Solution to Problem Set 8

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Given

$$P_t = E_t\left[\sum_{j=1}^{\infty}\delta^j \frac{u_C\left(C^*_{t+j}\right)}{u_C\left(C^*_t\right)} d_{t+j}\right] = E_t\left[\sum_{j=1}^{\infty}\delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} d_{t+j}\right],$$

$$\ln\left(C^*_{t+1}/C^*_t\right) = \mu_c + \sigma_c \eta_{t+1},$$

$$\ln\left(d_{t+1}/d_t\right) = \mu_d + \sigma_d \varepsilon_{t+1},$$

and

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

Show that

$$P_t = d_t \frac{\delta e^{\alpha}}{1-\delta e^{\alpha}},$$

where

$$\alpha \equiv \mu_d - (1-\gamma)\,\mu_c + \frac{1}{2}\left[(1-\gamma)^2\,\sigma_c^2 + \sigma_d^2\right] - (1-\gamma)\,\rho\sigma_c\sigma_d.$$

See textbook for the context.

Suggested Answer

Since $\ln\left(C_{t+1}^*/C_t^*\right) = \mu_c + \sigma_c \eta_{t+1}$, namely $C_{t+1}^*/C_t^* = e^{\mu_c + \sigma_c \eta_{t+1}}$, we could get

$$
\begin{aligned}
\ln\left(C_{t+j}^*/C_t^*\right) &= \ln\left(\frac{C_{t+j}^*}{C_{t+j-1}^*} \cdot \frac{C_{t+j-1}^*}{C_{t+j-2}^*} \cdot \cdots \cdot \frac{C_{t+1}^*}{C_t^*}\right) \\
&= \ln\left(C_{t+j}^*/C_{t+j-1}^*\right) + \ln\left(C_{t+j-1}^*/C_{t+j-2}^*\right) + \cdots + \ln\left(C_{t+1}^*/C_t^*\right) \\
&= \mu_c + \sigma_c \eta_{t+j} + \mu_c + \sigma_c \eta_{t+j-1} + \cdots + \mu_c + \sigma_c \eta_{t+2} + \mu_c + \sigma_c \eta_{t+1} \\
&= j\mu_c + \sigma_c\left(\eta_{t+1} + \eta_{t+2} + \cdots + \eta_{t+j}\right) = j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+i}
\end{aligned}
$$

or $C_{t+j}^*/C_t^* = \exp\left\{j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+i}\right\}$

Similarly, since $\ln\left(d_{t+1}/d_t\right) = \mu_d + \sigma_d \varepsilon_{t+1}$, we have $d_{t+j}/d_t = \exp\left\{j\mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+i}\right\}$. So

$$
\begin{aligned}
P_t &= E_t\left[\sum_{j=1}^{\infty} \delta^j \frac{u_C\left(C_{t+j}^*\right)}{u_C\left(C_t^*\right)} d_{t+j}\right] = E_t\left[d_t \cdot \sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}^*}{C_t^*}\right)^{\gamma-1} \frac{d_{t+j}}{d_t}\right] \\
&= E_t\left[d_t \cdot \sum_{j=1}^{\infty} \delta^j \cdot \left(\underbrace{\exp\left\{j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+i}\right\}}_{=C_{t+j}^*/C_t^*}\right)^{\gamma-1} \cdot \underbrace{\exp\left\{j\mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+i}\right\}}_{=d_{t+j}/d_t}\right] \\
&= d_t \sum_{j=1}^{\infty} \delta^j E_t\left[\exp\left\{(\gamma-1)\left(j\mu_c + \sigma_c \sum_{i=1}^{j} \eta_{t+i}\right) + \left(j\mu_d + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+i}\right)\right\}\right] \\
&= d_t \sum_{j=1}^{\infty} \delta^j \exp\{j \cdot \alpha\} \rightarrow (\text{ this is from the next box }) \\
&= d_t\left[\delta e^{\alpha} + \delta^2 e^{2\alpha} + \delta^3 e^{3\alpha} + \cdots + \delta^T e^{T\alpha}\right] \\
&= d_t \frac{\delta e^{\alpha}\left[1 - \left(\delta e^{\alpha}\right)^{\infty}\right]}{1 - \delta e^{\alpha}} = d_t \frac{\delta e^{\alpha}}{1 - \delta e^{\alpha}}
\end{aligned}
$$

Since $\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$,

$$\begin{aligned}
& \left[(\gamma-1)\, j\mu_c + j\mu_d\right] + \left[(\gamma-1)\,\sigma_c \sum_{i=1}^{j} \eta_{t+i} + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+i}\right] \\
\sim\ & N\left(\underbrace{\left[(\gamma-1)\, j\mu_c + j\mu_d\right]}_{\mu},\ \underbrace{\left[j\,(\gamma-1)^2\,\sigma_c^2 + j\sigma_d^2 + 2j\rho\,(\gamma-1)\,\rho\sigma_d\sigma_c\right]}_{\sigma^2}\right) \\
\sim\ & N\left(j \cdot \left[(\gamma-1)\,\mu_c + \mu_d\right],\quad j \cdot \left[(\gamma-1)^2\,\sigma_c^2 + \sigma_d^2 + 2\rho\,(\gamma-1)\,\rho\sigma_d\sigma_c\right]\right)
\end{aligned}$$

and if $\tilde{x} \sim N\left(\mu, \sigma^2\right)$, $\tilde{z} = e^{\tilde{x}}$, then the expected value of $\tilde{z}$ is

$$E\left[\tilde{z}\right] = e^{\mu + \frac{1}{2}\sigma^2}$$

We can get

$$\begin{aligned}
& E_t\left[\exp\left\{\left[(\gamma-1)\, j\mu_c + j\mu_d\right] + \left[(\gamma-1)\,\sigma_c \sum_{i=1}^{j} \eta_{t+i} + \sigma_d \sum_{i=1}^{j} \varepsilon_{t+i}\right]\right\}\right] \\
=\ & \exp\left\{\underbrace{j \cdot \left[(\gamma-1)\,\mu_c + \mu_d\right]}_{\mu} + \frac{1}{2} \cdot \underbrace{j \cdot \left[(\gamma-1)^2\,\sigma_c^2 + \sigma_d^2 + 2\rho\,(\gamma-1)\,\rho\sigma_d\sigma_c\right]}_{\sigma^2}\right\} \\
=\ & \exp\left\{j \cdot \left[(\gamma-1)\,\mu_c + \mu_d\right] + \frac{j}{2} \cdot \left[(\gamma-1)^2\,\sigma_c^2 + \sigma_d^2\right] + j \cdot \rho\,(\gamma-1)\,\rho\sigma_d\sigma_c\right\} \\
=\ & \exp\left\{j \cdot \left[(\gamma-1)\,\mu_c + \mu_d + \frac{1}{2}\left[(\gamma-1)^2\sigma_c^2 + \sigma_d^2\right] + (\gamma-1)\,\rho\sigma_d\sigma_c\right]\right\} \\
=\ & \exp\left\{j \cdot \alpha\right\}
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= (\gamma-1)\,\mu_c + \mu_d + \frac{1}{2}\left[(\gamma-1)^2\sigma_c^2 + \sigma_d^2\right] + (\gamma-1)\,\rho\sigma_d\sigma_c \\
&= \mu_d - (1-\gamma)\,\mu_c + \frac{1}{2}\left[(1-\gamma)^2\,\sigma_c^2 + \sigma_d^2\right] - (1-\gamma)\,\rho\sigma_c\sigma_d
\end{aligned}$$