3.16

(a)

$$\begin{aligned}\mathbb{E}(\hat{\beta^*}|X) &= \mathbb{E}[(X'V^{-1}X)^{-1}X'V^{-1}(X\beta+\epsilon)] \\ &= \mathbb{E}[\beta^0 + (X'V^{-1}X)^{-1}X'V^{-1}\mathbb{E}(\epsilon|X)) \\ &= \beta^0 \qquad (1)\end{aligned}$$

Define$D^{*'} = D'C^{-1}$,let $\hat{b}$ is another linear unbiased estimator

$$\hat{b} = D'Y = D'C^{-1}CY = D^{*'}Y^*$$

$\hat{b}$ is unbiased

$$\Rightarrow \beta^0 = \mathbb{E}(\hat{b}|X) = D^{*'}(X^*\beta^0) + D^{*'}\mathbb{E}(\epsilon^*|X) = D^{*'}X^*\beta^0 \Rightarrow D^{*'}X^* = I$$

$$\Rightarrow \hat{b} = D^{*'}Y^* = D^{*'}(X^*\beta + \epsilon^*) = \beta^0 + D^{*'}\epsilon^*)$$

$$\begin{aligned}Var(\hat{b}|X) - Var(\hat{\beta^*}|X) &= D^{*'}D^* - (X^{*'}X^{*'})^{-1} \\ &= D^{*'}D^* - D^{*'}X^*(X^{*'}X^{*'})^{-1}X^{*'}D^* \\ &= D^{*'}[I - X^*(X^{*'}X^{*'})^{-1}X^{*'}]D^* \\ &= D^{*'}M^*D^* \\ &= D^{*'}M^*M^{*'}D^* \\ &= P.S.D \qquad (2)\end{aligned}$$

where $M^* = I - X^*(X^{*'}X^{*'})^{-1}X^{*'}$
From (1) and (2), β^* is BLUE

(b)

$$\epsilon|X \sim N(0,V) \Rightarrow C'\epsilon C|X = \epsilon^*|X \sim N(0, C'VC) = N(0,I)$$

$$\begin{aligned}S^* &= \frac{e^{*'}e^*}{n-k} \\ &= \frac{1}{n-k}\epsilon^*M^*\epsilon^* \\ &= \frac{1}{n-k}\chi^2_{n-k} \qquad (3)\end{aligned}$$

$$
\begin{aligned}
Cov(\hat{\beta^*}, e^*|X) &= \mathbb{E}[(\hat{\beta^*} - \mathbb{E}\beta)(e^* - \mathbb{E}e^*)'] \\
&= \mathbb{E}[(\hat{\beta^*} - \mathbb{E}\beta)e^{*'}] \\
&= \mathbb{E}[(X^{*'}X^*)^{-1}X^*e^*|X] \\
&= \mathbb{E}[(X^{*'}X^*)^{-1}X^*M^*\epsilon^*|X] \\
&= (X^{*'}X^*)^{-1}X^*M^*\mathbb{E}[\epsilon^*|X] \\
&= 0 \qquad (4)
\end{aligned}
$$

$$
e^* = M^*\epsilon^* \sim N(0, M^*) \qquad (5)
$$

$$
\begin{aligned}
\hat{\beta^*} - \beta^0 &= (X^{*'}X^*)^{-1}X^{*'}Y^* - \beta^0 \\
&= (X^{*'}X^*)^{-1}X^{*'}(X^*\beta^0 + \epsilon^*) - \beta^0 \\
&= (X^{*'}X^*)^{-1}X^{*'}X^*\epsilon^* \\
&\sim N(0, (X^{*'}X^*)^{-1}) \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
R\hat{\beta^*} - r &= R'(\hat{\beta^*} - \beta^0) \\
&\sim N(0, R'(X^{*'}X^*)^{-1}R) \qquad (7)
\end{aligned}
$$

For the nomal distrubution, co-varance=0 $\Rightarrow$ independent $\Rightarrow$ So the $\hat{\beta^*} - \beta^0$ is independent of e^* $\Rightarrow$ $R\hat{\beta^*} - r$ is independent of S^*
From (6)

$$
\Rightarrow \tilde{T}^* \frac{R\hat{\beta} - r}{\sqrt{R'(X^{*'}X*)^{-1}R}} \sim N(0,1) \ for J = 1 \qquad (8)
$$

From (3)(8) and the independence,

$$
\begin{aligned}
\frac{R\hat{\beta} - r}{\sqrt{S^{*2}R'(X^{*'}X*)^{-1}R}} &= \frac{\frac{(\hat{\beta} - r)}{\sqrt{R'(X^{*'}X^*)^{-1}R}}}{\sqrt{S^{*2}}} \\
&\sim \frac{N(0,1)}{\sqrt{\frac{1}{n-k}\chi^2_{n-k}}} \\
&\sim t_{n-k} \qquad (9)
\end{aligned}
$$

From (7),

$$
\tilde{Q}^* = (R\hat{\beta} - r)'[R'(X^{*'}X^*)^{-1}R]^{-1}(R\hat{\beta} - r)' \sim \chi^2_J \qquad (10)
$$

From (3) and (10), joint the independence, we can get

$$F^* = \frac{(R\hat{\beta} - r)'[R'(X^{*'}X^*)^{-1}R]^{-1}(R\hat{\beta} - r)'/J}{S^{*2}} = \frac{\frac{1}{J}\chi_J^2}{\frac{1}{n-k}\chi_{n-k}^2} \sim F(J, n-K) \tag{11}$$

(c) (8) and (10) is the answer.

(d) As the hint, the t-distribution has more heavier tail than N(0,1) and so has a larger critical value at a given significance level. For the F distribution, when $n-k \to \infty$ the $F(J, n-k) \to \chi_J^2$. For the case of finite n-k, the $F(J, n-k)$ has more heavier tail than the χ_{n-k}^2, so has a larger critical value at a given significance level. Those mean that (T^*, F^*) has less probability to reject H_0 when H_0 is not true. So the the $(\tilde{T}^*, \tilde{F}^*)$ are more powerful than (T^*, F^*)