Iterated Elimination and Rationalizable Actions

Proposition

If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\{N, (A_i), (u_i)\}$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

($\Leftarrow$) First show that if a_i is rationalizable then $a_i \in X_i^T$.

- Let $(Z_j)_{j \in N}$ be the profile of sets that supports a_i.
- For any t, $Z_j \subset X_j^t$ since each action in Z_j is a best response to some belief over Z_{-j}.

($\Rightarrow$) Now show that for any player i any action in X_i^T is rationalizable.

- By definition if $a_i \in X_i^T$ then it is not strictly dominated and is a best response among actions in X_i^T to some belief $\mu_i(a_i)$ over X_{-i}^T.
- It must also be a best response among the actions in A_i.
    - Otherwise $\exists t$, a_i is a best response over X_{-i}^t but not over X_{-i}^{t-1}.
    - $\exists b_i \in X_i^{t-1} \backslash X_i^t$ which is a best response to $\mu_i(a_i)$ over X_{-i}^{t-1}.
    - b_i cannot be strictly dominated in tth round.
- Note that order is not important.