Chapter 4 Stochastic Calculus

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Lectures on Stochastic Calculus, 2013

Outline

Itô's Integral for Simple Integrands

- Fix a positive number T,

$$\int_0^T \Delta(t)\, dW(t) \tag{1}$$

- The basic ingredients here are a Brownian motion $W(t)$, $t \geq 0$, together with a filtration $\mathscr{F}(t)$, $t \geq 0$, for this Brownian motion.
- Let the integrand $\Delta(t)$ be an adapted stochastic process.
- The problem we face when trying to assign meaning to the Itô integral (1) is that Brownian motion paths cannot be differentiated with respect to time.

Construction of the Integral

- Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$; i.e., $0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$.
- Assume that $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$.
- In general, if $t_k \leq t \leq t_{k+1}$, then

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)] \tag{2}$$

- The process $I(t)$ in (2) is the Itô integral of the simple process $\Delta(t)$, a fact that we write as

$$I(t) = \int_0^t \Delta(u)\, dW(u).$$

Properties of the Integral

Theorem 4.2.1

The Itô integral defined by (2) is a martingale.

Properties of the Integral

Theorem 4.2.2

The Itô integral defined by (2) satisfies

$$\mathbb{E}\left[I^2(t)\right] = \mathbb{E}\left[\int_0^t \Delta^2(u)\,du\right] \tag{3}$$

Properties of the Integral

Theorem 4.2.3

The quadratic variation accumulated up to time t by the Itô integral (2) is

$$[I, I](t) = \int_0^t \Delta^2(u)\, du \tag{4}$$

Itô's Integral for General Integrands

- $\Delta(t)$ is allowed to vary continuously with time and also to jump.
- Assume that $\Delta(t)$, $t \geq 0$, is adapted to the filtration $\mathscr{F}(t)$, $t \geq 0$.
- Assume the square-integrability condition

$$\mathbb{E}\left[\int_0^T \Delta^2(t)\,dt\right] < \infty. \quad (5)$$

Itô's Integral for General Integrands

- In order to define $\int_0^T \Delta(t)\, dW(t)$, we approximate $\Delta(t)$ by simple processes.
- In general, then, it is possible to choose a sequence $\Delta_n(t)$ of simple processes such that as $n \to \infty$ these processes converge to the continuously varying $\Delta(t)$. By "converge,"

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |\Delta_n(t) - \Delta(t)|^2\, dt\right] = 0. \qquad (6)$$

  For each $\Delta_n(t)$, the Itô integral $\int_0^t \Delta_n(u)\, dW(u)$ has already been defined for $0 \le t \le T$.
- The Itô integral for the continuously varying integrand $\Delta(t)$

$$\int_0^t \Delta(u)\, dW(u) = \lim_{n\to\infty} \int_0^t \Delta_n(u)\, dW(u), 0 \le t \le T. \qquad (7)$$

Itô's Integral for General Integrands

Theorem 4.3.1

Let T be a positive constant and let $\Delta(t)$, $0 \le t \le T$, be an adapted stochastic process that satisfies (5). Then $I(t) = \int_0^t \Delta(u)\, dW(u)$ defined by (7) has the following properties.
(i) (Continuity) As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.
(ii) (Adaptivity) For each t, $I(t)$ is $\mathscr{F}(t)$-measurable.

Itô's Integral for General Integrands

Theorem 4.3.1

(iii) (Linearity) If $I(t) = \int_0^t \Delta(u)\, dW(u)$ and $J(t) = \int_0^t \Gamma(u)\, dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u))\, dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)\, dW(u)$.

(iv) (Martingale) $I(t)$ is a martingale.

(v) (Itô isometry) $\mathbb{E}\left[I^2(t)\right] = \mathbb{E}\left[\int_0^t \Delta^2(u)\, du\right]$.

(vi) (Quadratic variation) $[I, I](t) = \int_0^t \Delta^2(u)\, du$.

Formula for Brownian Motion

- "Differentiate" expressions of the form $f(W(t))$, where $f(x)$ is a differentiable function of and $W(t)$ is a Brownian motion.

$$df(W(t)) = f'(W(t))\,W'(t)\,dt = f'(W(t))\,dW(t)$$

- Because W has nonzero quadratic variation,

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2}f''(W(t))\,dt \tag{8}$$

- The Itô-Doeblin formula in integral form:

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(u))\,dW(u) + \frac{1}{2}\int_0^t f''(W(u))\,du \tag{9}$$

Formula for Brownian Motion

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion)

Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then for every $T \geq 0$,

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t))\, dt + \int_0^T f_x(t, W(t))\, dWt + \frac{1}{2} f_{xx}(t, W(t))\, dt. \tag{10}$$

Formula for Itô Processes

Definition 4.4.3

Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathscr{F}(t)$, $t \geq 0$, be an associated filtration. An Itô process is a stochastic process of the form

$$X(T) = X(0) + \int_0^t \Delta(u)\, dW(u) + \int_0^t \Theta(u)\, du \tag{11}$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes.

Formula for Itô Processes

Lemma 4.4.4

The quadratic variation of the Itô process (11) is

$$[X,X](t) = \int_0^t \Delta^2(u)\,du \tag{12}$$

Formula for Itô Processes

Definition 4.4.5

Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $\Gamma(t)$, $t \geq 0$, be an adapted process. We define the integral with respect to an Itô process[a]

$$\int_0^t \Gamma(u)\,dX(u) = \int_0^t \Gamma(u)\,\Delta(u)\,dW(u) + \int_0^t \Gamma(u)\,\Theta(u)\,du \quad (13)$$

[a] We assume that $\mathbb{E}\left[\int_0^t \Gamma^2(u)\,\Delta^2(u)\,du\right]$ and $\int_0^t |\Gamma(u)\,\Theta(u)|\,du$ are finite for every $t > 0$ so that the integrals on the right-hand side of (13) are defined.

Formula for Itô Processes

Theorem 4.4.6 (Itô-Doeblin formula for an Itô process)

Let $X(t)$, $t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $f(t,x)$ be a function which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,

$$
\begin{aligned}
f(T, X(T)) = f(0, X(0)) &+ \int_0^T f_t(t, X(t))\, dt \\
&+ \int_0^T f_x(t, X(t))\, \Delta(t)\, dW(t) \\
&+ \int_0^T f_x(t, X(t))\, \Theta(t)\, dt + \frac{1}{2} \int_0^T f_{xx}(t, X(t))\, \Delta^2(t)\, dt
\end{aligned}
\tag{14}
$$

Formula for Itô Processes

Theorem 4.4.9 (Itô integral of a deterministic integrand)

Let $W(s)$, $s \geq 0$, be a Brownian motion, and let $\Delta(s)$ be a nonrandom function of time. Define $I(t) = \int_0^t \Delta(s)\, dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s)\, ds$.

Examples

Generalized geometric Brownian motion

Let $W(t)$, $t \geq 0$, be a Brownian motion, let $\mathscr{F}(t)$, $t \geq 0$, be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Itô process

$$X(t) = \int_0^t \sigma(s)\,dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right) ds \tag{15}$$

$$dX(t) = \sigma(t)\,dW(t) + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right) dt$$

$$dX(t)\,dX(t) = \sigma^2(t)\,dW(t)\,dW(t) = \sigma^2(t)\,dt$$

Examples

Generalized geometric Brownian motion

Consider an asset price process given by

$$\begin{aligned} S(t) &= S(0)\, e^{X(t)} \\ &= S(0) \exp\left\{ \int_0^t \sigma(s)\, dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s) \right) ds \right\} \end{aligned} \tag{16}$$

where $S(0)$ is nonrandom and positive. We may write $S(t) = f(X(t))$, where $f(x) = S(0)\, e^x$, $f'(x) = S(0)\, e^x$, and $f''(x) = S(0)\, e^x$.

Examples

Generalized geometric Brownian motion

According to the Itô-Doeblin formula

$$\begin{aligned} dS(t) &= df(X(t)) \\ &= f'(X(t))\,dX(t) + \frac{1}{2} f''(X(t))\,dX(t)\,dX(t) \\ &= S(0)\,e^{X(t)}\,dX(t) + \frac{1}{2} S(0)\,e^{X(t)}\,dX(t)\,dX(t) \\ &= S(t)\,dX(t) + \frac{1}{2} S(t)\,dX(t)\,dX(t) \\ &= \alpha(t)\,S(t)\,dt + \sigma(t)\,S(t)\,dW(t) \end{aligned} \tag{17}$$

Examples

Vasicek interest rate model

Let $W(t)$, $t \geq 0$, be a Brownian motion. The Vasicek model for the interest rate process $R(t)$ is

$$dR(t) = (\alpha - \beta R(t))\,dt + \sigma\,dW(t) \tag{18}$$

where α, β, and σ are positive constants.
The solution to the stochastic differential equation (18) can be determined in closed form

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW(s) \tag{19}$$

Examples

Vasicek interest rate model

Let $W(t)$, $t \geq 0$, be a Brownian motion. The Vasicek model for the interest rate process $R(t)$ is

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where α, β, and σ are positive constants.
The solution to the stochastic differential equation (18) can be determined in closed form

$$R(t) = e^{-\beta t} R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \int_0^t e^{\beta s}\,dW(s) \tag{19}$$

Examples

Cox-Ingersoll-Ross (CIR) interest rate model

Let $W(t)$, $t \geq 0$, be a Brownian motion. The Cox-Ingersoll-Ross model for the interest rate process $R(t)$ is

$$dR(t) = (\alpha - \beta R(t))\,dt + \sigma\sqrt{R(t)}dW(t) \tag{20}$$

where α, β, and σ are positive constants.

$$\begin{aligned}
\mathbb{E}[R(t)] &= e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)\\
\mathbb{V}ar[R(t)] &= \mathbb{E}\left[R^2(t)\right] - (\mathbb{E}[R(t)])^2\\
&= \frac{\sigma^2}{\beta}R(0)\left(e^{-\beta t} - e^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(1 - 2e^{-\beta t} + e^{-2\beta t}\right)
\end{aligned}$$

Evolution of Portfolio Value

- Consider an agent who at each time t has a portfolio valued at $X(t)$.
- This portfolio invests in a money market account paying a constant rate of interest r and in a stock modeled by the geometric Brownian motion

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t) \tag{21}$$

- Suppose at each time t, the investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t)$, $t \geq 0$. The remainder of the portfolio value, $X(t) - \Delta(t)S(t)$, is invested in the money market account.

Evolution of Portfolio Value

- The differential $dX(t)$ for the investor's portfolio value at each time t is

$$\begin{aligned} dX(t) &= \Delta(t)\,dS(t) + r\left(X(t) - \Delta(t)\,S(t)\right)dt \\ &= \Delta(t)\left(\alpha S(t)\,dt + \sigma S(t)\,dW(t)\right) + r\left(X(t) - \Delta(t)\,S(t)\right)dt \\ &= rX(t)\,dt + \Delta(t)(\alpha - r)\,S(t)\,dt + \Delta(t)\,\sigma S(t)\,dW(t) \end{aligned} \tag{22}$$

  including

  1. an average underlying rate of return r on the portfolio, which is reflected by the term $rX(t)\,dt$,
  2. a risk premium $\alpha - r$ for investing in the stock, which is reflected by the term $\Delta(t)(\alpha - r)\,S(t)\,dt$, and
  3. a volatility term proportional to the size of the stock investment, which is the term $\Delta(t)\,\sigma S(t)\,dW(t)$.

Evolution of Portfolio Value

- The differential of the discounted portfolio value is

$$\begin{aligned} d\left(e^{-rt}X(t)\right) &= df(t, X(t)) \\ &= f_t(t, X(t))\,dt + f_x(t, X(t))\,dX(t) \\ &+ \frac{1}{2} f_{xx}(t, X(t))\,dX(t)\,dX(t) \\ &= -re^{-rt}X(t)\,dt + e^{-rt}dX(t) \\ &= \Delta(t)(\alpha - r)\,e^{-rt}S(t)\,dt + \Delta(t)\,\sigma e^{-rt}S(t)\,dW(t) \\ &= \Delta(t)\,d\left(e^{-rt}S(t)\right) \end{aligned} \tag{23}$$

Evolution of Option Value

- According to the Itô-Doeblin,

$$
\begin{aligned}
& dc(t, S(t)) \\
=& c_t(t, S(t))\, dt + c_x(t, S(t))\, dS(t) + \frac{1}{2} c_{xx}(t, S(t))\, dS(t)\, dS(t) \\
=& \left[c_t(t, S(t)) + \alpha S(t)\, c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)\, c_{xx}(t, S(t)) \right] dt \qquad (24) \\
& + \sigma S(t)\, c_x(t, S(t))\, dW(t)
\end{aligned}
$$

Evolution of Option Value

- The differential of the discounted option price

$$
\begin{aligned}
& de^{-rt}c(t,S(t)) \\
=& e^{-rt}\Big[-rc(t,S(t))+c_t(t,S(t))+\alpha S(t)c_x(t,S(t)) \\
& +\frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\Big]dt+e^{-rt}\sigma S(t)c_x(t,S(t))dW(t)
\end{aligned}
\tag{25}
$$

Equating the Evolutions

- A (short option) hedging portfolio starts with some initial capital $X(0)$ and invests in the stock and money market account so that the portfolio value $X(t)$ at each time $t \in [0, T]$ agrees with $c(t, S(t))$.
- This happens if and only if

$$d\left(e^{-rt}X(t)\right) = d\left(e^{-rt}c(t, S(t))\right) \text{ for all } t \in [0, T) \qquad (26)$$

Equating the Evolutions

- Comparing (23) and (25), we see that (26) holds if and only if

$$\begin{aligned}
&\Delta(t)(\alpha - t)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t) \\
&= \Big[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) \\
&\quad + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\Big]dt + \sigma S(t)c_x(t,S(t))\,dW(t)
\end{aligned} \tag{27}$$

Equating the Evolutions

- We first equate the $dW(t)$ terms in (27), which gives

$$\Delta(t) = c_x(t, S(t)) \text{ for all } t \in [0, T) \tag{28}$$

This is called the delta-hedging rule.

Equating the Evolutions

- We next equate the dt terms in (27), using (28), to obtain

$$
\begin{aligned}
&(\alpha - t)\,S(t)\,c_x(t,S(t)) \\
=& -rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)\,c_x(t,S(t)) \\
&+ \frac{1}{2}\sigma^2 S^2(t)\,c_{xx}(t,S(t)) \text{ for all } t \in [0,T)
\end{aligned}
\tag{29}
$$

Equating the Evolutions

- In conclusion, we should seek a continuous function $c(t,x)$ that is a solution to the Black-Scholes-Merton partial differential equation

$$rc(t,x) = c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2(t)\, c_{xx}(t,x) \qquad (30)$$

for all $t \in [0, T)$, $x \geq 0$ and that satisfies the terminal condition

$$c(T,x) = (x-K)^+ \qquad (31)$$

Solution to the Black-Scholes-Merton Equation

- Boundary conditions

$$c(t,0) = 0 \text{ for all } t \in [0,T] \tag{32}$$

$$\lim_{x\to\infty}\left[c(t,x) - \left(x - e^{-r(T-t)}K\right)\right] = 0 \text{ for all } t \in [0,T] \tag{33}$$

Solution to the Black-Scholes-Merton Equation

- The solution to the Black-Scholes-Merton equation (30) with terminal condition (31) and boundary conditions (32) and (33) is

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)), \quad 0 \le t < T, x > 0 \qquad (34)$$

where

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right] \qquad (35)$$

and N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz. \qquad (36)$$

The Greeks

- Delta

$$c_x(t,x) = N(d_+(T-t,x)) \tag{37}$$

- Gamma

$$\begin{aligned} c_{xx}(t,x) &= N'(d_+(T-t,x))\frac{\partial}{\partial x}d_+(T-t,x) \\ &= \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)) \end{aligned} \tag{38}$$

Put-Call Parity

- The value of a forward contract

$$f(t,x) = x - e^{-r(T-t)}K \tag{39}$$

- Put-Call Parity

$$f(t,x) = c(t,x) - p(t,x), x \geq 0, 0 \leq t \leq T \tag{40}$$

Multiple Brownian Motions

Definition 4.6.1

A d-dimensional Brownian motion is a process

$$\boldsymbol{W}(t) = \begin{pmatrix} W_1(t) & \dots & W_d(t) \end{pmatrix}$$

with the following properties.
(i) Each $W_i(t)$ is a one-dimensional Brownian motion.
(ii) If $i \neq j$, then processes $W_i(t)$ and $W_j(t)$ are independent.

Multiple Brownian Motions

Definition 4.6.1

Associated with a d-dimensional Brownian motion, we have a filtration $\mathscr{F}(t)$, $t \geq 0$, such that the following holds.

(iii) (Information accumulates) For $0 \leq s < t$, every set in $\mathscr{F}(s)$ is also in $\mathscr{F}(t)$.

(iv) (Adaptivity) For each $t \geq 0$, the random vector $\boldsymbol{W}(t)$ is $\mathscr{F}(t)$-measurable.

(v) (Independence of future increments) For $0 \leq t < u$, the vector of increments $\boldsymbol{W}(u) - \boldsymbol{W}(t)$ is independent of $\mathscr{F}(t)$.

Itô-Doeblin Formula for Multiple Processes

Theorem 4.6.2 (Two-dimensional Itô-Doeblin formula)

Let $f(t,x,y)$ be a function whose partial derivatives f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx} and f_{yy} are defined and are continuous. Let $X(t)$ and $Y(t)$ be Itô processes as discussed above. The two-dimensional Itô-Doeblin formula in differential form is

$$\begin{aligned} df &= f_t dt + f_x dX + f_y dY \\ &\quad + \frac{1}{2} f_{xx} dX dX + f_{xy} dX dY + \frac{1}{2} f_{yy} dY dY \end{aligned} \tag{41}$$

Itô-Doeblin Formula for Multiple Processes

Corollary 4.6.3 (Itô product rule)

Let $X(t)$ and $Y(t)$ be Itô processes. Then

$$dX(t)\,dY(t) = X(t)\,dY(t) + Y(t)\,dX(t) + dX(t)\,dY(t)$$

Recognizing a Brownian Motion

Theorem 4.6.4 (Lévy, one dimension)

Let $M(t)$, $t \geq 0$, be a martingale relative to a filtration $\mathscr{F}(t)$, $t \geq 0$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and $[M, M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.

Recognizing a Brownian Motion

Theorem 4.6.5 (Lévy, two dimensions)

Let $M_1(t)$ and $M_2(t)$, $t \geq 0$, be martingales relative to a filtration $\mathscr{F}(t)$, $t \geq 0$. Assume that for $i = 1, 2$, we have $M_i(0) = 0$, $M_i(t)$ has continuous paths, and $[M_i, M_i](t) = t$ for all $t \geq 0$. If in addition, $[M_1, M_2](t) = 0$ for all $t \geq 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions.

Gaussian Processes

Definition 4.7.1

A Gaussian process $X(t)$, $t \geq 0$, is a stochastic process that has the property that, for arbitrary times $0 < t_1 < t_2 < \cdots < t_n$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normally distributed.

Brownian Bridge as a Gaussian Process

Definition 4.7.4

Let $W(t)$ be a Brownian motion. Fix $T > 0$. We define the Brownian bridge from 0 to 0 on $[0, T]$ to be the process

$$X(t) = W(t) - \frac{t}{T} W(T), 0 \leq t \leq T \tag{42}$$

Brownian Bridge as a Gaussian Process

Definition 4.7.5

Let $W(t)$ be a Brownian motion. Fix $T > 0$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$. We define the Brownian bridge from a to b on $[0, T]$ to be the process

$$X^{a\to b}(t) = a + \frac{(b-a)t}{T} + X(t), 0 \le t \le T$$

where $X(t) = X^{0\to 0}$ is Brownian bridge from 0 to 0 of Definition 4.7.4.

Brownian Bridge as a Scaled Stochastic Integral

Theorem 4.7.6

Define the process

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-u} dW(u) & \text{for } 0 \le t < T \\ 0 & \text{for } t = T \end{cases}$$

Then $Y(t)$ is a continuous Gaussian process on $[0, T]$ and has mean and covariance functions

$$m^Y(t) = 0, t \in [0, T]$$
$$c^Y(s,t) = s \wedge t - \frac{st}{T} \text{ for all } s, t \in [0, T]$$

Brownian Bridge as a Scaled Stochastic Integral

Theorem 4.7.6

In particular, the process $Y(t)$ has the same distribution as the Brownian bridge from 0 to 0 on $[0, T]$ (Definition 4.7.5).

Multidimensional Distribution of the Brownian Bridge

- We fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and let $X^{a \to b}(t)$ denote the Brownian bridge from a to b on $[0, T]$.
- We also fix $0 = t_0 < t_1 < t_2 < \cdots < t_n < T$.
- In this section, we compute the joint density of $X^{a \to b}(t_1), \ldots, X^{a \to b}(t_n)$.

Multidimensional Distribution of the Brownian Bridge

- The density for $X^{a\to b}(t_1), \ldots, X^{a\to b}(t_n)$,

$$
\begin{aligned}
& f_{X^{a\to b}(t_1),\ldots,X^{a\to b}(t_n)}(x_1,\ldots,x_n) \\
=& \sqrt{\frac{T}{T-t_n}} \cdot \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \\
& \cdot \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b-x_n)^2}{2(T-t_n)} + \frac{(b-a)^2}{2T} \right\} \\
=& \frac{p(T-t_n, x_n, b)}{p(T,a,b)} \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j)
\end{aligned}
\tag{43}
$$

Multidimensional Distribution of the Brownian Bridge

- In (43)

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{ -\frac{(y-x)^2}{2\tau} \right\}$$

is the transition density for Brownian motion.

Brownian Bridge as a Conditioned Brownian Motion

- The joint density of $W(t_1), \ldots, W(t_n), W(T)$ is

$$f_{W(t_1),\ldots,W(t_n),W(T)}(x_1, \ldots, x_n, b) = p(T - t_n, x_n, b) \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j) \quad (44)$$

- The density of $W(t_1), \ldots, W(t_n)$ conditioned on $W(T) = b$ is thus the quotient

$$\frac{p(T - t_n, x_n, b)}{p(T, a, b)} \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j)$$

and this is $f_{X^{a \to b}(t_1),\ldots,X^{a \to b}(t_n)}(x_1, \ldots, x_n)$ of (43).

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- Define

$$M^{a\to b}(T) = \max_{0\le t\le T} X^{a\to b}(t)$$

Corollary 4.7.7

The density of $M^{a\to b}(T)$ is

$$f_{M^{a\to b}(T)}(y) = \frac{2(2y-b-a)}{T} e^{-\frac{2}{T}(y-a)(y-b)}, y > \max\{a, b\} \quad (45)$$