Homework 2 solution
Econometrics II Spring, 2013

3.1 Answer

(a) $\binom{\hat\beta_0}{\hat\beta_1} = \arg\min_{\beta} \sum_{t=1}^{n} (Y_t - \beta_0 - X_{1t}\beta_1)^2$

FOC:

$$\frac{\partial \sum_{t=1}^{n} (Y_t - \beta_0 - X_{1t}\beta_1)^2}{\partial \beta_1} = -2\sum_{t=1}^{n} (Y_t - \hat\beta_0 - X_{1t}\hat\beta_1)X_{1t} = 0 \tag{1}$$

$$\frac{\partial \sum_{t=1}^{n} (Y_t - \beta_0 - X_{1t}\beta_1)^2}{\partial \beta_0} = -2\sum_{t=1}^{n} (Y_t - \hat\beta_0 - X_{1t}\hat\beta_1) = 0 \tag{2}$$

From (2),

$$n\hat\beta_0 = \sum_{t=1}^{n} Y_t - \hat\beta_1 \sum_{t=1}^{n} X_{1t}$$

Hence,

$$\hat\beta_0 = \bar{Y} - \hat\beta_1 \bar{X}_1 \tag{3}$$

Plug (3) into (1), we get

$$\begin{aligned}
\hat\beta_1 &= \frac{\sum_{t=1}^{n} (Y_t - \bar{Y})X_{1t}}{\sum_{t=1}^{n} (X_{1t} - \bar{X}_1)X_{1t}} \\
&= \frac{\sum_{t=1}^{n} (Y_t - \bar{Y})(X_{1t} - \bar{X}_1)}{\sum_{t=1}^{n} (X_{1t} - \bar{X}_1)^2} \\
&= \frac{\sum_{t=1}^{n} (X_{1t} - \bar{X}_1)Y_t}{\sum_{t=1}^{n} (X_{1t} - \bar{X}_1)^2} \\
&= \sum_{t=1}^{n} C_t Y_t
\end{aligned}$$

(b)

$$\begin{aligned}
var(\hat{\beta}_1|X) &= var(\sum_{t=1}^{n} C_t Y_t|X) \overset{independent}{=} \sum_{t=1}^{n} C_t^2 var(Y_t|X) \\
&= \sigma_\varepsilon^2 \sum_{t=1}^{n} C_t^2 = \sigma_\varepsilon^2 \sum_{t=1}^{n} [\frac{(X_{1t} - \bar{X_1})}{\sum_{t=1}^{n}(X_{1t} - \bar{X_1})^2}]^2 \\
&= \frac{\sigma_\varepsilon^2}{(n-1)S_{X_1}^2}
\end{aligned}$$

(c)

$$\begin{aligned}
R^2 &= \frac{\sum(\hat{Y}_t - \bar{Y})^2}{\sum(Y_t - \bar{Y})^2} \\
&= \frac{\sum(\hat{Y}_t - \bar{Y})(Y_t - \bar{Y})}{\sum(Y_t - \bar{Y})^2} \quad \text{(cf. solution to quiz 2)} \\
&= \frac{\sum(\hat{\beta}_0 + X_{1t}\hat{\beta}_1 - \hat{\beta}_0 - \hat{\beta}_1 \bar{X_1})(Y_t - \bar{Y})}{\sum(Y_t - \bar{Y})^2} \\
&= \frac{\hat{\beta}_1 \sum(X_{1t} - \bar{X_1})(Y_t - \bar{Y})}{\sum(Y_t - \bar{Y})^2} \\
&= \frac{\hat{\beta}_1 [\sum(X_{1t} - \bar{X_1})(Y_t - \bar{Y})]^2}{\sum(Y_t - \bar{Y})^2 [\sum(X_{1t} - \bar{X_1})(Y_t - \bar{Y})]} \\
&= \hat{\rho}^2
\end{aligned}$$

where we use the fact that

$$\begin{aligned}
\sum(X_{1t} - \bar{X_1})(Y_t - \bar{Y}) &= \sum(X_{1t} - \bar{X_1})(X_{1t}\hat{\beta}_1 + e_t - \hat{\beta}_1 \bar{X_1}) \\
&= \hat{\beta}_1 \sum(X_{1t} - \bar{X_1})^2
\end{aligned}$$

3.4 Answer

The model is

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \varepsilon_t$$

Let $X = (\mathbf{1}, X_2, X_1)$, where $X_i = (X_{i1}, X_{i2}, \cdots, X_{iN})$ for $i = 1, 2$, and $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_2, \hat{\beta}_1)^\intercal$, then $var(\hat{\beta}|X) = \sigma^2 (X^\intercal X)^{-1}$. To compute $var(\hat{\beta}_1|X)$, we are interested in the lower right element in $\sigma^2 (X^\intercal X)^{-1}$.

$$X^\intercal X = \begin{pmatrix} \mathbf{1}^\intercal \\ X_2^\intercal \\ X_1^\intercal \end{pmatrix} (\mathbf{1}, X_2, X_1) = \begin{pmatrix} N & N\bar{X}_2 & N\bar{X}_1 \\ N\bar{X}_2 & X_2^\intercal X_2 & X_2^\intercal X_1 \\ N\bar{X}_1 & X_1^\intercal X_2 & X_1^\intercal X_1 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where $I_{11} = \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\intercal X_2 \end{pmatrix}, I_{12} = \begin{pmatrix} N\bar{X}_1 \\ X_2^\intercal X_1 \end{pmatrix} = I_{21}^\intercal, I_{22} = X_1^\intercal X_1.$
So the lower right element in $(X^\intercal X)^{-1}$, denoted as I^{22}, is $(I_{22}-I_{21}I_{11}^{-1}I_{12})^{-1}$.

$$\begin{aligned}
I_{22} - I_{21}I_{11}^{-1}I_{12} &= [X_1^\intercal X_1 - (N\bar{X}_1, X_1^\intercal X_2) \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\intercal X_2 \end{pmatrix}^{-1} \begin{pmatrix} N\bar{X}_1 \\ X_2^\intercal X_1 \end{pmatrix}] && (4) \\
&= [X_1^\intercal X_1 - X_1^\intercal (\mathbf{1}, X_2) \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\intercal X_2 \end{pmatrix}^{-1} \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\intercal X_2 \end{pmatrix} \\
&\quad \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\intercal X_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1} \\ X_2^\intercal \end{pmatrix} X_1] && (5) \\
&= [X_1^\intercal X_1 - X_1^\intercal (\mathbf{1}, X_2) \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\intercal X_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1} \\ X_2^\intercal \end{pmatrix} (\mathbf{1}, X_2) \\
&\quad \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\intercal X_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1} \\ X_2^\intercal \end{pmatrix} X_1] && (6) \\
&= X_1^\intercal X_1 - \hat{X}_1^\intercal \hat{X}_1
\end{aligned}$$

where $\hat{X}_1$ is the fitting value of X_1 in the regresion of X_1 on $(\mathbf{1}, X_2)$.
Note that, in the regresion of X_1 on $(\mathbf{1}, X_2)$,

$$R^2 = 1 - \frac{(X_1^\intercal X_1 - \hat{X}_1^\intercal \hat{X}_1)}{\sum (X_{1t} - \bar{X}_1)^2}$$

Hence

$$X_1^\intercal X_1 - \hat{X}_1^\intercal \hat{X}_1 = (1 - R^2) \sum (X_{1t} - \bar{X}_1)^2 \tag{7}$$

And, using (c) in the previous problem, we have

$$R^2 = \hat{r}^2 \tag{8}$$

Combining (6), (7) and (8), we get

$$var(\hat{\beta}_1|X) = \frac{\sigma^2}{(1-\hat{r}^2)\sum(X_{1t}-\bar{X}_1)^2}$$

Following the same steps, you can get $var(\hat{\beta}_2|X)$.

3.5 Answer

The idea is similar to the previous problem. Firstly, we rearrange $X = (X_1, X_2, \cdots, X_N)^\intercal$ as (X_{-j}, X^j), where X_{-j} is the matrix which is obtained by deleting the j-th column of X and X^j is the j-th column of X. Also, rearrange $\hat{\beta}$ accordingly. We still use X and $\hat{\beta}$ to denote the new regressors matrix and the coefficients vector. Then

$$\begin{aligned} var(\hat{\beta}|X) &= \sigma^2(X^\intercal X)^{-1} \\ &= \sigma^2[(X_{-j}, X^j)^\intercal (X_{-j}, X^j)]^{-1} \\ &= \sigma^2 \begin{pmatrix} X_{-j}^\intercal X_{-j} & X_{-j}^\intercal X^j \\ X^{j\intercal} X_{-j} & X^{j\intercal} X^j \end{pmatrix}^{-1} \end{aligned}$$

We are interested in the lower right element in $\begin{pmatrix} X_{-j}^\intercal X_{-j} & X_{-j}^\intercal X^j \\ X^{j\intercal} X_{-j} & X^{j\intercal} X^j \end{pmatrix}^{-1} \equiv \begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix}$.

$$\begin{aligned} I^{22} &= [X^{j\intercal} X^j - X^{j\intercal} X_{-j}(X_{-j}^\intercal X_{-j})^{-1} X_{-j}^\intercal X^j]^{-1} \\ &= [X^{j\intercal}(I - X_{-j}(X_{-j}^\intercal X_{-j})^{-1} X_{-j}^\intercal) X^j]^{-1} \\ &= \frac{1}{X^{j\intercal} M_{-j} X^j} \end{aligned}$$

where M_{-j} is the residual maker in the regresion of which the regressor matrix is X_{-j}.

Now, consider the regresion of X^j on X_{-j}.

$$R_j^2 = 1 - \frac{X^{j\intercal} M_{-j} X^j}{\sum(X_t^j - \bar{X}^j)^2}$$

Hence,

$$var(\hat{\beta}_j|X) = \frac{\sigma^2}{X^{j\intercal} M_{-j} X^j} = \frac{\sigma^2}{(1-R_j^2)\sum(X_{jt}-\bar{X}_j)^2}$$