Mid-term Exam of Advanced Econometrics II

1. Consider the problem of testing conditional homoskedasticity $\left(\; H_0 : \quad E\left(\varepsilon_t^2 \left| X_t \right.\right) = \sigma^2 \; \right)$ for a linear regression model

$$Y_t = X'_t \beta^0 + \varepsilon_t,$$

where X_t is a $K \times 1$ vector consisting of an intercept and explanatory variables. To test conditional homoskedasticity, we consider the auxiliary regression

$$\begin{aligned} \varepsilon_t &= vech(X_t X'_t)' \gamma + v_t \\ &= U'_t \gamma + v_t \end{aligned}$$

Suppose ε_t is directly observable. Construct an test for the null hypothesis when $E\left(\varepsilon_t^4 \left| X_t \right.\right) \neq \mu_4$, that is $E\left(\varepsilon_t^4 \left| X_t \right.\right)$ is a function of X_t. Derive its limiting distribution.

ANSWER: Under H_0

$$\begin{aligned} \operatorname{var}\left(v_t \left| X_t \right.\right) &= \operatorname{var}\left(\varepsilon_t^2 \left| X_t \right.\right) \\ &= E\left(\varepsilon_t^4 \left| X_t \right.\right) - \sigma^4 \\ &\neq cons \tan t \end{aligned}$$

This means the disturbance v_t in the auxiliary regression model display conditional heteroskedasticity. According to the auxiliary regression model, the null hypothesis is equivalent to

$$H_0 : \quad R\gamma = 0$$

where $R = \left(\; 0 \quad I_J \; \right)$, I_J is the identity matrix, and $J = \frac{K(K+1)}{2} - 1$. Then,

$$\begin{aligned} \sqrt{n}\hat{\gamma} =& \sqrt{n}(\hat{\gamma} - \gamma_0) + \sqrt{n}(\gamma_0 - 0) \\ =& \sqrt{n}(\hat{\gamma} - \gamma_0) \\ &\xrightarrow{d} N\left(0, Q_u^{-1} V_v Q_u^{-1}\right) \end{aligned}$$

where $V_v = E\left(U_t U'_t v_t^2\right)$.
It follows that a robust Wald test statistic

$$W = \sqrt{n} R\hat{\gamma} \left[R Q_u^{-1} V_v Q_u^{-1} R'\right]^{-1} \sqrt{n} R\hat{\gamma} \xrightarrow{d} \chi_J^2$$

Because $\hat{Q}_u \xrightarrow{p} Q_u$and $\hat{V}_u \xrightarrow{p} V_u$, we have the Wald test statistic

$$W = \sqrt{n} R\hat{\gamma} \left[R \hat{Q}_u^{-1} \hat{V}_v \hat{Q}_u^{-1} R'\right]^{-1} \sqrt{n} R\hat{\gamma} \xrightarrow{d} \chi_J^2$$

by the Slutsky theorem.

2. Define

$$\Sigma = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}' \Omega^{-1} \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}$$

where Γ_1 is $(q+1)\times p$ and has rank p, Γ_2 is non-singular, and Ω is p.s. Let Σ^{11} be the upper-left $p\times p$ block of Σ^{-1}, and let Ω_{11} be the upper-left $(q+1)\times(q+1)$ block of Ω. Show that $\Sigma^{11} = \left(\Gamma'_1\Omega_{11}^{-1}\Gamma'_1\right)^{-1}$.($Hints:\quad \Sigma^{11} = \left(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)^{-1}$)

ANSWER:

$$
\begin{aligned}
\Sigma &= \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}' \Omega^{-1} \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \\
&= \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}' \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}^{-1} \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \\
&= \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}' \begin{bmatrix} \Omega^{11} & \Omega^{12} \\ \Omega^{21} & \Omega^{22} \end{bmatrix} \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \\
&= \begin{bmatrix} \Gamma'_1\Omega^{11}\Gamma_1 & \Gamma'_1\Omega^{12}\Gamma_2 \\ \Gamma'_2\Omega^{21}\Gamma_1 & \Gamma'_2\Omega^{22}\Gamma_2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^{11} &= \left(\Gamma'_1\Omega^{11}\Gamma_1 - \Gamma'_1\Omega^{12}\Gamma_2\left(\Gamma'_2\Omega^{22}\Gamma_2\right)^{-1}\Gamma'_2\Omega^{21}\Gamma_1\right)^{-1} \\
&= \left(\Gamma'_1\left(\Omega^{11} - \Omega^{12}\left(\Omega^{22}\right)^{-1}\Omega^{21}\right)\Gamma_1\right)^{-1} \\
&= \left(\Gamma'_1\Omega_{11}^{-1}\Gamma_1\right)^{-1}
\end{aligned}
$$

3. Consider the linear model

$$
\underset{T\times 1}{y} = \underset{T\times k k\times 1}{X\beta} + \underset{T\times 1}{\varepsilon} \tag{1}
$$

the disturbance vector ε is normally distributed with $E(\varepsilon)=0$ and $E(\varepsilon\varepsilon')=\Omega$. The hypothesis to be tested is

$$
\underset{p\times k}{R}\,\beta = \underset{p\times 1}{r} \tag{2}
$$

(a) Do we need to impose some assumptions of p and k?
The likelihood is given by

$$
L(\beta,\Omega) = (2\pi)^{-T/2}|\Omega|^{-1/2}\exp\left\{-\frac{1}{2}(y-X\beta)'\Omega^{-1}(y-X\beta)\right\} \tag{3}
$$

(b) Denote by $\left(\hat\beta_u,\hat\Omega\right)$ the estimates which jointly maximize (3). Show that $\hat\beta_u = \left(X'\hat\Omega^{-1}X\right)^{-1}X'\hat\Omega^{-1}y$.
(c) Denote by $\left(\tilde\beta_R,\tilde\Omega\right)$ the estimates which jointly maximize (3) while satisfying the hypothesized restrictions (2). Show that $\tilde\beta_R = \hat\beta_R - \left(X'\tilde\Omega^{-1}X\right)^{-1}R'\lambda$, where $\lambda = \left[R\left(X'\tilde\Omega^{-1}X\right)^{-1}R'\right]^{-1}\left(R\hat\beta_R - r\right)$, and $\hat\beta_R = \left(X'\tilde\Omega^{-1}X\right)^{-1}X'\tilde\Omega^{-1}y$.
Note that the hat indicates an unrestricted estimate while the tilde indicates an estimate from imposing the hypothesized restrictions. Subscripts on an estimate of β indicate that it is the ML

estimate conditional on the unrestricted (u) or restricted (R) estimate of Ω. Thus $\hat{\beta}_R$ is the value of β which maximizes the likelihood when the hypothesized restrictions are ignored but $\tilde{\Omega}$ is used for Ω in the likelihood. Similarly, $\tilde{\beta}_u$ satisfies the restriction while maximizing the likelihood when Ω is set equal to $\hat{\Omega}$. Define the residuals

$$\hat{\varepsilon}_u = y - X\hat{\beta}_u$$

$$\tilde{\varepsilon}_R = y - X\tilde{\beta}_R$$

$$\hat{\varepsilon}_R = y - X\hat{\beta}_R$$

and

$$\tilde{\varepsilon}_u = y - X\tilde{\beta}_u$$

(d) Show that

$$\tilde{\beta}_u = \hat{\beta}_u - \left(X'\hat{\Omega}^{-1}X\right)^{-1} R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right)$$

and

$$\tilde{\varepsilon}_u = \hat{\varepsilon}_u + X\left(X'\hat{\Omega}^{-1}X\right)^{-1} R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right)$$

where $\hat{A} = \left[R\left(X'\hat{\Omega}^{-1}X\right)^{-1}R'\right]$.

Note that the $Wald$ statistic is given by

$$W = \left(R\hat{\beta}_u - r\right)'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right).$$

(e) Show that $W = \tilde{\varepsilon}_u'\hat{\Omega}^{-1}\tilde{\varepsilon}_u - \hat{\varepsilon}_u'\hat{\Omega}^{-1}\hat{\varepsilon}_u$, and furthermore, show that $\tilde{\varepsilon}_u'\hat{\Omega}^{-1}\tilde{\varepsilon}_u - \hat{\varepsilon}_u'\hat{\Omega}^{-1}\hat{\varepsilon}_u = -2\left[\log \sup_{R\beta=r} L\left(\beta \middle| \hat{\Omega}\right) - \log \sup_{\beta} L\left(\beta \middle| \hat{\Omega}\right)\right]$.

Note that the LM statistic is given by

$$LM = \lambda'\tilde{A}\lambda,$$

where $\tilde{A} = \left[R\left(X'\tilde{\Omega}^{-1}X\right)^{-1}R'\right]$. (f) Show that

$$\begin{aligned} LM &= \tilde{\varepsilon}_R'\tilde{\Omega}^{-1}\tilde{\varepsilon}_R - \hat{\varepsilon}_R'\tilde{\Omega}^{-1}\hat{\varepsilon}_R \\ &= -2\left[\log \sup_{R\beta=r} L\left(\beta \middle| \tilde{\Omega}\right) - \log \sup_{\beta} L\left(\beta \middle| \tilde{\Omega}\right)\right] \end{aligned}$$

Note that

$$\begin{aligned} LR &= -2\left[\log \sup_{R\beta=r,\Omega} L\left(\beta \left| \Omega\right.\right) - \log \sup_{\beta,\Omega} L\left(\beta \left| \Omega\right.\right)\right] \\ &= \tilde{\varepsilon}_R'\tilde{\Omega}^{-1}\tilde{\varepsilon}_R - \hat{\varepsilon}_u'\hat{\Omega}^{-1}\hat{\varepsilon}_u \end{aligned}$$

(g) Show that $W \geq LR \geq LM$

ANSWER:
(a) $k < T$ and $p < k$.
(b) The likelihood function is

$$L(\beta, \Omega) = (2\pi)^{-T/2} |\Omega|^{-1/2} \exp\left\{-\frac{1}{2}(y - X\beta)'\Omega^{-1}(y - X\beta)\right\}$$

$$\left(\hat{\beta}_u, \hat{\Omega}\right) = \arg\max_{\beta,\Omega} \ln L(\beta, \Omega)$$

$$\begin{aligned}
\frac{\partial \ln L(\beta, \Omega)}{\partial \beta} &= -\frac{1}{2}\frac{\partial (y - X\beta)'\Omega^{-1}(y - X\beta)}{\partial \beta} \\
&= -\frac{1}{2}\left(2X'\Omega^{-1}X\beta - 2X'\Omega^{-1}y\right) \\
&= 0
\end{aligned}$$

Then, $\hat{\beta}_u = \left(X'\hat{\Omega}^{-1}X\right)^{-1} X'\hat{\Omega}^{-1}y$.
(c)

$$\max_{\beta,\Omega} \ln L(\beta, \Omega) \quad s.t. \quad R\beta = r$$

$$\mathcal{L} = \ln L(\beta, \Omega) + \lambda(r - R\beta)$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \beta} &= X'\Omega^{-1}y - X'\Omega^{-1}X\beta - R'\lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= r - R\beta = 0
\end{aligned}$$

So

$$\begin{aligned}
\tilde{\beta}_R &= \left(X'\tilde{\Omega}^{-1}X\right)^{-1}\left(X'\tilde{\Omega}^{-1}y - R'\lambda\right) \\
&= \left(X'\tilde{\Omega}^{-1}X\right)^{-1} X'\tilde{\Omega}^{-1}y - \left(X'\tilde{\Omega}^{-1}X\right)^{-1} R'\lambda \\
&= \hat{\beta}_R - \left(X'\tilde{\Omega}^{-1}X\right)^{-1} R'\lambda
\end{aligned}$$

where $\hat{\beta}_R = \left(X'\tilde{\Omega}^{-1}X\right)^{-1} X'\tilde{\Omega}^{-1}y$.
From $r - R\tilde{\beta}_R = 0$, we can get

$$\lambda = \left(RX'\tilde{\Omega}^{-1}XR'\right)^{-1}\left(R\hat{\beta}_R - r\right)$$

(d)

$$\begin{aligned}
\tilde{\beta}_u &= \left(X'\hat{\Omega}^{-1}X\right)^{-1} X'\hat{\Omega}^{-1}y - \left(X'\hat{\Omega}^{-1}X\right)^{-1} R'\hat{\lambda} \\
&= \hat{\beta}_u - \left(X'\hat{\Omega}^{-1}X\right)^{-1} R'\hat{\lambda}
\end{aligned}$$

where $\hat{\lambda} = \left(RX'\hat{\Omega}^{-1}XR'\right)^{-1}\left(R\hat{\beta}_u - r\right) = \hat{A}^{-1}\left(R\hat{\beta}_u - r\right)$, $\hat{A} = RX'\hat{\Omega}^{-1}XR'$. So

$$\tilde{\beta}_u = \hat{\beta}_u - \left(X'\hat{\Omega}^{-1}X\right)^{-1}R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right).$$

Notice that $\tilde{\varepsilon}_u = y - X\tilde{\beta}_u$, $\hat{\varepsilon}_u = y - X\hat{\beta}_u$

$$\tilde{\varepsilon}_u - \hat{\varepsilon}_u = X\left(\hat{\beta}_u - \tilde{\beta}_u\right) = X\left(X'\hat{\Omega}^{-1}X\right)^{-1}R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right)$$

$$\tilde{\varepsilon}_u = \hat{\varepsilon}_u + X\left(X'\hat{\Omega}^{-1}X\right)^{-1}R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right)$$

(e)

$$\begin{aligned}\tilde{\varepsilon}_u'\hat{\Omega}^{-1}\tilde{\varepsilon}_u &= \left(\hat{\varepsilon}_u + X\left(X'\hat{\Omega}^{-1}X\right)^{-1}R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right)\right)'\hat{\Omega}^{-1}\left(\hat{\varepsilon}_u + X\left(X'\hat{\Omega}^{-1}X\right)^{-1}R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right)\right)\\ &= \hat{\varepsilon}_u'\hat{\Omega}^{-1}\hat{\varepsilon}_u + \left(R\hat{\beta}_u - r\right)'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right)\end{aligned}$$

where the cross term $\hat{\varepsilon}_u'X\left(X'\hat{\Omega}^{-1}X\right)^{-1}R'\hat{A}^{-1}\left(R\hat{\beta}_u - r\right) = 0$.

$$\begin{aligned}-2\left[\log\sup_{R\beta=r}L\left(\beta\,\middle|\,\hat{\Omega}\right) - \log\sup_{\beta}L\left(\beta\,\middle|\,\hat{\Omega}\right)\right] &= \left(y - X\tilde{\beta}_u\right)'\hat{\Omega}^{-1}\left(y - X\tilde{\beta}_u\right) - \left(y - X\hat{\beta}_u\right)'\hat{\Omega}^{-1}\left(y - X\hat{\beta}_u\right)\\ &= \tilde{\varepsilon}_u'\hat{\Omega}^{-1}\tilde{\varepsilon}_u - \hat{\varepsilon}_u'\hat{\Omega}^{-1}\hat{\varepsilon}_u\\ &= W\end{aligned}$$

(f)

$$LM = \lambda'\tilde{A}\lambda$$

where $\tilde{A} = R\left(X'\tilde{\Omega}^{-1}X\right)^{-1}R'$, $\lambda = \left(R\left(X'\tilde{\Omega}^{-1}X\right)^{-1}R'\right)^{-1}\left(R\hat{\beta}_R - r\right)$. Then,

$$\begin{aligned}LM &= \lambda'\tilde{A}\lambda\\ &= \left(R\hat{\beta}_R - r\right)'\tilde{A}^{-1}\left(R\hat{\beta}_R - r\right)\\ &= \tilde{\varepsilon}_R'\tilde{\Omega}^{-1}\tilde{\varepsilon}_R - \hat{\varepsilon}_R'\tilde{\Omega}^{-1}\hat{\varepsilon}_R\end{aligned}$$

$$\begin{aligned}-2\left[\log\sup_{R\beta=r}L\left(\beta\,\middle|\,\tilde{\Omega}\right) - \log\sup_{\beta}L\left(\beta\,\middle|\,\tilde{\Omega}\right)\right] &= \left(y - X\tilde{\beta}_R\right)'\tilde{\Omega}^{-1}\left(y - X\tilde{\beta}_R\right) - \left(y - X\hat{\beta}_R\right)'\tilde{\Omega}^{-1}\left(y - X\hat{\beta}_R\right)\\ &= \tilde{\varepsilon}_R'\tilde{\Omega}^{-1}\tilde{\varepsilon}_R - \hat{\varepsilon}_R'\tilde{\Omega}^{-1}\hat{\varepsilon}_R\\ &= LW\end{aligned}$$

(g) We have

$$
\begin{aligned}
W &= -2\left[\log \sup_{R\beta=r} L\left(\beta \left| \hat{\Omega}\right.\right) - \log\sup_{\beta} L\left(\beta \left| \hat{\Omega}\right.\right)\right] = \tilde{\varepsilon}_u' \hat{\Omega}^{-1} \tilde{\varepsilon}_u - \hat{\varepsilon}_u' \hat{\Omega}^{-1} \hat{\varepsilon}_u \\
LM &= -2\left[\log \sup_{R\beta=r} L\left(\beta \left| \tilde{\Omega}\right.\right) - \log\sup_{\beta} L\left(\beta \left| \tilde{\Omega}\right.\right)\right] = \tilde{\varepsilon}_R' \tilde{\Omega}^{-1} \tilde{\varepsilon}_R - \hat{\varepsilon}_R' \tilde{\Omega}^{-1} \hat{\varepsilon}_R \\
LR &= -2\left[\log \sup_{R\beta=r} L\left(\beta \left| \Omega\right.\right) - \log\sup_{\beta} L\left(\beta \left| \Omega\right.\right)\right] = \tilde{\varepsilon}_R' \tilde{\Omega}^{-1} \tilde{\varepsilon}_R - \hat{\varepsilon}_u' \hat{\Omega}^{-1} \hat{\varepsilon}_u
\end{aligned}
$$

From

$$
\left.\begin{array}{r}
\log\sup_{\beta} L\left(\beta \left| \hat{\Omega}\right.\right) = \log\sup_{\beta} L\left(\beta \left| \Omega\right.\right) \\
\log \sup_{R\beta=r} L\left(\beta \left| \hat{\Omega}\right.\right) \leq \log \sup_{R\beta=r} L\left(\beta \left| \Omega\right.\right)
\end{array}\right\} \Rightarrow W \geq LR
$$

$$
\left.\begin{array}{r}
\log \sup_{R\beta=r} L\left(\beta \left| \tilde{\Omega}\right.\right) = \log \sup_{R\beta=r} L\left(\beta \left| \Omega\right.\right) \\
\log\sup_{\beta} L\left(\beta \left| \tilde{\Omega}\right.\right) \leq \log\sup_{\beta} L\left(\beta \left| \Omega\right.\right)
\end{array}\right\} \Rightarrow LR \geq LM
$$

So,

$$
W \geq LR \geq LM
$$