

Advanced Microeconomics II

Expected Utility Theory

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A Gamble

Lottery A			
Prize	\$2,500,000	\$500,000	\$0
Probability	0	1	0

Lottery B			
Prize	\$2,500,000	\$500,000	\$0
Probability	0.1	0.89	0.01

Which lottery would you prefer?

Another Gamble

Lottery C			
Prize	\$2,500,000	\$500,000	\$0
Probability	0	0.11	0.89

Lottery D			
Prize	\$2,500,000	\$500,000	\$0
Probability	0.1	0	0.90

Which lottery would you prefer?

Choice Under Uncertainty

- Many important economic decisions involve risk.
- How do we model choice when it involves uncertainty?
- Standard choice theory works OK but uncertainty has a structure we can use to restrict preferences a “rational” person might hold.

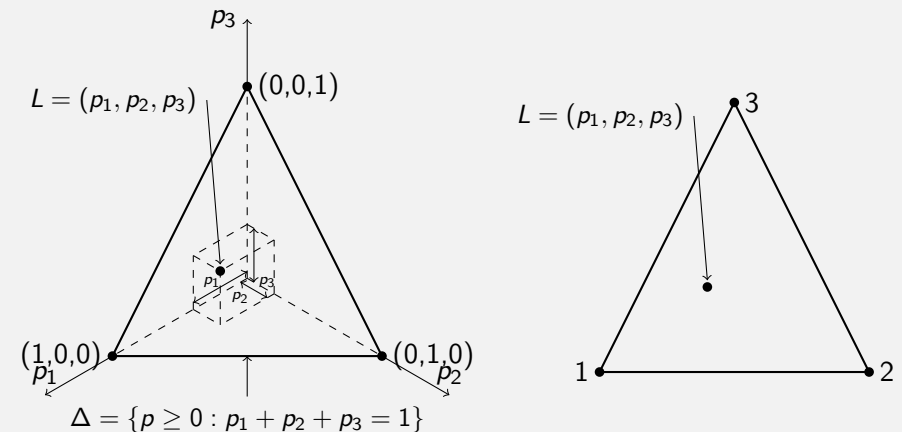
Modelling Risk

- C - set of all possible consequences (assume finite)
 - Index consequences by $n = 1, \dots, N$.
- Probabilities of outcomes are **objectively** known.

Definition

A **simple lottery** L is a list $L = (p_1, \dots, p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

Lottery Simplex

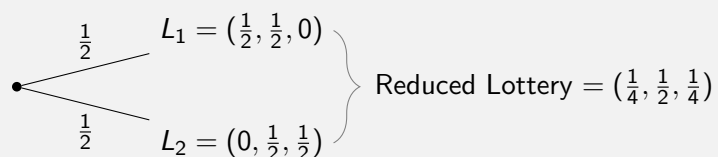


Compound Lotteries

Definition

Given K simple lotteries $L_k = (p_1^k, \dots, p_N^k, k = 1, \dots, K)$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the **compound lottery** $(L_1, \dots, L_K; \alpha_1, \dots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \dots, K$.

EXAMPLE $C = \{1, 2, 3\}$



Preferences Over Lotteries

- Assume consequentialism - only consequences matter.
- \mathcal{L} - the set of all simple lotteries over the set of outcomes C .
- \succeq is a rational preference relation on \mathcal{L} .

Definition

The preference relation \succeq on the space of simple lotteries \mathcal{L} is **continuous** if for any $L, L', L'' \in \mathcal{L}$ the sets

$$\{\alpha \in [0, 1] : \alpha L + (1 - \alpha)L' \succeq L''\} \subset [0, 1]$$

and

$$\{\alpha \in [0, 1] : L'' \succeq \alpha L + (1 - \alpha)L'\} \subset [0, 1]$$

are closed.

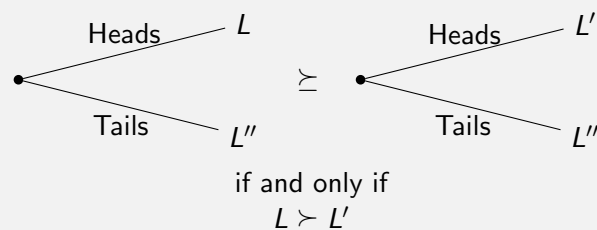
Independence Axiom

Definition

The preference relation \succeq on the space of simple lotteries \mathcal{L} satisfies the **independence axiom** if for any $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0, 1)$ we have

$$L \succeq L' \text{ iff } \alpha L + (1 - \alpha)L'' \succeq \alpha L' + (1 - \alpha)L''\}.$$

EXAMPLE



Expected Utility Form

Definition

The utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ has an **expected utility form** if there is an assignment of numbers (u_1, \dots, u_N) to the N outcomes such that for every simple lottery $L = (p_1, \dots, p_N) \in \mathcal{L}$ we have

$$U(L) = u_1 p_1 + \dots + u_N p_N$$

- Such a function is called a **von Neumann-Morgenstern (v.N-M) expected utility function**.
- Linear function in the probabilities.

Linearity of Expected Utility Form

Proposition

A utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ has an expected utility form if and only if it is **linear**, that is, if and only if

$$U\left(\sum_{k=1}^K \alpha_k L_k\right) = \sum_{k=1}^K \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathcal{L}, k = 1, \dots, K$ and probabilities $(\alpha_1, \dots, \alpha_K) \geq 0, \sum_k \alpha_k = 1$.

Proof

- Let L be a simple lottery $L = (p_1, \dots, p_N)$. L can be written as $\sum_n p_n L^n$, where $L^n = (\dots, 1, \dots)$. Let (u_1, \dots, u_N) be numbers such that $U(L^n) = u_n$.
Then $U(L) = U(\sum_n p_n L^n) = \sum_n p_n U(L^n) = \sum_n p_n u_n$.
So $U(\cdot)$ has expected utility form.
- In the other direction, consider a compound lottery $(L_1, \dots, L_K; \alpha_1, \dots, \alpha_K)$, where $L_k = (p_1^k, \dots, p_N^k)$.
Hence $U(\sum_k \alpha_k L_k) = \sum_n u_n (\sum_k \alpha_k p_n^k) = \sum_k \alpha_k (\sum_n u_n p_n^k) = \sum_k \alpha_k U(L_k)$.

Affine Transformations

Proposition

Suppose that $U : \mathcal{L} \rightarrow \mathbb{R}$ is a v.N-M expected utility function for the preference relation \succeq on \mathcal{L} . Then $\tilde{U} : \mathcal{L} \rightarrow \mathbb{R}$ is another v.N-M utility function for \succeq if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathcal{L}$.

Proof

W.L.O.G assume $\exists \bar{L}$ and \underline{L} , such that $\bar{L} \succeq L \succeq \underline{L}$ and $\bar{L} \succ \underline{L}$ for all $L \in \mathcal{L}$.

- Let $U(\cdot)$ be a v.N-M expected utility function and $\tilde{U}(L) = \beta U(L) + \gamma$, then

$$\begin{aligned} \tilde{U}\left(\sum_k \alpha_k L_k\right) &= \beta U\left(\sum_k \alpha_k L_k\right) + \gamma \\ &= \beta \left[\sum_k \alpha_k U(L_k) \right] + \gamma \\ &= \sum_k \alpha_k [\beta U(L_k) + \gamma] \\ &= \sum_k \alpha_k \tilde{U}(L_k) \end{aligned}$$

Proof Cont.

- In the reverse direction, assume $\tilde{U}(\cdot)$ and $U(\cdot)$ have the expected utility form. Let $L \in \mathcal{L}$ and define $\lambda_L \in [0, 1]$ by $U(L) = \lambda_L U(\bar{L}) + (1 - \lambda_L) U(\underline{L})$. Thus $\lambda_L = \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})}$. Since $\lambda_L U(\bar{L}) + (1 - \lambda_L) U(\underline{L}) = U(\lambda_L \bar{L} + (1 - \lambda_L) \underline{L})$, then $L \sim \lambda_L \bar{L} + (1 - \lambda_L) \underline{L}$, so

$$\begin{aligned} \tilde{U}(L) &= \tilde{U}(\lambda_L \bar{L} + (1 - \lambda_L) \underline{L}) \\ &= \lambda_L \tilde{U}(\bar{L}) + (1 - \lambda_L) \tilde{U}(\underline{L}) \\ &= \lambda_L (\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})) + \tilde{U}(\underline{L}) \end{aligned}$$

Thus $\tilde{U}(L) = \beta U(L) + \gamma$, where $\beta = \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})}$ and $\gamma = \tilde{U}(\underline{L}) - U(\underline{L}) \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\bar{L}) - U(\underline{L})}$

Implications

- Differences in utilities have meaning.
- Identify \succeq up to affine transformations.

EXAMPLE

There are 4 outcomes. If $u_1 - u_2 > u_3 - u_4$, then we have $\frac{1}{2}(u_1 + u_4) > \frac{1}{2}(u_2 + u_3)$, thus $L = (\frac{1}{2}, 0, 0, \frac{1}{2}) \succ L' = (0, \frac{1}{2}, \frac{1}{2}, 0)$

Expected Utility Theorem

Proposition

Suppose that the rational preference relation \succeq on the space of lotteries \mathcal{L} satisfies the continuity and independence axioms. Then \succeq admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \dots, N$ in such a manner that for any two lotteries $L = (p_1, \dots, p_N)$ and $L' = (p'_1, \dots, p'_N)$ we have

$$L \succeq L' \text{ if and only if } \sum_{n=1}^N u_n p_n \geq \sum_{n=1}^N u_n p'_n.$$

- Advantages - technically convenient, a useful guide to action
- Disadvantage - Poor descriptive theory.

Expected Utility Theorem Proof

Since \mathcal{L} is finite set and \succeq satisfies the independence axiom, $\exists \bar{L}$ and \underline{L} , such that $\bar{L} \succeq L \succeq \underline{L}$ and $\bar{L} \succ \underline{L}, \forall L \in \mathcal{L}$. W.l.o.g. $\bar{L} \succ \underline{L}$.

- Step 1. If $L \succ L'$ and $\alpha \in (0, 1)$, then $L \succ \alpha L + (1 - \alpha)L' \succ L'$ (Why?)

Expected Utility Theorem Proof Cont.

- Step 2. Let $\alpha, \beta \in [0, 1]$. Then $\beta \bar{L} + (1 - \beta)\underline{L} \succ \alpha \bar{L} + (1 - \alpha)\underline{L}$ if and only if $\beta > \alpha$.

- ▶ (\Leftarrow)
If $\beta > \alpha$, then

$$\beta \bar{L} + (1 - \beta)\underline{L} = \gamma \bar{L} + (1 - \gamma)[\alpha \bar{L} + (1 - \alpha)\underline{L}],$$

where $\gamma = \frac{\beta - \alpha}{1 - \alpha} \in (0, 1]$.

$$\begin{aligned} \bar{L} &\succ \alpha \bar{L} + (1 - \alpha)\underline{L} \\ \Rightarrow \gamma \bar{L} + (1 - \gamma)[\alpha \bar{L} + (1 - \alpha)\underline{L}] &\succ \alpha \bar{L} + (1 - \alpha)\underline{L} \text{ (Why?)} \\ \Rightarrow \beta \bar{L} + (1 - \beta)\underline{L} &\succ \alpha \bar{L} + (1 - \alpha)\underline{L}. \end{aligned}$$

- ▶ (\Rightarrow)
If $\beta = \alpha$, then $\beta \bar{L} + (1 - \beta)\underline{L} \sim \alpha \bar{L} + (1 - \alpha)\underline{L}$.
If $\alpha > \beta$, then from above $\alpha \bar{L} + (1 - \alpha)\underline{L} \succ \beta \bar{L} + (1 - \beta)\underline{L}$.

Expected Utility Theorem Proof Cont.

- Step 3. For any $L \in \mathcal{L}, \exists \alpha_L$ such that $\alpha_L \bar{L} + (1 - \alpha_L)\underline{L} \sim L$.

- ▶ Define

$$B = \{\alpha \in [0, 1] : \alpha \bar{L} + (1 - \alpha)\underline{L} \succeq L\}$$

$$W = \{\alpha \in [0, 1] : L \succeq \alpha \bar{L} + (1 - \alpha)\underline{L}\}$$

- ▶ B and W are non-empty, closed and cover $[0, 1]$. (Why?)
- ▶ $\exists \alpha$ belonging to both sets.
- ▶ Uniqueness follows from the result of step 2.

Expected Utility Theorem Proof Cont.

- Step 4. $U : \mathcal{L} \rightarrow \mathbb{R}$ that assigns $U(L) = \alpha_L$ for $\forall L \in \mathcal{L}$ represents \succeq .

► By step 3, we have $L \succeq L'$ if and only if

$$\alpha_L \bar{L} + (1 - \alpha_L) \underline{L} \succeq \alpha'_L \bar{L} + (1 - \alpha'_L) \underline{L}.$$

► By step 2,

$$\alpha_L \bar{L} + (1 - \alpha_L) \underline{L} \succeq \alpha'_L \bar{L} + (1 - \alpha'_L) \underline{L}$$

if and only if $\alpha_L \geq \alpha'_L$.

- What's left?

Expected Utility Theorem Proof Cont.

- Step 5. $U(\cdot)$ that assigns $U(L) = \alpha_L$ for $\forall L \in \mathcal{L}$ has the expected utility form.

► By definition

$$L \sim U(L)\bar{L} + (1 - U(L))\underline{L}$$

$$L' \sim U(L')\bar{L} + (1 - U(L'))\underline{L}$$

► So

$$\begin{aligned} \beta L + (1 - \beta)L' &\sim \beta[U(L)\bar{L} + (1 - U(L))\underline{L}] \\ &\quad + (1 - \beta)[U(L')\bar{L} + (1 - U(L'))\underline{L}] \\ &\sim [\beta U(L) + (1 - \beta)U(L')]\bar{L} \\ &\quad + [1 - \beta U(L) - (1 - \beta)U(L')]\underline{L} \end{aligned}$$

► By construction of $U(\cdot)$

$$U(\beta L + (1 - \beta)L') = \beta U(L) + (1 - \beta)U(L')$$

Allais Paradox

	Lottery A		
Prize	\$2,500,000	\$500,000	\$0
Probability	0	1	0

	Lottery B		
Prize	\$2,500,000	\$500,000	\$0
Probability	0.1	0.89	0.01

	Lottery C		
Prize	\$2,500,000	\$500,000	\$0
Probability	0	0.11	0.89

	Lottery D		
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Allais Paradox

- If $A \succeq_i B$ then

$$u_{05} > (.10)u_{25} + (.89)u_{05} + (.01)u_0.$$

- Add $(.89)u_0 - (.89)u_{05}$ to both sides.

$$(.11)u_{05} + (.89)u_0 > (.10)u_{25} + (.90)u_0.$$

- Any v.N-M utility function must have $C \succeq_i D$.
- Other paradoxes - St Petersburg, Ellsberg, Machina
- Alternative theories of choice under uncertainty - regret, prospect