Advanced Microeconomics II
Problem Set 3

WISE, Xiamen University

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Due 10:00 May 12, 2011

1. Two people have to share 100 dollars which come in 10-dollar bills. They do not have change, so what each player gets must be a multiple of 10 dollars. The procedure for sharing is follows. Player 1 makes a proposal (the number of 10-dollar bills she will receive). Hearing it, player 2 decides whether to accept the proposal, in which case it is implemented, or reject it. In the latter case it again becomes player 1's turn, and she now has to choose between "giving in", which yields both players zero payoff, and seeking "revenge", which yields player 1 zero payoff but costs player 2 one dollar.

   (a) Write down this problem as an extensive game with prefect information.

Solution: An extensive game that models the individual's predicament is $\langle N, H, P, (\succeq_i)\rangle$ where

- $N = \{1, 2\}$;
- Let $X = \{10, 20, \ldots, 100\}$. $H = \{\varnothing, X, X \times \{A, R\}, X \times \{A\} \times \{g, r\}$.
- $P(\varnothing) = 1, P(x) = 2$ and $\cancel{P(x, A)1}$;
- For all $x \in X$: $(x, A) \succeq_1 (y, A)$ iff $x \geq y$, $(x, A) \succeq_2 (y, A)$ iff $x \leq y$, $(0, A) \sim_1 (x, R, g) \sim_1 (x, R, r)$, $(100, A) \sim_2 (x, R, g) \succ_2 (x, R, r)$.

   (b) Find the set of Nash equilibria.

Solution: There are two types of Nash equilibria, 11 different Nash equilibrium payoffs.
Type 1: For all $x \in X$: $s_1(\varnothing) = x$; $s_2(y) = A$ if $y = x$, $s_2(y) = R$ if $y > x$ $s_2(y) \in \{A, R\}$ if $y < x$; $s_1(y, R) \in \{g, r\}$ for all $y \in X$. The payoff profile is $(x, 100 - x)$.
Type 2: For $x = 100$: $s_1(\varnothing) = 100$; $s_2(y) = R$ for all $y > 0$, $s_2(0) \in \{A, R\}$; $s_1(y, R) \in \{g, r\}$ for all $y < 100$, $s_1(100, R) = g$. The payoff profile is $(0, 0)$.

   (c) Find the set of sub-game perfect Nash equilibria.

Solution: There are two sub-game Nash equilibrium payoff profiles:
Type 1: $s_1(\varnothing) = 100$; $s_2(y) = A$ for all $y \in X$; $s_1(y, R) \in \{g, r\}$ for all $y \in X$. The payoff profile is $(100, 0)$.
Type 1: $s_1(\varnothing) = 90$; $s_2(y) = A$ for all $y < 100$, $s_2(100) = R$; $s_1(y, R) \in \{g, r\}$ for all $y < 100$, $s_1(100, R) = g$. The payoff profile is $(90, 10)$.

2. In the following bargaining game there are only three possible divisions of the pie $(1, 0)$, $(0.5, 0.5)$ and $(0, 1)$. Each player has the same discount rate $\delta = 0.9$.

   (a) Find all the stationary subgame equilibria.

Solution: Let $X = \{(1, 0), (0.5, 0.5), (0, 1)\}$. Each $x \in X$ is the payoff profile associated with a sub game perfect equilibrium. The strategies are:

- The player who makes the offer always offers x;
- Player 2 accepts any offer y iff $y_2 \geq x_2$;
- Player 1 accepts any offer y iff $y_1 \geq x_1$.

We just need to check that a player would rather wait and offer x than accept the next smallest portion. That is, we need

$$\delta 1 \geq 0.5; \quad \delta 0.5 \geq 0; \quad \delta 0 \geq 0.$$

Since $\delta = 0.9$ all these inequalities hold.

   (b) Construct a subgame perfect equilibrium where agreement is reached with delay, i.e., agreement is reached after the first period.

Solution:

- Player 1 offers $(1, 0)$ at $t = 0$;
- Player 2 rejects all offers;
- If player 1 offers $(1, 0)$ in period 0, then players offer $(0.5, 0.5)$ every period.
- If player 1 offers $(1, 0)$ in period 0, then players accept any offer that gives them at least half of the pie.
- If player 1 offers something else, then player 2 offers $(0, 1)$ every odd period.
- If player 1 offers something else, then player 1 offers $(0, 1)$ every even period.
- If player 1 offers something else, then player 2 only accepts offers of $(0, 1)$.
- If player 1 offers something else, then player 1 accepts any offer.

3. Prove the following proposition.

   Proposition 1. *Let w be a strictly enforceable feasible payoff profile of $G = \{N, (A_i), (u_i)\}$. For all $\epsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|w' - w| < \epsilon$.*

Solution: Let $w = \sum_{a\in A}(\beta_a/\gamma)u(a)$ be a strictly enforceable feasible payoff profile, where β_a for each $a \in A$ is an integer and $\gamma = \sum_{a\in A}(\beta_a)$, and let $(a^k)_{k=1}^{\gamma}$ be the cycling sequence of action profiles for which the cycle of length γ contains β_a repetitions of a for each $a \in A$. Let s_i be the strategy of player i in the repeated game that chooses a_i^t in each period t unless there was a previous period t' in which a single player other than i deviated from $a^{t'}$, in which case it chooses $(p_{-j})_i$, where j is the deviant in the first such period t' and (p_{-j}) is the profile of strategies that generate player j's minimax payoff v_j.

Let $\epsilon > 0$. Let $w'(\delta)$ be the payoff profile in the δ-discounted infinitely repeated game associated with the sequence $(a^k)_{k=1}^{\infty}$. Then,

$$w'(\delta) = (1-\delta)\sum_{i=0}^{\infty}\delta^{i\gamma}\sum_{k=1}^{\gamma}\delta^k u(a^k) = \frac{(1-\delta)\sum_{k=1}^{\gamma}\delta^k u(a^k)}{1-\delta^\gamma}.$$

Using L'Hôpital's rule shows that

$$\lim_{\delta\to 1} w'(\delta) = \frac{\sum_{k=1}^{\gamma} u(a^k)}{\gamma} = w.$$

Hence, there exists $\tilde{\delta}$ such that for all $\delta > \tilde{\delta}$, $\left|w'(\delta) - w\right| < \epsilon$. It remains to show that the strategies are equilibrium strategies.

Let M_i and m_i be the maximum and minimum stage game payoff for player i. Let δ_i solve

$$(1-\delta)M_i + \delta v_i \le (1-\delta)m_i + \delta w_i.$$

Such a δ exists since $w_i > v_i$ for each player i. For all $\delta > \max\{\delta_1, \dots, \delta_n\}$, $s = (s_i)_{i\in N}$ is a Nash equilibrium of the δ-discounted infinitely repeated game.

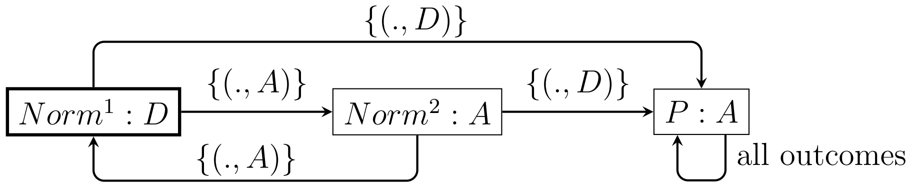
So for $\delta > \underline{\delta} = \max\{\tilde{\delta}, \delta_1, \dots, \delta_n\}$, the strategy profile s is a Nash equilibrium of the repeated game with a payoff profile w' within ϵ of w.

4. Consider the following stage game.

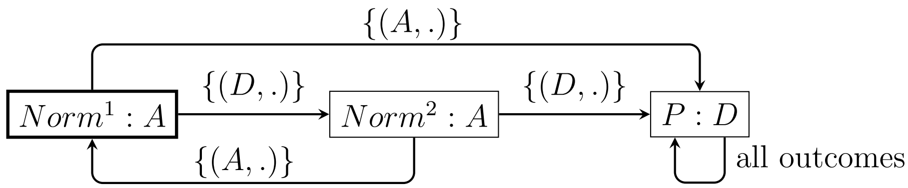
	A	D
A	$2, 2$	$0, 1$
D	$5, 4$	$1, 0$

(a) Construct a pair of strategies that generate the average per-period payoffs of $(3.5, 3)$, and are a Nash equilibrium but are not a subgame perfect equilibrium when players use the limit of means criterion to evaluate payoffs.

Solution: The sequence of outcomes that generate the average per-period payoffs of $(3.5, 3)$ are $(AA, AD, AA, AD, \ldots)$. The machine M_1 that supports this sequence of outcomes as a Nash equilibrium is

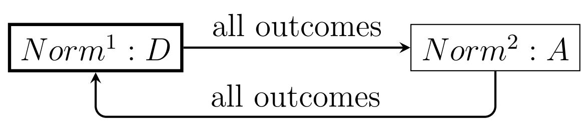


The machine M_2 that supports this sequence of outcomes as a Nash equilibrium is

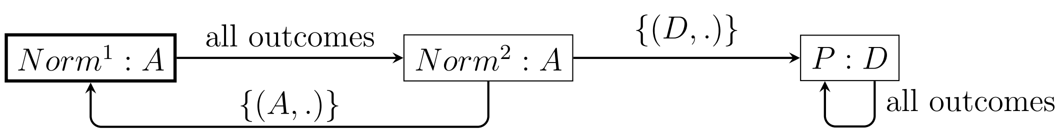


Since A is a dominant strategy for player 2, and player 2 always chooses A along the equilibrium path, there is never any incentive for player 2 to deviate. We also know D is a dominant strategy for player 1. Thus, player 1 never has any incentive to deviate when required to play ~~d~~. Thus, we can simplify the machines.

M_1 becomes



M_2 becomes

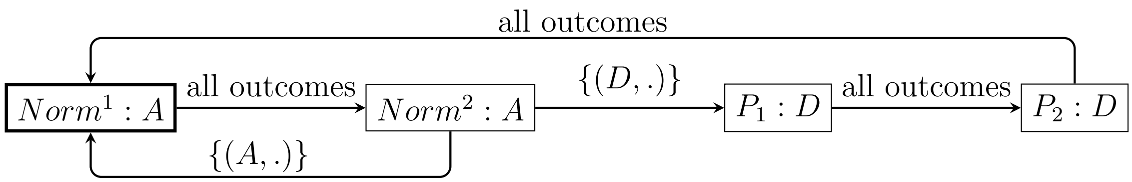


Note that these strategies are not subgame perfect equilibria, because it is not in player 2's interest to punish player 1 forever. Hence, in the subgame after a history in which player 1 deviates, the strategies do not form a Nash equilibrium.

(b) Construct a pair of strategies that generate average per-period payoffs of $(3.5, 3)$, and are a subgame perfect equilibrium when players use the limit of means criterion to evaluate payoffs but not a subgame perfect equilibrium when players use the overtaking criterion to evaluate payoffs.

Solution: To ensure that player 2 is willing to enforce the punishment for player 1, the punishment should be for a finite period of time. The player 1 stage game profit from deviating when the outcome should be (AA) equals 3. Player 1 needs to be punished for 2 periods to wipe out the gains (~~9;7~~; actually one period of punishment is sufficient if you restart the equilibrium payoffs with (AA) rather than (DA)).

The machine M_1 that represents these strategies is the same as in part (a). The machine M_2 that represents these strategies is



Note that player 2 has no incentive not to follow the punishment strategy since they revert to equilibrium payoffs once the punishment phase is over, so, in the limit, player 2 gets the same payoff whether or not he punishes player 1. However, this is not a sub-game perfect equilibrium if players use the overtaking criterion since player 2 could deviate when called to punish and generate the sequence of payoffs $(4, 4, 4, 2, \ldots)$ which is strictly preferred by player 2 to $(0, 0, 4, 2, \ldots)$.

(c) Construct a pair of strategies that generate the average per-period payoffs of $(3.5, 3)$, and are a subgame perfect equilibrium when players use overtaking criterion to evaluate payoffs.

Solution: Punishing player 1 for 2 periods as before will ensure player 1 has no profitable deviation along the equilibrium path. However, players must also have the incentive to apply punishments. If punishments are min-max punishments, which then transition back to the norm once finished, then no machine with finite states can describe such strategies that are SPE. Punishments are required to be of ever increasing (but still finite) length in order to be sub-game perfect.

The machines M_1 and M_2 each have the same states and transition functions: only their output functions differ.

- States:
  - $Norm^1$: Initial State
  - $Norm^2$: second period of $(a^k)_{k=1}^2$ cycle.
  - $P^1(j,t)$: Punishment phase of player 1 after a player 1 deviation with s periods remaining, where s is a positive integer.
  - $P^2(j,t)$: Punishment phase of player 2 after a player 2 deviation with s periods remaining, where s is a positive integer.
- Output function for player 1:
  - In $Norm^1$: choose D.
  - In $Norm^2$: choose A.
  - In $P^1(s)$: choose D.
  - In $P^2(s)$: choose A.
- Output function for player 2:
  - In $Norm^1$: choose A.
  - In $Norm^2$: choose A.
  - In $P^1(s)$: choose D.
  - In $P^2(s)$: choose A.

Solution:

- Transition function:
  - $\tau_i(Norm^1, a) = Norm^2$.
  - $\tau_i(Norm^2, a) = \begin{cases} Norm^1 & \text{if } a \neq (DA) \\ P^1(2) & \text{if } a = (DA). \end{cases}$
  - $\tau_i(P^1(s), a) = \begin{cases} P^1(s-1) & \text{if } 2 \leq s \leq 3, \text{ if } a \neq (DA) \\ Norm^1 & \text{if } s = 1, \text{ if } a \neq (DA) \\ P^2(T(1,s)) & \text{if } a = (DA), \end{cases}$

    where $T(1,t)$ is large enough that the sum of player 2's payoff in state $P^1(s)$ and his payoff in the subsequent $T(1,s)$ periods if he does not deviate is greater than his payoff in the deviation plus $2T(1,t)$. (such a number exists since after s periods the players were supposed to go back to the equilibrium sequence (DA, AA) which gives a two-period payoff of 6 to player 2 which is greater than 4.
  - $\tau_i(P^2(s), a) = \begin{cases} P^2(s-1) & \text{if } 2 \leq s \leq 3, \text{ if } a \neq (DA) \\ Norm^1 & \text{if } s = 1, \text{ if } a \neq (DA) \\ P^1(T(2,s)) & \text{if } a = (DA), \end{cases}$

    where $T(2,s)$ is large enough that the sum of player 1's payoff in state $P^2(s)$ and his payoff in the subsequent $T(2,s)$ periods if he does not deviate is greater than his payoff in the deviation plus $T(2,s)$. (such a number exists since after s periods the players were supposed to go back to the equilibrium sequence (DA, AA) which gives a two-period payoff of 7 to player 1 which is greater than 2.

Let me give an example for each player of how to calculate $T(j,s)$. Start at state $P^1(s)$, i.e., player 1 needs to be punished for s periods. Then player 2's sequence of payoffs with no deviation is $(0, \ldots, 0, 4, 2, 4, 2, \ldots)$, where 0 is repeated s times. If player 2 deviates then his sequence of payoffs is $(4, 2, 2, \ldots, 2, 4, 2, 4, 2, \ldots)$, where 2 is repeated $T(1,s)$ times. Assuming either s is odd and $T(1,s)$ is even, or s is even and $T(1,s)$ is odd, the gain from deviating equals $4 + 2(s-1) - (T(1,s) + 1 - s)$ which must be negative, i.e., $T(1,s) \geq 3s + 1$.

Now, start at state $P^2(s)$, i.e., player 2 needs to be punished for s periods. Then player 1's sequence of payoffs with no deviation is $(2, \ldots, 2, 5, 2, 5, 2, \ldots)$, where 2 is repeated s times. If player 1 deviates then his sequence of payoffs is $(5, 1, 1, \ldots, 1, 5, 2, 5, 2, \ldots)$, where 1 is repeated $T(2,s)$ times. Assuming either s is odd and $T(2,s)$ is even, or s is even and $T(2,s)$ is odd, the gain from deviating equals $3 - (s-1) - 5(T(2,s) + 1 - s)/2$ which must be negative, i.e., $T(2,s) \geq (3s-1)/5$.

5. Consider a game in which the following strategic game is repeated twice:

		Player 2		
		b_1	b_2	b_3
	a_1	$10, 10$	$2, 12$	$0, 13$
Player 1	a_2	$12, 2$	$5, 5$	$0, 0$
	a_3	$13, 0$	$0, 0$	$1, 1$

The players observe the actions chosen in the first play of the game prior to the second play.

(a) What are the pure strategy sub-game perfect Nash equilibrium payoffs of this game?

Solution: (a_2, b_2) and (a_3, b_3) are two pure strategy Nash equilibria of this simultaneous-move game. Thus, $(a_2b_2, a_2b_2), (a_3b_3, a_3b_3), (a_2b_2, a_3b_3)$ and (a_3b_3, a_2b_2) are sub-game perfect Nash equilibria outcomes of this game.

In addition, the payoff of each player in the stage game Nash equilibrium outcome of (a_2b_2) is 5 and the payoff of each player in the stage game Nash equilibrium outcome of (a_3b_3) is 1. The difference between the stage game Nash equilibrium payoffs for each player is 4. Hence, any stage game outcome where each player's most profitable deviation is no more than 4 is sustainable in the first period. The outcome is supported by punishing deviations with the lower Nash equilibrium in the second period. Hence, (a_1b_1, a_2b_2), (a_1b_2, a_2b_2), (a_1b_3, a_2b_2), (a_2b_1, a_2b_2) and (a_3b_1, a_2b_2) are additional sub-game equilibrium outcomes.

Thus the pure strategy sub-game perfect Nash equilibrium payoffs of this game are (2,2), (10,10), (6,6), (15,15), (7,17), (17,7), (5,18) and (18,5).

(b) What are the pure strategy subgame perfect Nash equilibria of this game? In particular, how many pure strategy sub-game perfect Nash equilibria are there?

Solution: The number of strategies that support each individual outcome is large. For example, take the first period outcome (a_2b_2), which is one of the two stage game Nash equilibrium outcomes mentioned above. Since this is a stage game Nash equilibrium, any stage game Nash equilibrium outcome can be selected as a result of a deviation to (a_2b_2) in the first period and there are only 2 Nash equilibrium outcomes in the stage game. As a result there are $2^9 = 512$ sub-game perfect equilibria associated with the first stage outcome (a_2b_2) (of these 512, 256 are associated with the outcome (a_2b_2, a_2b_2) and 256 are associated with the outcome (a_2b_2, a_3b_3). Similarly there are 512 outcomes associated with the first period outcome (a_3b_3).

More generally, for each first period outcome the number of sub-game perfect equilibria associated with such an outcome depends on the need for punishment. When the first stage outcome is a Nash equilibrium of the stage game, then there is no need for punishment off the equilibrium path, so there is no restriction on which second stage Nash equilibrium outcome to play off the equilibrium path. In contrast, the first stage outcome (a_1b_1) is sustained by punishment if 4 alternative outcomes occur $(a_1b_2), (a_1b_3), (a_2b_1), (a_3b_1)$. There is no restriction on which second stage Nash equilibrium outcome to play after the histories $(a_2b_2), (a_2b_3), (a_3b_2), (a_3b_3)$. So, there are $2^4 = 16$ sub-game perfect equilibria associated with first stage outcome a_1b_1. Similarly, we know that the first stage outcome (a_1b_2) is sustained by punishment if 2 alternative outcomes occur (a_1b_3) and (a_2b_2); (a_1b_3) is sustained by punishment if 1 alternative outcome occurs (a_3b_3); (a_2b_1) is sustained by punishment if 2 alternative outcomes occur (a_2b_2) and (a_3b_1); (a_3b_1) is sustained by punishment if 1 alternative outcome occurs (a_3b_3).

Hence, the total number of pure strategy sub-game perfect Nash equilibria is $2^9 + 2^9 + 2^4 + 2^7 + 2^7 + 2^8 + 2^8 = 1808$.