Solution to Exercises in Chapter 7 (part 2)

7.4 ANS: The derivation is given in page 13 of our textbook.

7.5 ANS: The derivation is given in page 14-16 of our textbook.

7.11 ANS: Write the augmented model in matrix form as:

$$Y = X\beta^0 + \hat{V}\rho + \tilde{u} = [X, \hat{V}] \begin{pmatrix} \beta^0 \\ \rho \end{pmatrix} + \tilde{u}$$

where

$$\hat{V} = \begin{pmatrix} \hat{v}_1' \\ \vdots \\ \hat{v}_n' \end{pmatrix}$$

Then

$$\begin{aligned} \hat{\alpha} &= \{[X, \hat{V}]'[X, \hat{V}]\}^{-1}[X, \hat{V}]'Y \\ &= \begin{pmatrix} X'X & X'\hat{V} \\ \hat{V}'X & \hat{V}'\hat{V} \end{pmatrix}^{-1} \begin{pmatrix} X'Y \\ \hat{V}'Y \end{pmatrix} \end{aligned}$$

Consider the inverse of above partitioned matrix:

$$I^{11} = (X'X - X'\hat{V}(\hat{V}'\hat{V})^{-1}\hat{V}'X)^{-1}$$

where

$$\begin{aligned} X'\hat{V}(\hat{V}'\hat{V})^{-1}\hat{V}'X &= (\hat{V} + \hat{X})'\hat{V}(\hat{V}'\hat{V})^{-1}\hat{V}'(\hat{V} + \hat{X}) \\ &= \hat{V}'\hat{V}(\hat{V}'\hat{V})^{-1}\hat{V}'\hat{V} \\ &= \hat{V}'\hat{V} \end{aligned}$$

(note that $\hat{X}'\hat{V} = 0$)
Hence,

$$\begin{aligned} I^{11} &= (X'X - \hat{V}'\hat{V})^{-1} \\ &= ((\hat{V} + \hat{X})'(\hat{V} + \hat{X}) - \hat{V}'\hat{V})^{-1} \\ &= (\hat{X}'\hat{X})^{-1} \end{aligned}$$

and

$$\begin{aligned} I^{12} &= -I^{11}I_{12}I_{22}^{-1} \\ &= -(\hat{X}'\hat{X})^{-1}X'\hat{V}(\hat{V}'\hat{V})^{-1} \\ &= -(\hat{X}'\hat{X})^{-1}(\hat{V} + \hat{X})'\hat{V}(\hat{V}'\hat{V})^{-1} \\ &= -(\hat{X}'\hat{X})^{-1} \end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\beta} &= (\hat{X}'\hat{X})^{-1}X'Y - (\hat{X}'\hat{X})^{-1}\hat{V}'Y \\
&= (\hat{X}'\hat{X})^{-1}(X' - \hat{V}')Y \\
&= (\hat{X}'\hat{X})^{-1}\hat{X}'Y \\
&\equiv \hat{\beta}_{2sls}
\end{aligned}$$

7.12 ANS: Define $P_Z = Z(Z'Z)^{-1}Z'$, then

$$\begin{aligned}
\hat{Y} &= P_Z Y \\
\hat{X} &= P_Z X
\end{aligned}$$

Consider the following model:

$$\hat{Y} = \hat{X}\beta + \varepsilon$$

i.e.,

$$P_Z Y = P_Z X \beta + \varepsilon$$

The OLS estimator of above model is:

$$\begin{aligned}
\hat{\beta} &= (X'P_Z P_Z X)^{-1}X'P_Z P_Z Y \\
&= (X'P_Z X)^{-1}X'P_Z Y \\
&\equiv \hat{\beta}_{2sls}
\end{aligned}$$