1. **Use Jensen's Inequality to show that a risk-averse individual whose preferences can be modeled by a von Neumann-Morgenstern expected utility function will not accept a fair lottery.**

   Solution: Risk aversion implies that the individual's utility function is concave. Hence if $U(\cdot)$ is some concave function, and $\tilde{x}$ is a random variable, then Jensen's inequality says that

$$\mathbb{E}\left[U(\tilde{x})\right] < U\left(\mathbb{E}\left[\tilde{x}\right]\right)$$

   Therefore, substituting $\tilde{x} = W + \tilde{\varepsilon}$, with $\mathbb{E}[\tilde{\varepsilon}] = 0$, we have

$$\mathbb{E}\left[U\left(W+\tilde{\varepsilon}\right)\right] < U\left(\mathbb{E}\left[W+\tilde{\varepsilon}\right]\right) = U(W)$$

2. **Suppose an individual has an utility funcion in the form of**

$$U(W) = \frac{1-\gamma}{\gamma}\left(\frac{\alpha W}{1-\gamma}+\beta\right)^{\gamma}, \gamma \neq 1$$

   subject to the restrictions $\gamma \neq 1$, $\alpha > 0$, $\dfrac{\alpha W}{1-\gamma} + \beta > 0$.

   (a) **Verify that the utility function becomes the constant absolute-risk-aversion utility function when $\beta = 1$ and $\gamma = -\infty$.**

   Solution

$$\begin{aligned}
&= \lim_{\gamma\to-\infty} \frac{1-\gamma}{\gamma}\left(\frac{\alpha W}{1-\gamma}+1\right)^{\gamma} \\
&= -\lim_{\gamma\to-\infty} \exp\left\{\gamma \ln\left(\frac{\alpha W}{1-\gamma}+1\right)\right\} = -\lim_{\gamma\to-\infty} \exp\left\{\frac{\ln\left(\frac{\alpha W}{1-\gamma}+1\right)}{1/\gamma}\right\} \\
&= -\lim_{\gamma\to-\infty} \exp\left\{\frac{d\ln\left(\frac{\alpha W}{1-\gamma}+1\right)/d\gamma}{d\left(1/\gamma\right)/d\gamma}\right\} = -\lim_{\gamma\to-\infty} \exp\left\{\frac{\frac{\alpha W(1-\gamma)^{-2}}{\frac{\alpha W}{1-\gamma}+1}}{-\gamma^{-2}}\right\} \\
&= -\lim_{\gamma\to-\infty} \exp\left\{\frac{-\alpha W\gamma^{2}}{\alpha W(1-\gamma)+(1-\gamma)^{2}}\right\} = -\lim_{\gamma\to-\infty} \exp\left\{-\alpha W\right\} \\
U(W) &= -e^{-\alpha W}
\end{aligned}$$

   The coefficient of absolute risk aversion is

$$R(W) = -\frac{U''(W)}{U'(W)} = -\frac{-\alpha^{2}e^{-\alpha W}}{\alpha e^{-\alpha W}} = \alpha$$

   which is a constant, i.e. the utility function is a constant absolute-risk-aversion utility function.

   (b) **Suppose a individual has expected utility of the form $E\left[U\left(\tilde{W}\right)\right]$, where $U(\cdot)$ is constant absolute-risk-aversion utility function. The individual's wealth is normally distributed as $N\left(\bar{W}, \sigma_W^2\right)$. What is this individual's certainty equivalent level of wealth?**

   Solution

$$E\left[U\left(\tilde{W}\right)\right] = E\left[-e^{-\alpha\tilde{W}}\right] = -e^{-\alpha\bar{W}+\frac{1}{2}\alpha^{2}\sigma_W^{2}} = -e^{-\alpha\left(\bar{W}-\frac{\alpha}{2}\sigma_W^{2}\right)} = U\left(\bar{W}-\frac{\alpha}{2}\sigma_W^{2}\right)$$

3. **Consider the standard Markowitz mean-variance portfolio choice problem where there are a riskless asset with return $R_f > 0$ and n risky asset.**
   The risky assets' $n \times 1$ vector of returns, $\tilde{R}$, have a multi-variate normal distribution $N\left(\bar{R}, V\right)$ where $\bar{R} = \begin{pmatrix} \bar{R}_1 & \bar{R}_2 & \dots & \bar{R}_n \end{pmatrix}'$ is the assets' $n\times 1$ vector of the expected returns and V is a nonsingular $n \times n$ covariance matrix. Let $\omega = \begin{pmatrix} \omega_1 & \omega_2 & \dots & \omega_n \end{pmatrix}'$ be an $n \times 1$ vector of portfolio proportions, such that ω_i is the proportion of total portfolio wealth invested in the i^{th} asset. Let $\tilde{R}_p$ represent the return on the portfolio, with the expected return $\bar{R}_p$ and variance σ_p^2.

(a) **Under the mean-variance framework, please write down the mathematical expression of an individual's optimization problem and show the first order conditions for this optimization problem.**

Solution

$$\bar{R}_p = (1 - \omega' e) R_f + \omega' \bar{R} = R_f - \omega' e R_f + \omega' \bar{R} = R_f + \omega' \left(\bar{R} - R_f e\right)$$
$$\sigma_p^2 = \omega' V \omega$$

The individual's optimization problem is :

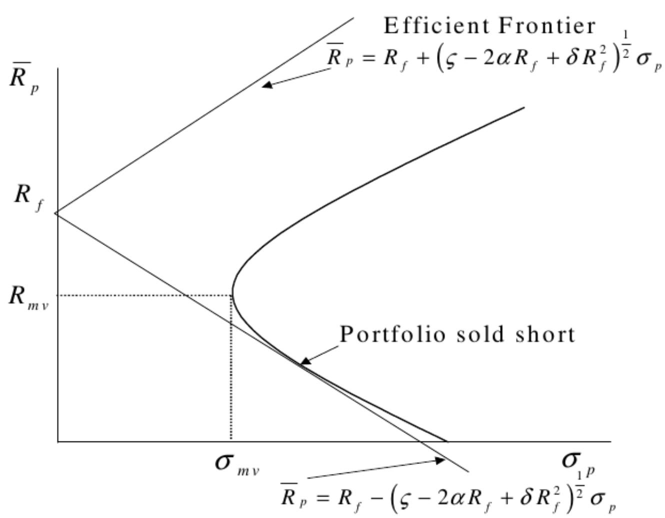
$$\min_w \frac{1}{2} \omega' V \omega + \lambda \left\{ \bar{R}_P - \left[R_f + \omega' \left(\bar{R} - R_f e \right) \right] \right\}$$

FOCs

$$\begin{aligned} \frac{\partial L(w, \lambda)}{\partial w} &= V \omega^* - \lambda \left(\bar{R} - R_f e \right) = 0 \\ \frac{\partial L(w, \lambda)}{\partial w} &= \bar{R}_P - \left[R_f + \omega' \left(\bar{R} - R_f e \right) \right] = 0 \end{aligned}$$

(b) **In portfolio standard deviation-expected return space, that is, $\sigma_P, \bar{R}_P$ space, graphically show the portfolio frontier (both efficient and inefficient portions of the frontier) for the case in which the return on the risk-free asset exceeds the return on the minimum variance risky asset portfolio, that is, $R_f > R_{mv}$.**

Solution



(c) **Also for the case, $R_f > R_{mv}$, explain the nature of the investor's portfolio along the efficient frontier. Would this situation be consistent with a market equilibrium assumed by the Capital Asset Pricing Model (CAPM) ? Why or why not?**

Solution

When $R_f > R_{mv}$, the efficient frontier is always above the risky-asset-only frontier. The efficient frontier is acheived by short positions in the tangency portfolio with the proceeds invested in the risk-free asset. This would not be consistent with a CAPM equilibrium because everyone could not have a short position in the same market portfolio. The aggregate position in the market portfolio would have to be a long position.

4. **Assume that individual investor k chooses between n risky assets in order to maximize the following utility function**

$$\max_{\{\omega_i^k\}} \bar{R}_k - \frac{1}{\theta_k} V_k$$

where the mean and variance of investor k's portfolio are $\bar{R}_k = \sum_{i=1}^n \omega_i^k \bar{R}_i$ and $V_k = \sum_{i=1}^n \sum_{j=1}^n \omega_i^k \omega_j^k \sigma_{ij}$, respectively, and where $\bar{R}_i$ is the expected return on risky asset i and σ_{ij} is the covariance between

the returns on risky asset i and risky asset j. ω_i^k is investor k's portfolio weight invested in risky asset i, so that $\sum_{i=1}^n \omega_i^k = 1$. θ_k is a positive constant and equals investor k's risk tolerance.

(a) **Write down the Lagrangian for this problem and show the first-order conditions.**
Solution

$$\mathcal{L} = \bar{R}_k - \frac{1}{\theta_k} V_k + \lambda_k \left(1 - \sum_{i=1}^n \omega_i^k\right) = \sum_{i=1}^n \omega_i^k \bar{R}_i - \frac{1}{\theta_k} \sum_{i=1}^n \sum_{j=1}^n \omega_i^k \omega_j^k \sigma_{ij} + \lambda_k \left(1 - \sum_{i=1}^n \omega_i^k\right) \tag{1}$$

The FOCs:

$$\frac{\partial \mathcal{L}}{\partial \omega_i^k} = \bar{R}_i - \frac{2}{\theta_k} \sum_{i=1}^n \omega_j^k \sigma_{ij} - \lambda_k = 0 \qquad i = 1, \ldots n$$

or

$$\bar{R}_i - \frac{2}{\theta_k} \sum_{i=1}^n \omega_j^k \sigma_{ij} = \lambda_k \qquad i = 1, \ldots n \tag{2}$$

(b) **Re-write the first-order condition to show that the expected return on asset i is a linear function of the covariance between risky asset i's return and the return on investor k's optimal portfolio.**
Solution

$$\bar{R}_i - \frac{2}{\theta_k} \sum_{i=1}^n \omega_j^k \sigma_{ij} = \lambda_k \qquad i = 1, \ldots n$$

where $\sum_{i=1}^n \omega_j^k \sigma_{ij}$ is the covariance of asset i with investor k's optimal portfolio, that is, $\mathbb{C}ov\left(\tilde{R}_i, \tilde{R}_p^k\right) = \sum_{i=1}^n \omega_j^k \sigma_{ij}$. Hence, the eqution above can be re-written as

$$\bar{R}_i = \lambda_k + \frac{2}{\theta_k} \mathbb{C}ov\left(\tilde{R}_i, \tilde{R}_p^k\right) \qquad i = 1, \ldots n \tag{3}$$

(c) **Assume that investor k has initial wealth equal to W_k and that there are $k = 1, \ldots, M$ total investors, each with different initial wealth and risk tolerance. Show that the equilibrium expected return on asset i is of a similar form to the first-order condition found in part (b), but depends on the wealth-weighted risk tolerances of investors and the covariance of the return on asset i with the market portfolio.**
Solution

Multiplying (2) by $W_k \theta_k$ gives

$$W_k \theta_k \overline{R}_i - 2W_k \sum_{j=1}^n w_j^k \sigma_{ij} = W_k \theta_k \lambda_k \tag{4}$$

Summing over all investors, we obtain

$$\sum_{k=1}^M W_k \theta_k \overline{R}_i - 2\sum_{k=1}^M W_k \sum_{j=1}^n w_j^k \sigma_{ij} = \sum_{k=1}^M W_k \theta_k \lambda_k \tag{5}$$

Let $\theta_M \equiv \sum_{k=1}^M W_k \theta_k$ be the wealth-weighted risk tolerances of the M investors. Note also that

$$2\sum_{k=1}^M W_k \sum_{j=1}^n w_j^k \sigma_{ij} = 2\sum_{j=1}^n \sum_{k=1}^M w_j^k W_k \sigma_{ij} = 2Cov\left(\widetilde{R}_i, \tilde{R}_M\right) \tag{6}$$

is equal to two times the covariance between asset i's return and the return on the market portfolio. Thus, (5) can be re-written as

$$\overline{R}_i = \frac{1}{\theta_M}\sum_{k=1}^{M} W_k\theta_k\lambda_k + \frac{2}{\theta_M}Cov\left(\widetilde{R}_i, \tilde{R}_M\right)$$

5. **Suppose that the Arbitrage Pricing Theory holds with $k = 2$ risk factors, so that the following model describes asset returns**

$$\tilde{r}_i = a_i + b_{i1}f_1 + b_{i2}f_2 + \tilde{\varepsilon}_i$$

Assume that the following three portfolios are observed.

Portfolio	**Expected returns**	b_{i1}	b_{i2}
A	12.0%	1	0.5
B	13.4%	3	0.2
D	12.0%	3	-0.5

(a) **Please find the values of the risk premium for risk factors.**
Solution
From the main APT equation and the problem data, one obtains the following system :

$$\begin{aligned} a_i &= \lambda_0 + b_{i1}\lambda_1 + b_{i2}\lambda_2 \\ 12.0\% &= \lambda_0 + \lambda_1 + 0.5\lambda_2 \\ 13.4\% &= \lambda_0 + 3\lambda_1 + 0.2\lambda_2 \\ 12.0\% &= \lambda_0 + 3\lambda_1 - 0.5\lambda_2 \end{aligned}$$

This system can easily be solved for

$$\lambda_0 = 10\%, \quad \lambda_1 = 1\%, \quad \lambda_2 = 2\%$$

Thus, the APT tells us that

$$\mathbb{E}\left[r_i\right] = 10\% + 1\% b_{i1} + 2\% b_{i2}$$

(b) **If $\tilde{r}_m - r_f = 4\%$, find the values for the following variables that would make the expected returns consistent with equilibrium determined by the CAPM.**

i. r_f
Solution
If there is a risk free asset one must have $\lambda_0 = r_f = 10\%$

ii. β_{pi}, **the market beta of the pure portfolio associated with factor i.**
Solution

Let P_i be the pure factor portfolio associated with factor i. One has $\lambda_i = \bar{r}_{Pi} - r_f$. Furthermore, if the CAPM holds one should have

$$\lambda_i = \bar{r}_{Pi} - r_f = \beta_{Pi}\left(\bar{r}_m - r_f\right)$$

Thus

$$\lambda_1 = 1\% = \beta_{P1}4\% \implies \beta_{P1} = \frac{1}{4}$$
$$\lambda_2 = 2\% = \beta_{P2}4\% \implies \beta_{P2} = \frac{1}{2}$$

6. **Consider an economy with $k = 2$ states of nature, a "good" state 1 and a "bad" state 2. There are two assets, a risk free asset with $R_f = 1.05$, and a second risky asset that pays cashflows $X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$. The current price of the risky asset is 6.**

(a) **Solve for the prices of the elementary securities, p_1 and p_2, and the risk-neutral probabilities of the two states.**

4.a Solve for the prices of the elementary securities, p_1 and p_2 and the risk-neutral probabilities of the two states.

Let

$$P = \begin{bmatrix} 1/1.05 \\ 6 \end{bmatrix}$$

and

$$X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}$$

Then

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} = P'X^{-1} = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0.2476 & 0.7048 \end{bmatrix}$$

Hence, the risk-neutral probabilities are $\widehat{\pi}_1 \equiv p_1 R_f = 0.26$ and $\widehat{\pi}_2 \equiv p_2 R_f = 0.74$.

(b) **Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?**

4.b Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

$m_1 = p_1/\pi_1 = 0.495$. $m_2 = p_2/\pi_2 = 1.410$.

7. **This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as**

$$\mathbb{E}[R_i] = R_f + \beta_i \gamma$$

where $\mathbb{E}[R_i]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

(a) **Derive an expression for γ as a function of a, b, $\mathbb{E}[R_m]$ and σ_m^2 .**

4.a Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

$$\begin{aligned} 0 &= E[m(R_i - R_f)] \\ &= E[(a + bR_m)(R_i - R_f)] \\ &= aE[R_i] - aR_f + bE[R_m R_i] - bR_f E[R_m] \\ &= a(E[R_i] - R_f) + b(E[R_m]E[R_i] + Cov[R_m, R_i] - R_f E[R_m]) \\ &= (E[R_i] - R_f)(a + bE[R_m]) + bCov[R_m, R_i] \end{aligned}$$

so

$$
\begin{aligned}
E[R_i] - R_f &= \frac{-bCov[R_m, R_i]}{a + bE[R_m]} \\
&= -\frac{Cov[R_m, R_i]}{\sigma_m^2} \frac{b\sigma_m^2}{a + bE[R_m]} \\
&= -\beta_i \frac{b\sigma_m^2}{a + bE[R_m]}
\end{aligned}
$$

so that

$$
\gamma = -\frac{b\sigma_m^2}{a + bE[R_m]}
$$

(b) **Note that the equation $1 = \mathbb{E}[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $\mathbb{E}[R_m]$, and σ_m^2 .**

4.b Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

For the risk-free asset, we have

$$
\frac{1}{R_f} = E[a + bm]
$$

or

$$
a = \frac{1}{R_f} - bE[m]
$$

For the market portfolio, we have

$$
\begin{aligned}
1 &= E[(a + bR_m) R_m] = aE[R_m] + bE\left[R_m^2\right] \\
&= aE[R_m] + b\left(\sigma_m^2 + E[R_m]^2\right)
\end{aligned}
$$

Substituting for a from the risk-free asset equation gives

$$
\begin{aligned}
1 &= \left(\frac{1}{R_f} - bE[m]\right) E[R_m] + b\left(\sigma_m^2 + E[R_m]^2\right) \\
&= \frac{E[R_m]}{R_f} + b\sigma_m^2
\end{aligned}
$$

or

$$
b = -\frac{E[R_m] - R_f}{R_f \sigma_m^2}
$$

so

$$
a = \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f \sigma_m^2}
$$

(c) **Using the formula for a and b in part (b), show that $\gamma = \mathbb{E}[R_m] - R_f$.**

4.c Using the formula for a and b in part b, show that $\gamma = E[R_m] - R_f$.

$$\begin{aligned} a + bE[R_m] &= \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f) - E[R_m](E[R_m] - R_f)}{R_f \sigma_m^2} \\ &= \frac{1}{R_f} \end{aligned}$$

$$\begin{aligned} \gamma &= -\frac{b\sigma_m^2}{a + bE[R_m]} \\ &= \frac{E[R_m] - R_f}{R_f \sigma_m^2} \sigma_m^2 R_f \\ &= E[R_m] - R_f \end{aligned}$$

8. **Consider a world with two states of nature. You have the following term structure of interest rates over two periods:**

$$r_1^1 = 11.1111\%, \quad r_2^1 = 25.0000\%, \quad r_1^2 = 13.2277\%, \quad r_2^2 = 21.2678\%$$

where the subscript denotes the state at the beginning of period 1, and the superscript denotes the period. For instance, $\frac{1}{\left(1 + r_j^2\right)^2}$ is the price at state j at the beginning of period 1 of a riskless asset paying 1 two periods later. Construct the stationary (same every period) Arrow-Debreu state price matrix.

Solution

- If today's state is state 1, to get \$1 for sure tomorrow using Arrow-Debreu prices, I need to pay $q_{11} + q_{12}$; thus

$$q_{11} + q_{12} = \frac{1}{1 + r_1^1} = 0.9$$

- Similarly, if today's state is state 2:

$$q_{21} + q_{22} = \frac{1}{1 + r_2^1} = 0.8$$

- To get \$1 for sure two periods from state 1 today, I need to pay

$$q_{11}q_{11} + q_{12}q_{21} + q_{11}q_{12} + q_{12}q_{22} = \frac{1}{\left(1 + r_1^2\right)^2} = 0.78$$

- If state 2 today

$$q_{21}q_{11} + q_{22}q_{21} + q_{21}q_{12} + q_{12}q_{22} = \frac{1}{\left(1 + r_1^2\right)^2} = 0.68$$

- The 4 equations can be solved for the 4 unknown Arrow-Debreu prices

$$q_{11} = 0.6, \quad q_{12} = 0.3, \quad q_{21} = 0.4, \quad q_{22} = 0.4$$

- So the matrix of Arrow-Debreu prices

$$q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.4 \end{pmatrix}$$