Solutions for Homework 1

1. Prove that a test with critical region

$$\left\{2\left(\ln(\hat{\theta}_n) - \ln(\tilde{\theta}_n^0)\right) \geq \chi^2_{1-\alpha}(r)\right\}$$

where $\tilde{\theta}_n^0 = \hat{\theta}_n - I(\hat{\theta}_n)^{-1}\frac{\partial g'(\hat{\theta}_n)}{\partial \theta}\left[\frac{\partial g(\hat{\theta}_n)}{\partial \theta'} I(\hat{\theta}_n)^{-1}\frac{\partial g'(\hat{\theta}_n)}{\partial \theta}\right]^{-1} g(\hat{\theta}_n)$ is asymptotically equivalent to W_n, LM_n and LR_n under $H_0: \quad g(\theta_0) = 0$ and r is the dimension of $g(\bullet)$.

ANSWER: Do Taylor expansion for $\ln(\tilde{\theta}_n^0)$ around $\hat{\theta}_n$, and notice that $\frac{\partial \ln(\hat{\theta}_n)}{\partial \theta'} = 0$, $\frac{\partial^2 \ln(\hat{\theta}_n)}{\partial \theta \partial \theta'} = -I(\hat{\theta}_n)$. Then,

$$\begin{aligned}
2\left(\ln(\hat{\theta}_n) - \ln(\tilde{\theta}_n^0)\right) &\approx n\left(\tilde{\theta}_n^0 - \hat{\theta}_n\right)' I(\hat{\theta}_n)\left(\tilde{\theta}_n^0 - \hat{\theta}_n\right) \\
&= ng'(\hat{\theta}_n)\left[\frac{\partial g(\hat{\theta}_n)}{\partial \theta'} I(\hat{\theta}_n)^{-1}\frac{\partial g'(\hat{\theta}_n)}{\partial \theta}\right]^{-1}\frac{\partial g(\hat{\theta}_n)}{\partial \theta'} I(\hat{\theta}_n)^{-1}\frac{\partial g'(\hat{\theta}_n)}{\partial \theta}\left[\frac{\partial g(\hat{\theta}_n)}{\partial \theta'} I(\hat{\theta}_n)^{-1}\frac{\partial g'(\hat{\theta}_n)}{\partial \theta}\right]^{-1} g(\hat{\theta}_n) \\
&= ng'(\hat{\theta}_n)\left[\frac{\partial g(\hat{\theta}_n)}{\partial \theta'} I(\hat{\theta}_n)^{-1}\frac{\partial g'(\hat{\theta}_n)}{\partial \theta}\right]^{-1} g(\hat{\theta}_n) \\
&= W_n
\end{aligned}$$

And since W_n, LM_n and LR_n are asymptotically equivalent, this statistic is asymptotically equivalent to all of them. Complete the proof.

2. Show that

$$M_X M_{X1} = M_{X1} M_X = M_X$$

where $X = [X_1, X_2]$.

ANSWER:From $M_X X = M_X[X_1, X_2] = [M_X X_1, M_X X_2] = 0$, we can get that $M_X X_1 = 0$. Then,

$$\begin{aligned}
0 &= M_X X_1 \\
&= M_X X_1 (X_1{}' X_1)^{-1} X_1{}' \\
&= M_X[I_n - I_n + X_1(X_1{}'X_1)^{-1}X_1{}'] \\
&= M_X - M_X[I_n - X_1(X_1{}'X_1)^{-1}X_1{}'] \\
&= M_X - M_X M_{X_1}
\end{aligned}$$

Thus,

$$M_X = M_X M_{X_1}$$

Similarly, we can proof that

$$M_X = M_{X_1} M_X$$

Complete the proof.

3. Consider a linear regression

$$y = \beta_1 l + X_2\beta_2 + u$$

where l is an n-vector of ones, and X_2 is an $n \times (k-1)$ matrix of observations on the remaining regressors. Show that the OLS estimators of β_1 and β_2 can be written as

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} n & l^T X_2 \\ 0 & X_2^T M_l X_2 \end{bmatrix}^{-1} \begin{bmatrix} l^T y \\ X_2^T M_l y \end{bmatrix}$$

where M_l is the matrix that takes deviations from the sample mean.

ANSWER: The OLS is

$$\min_{\hat{\beta}_1, \hat{\beta}_2} \hat{u}'\hat{u} = \min_{\hat{\beta}_1, \hat{\beta}_2} (y - \hat{\beta}_1 l + X_2\hat{\beta}_2)'(y - \hat{\beta}_1 l + X_2\hat{\beta}_2)$$

From first order condition, we can get

$$l'y - n\hat{\beta}_1 - l'X_2\hat{\beta}_2 = 0 \tag{1}$$

$$X'_2y - \hat{\beta}_1 X'_2 l - X'_2X_2\hat{\beta}_2 = 0 \tag{2}$$

from (1), we can get

$$\hat{\beta}_1 = \frac{1}{n}l'y - \frac{1}{n}l'X_2\hat{\beta}_2 \tag{3}$$

Substitute (3) to (2)

$$X'_2y - X'_2l(\frac{1}{n}l'y - \frac{1}{n}l'X_2\hat{\beta}_2) - X'_2X_2\hat{\beta}_2 = 0$$

$$X'_2M_lX_2\hat{\beta}_2 = X'_2M_ly \tag{4}$$

where $M_l = I - \frac{1}{n}ll'$. Combine (1) and (4) to get

$$\begin{pmatrix} n & l'X_2 \\ 0 & X'_2M_lX_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} l'y \\ X'_2M_ly \end{pmatrix}$$

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} n & l'X_2 \\ 0 & X'_2M_lX_2 \end{pmatrix}^{-1} \begin{pmatrix} l'y \\ X'_2M_ly \end{pmatrix}$$

Complete the proof.