Advanced Macroeconomics II
Gali (2008), Chapter 3

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Chapter 3 The Basic New Keynesian Model

- Workhorse for the analysis of monetary policy, fluctuations and welfare.
- Two departures from the classical monetary economy:
  - Imperfect competition in the goods market
    - Each firm produces a differentiated good for which it sets the price (instead of taking the price as given)
  - Price rigidity: some constraints are imposed on the price adjustment mechanism by assuming only a fraction of firms can reset their prices in any given period, e.g. Calvo (1983) staggered price setting.

3.1 Households

The introduction of differentiated goods requires that the household problem be modified slightly relative to the one considered in the previous chapter.

Assumptions:

- The economy is composed of a continuum of infinitely-lived individuals, whose total is normalized to unity.
- There exists a continuum of firms (goods) represented by the interval [0, 1].

The representative household's problem

A representative infinitely-lived household seeks to maximize

$$E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

where C_t is a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

with $\varepsilon > 1$: elasticity of substitution between any two given varieties of goods i and j.

In this economy,

- Each household i consumes a basket of all goods C_t.
- The household supplies labor and saves in the form of nominal state contingent securities.
- Each firm i produces a differentiated food, which enters the consumption basket, and demands labor in a competitive labor market given wage.

The representative household's problem

The problem can be solved in two steps.

- The agent allocates optimally his resources across each differentiated good in a purely static fashion.
- The agent solves a typical dynamic optimization problem, featuring intertemporal allocation of consumption and saving.

Step 1. Optimal (static) expenditure allocation

Define Z_t as the total nominal expenditure. The optimal allocation of any given expenditure level can be found by sloving:

$$\max_{C_t(i)} C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$s.t. \int_0^1 P_t(i) C_t(i) di \equiv Z_t$$

Step 1. Optimal (static) expenditure allocation

Set up a Lagrange function:

$$L: \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} - \lambda\left(\int_0^1 P_t(i)C_t(i)di - Z_t\right)$$

$$\frac{\partial L}{\partial C_t(i)}: C_t^{\frac{1}{\varepsilon}} C_t(i)^{-\frac{1}{\varepsilon}} - \lambda P_t(i) \overset{!}{=} 0, \ \ \forall i \in [0,1]$$

For any pair of goods (i,j),

$$C_t(i) = C_t(j)\left(\frac{P_t(i)}{P_t(j)}\right)^{-\varepsilon}$$

Substitute it into the consumption index, we can derive the isoelastic demand of good i as

$$C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t$$

where $P_t \equiv \left(\int_0^1 P_t(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$.

Isoelastic demand of good i

Proof:

$$
\begin{aligned}
C_t^{\frac{\varepsilon-1}{\varepsilon}} &= \int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \\
&= \int_0^1 \left[C_t(j) \left(\frac{P_t(i)}{P_t(j)} \right)^{-\varepsilon} \right]^{\frac{\varepsilon-1}{\varepsilon}} di \\
&= C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} P_t(j)^{\varepsilon-1} \int_0^1 P_t(i)^{1-\varepsilon} di
\end{aligned}
$$

Define a composite price index

$$
P_t \equiv \left(\int_0^1 P_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}},
$$

then

$$
\begin{aligned}
C_t^{\frac{\varepsilon-1}{\varepsilon}} &= C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} P_t(j)^{\varepsilon-1} P_t^{1-\varepsilon} \\
C_t(j) &= \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon} C_t
\end{aligned}
$$

The dual problem

Note that the expression for $P_t \equiv \left(\int_0^1 P_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$ can be derived by solving the problem dual to the one above, i.e.,

$$\min_{C_t(i)} Z_t = \int_0^1 P_t(i) C_t(i) di$$

$$s.t. \ \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = 1$$

That is, P_t is the expenditure for one unit of consumption.

With the price index and composite consumption thus constructed,

$$P_t C_t = \int_0^1 P_t(i) C_t(i) di$$

Price index $\cdot$ Consumption index $=$ Total consumption expenditure.

Step 2. Intertemporal problem

$$\max E_t \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

$$s.t. \int_0^1 P_t(i) C_t(i) di + Q_t B_t \leq B_{t-1} + W_t N_t + T_t \tag{1}$$

By utilizing

$$P_t C_t = \int_0^1 P_t(i) C_t(i) di,$$

the budget constraint becomes

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + T_t.$$

This is formally identical to the constraint faced by households in the single good economy analyzed in the previous classical monetary model.

Intertemporal problem

The same set of FONCs can be derived.

$$
\begin{aligned}
-\frac{U_{N,t}}{U_{C,t}} &= \frac{W_t}{P_t} \\
Q_t &= \beta E_t \left\{ \frac{U_{C,t+1}}{U_{C,t}} \frac{P_t}{P_{t+1}} \right\}.
\end{aligned}
$$

With a specific utilility function

$$
U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}
$$

Intertemporal problem and loglinearization

Log-linearize the household's FONCs, we have

$$\hat{w}_t - \hat{p}_t = \sigma \hat{c}_t + \varphi \hat{n}_t \tag{2}$$

$$0 = E_t \left[-\sigma(\hat{c}_{t+1} - \hat{c}_t) + \hat{\imath}_t - \hat{\pi}_{t+1} \right] \tag{3}$$

3.2 Firms

The market is populated by a continuum of firms acting as monopolistic competitors. Labor is the only input of production. The production function of firm i is given by

$$Y_t(i) = A_t F\left(N_t\left(i\right)\right)$$

- A_t: productivity (technology) level, common to all firms.
- $F(.)$ is a homogeneous function of degree one.
- Labor is an economy-wide competitive factor. W_t is the same to all.
- This also implies that all firms face a common nominal marginal cost MC_t.

Firms

All firms face an identical isoelastic demand curve

$$C_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon} C_t, \text{ given } P_t \text{ and } C_t.$$

Cost minimization implies the following efficiency condition for the choice of labor input:

$$W_t = MC_t A_t F_{N,t}(N_t(i)).$$

Under perfect competition, the price would equal marginal cost:

$$P_t(i) = MC_t = \frac{W_t}{A_t F_{N,t}(N_t(i))}.$$

Price setting under monopolistic competition and flexible prices

Firm chooses price, output and labor to maximize profit.

$$\max P_t(i) Y_t(i) - W_t N_t(i)$$
$$\textit{or} \max \frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t}{P_t} N_t(i)$$

with

$$Y_t(i) = C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t$$

$$N_t(i) = F^{-1}\left(\frac{Y_t(i)}{A_t}\right) = F^{-1}\left(\frac{C_t}{A_t}\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\right)$$

Substituting for $Y_t(i)$ and $N_t(i)$, the firm's problem becomes the one of choosing $P_t(i)$ to maximize

$$\left(\frac{P_t(i)}{P_t}\right)^{1-\varepsilon} C_t - \frac{W_t}{P_t} F^{-1}\left(\frac{C_t}{A_t}\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\right)$$

Price setting under monopolistic competition and flexible prices

The FONC with respect to $P_t(i)$:

$$(1-\varepsilon)\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\frac{C_t}{P_t}-\frac{W_t}{P_t}\frac{1}{F_N(N_t(i))}\frac{C_t}{A_t}(-\varepsilon)\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon}\frac{1}{P_t}\overset{!}{=}0$$

$$P_t(i)=\frac{1}{1-1/\varepsilon}\frac{W_t}{A_t F_{N,t}(N_t\,(i))}=\mu\cdot MC_t$$

When $\varepsilon\longrightarrow\infty$, $\mu\longrightarrow 1$, perfect competition, $P_t(i)=MC_t$. In general, ε is finite and $\mu>1$, $P_t(i)>MC_t$, leads to inefficiently low level of employment and output.

Price setting under sticky prices

Sticky price (Calvo pricing)

Calvo (1983): firms adjust their price infrequently and the opportunity to adjust follows an exogenous Poisson process. Each firm may reset its price with a constant probability $(1-\theta)$ independently of past history; with probability θ they cannot adjust price.

In this sense, the Calvo pricing rule is a time dependent rule, as opposed to being state-dependent.

- Thus, each period a measure $(1-\theta)$ of firms reset their prices, while a fraction θ keeps their prices unchanged.
- As a result, the average duration of a price is given by $\frac{1}{1-\theta}$.
- θ becomes a natural index of price stickiness.

3.2.1 Aggregate price dynamics

If the law of large number holds, a fraction $(1-\theta)$ of firms will reset the price at each point in time. The evolution of the aggregate price index therefore is

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)\left(P_t^*\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

In log-linear terms

$$\hat{p}_t = \theta \hat{p}_{t-1} + (1-\theta)\hat{p}_t^*. \tag{4}$$

Subtract $\hat{p}_{t-1}$ on both sides, it follows that...

Aggregate price dynamics

It follows that the rate of inflation is given by

$$\pi_t = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1}) \tag{5}$$

- In steady state with zero inflation $(\Pi = 1)$, $\bar{P}^* = \bar{P}_{t-1} = \bar{P}_t \forall t$.
- Positive inflation arises if and only if firms adjusting prices in any given period choose to charge prices that are on average above the average price level prevailed in the economy in the previous period.
- Thus understanding inflation implies an understanding of why firms may want to choose to adjust their relative price periodically.
- We have to examine how $\hat{p}_t^*$ is set in order to understand the inflation dynamics.

3.2.2 Optimal price setting

The problem of a firm i that is able to reset its price is the one of choosing $P_t^*(i)$ to maximize the expected present discounted stream of profits while that price remains effective. $P_t^*(i) = P_t^*$ for this type of firms.

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t \left[Q_{t,t+k} \left(P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) \right) \right]$$

$$s.t. \; Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k}$$

The FOC of firms that can reset prices

The FOC:

$$E_t \left\{ \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|t}}{\partial P_t^*} - \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial P_t^*} \right] \right\} \overset{!}{=} 0$$

with $\frac{\partial Y_{t+k|t}}{\partial P_t^*} = (-\varepsilon) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon-1} \frac{C_{t+k}}{P_{t+k}} = (-\varepsilon) \frac{Y_{t+k|t}}{P_t^*}$, and define

$$\psi_{t+k|t} \equiv \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}}$$

as the norminal marginal cost.

The FOC of firms that can reset prices

The FOC becomes

$$
\begin{aligned}
& E_t \left\{ \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[Y_{t+k|t} + \left(P_t^* - \psi_{t+k|t} \right) (-\varepsilon) \frac{Y_{t+k|t}}{P_t^*} \right] \right\} \\
= \; & E_t \left\{ \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} Y_{t+k|t} \left[1 + \left(P_t^* - \psi_{t+k|t} \right) (-\varepsilon) \frac{1}{P_t^*} \right] \right\} \\
= \; & E_t \left\{ \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} Y_{t+k|t} \left[1 - \varepsilon + \varepsilon \frac{\psi_{t+k|t}}{P_t^*} \right] \right\} \overset{!}{=} 0
\end{aligned}
$$

The FOC of firms that can reset prices

Rearrange, we get

$$P_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} Y_{t+k|t} \psi_{t+k|t}}{E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} Y_{t+k|t}}$$

- If $\theta = 0$, flexible price setting, $P_t^* = \frac{\varepsilon}{\varepsilon - 1} \psi_t$, i.e., firms set price as a simple (static) markup over the norminal marginal cost.
- The optimal price depends on a forecast of future values of aggregate demand conditions as well as on the future evolution of the marginal cost.

The FOC of firms that can reset prices

Define

- $Q_{t,t+k} \equiv \beta^k \tilde{Q}_{t,t+k}$, where $\tilde{Q}_{t,t+k} \equiv \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}$
- $Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} C_{t+k} = \left(P_t^*\right)^{-\varepsilon} P_{t+k}^{\varepsilon} Y_{t+k}$

Then

$$P_t^* = \underbrace{\frac{\varepsilon}{\varepsilon - 1}}_{\mu} \frac{E_t \sum_{k=0}^{\infty} (\beta\theta)^k \tilde{Q}_{t,t+k} P_{t+k}^{\varepsilon} Y_{t+k} \psi_{t+k|t}}{E_t \sum_{k=0}^{\infty} (\beta\theta)^k \tilde{Q}_{t,t+k} P_{t+k}^{\varepsilon} Y_{t+k}}$$

Equivalently,

$$\begin{aligned} & P_t^* E_t \sum_{k=0}^{\infty} (\beta\theta)^k \tilde{Q}_{t,t+k} P_{t+k}^{\varepsilon} Y_{t+k} \\ = \; & \mu E_t \sum_{k=0}^{\infty} (\beta\theta)^k \tilde{Q}_{t,t+k} P_{t+k}^{\varepsilon} Y_{t+k} \psi_{t+k|t} \end{aligned}$$

Log-linearization

Left-hand side (LHS):

$$LHS_t = P_t^* E_t \left[P_t^\varepsilon Y_t + \beta\theta \tilde{Q}_{t,t+1} P_{t+1}^\varepsilon Y_{t+1} + (\beta\theta)^2 \tilde{Q}_{t,t+2} P_{t+2}^\varepsilon Y_{t+2} \dots \right]$$

In steady state, assume

$$\bar{P}^* = \bar{P}, \quad \overline{\tilde{Q}} = 1$$

$$\begin{aligned}
\widehat{LHS}_t &\approx \bar{P}^{1+\varepsilon} \bar{Y} \left(\hat{p}_t^* + \varepsilon \hat{p}_t + \hat{y}_t \right) \\
&\quad + \bar{P}^{1+\varepsilon} \bar{Y} \beta\theta E_t \left(\hat{p}_t^* + \varepsilon \hat{p}_{t+1} + \hat{y}_{t+1} + \widehat{\tilde{q}}_{t,t+1} \right) + \dots \\
&= \bar{P}^{1+\varepsilon} \bar{Y} \sum_{k=0}^{\infty} (\beta\theta)^k \hat{p}_t^* \\
&\quad + \bar{P}^{1+\varepsilon} \bar{Y} E_t \left\{ \sum_{k=0}^{\infty} (\beta\theta)^k \left(\varepsilon \hat{p}_{t+k} + \hat{y}_{t+k} + \widehat{\tilde{q}}_{t,t+k} \right) \right\} \\
&= \left(\bar{P}^{1+\varepsilon} \bar{Y} \right) \frac{1}{1-\beta\theta} \hat{p}_t^* \\
&\quad + \bar{P}^{1+\varepsilon} \bar{Y} E_t \left\{ \sum_{k=0}^{\infty} (\beta\theta)^k \left(\varepsilon \hat{p}_{t+k} + \hat{y}_{t+k} + \widehat{\tilde{q}}_{t,t+k} \right) \right\}
\end{aligned}$$

Log-linearization

Similarly for the right-hand side (RHS), define

$$\text{Real marginal cost: } MC_t \equiv \frac{\psi_t}{P_t}, \text{ so } \psi_t = MC_t \cdot P_t$$

In steady state, $\overline{MC} = \frac{1}{\mu}$, so that $\bar{P} = \mu\overline{\psi}$ where $\mu = \frac{\varepsilon}{\varepsilon-1}$.

$$\begin{aligned} RHS_t &= \mu E_t \sum_{k=0}^{\infty} (\beta\theta)^k \tilde{Q}_{t,t+k} P_{t+k}^{\varepsilon} Y_{t+k} \psi_{t+k|t} \\ &= \mu \left[\tilde{Q}_t MC_t P_t^{1+\varepsilon} Y_t + \beta\theta \left(\tilde{Q}_{t,t+1} MC_{t+1|t} P_t^{1+\varepsilon} Y_{t+1} \right) + \ldots \right] \end{aligned}$$

Log-linearization

In log-linear terms,

$$
\begin{aligned}
\widehat{RHS}_t &= \mu \overline{MC} \bar{P}^{1+\varepsilon} \bar{Y} \left[\widehat{\tilde{q}}_t + \widehat{mc}_t + (1+\varepsilon)\hat{p}_t + \hat{y}_t \right] \\
&\quad + \mu \overline{MC} \bar{P}^{1+\varepsilon} \bar{Y} \beta\theta \left[\widehat{\tilde{q}}_{t,t+1} + \widehat{mc}_{t+1|t} + (1+\varepsilon)\hat{p}_{t+1} + \hat{y}_{t+1} \right] + \ldots \\
&= \bar{P}^{1+\varepsilon} \bar{Y} E_t \sum_{k=0}^{\infty} (\beta\theta)^k \left[\widehat{\tilde{q}}_{t,t+1} + \widehat{mc}_{t+k|t} + (1+\varepsilon)\hat{p}_{t+k} + \hat{y}_{t+k} \right]
\end{aligned}
$$

Log-linearization

By combining the log-linearized LHS and RHS, we obtain

$$\begin{aligned}
\hat{p}_t^* &= (1-\beta\theta)\, E_t \sum\nolimits_{k=0}^{\infty} (\beta\theta)^k \left(\widehat{mc}_{t+k|t} + \hat{p}_{t+k}\right) && (6) \\
&= (1-\beta\theta)\left(\widehat{mc}_{t|t} + \hat{p}_t\right) + \beta\theta E_t \hat{p}_{t+1}^* && (7)
\end{aligned}$$

Implications:

- Firms that are allowed to reset the price do so as a weighted average over the expected future nominal marginal cost.
- The presence of the aggregate price level denotes the willingness to maintain (in expectation) the relative price unchanged.
- The term involving $\widehat{mc}_{t+k|t}$ denotes the desire to change the expected relative price in order to avoid any gap that may emerge between expected and desired markup.

Inflation dynamics

Combine equation (4) with equation (7),

$$\begin{aligned}\hat{p}_t &= \theta\hat{p}_{t-1} + (1-\theta)\hat{p}_t^* \\ &= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)\left(\widehat{mc}_{t|t} + \hat{p}_t\right) + (1-\theta)\beta\theta\frac{E_t\hat{p}_{t+1} - \theta\hat{p}_t}{1-\theta}\end{aligned}$$

$$\begin{aligned}\theta\hat{p}_t &= \theta\hat{p}_{t-1} - (1-\theta)\hat{p}_t + (1-\theta)(1-\beta\theta)\left(\widehat{mc}_{t|t} + \hat{p}_t\right) \\ &\quad +\beta\theta E_t\hat{p}_{t+1} - \beta\theta^2\hat{p}_t \\ \theta\left(\hat{p}_t - \hat{p}_{t-1}\right) &= -(1-\theta)\hat{p}_t + (1-\theta)(1-\beta\theta)\hat{p}_t - \beta\theta_t^2\hat{p}_t \\ &\quad +(1-\theta)(1-\beta\theta)\widehat{mc}_{t|t} + \beta\theta E_t\hat{p}_{t+1} \\ &= (1-\theta)(1-\beta\theta)\widehat{mc}_{t|t} + \beta\theta\left(E_t\hat{p}_{t+1} - \hat{p}_t\right)\end{aligned}$$

Inflation dynamics

By rearrangement,

$$\theta \hat{\pi}_t = (1-\theta)(1-\beta\theta)\widehat{mc}_{t|t} + \beta\theta E_t\{\hat{\pi}_{t+1}\}$$

$$\hat{\pi}_t = \beta E_t\{\hat{\pi}_{t+1}\} + \frac{(1-\theta)(1-\beta\theta)}{\theta}\widehat{mc}_{t|t} \tag{8}$$

This is a forward-looking equation for inflation, which links movements of current inflation to contemporaneous movements in the real marginal cost and expected future inflation.

- Intuition: An increase in demand implies a rise in output, labor demand and therefore the real wage and the real marginal cost. This triggers a rise in current inflation (for any given expectations on future inflation).
- The longer prices are fixed ($\theta \uparrow$), the less firms are sensitive to changes in the real marginal cost, as current demand conditions matter less.

3.3 Equilibrium

Goods market clearing

$$Y_t(i) = C_t(i) \ \forall i \in [0,1]$$

Define

$$Y_t \equiv \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

we get

$$Y_t = C_t$$

Log-linearize,

$$\hat{y}_t = \hat{c}_t.$$

Combine it with the Euler equation (3) of households,

$$\hat{y}_t = E_t\{\hat{y}_{t+1}\} - \frac{1}{\sigma}\left(\hat{\imath}_t - E_t\{\hat{\pi}_{t+1}\}\right). \tag{9}$$

Equilibrium

Labor market equilibrium

Define

$$N_t = \int_0^1 N_t(i)di.$$

Using

$$\begin{aligned}
Y_t(i) &= A_t N_t(i)^{1-\alpha}, \\
N_t &= \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di \\
&= \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di \\
N_t^{1-\alpha} &= \frac{Y_t}{A_t} \underbrace{\left[\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di\right]^{1-\alpha}}_{D_t = \exp(d_t)}
\end{aligned}$$

Equilibrium

Labor market equilibrium and price dispersion

$$Y_t = A_t N_t^{1-\alpha} \exp(-d_t)$$

If $D_t = \exp(d_t) > 1$, $Y_t < A_t N_t^{1-\alpha}$, inefficiency in output level.
Take log,

$$y_t = a_t + (1-\alpha)\, n_t - d_t,$$

where

$$d_t = (1-\alpha) \log \left[\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} di \right].$$

Under flexible price, $P_t(i) = P_t$, $d_t = 0$, $D_t = 1$.

Equilibrium

Price dispersion

Under sticky price, see Appendix 3.3,

$$d_t \simeq \frac{1}{2}\frac{\varepsilon}{\Theta} var\left\{p_t(i)\right\}$$

in a neighborhood of $\pi = 0$. But d_t is equal to zero up to a first-order approximation, such that $\hat{d}_t \simeq 0$. So

$$\begin{aligned}
y_t &\simeq a_t + (1-\alpha)\, n_t \\
\hat{y}_t &= \hat{a}_t + (1-\alpha)\, \hat{n}_t \\
\hat{n}_t &= \frac{1}{1-\alpha}\left(\hat{y}_t - \hat{a}_t\right)
\end{aligned}$$

Equilibrium

Average real marginal cost

For an average firm, the economy's average real marginal cost

$$mc_t = \frac{W_t}{P_t}\frac{1}{A_t F_{N,t}(N_t)} = \frac{W_t}{P_t}\frac{1}{A_t(1-\alpha)N_t^{-\alpha}}$$

$$\begin{aligned}\widehat{mc}_t &= \hat{w}_t - \hat{p}_t - \widehat{mpn}_t \\ &= \hat{w}_t - \hat{p}_t - (\hat{a}_t - \alpha\hat{n}_t) \\ &= \hat{w}_t - \hat{p}_t - \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t)\end{aligned}$$

Equilibrium

Real marginal cost for firms resetting price

For firms that set price at t and remains it at $t+k$,

$$\begin{aligned}
\widehat{mc}_{t+k|t} &= \hat{w}_{t+k} - \hat{p}_{t+k} - \widehat{mpn}_{t+k|t} \\
&= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}\left(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}\right) \\
&= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}\left(\hat{a}_{t+k} - \alpha\hat{y}_{t+k} + \alpha\hat{y}_{t+k} - \alpha\hat{y}_{t+k|t}\right) \\
&= \widehat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}(\hat{p}_t^* - \hat{p}_{t+k}) \qquad (10)
\end{aligned}$$

where the last line uses

$$\begin{aligned}
Y_{t+k|k} &= \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k} \\
\hat{y}_{t+k} - \hat{y}_{t+k|t} &= \varepsilon(\hat{p}_t^* - \hat{p}_{t+k})
\end{aligned}$$

Equilibrium

Real marginal cost: the special case

$$\widehat{mc}_{t+k|t} = \widehat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}(\hat{p}_t^* - \hat{p}_{t+k})$$

In the special case of constant returns to scale ($\alpha = 0$, so that labor share $1-\alpha = 1$), for example, $Y_t = A_t N_t$, it happens that $\widehat{mc}_{t+k|t} = \widehat{mc}_{t+k}$, i.e., the marginal cost is independent of the level of production and hence, it is common across firms whether they can set the price at t.

Equilibrium

Optimal price in terms of average real marginal cost

Now substitute (10) into equation (6),

$$
\begin{aligned}
\hat{p}_t^* &= (1-\beta\theta)\, E_t \sum_{k=0}^{\infty} (\beta\theta)^k \left(\widehat{mc}_{t+k|t} + \hat{p}_{t+k}\right) \\
&= (1-\beta\theta)\, E_t \left\{ \sum_{k=0}^{\infty} (\beta\theta)^k \left[\widehat{mc}_{t+k} - \frac{\alpha\varepsilon}{1-\alpha} (\hat{p}_t^* - \hat{p}_{t+k}) + \hat{p}_{t+k} \right] \right\} \\
&= -\frac{\alpha\varepsilon}{1-\alpha} \hat{p}_t^* + (1-\beta\theta)\, E_t \left\{ \sum_{k=0}^{\infty} (\beta\theta)^k \left[\widehat{mc}_{t+k} + \frac{1-\alpha+\alpha\beta}{1-\alpha} \hat{p}_{t+k} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1-\alpha+\alpha\beta}{1-\alpha} \hat{p}_t^* &= (1-\beta\theta)\, E_t \left\{ \sum_{k=0}^{\infty} (\beta\theta)^k \left[\widehat{mc}_{t+k} + \frac{1-\alpha+\alpha\beta}{1-\alpha} \hat{p}_{t+k} \right] \right\} \\
\hat{p}_t^* &= (1-\beta\theta)\, E_t \left\{ \sum_{k=0}^{\infty} (\beta\theta)^k \left[\Theta \widehat{mc}_{t+k} + \hat{p}_{t+k} \right] \right\} \qquad (11)
\end{aligned}
$$

with $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\beta} \leq 1$.

Equilibrium

Optimal price in terms of average real marginal cost

Rewrite this equation, we get

$$\hat{p}_t^* = (1 - \beta\theta)\left(\Theta\widehat{mc}_t + \hat{p}_t\right) + \beta\theta E_t \hat{p}_{t+1}^* \tag{12}$$

Combine (12) and the aggregate price dynamics,

$$\begin{aligned}
\hat{p}_t &= \theta\hat{p}_{t-1} + (1-\theta)\hat{p}_t^* \\
&= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)\left(\Theta\widehat{mc}_t + \hat{p}_t\right) \\
&\quad + (1-\theta)\beta\theta\frac{E_t\{\hat{p}_{t+1}\} - \theta\hat{p}_t}{1-\theta} \\
\theta\hat{p}_t + (1-\theta)\hat{p}_t &= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)\Theta\widehat{mc}_t \\
&\quad + (1-\theta-\beta\theta)\hat{p}_t + \beta\theta E_t\{\hat{p}_{t+1}\}
\end{aligned}$$

Equilibrium

Inflation in terms of the average real marginal cost

$$
\begin{aligned}
\theta\hat{\pi}_t &= (1-\theta)(1-\beta\theta)\,\Theta\widehat{mc}_t - \beta\theta\hat{p}_t + \beta\theta E_t\hat{p}_{t+1} \\
\hat{\pi}_t &= \beta E_t\{\hat{\pi}_{t+1}\} + \underbrace{\frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta}_{\lambda}\cdot\widehat{mc}_t && (13) \\
&= \beta E_t\{\hat{\pi}_{t+1}\} + \lambda\widehat{mc}_t \\
&= \beta E_t\{\beta\hat{\pi}_{t+2} + \lambda\widehat{mc}_{t+1}\} + \lambda\widehat{mc}_t && (14) \\
& \quad \ldots && (15) \\
&= \lambda\sum_{k=0}^{\infty}\beta^k E_t\{\widehat{mc}_{t+k}\} && (16)
\end{aligned}
$$

Equilibrium

Inflation in terms of the average real marginal cost

$$\hat{\pi}_t = \lambda \sum_{k=0}^{\infty} \beta^k E_t \{\widehat{mc}_{t+k}\}$$

In this model, inflation results from the aggregate consequences of purposeful price-setting decisions by firms, which adjust their prices in light of current and anticipated cost conditions.

Equilibrium

To relate the marginal cost with the output gap

$$\begin{aligned}\widehat{mc}_t &= (\hat{w}_t - \hat{p}_t) - \widehat{mpn}_t \\ &= (\sigma \hat{y}_t + \varphi \hat{n}_t) - (\hat{a}_t - \alpha \hat{n}_t)\end{aligned}$$

Using

$$\begin{aligned}\hat{a}_t - \alpha \hat{n}_t &= \hat{y}_t - \hat{n}_t \\ \hat{n}_t &= \frac{1}{1-\alpha}(\hat{y}_t - \hat{a}_t),\end{aligned}$$

then we obtain

$$\widehat{mc}_t = \frac{\varphi + \alpha + \sigma - \alpha\sigma}{1-\alpha}\hat{y}_t - \frac{\varphi + 1}{1-\alpha}\hat{a}_t$$

Equilibrium

To relate the marginal cost with the output gap

$$\widehat{mc}_t = \frac{\varphi + \alpha + \sigma - \alpha\sigma}{1-\alpha}\hat{y}_t - \frac{\varphi + 1}{1-\alpha}\hat{a}_t$$

Under the flexible price,

$$P_t(i) = \frac{1}{1 - 1/\varepsilon}\frac{W_t}{A_t F_{N,t}(N_t(i))} = \mu \cdot \psi_t = P_t$$

The real marginal cost is constant and its variation is zero,

$$MC_t \equiv \frac{\psi_t}{P_t} = \mu; \;\; \widehat{mc}_t = 0$$

Define the output under the flexible price as the natural output Y_t^n and its log deviation as $\hat{y}_t^n$,

$$\hat{y}_t^n = \frac{\varphi + 1}{\varphi + \alpha + \sigma - \alpha\sigma}\hat{a}_t.$$

Equilibrium

To relate the marginal cost with the output gap

Definition

Output gap: difference between the real output, Y_t, and the natural output, Y_t^n. In log terms, it is defined as

$$\tilde{y}_t = \hat{y}_t - \hat{y}_t^n,$$

which is resulting from the sticky price.

$$\begin{aligned} \widehat{mc}_t &= \frac{\varphi + \alpha + \sigma - \alpha\sigma}{1-\alpha}(\hat{y}_t^n + \tilde{y}_t) - \frac{\varphi + 1}{1-\alpha}\hat{a}_t \\ &= \frac{\varphi + \alpha + \sigma(1-\alpha)}{1-\alpha}\tilde{y}_t \\ &= (\sigma + \frac{\varphi + \alpha}{1-\alpha})\tilde{y}_t \end{aligned} \tag{17}$$

Equilibrium

The NKPC in terms of the output gap

$$\hat{\pi}_t = \beta E_t \left\{ \hat{\pi}_{t+1} \right\} + \lambda \widehat{mc}_t \tag{18}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$.
Combine it with Equation (17), we get the NKPC

$$\hat{\pi}_t = \beta E_t \left\{ \hat{\pi}_{t+1} \right\} + \kappa \tilde{y}_t$$

where $\kappa \equiv \lambda(\sigma + \frac{\varphi+\alpha}{1-\alpha})$.

Equilibrium

The dynamic IS (DIS) curve

With the definition of output gap, the Euler equation (9) becomes

$$\begin{aligned}\hat{y}_t^n + \tilde{y}_t &= E_t\left\{\hat{y}_{t+1}^n + \tilde{y}_{t+1}\right\} - \frac{1}{\sigma}\left(\hat{\imath}_t - E_t\left\{\hat{\pi}_{t+1}\right\}\right) \\ \tilde{y}_t &= E_t\left\{\hat{y}_{t+1}^n - \hat{y}_t^n\right\} + E_t\left\{\tilde{y}_{t+1}\right\} - \frac{1}{\sigma}\left(\hat{\imath}_t - E_t\left\{\hat{\pi}_{t+1}\right\}\right).\end{aligned}$$

Define natural rate of interest (deviation) as

$$\begin{aligned}\hat{r}_t^n &\equiv \sigma E_t\left\{\hat{y}_{t+1}^n - \hat{y}_t^n\right\} = \frac{\sigma(1+\varphi)}{\varphi+\alpha+\sigma(1-\alpha)} E_t\left\{\Delta\hat{a}_{t+1}\right\} \\ &= \sigma\psi_{ya}^n E_t\left\{\Delta\hat{a}_{t+1}\right\},\end{aligned}$$

then we get the dynamic IS (DIS) equation

$$\tilde{y}_t = -\frac{1}{\sigma}\left(\hat{\imath}_t - E_t\left\{\hat{\pi}_{t+1}\right\} - \hat{r}_t^n\right) + E_t\left\{\tilde{y}_{t+1}\right\}. \tag{19}$$

The dynamic IS (DIS) curve

Define the expected real return as

$$\hat{r}_t \equiv \hat{\imath}_t - E_t\left\{\hat{\pi}_{t+1}\right\},$$

and assume

$$\lim_{T\to\infty} E_t\left\{\tilde{y}_{t+T}\right\} = 0,$$

we get the iterated solution for $\tilde{y}_t$ as

$$\tilde{y}_t = -\frac{1}{\sigma}\sum_{k=0}^{\infty}\left(\hat{r}_{t+k} - \hat{r}^n_{t+k}\right). \tag{20}$$

The output gap is proportional to the sum of current and anticipated deviations between the real interest rate and its natural counterpart.

The NKPC and DIS curves

The NKPC and DIS equations are the non-policy block of the basic New Keynesian model.

- The NKPC determines inflation given a path for the output gap.
- The DIS determines output gap given a path for the (exogenous) natural rate and the actural real rate.

In order to close the model, we need to supplement these two equations with one or more equations determining how the nominal interest rate i_t ($\hat{\imath}_t$) evolves over time, i.e., with a description of how the monetary policy is conducted.
In new Keynesian models, monetary policy is non-neutral.

$$\hat{\imath}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + v_t \tag{21}$$

v_t: exogenous component with zero mean. $\phi_\pi > 0$, $\phi_y > 0$ and assume steady state inflation $\bar{\pi} = 0$.

Combine Equation (21) with (19), then the DIS becomes

$$\tilde{y}_t = -\frac{1}{\sigma}\left(\phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t - E_t\left\{\hat{\pi}_{t+1}\right\} - \left(\hat{r}_t^n - v_t\right)\right) + E_t\left\{\tilde{y}_{t+1}\right\}. \tag{22}$$

Equilibrium under an interest rate rule

Rewrite the NKPC and DIS

$$\begin{aligned}\left(1+\frac{\phi_y}{\sigma}\right)\tilde{y}_t+\frac{\phi_\pi}{\sigma}\hat{\pi}_t &= E_t\left\{\tilde{y}_{t+1}\right\}+\frac{1}{\sigma}E_t\left\{\hat{\pi}_{t+1}\right\}+\frac{1}{\sigma}\left(\hat{r}_t^n-v_t\right)\\ \hat{\pi}_t &= \beta E_t\left\{\hat{\pi}_{t+1}\right\}+\kappa\tilde{y}_t\end{aligned}$$

into a system of difference equations of $\left[\tilde{y}_t,\ \hat{\pi}_t\right]'$ in a matrix form

$$\underbrace{\begin{bmatrix} 1+\frac{\phi_y}{\sigma} & \frac{\phi_\pi}{\sigma} \\ -\kappa & 1 \end{bmatrix}}_{M}\begin{bmatrix} \tilde{y}_t \\ \hat{\pi}_t \end{bmatrix}=\begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix}\begin{bmatrix} E_t\left\{\tilde{y}_{t+1}\right\} \\ E_t\left\{\hat{\pi}_{t+1}\right\} \end{bmatrix}+\begin{bmatrix} \frac{1}{\sigma} \\ 0 \end{bmatrix}\left(\hat{r}_t^n-v_t\right)$$

Equilibrium under an interest rate rule

$$\begin{bmatrix} \tilde{y}_t \\ \hat{\pi}_t \end{bmatrix} = M^{-1} \begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} E_t\{\tilde{y}_{t+1}\} \\ E_t\{\hat{\pi}_{t+1}\} \end{bmatrix} + M^{-1} \begin{bmatrix} \frac{1}{\sigma} \\ 0 \end{bmatrix} (\hat{r}_t^n - v_t)$$

$$\begin{aligned} M^{-1} &= \frac{\begin{bmatrix} 1 & -\frac{\phi_\pi}{\sigma} \\ \kappa & 1+\frac{\phi_y}{\sigma} \end{bmatrix}}{\det(M)} \\ &= \begin{bmatrix} \sigma & -\phi_\pi \\ \kappa & 1+\frac{\phi_y}{\sigma} \end{bmatrix} \cdot \underbrace{\frac{1}{\sigma+\phi_y+\kappa\phi_\pi}}_{\Omega} \end{aligned}$$

The system as difference equations

$$
\begin{aligned}
A_T &\equiv \Omega \begin{bmatrix} \sigma & -\phi_\pi \\ \kappa & 1+\frac{\phi_y}{\sigma} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{bmatrix} \\
&= \begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa+\beta(\sigma+\phi_y) \end{bmatrix} \Omega
\end{aligned}
$$

$$
\begin{aligned}
B_T &\equiv \Omega \begin{bmatrix} \sigma & -\phi_\pi \\ \kappa & 1+\frac{\phi_y}{\sigma} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \kappa \end{bmatrix} \Omega
\end{aligned}
$$

The system can be written as

$$
\begin{bmatrix} \tilde{y}_t \\ \hat{\pi}_t \end{bmatrix} = A_T \begin{bmatrix} E_t\left\{\tilde{y}_{t+1}\right\} \\ E_t\left\{\hat{\pi}_{t+1}\right\} \end{bmatrix} + B_T\left(\hat{r}_t^n - v_t\right) \tag{23}
$$

The unique stable solution

Given that both $\tilde{y}_t$ and $\hat{\pi}_t$ are nonpredetermined, the solution to (23) is unique if and only if A_T has both eigenvalues within the unit circle. This results in the following constraint

$$\kappa\left(\phi_\pi - 1\right) + (1-\beta)\phi_y > 0.$$

If $\phi_y = 0$, the constraint becomes $\phi_\pi > 1$, consistent to the Taylor principle.

3.4.1.1 The effects of a monetary policy shock

Assume an AR(1) process of v_t,

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v, \;\; \rho_v \in [0,1)$$

- $\varepsilon_t^v > 0$: contractionary monetary policy shock.
- $\varepsilon_t^v < 0$: expansionary monetary policy shock.

For simplicity, shut down technology shock, such that $\hat{y}_t^n = 0$ and $\hat{r}_t^n = 0$ $\forall t$.
Using undetermined coefficient method, guess that

$$\begin{aligned}\tilde{y}_t &= \psi_{yv} v_t \\ \hat{\pi}_t &= \psi_{\pi v} v_t.\end{aligned}$$

The effects of a monetary policy shock

Insert the guessed solutions into Equation (22),

$$\tilde{y}_t = -\frac{1}{\sigma}\left(\phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t - E_t\left\{\hat{\pi}_{t+1}\right\} - (\hat{r}_t^n - v_t)\right) + E_t\left\{\tilde{y}_{t+1}\right\}. \quad (24)$$

$$\psi_{yv} v_t = -\frac{1}{\sigma}\left(\phi_\pi \psi_{\pi v} v_t + \phi_y \psi_{yv} v_t + v_t - \rho_v \psi_{\pi v} v_t\right) + \rho_v \psi_{yv} v_t$$

$$[\sigma(1-\rho_v) + \phi_y]\psi_{yv} = (\rho_v - \phi_\pi)\psi_{\pi v} - 1 \quad (25)$$

For the NKPC equation,

$$\begin{aligned} \hat{\pi}_t &= \beta E_t\left\{\hat{\pi}_{t+1}\right\} + \kappa \tilde{y}_t \\ \psi_{\pi v} v_t &= \beta \rho_v \psi_{\pi v} v_t + \kappa \psi_{yv} v_t \\ \psi_{yv} &= \frac{1-\beta\rho_v}{\kappa}\psi_{\pi v} \quad (26) \end{aligned}$$

The effects of a monetary policy shock

Impulse responses

Insert Equation (26) into (25),

$$[\sigma(1-\rho_v)+\phi_y]\frac{1-\beta\rho_v}{\kappa}\psi_{\pi v} = (\rho_v - \phi_\pi)\psi_{\pi v} - 1$$

$$\psi_{\pi v} = -\kappa \cdot \underbrace{\frac{1}{(1-\beta\rho_v)[\sigma(1-\rho_v)+\phi_y]+\kappa(\phi_\pi - \rho_v)}}_{\Lambda_v > 0}$$

$$\hat{\pi}_t = \underbrace{-\kappa\Lambda_v}_{<0} \cdot v_t$$

where $\kappa \equiv (\sigma + \frac{\varphi+\alpha}{1-\alpha})\lambda = (\sigma + \frac{\varphi+\alpha}{1-\alpha})\frac{(1-\theta)(1-\beta\theta)}{\theta}\frac{1-\alpha}{1-\alpha+\alpha\varepsilon} > 0$.

The effects of a monetary policy shock

Impulse responses

Then, the response of the output gap is

$$\tilde{y}_t = \underbrace{-(1-\beta\rho_v)\Lambda_v}_{<0} \cdot v_t.$$

The effects of a monetary policy shock

Impulse responses

$$
\begin{aligned}
\hat{r}_t &= \hat{\imath}_t - E_t\{\hat{\pi}_{t+1}\} \\
&= \phi_\pi \psi_{\pi v} v_t + \phi_y \psi_{yv} v_t + v_t + \kappa \Lambda_v \rho_v v_t \\
&= \left\{-\Lambda_v \left[\kappa\left(\phi_\pi - \rho_v\right) + (1-\beta\rho_v)\phi_y\right] + 1\right\} v_t \\
&= \left\{-\Lambda_v \left[\frac{1}{\Lambda_v} - \sigma\left(1-\rho_v\right)\left(1-\beta\rho_v\right)\right] + 1\right\} v_t \\
&= \underbrace{\sigma\left(1-\rho_v\right)\left(1-\beta\rho_v\right)\Lambda_v}_{>0} \cdot v_t. \\
\hat{\imath}_t &= \left[\sigma\left(1-\rho_v\right)\left(1-\beta\rho_v\right) - \kappa\rho_v\right]\Lambda_v \cdot v_t,
\end{aligned}
$$

where the sign is ambiguous upon the value of ρ_v. $\rho_v \uparrow$, $\hat{\imath}_t \downarrow$.

The effects of a monetary policy shock

Summary: Equilibrium under an interest rate rule

$$\tilde{y}_t = -\frac{1}{\sigma}\left(\hat{\imath}_t - E_t\{\hat{\pi}_{t+1}\} - \hat{r}_t^n\right) + E_t\{\tilde{y}_{t+1}\}. \quad (27)$$

$$\hat{\pi}_t = \beta E_t\{\hat{\pi}_{t+1}\} + \kappa \tilde{y}_t \quad (28)$$

$$\hat{\imath}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + v_t \quad (29)$$

The effects of a monetary policy shock

Summary: Equilibrium under an interest rate rule

Assume an AR(1) process of v_t,

$$
\begin{aligned}
v_t &= \rho_v v_{t-1} + \varepsilon_t^v, \ \ \rho_v \in [0,1) \\
\hat{\pi}_t &= -\kappa \Lambda_v \cdot v_t \\
\tilde{y}_t &= -(1-\beta\rho_v)\Lambda_v \cdot v_t \\
\hat{r}_t &= \sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v \cdot v_t \\
\hat{\imath}_t &= [\sigma(1-\rho_v)(1-\beta\rho_v) - \kappa\rho_v]\,\Lambda_v \cdot v_t
\end{aligned}
$$

Calibration: $\beta = 0.99$ (quarterly data), $\sigma = 1$ (log utility), $\varphi = 1$, $\alpha = 1/3$, $\varepsilon = 6$, $\theta = 2/3$, $\eta = 4$, $\phi_\pi = 1.5$, $\phi_y = 0.5$, $\rho_v = 0.5$.

For $\varepsilon_t^v = 25bp$, $\hat{\pi}_t < 0$, $\tilde{y}_t < 0$, $\hat{y}_t < 0$, $\hat{\imath}_t > 0$ (as $\rho_v = 0.5$, the shock is not persistent enough), $\hat{m}_t < 0$.

The responses are consistent with CEE (2005) VAR analysis qualitatively.

3.4.1.2 The effects of a technology shock

Assume an AR(1) process of a_t,

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a, \;\; \rho_a \in [0, 1)$$

In the DIS equation,

$$\begin{aligned}
\hat{r}_t^n &\equiv \sigma E_t \left\{ \Delta \hat{y}_{t+1}^n \right\} \\
&= \sigma \psi_{ya}^n E_t \left\{ \Delta a_{t+1} \right\} \\
&= -\sigma \psi_{ya}^n (1 - \rho_a) a_t
\end{aligned}$$

The effects of a technology shock

Setting $v_t = 0 \ \forall \ t$,

- Use the undetermined coefficient method w.r.t. $\hat{r}_t^n$,

$$\begin{aligned} \tilde{y}_t &= \psi_{yr} \hat{r}_t^n \\ \hat{\pi}_t &= \psi_{\pi r} \hat{r}_t^n \end{aligned}$$

- $\hat{r}_t^n$ enters the equilibrium conditions in a way symmetric to v_t but with an opposite sign.

$$\begin{bmatrix} \tilde{y}_t \\ \hat{\pi}_t \end{bmatrix} = A_T \begin{bmatrix} E_t \{\tilde{y}_{t+1}\} \\ E_t \{\hat{\pi}_{t+1}\} \end{bmatrix} + B_T \left(\hat{r}_t^n - v_t\right)$$

$$\psi_{yr} = -\psi_{yv}, \quad \psi_{\pi r} = -\psi_{\pi v}$$

The impulse responses to a technology shock

$$\begin{aligned}\tilde{y}_t &= (1-\beta\rho_v)\Lambda_a \hat{r}_t^n \\ &= -\sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a a_t \\ \hat{\pi}_t &= \kappa\Lambda_a v_t \\ &= -\sigma\psi_{ya}^n(1-\rho_a)\kappa\Lambda_a a_t\end{aligned}$$

where

$$\Lambda_a \equiv \frac{1}{(1-\beta\rho_a)[\sigma(1-\rho_a)+\phi_y]+\kappa(\phi_\pi-\rho_a)} > 0$$

The impulse responses to a technology shock

$$
\begin{aligned}
\hat{y}_t &= \hat{y}_t^n + \tilde{y}_t \\
&= \psi_{ya}^n a_t - \sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a a_t \\
&= \psi_{ya}^n \left[1-\sigma(1-\rho_a)(1-\beta\rho_a)\Lambda_a\right] a_t
\end{aligned}
$$

Ambiguous sign.

$$
\begin{aligned}
(1-\alpha)\hat{n}_t &= \hat{y}_t - a_t \\
&= \left[(\psi_{ya}^n - 1) - \sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a\right] a_t.
\end{aligned}
$$

In the baseline calibration: $\sigma = 1$, $\psi_{ya}^n = \frac{(1+\varphi)}{\varphi+\alpha+\sigma(1-\alpha)} = 1$. Employment declines for $a_t > 0$. Consistency with much empirical evidence.

3.4.2 Equilibrium under an exogenous money supply

Exogenous money supply: $\Delta \hat{m}_t$
Define log real money balance as

$$
\begin{aligned}
\hat{l}_t &\equiv \hat{m}_t - \hat{p}_t \\
\hat{l}_t &= \hat{y}_t - \eta \hat{\imath}_t \\
\tilde{y}_t - \eta \hat{\imath}_t &= \hat{l}_t - \hat{y}_t^n \\
\hat{\imath}_t &= \frac{1}{\eta}\left(\tilde{y}_t + \hat{y}_t^n - \hat{l}_t\right)
\end{aligned}
$$

Rewrite the DIS equation as

$$
(1+\sigma\eta)\tilde{y}_t = \sigma\eta E_t\left\{\tilde{y}_{t+1}\right\} + \hat{l}_t + \eta E_t\left\{\hat{\pi}_{t+1}\right\} + \eta \hat{r}_t^n - \hat{y}_t^n \qquad (30)
$$

Equilibrium under an exogenous money supply

From the definition of $\hat{l}_t$,

$$\begin{aligned} \hat{l}_t - \hat{l}_{t-1} &= \Delta \hat{m}_t - \hat{\pi}_t \\ \hat{l}_{t-1} &= \hat{l}_t + \hat{\pi}_t - \Delta \hat{m}_t \end{aligned} \tag{31}$$

Now the equilibrium dynamics for real balance, output gap and inflation can be formed as

$$A_{M,0} \begin{bmatrix} \tilde{y}_t \\ \hat{\pi}_t \\ \hat{l}_{t-1} \end{bmatrix} = A_{M,1} \begin{bmatrix} E_t \{\tilde{y}_{t+1}\} \\ E_t \{\hat{\pi}_{t+1}\} \\ \hat{l}_t \end{bmatrix} + B_M \begin{bmatrix} \hat{r}_t^n \\ \hat{y}_t^n \\ \Delta \hat{m}_t \end{bmatrix} \tag{32}$$

Equilibrium under an exogenous money supply

With one predetermined variable $\hat{l}_{t-1}$ and two nonpredetermined variables $\tilde{y}_t$ and $\hat{\pi}_t$, a stationary solution will exist and unique, if and only if

$$A_M \equiv A_{M,0}^{-1} A_{M,1}$$

has two eigenvalues inside the unit circle and one eigen value outside (or on) the unit circle.

This condition can be satisfied by a broad range of calibration.

3.4.2.1 The effects of a monetary policy shock

$$\Delta \hat{m}_t = \rho_m \Delta \hat{m}_{t-1} + \varepsilon_t^m, \;\; \rho_m \in [0, 1)$$

Setting

- $\hat{r}_t^n = \hat{y}_t^n = 0$, and
- $\varepsilon_t^m = 0.25 \; \forall \; t = 0$, $\varepsilon_t^m = 0 \; \forall \; t \neq 0$, an expansionary monetary policy shock.

$$\begin{aligned} \Delta \hat{m}_t &\uparrow \;\; , \tilde{y}_t \uparrow , \hat{\pi}_t \uparrow , \hat{l}_t \uparrow \\ \hat{\imath}_t &\uparrow \;\; : \text{no liquidity effect} \\ \hat{r}_t &\downarrow \end{aligned}$$

- The absence of a liquidity effect is the consequence of calibrated parameter $\sigma = 1$. If $\sigma \gg 1$, $\rho_m \ll 1$, liquidity effect presents.

3.4.2.2 The effects of a technology shock

$$
\begin{array}{lll}
a_t & \uparrow & \\
\hat{r}_t^n & \downarrow & \\
\hat{y}_t & \uparrow & , \hat{\pi}_t \uparrow , \hat{n}_t \downarrow , \hat{r}_t \uparrow \text{(contractionary)}, \hat{\imath}_t \simeq 0 \\
\hat{y}_t^n & \uparrow & \uparrow , \tilde{y}_t \downarrow
\end{array}
$$