Advanced Microeconomics II

Infinitely Repeated Games

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Infinite versus Finite

- Recall the set of SPE in the Finitely Repeated Prisoner's Dilemma game.
- Does the same hold true in the Infinitely Repeated game version?

Infinitely Repeated Game

Definition

Let $G = \{N, (A_i), (\succeq_i)\}$ be a strategic game; let $A = \times_{i \in N} A_i$. An infinitely repeated game of G is an extensive game with perfect information and and simultaneous moves $\{N, H, P, (\succeq_i^*)\}$ in which

- $H = \{\varnothing\} \cup (\{\cup_{t=1}^{\infty} A^t) \cup A^{\infty}\}$ (where A^{∞} is the set of infinite sequences $(a^t)_{t=1}^{\infty}$ of action profiles in G)
- $P(h) = N$ for each nonterminal history $h \in H$.
- $\succeq_i^*$ is a preference relation on A^{∞} that extends the preference relation $\succeq_i$ such that it satisfies the condition of weak separability: if $(a^t) \in A^{\infty}, a \in A, a' \in A$, and $a \succeq_i a'$ then

$$(a^1, \ldots, a^{t-1}, a, a^{t+1}, \ldots) \succeq_i^* (a^1, \ldots, a^{t-1}, a', a^{t+1}, \ldots)$$

for all values of t.

Prisoner's Dilemma Example

- A history is terminal if and only if it is infinite.
- A strategy of player i is a function that assigns an action $a_i \in A_i$ to every finite sequence of outcomes in G.

	C	D
C	$3, 3$	$1, 4$
D	$4, 1$	$2, 2$

- Players play the Prisoner's dilemma forever.
- How should we evaluate preferences over terminal histories?

Discounting

Three possible methods to evaluate terminal histories:

Definition

Discounting: There is some number $\delta \in (0, 1)$ (the discount factor) such that the sequence (v_i^t) is at least as good as the sequence (w_i^t) if and only if $\sum_{t=1}^{\infty} \delta^{t-1}(v_i^t - w_i^t) \geq 0$.

The payoff profile of v_i^t is $((1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} v_i^t)_{i \in N}$ ("average period payoffs").

- Per-period payoff values diminish over time.
- Changes in a single period payoffs affect preferences.

Overtaking

Definition

Overtaking: The sequence (v_i^t) is preferred to the sequence (w_i^t) if and only if $\liminf \sum_{t=1}^{T}(v_i^t - w_i^t) > 0$.

$$\liminf_{T\to\infty} \sum_{t=1}^{T}(v_i^t - w_i^t) = \lim_{T\to\infty}\left(\inf_{T'\geq T} \sum_{t=1}^{T'}(v_i^t - w_i^t)\right)$$

Example: $v = (1, 0, 2, 0, 2, 0, \ldots)$ and $w = (0, 2, 0, 2, 0, 2, \ldots)$

T	1	2	3	4	5	6	$\ldots$
$\sum_{t=1}^{T}(v_i^t - w_i^t)$	1	-1	1	-1	1	-1	$\ldots$
$\inf_{T'\geq T} \sum_{t=1}^{T'}(v_i^t - w_i^t)$	-1	-1	-1	-1	-1	-1	$\ldots$

- Per-period payoff values do not diminish over time.
- Changes in a single period payoffs affect preferences.

Limit of means

Definition

Limit of means: The sequence (v_i^t) is preferred to the sequence (w_i^t) if and only if $\liminf \sum_{t=1}^{T}(v_i^t - w_i^t)/T > 0$ (i.e. if and only if there exists $\epsilon > 0$ such that $\sum_{t=1}^{T}(v_i^t - w_i^t)/T > \epsilon$ for all but a finite number of periods T).

Example: $v = (1, 0, 2, 0, 2, 0, \ldots)$ and $w = (0, 2, 0, 2, 0, 2, \ldots)$

T	1	2	3	4	5	6	...
$\sum_{t=1}^{T}(v_i^t - w_i^t)/T$	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$-\frac{1}{6}$	...
$\inf_{T' \geq T} \sum_{t=1}^{T'}(v_i^t - w_i^t)/T$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{6}$	...

The payoff profile of v_i^t is $\lim_{T\to\infty}(\sum_{t=1}^{T} v_i^t/T)_{i\in N}$, if it exists.

- Per-period payoff values do not diminish over time.
- Changes in a single period payoffs do not affect preferences.

Examples

Rank the following streams of payoffs according to each criteria.

- $v_1 = (1, -1, 0, 0, \ldots)$ and $w_1 = (0, 0, \ldots)$
- $v_2 = (-1, 2, 0, 0, \ldots)$ and $w_2 = (0, 0, \ldots)$
- $v_3 = (1, 0, \ldots)$ and $w_3 = (0, \ldots, 0, 1, 1, \ldots)$ where there are M zeros.

Feasible Payoff Profiles

- Recall that $u(a)$ is the vector $(u_i(a))_{i \in N}$.

Definition

$v \in \mathcal{R}^N$ is a payoff profile of $\{N, (A_i), (u_i)\}$ if there is an outcome $a \in A$ for which $v = u(a)$. A vector $v \in \mathcal{R}^N$ is a feasible payoff profile of $\{N, (A_i), (u_i)\}$ if it is a convex combination of payoff profiles of outcomes in A: that is, if $v = \sum_{a \in A} \alpha_a u(a)$ for some collection $(\alpha_a)_{a \in A}$ of nonnegative rational numbers α_a with $\sum_{a \in A} \alpha_a = 1$.

Recall: Enforceable Outcomes

Definition

Player i's minmax payoff in G (denoted v_i) is

$$v_i = \min_{a-i \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

Definition

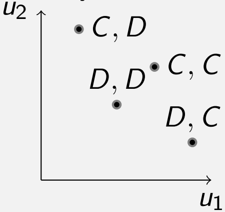
A payoff profile w is enforceable if $w_i \geq v_i$ for all $i \in N$. A payoff profile w is strictly enforceable if $w_i > v_i$ for all $i \in N$. An outcome $a \in A$ is a (strictly) enforceable outcome of G if $u(a)$ is (strictly) enforceable.

- Let $p_{-i} \in A_{-i}$ be a solution to the minimization problem above.
- Let $b_i(p_{-i}) \in A_i$ be a best response of player i to $p_{-i} \in A_{-i}$.
- Denote (p_i) as the action profile $(b_i(p_{-i}), p_{-i})$ for each $i \in N$.

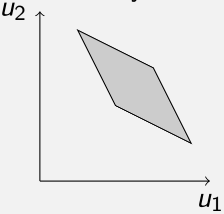
Feasible Payoff Profiles Example

	C	D
C	$3, 3$	$1, 4$
D	$4, 1$	$2, 2$

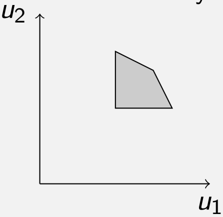
Payoff Profiles



Feasible Payoff Profiles



Enforcible Feasible Payoff Profiles



Strategies as Machines

Definition

A machine for player i of the infinitely repeated game G has the following components.

- A set Q_i (the set of states).
- An element $q_i^0 \in Q_i$ (the initial state).
- A function $f_i : Q_i \to A_i$ that assigns an action to every state (the output function).
- A function $\tau_i; Q_i \times A \to Q_i$ that assigns a state to every pair consisting of a state and an action profile (the transition function).

Always Cooperate Machine

- $Q_i = \{\mathcal{C}\}$.
- $q_i^0 = \mathcal{C}$.
- $f_i(\mathcal{C}) = C$.
- $\tau_i(\mathcal{X}, (Y, Z)) = \mathcal{C}$ for all $(\mathcal{X}, (Y, Z)) \in \{\mathcal{C}\} \times \{C, D\}^2$.

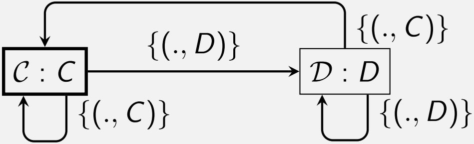
Never Cooperate Machine

- $Q_i = \{\mathcal{D}\}$.
- $q_i^0 = \mathcal{D}$.
- $f_i(\mathcal{D}) = D$.
- $\tau_i(\mathcal{X}, (Y, Z)) = \mathcal{D}$ for all $(\mathcal{X}, (Y, Z)) \in \{\mathcal{D}\} \times \{C, D\}^2$.

$$\begin{tikzcd} \boxed{\mathcal{D} : D} \arrow[loop below, "\text{all outcomes}"] \end{tikzcd}$$

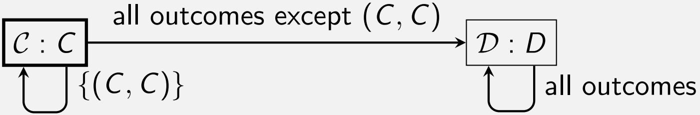
Tit-for-Tat Machine

- $Q_i = \{\mathcal{C}, \mathcal{D}\}$.
- $q_i^0 = \mathcal{C}$.
- $f_i(\mathcal{C}) = C, f_i(\mathcal{D}) = D$.
- $\tau_i(\mathcal{X}, (Y, Z)) = \mathcal{D}$ if $A_{-i} = D$, $\tau_i(\mathcal{X}, (Y, Z)) = \mathcal{C}$ if $A_{-i} = C$.



Grim Trigger Machine

- $Q_i = \{\mathcal{C}, \mathcal{D}\}$.
- $q_i^0 = \mathcal{C}$.
- $f_i(\mathcal{C}) = C, f_i(\mathcal{D}) = D$.
- $\tau_i(\mathcal{C}, (C, C)) = \mathcal{C}$ and $\tau_i(\mathcal{X}, (Y, Z)) = \mathcal{D}$ if $(\mathcal{X}, (Y, Z)) \neq (\mathcal{C}, (C, C))$.



Enforceable Outcomes and Nash Equilibria

Proposition

Every Nash equilibrium payoff profile of the limit of means infinitely repeated game of $G = \{N, (A_i), (u_i)\}$ is an enforceable payoff profile of G. The same is true, for any $\delta \in (0,1)$, of every Nash equilibrium payoff profile of the δ-discounted infinitely repeated game of G.

- If $w_i < v_i$ then player i has a profitable deviation.
- For each history, play $b_i(s_{-i}(h))$.
- This generates a payoff of at least v_i in each period and thus v_i in the game.

Enforceable Payoff Profile as a Machine

This machine guarantees player 1 no less than his minmax payoff v_1 given a machine of player 2.

- $Q_1 = Q_2$.
- $q_1^0 = q_2^0$.
- $f_1(q) = b_1(f_2(q))$ for all $q \in Q_2$.
- $\tau_1(q, a) = \tau_2(q, a)$ for all $q \in Q_2$ and $a \in A$.

Nash Folk Theorem for the Limit of Means Criterion

Proposition

Every feasible enforceable payoff profile of $G = \{N, (A_i), (u_i)\}$ is a Nash equilibrium payoff profile of the limit of means infinitely repeated game of G.

- Let $w = \sum_{a \in A} (\beta_a / \gamma) u(a)$ be a feasible enforceable payoff profile:
  - β_a is an integer, $\gamma = \sum_{a \in A} \beta_a$.
  - (a^t) is a cycle of action profiles which contains β_a repetitions of a for each $a \in A$.
- Player i's strategy:
  - Choose a_i^t in period t unless there was a previous t' where a single player other than i deviated.
  - Otherwise choose $(p_j)_i$, where j is the first single player deviant from $a^{t'}$.
- Any player j who deviates receive his minmax payoff j.

Nash Folk Theorem as a Machine

- $Q_i = \{S_1, \ldots, S_\gamma, P_1, \ldots, P_n\}$.
- $q_i^0 = S_1$.
- $f_i(q) = \begin{cases} a_i^l & \text{if } q = S_l \\ (p_j)_i & \text{if } q = P_j \end{cases}$
- $\tau_i(S_l, a) = \begin{cases} P_j & \text{if } a_j \neq a_j^l \text{ and } a_i = a_i^l \text{ for all } i \neq j \\ S_{l+1 (\text{mod } \gamma)} & \text{otherwise} \end{cases}$
- $\tau_i(P_j, a) = P_j$ for all $a \in A$.

$m(\text{mod } \gamma)$ is the integer q with $1 \leq q \leq \gamma$ satisfying $m = l\gamma + q$ for some integer l. Examples: $4(\text{mod } 5) = 4$, $5(\text{mod } 5) = 5$, $6(\text{mod } 5) = 1$

Nash Folk Theorem for the Discounting Criterion

Proposition

Let w be a feasible strictly enforceable payoff profile of $G = \{N, (A_i), (u_i)\}$. For all $\epsilon > 0$ there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the δ-discounted infinitely repeated game of G has a Nash equilibrium whose payoff profile w' satisfies $|w' - w| < \epsilon$.

- Proof is similar (Homework).

Trigger Strategies May Not Be SPE

		Player 2	
		A	D
Player 1	A	$2, 3$	$1, 5$
	D	$0, 0$	$0, 1$

Player payoffs are defined by the limit of means criterion.

- What is player 1's minmax payoff?
- What is player 2's minmax payoff?
- What are the equilibrium strategies from the proof that support $((A, A), (A, A), \ldots)$ as a Nash equilibrium outcome?
- Find a history for which the strategies are not SPE.

Perfect Folk Theorem For Limit of Means Criterion

Proposition

Every feasible strictly enforceable payoff profile of G is a subgame perfect equilibrium payoff profile of the limit of means infinitely repeated game of G.

- Let $w = \sum_{a \in A} (\beta_a / \gamma) u(a)$ be a feasible strictly enforceable payoff profile:
  - β_a is an integer, $\gamma = \sum_{a \in A} \beta_a$.
  - $(a^k)_{k=1}^{\gamma}$ is a sequence of action profiles which contains β_a repetitions of a for each $a \in A$.
- $g^* = \max_{i \in N, a'_i \in A_i, a \in A} [u_i(a'_i, a_{-i}) - u_i(a, a_{-i})]$
- Since $w_i > v_i$ there exists a positive integral multiple of γ, m^* such that

$$\gamma g^* + m^* v_i \leq m^* w_i \text{ for all } i \in N.$$

Perfect Folk Theorem Strategies

The set of strategies for each player is given by the following machine:

- States:
  - $(Norm^k, 0)$: k^{th} period of $(a^k)_{k=1}^{\gamma}$ cycle, with no previous deviation ($(Norm^1, 0)$ is the initial state)
  - $(Norm^k, j)$: k^{th} period of $(a^k)_{k=1}^{\gamma}$ cycle, with a previous single player deviation, the first by player $j \in N$
  - $P(j, t)$: Punishment phase of player $j \in N$ with $t \in \{1, \ldots, m^*\}$ periods remaining.
- Output function:
  - In $(Norm^k, 0)$ or $(Norm^k, j))$: choose a_i^k.
  - In $P(j, t)$: choose $(p_j)_i$.

Perfect Folk Theorem Strategies (ctd)

- Transition function:
  - $\tau_i((Norm^k, 0), a) =$
    $$\begin{cases} (Norm^{k+1 (\text{mod } \gamma)}, 0) & \text{if no single player deviated} \\ (Norm^{k+1}, j) & \text{if a single player } j \text{ deviated and } k \leq \gamma - 1 \\ P(j, m^*) & \text{if single player } j \text{ deviation and } k = \gamma. \end{cases}$$
  - $\tau_i((Norm^k, j), a) = \begin{cases} (Norm^{k+1}, j) & \text{if } k \leq \gamma - 1, \text{ for all } a \in A \\ P(j, m^*) & \text{if } k = \gamma, \text{ for all } a \in A. \end{cases}$
  - $\tau_i(P(j, t), a) = \begin{cases} P(j, t-1) & \text{if } 2 \leq t \leq m^*, \text{ for all } a \in A \\ (Norm^1, 0) & \text{if } t = 1, \text{ for all } a \in A. \end{cases}$

Example

		Player 2	
		A	D
Player 1	A	$2, 3$	$1, 5$
	D	$0, 0$	$0, 1$

Player payoffs are defined by the limit of means criterion.

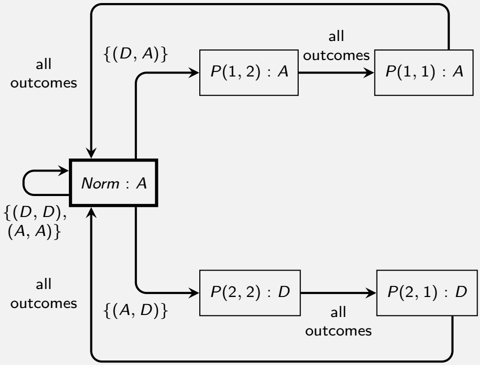
What is g^*?

- Is $(2, 3)$ an SPE payoff profile?
  - What is γ?
  - What is m^*?
- Is $(1.5, 4)$ an SPE payoff profile?
  - What is γ?
  - What is m^*?

Machine Example

	A	D
A	$2,3$	$1,5$
D	$0,0$	$0,1$

Player 1



Example

		Player 2	
		A	D
Player 1	A	$2,3$	$1,5$
	D	$0,0$	$0,1$

Player payoffs are defined by the overtaking criterion.

- Take the previous strategies that supported $(2,3)$ as an SPE in the limit of means infinitely repeated game.
- These strategies do not support $(2,3)$ in the overtaking criterion infinitely repeated game?
- After a history in which player 2 deviates, player 1 has a profitable deviation.
  - $(1,1,2,2,\ldots) \succeq_1 (0,0,2,2,\ldots)$
- Same for discounting criterion.

Perfect Folk Theorem For Overtaking Criterion

Proposition

For any strictly enforceable outcome a^* of G there is a subgame perfect equilibrium of the overtaking infinitely repeated game of G that generates the path (a^t) in which $a^t = a^*$ for all t.

- For simplicity we restrict attention to strictly enforceable outcomes rather than payoff profiles.
- $M = \max_{i \in N, a \in A} u_i(a)$
- Any deviation generates a punishment phase long enough to wipe out the gain.
  - Length of phase is finite since $a_i^* > v_i$

Perfect Folk Theorem Strategies

Each player uses the following machine:

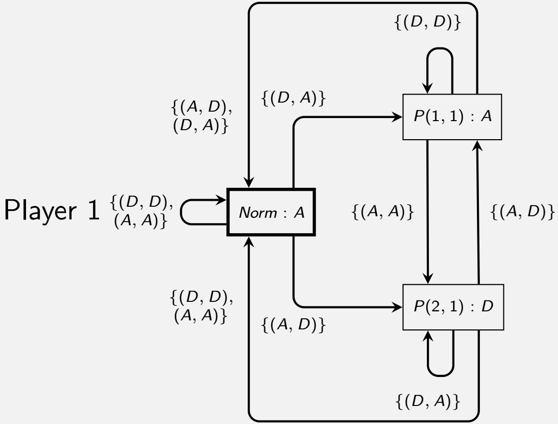
- States:
  - *Norm*: *Norm* is the initial state
  - $P(j,t)$: Punishment phase of player $j \in N$ with $t \in N$ periods remaining.
- Output function:
  - In *Norm*: choose a_i^*.
  - In $P(j,t)$: choose $(p_j)_i$.

Perfect Folk Theorem Strategies (ctd)

- Transition function:
  - $\tau_i(Norm, a) =$
    $$\begin{cases} Norm & \text{if no single player deviation} \\ P(j, \bar{t}) & \text{if single player } j \text{ deviates. where } \bar{t}_j \text{ is the smallest} \\ & \text{integer such that } M + \bar{t}_j v_j < (\bar{t}_j + 1)u_j(a^*). \end{cases}$$
  - $\tau_i(P(j,t), a) =$
    $$\begin{cases} P(j, t-1) & \text{if no single player deviation and } t \geq 2 \\ Norm & \text{if no single player deviation and } t = 1 \\ P(k, T(j,t)) & \text{if single player } k \text{ deviates, where } T(j,t) \text{ is large} \\ & \text{enough that sum of } k\text{'s payoffs in state } P(j,t) \\ & \text{and his payoff in the subsequent } T(j,t) \text{ periods} \\ & \text{if he does not deviate is greater than his payoff} \\ & \text{in the deviation plus } T(j,t)v_k. \end{cases}$$

Example

	A	D
A	$2, 3$	$1, 5$
D	$0, 0$	$0, 1$



Perfect Folk Theorem For Discounting Criterion

Proposition

Let a^* be a strictly enforceable outcome of G. Assume that there is a collection $(a(i))_{i \in N}$ of strictly enforceable outcomes of G such that for every player $i \in N$ we have $a^* \succ_i a(i)$ and $a(j) \succ_i a(i)$ for all $j \in N \backslash \{i\}$. Then there exists $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$ there is a subgame perfect equilibrium of the δ-discounted infinitely repeated game of G that generates the path (a^t) in which $a^t = a^*$ for all t.

	C	D	E
C	$3,3$	$1,4$	$0,0$
D	$4,1$	$2,2$	$0.5,0$
E	$0,0$	$0,0.5$	$0,0$

- Which outcomes satisfy the conditions of the proposition?
- What are $a(1)$ and $a(2)$?

Discounting Criterion Machine

- States: $\{C(j) : j \in \{0\} \cup N\} \cup \{P(j,t) : j \in N \text{ and } 1 \leq t \leq L\}$.
- Initial state: $C(0)$.
- Output function: In $C(j)$ choose $(a(j))_i$. In $P(j,t)$ choose $(p_j)_i$.
- Transition function:
  - $\tau_i(C(j), a) =$
    $$\begin{cases} C(j) & \text{if no single player deviation from } a(j) \ (a(0) = a^*) \\ P(k, L) & \text{if single player } k \text{ deviates.} \end{cases}$$
  - $\tau_i(P(j,t), a) =$
    $$\begin{cases} P(j, t-1) & \text{if no single player deviation and } 2 \leq t \leq L \\ C(j) & \text{if no single player deviation and } t = 1 \\ P(k, L) & \text{if single player } k \text{ deviates} \end{cases}$$

How To Deter Deviations In State $C(j)$

- Let $M = \max_{i \in N, a \in A} u_i(a)$, $m = \min_{i \in N, a \in A} u_i(a)$.
- Payoff from deviating:

$$\max_{a'_i \in A_i} u(a'_i, a(j)_{-i}) + \sum_{k=2}^{L+1} \delta^{k-1} v_i + \sum_{k=L+2}^{\infty} \delta^{k-1} u_i(a(i))$$

- Payoff from no deviation:

$$u_i(a(j)) + \sum_{k=2}^{L+1} \delta^{k-1} u_i(a(j)) + \sum_{k=L+2}^{\infty} \delta^{k-1} u_i(a(j))$$

- Choose L such that $M - m < L(u_i(a(j)) - v_i)$
- This ensures there exists δ^* such that for all $\delta > \delta^*$

$$\max_{a'_i \in A_i} u(a'_i, a(j)_{-i}) - u_i(a(j)) < \sum_{k=2}^{L+1} \delta^{k-1} (u_i(a(j)) - v_i).$$

How To Deter Deviations In State $P(j, t)$

- Payoff from deviating:

$$\max_{a_i' \in A_i} u(a_i', p(j)_{-i}) + \sum_{k=2}^{L+1} \delta^{k-1} v_i + \sum_{k=L+2}^{\infty} \delta^{k-1} u_i(a(i))$$

- Payoff from no deviation:

$$\sum_{k=1}^{t} \delta^{k-1} u_i(p(j)) + \sum_{k=t+1}^{\infty} \delta^{k-1} u_i(a(j))$$

- Since $v_i < u_i(a(i))$ it is sufficient that

$$\sum_{k=1}^{L+1} \delta^{k-1}(M - m) < \sum_{k=L+2}^{\infty} \delta^{k-1}(u_i(a(j)) - u_i(a(i)))$$

- For δ close to 1 this is satisfied since $u_i(a(j)) > u_i(a(i))$.

Simple Supporting Strategies

- Credible punishment relies only on the identity of deviant, not on the history that preceded the deviation.
- Such a strategy can be used to support any SPE outcome.
- For each player i punish his deviation with his worst possible SPE payoff.
  - Need to show that worst payoff exists (set of SPE payoffs is closed).
  - Denote player i's worst SPE payoff by $m(i)$.
  - Let $(a(i)^t)$ to be the outcome of a subgame perfect equilibrium in which player i's payoff is $m(i)$.

Simple Supporting Strategies

Proposition

Let (a^t) be the outcome of a subgame perfect equilibrium of the δ-discounted infinitely repeated game of $G = \{N, (A_i), (u_i)\}$. Then the strategy profile in which each player i uses the following machine is a subgame perfect equilibrium with the same outcome (a^t).

- *Set of states:* $\{Norm^t : t \text{ is a positive integer}\} \cup \{P(j,t) : j \in N \text{ and } t \text{ is positive integer}\}$.
- *Initial state: $Norm^1$.*
- *Output function: In state $Norm^t$ play a_i^t. In state $P(j,t)$ play $a(j)_i^t$.*
- *Transition function:*
  - *In state $Norm^t$ move to $Norm^{t+1}$ unless exactly one player, say j deviated from a^t, in which case move to $P(j,1)$.*
  - *In state $P(j,t)$ move to $P(j,t+1)$ unless exactly one player, say j' deviated from $a(j)^t$, in which case move to $P(j',1)$.*