1. The moment generating function (mgf) of a random variable X is defined as the expected value of an exponential function of X, $\mathbb{E}[\exp(\theta X)]$, where θ (or any other letter) is a dummy parameter.

   (a) Please show that the k^{th} moment of X is the k^{th} derivative of the mgf of X, at $\theta = 0$

$$\mathbb{E}\left[X^k\right] = \left.\frac{d^k \text{mgf of } X}{d\theta^k}\right|_{\theta=0}$$

   Solution

$$\mathbb{E}[\exp(\theta X)] = 1 + \theta\mathbb{E}[X] + \frac{1}{2!}\theta^2\mathbb{E}\left[X^2\right] + \cdots + \frac{1}{k!}\theta^k\mathbb{E}\left[X^k\right] + \cdots$$

$$\frac{d\{\mathbb{E}[exp(\theta X)]\}}{d\theta}|_{\theta=0} = \left\{0 + \mathbb{E}[X] + \cdots + \frac{k}{k!}\theta^{k-1}\mathbb{E}\left[X^k\right] + \cdots\right\}|_{\theta=0} = \mathbb{E}[X]$$

$$\frac{d^2\{\mathbb{E}[exp(\theta X)]\}}{d\theta^2}|_{\theta=0} = \left\{0 + 0 + \frac{2\times 1}{2!}\mathbb{E}\left[X^2\right] + \cdots + \frac{k(k-1)}{k!}\theta^{k-2}\mathbb{E}\left[X^k\right] + \cdots\right\}|_{\theta=0} = \mathbb{E}\left[X^2\right]$$

$$\vdots$$

$$\begin{aligned}\frac{d^k\{\mathbb{E}[exp(\theta X)]\}}{d\theta^k}|_{\theta=0} &= \left\{0 + \cdots + \frac{\overbrace{k(k-1)(k-2)\ldots 2\times 1}^{k!}}{k!}\overbrace{\theta^{k-k}}^{=1}\mathbb{E}\left[X^k\right] + \cdots\right\}|_{\theta=0} \\ &= \mathbb{E}\left[X^k\right]\end{aligned}$$

   (b) Please derive that the mgf of $B(t)$ is

$$\mathbb{E}[\exp(\theta B(t))] = \exp\frac{1}{2}\theta^2 t$$

   where $B(t)$ is a Brownian motion.
   Solution

$$\mathbb{E}[\exp(\theta B(t))] = \int_{x=-\infty}^{\infty}\exp(\theta x)\frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2\right]dx$$

   Define $z \stackrel{def}{=} x/\sqrt{t}$.

$$\begin{aligned}\mathbb{E}[\exp(\theta B(t))] &= \int_{z=-\infty}^{\infty}\exp\left(\theta\sqrt{t}z\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)dz \\ &= \exp\left(\frac{1}{2}\theta^2 t\right)\int_{z=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(z-\theta\sqrt{t}\right)^2\right)dz\end{aligned}$$

where $\int$ = area under a normal density = 1. Thus the mgf of $B(t)$ is

$$\mathbb{E}\left[\exp\left(\theta B(t)\right)\right] = \exp\frac{1}{2}\theta^2 t$$

2. Show that the covariance between Brownian motion positions at any times s and t is

$$\mathbb{C}ov\left[B(s), B(t)\right] = \min(s,t).$$

Solution

Without lossing generality, consider the covariance between Brownian motion positions at any times s and t, where $s < t$.

$$\begin{aligned}
\mathbb{C}ov\left[B(s), B(t)\right] &= \mathbb{E}\left[\{B(s) - \mathbb{E}[B(s)]\}\{B(t) - \mathbb{E}[B(t)]\}\right] \\
&= \mathbb{E}\left[B(s)B(t)\right] \\
&= \mathbb{E}\Big[B(s)\left(B(s) + B(t) - B(s)\right)\Big] \\
&= \mathbb{E}\Big[B(s)^2 + B(s)\{B(t) - B(s)\} \\
&= \mathbb{E}\Big[B(s)^2\Big] + \mathbb{E}\Big[B(s)\{B(t) - B(s)\} \\
&= \mathbb{E}\Big[B(s)^2\Big] + \mathbb{E}\Big[B(s)\Big]\mathbb{E}\Big[B(t) - B(s)\Big] = s + 00 = s
\end{aligned}$$

If the time notation was $t < s$ then $\mathbb{E}\left[B(s)B(t)\right] = t$. Generally for any times s and t

$$\mathbb{E}\left[B(s)B(t)\right] = \min(s,t)$$

3. A random process is the absolute value of a standard Brownian motion, $Z(t) \stackrel{def}{=} |B(t)|$.

   (a) Please derive the probability distribution of $Z(t)$ at time t, $\mathbb{P}[Z(t) \leq z]$, and the probability density function of $Z(t)$ at time t.

   Solution

   The probability distribution with respect to z is

$$\begin{aligned}
\mathbb{P}[Z(t) \leq z] &= \mathbb{P}[|B(t)| \leq z] = \mathbb{P}[B(t) \leq z] - \mathbb{P}[B(t) \leq -z] \\
&= \mathbb{P}[B(t) \leq z] - \{1 - \mathbb{P}[B(t) \leq z]\} = 2\mathbb{P}[B(t) \leq z] - 1 \\
&= 2\int_{x=-\infty}^{z} \frac{1}{\sqrt{t}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2\right] dx - 1
\end{aligned}$$

Thus, the probability density function of $Z(t)$ at time t is

$$\frac{d\left\{2\int_{x=-\infty}^{z}\frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2\right]dx-1\right\}}{dz}=2\frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{z}{\sqrt{t}}\right)^2\right]$$

(b) Please calculate the expectation and variance of $Z(t)$ at time t.
Solution
The expectation

$$\mathbb{E}\left[|B(t)|\right]=\int_{z=0}^{\infty}z\cdot 2\frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{z}{\sqrt{t}}\right)^2\right]dz$$

Define $y \overset{def}{=} z/\sqrt{t}$

$$\begin{aligned}
\mathbb{E}\left[|B(t)|\right] &= \int_{z=0}^{\infty}z\cdot 2\frac{1}{\sqrt{t}\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{z}{\sqrt{t}}\right)^2\right]dz \\
&= -2\sqrt{t}\frac{1}{\sqrt{2\pi}}\int_{y=0}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}y^2\right]d\left(-\frac{1}{2}y^2\right) \\
&= -2\sqrt{t}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}y^2\right]\Big|_{y=0}^{\infty} = -2\sqrt{t}\frac{1}{\sqrt{2\pi}}(0-1) \\
&= \sqrt{\frac{2t}{\pi}}
\end{aligned}$$

The variance

$$\mathbb{V}ar\left[Z(t)\right]=\mathbb{E}\left[Z(t)^2\right]-\left\{\mathbb{E}\left[Z(t)\right]\right\}^2=\mathbb{E}\left[Z(t)^2\right]-\frac{2t}{\pi}$$

where

$$\mathbb{E}\left[Z(t)^2\right]=\mathbb{E}\left[|B(t)|^2\right]=\mathbb{E}\left[B(t)^2\right]=t$$

Thus

$$\mathbb{V}ar\left[Z(t)\right]=\left(1-\frac{2}{\pi}\right)t$$

4. Partition the time period $[0,T]$ into n intervals of equal length $\Delta t=T/n$. The time points in the partition are $t_k \overset{def}{=} k\Delta t$. The sum of the squared Brownian motion increments over these intervals is $\sum_{k=0}^{n-1}\left[B(t_{k+1})-B(t_k)\right]^2$. Please show that

$$\mathbb{E}\left\{\left[\sum_{k=0}^{n-1}\left[B(t_{k+1})-B(t_k)\right]^2-T\right]^2\right\}\to 0$$

as $n \to \infty$.

Solution

Replace T by $\sum_{k=0}^{n-1} \Delta t$. We get

$$\mathbb{E}\left\{\left[\sum_{k=0}^{n-1} \left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2 - T\right]^2\right\} = \mathbb{E}\left\{\left[\sum_{k=0}^{n-1} \left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2 - \Delta t\right]^2\right\}$$

Expanding the square, the resulting expression consists of full square terms $\left(\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2 - \Delta t\right)^2$ and cross terms $\left(\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2 - \Delta t\right)\left(\left[B\left(t_{m+1}\right) - B\left(t_m\right)\right]^2 - \Delta t\right)$ where $m \neq k$.

The two parts of a cross term are independent random variables, so the expected value of a cross term can be written as the product of the expected value of each of the parts. As these are each zero, what remains is the expected value of the full square terms. Thus

$$\begin{aligned} \mathbb{E}\left\{\left[\sum_{k=0}^{n-1} \left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2 - \Delta t\right]^2\right\} &= \mathbb{E}\left\{\sum_{k=0}^{n-1} \left(\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2 - \Delta t\right)^2\right\} \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left\{\left(\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2 - \Delta t\right)^2\right\} \end{aligned}$$

Expanding the term in the sum gives

$$\mathbb{E}\left\{\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^4\right\} - 2\mathbb{E}\left\{\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2\right\}\Delta t + (\Delta t)^2$$

- For the first term, use $\mathbb{E}\left[B(t)^4\right] = 3t^2$ so $\mathbb{E}\left\{\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^4\right\} = 3(\Delta t)^2$.
- In the second term, $\mathbb{E}\left\{\left[B\left(t_{k+1}\right) - B\left(t_k\right)\right]^2\right\} = \Delta t$.
- Putting it together gives $3(\Delta t)^2 - 2\Delta t\Delta t + (\Delta t)^2 = 2(\Delta t)^2 = 2(T/n)^2$.

Summing over all terms gives $\sum_{k=0}^{n-1} 2(T/n)^2 = n2(T/n)^2 = 2T^2/n$ which tends to zero as $n \to \infty$.

5. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent identically distributed random variables with mean zero. The random variable X_i could be the numerical outcome of the i^{th} step in a random walk where the movement is ± 1 with equal probability. The position after n steps is then

$$S_n \stackrel{\text{def}}{=} X_1 + X_2 + \cdots + X_n.$$

The filtration which contains the results of the first n increments is denoted $\mathfrak{F}_n$.

Verify whether the discrete process $S_n^2 - n$ is a martingale.

Solution

Let $Y_n = S_n^2 - n$, $\Rightarrow Y_{n+1} = S_{n+1}^2 - (n+1)$

According to the definition of S_n, we have$S_{n+1} = S_n + X_{n+1}$. Thus

$$S_{n+1}^2 = (S_n + X_{n+1})^2 = S_n^2 + 2S_nX_{n+1} + X_{n+1}^2$$

So

$$\begin{aligned}
\mathbb{E}[Y_{n+1} \mid \mathfrak{F}_n] &= \mathbb{E}\left[S_{n+1}^2 - (n+1) \mid \mathfrak{F}_n\right] \\
&= \mathbb{E}\left[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - (n+1) \mid \mathfrak{F}_n\right] \\
&= S_n^2 + 2S_n\mathbb{E}[X_{n+1} \mid \mathfrak{F}_n] + \mathbb{E}\left[X_{n+1}^2 \mid \mathfrak{F}_n\right] - n - 1 \\
&= S_n^2 + 2S_n \times 0 + \mathbb{E}\left[X_{n+1}^2\right] - n - 1 \\
&= S_n^2 + \left[\frac{1}{2} \times 1^2 + \frac{1}{2} \times (-1)^2\right] - n - 1 \\
&= S_n^2 - n = Y_n
\end{aligned}$$

which means the discrete process $S_n^2 - n$ is a martingale.

6. Verify whether the random process $\exp\left[B(t) - \frac{1}{2}t\right]$ is a martingale, where $B(t)$ is a Brownian motion.

Solution

It is to be verified whether

$$\mathbb{E}\left[\exp\left(B(t) - \frac{1}{2}t\right)\middle|\mathfrak{F}(s)\right] = \exp\left(B(s) - \frac{1}{2}s\right)$$

Write $B(t)$ as the known value $B(s)$ plus the random variable $[B(t) - B(s)]$, and t as $[s + (t - s)]$. The left-hand side is then

$$\begin{aligned}
&\mathbb{E}\left[\exp\left([B(s) + B(t) - B(s)] - \frac{1}{2}[s + (t-s)]\right)\middle|\mathfrak{F}(s)\right] \\
= \; &\mathbb{E}\left[\exp\left\{\left[B(s) - \frac{1}{2}s\right] + \left[B(t) - B(s) - \frac{1}{2}(t-s)\right]\right\}\middle|\mathfrak{F}(s)\right] \\
= \; &\exp\left(B(s) - \frac{1}{2}s\right)\mathbb{E}\left[\exp\left([B(t) - B(s)] - \frac{1}{2}(t-s)\right)\middle|\mathfrak{F}(s)\right]
\end{aligned}$$

Consider the exponent of the second term and call it Y, so

$$Y = [B(t) - B(s)] - \frac{1}{2}(t-s)$$

which is normally distributed. Then

$$\mathbb{E}\left[\exp\left([B(t)-B(s)]-\frac{1}{2}(t-s)\right)\middle|\mathfrak{F}(s)\right]=\mathbb{E}[\exp(Y)]=\exp\left\{\mathbb{E}[Y]+\frac{1}{2}\mathbb{V}ar[Y]\right\}$$

As

$$\begin{aligned}\mathbb{E}[Y] &= -\frac{1}{2}(t-s)\\ \mathbb{V}ar[Y] &= (t-s)\end{aligned}$$

So

$$\exp\left\{\mathbb{E}[Y]+\frac{1}{2}\mathbb{V}ar[Y]\right\}=\exp\left\{-\frac{1}{2}(t-s)+\frac{1}{2}(t-s)\right\}=\exp(0)=1$$

Thus

$$\mathbb{E}\left[\exp\left(B(t)-\frac{1}{2}t\right)\middle|\mathfrak{F}(s)\right]=\exp\left(B(s)-\frac{1}{2}s\right)$$

which means $\exp\left(B(t)-\frac{1}{2}t\right)$ is a martingale.

7. Find out whether the process $[B(t)]^2-t$ is a martingale, where $B(t)$ is a Brownian motion.
   Solution

$$\begin{aligned}
& \mathbb{E}\left\{[B(s)+B(t)-B(s)]^2-[s+(t-s)]\middle|\mathfrak{F}\right\}(s)\\
=\; & \mathbb{E}\left\{B(s)^2+2B(s)[B(t)-B(s)]+[B(t)-B(s)]^2-[s+(t-s)]\middle|\mathfrak{F}(s)\right\}\\
=\; & \mathbb{E}\left\{B(s)^2-s\middle|\mathfrak{F}(s)\right\}+\mathbb{E}\left\{2B(s)[B(t)-B(s)]|\mathfrak{F}(s)\right\}+\mathbb{E}\left\{[B(t)-B(s)]^2-(t-s)\middle|\mathfrak{F}(s)\right\}\\
=\; & B(s)^2-s+2B(s)\mathbb{E}\left\{[B(t)-B(s)]|\mathfrak{F}(s)\right\}+[(t-s)-(t-s)]\\
=\; & B(s)^2-s+2B(s)\times 0+0\\
=\; & B(s)^2-s
\end{aligned}$$

   Thus the process $[B(t)]^2-t$ is a martingale.

8. Find a closed form expression for $I(T)\stackrel{\text{def}}{=}\int_{t=0}^{T}B(t)\,dB(t)$, where $B(t)$ is a Brownian motion.
   Solution
   Let $Y(T)\stackrel{\text{def}}{=}\frac{1}{2}B(T)^2$ be a trial solution. As $Y(T)$ is a function of the single variable $B(T)$,

$$dY=\frac{dY}{dB}dB+\frac{1}{2}\frac{d^2Y}{dB^2}(dB)^2=BdB+\frac{1}{2}dT$$

In equivalent integral form,

$$\int_{t=0}^{T} dY(t) = \underbrace{\int_{t=0}^{T} B(t)\, dB(t)}_{\text{required integral}} \int_{t=0}^{T} \frac{1}{2} dt$$

Rearranging gives

$$\int_{t=0}^{T} B(t)\, dB(t) = Y(T) - Y(0) - \frac{1}{2}T = Y(T) - \frac{1}{2}T$$

Substituting $Y(T) = \frac{1}{2}B(T)^2$ then gives the result

$$\int_{t=0}^{T} B(t)\, dB(t) = \frac{1}{2}B(T)^2 - \frac{1}{2}T$$

9. Derive the variance of

$$\int_{t=0}^{T} [B(t) + t]^2\, dB(t)$$

Solution

Define $X = \int_{t=0}^{T} [B(t) + t]^2\, dB(t)$. As X is an Ito stochastic integral, $\mathbb{E}[X] = 0$. Thus

$$\begin{aligned} \mathbb{V}\mathrm{ar}[X] &= \mathbb{E}\left[X^2\right] = \mathbb{E}\left[\left\{\int_{t=0}^{T} [B(t) + t]^2\, dB(t)\right\}^2\right] \\ &= \int_{t=0}^{T} \mathbb{E}\left[\{B(t) + t\}^4\right] dt \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}\left[\{B(t) + t\}^4\right] &= \mathbb{E}\left[B(t)^4 + 4B(t)^3 t + 6B(t)^2 t^2 + 4B(t)\, t^3 + t^4\right] \\ &= \mathbb{E}\left[B(t)^4\right] + 4t\mathbb{E}\left[B(t)^3\right] + 6t^2\mathbb{E}\left[B(t)^2\right] + 4t^3\mathbb{E}[B(t)] + t^4 \\ &= 3t^2 + 4t \times 0 + 6t^2 \times t + 4t^3 \times 0 + t^4 \\ &= 3t^2 + 6t^3 + t^4 \end{aligned}$$

Integrating this from $t = 0$ to $t = T$ gives the answer

$$\begin{aligned} \mathbb{V}\mathrm{ar}\left[\int_{t=0}^{T} [B(t) + t]^2\, dB(t)\right] &= \int_{t=0}^{T} \left[3t^2 + 6t^3 + t^4\right] dt \\ &= T^3 + \frac{3}{2}T^4 + \frac{1}{5}T^5 \end{aligned}$$

10. $B(t)$ is a Brownian motion.

(a) f is a function of $B(t)$

$$f[B(t)] = [B(t)]^4$$

Please derive the stochastic differential expression for f via Ito lemma.

Solution

$$\frac{df}{dB} = 4B^3,\ \frac{d^2 f}{dB^2} = 12B^2,\ (dB)^2 = dt$$

$$\begin{aligned} df &= \frac{1}{2}\frac{d^2 f}{dB^2}dt + \frac{df}{dB}dB \\ &= \frac{1}{2}\left(12B^2\right)dt + 4B^3 dB \\ d[B(t)]^4 &= 6B^2 dt + 4B^3 dB \end{aligned}$$

Substituting f gives $d[B]^4 = 6B^2 dt + 4B^3 dB$.

(b) f is a function of $B(t)$ and t.

$$f[t, B(t)] = \exp[\mu t + \sigma B(t)]$$

Please derive the stochastic differential expression for f via Ito lemma.

Solution

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial B}dB + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\underbrace{(dt)^2}_{=0} + \frac{\partial^2 f}{\partial t \partial B}\underbrace{dtdB}_{=0} + \frac{1}{2}\frac{\partial^2 f}{\partial B^2}\underbrace{(dB)^2}_{=dt} \\ &= \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial B^2}\right]dt + \frac{\partial f}{\partial B}dB \end{aligned}$$

where

$$\begin{aligned} \frac{\partial f}{\partial t} &= \mu \exp[\mu t + \sigma B] = \mu f, \qquad \frac{\partial f}{\partial B} = \sigma \exp[\mu t + \sigma B] = \sigma f \\ \partial^2 f / \partial B^2 &= \sigma^2 \exp[\mu t + \sigma B] = \sigma^2 f \end{aligned}$$

Thus

$$\begin{aligned} df &= \left\{\mu f + \frac{1}{2}\sigma^2 f\right\} dt + \sigma f dB \\ &= f\left\{\left(\mu + \frac{1}{2}\sigma^2\right) dt + \sigma dB\right\} \end{aligned}$$

(c) Suppose that the stock price $S(t)$ follows a geometric Brownian motion,

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t)$$

Please find the stochastic differential expression for $\ln S(t)$.

Solution

Let $f = \ln S(t)$

$$\begin{aligned} df &= \frac{\partial f}{\partial S} dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 \\ &= \frac{1}{S}[\mu S dt + \sigma S dB(t)] + \frac{1}{2}\left[-\frac{1}{S^2}\right][\mu S dt + \sigma S dB(t)]^2 \\ &= \mu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB(t) \end{aligned}$$

11. Arithmetic Brownian motion (ABM) is the name for a random process, say X, specified by the stochastic differential equation

$$dX(t) = \mu dt + \sigma dB(t)$$

μ and σ known constants, $\sigma > 0$, and $B(t)$ is a Brownian motion. $X(0)$ is known.

(a) Please derive the mathematical expression of $X(t)$ with the help of stochastic integral.

(b) Please specify the distribution of $X(t)$. (distribution, mean, variance)

Solution

$$\int_{t=0}^{T} dX(t) = \int_{t=0}^{T} \mu dt + \int_{t=0}^{T} \sigma dB(t)$$

which can be written as

$$\begin{aligned} X(T) - X(0) &= \mu[T - 0] + \sigma[B(T) - B(0)] \\ X(T) &= X(0) + \mu T + \sigma B(T) \end{aligned}$$

$X(T)$ equals a non-random term, $X(0) + \mu T$, plus a constant times the normally distributed random variable $B(T)$, so it is normally distributed. The distribution parameters are

$$\begin{aligned}
\mathbb{E}\left[X\left(T\right)\right] &= \mathbb{E}\left[X\left(0\right)+\mu T+\sigma B\left(T\right)\right]=X\left(0\right)+\mu T+\sigma \mathbb{E}\left[B\left(T\right)\right] \\
&= X\left(0\right)+\mu T
\end{aligned}$$

$$\mathbb{V}\mathrm{ar}\left[X\left(T\right)\right]=\mathbb{V}\mathrm{ar}\left[X\left(0\right)+\mu T+\sigma B\left(T\right)\right]=\mathbb{V}\mathrm{ar}\left[\sigma B\left(T\right)\right]=\sigma^2 T$$

12. A random process $X\left(t\right)$'s stochastic differential equation is given by

$$dX\left(t\right)=-\lambda X\left(t\right)dt+\sigma dB\left(t\right)$$

λ and σ known constants both positive. $B\left(t\right)$ is a Brownian motion. $X\left(0\right)$ is known.

(a) Please find an explicit expression for $X\left(t\right)$ with the help of stochastic integral.

(b) Please specify the distribution of $X\left(t\right)$. (distribution, mean, variance)
Solution
The method of solution for this SDE uses a technique from ordinary differential equations to eliminate the drift. It is the transformation

$$Y\left(t\right)\stackrel{\mathrm{def}}{=}X\left(t\right)\exp\left(\lambda t\right).$$

Applying Ito's formula to Y, as a function of X and t, gives the dynamics of Y as

$$dY\left(t\right)=\frac{\partial Y}{\partial t}dt+\frac{\partial Y}{\partial X}dX+\frac{1}{2}\frac{\partial^2 Y}{\partial X^2}\left(dX\right)^2+\frac{\partial^2 Y}{\partial X\partial t}dtdX$$

where

$$\begin{aligned}
\frac{\partial Y}{\partial t} &= X\exp\left(\lambda t\right)\lambda=Y\lambda \\
\frac{\partial Y}{\partial X} &= \exp\left(\lambda t\right), \quad \frac{\partial^2 Y}{\partial X^2}=0,\ \text{so}\ \left(dX\right)^2\ \text{is not needed} \\
dtdX &= dt\left(-\lambda X dt+\sigma dB\right)=0,\ \text{so}\ \frac{\partial^2 Y}{\partial X\partial t}\ \text{is not needed}
\end{aligned}$$

Substituting the above gives

$$dY\left(t\right) = Y\lambda dt+\exp\left(\lambda t\right)dX$$

Substituting dX gives

$$\begin{aligned} dY(t) &= Y\lambda dt + \exp(\lambda t)(-\lambda X dt + \sigma dB) \\ &= Y\lambda dt - \lambda \underbrace{X \exp(\lambda t)}_{=Y} dt + \sigma \exp(\lambda t) dB \\ &= \sigma \exp(\lambda t) dB \end{aligned}$$

In integral form

$$Y(T) = Y(0) + \sigma \int_{t=0}^{T} \exp(\lambda t) dB(t)$$

To express this in terms of X, use $X(T) = \exp(-\lambda T) Y(T)$, and $X(0) = \exp(\lambda \cdot 0) Y(0) = Y(0)$

$$\begin{aligned} X(T) &= \exp(-\lambda T) Y(T) = \exp(-\lambda T)\left[Y(0) + \sigma \int_{t=0}^{T} \exp(\lambda t) dB(t)\right] \\ &= \exp(-\lambda T)\left[X(0) + \sigma \int_{t=0}^{T} \exp(\lambda t) dB(t)\right] \\ &= \exp(-\lambda T) X(0) + \exp(-\lambda T) \sigma \underbrace{\int_{t=0}^{T} \exp(\lambda t) dB(t)}_{\text{normal distributed}} \end{aligned}$$

As the first right-hand term is non-random, $X(T)$ has a normal distribution.

- Given $X(0)$, its mean is

$$\mathbb{E}[X(T)] = \mathbb{E}[\exp(-\lambda T) X(0)] + \mathbb{E}\left[\exp(-\lambda T) \sigma \int_{t=0}^{T} \exp(\lambda t) dB(t)\right]$$

  As the Ito stochastic integral has mean zero

$$\mathbb{E}[X(T)] = \exp(-\lambda T) X(0)$$

  For large T the expected value $\mathbb{E}[X(T)]$ approaches zero.

- Its variance is

$$\mathbb{V}\text{ar}[X(T)] = \mathbb{E}\left\{[X(T) - \text{mean}]^2\right\}$$

i.e.

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left[X\left(T\right)\right] &= \mathbb{E}\left\{\left[\exp\left(-\lambda T\right)\sigma\int_{t=0}^{T}\exp\left(\lambda t\right)dB\left(t\right)\right]^{2}\right\} \\
&= \left[\exp\left(-\lambda T\right)\sigma\right]^{2}\mathbb{E}\left\{\left[\int_{t=0}^{T}\exp\left(\lambda t\right)dB\left(t\right)\right]^{2}\right\} \\
&= \exp\left(-2\lambda T\right)\sigma^{2}\int_{t=0}^{T}\exp\left(2\lambda t\right)dt \\
&= \exp\left(-2\lambda T\right)\sigma^{2}\left.\left[\frac{1}{2\lambda}\exp\left(2\lambda t\right)\right]\right|_{t=0}^{T} \\
&= \frac{\sigma^{2}}{2\lambda}\exp\left(-2\lambda T\right)\left[\exp\left(2\lambda T\right)-1\right] \\
&= \frac{\sigma^{2}}{2\lambda}\left[1-\exp\left(-2\lambda T\right)\right]
\end{aligned}$$

For large T the variance approaches $\sigma^{2}\left(1/2\lambda\right)$.