Advanced Microeconomics II
Problem Set 2

WISE, Xiamen University

Spring 2011

Due 10:00 Apr 9, 2011

1. (Gibbons 1.2) Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, s_1 and s_2, where $0 \leq s_i \leq 1, i = 1, 2$. If $s_1 + s_2 \leq 1$, then the players receive the shares they named; if $s_1 + s_2 > 1$, then both players receive zero.

   (a) Formulate this as a strategic game.

Solution: $G = \{N, (S_i)_{i=1}^n, (u_i)_{i=1}^n\}$, $N = \{1, 2\}$, $S_i = [0, 1], i = 1, 2.$

$$u_i(s_i, s_j) = \begin{cases} s_i & \text{if } s_i + s_j \leq 1 \\ 0 & \text{if } s_i + s_j > 1. \end{cases}$$

   (b) What are the pure-strategy Nash equilibria of this game?

Solution: From a, we can get the best response function of each player.

$$B_i(s_j) = 1 - s_j, \ i \neq j, i, j \in \{1, 2\}$$

Thus we know the pure-strategy Nash equilibria of this game are $(s_1, 1 - s_1)$, where $s_1 \in [0, 1]$ and $(1, 1)$.

   (c) Derive a mixed strategy equilibrium where each player randomly choose one of two numbers.

Solution: Consider a symmetric mixed strategy equilibrium. Let $a_1 \in (0, 1/2)$ and let $a_2 = 1 - a_1$. Denote α as the probability of choosing a_1. To be an equilibrium the player must be indifferent between choosing between the two numbers. Hence,

$$a_1 = \alpha a_2 \Rightarrow \alpha = \frac{a_1}{a_2}.$$

Note that any other action gives a payoff strictly less than a_1.

   (d) Consider a symmetric mixed strategy equilibrium $F(.)$, where the support of the mixed strategy is an interval $[a, b]$.

      i. Show that to be a mixed strategy equilibrium, a and b should satisfy $a + b \leq 1$.

Solution: Note that $a < b$. Otherwise, we have pure strategy equilibrium that are have been identified above. Suppose $a + b > 1$, then $E(U_i(b)) = 0$. Since every action in the support of each player's equilibrium mixed strategy yields that player the same payoff, then $U_i = 0$. If $b < 1$ a player could choose $s = 1 - b > 0$ which guarantees a strictly positive payoff. If $b = 1$ then there exists ϵ such that $F(b - \epsilon) > 0$. Hence, a player could choose $s = \epsilon$ which guarantees a strictly positive payoff of $\epsilon F(b - \epsilon)$.

ii. Show that to be a mixed strategy equilibrium, a and b should satisfy $a + b \geq 1$.

Solution: Suppose $a + b < 1$, then $E(U_i(a)) = a$ and $b < 1$. Since every action in the support of each player's equilibrium mixed strategy yields that player the same payoff, then $U_i = a$. But if player i chooses $1 - b$, they guarantee a payoff of $1 - b > a$.

iii. Show that to be a mixed strategy equilibrium the probability that $s_i = a$ is not atomless, i.e. $F(a) > 0$.

Solution: From i and ii, we know that if there is a mixed strategy equilibrium, $a + b = 1$ and that $E(U_i(a)) = a$. Also, if $F(a) = 0$, then the answer to the last two questions shows that $E(U_i(b)) = 0 \neq a$. This cannot be an equilibrium since every action in the support of each player's equilibrium mixed strategy yields that player the same payoff.

iv. Derive such a mixed strategy equilibrium.

Solution: From iii, $E(U_i(a)) = a = E(U_i(b)) = bF(a)$, $F(a) = \frac{a}{b}$. To be an equilibrium strategy,

$$a \equiv sF(1 - s) \quad \text{for all } s \in [a, b].$$

Differentiating gives $F(1 - s) - f(1 - s)s = 0$, which implies $F(s) = \frac{c}{(1-s)}$, where c is the constant of integration. To solve the constant of integration which use the properties of a probability distribution function, i.e. $F(b) = 1$, which implies $c = 1 - b = a$. So the symmetric mixed strategy equilibrium is

$$F(s_i) = \frac{a}{1 - s_i}$$

where $i \in \{1, 2\}$.

2. (Gibbons 1.5) Consider the following two finite versions of the Cournot duopoly model. $P(Q) = a - Q$ is the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. The total cost to firm i of producing quantity q_i is cq_i, where $c < a$.

   (a) First, suppose each firm must choose either half the monopoly quantity, $q_m/2 = (a-c)/4$, or the Cournot equilibrium quantity, $q_c = (a - c)/3$. No other quantities are feasible. Show that this two-action game is equivalent to the Prisoner's Dilemma: each firm has a strictly dominated strategy, and both are worse off in equilibrium than they would be if they cooperated.

Solution: We can easily get the payoff matrix,

	$q_m/2$	q_c
$q_m/2$	$\frac{(a-c)^2}{8}, \frac{(a-c)^2}{8}$	$\frac{5(a-c)^2}{48}, \frac{5(a-c)^2}{36}$
q_c	$\frac{5(a-c)^2}{36}, \frac{5(a-c)^2}{48}$	$\frac{(a-c)^2}{9}, \frac{(a-c)^2}{9}$

Since $\frac{(a-c)^2}{8} < \frac{5(a-c)^2}{36}$ and $\frac{5(a-c)^2}{48} < \frac{(a-c)^2}{9}$, furthermore, $\frac{(a-c)^2}{8} > \frac{(a-c)^2}{9}$, thus this two-action game is equivalent to the Prisoner's Dilemma.

(b) Second, suppose each firm can choose either $q_m/2$, or q_c, or a third quantity, q'. Find a value for q' such that the game is equivalent to the Cournot model presented in class, in the sense that (q_c, q_c) is a unique Nash equilibrium and both firms are worse off in equilibrium than they could be if they cooperated, but neither firm has a strictly dominated strategy.

Solution: Recall from class that a strategy is strictly dominated if and only if it is a never-best response. Hence, for $q_m/2$ to not be strictly dominated it must be a best response to q'. In particular, we require that

$$\Pi_1(q_m/2, q') \geq \Pi_1(q', q').$$

For q' to not be strictly dominated we require that q' be a best response to either q_c, $q_m/2$ or q'. Since q_c is the unique best response to q_c, q' must be a best response to either q' or $q_m/2$. If q' is a best response to $q_m/2$ this would generate a second Nash equilibrium, since we also require that $q_m/2$ is a best response to q'. Hence q' must be a best response to q'. In particular, we require that

$$\Pi_1(q', q') \geq \Pi_1(q_m/2, q').$$

The two inequalities imply that

$$\Pi_1(q', q') = \Pi_1(q_m/2, q').$$

This implies that $q' = 3(a-c)/8$. However, $3(a-c)/8$ is a best response to $q_m/2$. Hence, no such q' can be found.

Note that in the original Cournot game there were many actions that were strictly dominated. If we allow q' itself to be strictly dominated then we only require that

- $\Pi_1(q_m/2, q') \geq \max\{\Pi_1(q', q'), \Pi_1(q_c, q')\}$ to ensure that $q_m/2$ is not strictly dominated by another strategy and;
- $\Pi_1(q', q_m/2) < \max\{\Pi_1(q_c, q_m/2), \Pi_1(q_m/2, q_m/2)\}$ to ensure that $(q', q_m/2)$ is not a Nash equilibrium.

The first requirement implies that $q' \geq 5(a-c)/12$ and $(0 \geq q' \leq (a-c)/4$ or $q' \geq 3(a-c)/8$. Hence, $q' \geq 5(a-c)/12$ The second requirement implies that $(q' > 5(a-c)/12$ or $q' < (a-c)/3)$ or $(q' > (a-c)/2$ or $q' < (a-c)/4)$. All these constraints together implies that any $q' > 5(a-c)/12$ will satisfy the constraints.

3. Consider the strategic game $G = \{N, (A_i), (u_i)\}$. For each $i \in N$, let A_i be a nonempty compact convex subset of Euclidean space and the utility function u_i be continuous and quasi-concave on A_i.
   (a) Prove that $B(a) = \times_{i \in N} B_i(a_{-i})$ is convex, where $B_i(a_{-i})$ is the best response function of player i, i.e. show that if $b \in B(a)$ and $c \in B(a)$ then for any $\lambda \in [0,1]$, $\lambda b + (1-\lambda)c \in B(a)$.

Solution: $B(a) = \times_{i \in N} B_i(a_{-i})$, where $B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a_i') \text{ for all } a_i' \in A_i\}$. Since u_i is quasi-concave on A_i, $b \in B(a)$ and $c \in B(a)$, then

$$u_i\left(\lambda b_i + (1-\lambda)c_i, a_{-i}\right) \geq \min\left\{u_i(b_i, a_{-i}), u_i(c_i, a_{-i})\right\} \geq u_i(a_i', a_{-i})$$

So for any $\lambda \in [0,1]$, $\lambda b + (1-\lambda)c \in B(a)$.

(b) Let A_i be finite for each $i \in N$. Prove that for each player i, the U_i associated with the mixed extension of G is quasi-concave over $\times_{j \in N} \Delta(A_j)$.

Solution: Since expected payoff is linear in the probabilities, then

$$U_i(\lambda\alpha + (1-\lambda)\beta) = \lambda U_i(\alpha) + (1-\lambda)U_i(\beta) \geq \min\{U_i(\alpha), U_i(\beta)\}$$

thus each player's payoff function in the mixed extension of G is quasi-concave.

4. Consider the following 2 player game.

	L	R
U	$6,6$	$2,7$
D	$7,2$	$0,0$

(a) Find the correlated equilibrium that maximizes the sum of the two players payoffs.

Solution: From proposition 47.1 in the textbook, we can restrict attention to equilibria where $\Omega = A$, and for each $i \in N$ information partitions that consist of all sets of the form $\{a \in A : a_i = b_i\}$. Let α be the probability of outcome (U, L), β be the probability of outcome (U, R), δ be the probability of outcome (D, L) and γ be the probability of outcome (D, R). By symmetry of the game we can restrict attention to correlated equilibria for which $\beta = \delta$. We need to make sure that each player has the incentive to follow their signal. This requires that when player 1 is required to play U he cannot do better by playing D. This requires that

$$6\alpha + 2\beta \geq 7\alpha + 0\beta.$$

Similarly, it requires that when player 1 is required to play D he cannot do better by playing U. This requires that

$$7\beta + 0\gamma \geq 6\beta + 2\gamma.$$

So, to find the correlated equilibrium that maximizes the sum of the two players payoffs, we need to solve the following linear programming problem:

$$\max_{\alpha,\beta} 6\alpha + 9\beta \text{ subject to}$$

$$\begin{aligned} 2\beta &\geq \alpha \\ \beta &\geq 2\gamma \\ 0 \leq \alpha &\leq 1 \\ 0 \leq \beta &\leq 1 \\ 0 \leq \gamma &\leq 1 \\ \alpha + \beta + \gamma &= 1 \end{aligned}$$

It is clear that $\gamma^* = 0$, and that the first constraint is binding, i.e. $2\beta = \alpha$. Hence, $\alpha^* = 1/2, \beta^* = 1/4, \gamma^* = 0$ and the payoffs to each player are 21/4.

(b) Construct a correlated equilibrium that generates a payoff for both players of $(19/4, 19/4)$.

Solution: See page 47 of the textbook or use proposition 46.2. To do so, we want to play the mixed strategy equilibrium, and the correlated equilibrium constructed in the last question to generate the appropriate payoffs.

	L	R
U	4/9	2/9
D	2/9	1/9

Mixed strategy distribution

	L	R
U	1/2	1/4
D	1/4	0

Maximal payoff distribution

Let λ be the probability of playing the mixed strategy equilibrium. Each player's payoff of the mixed strategy equilibrium are 14/3. Then to generate an equilibrium payoff of $(19/4, 19/4)$ we require that

$$\lambda \frac{14}{3} + (1-\lambda)\frac{21}{4} = 19/4 \Rightarrow \lambda = 6/7.$$

Then the distribution over outcomes generated by this constructed equilibrium is

	L	R
U	19/42	19/84
D	19/84	2/21

It is left as an exercise to check that this distribution is indeed a correlated equilibrium.

5. Consider the following three-player game where $A_1 = \{U, D\}$, $A_2 = \{L, R\}$ and $A_3 = \{M_1, M_2, M_3, M_4\}$. The number in each box represents the (equal) payoff to each player.

	L	R
U	3	0
D	0	0

M_1

	L	R
U	0	3
D	3	0

M_2

	L	R
U	0	0
D	0	3

M_3

	L	R
U	2	0
D	0	2

M_4

(a) What are the set of rationalizable strategies if players beliefs allow correlation between opponents strategies?

Solution: If we allow correlation, the actions of player 1 that are rationalizable are U and D; those of player 2 are L and R; those of player 3 are M_1, M_2, M_3, M_4. All but M_4 are actions that belong to some pure strategy equilibrium: (U, L, M_1), (D, R, M_3), (D, L, M_2) and (U, R, M_2). M_4 is rationalizable since the correlated actions U, L and D, R, with probability equal to 1/2, make M_4 a best response.

(b) What are the set of rationalizable strategies if players beliefs are restricted to be products of independent probability distributions over opponents strategies?

Solution: If we do not allow correlation, the actions of player 1 that are rationalizable are U and D; those of player 2 are L and R; those of player 3 are M_1, M_2, M_3. Again, all these actions are part of some Nash equilibrium.

For the action M_4 to be rationalizable there must exist values of p and q, such that the inequality $2pq + 2(1-p)(1-q) \geq \max\{3pq, 3(1-p)(1-q), 3p(1-q) + 3(1-p)q\}$ is satisfied. $(p, 1-p)$ and $(q, 1-q)$ represent the mixed strategies of player 1 and 2 respectively. These inequalities are symmetric in p and q, so we can let $p = q = x, 0 \leq x \leq 1$. Thus, the 3 inequalities and their implications are

$$\begin{aligned} x^2 - 4x + 2 \geq 0 &\Rightarrow 0 \leq x \leq 2 - \sqrt{2} \\ x^2 + 2x - 1 \geq 0, &\Rightarrow \sqrt{2} - 1 \leq x \leq 1 \\ 5x^2 - 5x + 1 \geq 0, &\Rightarrow 1 \leq x \leq \frac{5 - \sqrt{5}}{10} \text{ or } \frac{5 + \sqrt{5}}{10} \leq x \leq 1 \end{aligned}$$

Since there is no such value of x that satisfies these inequalities, M_4 is not rationalizable.

6. Show that if there is a unique profile of actions that survives iterated elimination of strictly dominated actions, this profile is a Nash equilibrium.

Solution: Proof 1: Suppose that iterated elimination of strictly dominated actions eliminates all but the actions $(a_1^*, \ldots, a_n^*)$ but these actions are not a Nash equilibrium. Then there must exist some player i and $a_i \in A_i$ such that

$$u_i(a_1^*, \ldots, a_{i-1}^*, a_i^*, a_{i+1}^*, \ldots, a_n^*) < u_i(a_1^*, \ldots, a_{i-1}^*, a_i, a_{i+1}^*, \ldots, a_n^*).$$

and there exists a_i' in the set of player i's strategies remaining at some stage of the process such that

$$u_i(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) < u_i(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)$$

for each $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ that can be constructed from the actions remaining in the other player's action spaces at that stage of the process. Since the other player's actions $(a_1^*, \ldots, a_{i-1}^*, a_{i+1}^*, \ldots, a_n^*)$ are never eliminated, we have

$$u_i(a_1^*, \ldots, a_{i-1}^*, a_i, a_{i+1}^*, \ldots, a_n^*) < u_i(a_1^*, \ldots, a_{i-1}^*, a_i', a_{i+1}^*, \ldots, a_n^*)$$

If $a_i' = a_i^*$ then there exists a contradiction. If $a_i' \neq a_i^*$ then some other action a_i'' must later strictly a_i', since a_i' does not survive the process. Thus, inequalities analogous to the above inequalities hold with a_i' and a_i'' replacing a_i and a_i', respectively. Once again, if $a_i'' = a_i^*$ then the proof is complete; otherwise, two more analogous inequalities can be constructed. Since a_i^* is the only action from A_i to survive the process, repeating this argument in a finite game eventually completes the proof. (**from Gibbons's book**)

Proof 2: Let a^* be the unique outcome that survives iterated elimination of strictly dominated strategies. Since X^T is a singleton it is clear that a^* is a Nash equilibrium of $\{N, (X_i^T), (u_i^T)\}$. Now assume that a^* is a Nash equilibrium of $\{N, (X_i^{t+1}), (u_i^{t+1})\}$ where $0 \leq t < T$, i.e. $u_i^{t+1}(a_i^*, a_{-i}) < u_i^{t+1}(a_i, a_{-i}^*)$ for all $a_i \in X_i^{t+1}$. Let $a_i' \in X_i \backslash X_i^{t+1}$ for some $i \in N$. Since $a_i' \in X_i \backslash X_i^{t+1}$, a_i is strictly dominated in $\{N, (X_i^t), (u_i^t)\}$. Hence, a_i is a never-best response in $\{N, (X_i^t), (u_i^t)\}$. Hence, $u_i^t(a_i', a_{-i}) < u_i^t(a^*)$. Since a_i' was chosen arbitrarily it must be that $u_i^t(a_i^*, a_{-i}) \geq u_i^t(a_i, a_{-i}^*)$ for all $a_i \in X_i^t$, i.e. a_i^* is a best response to a_{-i}^* in $\{N, (X_i^t), (u_i^t)\}$. Hence, a^* is a Nash equilibrium for $\{N, (X_i^0), (u_i^0)\} = \{N, (A_i), (u_i)\}$.