# Advanced Microeconomics II

## Iterated Elimination of Dominated Strategies

Brett Graham

Wang Yanan Institute for Studies in Economics
Xiamen University, China

April 1, 2015

# Never-Best Response and Strictly Dominated

### Definition

An action $a_i \in A_i$ is a never-best response if it is not a best response to any belief $\mu_i$ of player $i$ where $\mu_i \in \Delta(\times A_{-i})$.

### Definition

The action $a_i \in A_i$ of player $i$ in the strategic game $\{N, (A_i), (u_i)\}$ is strictly dominated if there is a mixed strategy $\alpha_i$ of player $i$ such that $U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, where $U_i(\alpha_i, a_{-i})$ is the payoff of player $i$ if he uses the mixed strategy $\alpha_i$ and the other players' vector of actions is $a_{-i}$.

# Never-Best Response - Strictly Dominated Equivalence

## Lemma

*An action of a player in a finite strategic game is a never-best response if and only if it is strictly dominated.*

Fix a game $G = \{N, (A_i), (u_i)\}$ and a strategy $a_i^*$.

- Create an auxiliary zero-sum game $G' = \{\{1,2\}, (A_i')_{i=1}^2, (u_i')_{i=1}^2\}$ where
  - $A_1' = A_i \backslash \{a_i^*\}$ and $A_2' = A_{-i}$.
  - $u_1'(a_i, a_{-i}) = u_i(a_i, a_{-i}) - u_i(a_i^*, a_{-i})$

| | $L$ | $R$ |
|---|---|---|
| $T$ | $3,0$ | $0,1$ |
| $M$ | $0,0$ | $3,1$ |
| $B$ | $1,1$ | $1,0$ |

$G$

$\Rightarrow$

| | $L$ | $R$ |
|---|---|---|
| $T$ | $2,-2$ | $-1,1$ |
| $M$ | $-1,1$ | $2,-2$ |

$G'$

# Never-Best Response - Strictly Dominated Equivalence

$a_i^*$ is a never-best response in $G$ if and only if for any mixed strategy of player 2 in $G'$ there is an action of player 1 in $G'$ that yields player 1 a positive payoff.

$$
\begin{aligned}
&\forall m_{-i} \in \Delta(A_{-i}), \exists a_i \in A_i : U_i(a_i, m_{-i}) - U_i(a_i^*, m_{-i}) > 0 \\
\Leftrightarrow &\forall m_2 \in \Delta(A_2'), \exists a_1 \in A_1' : U_1'(a_1, m_2) > 0 \\
\Leftrightarrow &\forall m_2 \in \Delta(A_2'), \exists m_1 \in \Delta(A_1') : U_1'(m_1, m_2) > 0 \text{ (Why?)} \\
\Leftrightarrow &\min_{m_2 \in \Delta(A_2')} \max_{m_1 \in \Delta(A_1')} U_1'(m_1, m_2) > 0
\end{aligned}
$$

From previous results: $G'$ has a mixed strategy equilibrium $\Rightarrow$

$$
\begin{aligned}
&\min_{m_2 \in \Delta(A_2')} \max_{m_1 \in \Delta(A_1')} U_1'(m_1, m_2) > 0 \\
\Leftrightarrow &\max_{m_1 \in \Delta(A_1')} \min_{m_2 \in \Delta(A_2')} U_1'(m_1, m_2) > 0 \\
\Leftrightarrow &\exists m_1^* \in \Delta(A_1') : \forall m_2 \in \Delta(A_2'), U_1'(m_1^*, m_2) > 0 \\
\Leftrightarrow &\exists m_1^* \in \Delta(A_i) : \forall a_{-i} \in A_{-i}, U_i(m_1^*, a_{-i}) - u_i(a_i^*, a_{-i}) > 0
\end{aligned}
$$

# Iterated Elimination of Strictly Dominated Actions

## Definition

The set $X \subset A$ of outcomes of a finite strategic game $\{N, (A_i), (u_i)\}$ survives iterated elimination of strictly dominated strategies if $X = \times_{j \in N} X_j$ and there is a collection $((X_j^t)_{j \in N})_{t=0}^T$ of sets that satisfies the following conditions for each $j \in N$.

- $X_j^0 = A_j$ and $X_j^T = X_j$.
- $X_j^{t+1} \subset X_j^t$ for each $t = 0, \ldots, T-1$.
- For each $t = 0, \ldots, T-1$ every action of player $j$ in $X_j^t \backslash X_j^{t+1}$ is strictly dominated in the game $\{N, (X_i^t), (u_i^t)\}$ where $u_i^t$ for each $i \in N$ is the function $u_i$ restricted to $\times_{j \in N} X_j^t$.
- No action in $X_j^T$ is strictly dominated in the game $\{N, (X_i^T), (u_i^T)\}$.

# Iterated Elimination - Example

$X^0$:

| | $L$ | $R$ |
|---|---|---|
| $T$ | $3, 0$ | $0, 1$ |
| $M$ | $0, 0$ | $3, 1$ |
| $B$ | $1, 1$ | $1, 0$ |

$X^1$:

| | $L$ | $R$ |
|---|---|---|
| $T$ | $3, 0$ | $0, 1$ |
| $M$ | $0, 0$ | $3, 1$ |

$X^2$:

| | $R$ |
|---|---|
| $T$ | $0, 1$ |
| $M$ | $3, 1$ |

$X^3$:

| | $R$ |
|---|---|
| $M$ | $3, 1$ |

- $X^0 \to X^1$: $B$ is strictly dominated by $\alpha_1(T) = \alpha_1(M) = 1/2$.
- $X^1 \to X^2$: $L$ is strictly dominated by $R$.
- $X^2 \to X^3$: $T$ is strictly dominated by $M$.

# For You

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $a_1$ | $0,7$ | $2,5$ | $7,0$ | $0,1$ |
| $a_2$ | $5,2$ | $3,3$ | $5,2$ | $0,1$ |
| $a_3$ | $7,0$ | $2,5$ | $0,7$ | $0,1$ |
| $a_4$ | $0,0$ | $0,-2$ | $0,0$ | $10,-1$ |

- What are the set of strategies that survive iterated elimination of strictly dominated strategies.

# Iterated Elimination and Rationalizable Actions

## Proposition

*If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\{N, (A_i), (u_i)\}$ then $X_j$ is the set of player $j$'s rationalizable actions for each $j \in N$.*

($\Leftarrow$) First show that if $a_i$ is rationalizable then $a_i \in X_i^T$.

- Let $(Z_j)_{j \in N}$ be the profile of sets that supports $a_i$.
- For any $t$, $Z_j \subset X_j^t$ since each action in $Z_j$ is a best response to some belief over $Z_{-j}$.

($\Rightarrow$) Now show that for any player $i$ any action in $X_i^T$ is rationalizable.

- By definition if $a_i \in X_i^T$ then it is not strictly dominated and is a best response among actions in $X_i^T$ to some belief $\mu_i(a_i)$ over $X_{-i}^T$.
- It must also be a best response among the actions in $A_i$.
  - Otherwise $\exists t$, $a_i$ is a best response over $X_{-i}^t$ but not over $X_{-i}^{t-1}$.
  - $\exists b_i \in X_i^{t-1} \backslash X_i^t$ which is a best response to $\mu_i(a_i)$ over $X_{-i}^{t-1}$.
  - $b_i$ cannot be strictly dominated in $t$th round.
- Note that order is not important.

# Another Example

| | L | R |
|---|---|---|
| U | 8 | 0 |
| D | 0 | 0 |

$M_1$

| L | R |
|---|---|
| 4 | 0 |
| 0 | 4 |

$M_2$

| L | R |
|---|---|
| 0 | 0 |
| 0 | 8 |

$M_3$

| L | R |
|---|---|
| 3 | 3 |
| 3 | 3 |

$M_4$

- Since order is not important, in each round let's eliminate all strictly dominated strategies in that round.

# Iterated Elimination of Weakly Dominated Actions

**Definition**

The action $a_i \in A_i$ of player $i$ in the strategic game $\{N, (A_i), (u_i)\}$ is weakly dominated if there is a mixed strategy $\alpha_i$ of player $i$ such that $U_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$ and $U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for some $a_{-i} \in A_{-i}$ where $U_i(\alpha_i, a_{-i})$ is the payoff of player $i$ if he uses the mixed strategy $\alpha_i$ and the other players' vector of actions is $a_{-i}$.

- Is a weakly dominated action strictly dominated?
- Is a strictly dominated action weakly dominated?
- A weakly dominated action that is not strictly dominated is a best response to some belief.
- Order matters for iterated elimination of weakly dominated strategies.

# Weak Iterated Elimination - Example

$X^0$

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1, 1 | 0, 0 |
| $M$ | 1, 1 | 2, 1 |
| $B$ | 0, 0 | 2, 1 |

$X^1$

| | $L$ | $R$ |
|---|---|---|
| $M$ | 1, 1 | 2, 1 |
| $B$ | 0, 0 | 2, 1 |

$X^2$

| | $R$ |
|---|---|
| $M$ | 2, 1 |
| $B$ | 2, 1 |

- $X^0 \rightarrow X^1$: $T$ is weakly dominated by $M$.
- $X^1 \rightarrow X^2$: $L$ is weakly dominated by $R$.

$X^0$

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1, 1 | 0, 0 |
| $M$ | 1, 1 | 2, 1 |
| $B$ | 0, 0 | 2, 1 |

$X^1$

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1, 1 | 0, 0 |
| $M$ | 1, 1 | 2, 1 |

$X^2$

| | $L$ |
|---|---|
| $T$ | 1, 1 |
| $M$ | 1, 1 |

- $X^0 \rightarrow X^1$: $B$ is weakly dominated by $M$.
- $X^1 \rightarrow X^2$: $R$ is weakly dominated by $L$.

# Dominance Solvability

**Definition**

A strategic game is dominance solvable if all players are indifferent between all outcomes that survive the iterative procedure in which all the weakly dominated actions of each player are eliminated at each stage.

$X^0$:

| | $L$ | $R$ |
|---|---|---|
| $T$ | $1,1$ | $0,0$ |
| $M$ | $1,1$ | $2,1$ |
| $B$ | $0,0$ | $2,1$ |

$X^1$:

| | $L$ | $R$ |
|---|---|---|
| $M$ | $1,1$ | $2,1$ |

- The game is not dominance solvable.

# Example

Each of two players announces a non-negative integer equal to at most 100. If $a_1 + a_2 \leq 100$, where $a_i$ is the number announced by player $i$, then each player $i$ receives payoff of $a_i$. If $a_1 + a_2 > 100$ and $a_i < a_j$ then player $i$ receives $a_i$ and player $j$ receives $100 - a_i$; if $a_1 + a_2 > 100$ and $a_i = a_j$ then each player receives 50.

Formulate this as a normal form strategic game.