Chapter 5 Risk-Neutral Pricing

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Outline

Girsanov's Theorem for a Single Brownian Motion

- Assume a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a nonnegative random variable Z satisfy ing $\mathbb{E}Z = 1$.
- Defined a new probability measure $\tilde{\mathbb{P}}$ by the formula

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \text{ for all } A \in \mathscr{F}. \tag{1}$$

- Relate expectations

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ]$$

- Z is the Radon-Nikodym derivative of $\mathbb{P}$ with respect to$\tilde{\mathbb{P}}$

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}.$$

Girsanov's Theorem for a Single Brownian Motion

- Define the Radon-Nikodym derivative process

$$Z(t) = \mathbb{E}[Z|\mathscr{F}(t)], 0 \leq t \leq T$$

- The Radon-Nikodym derivative process is a martingale: for $0 \leq s \leq t \leq T$,

$$\mathbb{E}[Z(t)|\mathscr{F}(s)] = \mathbb{E}[\mathbb{E}[Z|\mathscr{F}(t)]|\mathscr{F}(s)] = \mathbb{E}[Z|\mathscr{F}(s)] = Z(s)$$

Girsanov's Theorem for a Single Brownian Motion

Lemma 5.2.1.

Let t satisfying $0 \le t \le T$ be given and let Y be an $\mathscr{F}(t)$measurable random variable. Then

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)]$$

Lemma 5.2.2.

Let s and t satisfying $0 \le s \le t \le T$ be given and let Y be an $\mathscr{F}(t)$ measurable random variable. Then

$$\tilde{\mathbb{E}}[Y|\mathscr{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathscr{F}(s)]$$

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Girsanov's Theorem for a Single Brownian Motion

Theorem 5.2.3 (Girsanov, one dimension).

Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $\mathscr{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \leq t \leq T$, be an adapted process. Define

$$Z(t) = \exp\{-\int_0^t \Theta(u)\, dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\, du\},$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)\, du,$$

and assume that $\mathbb{E}\left[\int_0^T \Theta^2(u) Z^2(u)\, du\right] < \infty$. Set $Z = Z(t)$. Then $\mathbb{E}[Z] = 1$ and under the probability measure $\tilde{\mathbb{P}}$ given by (1) the process $\tilde{W}(t), 0 \leq t \leq T$,is a Brownian motion.

Stock Under the Risk-Neutral Measure

- Consider a stock price process whose differential is

$$dS(t) = \alpha(t) S(t) dt + \sigma(t) S(t) dW(t), 0 \leq t \leq T.$$

  The mean rate of return $\alpha(t)$ and the volatility $\sigma(t)$ are allowed to be adapted processes.

- Suppose an adapted interest rate process $R(t)$. Define the discount process

$$D(t) = e^{-\int_0^t R(s) ds}$$

$$dD(t) = -R(t) D(t) dt$$

Stock Under the Risk-Neutral Measure

- The discounted stock price process is

$$
\begin{aligned}
D(t)S(t) = S(0)\exp\Big\{ & \int_0^t \sigma(s)\,dW(s) \\
& + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds \Big\}
\end{aligned}
$$

$$
\begin{aligned}
dD(t)S(t) &= (\alpha(t) - R(t))\,D(t)S(t)\,dt \\
&\quad + \sigma(t)D(t)S(t)\,dW(t) \\
&= \sigma(t)D(t)S(t)\left[\Theta(t)\,dt + dW(t)\right]
\end{aligned}
$$

where we define the market price of risk to be

$$
\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}
$$

Stock Under the Risk-Neutral Measure

- Introduce the probability measure $\tilde{\mathbb{P}}$

$$dD(t)S(t) = \sigma(t)D(t)S(t)d\tilde{W}(t)$$

- The discounted stock price is

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\tilde{W}(u),$$

and under $\tilde{\mathbb{P}}$ the process $\int_0^t \sigma(u)D(u)S(u)d\tilde{W}(u)$ is an Ito integral and hence a martingale.

- By $dW(t) = -\Theta(t)dt + d\tilde{W}(t)$

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)d\tilde{W}(s) + \int_0^t \left(R(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}$$

Value of Portfolio Process Under the Risk-Neutral Measure

- The wealth equation revisited

$$\begin{aligned} dX(t) &= \Delta(t)\,dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\ &= R(t)X(t)\,dt + \Delta(t)(\alpha(t) - R(t))S(t)\,dt \\ &+ \Delta(t)\sigma(t)S(t)\,dW(t) \\ &= R(t)X(t)\,dt + \Delta(t)\sigma(t)S(t)[\Theta(t)\,dt + dW(t)] \end{aligned}$$

- The discounted value

$$dD(t)X(t) = \Delta(t)\sigma(t)D(t)S(t)\,d\tilde{W}(t)$$

Pricing Under the Risk-Neutral Measure

- Our agent wishes to choose initial capital $X(0)$ and portfolio strategy $\Delta(t), 0 \leq t \leq T$, such that

$$X(t) = V(t) \text{ a.s.}$$

where $V(t)$ be an $\mathscr{F}(t)$-measurable random variable for the derivative security price.

- That $D(t)X(t)$ is a martingale under $\tilde{\mathbb{P}}$ implie

$$D(t)X(t) = \tilde{\mathbb{E}}[D(t)X(t)| \mid \mathscr{F}(t)] = \tilde{\mathbb{E}}[V(t)X(t)| \mid \mathscr{F}(t)]$$

$$D(t)V(t) = \tilde{\mathbb{E}}[D(t)X(t)| \mid \mathscr{F}(t)], 0 \leq t \leq T$$

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)du} V(t) \middle| \mathscr{F}(t)\right], 0 \leq t \leq T$$

Deriving the Black-Scholes-Merton Formula

- To obtain the Black-Scholes-Merton price of a European call, we as sume a constant volatility σ, constant interest rate r, and take the derivative security payoff to be $V(t) = (S(t) - K)^+$.
- The call price

$$c(t, S(t)) = \tilde{\mathbb{E}}\left[e^{-r(T-t)} (S(t) - K)^+ \middle| \mathscr{F}(t) \right]$$

- The BSM Formula

$$BSM(\tau, x; K, r, \sigma) = xN(d_+(\tau, x)) - e^{-r\tau} KN(d_-(\tau, x))$$

Martingale Representation with One Brownian Motion

Theorem 5.3.1 (Martingale representation, one dimension).

Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $\mathscr{F}(t), 0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $M(t), 0 \leq t \leq T$, be a martingale with respect to this filtration (i.e., for every t, $M(t)$ is $\mathscr{F}(t)$-measurable and for $0 \leq s \leq t \leq T$, $\mathbb{E}[M(t)|\mathscr{F}(s)] = M(s)$). Then there is an adapted process $\Gamma(u), 0 \leq u \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u)\, dW(u), 0 \leq t \leq T$$

Martingale Representation with One Brownian Motion

Corollary 5.3.2.

Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $\mathscr{F}(t), 0 \le t \le T$, be the filtration generated by this Brownian motion. Let $\Theta(t), 0 \le t \le T$, be an adapted pro cess, define

$$\begin{aligned} Z(t) &= \exp\left\{-\int_0^t \Theta(u)\,dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\,du\right\}, \\ \tilde{W}(t) &= W(t) + \int_0^t \Theta(u)\,du, \end{aligned}$$

and assume that $\tilde{\mathbb{E}}\left[\int_0^T \Theta^2(u) Z^2(u)\,du\right] < \infty$. Set $Z = Z(t)$. Then $\mathbb{E}Z = 1$, and under the probability measure $\tilde{\mathbb{P}}$ given by (1), the process $\tilde{W}(t), 0 \le t \le T$, is a Brownian motion.

Hedging with One Stock

- Let $V(t)$ be an $\mathscr{F}(t)$-measurable random variable and, for $0 \leq t \leq T$, define $V(t)$ by the risk-neutral pricing formula

$$D(t)V(t) = \tilde{\mathbb{E}}[D(t)V(t)|\mathscr{F}(t).]$$

$$D(t)V(t) = V(0) + \int_0^t \tilde{\Gamma}(u)\,d\tilde{W}(u), 0 \leq t \leq T$$

- On the other hand, for any portfolio process $\Delta(t)$, the differential of the discounted portfolio value is

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)\,d\tilde{W}(u), 0 \leq t \leq T.$$

Hedging with One Stock

- In order to have $X(t) = V(t)$ for all t, we should choose

$$X(0) = V(0)$$

and choose $\Delta(t)$ to satisfy

$$\Delta(t)\sigma(t)D(t)S(t) = \tilde{\Gamma}(t).0 \leq t \leq T$$

which is equiva lent to

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{\sigma(t)D(t)S(t)}, 0 \leq t \leq T$$

Girsanov and Martingale Representation Theorems

- Throughout this section,

$$W(t) = (W_1(t), ..., W_d(t))$$

is a multidimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Girsanov and Martingale Representation Theorems

Theorem 5.4.1 (Girsanov, multiple dimensions).

Let T be a fixed positive time, and let $\Theta(t) = (\Theta_1(t), \ldots, \Theta_d(t))$ be a d-dimensional adapted process. Define

$$Z(t) = \exp\left\{-\int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du\right\}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)\, du$$

and assume that

$$\mathbb{E}\int_0^T \|\Theta(u)\|^2 Z^2(u)\, du < \infty.$$

Girsanov and Martingale Representation Theorems

Theorem 5.4.1 (Girsanov, multiple dimensions).

Set $Z = Z(t)$. Then $\mathbb{E}Z = 1$, and under the prob ability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \text{ for all } A \in \mathscr{F},$$

the process $\tilde{W}(t)$ is a d-dimensional Brownian motion.

Girsanov and Martingale Representation Theorems

Theorem 5.4.2 (Martingale representation, multiple dimensions).

Let T be a fi xed positive time, and assume that $\mathscr{F}(t), 0 \leq t \leq T$, is the filtration generated by the d-dimensional Brownian motion $W(t), 0 \leq t \leq T$. Let $M(t), 0 \leq t \leq T$, be a martingale with respect to this filtration under $\mathbb{P}$. Then there is an adapted, d-dimensional process $\Gamma(u) = (\Gamma_1(u), ..., \Gamma_d(u)), 0 \leq u \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u)\, dW(u), 0 \leq t \leq T$$

Girsanov and Martingale Representation Theorems

Theorem 5.4.2 (Martingale representation, multiple dimensions).

If, in addition, we assume the notation and assumptions of Theorem 5.4.1 and if $\tilde{M}(t), 0 \leq t \leq T$, is a $\tilde{\mathbb{P}}$-martingale, then there is an adapted, d-dimensional process $\tilde{\Gamma}(u) = (\tilde{\Gamma}_1(u), \ldots, \tilde{\Gamma}_d(u))$ such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u) \cdot d\tilde{W}(u), 0 \leq t \leq T$$

Multidimensional Market Model

- Assume there are m stocks, each with stochastic differential

$$dS_i(t) = \alpha_i(t) S_i(t) dt + S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t), i = 1, \ldots, m.$$

- Set $\sigma_i(t) = \sqrt{\sum_{j=1}^{d} \sigma_{ij}^2(t)}$, define processes

$$B_i(t) = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u), i = 1, \ldots, m.$$

Multidimensional Market Model

- For $i \neq k$, the Brownian motions $B_i(t)$ and $B_k(t)$ are typically not independent.

$$dB_i(t)\,dB_k(t) = \sum_{j=1}^{d} \frac{\sigma_{ij}(t)\,\sigma_{kj}(t)}{\sigma_i(t)\,\sigma_k(t)}\,dt = \rho_{ik}(t)\,dt$$

where

$$\rho_{ik}(t) = \frac{1}{\sigma_i(t)\,\sigma_k(t)} \sum_{j=1}^{d} \sigma_{ij}(t)\,\sigma_{kj}(t)$$

- The covariance formula

$$\mathbb{C}ov\left[B_i(t), B_k(t)\right] = \mathbb{E}\int_0^t \rho_{ik}(u)\,du.$$

Multidimensional Market Model

- Equities

$$dS_i(t) = \alpha_i(t) S_i(t)\,dt + \sigma_i(t) S_i(t)\,dB_i(t)$$

$$\begin{aligned} dS_i(t)\,dS_k(t) &= \sigma_i(t)\sigma_k(t) S_i(t) S_k(t)\,dB_i(t)\,dB_k(t) \\ &= \rho_{ik}(t)\sigma_i(t)\sigma_k(t) S_i(t) S_k(t)\,dt. \end{aligned}$$

$$\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_k(t)}{S_k(t)} = \rho_{ik}(t)\sigma_i(t)\sigma_k(t)\,dt.$$

Multidimensional Market Model

- Define a discount process

$$D(t) = e^{-\int_0^t R(u)du}$$

$$\begin{aligned}
& dD(t)S_i(t) \\
=& D(t)\left[dS_i(t) - R(t)S_i(t)dt\right] \\
=& D(t)S_i(t)\left[(\alpha_i(t) - R(t))dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right] \\
=& D(t)S_i(t)[[(\alpha_i(t) - R(t))dt + \sigma_i(t)dB_i(t)], i = 1, \ldots, m.
\end{aligned}$$

Existence of the Risk-Neutral Measure

Definition 5.4.3.

A probability measure $\tilde{\mathbb{P}}$ is said to be risk-neutral if
(i) $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent (i.e., for every $A \in \mathscr{F}$, $\mathbb{P}(A) = 0$ if and only if $\tilde{\mathbb{P}}(A) = 0$), and
(ii)under $\tilde{\mathbb{P}}$, the discounted stock price $D(t)S_i(t)$ is a martingale for every $i = 1, ..., m$.

Existence of the Risk-Neutral Measure

- The discounted stock prices

$$dD(t)S_i(t) = D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\left[\Theta_j(t)\,dt + dW_j(t)\right]$$

$$dD(t)S_i(t) = D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\,d\tilde{W}_j(t)$$

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{ij}(t)\Theta_j(t),\, i = 1, \ldots, m.$$

Existence of the Risk-Neutral Measure

- An agent begins with initial capital $X(0)$ and choose adapted portfolio processes $\Delta_i(t)$, one for each stock $S_i(t)$.

$$\begin{aligned} dD(t)X(t) &= D(t)(dX(t) - R(t)X(t)\,dt) \\ &= \sum_{i=1}^{m} \Delta_i(t)\,d(D(t)S_i(t)). \end{aligned}$$

Existence of the Risk-Neutral Measure

Lemma 5.4.5.

Let $\tilde{\mathbb{P}}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\tilde{\mathbb{P}}$, the discounted portfolio value $D(t)X(t)$ is a martingale.

Existence of the Risk-Neutral Measure

Definition 5.4.6.

An arbitrage is a portfolio value pro cess $X(t)$ satisfying $X(0) = 0$ and also satisfying for some time $T > 0$

$$\mathbb{P}\{X(t) \geq 0\} = 1, \mathbb{P}\{X(t) > 0\} > 0$$

Theorem 5.4.7 (First fundamental theorem of asset pricing).

If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

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If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Uniqueness of the Risk-Neutral Measure

Definition 5.4.8.

A market model is complete if every derivative security can be hedged.

Theorem 5.4.9 (Second fundamental theorem of asset pricing).

Consider a market model that has a risk-neutral probability measure . The model is complete if and only if the risk-neutral probability measure is unique.

Uniqueness of the Risk-Neutral Measure

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A market model is complete if every derivative security can be hedged.

Theorem 5.4.9 (Second fundamental theorem of asset pricing).

Consider a market model that has a risk-neutral probability measure . The model is complete if and only if the risk-neutral probability measure is unique.