Mixed Strategy Nash Equilibrium and Best Response

Lemma

Let $G = \{N, (A_i), (u_i)\}$ be a finite game. Then $\alpha^* \in \times_{i \in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G if and only if for every player $i \in N$ every pure strategy in the support of α_i^* is a best response to α_{-i}^*.

- We can write $U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(e(a_i), \alpha_{-i})$, where $e(a_i)$ is the strategy that plays a_i with probability one.
- ($\Rightarrow$) If a_i in the support of α_i is not a best response, then transfer $\alpha_i(a_i)$ to a best response action.
- ($\Leftarrow$) If there exists another α_i' that gives a higher payoff then there must be at least one action in the support of α_i' that gives a higher payoff than some action in the support of α_i^*.

Implication: Every action in the support of α_i^* yields the same payoff.