Advanced Microeconomics II
Problem Set 4

WISE, Xiamen University

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Due 10:00 June 2, 2011

1. Two people are bidding for an object in a second-price auction. In a second-price auction the bidders submit non-negative bids. The bidder who submits the highest bid is the winning bidder and pays a price equal to the second highest bid. Player i has value v_i for the good, where v_i is drawn from a uniform distribution between zero and one.

   (a) Write down this problem as a Bayesian game of incomplete information.

Solution: The Bayesian game $\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succsim_i)\rangle$ is:

- the set N of players is $\{1, 2\}$
- $\Omega = [0, 1]^2$
- $A_i = [0, \infty)$
- $T_i = [0, 1]$
- $\tau_i(v_1, v_2) = v_i$ for all $(v_1, v_2) \in \Omega$.
- p_i of i is given by the continuous density function $f_i(v_1, v_2) = 1$ for all $(v_1, v_2) \in [0, 1]^2$
- $u_i(a_1, a_2, v_1, v_2) = v_i - b_j$ if $b_i > b_j$ $j \neq i$, $(v_i - b_i)/2$ if $b_i = b_j$, and 0 otherwise.

   (b) Prove that bidding your value is a dominant strategy for every possible player type.

Solution: Let player i's type be v_i and the other player's strategy be $b_j(v_j)$. A state by state comparison of payoffs when player i's action is v_i compared with payoffs when player i's action is b'_i shows that $b(v_i) = v_i$ is a weakly dominant strategy for player i. If v_j is such that $b_j(v_j) > v_i$ then for all $b_i < b_j(v_j)$ the payoff is the same as bidding v_i. If $b_i \geq b_j(v_j)$ then the payoff is lower than bidding v_i. If v_j is such that $b_j(v_j) < v_i$ then for all $b_i > b_j(v_j)$ the payoff is the same as bidding v_i. If $b_i \leq b_j(v_j)$ then the payoff is lower than bidding v_i. If v_j is such that $b_j(v_j) = v_i$ then for all b_i the payoff is the same as bidding v_i. Thus player i bidding their value v_i is a dominant strategy for every player type.

   (c) Find the set of Nash equilibria for this Bayesian game.

Solution: From the above analysis, we know that v_i is a weakly dominate strategy for every possible player type, thus each player bids her valuation is a Nash equilibrium.

Furthermore, player i ($i \in N$) bids 1 and the other player bids 0 is a Nash equilibrium. There are no other Nash equilibria of this game. To formally prove this is not hard but is somewhat tedious so I omit it here.

(d) For each of the equilibria identified above, what is the expected revenue of the seller? How does it compare to the expected revenue of the seller in the first-price auction?

Solution: In the first type of Nash equilibrium, the price paid by the winning bid is the second-order statistic $v_{(2)}$ of two uniformly distributed random variables.

$$F_{v_{(2)}}(x) = \Pr(v_{(2)} < x) = v^2 + 2v(1-v) \text{ and.}$$

$$f_{v_{(2)}}(x) = 2v + 2 - 4v = 2(1-v).$$

Hence,

$$E(v_{(2)}) = \int_0^1 2(1-v)v\,dv = 2(1/2 - 1/3) = 1/3.$$

In the second type Nash equilibrium, the expected revenue of the seller is 0.

In the first-price sealed bid auction symmetric Nash equilibrium, the price paid by the winning bid is the first-order statistic $v_{(1)}$ of two uniformly distributed random variables divided by two.

$$F_{v_{(1)}}(x) = \Pr(v_{(1)} < x) = v^2 \text{ and.}$$

$$f_{v_{(1)}}(x) = 2v.$$

Hence,

$$E(v_{(1)}/2) = \int_0^1 v^2\,dv = 1/3.$$

Thus the first type Nash equilibrium of the second-price sealed bid auction yields the same expected revenue to the seller as the Nash equilibrium of the first-price sealed bid auction .

2. In this question we model differences in players knowledge as a Bayesian game. There are two players and three possible states of the world, i.e. $\Omega = \{\alpha, \beta, \gamma\}$. The prior probability of each state is $p(\alpha) = 1/5, p(\beta) = 3/5, p(\gamma) = 1/5$. Each player has two types. In any state player 1 either knows the state is α or knows the state is β or γ, while player 2 either knows the state is α or β or knows the state is γ.

The payoffs for each action profile and state are shown in the following three payoff matrices, one for each state.

	L	R
L	$2,2$	$0,0$
R	$3,0$	$1,1$

State α

	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

State β

	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

State γ

(a) Write down this problem as a Bayesian game of incomplete information.

Solution: The Bayesian game $\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succsim_i)\rangle$ is:

- the set N of players is $\{1, 2\}$
- the set Ω of states is $\{\alpha, \beta, \gamma\}$
- the set A_i of actions of each player i is $\{L, R\}$
- the set T_1 of signals that player 1 can receives is $\{\{\alpha\}, \{\beta, \gamma\}\}$ and the set T_2 of signals that player 2 can receives is $\{\{\alpha, \beta\}, \{\gamma\}\}$
- the signal function $\tau_1(\alpha) = \{\alpha\}, \tau_1(\beta) = \tau_1(\gamma) = \{\beta, \gamma\}$; $\tau_2(\alpha) = \tau_2(\beta) = \{\alpha, \beta\}, \tau_2(\gamma) = \{\gamma\}$
- the prior belief p_i of i is given by $p_i(\alpha) = \frac{1}{5}, p_i(\beta) = \frac{3}{5}, p_i(\gamma) = \frac{1}{5}$
- the payoffs for each action profile and state are shown in the above three payoff matrices, one for each state.

(b) Solve for the set of Nash equilibria.

Solution: Denote by δ_1 the probability that player 1 of type $\{\alpha\}$ plays L. Denote by δ_2 the probability that player 1 of type $\{\beta, \gamma\}$ plays L. Denote by λ_1 the probability that player 2 of type $\{\alpha, \beta\}$ plays L. Denote by λ_2 the probability that player 2 of type $\{\gamma\}$ plays L.

R is the dominant strategy for player 1 of type $\{\alpha\}$, hence in any Nash equilibrium, $\delta_1^* = 0$.

Now consider player 2 of type $\{\alpha, \beta\}$. Given action a, his payoffs are:

$$a = L : \frac{1}{4}(0) + \frac{3}{4}(2\delta_2 + 0(1-\delta_2)) \text{ and}$$
$$a = R : \frac{1}{4}(1) + \frac{3}{4}(0\delta_2 + 1(1-\delta_2)).$$

Hence, if $\delta_2 > 4/9$, then $\lambda_1^* = 1$, if $\delta_2 < 4/9$, then $\lambda_1^* = 0$, and if $\delta_2 = 4/9$, then $\lambda_1^* \in [0, 1]$.

Now consider player 2 of type $\{\gamma\}$. Given action a, his payoffs are:

$$a = L : (3\delta_2 + 0(1-\delta_2)) \text{ and } a = R : (0\delta_2 + 1(1-\delta_2)).$$

Hence, if $\delta_2 > 1/3$, then $\lambda_2^* = 1$, if $\delta_2 < 1/3$, then $\lambda_2^* = 0$, and if $\delta_2 = 1/3$, then $\lambda_2^* \in [0, 1]$.

Finally, consider player 1 of type $\{\beta, \gamma\}$. Given action a, his payoffs are:

$$a = L : \frac{3}{4}(2\lambda_1 + 0(1-\lambda_1)) + \frac{1}{4}(2\lambda_2 + 0(1-\lambda_2)) \text{ and}$$
$$a = R : \frac{3}{4}(0\lambda_1 + 1(1-\lambda_1)) + \frac{1}{4}(0\lambda_2 + 1(1-\lambda_2)).$$

Hence, if $\lambda_1 + \lambda_2/3 > 4/9$, then $\delta_2^* = 1$, if $\lambda_1 + \lambda_2/3 < 4/9$, then $\delta_2^* = 0$, and if $\lambda_1 + \lambda_2/3 >= 4/9$, then $\delta_2^* \in [0, 1]1$.

Now we find fixed points. $0 \leq \delta_2^* < 1/3 \Rightarrow \lambda_1^* = \lambda_2^* = 0 \Rightarrow \delta_2^* = 0$. Hence, $(\delta_1^*, \delta_2^*, \lambda_1^*, \lambda_2^*) = (0, 0, 0, 0)$ is a Nash equilibrium. $\delta_2^* = 1/3 \Rightarrow \lambda_1^* = 0$ and $\lambda_2^* \in [0, 1] \Rightarrow \delta_2^* = 0$, which is a contradiction to the initial hypothesis. $1/3 < \delta_2^* < 4/9 \Rightarrow \lambda_1^* = 0$ and $\lambda_2^* = 1 \Rightarrow \delta_2^* = 0$, which is a contradiction to the initial hypothesis. $\delta_2^* = 4/9 \Rightarrow \lambda_1^* \in [0, 1]$ and $\lambda_2^* = 1 \Rightarrow \delta_2^* = 4/9$ only if $\lambda_1 = 1/9$. Hence, $(\delta_1^*, \delta_2^*, \lambda_1^*, \lambda_2^*) = (0, 4/9, 1/9, 1)$ is a Nash equilibrium. $4/9 < \delta_2^* \leq 1 \Rightarrow \lambda_1^* = \lambda_2^* = 1 \Rightarrow \delta_2^* = 1$. Hence, $(\delta_1^*, \delta_2^*, \lambda_1^*, \lambda_2^*) = (0, 1, 1, 1)$ is a Nash equilibrium.

3. A seller and buyer are considering a possible trade. The seller's value of the object is zero. The buyer's value, $v > 0$, is known to the buyer but not the seller. There are two periods of bargaining. At the start of each period the seller makes a take-it-or-leave-it offer which the buyer may accept or reject. The game ends when an offer is accepted or after two periods. Both players discount period two payoffs by $\delta \in (0, 1)$.

    (a) Characterize the pure strategy weak perfect Bayesian equilibria when v can take two values v_l and v_h with $v_h > v_L > 0$, and where $\gamma = \Pr(v_H)$.

Solution: Consider the second period strategies of the buyers. A buyer of type i rejects any $p_2 > v_i$ and accepts any $p_2 \leq v_i$ (If $p_2 = v_i$ the buyer is indifferent but to ensure an equilibrium the best response of the seller must be well-defined so we require that in such an instance the buyer accepts with probability 1. Thus the seller's optimal second period offer p_2 is v_h if $\mu(p_1, R)v_h > v_l$, is v_l if $\mu(p_1, R)v_h < v_l$ and he is indifferent between the strategies v_l and v_h otherwise, where $\mu(p_1, R)$ is the belief of the seller in the second period given the price p_1 in the first period and rejection in the first period by the buyer.

Now consider the first period strategies of the buyers. If $p_1 \leq v_l$, , both high value types must buy the good (since the best possible price they can get in the second period is $p_2 = v_l$), and the seller's expected payoff is x. Low value cannot buy the good if $p_1 > v_l$, since this gives them a negative payoff.

If high value type buyers buy the good in the first period at a price higher than v_l then the seller must infer in the second period, along the equilibrium path, that $\mu(p_1, R) = 0$ and offer $p_2 = v_l$. Hence, the price p_1 at which high type buyers accept must be low enough that they prefer to buy in period 1, i.e.,

$$v_h - p_1 \geq \delta(v_h - v_l).$$

Since the buyer of high type must accept any such price the optimal strategy of the seller is to offer the price $p_1^* = (1 - \delta)v_h + \delta v_l$. To be an equilibrium, the profit made by offer (p_1^*, v_l) must be at least as large as the profit made by offering (v_l, v_l), i.e.,

$$\gamma((1 - \delta)v_h + \delta v_L + (1 - \gamma)\delta v_l \geq v_l$$

which is true if $\gamma \geq v_l/v_h$.

Off the equilibrium path, in a weak perfect Bayesian equilibrium, the beliefs of the seller can be anything, but his offer must be sequentially rational given his beliefs. Similarly, the buyer type actions off the equilibrium path must be sequentially rational (no beliefs to define here) given the relationship between the first period and second period price offer made by the seller. In a perfect Bayesian equilibrium, off the equilibrium path the seller must maintain the same beliefs about the buyer if both types would reject such an offer, and must have degenerate beliefs if player 1 type actions differ.

(b) Characterize the pure strategy weak perfect Bayesian equilibria when v is uniformly distributed on $[\underline{v}, \overline{v}]$.

Solution: There are three possible solutions to the game depending on the parameter values v_l, v_h, and δ. In the first case, the seller sells positive amounts in both periods, but some consumers never buy the good. In the second case, the sellers sells positive amounts in both periods and all consumers buy the good. In the third case, all consumers buy in the first period. In this answer I will only show the equilibrium for the first case.

Denote the seller's pure strategy of as p_1, p_2, where $p_1, p_2 \in [\underline{v}, \overline{v}]$. In the second period, a buyer with value v who has not bought in the first period will buy if $p_2 \leq v$. There also exists a unique type v^* who is indifferent between buying the good in the first period or in the second period, $v^* - p_1 = \delta(v^* - p_2) \Rightarrow v^* = \frac{p_1 - \delta p_2}{1-\delta}$. Given that all types $v \geq v^*$ will buy the good in the first period, p_2^* must satisfy sequential rationality. Hence, p_2^* is the price p_2 that maximize the following equation,

$$p_2 \cdot \Pr\{v \geq p_2 \big| v < v^*\} = \frac{p_2(v^* - p_2)}{v^* - \underline{v}}$$

Assuming the first-order condition characterizes the solution to the problem (this will be true if there is sufficient distance between $\overline{v}$ and $\underline{v}$), then $p_2^* = \frac{v^*}{2}$. Thus, $v^* = \frac{p_1 - \delta \frac{v^*}{2}}{1-\delta} \Rightarrow v^* = \frac{2p_1}{2-\delta}$, $p_2^* = \frac{p_1}{2-\delta}$. The probability a consumer buys in the first period is $(\overline{v} - v^*)/(\overline{v} - \underline{v})$ and the probability a consumer buys in the second period is

$$\left(1 - \frac{\overline{v} - v^*}{\overline{v} - \underline{v}}\right) \frac{v^* - p_2^*}{v^* - \underline{v}} = \frac{v^* - p_2^*}{\overline{v} - \underline{v}}.$$

The equilibrium price p_1^* in the first period maximizes firm expected profit:

$$\Pi(p_1) = p_1 \frac{(2-\delta)^2 v_h - (4 - 3\delta)p_1}{(2-\delta)^2(\overline{v} - \underline{v})}$$

Hence, $p_1^* = \frac{\overline{v}(2-\delta)^2}{8-6\delta}$. Note that $p_2* \geq v_l$ if and only if $v_l \leq \frac{(2-\delta)\overline{v}}{8-6\delta}$.

Thus when $\underline{v} \leq \frac{(2-\delta)}{8-6\delta}\overline{v}$ the pure strategy weak perfect Bayesian equilibrium strategies are

- $p_1^* = \frac{(2-\delta)^2}{8-6\delta}\overline{v}$
- $p_2^*(p_1) = \frac{p_1}{2-\delta}$.
- Consumer of type v accepts an offer p_1 if and only if $p1 \leq (2-\delta)v/2$.
- Consumer of type v accepts an offer p_2 if and only if $p2 \leq v$.

Beliefs are updated using Bayes rule.Is there any freedom of beliefs off the equilibrium path?

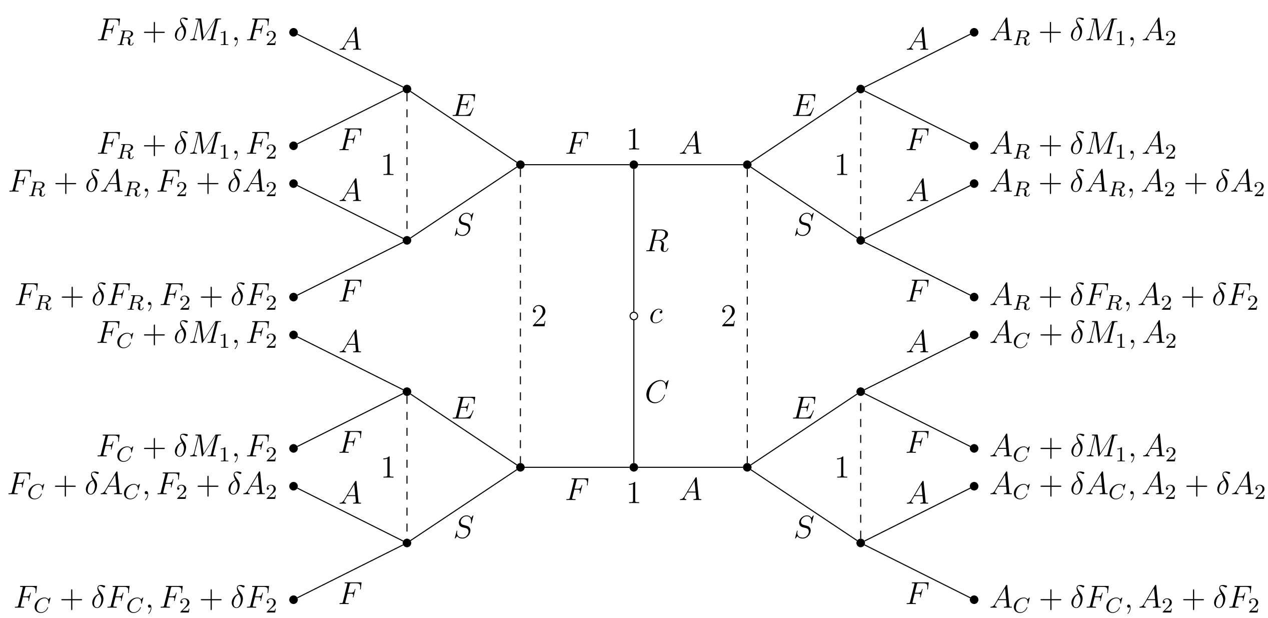
4. There are two firms, firm 1 is the incumbent and firm 2. At the start of the game chance, c, chooses the type of player 1 from one of two possible states. With probability λ firm 1 is "rational", R, and with probability $1-\lambda$ firm 1 is "crazy", C. The firms then interact in the market for two periods. In the first period, firm 1 takes one of two possible actions, fight F or accommodate, A. In the second period, firms simultaneously choose actions. Player 1

again chooses whether to fight or accommodate, while player 2 chooses one of two possible actions, stay, S, or exit, E.

In each period, firm profits are realized and firms discount the second period profits by the common discount factor δ. If both firms operate in the market then a rational firm 1 makes A_R if it accommodates and F_R if it fights, while a crazy firm 1 makes A_C if it accommodates and F_C if it fights. If only firm 1 operates in the market it makes monopoly profit M_1. Player 2 makes A_2 if he stays and player 1 accommodates, F_2 if if he stays and player one fights and 0 if he exits. Assume that $M_1 > A_R > F_R$, $F_C > M_1 > A_C$ and $A_2 > 0 > F_2$.

(a) Write down this problem as an extensive game of incomplete information.

Solution:



(b) Find parameter values for the payoffs for which there exists separating perfect Bayesian equilibria in this game.

Solution: Since $F_C > M_1 > A_C$, a firm of type crazy always chooses F. Furthermore, separating equilibrium requires the two types of firm 1 choose two different actions in period 1, thus, for firm 2, $\mu_1(R)(A) = 1$ and $\mu_1(R)(F) = 0$. Hence, the parameter values must satisfy $(1 + \delta)A_R \geq F_R + \delta M_1$.

(c) Find parameter values for the payoffs for which there exist pooling perfect Bayesian equilibria in this game.

Solution: Since a pooling equilibrium requires the two types of firm 1 choose the same actions in period 1 (and the crazy firm always fights), firm 2 beliefs along the equilibrium path must be $\mu_1(R)(F) = \lambda$. If firm 2 chooses S in period 2, the rational type firm 1 will choose A in period 2. Hence the parameter values must satisfy $\lambda A_2 + (1-\lambda)F_2 \leq 0$. In addition, it must be worthwhile for the rational firm to fight in the first period rather than accommodate in both periods, i.e., $(1+\delta)A_R \leq F_R + \delta M_1$. Sufficient off-the-equilibrium-path beliefs that support such an equilibrium are $\mu_1(R)(A) = 1$.

(d) Find parameter values for the payoffs for which there exist hybrid perfect Bayesian equilibria in this game.

Solution: In the hybrid equilibrium, the rational type firm randomizes between A and F, thus such a firm type must be indifferent between the two choices. Denote such a randomization by γ. For firm 2, $\mu(R)(A) = 1$ and $\mu(R)(F) = \frac{\lambda\gamma}{\lambda\gamma+(1-\lambda)} < \lambda$. To ensure that a rational firm 1 type is indifferent between fighting and accommodating in period 1, firm 2 randomizes between S and E. Denote the probability that firm 2 stays by α. The mixed strategy of a rational firm 1 type ensures that

$$\frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)} A_2 + \frac{1-\lambda}{\lambda\gamma + (1-\lambda)} F_2 = 0.$$

The mixed strategy of firm 2 ensures that

$$F_R + \delta[(1-\alpha)M_1 + \alpha A_R] = (1+\delta)A_R.$$

The requirement on the parameters are that the weights γ and α required to make these players indifferent are indeed probabilities. Hence, we require that $\lambda A_2 + (1-\lambda)F_2 \geq 0$ (player 1 can randomize appropriately to ensure player 2 is indifferent), and $(1+\delta)A_R < F_R + \delta M_1$ (player 2 can randomize to ensure player 1 is indifferent).