Advanced Microeconomics II
Problem Set 1

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1. Two Xiamen teenagers Wang and Li are playing Chicken. Wang drives his motorbike south down a one-lane road, and Li drives his motorbike north along the same road. Each has two strategies: Stay or Swerve. If one player chooses Swerve he looses face; if both Swerve, they both lose face. However, if both choose Stay, they are both killed. The payoff matrix for Chicken looks like this:

		Li	
		Stay	*Swerve*
Wang	*Stay*	$-3, -3$	$2, 0$
	Swerve	$0, 2$	$1, 1$

   (a) Find all pure strategy equilibria.

   Solution: The pure strategy equilibria are: (*Stay*,*Swerve*) and (*Swerve*,*Stay*).

   (b) Find all mixed strategy equilibria.

   Solution: The mixed strategy equilibrium is

$$\alpha_1(Stay) = \alpha_2(Stay) = 1/4, \alpha_1(Swerve) = \alpha_2(Swerve) = 3/4.$$

   (c) What is the probability that both teenagers will survive?

   Solution: Under the pure strategy equilibria, the probability of survival is 100%. Under the mixed-strategy equilibrium the probability of survivial is $1 - (1/4)^2 = 15/16$.

2. There are two firms, each with zero marginal costs and fixed costs of k, who compete over price. There are two types of consumers, informed consumers know the lowest price being charged and uninformed consumers simply choose a store at random. Suppose that there are I informed consumers and $2U$ uninformed consumers. Hence, each store will get U uninformed consumers each period for certain and will get the informed consumers only if they happen to have the lowest price. They evenly divide the market if they charge the same price. The reservation price of each consumer is r.

   (a) Formulate this as a strategic game.

   Solution: This strategic game consists of:

   - $N = \{1, 2\}$

- For each player $i \in N$, A_i is the interval $[0, \infty)$
- For each player $i \in N$, $j \in N$,and $j \neq i$:

$$U_i(p) = \begin{cases} -k & \text{if } r < p_i \\ Up_i - k & \text{if } p_j < p_i \leq r \\ (U + I/2)p_i - k & \text{if } p_i = p_j \leq r \\ (U + I)p_i - k & \text{if } p_i < \min\{p_j, r\}. \end{cases}$$

(b) Find the symmetric mixed-strategy Nash equilibrium of this game.

Solution: Let $F(p)$ be the cumulative distribution function of the equilibrium strategy. and $f(p)$ be the associated probability density function which we will assume is a continuous density function. (It can be shown that the probability of a tie in equilibrium is zero). Then the profit of player 1 when he charges the price p_1 is

$$\begin{aligned} \pi(p_1) &= \int_0^{p_1} (Up_1 - k) f_2(p_2) dp_2 + \int_{p_1}^{\infty} ((U + I)p_1 - k) f_2(p_2) dp_2 \\ &= p_1(U + (1 - F(p_1)I) - k. \end{aligned}$$

Recall that for a mixed-strategy equilibrium, every action in the support of the player's pure strategy gives the same payoff to that player. Hence,

$$p_1(U + (1 - F(p_1)I) - k = \bar{\pi}$$

Solving for $F(p_1)$ gives

$$F(p_1) = \frac{p_1(I + U) - k - \bar{\pi}}{p_1 I}.$$

It remains to determine $\bar{\pi}$. The upper limit of the support of $F(p)$ cannot be greater than r. It can never be optimal for a firm to charge a price above r since the firm would generate zero profit and could do better by charging a price r and obtaining profit equal to Ur. Similarly the upper limit of the support of $F(p)$ cannot be less than r. Charging an upper limit price $\bar{p} < r$ generates a profit $U\bar{p} < Ur$. Hence, $F(r) = 1$ and $\bar{\pi} = rU - k$ implying

$$F(p) = \frac{p(I + U) - rU}{pI}.$$

Note that the lower limit of the support of $F(p)$, $\underline{p} = rU/(I + U)$.

(c) As the ratio of uninformed consumers in the market increases what happens to the distribution of firm prices? Interpret this result.

Solution: Denote $u = U/I$. Then we can rewrite $F(p)$ as

$$F(p) = \begin{cases} 0 & \text{if } p \leq ru/(1+u) \\ (1+u) - \frac{ru}{p} & \text{if } ru/(1+u) < p \leq r \\ 1 & \text{if } r < p. \end{cases}$$

Note that if $u_1 > u_2$ then $F(u_1)$ first-order stochastically dominates $F(u_2)$. As an extreme case, when all the consumers are uninformed, the mixed-strategy Nash equilibrium will degenerate to (r, r). One cost of competing for informed consumers is the lost profit from charging a lower price to the uninformed consumers (instead of their reservation value r). It becomes more costly to compete for these informed consumers as there are relatively less of them in the market. As a result, the firms compete less aggresively for these consumers. Equilibrium prices (and profits) increase (in a stochastic sense).

3. Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows:

   (i) any outcome in which the good is provided and she does not contribute,

   (ii) any outcome in which the good is provided and she contributes,

   (iii) any outcome in which the good is not provided and she does not contribute,

   (iv) any outcome in which the good is not provided and she contributes.

   (a) Formulate this situation as a strategic game.

   Solution: This strategic game consists of:

   - $N = \{1, 2, \ldots, n\}$
   - For each player $i \in N$,A_i={0,1}
   - For each player $i \in N$,the preference $\succsim_i$ on A is defined as:

$$\left\{a : \sum_1^n a_j \geq k, a_i = 0\right\} \succsim_i \left\{a : \sum_1^n a_j \geq k, a_i = 1\right\}$$
$$\succsim_i \left\{a : \sum_1^n a_j < k, a_i = 0\right\} \succsim_i \left\{a : \sum_1^n a_j < k, a_i = 1\right\}.$$

   (b) Find its Nash equilibria.

Solution: There exist two kinds of NE:

- $\{a : \sum_1^n a_j = k\}$
- $\{a : \sum_1^n a_j = 0\}$

We can get the results by the following two propositions:
Proposition 1. *If a is a NE,then $\sum_1^n a_j \le k$.*

Proof. Suppose a is a NE,in which $\sum_1^n a_j > k$ and a_i=1.Then the best response of player i given others' actions will be $a_i = 0$. □

Proposition 2. *If a is a NE,then $\sum_1^n a_j$ can't be any number larger than 0 and smaller than k.*

Proof. Suppose a is a NE,in which $\sum_1^n a_j = s$ and $0 < s < k$ and a_i=1.Then the best response of player i given others' actions will be $a_i = 0$. □

Do there exist additional mixed-strategy Nash equilibria? To establish a mixed-strategy Nash equilibrium we need to define preferences over the set of lotteries over outcomes induced by player's mixed strategies. For simplicity we will assign values $U_1 > U_2 > U_3 > U_4$ to the appropriate outcomes so as to reflect the above preferences. Note that the particular values we assign are important since expected utility is no longer an ordinal concept—differences in utility matter.

For there to be additional mixed-strategy Nash Equilibria, a player needs to be indifferent between his two strategies. Consider the payoff for each action:

$$U_i(0, a_{-i}) = U_3 \Pr(\sum_{j \ne i} a_j \le k-1) + U_1(1 - \Pr(\sum_{j \ne i} a_j \le k-1)),$$
$$U_i(1, a_{-i}) = U_4 \Pr(\sum_{j \ne i} a_j \le k-2) + U_2(1 - \Pr(\sum_{j \ne i} a_j \le k-2))$$

If we can equate these payoffs, then additional mixed-strategy Nash equilibria exist. First let us consider a symmetric mixed-strategy equilibrium where each player contributes with probability α. Then for any integer x, $\Pr(\sum_{j \ne i} a_j = x)$ has a binomial distribution where the number of trials is n, the number of successes is x and the probability of success is α. Hence,

$$\Pr(\sum_{j \ne i} a_j \le x) = \sum_{r=0}^{x} \binom{n-1}{r} \alpha^r (1-\alpha)^{n-1-r}.$$

Whether or not additional mixed-strategy equilibria exist depends on the utilities chosen for each outcome. Consider, for example, when $n = k$. Then

$$U_i(0, \alpha) = U_3 \Pr(\sum_{j \ne i} a_j \le n-1) + U_1(1 - \Pr(\sum_{j \ne i} a_j \le n-1)) = U_3,$$
$$U_i(1, \alpha) = U_4 \Pr(\sum_{j \ne i} a_j \le n-2) + U_2(1 - \Pr(\sum_{j \ne i} a_j \le n-2)) = (1-\alpha^{n-1})U_4 + \alpha^{n-1}U_2.$$

These are equal if $\alpha = \left(\frac{U_3-U_4}{U_2-U_4}\right)^{\frac{1}{n-1}}$. So, when we require unanimity, an additional mixed-strategy Nash equilibrium always exists.

Now consider when $n = 3, k = 2$ and $U_4 = 0$. Then

$$U_i(0, \alpha) = U_3(1 - \alpha^2) + U_1\alpha^2; \quad U_i(1, \alpha) = U_2(1 - (1 - \alpha)^2).$$

These utilties are equal when

$$\alpha = \frac{U_2 \pm \sqrt{U_2(U_2 - U_3) - U_3(U_1 - U_3)}}{U_1 + U_2 - U_3}.$$

Clearly, if U_1 is sufficiently large then no such α can be found, while if $U_1 = 3$, $U_2 = 2$ and $U_3 = 1$, then $\alpha = 1/2$ is the strategy of each player in the symmetric mixed-strategy Nash equilibrium. Note that for the right choice of parameters there can be multiple non-degenerate mixed-strategy Nash equilibria.

You may find it useful to understand what about my "proof" in class was wrong.

4. Consider the strategic game $G = \{N, (A_i), (u_i)\}$. For each $i \in N$, let A_i be a nonempty compact convex subset of Euclidean space and the utility function u_i be continuous and quasi-concave on A_i.

   (a) Prove that $B(a) = \times_{i\in N} B_i(a_{-i})$ is convex, where $B_i(a_{-i})$ is the best response function of player i, i.e. show that if $b \in B(a)$ and $c \in B(a)$ then for any $\lambda \in [0, 1]$, $\lambda b+(1-\lambda)c \in B(a)$.

Solution: $B(a) = \times_{i\in N} B_i(a_{-i})$, where

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succeq_i (a_{-i}, a'_i) \text{ for all } a'_i \in A_i\}.$$

Since u_i is quasi-concave on A_i, $b \in B(a)$ and $c \in B(a)$, then

$$u_i\left(\lambda b_i + (1 - \lambda)c_i, a_{-i}\right) \geq \min\left\{u_i(b_i, a_{-i}), u_i(c_i, a_{-i})\right\} \geq u_i(a'_i, a_{-i})$$

So for any $\lambda \in [0, 1]$, $\lambda b + (1 - \lambda)c \in B(a)$.

   (b) Let A_i be finite for each $i \in N$. Prove that for each player i, the U_i associated with the mixed extension of G is quasi-concave over $\times_{j\in N}\Delta(A_j)$.

Solution: The statement is false. It is easy to construct a counter-example. Consider the following two-player game:

	A	B
A	$1, 1$	$-1, -1$
B	$-1, -1$	$1, 1$

Let α be the strategy where both players play A. Let β be the strategy where both players play B. The utility for both players from both strategies is 1. Let $\lambda = 1/2$.

Then, the strategy $\gamma = \lambda\alpha + (1-\lambda)\beta$ is the strategy where both players play A and B with equal probability. The payoff from such a strategy is 0.

5. Two people can perform a task if, and only if, they both exert effort. They are both better off if they both exert effort and perform the task than if neither exerts effort (and nothing is accomplished); the worst outcome for each person is that she exerts effort and the other does not (in which case again nothing is accomplished). Specifically, the players' preferences are represented by the expected value of the payoff functions in the following table, where c is a positive number less than 1 that can be interpreted as the cost of exerting effort.

	No effort	*Effort*
No effort	$0, 0$	$0, -c$
Effort	$-c, 0$	$1-c, 1-c$

(a) Find all the mixed strategy Nash equilibria of this game.

Solution: There are three Nash equilibria:

- Pure Strategy NE: (*No effort*,*No effort*),(*Effort*,*Effort*)
- Mixed Strategy NE:

$$\alpha_1(\textit{No effort}) = \alpha_2(\textit{No effort}) = 1 - c, \quad \alpha_2(\textit{Effort}) = \alpha_2(\textit{Effort}) = c.$$

(b) How do the equilibria change as c increases? Explain the reasons for the changes.

Solution: Under the pure strategy equilibria there is no effect of increasing c other than to reduce payoffs in the *Effort* equilibrium. Under the mixed-strategy equilibrium, as c increases, both players increase the probability of effort. In order to remain indifferent between the two actions as the cost of effort increases it is necessary to co-ordinate more often on the Pareto optimal equilibrium.