Advanced Microeconomics II
Problem Set 1

WISE, Xiamen University

Spring 2010

Due 10:00 Mar 10, 2011

1. (Varian 17.4) There are two consumers A and B with the following utility functions and endowments:

$$u_A(x_A^1, x_A^2) = a \ln x_A^1 + (1-a) \ln x_A^2 \quad \omega_A = (0, 1)$$
$$u_B(x_B^1, x_B^2) = \min\{x_B^1, x_B^2\} \quad \omega_B = (1, 0).$$

Calculate the strongly Pareto efficient set and the Walrasian equilibrium set.

Solution: Since $\frac{\partial u_A}{\partial x_A^1} = a\frac{1}{x_A^1} > 0$ and $\frac{\partial u_A}{\partial x_A^2} = (1-a)\frac{1}{x_A^2} > 0$, then if $x_B^1 \neq x_B^2$, we can make A better off without making B worse off. W.l.o.g, let $x_B^1 > x_B^2$, we can give $x_B^1 - x_B^2$ units of good 1 to A, then $A's$ utility is increased and $B's$ utility does not changed. So the strongly Pareto efficient set is $\left\{x_A^1, x_A^2, x_B^1, x_B^2 \middle| x_A^1 = x_A^2 = 1 - x_B^1 = 1 - x_B^2\right\}$.

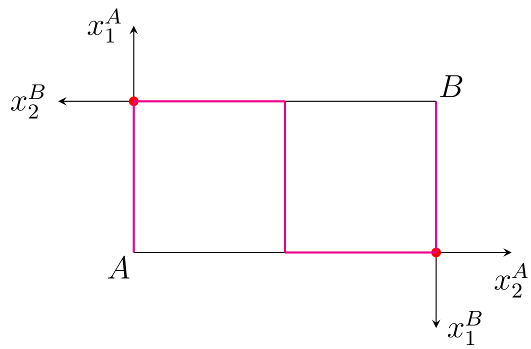
Let $p_1 = 1$. From class we know that $x_A^1(p) = ap$ and $x_B^1(p) = 1/(1+p)$. Hence, equilibrium is where

$$\begin{aligned}
0 =& z_1(p) - 1 \\
=& ap + \frac{1}{1+p} - 1 \\
=& \frac{ap + ap^2 - p}{1+p} \\
=& (ap + a - 1)\frac{p}{1+p}.
\end{aligned}$$

Hence $p = (1-a)/a$, $x_A^1 = x_A^2 = 1-a$, $x_B^1 = x_B^2 = a\}$.

2. (Varian 17.8) Suppose we have two consumers A and B with identical utility functions $u_A(x_1, x_2) = u_B(x_1, x_2) = \max\{x_1, x_2\}$. There are 1 unit of good 1 and 2 units of good 2. Draw an Edgeworth box that illustrates the strongly Pareto efficient and the (weakly) Pareto efficient sets.

Solution: According to the consumers' utility functions and the total units of goods, $\max u_A =$ 1 or 2, $\max u_B = 1$ or 2. Consumers do not need to consider the minimal good. Thus the strongly Pareto efficient set consists of 2 allocations (red circles): A gets all of good 1 and B gets all of good 2, the other Pareto efficient allocation is exactly the reverse of this. The weakly Pareto efficient set consists of three types allocation (magenta lines): first, one of the consumers has 1 unit of good 1 and the other consumer has at least 1 unit of good 2; second, one of the consumers has 2 unit of good 2 and the other consumer has any unit of good 1; third, both consumers have 1 unit of good 2.



3. (Varian 17.10) If we allow for the possibility of satiation, the consumer's budget constraint takes the form $px_i \leq p\omega_i$. Walras' law then becomes $pz(p) \leq 0$ for all $p \geq 0$. Show that the proof of existence of a Walrasian equilibrium in an exchange economy still applies for this generalized form of Walras' law.

Solution: Define the mapping $G : S^{k-1} \to S^{k-1}$ by

$$g_i(p) = \frac{p_i + \max\{0, z_i(p)\}}{1 + \sum_{j=1}^{k} \max\{0, z_j(p)\}} \text{ for } i = 1, \ldots, k.$$

Notice that this map is continuous since z and the max function are continuous functions. Furthermore, $g(p)$ is a point in the simplex S^{k-1} since $\sum_{i=1}^{k} g_i(p) = 1$. Then by the Brouwer's fixed-point theorem there is a p^* such that $p^* = g(p^*)$, i.e.,

$$p_i^* = \frac{p_i^* + \max\{0, z_i(p^*)\}}{1 + \sum_{j=1}^{k} \max\{0, z_j(p^*)\}} \text{ for } i = 1, \ldots, k.$$

We will show that p^* is a Walrasian equilibrium. Cross-multiply the above equation and rearrange to get

$$p_i^* \sum_{j=1}^{k} \max\{0, z_j(p^*)\} = \max\{0, z_i(p^*)\} \text{ for } i = 1, \ldots, k.$$

Now multiply each of these k equations by $z_i(p^*)$:

$$z_i(p^*) p_i^* \sum_{j=1}^{k} \max\{0, z_j(p^*)\} = z_i(p^*) \max\{0, z_i(p^*)\} \text{ for } i = 1, \ldots, k.$$

Summing these k equations gives

$$\left[\sum_{j=1}^{k} \max\{0, z_j(p^*)\} \right] \sum_{i=1}^{k} z_i(p^*) p_i^* = \sum_{i=1}^{k} z_i(p^*) \max\{0, z_i(p^*)\}.$$

Since $pz(p) \leq 0$ and $\sum_{j=1}^{k} \max\{0, z_j(p^*)\} \geq 0$, then $\sum_{i=1}^{k} z_i(p^*) \max\{0, z_i(p^*)\} \leq 0$, hence $z_i(p^*) \leq 0$ for $i = 1, \ldots, k$.

4. (Varian 18.2) Consider an economy with two firms and two consumers. Firm 1 is entirely owned by consumer 1. It produces guns from oil via the production function $g = 2x$. Firm 2 is entirely owned by consumer 2; it produces butter from oil via the production function $b = 3x$. Each consumer owns 10 units of oil. Consumer 1's utility function is $u(g, b) = g^{0.4} b^{0.6}$ and consumer 2's utility function is $u(g, b) = 10 + 0.5 \ln g + 0.5 \ln b$.

   (a) Find the market clearing prices for guns, butter, and oil.

Solution: Let p_g, p_b and p_o be the price of guns, butter and oil respectively, let $p_o = 1$. Then the zero-profit condition implies that $p_g 2x - x = 0$ and $p_b 3x - x = 0$. These mean that $p_g = \frac{1}{2}$ and $p_b = \frac{1}{3}$.

(b) How many guns and how much butter does each consumer consume?

Solution: Since $u_1(g, b) = g^{0.4}b^{0.6}$ and $u_2(g, b) = 10 + 0.5 \ln g + 0.5 \ln b$, we know both utility functions are Cobb-Douglas. Moreover, both consumers have an same endowment worth 10 ($p_o = 1$). From this we can easily calculate that $x_1^g = \frac{0.4\times 10}{1/2} = 8$, similarly, we get $x_1^b = 18, x_2^g = 10$ and $x_2^b = 15$.

(c) How much oil does each firm use?

Solution: To make 18 guns, firm 1 needs 9 barrels of oil. To make 33 butter, firm 2 needs 11 barrels of oil.

5. Complete the proof of the second welfare theorem with production.

Solution: Let

$$P_i = \{x_i \in R^k : x_i \succ_i x_i^*\}.$$

The define

$$P = \sum_{i=1}^{n} P_i = \{z : z = \sum_{i=1}^{n} x_i \text{ with } x_i \in P_i\}.$$

Since each P_i is a convex set by hypothesis, it follows that P is a convex set. Let F be the set of all feasible aggregate bundles, that is,

$$F = \{\omega + \sum_{j=1}^{m} y_j : y_j \text{ is in } Y_j\}.$$

Then P and F are both convex sets, and, since (x^*, y^*) is Pareto efficient, P and F are disjoint. By the separating hyperplane theorem there exists a $p \neq 0$ and scalar r such that

$$pz \geq r \geq pz' \text{ for all } z \in P, \text{ for all } z' \in F$$

By continuity, for each i there exists x_i' arbitrarily close to x_i such that $x_i' \succ x_i^*$, i.e. $x_i' \in P_i$. Hence, $\sum_{i=1}^{n} x_i' \in P$ and $p\sum_{i=1}^{n} x_i' \geq r$. Taking limits as x_i' converges to x_i establishes that $p\sum_{i=1}^{n} x_i^* \geq r$.

Also, since $\sum_{i=1}^{n} x_i^* = \omega + \sum_{j=1}^{m} y_j^* \in F$ then p$\sum_{i=1}^{n} x_i^* = p(\omega + \sum_{j=1}^{m} y_j^*) = r$.

Hence,

$$pz \geq p(\omega + \sum_{j=1}^{m} y_j^*) = p\sum_{i=1}^{n} x_i^*$$

Let $e_i = (0, \ldots, 1, \ldots, 0)$ with a 1 in the i^{th} component. Since preferences are strongly monotonic, $\omega + \sum_{j=1}^{m} y_j + e_i \in P$. Since if we have one more unit of any good, it is possible to redistribute it to make everyone better off. Thus we have

$$p\left(\omega + \sum_{j=1}^{m} y_j^* + e_i - \omega - \sum_{j=1}^{m} y_j^*\right) \geq 0 \text{ for } i = 1, \ldots, k.$$

Then,

$$pe_i \geq 0 \text{ for } i = 1, \ldots, k.$$

So p is nonnegative.

Solution: CONTINUED

Suppose that $x_j \succ_j x_j^*$. Let

$$z_j = (1-\theta)x_j, z_i = x_i^* + \frac{\theta x_j}{n-1}, i \neq j.$$

For sufficiently small θ, z is Pareto preferred to x^*, so that $\sum_{i=1}^n z_i \in P$. Thus

$$p\sum_{i=1}^n z_i \geq p\sum_{i=1}^n x_i^* \text{ for } i = 1, \ldots, k.$$

Then,

$$p\left((1-\theta)x_j + \sum_{i=1, i\neq j}^n x_i^* + \theta x_j\right) \geq p\left(x_j^* + \sum_{i=1, i\neq j}^n x_i^*\right) \text{ for } i = 1, \ldots, k$$

So $px_j \geq px_j^*$.

Assume that $px_j = px_j^*$. There exists some θ close to 1 such that θx_j is strictly preferred to x_j^*. By repeating the above argument, we know that

$$p\theta x_j \geq px_j^*.$$

By assumption every component of $x_j^* > 0$. Hence, $px_j^* > 0$. If $px_j = px_j^*$, then $\theta px_j < px_j^*$. This is a contradiction. Thus if $x_i' \succ_i x^*$, $px_i' > p_x^*$ for $i = 1, \ldots, n$.

Suppose y_j' is in Y_j and $py_j' > py_j^*$. Let

$$y_j'' = y_j', y_i'' = y_i^*, i \neq j.$$

Thus, $\omega + \sum_{j=1}^m y_j'' \in F$ and

$$p(\omega + \sum_{j=1}^m y_j'') \leq r = p(\omega + \sum_{j=1}^m y_j^*) = r < p(\omega + \sum_{j=1}^m y_j'')$$

This is a contradiction. Thus if y_j' is in Y_j, $py_j^* \geq py_j'$ for $j = 1, \ldots, m$.

6. In a "house economy" there are five individuals, each of one owns precisely one house. Individual 1 owns house a, individual 2 owns house b and so on. The individuals want to consume precisely one house each, and have the following preferences (rankings) given by the table below.

1	2	3	4	5
b	a	e	a	a
c	d	b	d	b
e	c	d	e	d
a	b	a	c	e
d	e	c	b	c

Hence for individual 1, the house b is best, the house c second best and so on. No indifferences prevail. Now determine all core allocations in this economy.

Solution: According to their preferences and endowments, no individuals want to get the house which is not preferred to the initial own house. So we can only consider the preferences given by the following table.

1	2	3	4	5
b	a	e	a	a
c	d	b	d	b
e	c	d		d
a	b	a		e
		c		

Note that a coalition of agents 1 and 2 generate the best possible outcome for both, so that at least one must receive their most preferred house in a core allocation. If we give agent 1 e then agent 2 must receive a agent 4 must receive d, agent 5 must receive e and agent 3 must receive c. But such an allocation is blocked by a coalition of agents 1 and 3. Hence, agent 1 must receive either b or c in a core allocation. This leaves the following eight allocations as possible core allocations (b, a, c, d, e), (b, d, c, a, e), (b, c, e, a, d), (b, c, e, d, a), (b, c, d, a, e), (b, c, a, d, e), (c, a, b, d, e), and (c, a, e, d, b).

Note that all the above allocations are weakly Pareto efficient but not strongly Pareto efficient. Since our core concept requires strict preference by all agents in the coalition it is sometimes referred to as the weak core. The strong core requires that each coalition member be no worse off and at least one member be strictly better off. The strong core of this economy is (b, a, c, d, e).

7. Suppose we have two consumers A and B and two goods x_1 and x_2. Consumer 1's utility function is $u_A(x_1, x_2) = \min\{x_1, 2x_2\}$ and consumer 2's utility function is $u_B(x_1, x_2) = \min\{2x_1, x_2\}$. Initial endowments are $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$.

   (a) Are x_1 and x_2 gross substitutes.

Solution: Let the price of good 1 be 1 and the price of good 2 be p. Since $u_A(x_1, x_2) = \min\{x_1, 2x_2\}$, $u_B(x_1, x_2) = \min\{2x_1, x_2\}$, $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$, we get $z_1 = \frac{2}{2+p} - 1 + \frac{p}{1+2p}$, thus $\frac{\partial z_1}{\partial p} = \frac{-7p^2-4p+2}{(1+2p)^2(2+p)^2}$. Hence, x_1 and x_2 are gross substitutes if and only if $p \in [0, \frac{3\sqrt{2}-2}{7})$, .

(b) Find the set of Walrasian equilibrium.

Solution: From a, we know $z_1 = \frac{2}{2+p} - 1 + \frac{p}{1+2p}$ for $0 < p < \infty$. In the Walrasian equilibrium, $z_1 = 0$. Thus, $p = 1$, $\{x_A^1 = \frac{2}{3}, x_A^2 = \frac{1}{3}, x_B^1 = \frac{1}{3}, x_B^2 = \frac{2}{3}\}$ is a Walrasian equilibrium.

$p = 0$ is also an equilibrium both agents are maximizing their utility subject to their budget constraint. Hence, $p = 0$, $\{x_A^1 = 1, x_A^2 \in [1/2, 1], x_B^1 = 0, x_B^2 = 1 - x_A^2\}$ is a Walrasian equilibrium.

$p = \infty$ is also an equilibrium both agents are maximizing their utility subject to their budget constraint. Hence, $p = \infty$, $\{x_A^1 \in [0, 1/2], x_A^2 = 0, x_B^1 = 1 - x_A^1, x_B^2 = 1\}$ is a Walrasian equilibrium.

(c) Assume the following price adjustment rule $\dot{p}_i = z_i(p)$ for $i = 1 \ldots, k$. Find an unstable Walrasian equilibrium for this economy.

Solution: From a, $\dot{p}_1 = z_1(p) = \frac{2}{2+p} - 1 + \frac{p}{1+2p} = \frac{p-p^2}{(2+p)(1+2p)}$. Hence, if $p > 1$, then $\dot{p}_1 < 0$, i.e. p increases (since p_1 is the numeraire); if $p < 1$, then $\dot{p}_1 > 0$; if $p = 1$, then $\dot{p} = 0$. Thus, p always converges to 0 or ∞ unless $p = 1$. $p = 1$ is an unstable Walrasian equilibrium price for this economy.

(d) Does the aggregate excess demand function satisfy the Weak Axiom of Revealed Preference?

Solution: Let $p^* = (1, 1)$.

$$\begin{aligned} p^* z(p) =& \frac{2}{2+p} + \frac{p}{1+2p} - 1 + \frac{1}{2+p} + \frac{2p}{1+2p} - 1 \\ =& \frac{-(1-p)^2}{(2+p)(1+2p)} \\ <& 0 \text{ for all } p \neq p^*. \end{aligned}$$

So the aggregate excess demand function does not satisfy the Weak Axiom of Revealed Preference.

Let $p^* = (1, 0)$.

$$\begin{aligned} p^* z(p) =& \frac{2}{2+p} + \frac{p}{1+2p} - 1 \\ =& \frac{p(1-p)}{(2+p)(1+2p)} \\ >& 0 \text{ if and only if } p < 1. \end{aligned}$$

Let $p^* = (0, 1)$.

$$\begin{aligned} p^* z(p) =& \frac{1}{2+p} + \frac{2p}{1+2p} - 1 \\ =& \frac{p-1}{(2+p)(1+2p)} \\ >& 0 \text{ if and only if } p > 1. \end{aligned}$$

This is consistent with what we found in part c; when $p < 1$, p converges to 0, when $p > 1$, p converges to ∞.