Chapter 3 Brownian Motion

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Outline

Symmetric Random Walk

- Denote ω the infinite sequence of tosses, and ω_n is the outcome of the n^{th} toss.
- Let

$$X_j = \begin{cases} 1 & \text{if } \omega_j = H \\ -1 & \text{if } \omega_j = T \end{cases}$$

- Define $M_0 = 0$,

$$M_k = \sum_{j=1}^{k} X_j, k = 1, 2, \ldots$$

The process $M_k, k = 0, 1, 2, \ldots$ is a *symmetric random walk.*

Increments of the Symmetric Random Walk

- A random walk has *independent increments*.

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

- Each increment $M_{k_{i+1}} - M_{k_i}$, has expected value 0 and variance $k_{i+1} - k_i$.

$$\mathbb{V}ar\left[M_{k_{i+1}} - M_{k_i}\right] = \sum_{j=k_i+1}^{k_{i+1}} \mathbb{V}ar\left[X_j\right] = \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i.$$

Martingale Property for the Symmetric Random Walk

- The symmetric random walk is a martingale. For $k < l$,

$$\begin{aligned}
\mathbb{E}[M_l | \mathscr{F}_k] &= \mathbb{E}[(M_l - M_k) + M_k | \mathscr{F}_k] \\
&= \mathbb{E}[M_l - M_k | \mathscr{F}_k] + \mathbb{E}[M_k | \mathscr{F}_k] \\
&= \mathbb{E}[M_l - M_k | \mathscr{F}_k] + M_k \\
&= \mathbb{E}[M_l - M_k] + M_k = M_k
\end{aligned}$$

Quadratic Variation of the Symmetric Random Walk

- The quadratic variation up to time k is defined to be

$$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k.$$

- $[M, M]_k$ is computed along a single path, and the probabilities of up and down steps do not enter the computation.

Scaled Symmetric Random Walk

- The scaled symmetric random walk

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

- The scaled random walk has independent increments.
- Let $0 \le s \le t$ be given, and decompose $W^{(n)}(t)$ as

$$W^{(n)}(t) = \left(W^{(n)}(t) - W^{(n)}(s)\right) + W^{(n)}(s).$$

Scaled Symmetric Random Walk

- The martingale property

$$\mathbb{E}\left[W^{(n)}(t) \middle| \mathscr{F}(s) \right] = W^{(n)}(s)$$

- The quadratic variation

$$\begin{aligned}
\left[W^{(n)}, W^{(n)} \right](t) &= \sum_{j=1}^{nt} \left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right) \right]^2 \\
&= \sum_{j=1}^{nt} \left[\frac{1}{\sqrt{n}} X_j \right]^2 = \sum_{j=1}^{nt} \frac{1}{n} = t
\end{aligned}$$

Limiting Distribution of the Scaled Random Walk

Theorem 3.2.1 (Central Limit)

Fix $t \geq 0$. As $n \to \infty$, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t.

- Outline of Proof

$$\varphi_n(u) = \mathbb{E}\left[e^{uW^{(n)}(t)}\right] = \left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)^{nt}$$

$$\lim_{n\to\infty} \log \varphi_n(u) = t \lim_{x\downarrow 0} \frac{\log\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x^2} = \frac{1}{2}u^2 t$$

Log-Normal Distribution as the Limit of the Binomial Model

- Take the up factor to be $u_n = 1 + \frac{\sigma}{\sqrt{n}}$ and the down factor to be $d_n = 1 - \frac{\sigma}{\sqrt{n}}$.
- The risk-neutral probabilities $\tilde{p} = \tilde{q} = \frac{1}{2}$ with $r = 0$.
- The random walk M_{nt} is the number of heads minus the number of tails in these nt coin tosses:

$$M_{nt} = H_{nt} - T_{nt}.$$

- The stock price at time t is

$$\begin{aligned} S_n(t) &= S(0)\, u_n^{H_{nt}} d_n^{T_{nt}} \qquad (1) \\ &= S(0) \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt + M_{nt})} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt - M_{nt})} \end{aligned}$$

Log-Normal Distribution as the Limit of the Binomial Model

Theorem 3.2.2

As $n \to \infty$, the distribution of $S_n(t)$ in (1) converges to the distribution of

$$S(t) = S(0) \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\},$$

where $W(t)$ is a normal random variable with mean zero and variance t.

Log-Normal Distribution as the Limit of the Binomial Model

- Sketch of Proof

$$\log S(t) = \log S(0) + \frac{1}{2}(nt + M_{nt})\left(\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n} + O\left(n^{-\frac{3}{2}}\right)\right)$$
$$+ \frac{1}{2}(nt - M_{nt})\left(-\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n} + O\left(n^{-\frac{3}{2}}\right)\right)$$

Definition of Brownian Motion

Definition 3.3.1

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $W(t)$, $t \geq 0$, is a Brownian motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1}),$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0, \tag{2}$$

$$\mathbb{V}ar[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i. \tag{3}$$

Distribution of Brownian Motion

Theorem 3.3.2 (Alternative characterizations of Brownian motion)

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. The following three properties are equivalent.

(i) For all $0 = t_0 < t_1 < \cdots < t_m$, the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with mean and variance given by (2) and (3).

Distribution of Brownian Motion

Theorem 3.3.2 (Alternative characterizations of Brownian motion)

(ii) For all $0 = t_0 < t_1 < \cdots < t_m$, the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed with means equal to zero and covariance matrix

$$\begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}.$$

Distribution of Brownian Motion

Theorem 3.3.2 (Alternative characterizations of Brownian motion)

(iii) For all $0 = t_0 < t_1 < \ldots < t_m$, the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ have the joint moment-generating function

$$
\begin{aligned}
&\varphi(u_1, u_2, \ldots, u_m) \\
=&\mathbb{E}\left[\exp\left\{u_m W(t_m) + u_{m-1} W(t_{m-1}) + \cdots + u_1 W(t_1)\right\}\right] \\
=&\exp\left\{\frac{1}{2}(u_1 + u_2 + \cdots + u_m)^2 t_1 + \cdots + \frac{1}{2} u_m^2 (t_m - t_{m-1})\right\}.
\end{aligned}
$$

If any of (i), (ii), or (iii) holds (and hence they all hold), then $W(t)$, $t \geq 0$, is a Brownian motion.

Filtration for Brownian Motion

Definition 3.3.3

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t)$, $t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathscr{F}(t)$, $t \geq 0$, satisfying:

(i) (Information accumulates) For $0 \leq s < t$, every set in $\mathscr{F}(s)$ is also in $\mathscr{F}(t)$. In other words, there is at least as much information available at the later time $\mathscr{F}(t)$ as there is at the earlier time $\mathscr{F}(s)$.

Filtration for Brownian Motion

Definition 3.3.3

(ii) (Adaptivity) For each $t \geq 0$, the Brownian motion $W(t)$ at time t is $\mathscr{F}(t)$-measurable. In other words, the information available at time t is sufficient to evaluate the Brownian motion $W(t)$ at that time.

Filtration for Brownian Motion

Definition 3.3.3

(iii) (Independence of future increments) For $0 \leq t < u$, the increment $W(u) - W(t)$ is independent of $\mathscr{F}(t)$. In other words, any increment of the Brownian motion after time t is independent of the information available at time t.

Let $\Delta(t)$, $t \geq 0$, be a stochastic process. We say that $\Delta(t)$ is adapted to the filtration $\mathscr{F}(t)$ if for each $t \geq 0$ the random variable $\Delta(t)$ is $\mathscr{F}(t)$-measurable.

Martingale Property for Brownian Motion

Theorem 3.3.4

Brownian motion is a martingale.

First-Order Variation

- The first-order variation

$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} \left| f'\left(t_j^*\right) \right| \left(t_{j+1} - t_j\right) = \int_0^T \left| f'(t) \right| dt$$

Quadratic Variation

Definition 3.4.1

Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variation of f up to time T is

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1})) - f(t_j)]^2,$$

where $\Pi = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$.

Quadratic Variation

Theorem 3.4.3

Let W be a Brownian motion. Then $[W, W](T) = T$ for all $T \geq 0$ almost surely.

- Proof

$$Q_\Pi = \sum_{j=0}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2$$

$$\lim_{\|\Pi\| \to 0} \mathbb{V}ar\left[Q_\Pi\right] = 0 \Rightarrow \lim_{\|\Pi\| \to 0} Q_\Pi = \mathbb{E}\left[Q_\Pi\right] = T$$

Quadratic Variation

- Write informally

$$dW(t)\,dW(t) = dt.$$

- *Brownian motion accumulates quadratic variation at rate one per unit time.*
- Similarly,

$$dW(t)\,dt = 0, dtdt = 0.$$

Volatility of Geometric Brownian Motion

- Let α and $\sigma > 0$ be constants, and define the *geometric Brownian motion*

$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}.$$

- The realized volatility

$$\frac{1}{T_2 - T_1}\sum_{j=0}^{m-1}\left(\log\frac{S(t_{j+1})}{S(t_j)}\right)^2 \approx \sigma^2.$$

Markov Property

Theorem 3.5.1

Let $W(t)$, $t \geq 0$ be a Brownian motion and let $\mathscr{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion (see Definition 3.3.3). Then $W(t)$, $t \geq 0$ is a Markov process.

- Proof

$$\mathbb{E}[f(W(t))||\mathscr{F}(s)] = \mathbb{E}[f(W(t) - W(s) + W(s))||\mathscr{F}(s)]$$

First Passage Time Distribution

Theorem 3.6.1

Let $W(t)$, $t \geq 0$, be a Brownian motion with a filtmtion $\mathscr{F}(t)$, $t \geq 0$, and let σ be a constant. The process $Z(t)$, $t \geq 0$, of

$$Z(t) = \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\},$$

is a martingale.

First Passage Time Distribution

- Let m be a real number, and define the *first passage time* to level m

$$\tau_m = \min\{t \geq 0; W(t) = m\}.$$

Theorem 3.6.2

For $m \in \mathbb{R}$, the first passage time of Brownian motion to level m is finite almost surely, and the Laplace transform of its distribution is given by

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{-|m|\sqrt{2\alpha}} \text{ for all } \alpha > 0.$$

Reflection Equality

- The reflection equality

$$\mathbb{P}\{\tau_m \leq t, W(t) \leq w\} = \mathbb{P}\{W(t) \geq 2m - w\}, w \leq m, m > 0.$$

First Passage Time Distribution

Theorem 3.7.1

For all $m \neq 0$, the random variable τ_m has cumulative distribution function

$$\mathbb{P}\{\tau_m \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy, t \geq 0,$$

and density

$$f_{\tau_m}(t) = \frac{d}{dt}\mathbb{P}\{\tau_m \leq t\} = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, t \geq 0.$$

Distribution of Brownian Motion and Its Maximum

- Define the maximum to date for Brownian motion to be

$$M(t) = \max_{0 \le s \le t} W(s).$$

Distribution of Brownian Motion and Its Maximum

Theorem 3.7.3

For $t > 0$, the joint density of $(M(t), W(t))$ is

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, w \leq m, m > 0.$$

Corollary 3.7.4

The conditional distribution of $M(t)$ given $W(t) = w$ is

$$f_{M(t)|W(t)}(m|w) = \frac{2(2m - w)}{t} e^{-\frac{2m(m-w)}{t}}, w \leq m, m > 0.$$

Distribution of Brownian Motion and Its Maximum

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