Advanced Microeconomics II

Nash Equilibrium

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Behavioral Assumptions

A model of individual rational choice for environments without uncertainty involves

- a set of possible actions A,
- a set of possible consequences C,
- a consequence function that maps actions to consequences, $g : A \to C$.
- a preference relation, $\succeq$, on consequences.

From any subset B of A, a rational decision-maker chooses an action a^* that is

- feasible (belongs to B) and optimal ($g(a^*) \succeq g(a)$ for all $a \in B$).

Strategic Game

Definition

A strategic game $G = \{N, (A_i)_{i=1}^N, (\succeq_i)_{i=1}^N\}$, where

- N is the set of players,
- for each i, A_i is the set of actions available to player i and
- for each i, $\succeq_i$ is a preference relation on $A = \times_{j \in N} A_j$.

If A is finite then the game is finite.

Matching Pennies Example

Each player has a penny. They each secretly choose a side of the coin to reveal and then they reveal their coins simultaneously.

- If the penny faces match, player 2 gives the player 1 \$1.
- If the penny faces do not match, player 1 gives player 2 \$1.

$G = \{N, (A_i)_{i=1}^N, (\succeq_i)_{i=1}^N\}$, where

- $N = \{1, 2\}$,
- $A_1 = A_2 = \{H, T\}$,
- $\succeq_1 = \{\{(H,H),(T,T)\},\{(H,H),(H,T)\},\{(H,H),(T,H)\},\{(H,H),(H,H)\},$ $\{(T,T),(T,T)\},\{(T,T),(H,T)\},\{(T,T),(T,H)\},\{(T,T),(H,H)\},$ $\{(T,H),(T,H)\},\{(T,H),(H,T)\},\{(H,T),(H,T)\},\{(H,T),(T,H)\}\}.$
- $\succeq_2 = \{\{(H,T),(T,T)\},\{(H,T),(H,T)\},\{(H,T),(T,H)\},\{(H,T),(H,H)\},$ $\{(T,H),(T,T)\},\{(T,H),(H,T)\},\{(T,H),(T,H)\},\{(T,H),(H,H)\},$ $\{(H,H),(H,H)\},\{(H,H),(T,T)\},\{(T,T),(H,H)\},\{(T,T),(T,T)\}\}.$

Utility

In general, we will assume that preferences over outcomes for each player i can be represented by a payoff (utility) function $u_i : A \to R$. In this case

the game is denoted $G = \{N, (A_i)_{i=1}^N, (u_i)_{i=1}^N\}$. Implied assumptions?

Matching Pennies Example

Each player has a penny. They each secretly choose a side of the coin to reveal and then they reveal their coins simultaneously.

- If the penny faces match, the second player gives the first player $1.
- If the penny faces do not match, the first player gives the second player $1.

$G = \{N, (A_i)_{i=1}^N, (\succeq_i)_{i=1}^N\}$, where

- $N = \{1, 2\}$,
- $A_1 = A_2 = \{H, T\}$,
- $u_1(H, H) = u_1(T, T) = 1, u_1(H, T) = u_1(T, H) = -1.$
- $u_2(H, H) = u_2(T, T) = -1, u_2(H, T) = u_2(T, H) = 1.$

Strategic Game Representation

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & L & R \\ \cline{3-4} \text{Player 1} & U & w_1, w_2 & x_1, x_2 \\ \cline{3-4} & D & y_1, y_2 & z_1, z_2 \\ \cline{3-4} \end{array}$$

Matching Pennies Example

Each player has a penny. They each secretly choose a side of the coin to reveal and then they reveal their coins simultaneously.

- If the penny faces match, the second player gives the first player $1.
- If the penny faces do not match, the first player gives the second player $1.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & H & T \\ \cline{3-4} \text{Player 1} & H & 1, -1 & -1, 1 \\ \cline{3-4} & T & -1, 1 & 1, -1 \\ \cline{3-4} \end{array}$$

Prisoner's Dilemma

Two criminal are being questioned by police in separate rooms about a burglary.

- If they both confess to the crime, they receive 5 years in jail.
- If one confesses and one denies, the confessor is released and the denier receives 10 years in jail.
- If the both deny, they both receive 1 year in jail for a lesser charge.

		Player 2	
		D	C
Player 1	D	$-1, -1$	$-10, 0$
	C	$0, -10$	$-5, -5$

Pure Coordination Game

My wife and I both like watching movies and football together. We like watching movies more than football.

- If we watch movies together we both receive \$3.
- If we watch football together we both receive \$1.
- If we watch different things, neither of us receive any benefit.

Derive the strategic game representation.

Games of Consequence

An alternative definition defines preferences over consequences rather than actions. This definition includes

- a set of players N,
- a set of actions A_i for each player i,
- a set of consequences C,
- a consequence function $g : A \to C$,
- a preference relation $\succeq_i^*$ defined over C for each player i.

We can map this to an equivalent strategic game by deriving a preference relation over A as

$$a \succeq_i b \text{ iff } g(a) \succeq_i^* g(b).$$

Cournot Competition

Two firm's, firm 1 and firm 2, compete in the same market.

- They each choose a quantity q_i which can be produced costlessly.
- The market demand schedule is $P(q_1, q_2) = \max\{0, 1 - q_1 - q_2\}$.
- The consequence of each firm's action is firm profits; each firm cares only about maximizing its own profit; $\pi_i = q_i P(q_1, q_2)$.

$N = \{1, 2\}$, $A_1 = A_2 = [0, \infty)$,
$C = \{(c_1, c_2) : c_1 + c_2 \leq 1/4 \text{ and } c_1 \geq 0 \text{ and } c_2 \geq 0\}$. $c \succeq_i^* d$ iff $c_i \geq d_i$
$g(a_1, a_2) = (\max\{(1 - a_1 - a_2)a_1, 0\}, \max\{(1 - a_1 - a_2)a_2, 0\})$
$G = \{N, (A_i)_{i=1}^N, (u_i)_{i=1}^N\}$, where

$N = \{1, 2\}$, $A_1 = A_2 = [0, \infty)$,
$u_i(a_1, a_2) = g_i(a_1, a_2)$

Games of Exogenous Uncertainty

There may be uncertainty about the consequences of a given action profile. These games have

- a probability space Ω
- a mapping g from actions and probability to consequences, $g : A \times \Omega \to C$.
- a preference relation $\succeq_i^*$ over all lotteries on C induced by each action a for each player i.

We can map this to an equivalent strategic game by deriving a preference relation over A as

$$a \succeq_i b \text{ iff } \mathcal{L}(g(a,.)) \succeq_i^* \mathcal{L}(g(b,.))$$

where $\mathcal{L}(g(a,.)$ is the lottery over C induced by $g(a)$.
In this case, the decision-maker maximizes the expected value of a von Neumann-Morgenstern expected utility function.

von Neumann-Morgenstern Utility

Definition

A simple lottery L is a list $L = (p_1, \ldots, p_{|C|})$ with $p_c \geq 0$ for all c and $\sum_c p_c = 1$.

Denote by $\mathcal{L}$ the set of all simple lotteries over the set of outcomes C.

Definition

A von Neumann-Morgenstern utility function is a function $U : \mathcal{L} \to R$ such that there exists an assignment of numbers $(u_1, \ldots, u_{|C|})$ to the $|C|$ outcomes such that for every L we have

$$U(L) = u_1 p_1 + \ldots + u_{|C|} p_{|C|}$$

and that for any two lotteries L, $L' \in \mathcal{L}$, $U(L) \geq U(L')$ iff $L \succeq L'$.

Implied assumptions?

Cournot Competition With Uncertain Demand

Two firm compete in the same market choosing how much to produce.

- With probability 1/2 they face a market demand schedule, $P(Q) = 1 - Q$.
- With probability 1/2 they face a market demand schedule, $P(Q) = 2 - Q$.

$G = \{N, (A_i)_{i=1}^N, (u_i)_{i=1}^N\}$, where
$N = \{1, 2\}$,
$A_1 = A_2 = [0, \infty)$,
$u_i(a_1, a_2) = \frac{1}{2}\max\{(1 - a_1 - a_2)a_i, 0\} + \frac{1}{2}\max\{(2 - a_1 - a_2)a_i, 0\}$

Solution Concepts

A solution to a game is a systematic description of the outcomes that may emerge in a family of games.

- Game theory suggests reasonable solutions for classes of games and examines their properties.

Two alternative interpretations of a solution concept

- Steady state
- Deductive

Nash Equilibrium

Definition

A Nash equilibrium of a strategic game G is a profile $a^* \in A$ of actions such that for every $i \in N$ we have

$$(a_i^*, a_{-i}^*) \succeq_i (a_i, a_{-i}^*) \text{ for all } a_i \in A_i.$$

Nash Equilibrium Discussion

Why Nash?

- A consequence of rational inference.
- A necessary condition if there is a unique predicted outcome.
- Focal point.
- A self-enforcing agreement.
- A stable social convention.

Best Response Function

Definition

A best-response function B_i for player i is a mapping from $A_{-i} \rightarrow A_i$ such that

$$B_i(a_{-i}) = \{a_i \in A_i : (a_i, a_{-i}) \succeq_i (a_i', a_{-i}) \text{ for all } a_i' \in A_i\}.$$

Matching Pennies Example

		Player 2	
		H	T
Player 1	H	1,−1	−1,1
	T	−1,1	1,−1

Prisoner's Dilemma Example

		Player 2	
		D	C
Player 1	D	$-1,-1$	$-10,0$
	C	$0,-10$	$-5,-5$

Cournot Game

$G = \{N, (A_i)_{i=1}^N, (\succeq_i)_{i=1}^N\}$, where
$N = \{1, 2\}$, $A_1 = A_2 = [0, \infty)$,

$$u_i(a_1, a_2) = \max\{(1 - a_1 - a_2)a_i, 0\}$$

Pure Coordination Example

		My wife	
		M	F
Me	M	$3,3$	$0,0$
	F	$0,0$	$1,1$

Derive the best response function for each player.

Non-Unique Best Response

		My wife	
		M	F
Me	M	$3,3$	$0,0$
	F	$3,0$	$1,1$

Nash Equilibrium

Definition

A Nash equilibrium of a strategic game G is a profile $a^* \in A$ of actions such that for every $i \in N$ we have

$$a_i^* \in B_i(a_{-i}^*) \text{ for all } i \in N.$$

Prisoner's Dilemma Example

		Player 2	
		D	C
Player 1	D	$-1, -1$	$-10, 0$
	C	$0, -10$	$-5, -5$

Cournot Game

$G = \{N, (A_i)_{i=1}^N, (\succeq_i)_{i=1}^N\}$, where
$N = \{1, 2\}$, $A_1 = A_2 = [0, \infty)$,

$$u_i(a_1, a_2) = \max\{(1 - a_1 - a_2)a_i, 0\}$$

$$B_i(a_j) = \begin{cases} \frac{1-a_j}{2} & \text{if } 0 \leq a_j \leq 1 \\ x & \text{where } x \in [0, \infty) \text{ otherwise.} \end{cases}$$

Pure Coordination Example

		My wife	
		M	F
Me	M	$3, 3$	$0, 0$
	F	$0, 0$	$1, 1$

Derive the set of Nash equilibria for this game.

Matching Pennies Example

		Player 2	
		H	T
Player 1	H	1,−1	−1,1
	T	−1,1	1,−1

Existence of Nash Equilibrium

Proposition

The strategic game $\{N, (A_i), (\succeq_i)\}$ has a Nash equilibrium if for all $i \in N$

- *the set A_i of actions of player i is a nonempty compact convex subset of Euclidean space and*
- *the preference relation $\succeq_i$ is*
  - *continuous and*
  - *quasi-concave on A_i.*

Kakutani's Fixed Point Theorem

Lemma

Let X be a compact convex subset of $\mathbf{R^n}$ and let $f : X \to X$ be a set-valued function for which

- *for all $x \in X$ the set $f(x)$ is nonempty and convex*
- *the graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$, $y_n \to y$, we have $y \in f(x)$).*

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Nash Equilibrium Existence Proof

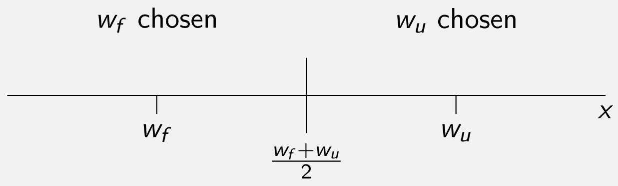
- Use Kakutani's fixed point theorem. We need
  - $f : A \to A$ such that
  - for all $a \in A$ the set $f(a)$ is nonempty and convex
  - the graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$, $y_n \to y$, we have $y \in f(x)$).
- Use $B(a) = \times_{i \in N} B_i(a_{-i})$.
  - $B : A \to A$.
  - For all $a \in A$ the set $B(a)$ is nonempty (Why?).
  - For all $a \in A$ the set $B(a)$ is convex (Why?).
  - B has a closed graph (Why?).
- Thus B has a fixed point, which is a Nash equilibrium.

Examples

- Does there exist a Nash equilibrium to the Cournot game?
  - Is A_i a non-empty, compact convex subset of a Euclidean space?
  - Is $\succeq_i$ continuous?
  - Is $\succeq_i$ quasi-concave on A_i?
- Does there exist a Nash equilibrium to the Bertrand game?
  - Is A_i a non-empty, compact convex subset of a Euclidean space?
  - Is $\succeq_i$ continuous?
  - Is $\succeq_i$ quasi-concave on A_i?

Wage Bargaining

- Many firm/union wage disputes are settled by arbitration.
- In *final offer* arbitration
  - The firm and union simultaneously make offers, w_f and w_u.
  - The arbitrator then chooses one of the offers.
- The arbitrator has an ideal settlement point x.
  - The arbitrator chooses the offer closest to x.



The firm and union believe x is randomly distributed according to $F(x)$ with associated density $f(x)$.

Strategic Game

- $N = \{f, u\}$
- $A_i = \mathcal{R}$; $w_u \in A_u$; $w_f \in A_f$;
- $u_u(w_f, w_u) = \begin{cases} w_f F\left(\frac{w_f+w_u}{2}\right) + w_u\left(1 - F\left(\frac{w_f+w_u}{2}\right)\right) & \text{if } w_u \geq w_f \\ w_u F\left(\frac{w_f+w_u}{2}\right) + w_f\left(1 - F\left(\frac{w_f+w_u}{2}\right)\right) & \text{if } w_u < w_f \end{cases};$
- $u_f(w_f, w_u) = -u_u(w_f, w_u)$.

Nash Equilibrium

Union's Objective (assuming $w_u > w_f$):

$$\max_{w_u} w_f F\left(\frac{w_f + w_u}{2}\right) + w_u\left(1 - F\left(\frac{w_f + w_u}{2}\right)\right)$$

FOC:

$$(w_u^* - w_f)\frac{1}{2}f\left(\frac{w_f + w_u^*}{2}\right) = 1 - F\left(\frac{w_f + w_u^*}{2}\right)$$

Firms's Objective (assuming $w_u > w_f$):

$$\max_{w_f} -w_f F\left(\frac{w_f + w_u}{2}\right) - w_u\left(1 - F\left(\frac{w_f + w_u}{2}\right)\right)$$

FOC:

$$(w_u - w_f^*)\frac{1}{2}f\left(\frac{w_f^* + w_u}{2}\right) = F\left(\frac{w_f^* + w_u}{2}\right)$$

Hence,

$$F\left(\frac{w_f^* + w_u^*}{2}\right) = \frac{1}{2} \Rightarrow w_u^* - w_f^* = \frac{1}{f\left(\frac{w_f^*+w_u^*}{2}\right)}.$$

Example

Let $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{1}{2\sigma^2}(x-m)^2\right\}$.

$$\frac{w_f^* + w_u^*}{2} = m$$

$$w_u^* = m + \sqrt{\frac{\pi\sigma^2}{2}}; w_f^* = m - \sqrt{\frac{\pi\sigma^2}{2}}.$$

Problem of the Commons

There a n fishermen in Xiamen who go out fishing each day.

- Every day, each fisherman i chooses how much time t_i to spend catching fish.
- The amount of fish they catch per hour is determined by the function $f(T)$ where $T = t_1 + \ldots + t_n$ is the aggregate number of hours of fishing.
- For $T > T_{max}$, $f(T) = 0$, otherwise $f(T) > 0$, $f'(T) < 0$, $f''(T) < 0$.
- The cost of fishing per hour is c fish (assume $f(0) > c$).

Strategic Game

- $N = \{1, \ldots, n\}$
- $A_i = [0, 24]$; $t_i \in A_i$; $t \in A = \times_{j=1}^n A_i$.
- $u_i(t) = t_i f(\sum_{j=1}^n t_j) - ct_i$

Nash Equilibrium

Each individual fishermen solves

$$\max_{t_i} t_i f(\sum_{j=1}^n t_j) - ct_i$$

First-order condition

$$f(\sum_{j=1}^n t_j) + t_i f'(\sum_{j=1}^n t_j) - c = 0$$

Sum over all first-order conditions and divide by n:

$$f(T^*) + \frac{T^* f'(T^*)}{n} - c = 0$$

where $T^* = \sum_{j=1}^n t_j^*$.

Social Optimum

Social planner optimizes

$$\max_{T} Tf(T) - cT$$

First-order condition

$$f(T^{**}) + T^{**}f'(T^{**}) - c = 0$$

Contrast with the Nash Equilibrium first-order condition:

$$f(T^{**}) + \frac{T^{**}f'(T^{**})}{n} - c > 0$$

$T^* > T^{**}$ - the resource is overutilized.

Production Subsidy

- Three sugar farmers have each harvested 6 tonnes of sugar.
- The demand for sugar is given by $q = 10 - p$ where p is the price per kilogram.
- There is a government price support program for sugar that ensures that the price cannot fall below 0.25 RMB per kilogram.
- Each producer must independently decide how much sugar to ship to the market and how much to discard.

Strategic Game

- $N = \{1, 2, 3\}$
- $A_i = [0, 10]$; $q_i \in A_i$; $q \in A = [0, 10]^3$.
- $u_i(q) = \begin{cases} (10 - q_1 - q_2 - q_3)q_i & \text{if } q_1 + q_2 + q_3 \leq 9.75 \\ 0.25q_i & \text{otherwise} \end{cases}$

Nash Equilibrium

First assume that $q_1 + q_2 + q_3 \leq 9.75$

- First order condition: $10 - 2q_1 - q_2 - q_3 = 0$.
- Impose symmetry: $q_1 = q_2 = q_3 = 2.5$; $\pi_1 = \pi_2 = \pi_3 = 6.25$
- Are the quantities consistent with assumption?
- What about a unilateral deviation by one player?

Now assume that $q_1 + q_2 + q_3 > 9.75$

- $q_1 = q_2 = q_3 = 6$; $\pi_1 = \pi_2 = \pi_3 = 1.5$.
- Are the quantities consistent with assumption?
- What about a unilateral deviation by one player?

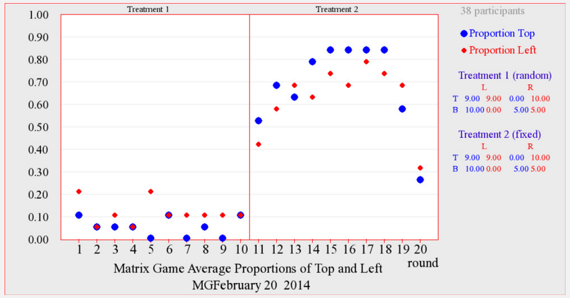
Are subsidies good or bad?

Predictive Ability

Recall the following game from the class experiment.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & L & R \\ \cline{3-4} \text{Player 1} & T & 9,9 & 0,10 \\ \cline{3-4} & B & 10,0 & 5,5 \\ \cline{3-4} \end{array}$$

Are there Nash equilibria? What are they?



Predictive Ability

Recall the following game from the class experiment.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & L & R \\ \cline{3-4} \text{Player 1} & T & 10,10 & 0,3 \\ \cline{3-4} & B & 3,0 & 3,3 \\ \cline{3-4} \end{array}$$

Are there Nash equilibria? What are they?