Facts.

- The rank of $A^T A$ is the same as the rank of A.
- $r(AB) \leq \min(r(A), r(B))$.
- The ranks of A and A^T are the same—it does not matter whether we look at columns or rows.
- For the $m \times n$ matrix A it is always true that $r(A) \leq \min(m, n)$.

1 The Identity Matrix and Arrow–Debreu Securities

A square matrix of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is called the identity matrix and is denoted I (or sometimes I_n to denote the dimension). The identity matrix is closely linked to *Arrow–Debreu securities*.

There are as many Arrow–Debreu securities (also called pure securities or elementary state securities) as there are states of the world. The Arrow–Debreu security for state j (denoted e_j) pays 1 in state j and 0 in all other states. Ordering all Arrow–Debreu securities into a matrix $\begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix}$ gives

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

an $m \times m$ identity matrix.

2 Inverse Matrix and Replicating Portfolios

Remember that a matrix A must be square with linearly independent columns to have an inverse. In this section we are interested in the interpretation of the inverse matrix. Let us begin with the definition:

$$AA^{-1} = I \tag{1}$$

If we divide the matrices A^{-1} and I into n columns, the matrix equality (1) is split into n systems of the form

$$AA^{-1}_{\bullet j} = e_j \tag{2}$$

where $A^{-1}_{\bullet j}$ is the jth column of the inverse matrix and e_j is the jth column of the identity matrix, $j = 1, 2, \ldots, n$. Thus, for example, the solution x of the system

$$Ax = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

gives us the first column of the inverse matrix.

Again, if we think of A as containing pay-offs of n basis assets in n states, then solving

$$Ax = e_j$$

means finding a portfolio x that replicates the Arrow–Debreu security for state j. Existence of the inverse matrix therefore requires *existence of the replicating portfolio* for each Arrow–Debreu security

3 Hedging with Redundant Securities and Incomplete Market

We have seen that in a complete market without redundant assets any focus asset can be hedged perfectly and the replicating portfolio is given by $x = A^{-1}b$. In general, the market need not be complete, and at the same time redundant basis assets may be present. There four possible combinations (complete, incomplete) $\times$ (no redundant basis assets, redundant basis assets) that are described below.

Case 1 $r(A) = m = n$ (Complete Market, Basis Assets Are Linearly Independent)

Matrix A is square and has full rank, therefore the inverse exists. Applying A^{-1} from the left, $A^{-1}Ax = A^{-1}b$, $x = A^{-1}b$. $A^{-1}b$ is the unique solution.

Case 2 $r(A) = m < n$ (Market is Complete, but There Are $n-m$ Redundant Basis Assets)

Then A can be partitioned into linearly independent and redundant assets, that is, it can be divided into two matrices A_1, A_2 with m and $n-m$ columns such that $r(A_1) = m$ and there is an $m \times (n-m)$ matrix C of replicating portfolios such that $A_2 = A_1C$.

The vector x too must be partitioned correspondingly, $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}$, where $x^{(1)}$ denotes the portfolio of linearly independent basis assets and $x^{(2)}$ in the portfolio of redundant basis asset. Now the system can be written as $A_1x^{(1)} + A_2x^{(2)} = b$.

Since A_2 contains redundant assets, we can express assets in A_2 as portfolios of linearly independent assets in A_1

$$A_1 x^{(1)} + A_1 C x^{(2)} = b,$$

and factor out A_1

$$A_1(x^{(1)} + Cx^{(2)}) = b. \tag{3}$$

Matrix $A_1$is square with full rank, therefore it is invertible. Multiplying both sides of (3) by A_1^{-1} we have

$$x^{(1)} + Cx^{(2)} = A_1^{-1} b \tag{4}$$

Now we can choose the portfolio of redundant basis asets $x^{(2)}$ arbitrarily and calculate the required portfolio of linearly independent basis assets from (4):

$$x^{(1)} = A_1^{-1} b - Cx^{(2)}.$$

Since we have $n-m$ redundant basis assets, the vector $x^{(2)}$ has $n-m$ entries and therefore the solution has $n-m$ free parameters.

Case 3 $r(A) = n < m$ (Market Is Incomplete, all Basis Assets Are Linearly Independent)

Multiplying by A^T from the left we have

$$A^T A x = A^T b. \tag{5}$$

Matrix $A^T A$ is a square $n \times n$ and it has rank n, therefore it is invertible. Now apply $(A^T A)^{-1}$from the left in (5)

$$(A^T A)^{-1} A^T A x = (A^T A)^{-1} A^T b,$$

$$\hat{x} = (A^T A)^{-1} A^T b. \tag{6}$$

This is a solution to the modified equation (5). To verify whether $\hat{x}$ solves the original equation, substitute $\hat{x}$ back into $Ax = b$; this gives us the following condition:

$$A\underbrace{(A^TA)^{-1}A^Tb}_{\hat{x}} = b \quad . \tag{7}$$

If (7) is satisfied, then b is a redundant security and we have a unique perfect hedge $x = (A^TA)^{-1}A^Tb$. Otherwise, there is no solution (b and basis assets are linearly independent).

Case 4 $r(A) < m$, $r(A) < n$ (Market Is Incomplete, There Are $n - r(A)$ Redundant Assets)

As Case 2 the original problem boils down to

$$A_1(x^{(1)} + Cx^{(2)}) = b. \tag{8}$$

Matrix A_1 is not square but it has full rank; we can therefore use the trick from Case 3, that is, multiply by A_1^T and then by $(A_1^TA_1)^{-1}$:

$$x^{(1)} + Cx^{(2)} = (A_1^TA_1)^{-1}A_1^Tb. \tag{9}$$

The portfolio of redundant assets $x^{(2)}$ can be chosen arbitrarily.

As in Case 3 we need to verify that this solution indeed solves the original problem (8). On substituting (9) into (8) we obtain the condition,

$$A_1(A_1^TA_1)^{-1}A_1^Tb = b. \tag{10}$$

In conclusion, if condition (10) is satisfied, then b can be perfectly hedged and the replicating portfolio can be constructed in infinitely many ways according to (9) with $x^{(2)}$ arbitrary.

Mathematically, there are infinitely many solutions with $n - r(A)$ free parameters. If the condition (10) is violated, then there is no solution—the focus asset b is not in the marketed subspace generated by the basis assets in matrix A.

4 Finding the Best Approximate Hedge

When the basis assets do not span the whole market (the market is not complete, $r(A) < m$), then some focus assets cannot be hedged perfectly. That is, the solution to $Ax = b$ does not always exist. Nevertheless, we would like to find at least the best approximate hedge. The deviation of the basis asset portfolio payoff from the focus asset is called the *replication error*: $\varepsilon = Ax - b$. A frequently used criterion is to minimize the *sum of squared replication errors* (SSREs) over all states:

$$\text{SSRE} = \varepsilon_1^2 + \varepsilon_2^2 + ... + \varepsilon_m^2 = (A_{1\bullet}x - b_1)^2 + ... + (A_{m\bullet}x - b_m)^2.$$

Fact. Assuming that the securities in A are linearly independent, the portfolio minimizing the squared replication error is given as

$$\hat{x} = (A^T A)^{-1} A^T b. \tag{11}$$

Note that this portfolio has already come up as the candidate for the perfect hedge in Case 3. Now we know that even if it is not the perfect hedge, it is in some sense the best hedge that one can find.
The payoff of the best hedge is $A\hat{x} = A(A^T A)^{-1} A^T b$. The procedure of finding x by minimizing the sum of squared errors is known as the *least-squares method.*

4.1 Geometric Interpretation of the Best Hedge

We say that two column vectors x and y are at right angles (are orthogonal or perpendicular to each other) if $x^T y = 0$. The quantity $\|x\| = \sqrt{x^T x}$ is called the length (*norm*) of the vector x. We can reinterpret the sum of the squared hedging errors as the squared length of the vector ε, SSRE$=\|\varepsilon\|^2$. Since $\varepsilon = Ax - b$, the best approximate hedge achieves the shortest distance between the focus asset b and the marketed subspace Span (A). Using three-dimensional examples it is easy to verify that the shortest distance is achieved when the hedging error ε is perpendicular to the marketed subspace Span (A). Thus the optimality condition requires that ε is at right angles to each column in A

$$A^T \hat{\varepsilon} = 0.$$

Substituting for $\hat{\varepsilon}$we obtain $A^T(A\hat{x} - b) = 0$,

$$A^T A\hat{x} = A^T b,$$

$\hat{x} = (A^T A)^{-1} A^T b$, which is exactly the equation (11).