Advanced Microeconomics II
Solution of 2010 Midterm

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1. Define the following concepts.

   (a) (5 points) A strategic game.

Solution: A strategic game $G = \{N, (A_i)_{i=1}^N, (\succeq_i)_{i=1}^N\}$, where

- N is the set of players,
- for each i, A_i is the set of actions available to player i and
- for each i, $\succeq_i$ is a preference relation on $A = \times_{j \in N} A_j$.

   (b) (5 points) A Bayesian game.

Solution: A Bayesian game consists of

- a finite set N (the set of players)
- a finite set Ω (the set of states)

and for each player $i \in N$

- a set A_i (the set of actions available to player i)
- a finite set T_i (the set of signals that may be observed by player i) and a function $\tau_i : \Omega \to T_i$ (the signal function of player i)
- a probability measure p_i on Ω (the prior belief of player i) for which $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
- a preference relation $\succeq_i$ on the set of probability measures over $A \times \Omega$ (the preference relation of player i), where $A = \times_{j \in N} A_j$.

   (c) (5 points) The mixed extension of a strategic game.

Solution: The mixed extension of the strategic game $\{N, (A_i), (u_i)\}$ is the strategic game $\{N, (\Delta(A_i)), (U_i)\}$ where

- $\Delta(A_i)$ is the set of probability distributions over A_i,
- $U_i : \times_{j \in N} \Delta(A_j) \rightarrow \mathcal{R}$ is a von Neumann-Morgenstern utility function for player i that represents preferences over $\times_{j \in N} \Delta(A_j)$. For finite A, $U_i(\alpha) = \sum_{a \in A} (\Pi_{j \in N} \alpha_j(a_j)) u_i(a)$, where for each $i \in N$, $\alpha_i(a_i)$ as the probability that α_i assigns to a_i.

(d) (5 points) A weakly dominated strategy.

Solution: A strategy $\alpha_i \in \Delta(S_i)$ is weakly dominated for player i in game $\langle N, \Delta(S_i), (U_i) \rangle$ if there exist another strategy $\alpha_i' \in \Delta(S_i)$ such that for all $\alpha_{-i} \in \Pi_{j \neq i} \Delta(S_j)$,

$$U_i(\alpha_i', \alpha_{-i}) \geq U_i(\alpha_i, \alpha_{-i}),$$

with strict inequality for some α_{-i}.

I also accepted the definition of a weakly dominated action:

The action $a_i \in A_i$ of player i in the strategic game $\{N, (A_i), (u_i)\}$ is strictly dominated if there is a mixed strategy α_i of player i such that $U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, where $U_i(\alpha_i, a_{-i})$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

2. (20 points) Consider the Cournot game with n players, i.e., $G = \{N, (A_i), (u_i)\}$ where $N = \{1, \ldots, n\}$, and for each $i \in N$, $A_i = [0, \infty)$, and

$$u_i(a) = \begin{cases} a_i(1 - \sum_{k=1}^n a_k) & \text{if } \sum_{k=1}^n a_k \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What are the set of rationalizable strategies for each player?

Solution: Since the game is symmetric, the set of rationalizable strategies is the same for every player. Denote the set of rationalizable strategies by Z. Let $m = \inf Z$ and $M = \sup Z$. If a_i is rationalizable then $m \leq a_i \leq M$.

Given a belief $\mu_i(a_{-i})$ over Z^{n-1}, player i's best response maximizes

$$E[a_i(1 - a_i - \sum_{j \neq i} a_j)] = a_i[1 - a_i - E(\sum_{j \neq i} a_j)].$$

Thus,

$$B_i(E[\sum_{j \neq i} a_j]) = \begin{cases} \frac{1 - E(\sum_{j \neq i} a_j)}{2} & \text{if } E(\sum_{j \neq i} a_j < 1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, B_i is a monotonically decreasing function of $E[\sum_{j=1}^{n} a_j]$, which implies

$$B_i(E[\sum_{j \neq i} a_j]) \in [B_i(\infty), B_i(0)] = [0, \frac{1}{2}].$$

Thus $Z \subseteq [0, \frac{1}{2}]$.

Further, $(n-1)m \leq E[\sum_{j \neq i} a_j] \leq (n-1)M$. Hence,

$$2m \geq 1 - (n-1)M \text{ and } 2M \leq 1 - (n-1)m.$$

When $n = 2$ these inequalities reduce to $m \geq 1/3$ and $M \leq 1/3$. Hence, $M = m = 1/3$. If $n \geq 3$, $Z = [0, 1/2]$. To see this, let player i believe the other players are playing action a with probability 1, where $a \in [0, 1/2]$. Then $E[\sum_{j \neq i} a_j] = (n-1)a$ and

$$B_i((n-1)a) = \max\{0, \frac{1 - (n-1)a}{2}\}.$$

Hence, if $n \geq 3$ and $a \in [0, 1/2]$ then $B_i((n-1)a)$ ranges from $[0, 1/2]$.

3. Consider the following auction process. There are 2 people bidding for an item with value V. The person who makes the highest bid wins the item. If both players make the same bid then they each win the item with probability 1/2. Each player must pay whatever they bid.
   (a) (5 points) Formulate this as a strategic game.

Solution: $N = \{1, 2\}$, $A_i = [0, +\infty), i \in N$.

$$u_i(a_1, a_2) = \begin{cases} V - a_i & \text{if } a_i > a_j \\ \frac{V}{2} - a_i & \text{if } a_i = a_j \\ -a_i & \text{if } a_i < a_j, \end{cases}$$

where j is the other bidder.

(b) (15 points) Find a mixed strategy Nash equilibrium for this game.

Solution: This game has no pure strategy Nash equilibrium (why?). Since this game is symmetric, assume (F, F) is the mixed strategy Nash equilibrium, where F is a continuous cumulative probability distribution over $[\underline{a}, \overline{a}]$. Any action in the support of F must give the same payoff. Since the probability of winning the item is zero when a player bids $\underline{a}$, $U_i(\underline{a}) = -\underline{a}$, which implies that $\underline{a} = 0$. Otherwise a player could profitably deviate and play 0 instead.

When bidder i uses the action a, his expected payoff is

$$(V - a)F(a) + (\frac{V}{2} - a) \times F'(a) + (-a)(1 - F(a)) = VF(a) - a = 0.$$

This $F(a) = \frac{a}{V}$. Since $F(\overline{a}) = 1$ we have $\overline{a} = V$. Thus the game has a mixed strategy Nash equilibrium in which each player randomizes uniformly over $[0, V]$.

4. Consider the strategic game described in the following table.

	L	R
U	$3, 3$	$1, 4$
D	$4, 1$	$0, 0$

(a) (5 points) What are the set of mixed strategy Nash equilibria for this game.

Solution: Note that the set of pure strategy Nash equilibria are a subset of the mixed strategy Nash equilibria. It is obvious that there are 2 pure strategy Nash equilibria in this game, one is (U, R) and the other is (D, L). In order to find the other mixed strategy Nash equilibrium, assume (α, β) is the mixed strategy Nash equilibrium. Consider player 1, to be indifferent between U and D, we must have

$$3\alpha + 1 \times (1 - \alpha) = 4\alpha + 0 \times (1 - \alpha) \Rightarrow \alpha = \frac{1}{2}$$

Since the game is symmetric, we have $\beta = \frac{1}{2}$, thus $(\frac{1}{2}, \frac{1}{2})$ is a mixed strategy Nash equilibrium. Thus the full set of mixed strategy Nash equilibria is

$$\{\{(1/2, 1/2), (1/2, 1/2)\}, \{(0, 1), (1, 0)\}, \{(1, 0), (0, 1)\}\}.$$

(b) (5 points) Construct a correlated equilibrium for this game with payoffs that are equal to the payoffs in one of the Nash equilibria you constructed in (a).

Solution: Three possible solutions:

1. Let $\Omega = \{x\}$, $\pi(x) = 1$, $P_1 = P_2 = \{x\}$, $\sigma_1(x) = U$ and $\sigma_2(x) = R$. This correlated equilibrium yields the payoff profile $(1, 4)$.

2. Let $\Omega = \{x\}$, $\pi(x) = 1$, $P_1 = P_2 = \{x\}$, $\sigma_1(x) = D$ and $\sigma_2(x) = L$. This correlated equilibrium yields the payoff profile $(4, 1)$.

3. Let $\Omega = \{w, x, y, z\}$, $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1/4$, $P_1 = \{\{w, x\}, \{y, z\}\}$, $P_2 = \{\{w, y\}, \{x, z\}\}$, $\sigma_1(w) = \sigma_1(x) = U$, $\sigma_1(y) = \sigma_1(z) = D$, $\sigma_2(w) = \sigma_2(y) = L$ and $\sigma_2(x) = \sigma_2(z) = R$. This correlated equilibrium yields the payoff profile $(2, 2)$.

(c) (10 points) Construct a correlated equilibrium where payoffs for each player are equal to 2.5.

Solution: Solution 1: Note that the payoff 2.5 is a convex combination of two Nash equilibrium profiles. Let $\Omega = \{x, y\}$, $\pi(x) = \pi(y) = 1/2$, $P_1 = P_2 = \{\{x\}, \{y\}\}$, $\sigma_1(x) = D$, $\sigma_1(y) = U$, $\sigma_2(x) = L$ and $\sigma_2(y) = R$.

Solution 2: First find the correlated equilibrium with the highest joint payoff. Let $\Omega = \{x, y, z\}$, $\pi(x) = \alpha = \pi(z)$, $\pi_(y) = \beta$, $P_1 = \{\{x\}, \{y, z\}\}$, $P_2 = \{\{x, y\}, \{z\}\}$, $\sigma_1(x) = D$, $\sigma_1(y) = \sigma_1(z) = U$, $\sigma_2(x) = \sigma_2(y) = L$ and $\sigma_2(z) = R$. The correlated equilibrium that maximizes joint payoffs solve

$$\max_{\alpha,\beta} 5\alpha + 3\beta \text{ s.t. } 2\alpha + \beta = 1 \text{ and } \alpha \geq \beta.$$

Hence $\alpha = \beta = 1/3$. This correlated equilibrium generates payoffs of $8/3, 8/3$.

To construct a correlated equilibrium with payoffs of $2, 2$, find the convex combination of the correlated equilibrium above and the mixed strategy Nash equilibrium that generate the appropriate payoffs. We need to find λ that solves

$$\lambda 2 + (1 - \lambda)8/3 = 5/2.$$

Hence, $\lambda = 1/4$. So the correlated equilibrium constructed in this way is as follows. Let $\Omega = \{w, x, y, z\}$, $\pi(w) = \pi(x) = \pi(y) = 5/16, \pi(z) = 1/16$, $P_1 = \{\{w, x\}, \{y, z\}\}$, $P_2 = \{\{w, y\}, \{x, z\}\}$, $\sigma_1(w) = \sigma_1(x) = U$, $\sigma_1(y) = \sigma_1(z) = D$, $\sigma_2(w) = \sigma_2(y) = L$ and $\sigma_2(x) = \sigma_2(z) = R$.

5. (a) (5 points) Give an example of a Nash equilibrium that does not survive iterated elimination of weakly dominated strategies.

Solution: Consider the strategic game described in the following table.

	L	R
U	$1, 1$	$0, 1$
D	$1, 0$	$2, 2$

Since U is weakly dominated by D, so the Nash equilibrium (U, L) does not survive iterated elimination of weakly dominated strategies.

(b) (15 points) Prove that if a game is solvable by iterated elimination of *strictly* dominated strategies then it has a unique Nash equilibrium.

Solution: There is a potential source of confusion here in that I did not define the concept of a game that is solvable by iterated elimination of *strictly* dominated strategies. One approach is to extend the definition given in class of dominance solvable:

Definition: A strategic game is solvable by iterated elimination of *strictly* dominated strategies (IESDS) if all players are indifferent between all outcomes that survive IESDS.

If one were to use this approach then the statement above cannot be true. This is easily shown with a counterexample.

	L	R
U	$1, 1$	$1, 1$

Clearly, this has multiple equilibria.

Alternatively, you might use the definition above to prove the following statement: If a game is solvable by IESDS then it has a unique Nash equilibrium payoff.

To show this, first show that any set of Nash equilibrium strategies survives IESDS. Recall the definition of IESDS.

The $X \subseteq A$ of outcomes of a finite strategic game $\{N, (A_i), (u_i)\}$ survives IESDS if $X = \times_{j \in N} X_j$ and there is a collection $((X_j^t)_{j \in N})_{t=0}^T$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.
- $X_j^{t+1} \subseteq X_j^t$ for each $t = 0, \ldots, T-1$.
- For each $t = 0, \ldots, T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is strictly dominated in the game $\{N, (X_i^t), (u_i^t)\}$ where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j \in N} X_j^t$.
- No action in X_j^T is strictly dominated in the game $\{N, (X_i^T), (u_i^T)\}$.

Let a^* be a Nash equilibrium profile of strategies. First we show that a^* survives IESDS. Clearly $a \in X^0 = \times_{j \in N} X_j^0$. Now assume that $a \in X^t = \times_{j \in N} X_j^t$ for some $t \in \{0, \ldots, T-1\}$. Since for each $i \in N$, $u(a_i^*, a_{-i}^*) \geq u(a_i, a_{-i}^*)$ for all $a_i \in A_i$ we have $u(a_i^*, a_{-i}^*) \geq u(a_i, a_{-i}^*)$ for all $a_i \in X_i^t$. Hence, $a \in X^{t+1} = \times_{j \in N} X_j^{t+1}$.

Now we show that the payoff profile from a game that is solvable by IESDS is a Nash equilibrium payoff profile. Suppose not. Let a be an action profile that survives IESDS. Since the payoff profile of a is not a Nash equilibrium payoff profile there exists some player $i \in N$ and some action $b_i \neq a_i$ such that $u_i(b_i, a_{-i}^*) > u_i(a_i^*, a_{-i}^*)$ for all $a_i \in A_i$. Since a_{-i}^* survives IESDS, so must b_i. This contradicts the assumption that the game is solvable by IESDS since $u_i(b_i, a_{-i}^*) \neq u_i(a_i^*, a_{-i}^*)$.

An alternative definition (and more standard in the literature) is the following.

Definition: A game is solvable by iterated elimination of strictly dominated strategies if, for each player i, X_i^T is a singleton (i.e., a one-element set).

With this definition we can prove the original statement. We can just follow the previous proof, and note that by the definition of a solvable game by IESDS the outcome that survives must be unique.

6. Consider the game at the top of Figure 1. Let $\Omega = \{\omega_\alpha, \omega_\beta, \omega_\gamma, \omega_\delta, \omega_\epsilon, \omega_\zeta, \omega_\eta\}$ be the set of states and let the player's action functions and information functions be those given in the table at the bottom of the figure. Player's posterior beliefs are updated using the prior and the partition induced by their information function.

   (a) (4 points) What are the set of mixed strategy Nash equilibria in this game?

Solution: It is obvious that there are 2 pure strategy Nash equilibria in this game, one is (U, L, A) and the other is (D, R, B). In order to find the mixed strategy Nash equilibria, assume (p, q, r) is the mixed strategy Nash equilibrium, consider player 1, he is indifferent between U and D, thus we have

$$2qr + 1 \times q(1-r) = 1 \times (1-q)r + 2(1-q)(1-r) \Rightarrow 3q + r = 2$$

Similarly, for player 2, we get

$$2pr + 1 \times p(1-r) = 1 \times (1-p)r + 2(1-p)(1-r) \Rightarrow 3p + r = 2$$

for player 3, we get

$$2pq + 1 \times (1-p)(1-q) = 1 \times pq + 2(1-p)(1-q) \Rightarrow p + q = 1$$

Solve this simultaneous system of equations, we get $p = q = r = \frac{1}{2}$.

   (b) (2 points) Under what conditions do player 1 and player 2 know player 3's beliefs in state ω_δ?

Solution: Since $P_1(\omega_\delta) = \{\omega_\gamma, \omega_\delta\} \subseteq \{\omega_\gamma, \omega_\delta, \omega_\epsilon\}$ and $P_2(\omega_\delta) = \{\omega_\delta, \omega_\epsilon\} \subseteq \{\omega_\gamma, \omega_\delta, \omega_\epsilon\}$, player 1 and player 2 know player 3's beliefs in state ω_δ under any conditions.

   (c) (2 points) Under what conditions do player 1 and player 3 know player 2's beliefs in state ω_δ?

Solution: Since $P_1(\omega_\delta) = \{\omega_\gamma, \omega_\delta\}$ and $P_3(\omega_\delta) = \{\omega_\gamma, \omega_\delta, \omega_\epsilon\}$, we need $\mu_2(\omega_\gamma) = \mu_1(\omega_\delta)$, thus $\frac{\gamma}{\beta+\gamma} = \frac{\delta}{\delta+\epsilon}$.

   (d) (2 points) Under what conditions do player 2 and player 3 know player 1's beliefs in state ω_δ?

Solution: Since $P_2(\omega_\delta) = \{\omega_\delta, \omega_\epsilon\}$ and $P_3(\omega_\delta) = \{\omega_\gamma, \omega_\delta, \omega_\epsilon\}$, we need $\mu_1(\omega_\delta) = \mu_1(\omega_\epsilon)$, thus $\frac{\delta}{\gamma+\delta} = \frac{\epsilon}{\epsilon+\zeta}$.

   (e) (3 points) Under what conditions do player 1 and player 2 know that player 3 is rational?

Solution: B is the best response of player 3 to $\mu_3(\omega_\alpha)$, A is a best response of player 3 to $\mu_3(\omega_\beta)$, $\mu_3(\omega_\zeta)$ and $\mu_3(\omega_\eta)$. A is a best response of player 3 to $\mu_3(\omega_\gamma)$ if

$$2\times\frac{\gamma}{\gamma+\delta+\epsilon}+0\times\frac{\delta}{\gamma+\delta+\epsilon}+1\times\frac{\epsilon}{\gamma+\delta+\epsilon}\geq 1\times\frac{\gamma}{\gamma+\delta+\epsilon}+0\times\frac{\delta}{\gamma+\delta+\epsilon}+2\times\frac{\epsilon}{\gamma+\delta+\epsilon}\Rightarrow\gamma\geq\epsilon$$

Thus player 1 and player 2 know that player 3 is rational if $\gamma \geq \epsilon$.

(f) (3 points) Under what conditions do player 1 and player 3 know that player 2 is rational?

Solution: R is the best response of player 2 to $\mu_2(\omega_\alpha)$. L is a best response of player 2 to $\mu_2(\omega_\beta)$ and $\mu_3(\omega_\zeta)$ if

$$0\times\frac{\beta}{\beta+\gamma}+2\times\frac{\gamma}{\beta+\gamma}\geq 1\times\frac{\beta}{\beta+\gamma}+0\times\frac{\gamma}{\beta+\gamma}\Rightarrow 2\gamma\geq\beta$$
$$0\times\frac{\zeta}{\zeta+\eta}+2\times\frac{\eta}{\zeta+\eta}\geq 1\times\frac{\zeta}{\zeta+\eta}+0\times\frac{\eta}{\zeta+\eta}\Rightarrow 2\eta\geq\zeta$$

R is a best response of player 2 to $\mu_2(\omega_\delta)$ if

$$0\times\frac{\delta}{\delta+\epsilon}+1\times\frac{\epsilon}{\delta+\epsilon}\geq 2\times\frac{\delta}{\delta+\epsilon}+0\times\frac{\epsilon}{\delta+\epsilon}\Rightarrow\epsilon\geq 2\delta$$

Thus player 1 and player 3 know that player 3 is rational if $2\gamma \geq \beta, 2\eta \geq \zeta$ and $\epsilon \geq 2\delta$.

(g) (3 points) Under what conditions do player 2 and player 3 know that player 1 is rational?

Solution: U is the best response of player 1 to $\mu_1(\omega_\eta)$. D is a best response of player 1 to $\mu_1(\omega_\alpha)$ and $\mu_3(\omega_\epsilon)$ if

$$2\times\frac{\alpha}{\alpha+\beta}+0\times\frac{\beta}{\alpha+\beta}\geq 0\times\frac{\alpha}{\alpha+\beta}+2\times\frac{\beta}{\alpha+\beta}\Rightarrow\alpha\geq\beta$$
$$1\times\frac{\epsilon}{\epsilon+\zeta}+0\times\frac{\zeta}{\epsilon+\zeta}\geq 0\times\frac{\epsilon}{\epsilon+\zeta}+2\times\frac{\zeta}{\epsilon+\zeta}\Rightarrow\epsilon\geq 2\zeta$$

U is a best response of player 1 to $\mu_1(\omega_\gamma)$ if

$$2\times\frac{\gamma}{\gamma+\delta}+0\times\frac{\delta}{\gamma+\delta}\geq 0\times\frac{\gamma}{\gamma+\delta}+1\times\frac{\delta}{\gamma+\delta}\Rightarrow 2\gamma\geq 2\delta$$

Thus player 2 and player 3 know that player 1 is rational if $\alpha \geq \beta, \epsilon \geq 2\zeta$ and $2\gamma \geq \delta$.

(h) (1 point) Under what conditions is each player's belief consistent with his knowledge?

	L	R
U	$2,2,2$	$0,0,0$
D	$0,0,0$	$1,1,1$

A

	L	R
U	$1,1,1$	$0,0,0$
D	$0,0,0$	$2,2,2$

B

State	ω_α	ω_β	ω_γ	ω_δ	ω_ϵ	ω_ζ	ω_η
Probability	α	β	γ	δ	ϵ	ζ	η
1's action	D	D	U	U	D	D	U
2's action	R	L	L	R	R	L	L
3's action	B	A	A	A	A	A	A
1's partition	$\{\omega_\alpha$	$\omega_\beta\}$	$\{\omega_\gamma$	$\omega_\delta\}$	$\{\omega_\epsilon$	$\omega_\zeta\}$	$\{\omega_\eta\}$
2's partition	$\{\omega_\alpha\}$	$\{\omega_\beta$	$\omega_\gamma\}$	$\{\omega_\delta$	$\omega_\epsilon\}$	$\{\omega_\zeta$	$\omega_\eta\}$
3's partition	$\{\omega_\alpha\}$	$\{\omega_\beta\}$	$\{\omega_\gamma$	ω_δ	$\omega_\epsilon\}$	$\{\omega_\zeta\}$	$\{\omega_\eta\}$

Figure 1: At the top is a three-player game in which player 1 chooses one of the two rows, player 2 chooses one of the two columns, and player 3 chooses one of the two tables. At the bottom are the common prior distribution, action functions and information functions for each player in the game. $\alpha, \beta, \gamma, \delta, \epsilon, \eta, \zeta$ are all strictly positive.

Solution: Since beliefs are defined as the posterior distribution based on the player's partition function, each player has a belief that is consistent with his knowledge in every state under any conditions.