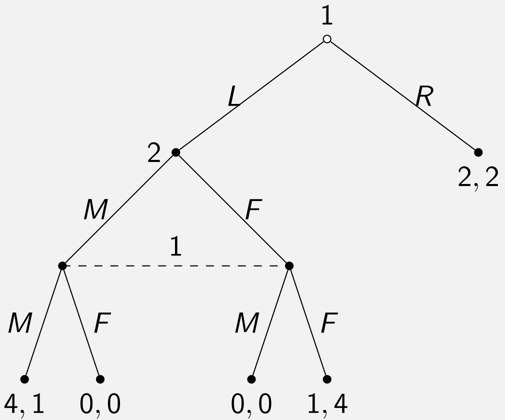
A Nash equilibrium in mixed strategies of an extensive game is a profile σ^* of mixed strategies with the property that for every player $i \in N$ we have

$$O(\sigma_i^*, \sigma_{-i}^*) \succeq_i O(\sigma_i, \sigma_{-i}^*) \text{ for every mixed strategy } \sigma_i \text{ of player } i.$$

A Nash equilibrium in behavioural strategies is defined analogously.

- Again, off the equilibrium path, Nash equilibrium allows lots of freedom.

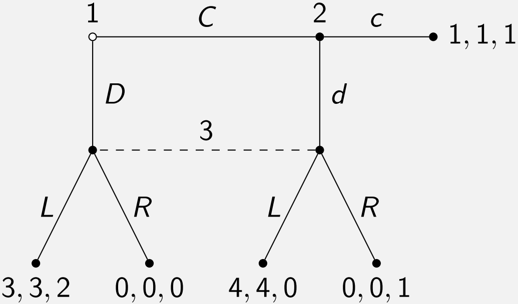
Example - Battle of the Sexes With an Outside Option



- Nash equilibria?

Test

Example - Selten's Horse



- Nash equilibria?

Test

Sub Games

Definition

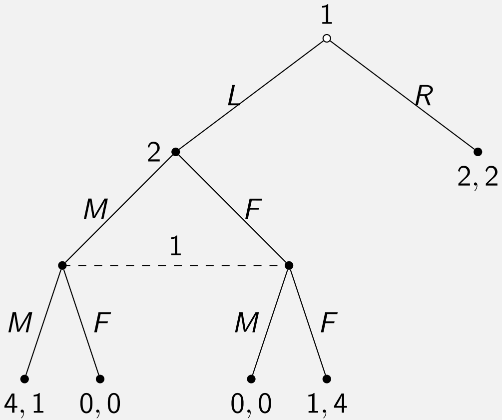
A subgame of $\Gamma = \{N, H, P, f_c, (\mathcal{I}_i)_{i \in N}\}$ has the following properties:

- it begins with an information set containing a single history $h \in H$, and contains all histories $h' \in H$ for which there exists $\tilde{h}$ such that $h' = (h, \tilde{h})$ and no other histories.
- If history $h \in I_i$ is in the subgame then every $h' \in I_i$ is also in the subgame.

Definition

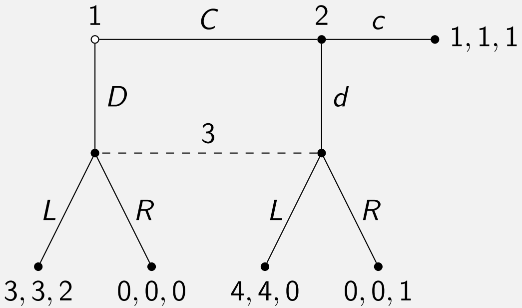
A profile of strategies $\sigma = (\sigma)_{i \in N}$ is a subgame perfect equilibrium of $\Gamma = \{N, H, P, f_c, (\mathcal{I}_i)_{i \in N}\}$ if it induces a Nash equilibrium in every subgame of Γ.

Example - Battle of the Sexes With an Outside Option



- Subgame perfect equilibria?

Example - Selten's Horse



- Subgame perfect equilibria?

Test