# Advanced Microeconomics II
# Quiz 3

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1. Consider the $T$-period repeated game of $G$ where $G$ is described by the following matrix:

| | | Player 2 | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| Player 1 | $a_1$ | $10, 10$ | $2, 2$ | $0, 3$ |
| | $a_2$ | $2, 2$ | $4, 4$ | $0, 2$ |
| | $a_3$ | $3, 0$ | $2, 0$ | $1, 1$ |

   (a) (10 points) When $T = 2$ how many subgame perfect equilibrium outcomes of this game are there where the outcome in the second stage is $(a_3b_3)$?

   **Solution:** The three pure strategy Nash equilibria of this simultaneous-move game are $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$. The payoffs associated with $(a_3b_3)$ are the lowest Nash equilibrium payoffs. Thus it is impossible to punish if we require that $(a_3, b_3)$ be the outcome in the last period. Thus, there are three sub-game perfect Nash equilibria outcomes of this game where the outcome in the second stage is $(a_3b_3)$), each associated with a different first-stage game Nash equilibrium outcome.

   (b) (10 points) When $T = 2$ how many subgame perfect equilibrium outcomes of this game are there where the outcome in the second stage is $(a_2b_2)$?

   **Solution:** There are three sub-game perfect Nash equilibria outcomes of this game where the outcome in the second stage is $(a_2b_2)$), each associated with a different first-stage game Nash equilibrium outcome.

   In addition, the payoff of each player in the stage game Nash equilibrium outcome of $(a_2b_2)$ is 4 and the payoff of each player in the stage game Nash equilibrium outcome of $(a_1b_1)$ is 1. The difference between the stage game Nash equilibrium payoffs for each player is 3. Hence, any stage game outcome where each player's most profitable deviation is no more than 3 is sustainable in the first period. The outcome is supported by punishing deviations with the lowest Nash equilibrium payoffs in the second period. Hence, $(a_3b_2, a_2b_2)$ and $(a_2b_3, a_2b_3)$ are additional sub-game equilibrium outcomes, bringing the total to 5.

   (c) (10 points) When $T = 2$ how many subgame perfect equilibrium outcomes of this game are there where the outcome in the second stage is $(a_1b_1)$?

   **Solution:** There are three sub-game perfect Nash equilibria outcomes of this game where the outcome in the second stage is $(a_1b_1)$), each associated with a different first-stage game Nash equilibrium outcome.

   In addition, the payoff of each player in the stage game Nash equilibrium outcome of $(a_1b_1)$ is 10 and the payoff of each player in the stage game Nash equilibrium outcome of $(a_1b_1)$ is 1. The difference between the stage game Nash equilibrium payoffs for each player is 9. Hence, any stage game outcome where each player's most profitable deviation is no more than 9 is sustainable in the first period. The outcome is supported by punishing deviations with the lowest Nash equilibrium payoffs in the second period. Hence, $(a_2b_1, a_1b_1)$, $(a_3b_1, a_1b_1)$, $(a_1b_2, a_1b_1)$, $(a_1b_3, a_1b_1)$, $(a_3b_2, a_1b_1)$ and $(a_2b_3, a_1b_1)$ are additional sub-game equilibrium outcomes, bringing the total to 9.

(d) (10 points) How long does $T$ need to be for there to exist a subgame perfect equilibrium payoff profile where the (average per period) equilibrium payoff profile is within $\epsilon$ of $(2, 2)$? Be clear about the strategies of both players.

**Solution:** Define $V = \{(2, 3), (1, 3)\}$. Let $(k, j) \in V$ and define the strategies in this way.

The strategy for player 1 is to play $a_2$ in period 1 and to play $a_2$ for every period until $T(k, l) - L(k, l)$ if the history only contains outcomes $(a_2b_1)$; otherwise play $a_j$. In the last $L(k, l)$ periods play $a_k$ if the history from 1 to $L(k, l)$ only contains the outcomes $(a_2b_1)$; otherwise play $a_j$.

The strategy for player 2 is to play $b_1$ in period 1 and to play $b_1$ for every period until $T(k, l) - L(k, l)$ if the history only contains outcomes $(a_2b_1)$; otherwise play $b_j$. In the last $L(k, l)$ periods play $b_k$ if the history from 1 to $L(k, l)$ only contains the outcomes $(a_2b_1)$; otherwise play $b_j$.

It is left to define $L(k, l)$ and $T(k, l)$. We require $L(k, l)$ to be long enough to make any one period deviation unprofitable. The most profitable stage game one-player deviation is 8. When $(k, l) = (2, 3)$ each period of punishment costs a player 3. Hence it takes three periods of punishment to make this deviation unprofitable so $L(2, 3) = 3$. When $(k, l) = (1, 3)$ each period of punishment costs a player 9. Hence it takes one period of punishment to make this deviation unprofitable so $L(1, 3) = 1$.

We require $T$ to be long enough so that the payoffs are within $\epsilon$ of $(2, 2)$. In particular, when $(k, l) = (2, 3)$ we require that

$$\frac{(T(2,3) - 3)2 + 3 \times 4}{T(2,3)} - 2 < \epsilon \Rightarrow T(2,3) > 6/\epsilon.$$

When $(k, l) = (1, 3)$ we require that

$$\frac{(T(1,3) - 1)2 + 1 \times 10}{T(1,3)} - 2 < \epsilon \Rightarrow T(1,3) > 8/\epsilon.$$

Note that $(k, l) = (1, 2)$ does not work here since the payoffs in the 'bad' stage game Nash equilibrium are 4 so a player who deviated in the first period would get an average payoff equal to approximately 4 rather than 2. Also note that when $T = 3$, the outcome $(2, 0), (0, 2), (4, 4)$ is a subgame perfect equilibrium outcome and has the equilibrium payoff profile of exactly $(2, 2)$. Player 1's strategy is $s_1(\varnothing) = a_3$; $s_1(a_3b_2) = a_2$ and $s_1(h) = a_3$ for any other $h$ of length one; $s_1(a_3b_2, a_2b_3) = a_2$

and $s_1(h) = b_3$ for any other $h$ of length two. Player 2's strategy is $s_2(\varnothing) = b_2$; $s_2(a_3b_2) = b_3$ and $s_2(h) = b_3$ for any other $h$ of length one; $s_2(a_3b_2, a_2b_3) = b_2$ and $s_2(h) = b_3$ for any other $h$ of length two. I leave to you to check that this is indeed a subgame perfect equilibrium.