A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily

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ABSTRACT

Often, researchers wish to analyze nonlinear dynamic discrete-time stochastic models. This chapter provides a toolkit for solving such models easily, building on log-linearizing the necessary equations characterizing the equilibrium and solving for the recursive equilibrium law of motion with the method of undetermined coefficients. This chapter contains nothing substantially new. Instead, the chapter simplifies and unifies existing approaches to make them accessible for a wide audience, showing how to log-linearizing the nonlinear equations without the need for explicit differentiation, how to use the method of undetermined coefficients for models with a vector of endogenous state variables, to provide a general solution by characterizing the solution with a matrix quadratic equation and solving it, and to provide frequency-domain techniques to calculate the second order properties of the model in its HP-filtered version without resorting to simulations. Since the method is an Euler-equation based approach rather than an approach based on solving a social planners problem, models with externalities or distortionary taxation do not pose additional problems. MATLAB programs to carry out the calculations in this chapter are made available. This chapter should be useful for researchers and Ph.D. students alike.

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1 Introduction

Often, researchers wish to analyze nonlinear dynamic discrete-time stochastic models. This chapter provides a toolkit for solving such models easily, building on log-linearizing the necessary equations characterizing the equilibrium and solving for the recursive equilibrium law of motion with the method of undetermined coefficients[1]

This chapter contains nothing substantially new. Instead, the point of this chapter is to simplify and unify existing methods in order to make them accessible to a large audience of researchers, who may have always been interested in analyzing, say, real business cycle models on their own, but hesitated to make the step of learning the numerical tools involved. This chapter reduces the pain from taking that step. The methods here can be used to analyze most of the models studied in the literature. We discuss how to log-linearizing the nonlinear equations without the need for explicit differentiation and how to use the method of undetermined coefficients for models with a vector of endogenous state variables. The methods explained here follow directly from McCallum (1983), King, Plosser and Rebelo (1987) and Campbell (1994), among others[2]. We provide a general solution built on solving matrix-quadratic equations, see also Binder and Pesaran (1996), and provide frequency-domain techniques, building on results in King and Rebelo (1993), to calculate the second-order moments of the model in its HP-filtered version without resorting to simulations. Since the method is an Euler-equation based approach rather than an approach based on solving a social planners problem, solving models with externalities or distortionary taxation does not pose additional problems. Since the (nonlinear) Euler equations usually need to be calculated in any case in order to find the steady state, applying the method described in this chapter requires little in terms of additional manipulations by hand, given some preprogrammed routines to carry out the matrix calculations of section 6. MATLAB programs to carry out these calculations, given the log-linearized system, are available at my home page[3]. The method in this chapter therefore allows to solve nonlinear dynamic stochastic models easily.

Numerical solution methods for solving nonlinear stochastic dynamic models have been studied extensively in the literature, see in particular Kydland and Prescott (1982),

[1] Note that the nonlinear model is thus replaced by a linearized approximate model. "Essential" nonlinearities like chaotic systems are unlikely to be handled well by the methods in this chapter.

[2] Campbell even touts the approach followed in his paper as "analytical", but note that in his case as well as in our case, one needs to linearize equations and solve quadratic equations. Campbell presumably attaches the attribute "analytical" to this numerical procedure, since it is rather straightforward indeed and carrying it out by hand is actually feasible in many cases. Otherwise, every numerical calculation anywhere could be called "analytical", since it could in principle be carried out and analyzed by hand - it would just take very long.

[3] http://cwis.kub.nl/~few5/center/STAFF/uhlig/toolkit.dir/toolkit.htm is the address of the web site for the programs.

the comparison by Taylor and Uhlig (1990) and the methods proposed by various authors in the same issue, Judd (1991), Hansen and Prescott (1995) and Danthine and Donaldson (1995). The literature on solving linear-quadratic dynamic stochastic models or linear stochastic difference equations is even larger. The key paper here is Blanchard and Kahn (1980). Furthermore, there are the textbook treatment in Sargent (1987), Chapters IX and XI, as well as, say, Muth (1961), McGrattan (1994) or Hansen, McGrattan and Sargent (1994), to name a random few. Subject to applicability, all the methods relying on a log-linear approximation to the steady state have in common that they will find the same recursive equilibrium law of motion as the method described in this chapter, since the linear space approximating a nonlinear differentiable function is unique and "immune" to differentiable transformations of the parameter space. But while McGrattan (1994) and Hansen, McGrattan and Sargent (1994) focus on solving models via maximizing a quadratic objective function, and while Blanchard and Kahn (1980) solve linear systems by searching for the stable manifold in the entire system of necessary equations describing the equilibrium relationships, this chapter by contrast solves directly for the desired recursive equilibrium law of motion. This approach is very natural. The stability condition is imposed at the point, where a certain matrix quadratic equation is solved. It is shown how this matrix quadratic equation can be reduced to a standard eigenvalue problem of another matrix with twice as many dimensions.

Three related contributions are McCallum (1983), which is the key reference for the method of undetermined coefficients, Ceria and Rios-Rull (1992) and Binder and Pesaran (1996). These contributions also derive the recursive equilibrium law of motion. McCallum (1983) reduces the coefficient-finding problem to a problem solvable with the methods in Blanchard and Kahn (1980), whereas Ceria and Rios-Rull (1992) reduce the problem to one of solving a matrix-quadratic equation as do we, but do not reduce the matrix-quadratic equation problem to a standard eigenvalue problem. Binder and Pesaran (1996) finally may be most closely related in that they reduce the matrix quadratic equation characterizing the solution to an eigenvalue problem as we do. These three contributions, however, for most parts do not distinguish between endogenous variables which have to be part of the state vector, and other endogenous variables. Thus applying these models in somewhat larger system can either result in unnecessary large and computationally demanding eigenvalue problems in which "bubble solutions" have to be removed in a painstaking fashion, or one is always forced to reduce the system beforehand to make it fit their description[4].

But all these technical differences to the existing literature are not in any way

[4] Furthermore, McCallum (1983) uses eigenvalue methods also to solve some other equations in his method, which are solved here by a simple linear-equation-solution techniques, compare his solution for equation (A.6) in his paper to equation (6.14).

essential. It shall be stressed again that the main purpose and merit of this chapter is to make solving nonlinear dynamic stochastic models easy. In fact, this chapter describes the entire method as a "cookbook recipe", which should be of great practical use to Ph.D. students and researchers alike. Since the focus here is entirely on the computational aspect of studying these models, some issues are left aside entirely. In particular, the issue of existence or multiplicity of equilibria as well as the reasons for concentrating on stable solutions is not discussed. The methods in this chapter should therefore not be applied blindly, but only in light of, say, McCallum (1983), Stokey, Lucas with Prescott (1989) and the related literature.

The outline of the chapter will be evident from the description of the general procedure in the next section. In particular, section 4 shows, how to do everything by hand in the stochastic neoclassical growth model.

2 The general procedure

The general procedure to solve and analyze nonlinear dynamic stochastic models takes the following steps.

1. Find the necessary equations characterizing the equilibrium, i.e. constraints, first-order conditions, etc., see sections 4 and 5.

2. Pick parameters and find the steady state(s), see sections 4 and 5.

3. Log-linearize the necessary equations characterizing the equilibrium of the system to make the equations approximately linear in the log-deviations from the steady state, see sections 3, 4 and 5.

4. Solve for the recursive equilibrium law of motion via the method of undetermined coefficients, employing the formulas of section 6. Also, see section 4, where all the calculations are done "by hand" and explained in detail.

5. Analyze the solution via impulse-response analysis, see section 4 and 7, and second-order-properties, possibly taking account of, say, the Hodrick-Prescott-Filter. This can be done without having to simulate the model, see section 7.

The next section skips directly to step 3 of the procedure outlined above and describes how to log-linearize nonlinear equations without explicit differentiation. Sections 4 and 5 then provide two prototype examples, in which calculating the Euler equations, the steady state and the log-linearization is carried out to see how this method works. Section 4 analyzes the stochastic neoclassical growth model and states and explains the general modelling approach, all the details of the calculations including the calculation of the recursive equilibrium law of motion "by hand",

whereas section 5 studies the real business cycle model of Hansen (1985), deriving the log-linearized version fairly quickly: once, a linearized system has been obtained, the methods in section 6 provide the desired recursive equilibrium law of motion.

Those, who wish faster access should skip section 4 and go to section 5 after reading section 3. Readers who are familiar enough with log-linearization are advised to skip even more and go directly to section 6 now.

3 Log-linearization

Log-linearizing the necessary equations characterizing the equilibrium is a well-known technique. In the context of real business cycle models, log-linearization has been proposed in particular by King, Plosser and Rebelo (1987) and Campbell (1994). Log-linearization also appears frequently in text books, see e.g. Obstfeld and Rogoff, p. 503-505. Nonetheless, the technique often seems to create more headaches than it should. It is thus useful for the purpose of this chapter to review how it is done. The next two sections simplify the approach of Campbell (1994). Looking ahead at the many equations in particular of section 4 to follow, this claim may not seem entirely credible. However, these equations were stated to spell out each step in detail. When studying Campbell (1994), one might be under the impression, that magic and quite a bit of cleverness is involved in deriving the results. The point of in particular sections 3, 4 and 5 is to show, that one does not need to be as clever as John Campbell to use these methods. On the contrary, everything is remarkably straightforward, and, as long as one proceeds carefully, practically nothing can go wrong. Different choices in places where choices can be made still result in the same final outcome.

The principle of log-linearization is to use a Taylor approximation around the steady state to replace all equations by approximations, which are linear functions in the log-deviations of the variables.

Formally, let X_t be the vector of variables, $\bar{X}$ their steady state and

$$x_t = \log X_t - \log \bar{X}$$

the vector of log-deviations. The vector $100 \cdot x_t$ tells us, by how much the variables differ from their steady state levels in period t in per cent. The necessary equations characterizing the equilibrium can be written as

$$1 = f(x_t, x_{t-1}) \tag{3.1}$$

$$1 = E_t\left[g(x_{t+1}, x_t)\right] \tag{3.2}$$

where $f(0,0) = 1$ and $g(0,0) = 1$, i.e. the left-hand side of (3.1) and (3.2). Taking first-order approximations around $(x_t, x_{t-1}) = (0,0)$ yields[5]

$$\begin{aligned} 0 &\approx f_1 \cdot x_t + f_2 \cdot x_{t-1} \\ 0 &\approx E_t\left[g_1 \cdot x_{t+1} + g_2 \cdot x_t\right] \end{aligned}$$

One obtains a linear system in x_t and x_{t-1} in the deterministic equations and x_{t+1} and x_t in the expectational equations. This linear system can be solved with the method of undetermined coefficients, described in section 6.

In the large majority of cases, there is no need to differentiate the functions f and g explicitely. Instead, the log-linearized system can usually be obtained as follows. Multiply out everything before log-linearizing. Replace a variable X_t with $X_t = \bar{X}e^{x_t}$, where x_t is a real number close to zero. Let likewise y_t be a real number close to zero. Take logarithms, where both sides of an equation only involve products, or use the following three building blocks, where a is some constant:

$$\begin{aligned} e^{x_t + a y_t} &\approx 1 + x_t + a y_t \\ x_t y_t &\approx 0 \\ E_t\left[a e^{x_{t+1}}\right] &\approx E_t\left[a x_{t+1}\right] \text{ up to a constant .} \end{aligned}$$

For example, these building blocks yield

$$\begin{aligned} e^{x_t} &\approx 1 + x_t \\ aX_t &\approx a\bar{X}x_t \text{ up to a constant} \\ (X_t + a)Y_t &\approx \bar{X}\bar{Y}x_t + (\bar{X} + a)\bar{Y}y_t \text{ up to a constant} \end{aligned}$$

Constants drop out of each equation in the end, since they satisfy steady state relationships, but they are important in intermediate steps: compare for example the

[5] An alternative to approximate (3.2) rewrites it as

$$0 = \log\left(E_t\left[\exp\left(\tilde{g}(x_{t+1}, x_t)\right)\right]\right)$$

where $\tilde{g} = \log g$. Assuming x_t and x_{t+1} to be (approximately) conditionally jointly normally distributed with an (approximately) constant conditional variance-covariance matrix, and assuming that

$$\log g(0,0) \approx \frac{1}{2}\mathrm{Var}_t\left[\tilde{g}_1 \cdot x_{t+1} + \tilde{g}_2 \cdot x_t\right], \tag{3.3}$$

independent of t (rather than $\log g(0,0) = 0$) yields

$$\begin{aligned} 0 &\approx \log E_t\left[\exp\left(\tilde{g}(0,0) + \tilde{g}_1 \cdot x_{t+1} + \tilde{g}_2 \cdot x_t\right)\right] \\ &\approx E_t\left[\tilde{g}_1 \cdot x_{t+1} + \tilde{g}_2 \cdot x_t\right], \end{aligned}$$

using $E[e^X] = e^{E[X] + \mathrm{Var}[X]/2}$ for normally distributed variables. The two ways of approximating (3.2) differ essentially only in their choice for $g(0,0)$, since $g_1 = \tilde{g}_1$, if $g(0,0) = 1$.

two equations above. Rather than describing the general principles further, it is fruitful to consider specific examples instead. The first example in section 4 studies the neoclassical growth model in great detail and performs all the calculations "by hand". That section can also be used as a supplement to introducing students into modern dynamic macroeconomic theory. Advanced readers may wish to skip instead right away to section 5, which analyzes Hansens (1985) real business cycle model and which is more compact than section 4.

4 Doing by hand: the neoclassical growth model.

In this section, the stochastic neoclassical growth model shall be studied. This is useful, since all the calculations for this model can actually be done "by hand," i.e. with just pencil, paper and perhaps a pocket calculator. Furthermore, it serves as a benchmark paradigm in much of the modern macroeconomic literature. We therefore also take this opportunity to review the modelling principles for this literature before returning to the computational focus of this chapter. For a book-length perspective on these principles, the reader is advised to study Sargent (1987).

4.1 Modelling principles.

Theories are usually analyzed in order to answer a particular question or to theoretically understand a particularly interesting fact or set of facts. Modern macroeconomic theory is applied dynamic general equilibrium analysis. To spell out such a theory, one needs to explicitely specify the *environment*:

1. preferences,
2. technologies,
3. endowments,
4. and information.

Furthermore, one needs to state the object of study. Available choices are usually

1. The *social planners problem*. In that case, one needs to specify the planners *objective function*.
2. The *competitive equilibrium*. In that case, one needs to specify the markets and provide a definition of an equilibrium. In particular, one needs to spell out the precise extent of market powers.
3. The *game*. In that case, one needs to specify the *rules* and to provide a definition of an equilibrium.

4.2 The environment

For the stochastic neoclassical growth model, the environment is as follows.

1. *Preferences:* The *representative agent* experiences utility according to

$$U = E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta}\right],$$

where C_t is consumption[6], $0 < \beta < 1$ is the discount factor and $\eta > 0$ is the coefficient of relative risk aversion.

2. *Technologies:* We assume a Cobb-Douglas production function[7]

$$C_t + K_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} + (1-\delta) K_{t-1}$$

where K_t is capital, N_t is labor, $0 < \rho < 1$ ("*capital share*") and $0 < \delta < 1$ ("*depreciation rate*") are parameters and where Z_t, the total factor productivity, is exogenously evolving according to ,

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.\mathcal{N}(0;\sigma^2),$$

Here, $0 < \psi < 1$, $\bar{Z}$ are parameters.

3. *Endowment:* Each period, the representative agent is endowed with one unit of time, $N_t = 1$. Furthermore, he is endowed with capital K_{-1} before $t = 0$.

4. *Information:* C_t, N_t and K_t need to be chosen based on all information $\mathcal{I}_t$ up to time t.

4.3 The social planners problem.

The objective of the social planner is to maximize the utility of the representative agent subject to feasibility, i.e.

$$\max_{(C_t,K_t)_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta}\right]$$

[6] We use capital letters to denote "levels" of variables, and use small letters to denote log-deviations. This should not be confused with the more common notational usage in other parts of the literature, where capital letters are usually reserved for aggregate variables, while small letters denote individual variables.

[7] We use the date $t-1$ rather than the more commonly used date t as subscript for capital in the production function. This is just a notational difference, which we find useful, however, With the notation here, the date of a variable refers to the point in time, when it is actually chosen. Put differently, it refers to the information, with respect to which a variable is measurable. This turns out to be particularly convenient, once one needs to solve for the dynamics with the theorems in section 6. If the more commonly used notation is used instead, one needs to much more careful in order to not introduce mistakes at that point

$$\begin{aligned} & \text{s.t.} \quad K_{-1},\, Z_0, \\ C_t + K_t &= Z_t K_{t-1}^{\rho} + (1-\delta) K_{t-1} \\ \log Z_t &= (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \\ \epsilon_t &\sim i.i.d. \mathcal{N}(0;\sigma^2) \end{aligned}$$

To solve it, one should use the techniques of dynamic programming. Stokey, Lucas, with Prescott (1989) provide the standard textbook on this technique. Here, we bypass the dynamic programming foundations, and proceed directly to the necessary first order conditions of optimality. To calculate them, form the *Lagrangian*:

$$\begin{aligned} L = \max_{(C_t,K_t)_{t=0}^{\infty}} E[\sum_{t=0}^{\infty} \beta^t (& \frac{C_t^{1-\eta}-1}{1-\eta} \\ & -\lambda_t \left(C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta)K_{t-1}\right))] \end{aligned}$$

The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial \lambda_t}: \quad 0 &= C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta)K_{t-1} \\ \frac{\partial L}{\partial C_t}: \quad 0 &= C_t^{-\eta} - \lambda_t \\ \frac{\partial L}{\partial K_t}: \quad 0 &= -\lambda_t + \beta E_t \left[\lambda_{t+1}\left(\rho Z_{t+1} K_t^{\rho-1} + (1-\delta)\right)\right] \end{aligned} \tag{4.4}$$

To the uninitiated, the equation (4.4) for $\frac{\partial L}{\partial K_t}$ may seem tricky. To check it, write out the terms for t and $t+1$ in the objective function,

$$\begin{aligned} \ldots \quad + \quad & \beta^t \left(\frac{C_t^{1-\eta}-1}{1-\eta} - \lambda_t \left(C_t + K_t - Z_t K_{t-1}^{\rho} - (1-\delta)K_{t-1}\right)\right) \\ \ldots \quad + \quad & \beta^{t+1} \left(\frac{C_{t+1}^{1-\eta}-1}{1-\eta} - \lambda_{t+1} \left(C_{t+1} + K_{t+1} - Z_{t+1} K_t^{\rho} - (1-\delta)K_t\right)\right) \\ + \quad & \ldots \end{aligned}$$

and differentiate with respect to K_t to get (4.4). The expectation E_t comes in, because information of date $t+1$ is not yet known at date t, when choosing K_t. The first-order conditions are often also called *Euler equations*.

One also obtains the *transversality condition*

$$0 = \lim_{T\to\infty} E_0[\beta^T C_T^{-\eta} K_T] \tag{4.5}$$

obtained from a limiting Kuhn-Tucker condition, i.e. from summing just to T rather than ∞ in the social planners solution, substituting C_t with $Z_t K_{t-1}^{\rho} - (1-\delta)K_{t-1} - K_t$ everywhere, taking the derivative with respect to K_T, multiplying with K_T, and

setting the result to zero while taking the limit for $T \to \infty$. Another interpretation is given in the next subsection 4.4. It is the transversality condition which (essentially) rules out explosive solutions: this is what we shall keep in mind.

To solve for the steady state, rewrite the necessary conditions:

1.

$$C_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t$$

2.

$$R_t = \rho Z_t K_{t-1}^{\rho-1} + (1-\delta)$$

3.

$$1 = E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \tag{4.6}$$

4.

$$\log Z_t = (1-\psi)\log\bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \;\; \epsilon_t \sim i.i.d.\mathcal{N}(0;\sigma^2)$$

Equation (4.6) is the *Lucas asset pricing equation*, see Lucas (1978), which typically arises in these models. Dropping the time indices yields

$$\begin{aligned} \bar{C} &= \bar{Z}\bar{K}^{\rho} + (1-\delta)\bar{K} - \bar{K} \\ \bar{R} &= \rho\bar{Z}\bar{K}^{\rho-1} + (1-\delta) \\ 1 &= \beta\bar{R} \end{aligned}$$

or

$$\begin{aligned} \bar{R} &= \frac{1}{\beta} \\ \bar{K} &= \left(\frac{\rho\bar{Z}}{\bar{R}-1+\delta}\right)^{1/(1-\rho)} \\ (\text{ hence: } \bar{Y} &= \bar{Z}\bar{K}^{\rho}) \\ \bar{C} &= \bar{Y} - \delta\bar{K} \end{aligned}$$

It is possible to reduce the first three of these equations to just two or just one by eliminating some of the variables. Quite popular is the reduction to a system in C_t and K_{t-1}, which we will discuss in section 4.6, or to a system in just K_t at leads and lags, which we will discuss in subsection 4.7. However, there is no particular

reason to make such a reduction: we therefore choose to carry all the equations with us, since it will then also be easier to keep seeing the economic interpretation of the log-linearizations.

While one could now start to analyze the dynamics, it may be interesting to do a "detour" via studying the competitive equilibrium: as one shall expect from the welfare theorems, the solution to the competitive equilibrium yields the same allocation as the solution to the social planners problem. A reader who is just interested in analyzing the dynamics of the social planners problem should skip directly to subsection 4.5.

4.4 The competitive equilibrium

Let us define a competitive equilibrium to be a sequence $(C_t, N_t, K_t, R_t, W_t)_{t=0}^{\infty}$, so that

1. Given[8] $K_{-1}^{(s)}$ and market wages W_t and returns R_t, the representative agent solves

$$\begin{aligned} \max_{(C_t, K_t^{(s)})_{t=0}^{\infty}} \quad & E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta}-1}{1-\eta}\right] \\ \text{s.t.} \quad & N_t^{(s)} = 1, \\ C_t + K_t^{(s)} \quad = \quad & W_t N_t^{(s)} + R_t K_{t-1}^{(s)} \end{aligned}$$

   plus the *no-Ponzi-game condition*

$$0 = \lim_{t\to\infty} E_0 \prod_{s=1}^{t} R_t^{-1} K_t.$$

2. Given $(W_t, R_t)_{t=0}^{\infty}$, the representative firm solves[9]

$$\max_{K_{t-1}^{(d)}, N_t^{(d)}} Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta) K_{t-1}^{(d)} - W_t N_t^{(d)} - R_t K_{t-1}$$

   where

$$\log Z_t = (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.\mathcal{N}(0;\sigma^2),$$

   is exogeneous.

3. Markets clear :

[8] The superscript $^{(s)}$ on $K_{t-1}^{(s)}$ and $N_t^{(s)}$ is meant to indicate "supply".

[9] The superscript $^{(d)}$ on $K_{t-1}^{(d)}$ and $N_t^{(d)}$ is meant to indicate "demand".

(a) The labor market,

$$N_t^{(d)} = N_t^{(s)} = N_t$$

(b) The capital market,

$$K_{t-1}^{(d)} = K_{t-1}^{(s)} = K_{t-1}$$

(c) The goods market,

$$C_t + K_t \quad = \quad Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1}$$

We need only two out of these three conditions by *Walras' law*.

Another way to define a competitive equilibrium is to drop R_t and introduce history-contingent prices P_t for consumption goods of time t in terms of consumption goods at date = 0. This has the advantage of turning the sequence of budget constraints of the consumer into one infinite-horizon budget constraint, clarifying the role of the no-Ponzi-game condition: the no-Ponzi-game condition stipulates, that in net present value terms, the agent should neither have capital left over at infinity or borrow anything at infinity. Using the first order conditions below, a close look at the no-Ponzi-game condition reveals, that it is essentially nothing but the transversality condition 4.5 of the social planners problem.

To analyze the competitive equilibrium, proceed as follows. The representative firm solves

$$\max_{K_{t-1}^{(d)}, N_t^{(d)}} Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta)K_{t-1}^{(d)} - W_t N_t^{(d)} - R_t K_{t-1}$$

The first order conditions of the firm ("demand curves") are

$$\begin{aligned} W_t &= (1-\rho)Z_t \left(K_{t-1}^{(d)}\right)^{\rho} \left(N_t^{(d)}\right)^{-\rho} \\ R_t &= \rho Z_t \left(K_{t-1}^{(d)}\right)^{\rho-1} \left(N_t^{(d)}\right)^{1-\rho} + (1-\delta) \end{aligned}$$

Rewrite this, dropping $^{(d)}$ and using

$$Y_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho}$$

on obtains, as usual for Cobb-Douglas

1. that the wage payments equal the labor share,

$$W_t N_t \quad = \quad (1-\rho)Y_t$$

2. and that the returns equal the capital share plus one minus depreciation,

$$R_t K_{t-1} = \rho Y_t + (1-\delta)K_{t-1}$$

The interest rate is $R_t - 1$:

$$r_t \approx R_t - 1 = \rho \frac{Y_t}{K_{t-1}} - \delta$$

For the representative agent, form the Lagrangian:

$$L = \max_{(C_t,K_t)_{t=0}^{\infty}} E[\sum_{t=0}^{\infty} \beta^t (\frac{C_t^{1-\eta}-1}{1-\eta} - \lambda_t (C_t + K_t - W_t - R_t K_{t-1}))]$$

The first order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial \lambda_t}: & \quad 0 = C_t + K_t - W_t - R_t K_{t-1} \\
\frac{\partial L}{\partial C_t}: & \quad 0 = C_t^{-\eta} - \lambda_t \\
\frac{\partial L}{\partial K_t}: & \quad 0 = -\lambda_t + \beta E_t [\lambda_{t+1} R_{t+1}]
\end{aligned}$$

Using, what one already knows for R_t and W_t yields

1.

$$C_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t$$

2.

$$R_t = \rho Z_t K_{t-1}^{\rho-1} + (1-\delta)$$

3.

$$1 = E_t \left[\beta \left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right]$$

4.

$$\log Z_t = (1-\psi) \log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d. \mathcal{N}(0; \sigma^2)$$

These are the same equations as for social planners problem! Thus, whether one studies a competitive equilibrium or the social planners problem, one ends up with the same allocation of resources.

4.5 Solving for the dynamics.

Let us return to the problem of solving for the dynamics in the stochastic neoclassical growth model. As stated in section 3, one needs to do five things:

1. Find the constraints and the first-order conditions: done!
2. Find the steady state: done!
3. Log-linearize the constraints and the first-order conditions.
4. Solve for the recursive equilibrium law of motion via the method of undetermined coefficients.
5. Analyze the solution via impulse-response analysis and second-order-properties.

4.5.1 Log-Linearization

To apply what was stated already in section 3, let e.g. c_t denote the logarithmic deviation of C_t from its steady state value $\bar{C}$. Formally:

$$c_t = \log(C_t) - \log(\bar{C}).$$

Interpretation: If $c_t = 0.03$, then C_t is approximately 3 percent above its steady state value. Write

$$C_t = \bar{C}e^{c_t} \approx \bar{C}(1 + c_t)$$

If there is a magic trick, then this is it! More examples:

$$\begin{aligned} Z_t K_{t-1}^{\rho} &= \bar{Z}\bar{K}^{\rho} e^{z_t + \rho k_{t-1}} \approx \bar{Z}\bar{K}^{\rho}(1 + z_t + \rho k_{t-1}) \\ C_t + K_t &= \bar{C}e^{c_t} + \bar{K}e^{k_t} \approx \bar{C} + \bar{K} + \bar{C}c_t + \bar{K}k_t \end{aligned} \tag{4.7}$$

If there are products, then it is easier to first multiply them out and to combine products of exponential terms before one log-linearizes. E.g., equation (4.7 is easier than

$$\begin{aligned} Z_t K_{t-1}^{\rho} &= \bar{Z}\bar{K}^{\rho} e^{z_t} e^{\rho k_{t-1}} \\ &\approx \bar{Z}\bar{K}^{\rho}(1 + z_t)(1 + \rho k_{t-1}) \\ &\approx \bar{Z}\bar{K}^{\rho}(1 + z_t + \rho k_{t-1}), \end{aligned}$$

although one gets the same final result, of course. Just in case, one needs to keep in mind, that products of "small letters" are approximately zero, e.g.

$$z_t k_{t-1} \approx 0.$$

Doing this for the constraints and the first-order conditions of the model yields the following.

1. For the first equation, the feasibility constraint, one obtains:

$$\begin{aligned} C_t &= Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t \\ \bar{C}e^{c_t} &= \bar{Z}\bar{K}^{\rho}e^{z_t+\rho k_{t-1}} + (1-\delta)\bar{K}e^{k_{t-1}} - \bar{K}e^{k_t} \\ \bar{C} + \bar{C}c_t &\approx \bar{Z}\bar{K}^{\rho} + (1-\delta)\bar{K} - \bar{K} \\ &\quad + \bar{Z}\bar{K}^{\rho}(z_t + \rho k_{t-1}) + (1-\delta)\bar{K}k_{t-1} - \bar{K}k_t \end{aligned}$$

   Use the steady state relationships

$$\begin{aligned} \bar{Y} &= \bar{Z}\bar{K}^{\rho} \\ \bar{C} &= \bar{Y} - \delta\bar{K} \end{aligned}$$

   to get

$$\bar{C}c_t \approx \bar{Z}\bar{K}^{\rho}(z_t + \rho k_{t-1}) + (1-\delta)\bar{K}k_{t-1} - \bar{K}k_t$$

   or, simplified, because we want to solve for the dynamics by hand,

$$c_t \approx \frac{\bar{Y}}{\bar{C}}z_t + \frac{\bar{K}}{\bar{C}}\bar{R}k_{t-1} - \frac{\bar{K}}{\bar{C}}k_t$$

   One can still see the economic interpretation of this equation. If productivity z_t or productive capital k_{t-1} is above its steady state level, total production is higher, and thus, higher consumption can be afforded. On the other hand, higher investment in the form of higher k_t decrease consumption ceteris paribus. To convert percentage changes of any of these variables into percentage changes of consumption, one needs to multiply with the corresponding steady state ratios of the levels.

2. For the second equation, the calculation of the return, one gets

$$\begin{aligned} R_t &= \rho Z_t K_{t-1}^{\rho-1} + 1 - \delta \\ \bar{R}e^{r_t} &= \rho\bar{Z}\bar{K}^{\rho-1}e^{z_t+(\rho-1)k_{t-1}} + 1 - \delta \\ \bar{R} + \bar{R}r_t &\approx \rho\bar{Z}\bar{K}^{\rho-1} + 1 - \delta \\ &\quad + \rho\bar{Z}\bar{K}^{\rho-1}(z_t + (\rho-1)k_{t-1}) \end{aligned}$$

   Use the steady state relationship

$$\frac{1}{\beta} = \bar{R} = \rho\bar{Z}\bar{K}^{\rho} + 1 - \delta$$

   to get

$$\bar{R}r_t \approx \rho\bar{Z}\bar{K}^{\rho-1}(z_t + (\rho-1)k_{t-1})$$

or, simplified, because we want to solve for the dynamics by hand,

$$r_t \approx (1-\beta(1-\delta))(z_t-(1-\rho)k_{t-1})$$

Economically, this equation states a relationship between the interest rate on the left hand side and the marginal product of capital on the right-hand side, which is increasing in z_t and decreasing in k_{t-1}. This is exactly what one should expect.

3. For the third equation, the the Lucas asset pricing equation, one gets

$$\begin{aligned} 1 &= E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \\ 1 &= E_t\left[\beta\left(\frac{\bar{C}e^{c_t-c_{t+1}}}{\bar{C}}\right)^{\eta} \bar{R}e^{r_{t+1}}\right] \\ 1 &\approx E_t\left[\beta\bar{R}+\beta\bar{R}(\eta(c_t-c_{t+1})+r_{t+1})\right] \end{aligned}$$

Use the steady state relationship

$$1 = \beta\bar{R}$$

to get

$$0 \approx E_t\left[\eta(c_t-c_{t+1})+r_{t+1}\right]$$

One can see that percentage deviations of the marginal rate of substitution from its steady state level, given by $\eta(c_t-c_{t+1})$, need to equal the negative of the interest rate r_{t+1} in expectation. In particular, high expected interest rates coincide with low marginal rates of substitution, i.e. with an expected rise in consumption. This makes sense: if a rise in consumption is expected, only a high interest rate can prevent agents from borrowing against that future rise.

4. For the fourth equation:

$$\begin{aligned} \log Z_t &= (1-\psi)\log\bar{Z}+\psi\log Z_{t-1}+\epsilon_t, \\ \log(\bar{Z}e^{z_t}) &= (1-\psi)\log\bar{Z}+\psi\log(\bar{Z}e^{z_{t-1}})+\epsilon_t, \\ z_t &= \psi z_{t-1}+\epsilon_t, \end{aligned}$$

holding exactly.

Collect the equations obtained:

1.

$$c_t = \frac{\bar{Y}}{\bar{C}}z_t+\frac{\bar{K}}{\beta\bar{C}}k_{t-1}-\frac{\bar{K}}{\bar{C}}k_t$$

2.

$$r_t = (1-\beta(1-\delta))(z_t-(1-\rho)k_{t-1})$$

3.

$$0 = E_t\left[\eta(c_t-c_{t+1})+r_{t+1}\right]$$

4.

$$z_t = \psi z_{t-1}+\epsilon_t$$

Here too, it is possible to reduce the first three of these equations to just two or just one by eliminating some of the variables. In particular, we will discuss the popular reduction to a system in c_t and k_{t-1} in subsection 4.6, and the reduction to a second-order difference equation in just k_t in subsection 4.7. However, there is no particular reason to make such a reduction here: we therefore keep on carrying all the equations with us. The final result is, of course, the same.

4.5.2 Solve for the dynamics with the method of undetermined coefficients.

What is given at time t are the *state variables* k_{t-1} and z_t. What we need to find are k_t, r_t and c_t. We postulate a *linear recursive law of motion*,

$$\begin{aligned} k_t &= \nu_{kk}k_{t-1}+\nu_{kz}z_t \\ r_t &= \nu_{rk}k_{t-1}+\nu_{rz}z_t \\ c_t &= \nu_{ck}k_{t-1}+\nu_{cz}z_t \end{aligned}$$

The task is to solve for the as of yet "undetermined" coefficients

$$\nu_{kk},\nu_{kz},\nu_{rk},\nu_{rz},\nu_{ck},\nu_{cz}$$

This can be done directly, employing the formulas of section 6, but it is instructive to go through this example "by hand" to get a feel for the details. These coefficients can be interpreted as *elasticities*: if e.g. $\nu_{ck}=0.5$ and K_{t-1} is 10 percent above its steady state level, then C_t should be set 5 percent above its steady state level.

To solve for the coefficients $\nu_{kk},\nu_{kz},\nu_{rk},\nu_{rz},\nu_{ck},\nu_{cz}$, substitute the postulated linear recursive law of motion into the equations we have obtained until only k_{t-1} and z_t remain and compare coefficients, noting that

$$E_t[z_{t+1}]=\psi z_t$$

Thus,

1. for the first equation ("feasibility"):

$$\begin{aligned} c_t &= \left(1+\delta\frac{\bar{K}}{\bar{C}}\right)z_t + \frac{\bar{K}}{\beta\bar{C}}k_{t-1} - \frac{\bar{K}}{\bar{C}}k_t \\ \nu_{ck}k_{t-1}+\nu_{cz}z_t &= \frac{\bar{Y}}{\bar{C}}z_t + \left(\frac{1}{\beta}-\nu_{kk}\right)\frac{\bar{K}}{\bar{C}}k_{t-1} - \frac{\bar{K}}{\bar{C}}\nu_{kz}z_t \end{aligned}$$

   Since this needs to be satisfied for *any* value of k_{t-1} and z_t, we must have

$$\begin{aligned} \nu_{ck} &= \left(\frac{1}{\beta}-\nu_{kk}\right)\frac{\bar{K}}{\bar{C}} \\ \nu_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz} \end{aligned}$$

2. For the second equation ("calculation of the return"),

$$\begin{aligned} r_t &= (1-\beta(1-\delta))(z_t-(1-\rho)k_{t-1}) \\ \nu_{rk}k_{t-1}+\nu_{rz}z_t &= (1-\beta(1-\delta))(z_t-(1-\rho)k_{t-1}) \end{aligned}$$

   Comparing coefficients, we get

$$\begin{aligned} \nu_{rk} &= -(1-\beta(1-\delta))(1-\rho) \\ \nu_{rz} &= 1-\beta(1-\delta) \end{aligned}$$

3. For the third equation ("asset pricing"),

$$\begin{aligned} 0 &= E_t\left[\eta(c_t-c_{t+1})+r_{t+1}\right] \\ 0 &= E_t[\eta((\nu_{ck}k_{t-1}+\nu_{cz}z_t)-(\nu_{ck}k_t+\nu_{cz}z_{t+1})) \\ &\quad +\nu_{rk}k_t+\nu_{rz}z_{t+1}] \\ &= (\nu_{rk}-\eta\nu_{ck})k_t+\eta\nu_{ck}k_{t-1}+((\nu_{rz}-\eta\nu_{cz})\psi+\eta\nu_{cz})z_t \\ &= ((\nu_{rk}-\eta\nu_{ck})\nu_{kk}+\eta\nu_{ck})k_{t-1} \\ &\quad +((\nu_{rk}-\eta\nu_{ck})\nu_{kz}+(\nu_{rz}-\eta\nu_{cz})\psi+\eta\nu_{cz})z_t \end{aligned}$$

   Note, that we needed to plug things in *twice* here! This is typical for the log-linearized Lucas asset pricing equation. Comparing coefficients, we get

$$\begin{aligned} 0 &= (\nu_{rk}-\eta\nu_{ck})\nu_{kk}+\eta\nu_{ck} \\ 0 &= (\nu_{rk}-\eta\nu_{ck})\nu_{kz}+(\nu_{rz}-\eta\nu_{cz})\psi+\eta\nu_{cz} \end{aligned}$$

Collecting, we get the equations from comparing the coefficients on k_{t-1},

$$\begin{aligned} \nu_{ck} &= \left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\bar{K}}{\bar{C}} && (4.8)\\ \nu_{rk} &= -(1-\beta(1-\delta))(1-\rho) && (4.9)\\ 0 &= (\nu_{rk} - \eta\nu_{ck})\nu_{kk} + \eta\nu_{ck} && (4.10)\\ & && (4.11) \end{aligned}$$

and the equations from comparing the coefficients on z_t,

$$\begin{aligned} \nu_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz} && (4.12)\\ \nu_{rz} &= 1-\beta(1-\delta) && (4.13)\\ 0 &= (\nu_{rk} - \eta\nu_{ck})\nu_{kz} + (\nu_{rz} - \eta\nu_{cz})\psi + \eta\nu_{cz} && (4.14) \end{aligned}$$

One now needs to solve for ν_{kk}. This is indeed the "crucial" coefficient, since it relates the new value k_t of the endogenous state variable to its old value k_{t-1}, i.e. captures the essence of the dynamics of the system. Once ν_{kk} is known, all other coefficients can easily be computed, as we shall see.

To solve for ν_{kk}, substitute out ν_{ck} and ν_{rk} in equation (4.10) with (4.9) and (4.8):

$$0 = (-(1-\beta(1-\delta))(1-\rho) - \eta\left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\bar{K}}{\bar{C}})\nu_{kk} + \eta\left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\bar{K}}{\bar{C}}$$

Simplify: divide by $\eta\bar{K}/\bar{C}$, sort powers of ν_{kk} to get

$$0 = \nu_{kk}^2 - \gamma\nu_{kk} + \frac{1}{\beta}$$

where

$$\begin{aligned} \gamma &= (1-\beta(1-\delta))(1-\rho)\frac{\bar{C}}{\eta\bar{K}} + 1 + \frac{1}{\beta} && (4.15)\\ &= \frac{(1-\beta(1-\delta))(1-\rho)(1-\beta+\beta\delta(1-\rho))}{\eta\rho\beta} + 1 + \frac{1}{\beta} \end{aligned}$$

The solution to this quadratic equation is given by

$$\nu_{kk} = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}}$$

Note that $\gamma > 0$. The product of the two roots is $1/\beta$. We are looking for a root which is stable, i.e. is smaller than one in absolute value. The stable root must therefore be the smaller of the two roots.

In order to solve for the other coefficients, proceed as follows.

1. The other coefficients ν_{rk} and ν_{ck} on k_{t-1} can be found from rewriting euuations (4.8) and (4.9) as

$$\begin{aligned} \nu_{rk} &= -(1-\beta(1-\delta))(1-\rho) \\ \nu_{ck} &= \left(\frac{1}{\beta} - \nu_{kk}\right)\frac{\bar{K}}{\bar{C}} \end{aligned}$$

2. For the coefficients on z_t, directly calculate

$$\nu_{rz} = 1-\beta(1-\delta)$$

Now, equations (4.14) and (4.12) are a system of two linear equations in the two unknowns ν_{cz} and ν_{kz}, which can be solved easily. The solution is perhaps a bit ugly, but can be stated without much problem:

$$\begin{aligned} \nu_{kz} &= \frac{\nu_{rz}\psi + \eta(1-\psi)\frac{\bar{Y}}{\bar{C}}}{-\nu_{rk} + \eta\nu_{ck} + \eta(1-\psi)\frac{\bar{K}}{\bar{C}}} \\ \nu_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz} \end{aligned}$$

4.5.3 Some results

After all this hard work, here are some results. "Calibrated" parameters are ("quarterly data"): $\beta = 1/1.01 \approx 0.99$, $\rho = 0.36$, $\eta = 1.0$, $\delta = 0.025$, $\bar{Z} = 1$. We get

$$\begin{aligned} \nu_{kk} &= 0.965, & \nu_{kz} &= 0.075 \\ \nu_{ck} &= 0.618, & \nu_{cz} &= 0.305 \\ \nu_{rk} &= -0.022, & \nu_{rz} &= 0.035 \end{aligned}$$

Using the formulas obtained above, one can do some sensitivity analysis, see tables 1 and 2.

What one can do now is to

1. trace out, what happens if the initial capital is, say, approximately 20 percent below steady state, $k_{-1} = -0.2$, and there are otherwise no shocks ($z_t \equiv 0$). Then, $k_t = \nu_{kk}^{t+1} k_{-1}$. With $\mid \nu_{kk} \mid < 1$, we get convergence back to the steady state.

2. trace out what happens to all the other variables along the way. This can be done in two different ways. Either, one uses the log-linearized system and calculates $c_t = \nu_{ck} k_{t-1}$, for example. This is always done in the programs described in the appendix A. Or, one calculates the level $K_t = \bar{K}\exp(k_t)$ from the obtained path for k_t and likewise the level of Z_t, and computes the level for the original variables using the original nonlinear equations. For example, one gets $C_t = Z_t K_{t-1}^{\rho} + (1-\delta)K_{t-1} - K_t$.

ν_{kk} :	$\eta = 0.01$	$\eta = 0.5$	$\eta = 1$	$\eta = 2$	$\eta = 1000$
$\delta = 0$	0.8804	0.9857	0.9909	0.9944	1.0000
$\delta = 0.025$	0.6759	0.9496	0.9654	0.9766	0.9998
$\delta = 0.1$	0.3238	0.8489	0.8918	0.9235	0.9987
$\delta = 1.0$	0.0086	0.2480	0.3600	0.4789	0.9711

Table 1: *Some sensitivity analysis in the neoclassical growth model. If depreciation δ is less or if the intertemporal elasticity of substitution $1/\eta$ is smaller, the speed $1-\eta_{kk}$ of convergence back to the steady state is slower.*

ν_{kz} :	$\eta = 0.01$	$\eta = 0.5$	$\eta = 1$	$\eta = 2$	$\eta = 1000$
$\delta = 0$	0.1395	0.0256	0.0238	0.0231	0.0231
$\delta = 0.025$	0.4458	0.0847	0.0752	0.0718	0.0808
$\delta = 0.1$	0.9876	0.2412	0.2003	0.1804	0.2496
$\delta = 1.0$	1.4722	1.1433	1.0000	0.8611	1.5772

Table 2: *Some sensitivity analysis in the neoclassical growth model. If depreciation δ is less or if the intertemporal elasticity of substitution $1/\eta$ is smaller, the reaction η_{kz} of the new capital stock, i.e. of investment, is generally smaller too, except for very low levels of $1/\eta$ (compare the last two columns).*

3. simulate the model: simulate $\epsilon_t's$, pick some initial k_{-1} and z_0. Then, calculate recursively

$$\begin{aligned} z_t &= \psi z_{t-1} + \epsilon_t \\ k_t &= \nu_{kk} k_{t-1} + \nu_{kz} z_t. \end{aligned}$$

With that, obtain all other variables.

4. trace out what happens to all the variables after $\epsilon_0 = 1$, $\epsilon_t = 0$ for $t > 1$, when starting from the steady state. This is called an *impulse response* analysis. Impulse responses for the neoclassical growth model are shown in figure 1.

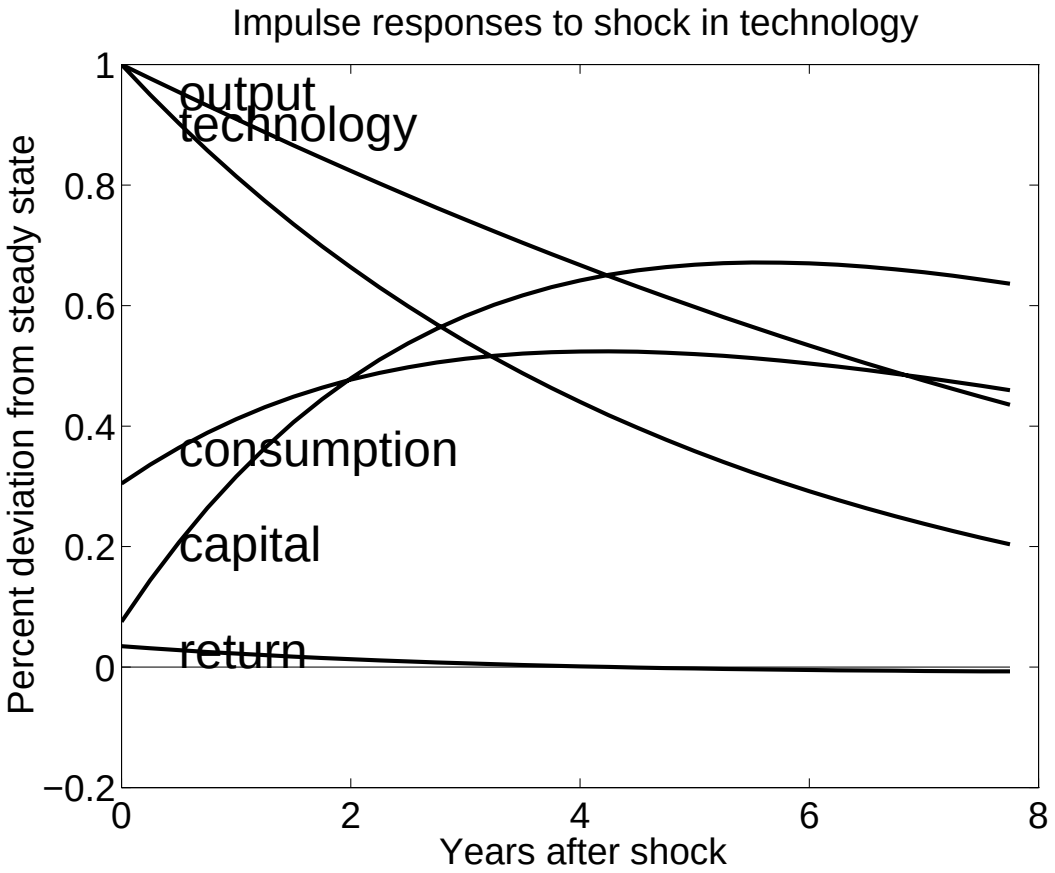

Figure 1: *This figure shows the impulse response for the stochastic neoclassical growth model. The parameters are as stated in the text.*

4.6 The relationship to a state-space approach.

In this section, we will discuss the popular reduction to a system in c_t and k_{t-1} for the log-linearized system or to a system in C_t and K_{t-1} in the original system: this

yields the *state-space* approach. To start with the log-linearized system, eliminate r_t from the first three of the four equations characterizing the dynamics, and set $z_t \equiv 0$ for the purpose of this discussion. We get the two equations

$$c_t = \frac{\bar{K}}{\beta \bar{C}} k_{t-1} - \frac{\bar{K}}{\bar{C}} k_t \tag{4.16}$$

$$0 = \eta(c_t - c_{t+1}) - (1 - \beta(1-\delta))(1-\rho)k_t \tag{4.17}$$

Furthermore, for the purposes of this subsection, it is convenient to solve the first equation for k_t and use it to eliminate k_t in the second[10]. Slightly rewriting the result, one gets

$$k_t - k_{t-1} = \left(\frac{1}{\beta} - 1\right) k_{t-1} - \frac{\bar{C}}{\bar{K}} c_t \tag{4.18}$$

$$c_{t+1} - c_t = \frac{1}{\eta}(1 - \beta(1-\delta))(1-\rho)\left(\frac{\bar{C}}{\bar{K}} c_t - \frac{1}{\beta} k_{t-1}\right) \tag{4.19}$$

In the state space approach, one looks at the equations (4.18) and (4.19) as a dynamic system in the two-dimensional vector (k_{t-1}, c_t), and analyzes its properties as follows.

First, one needs to solve for the steady state from these two equations: as we know already, it is given by $c_t = 0$ and $k_t = 0$. Seen differently, set $k_t = k_{t-1} = k$ and $c_t = c$ in (4.18) to get the first steady state equation,

$$c = \left(\frac{1}{\beta} - 1\right) \frac{\bar{K}}{\beta \bar{C}} k \tag{4.20}$$

Proceed likewise with $c_{t-1} = c_t = c$ in (4.19) to get the second steady state equation

$$c = \frac{\bar{K}}{\beta \bar{C}} k \tag{4.21}$$

These two steady state equations describe two curves in the two-dimensional (k_{t-1}, c_t)-plane, cutting that plane into four quadrants: see figure 2.

Any point (k_{t-1}, c_t) in that plane can in principle occur from the perspective of the state space approach. Next, one thus seeks to predict the changes $k_t - k_{t-1}$ and $c_{t+1} - c_t$ when starting from any such point. The signs of these changes depend on the quadrant in which the point lies. For example, in the upper left quadrant, we are "above" the curve describing the first equation (4.20). Thus, for a point (k_{t-1}, c_t) above that curve, we get $k_t - k_{t-1} < 0$ from equation (4.18). This is indicated by an

[10] This manipulation is necessary because we are in a discrete-time framework, while in a continuous-time framework, one essentially has $k_t \approx k_{t-dt}$. For the same reason, the state space diagram for the discrete-time framework looks slightly different from those familiar from the continuous-time analysis.

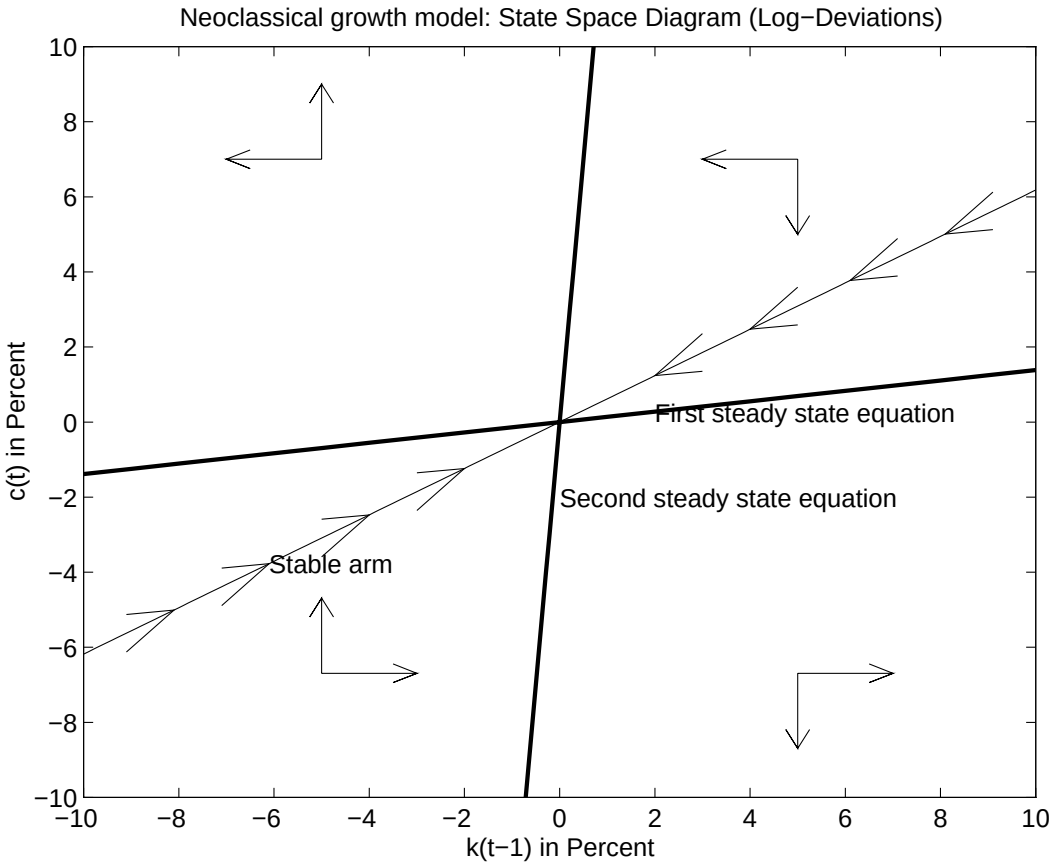

Figure 2: *This figure shows the state space diagram for the log-linearized neoclassical growth model. The two steady state equations cut the plane into four quadrants, which differ qualitatively in their dynamics as indicated by the arrows at right angles. The stable arm is the function* $c_t = \nu_{ck} k_{t-1}$*, which was derived with the method of undetermined coefficients.*

arrow pointing to the left. Furthermore, in the upper left quadrant, we are "to the left" of the curve describing the second equation (4.21). Thus, for a point (k_{t-1}, c_t) to the left of that curve, we get $c_{t+1} - c_t > 0$ from equation (4.19). Thus, consumption is increasing there, indicated by the arrows pointing upwards. In this manner, one can analyze the dynamic behaviour at every point in the plane. Looking at these arrows, one can see that the system is *saddle-point stable*: it diverges away from the origin in the upper left quadrant and the lower right quadrant, and may have a chance to converge towards it in the lower left quadrant and the upper right quadrant. Finally, one can trace out trajectories of the dynamic system, starting it at any point (k_{t-1}, c_t) and letting it evolve according to the equations (4.18) and (4.19). It turns out that these trajectories will converge to the steady state $k = 0, c = 0$, if and only if the

trajectories were started from a point on the *stable arm*. Further analysis reveals, that the stable arm is given by $c_t = \eta_{ck} k_{t-1}$. In other words, the method of undetermined coefficients delivers the calculation of the stable arm for saddle-point stable systems.

Rather than looking at the system in log-deviation form, one can also look at the original, nonlinear system and reduce it to a system in C_t and K_{t-1}, setting $Z_t \equiv \bar{Z}$ for the sake of this argument:

$$\begin{aligned} C_t &= \bar{Z} K_{t-1}^{\rho} + (1-\delta) K_{t-1} - K_t \\ 1 &= \beta \left(\frac{C_t}{C_{t+1}} \right)^{\eta} \left(\rho \bar{Z} K_t^{\rho-1} + (1-\delta) \right) \end{aligned}$$

As above, solve the first equation for K_t and use the result to replace K_t in the second equation[11], yielding with slight rewriting

$$\begin{aligned} K_t - K_{t-1} &= \bar{Z} K_{t-1}^{\rho} - \delta K_{t-1} - C_t \\ \frac{C_{t+1}}{C_t} &= \left(\beta \left(\rho \bar{Z} \left(\bar{Z} K_{t-1}^{\rho} + (1-\delta) K_{t-1} - C_t \right)^{\rho-1} + (1-\delta) \right) \right)^{1/\eta} \end{aligned}$$

Again, one obtains two steady state relationships for $K_{t-1} = K_t = K$ and $C_{t+1} = C_t = C$:

$$\begin{aligned} C &= \bar{Z} K^{\rho} - \delta K \\ C &= \bar{Z} K^{\rho} + (1-\delta) K - \left(\frac{\rho \bar{Z}}{(1/\beta) - 1 + \delta} \right)^{1/(1-\rho)} \end{aligned}$$

These two relationships can be plotted into the (K_{t-1}, C_t)-plane, dissecting that plane into four quadrants, see figure 3. The analysis proceeds exactly as above. As stable arm, we have used the relationship $C_t = \bar{C} \exp(\nu_{ck} \log(K_{t-1}/\bar{K}))$, which according to our log-linear analysis is approximately correct.

The state-space approach is certainly useful for gaining insights into small systems such as the neoclassical growth model we have studied here. However, for larger models, it becomes impractical very quickly.

4.7 The relationship to second-order difference equations.

In this subsection, we will discuss the popular reduction to a *second-order difference equation*. Further discussion can also be found in subsection 6.4. As in the previous subsection, we ignore the stochastic term z_t for the purpose of the discussion here by setting it identical to zero. The four log-linearized equations characterizing the dynamics can be reduced to a single second-order equation in k_t. One way of seeing

[11] Again, this manipulation is not necessary in a continuous-time framework.

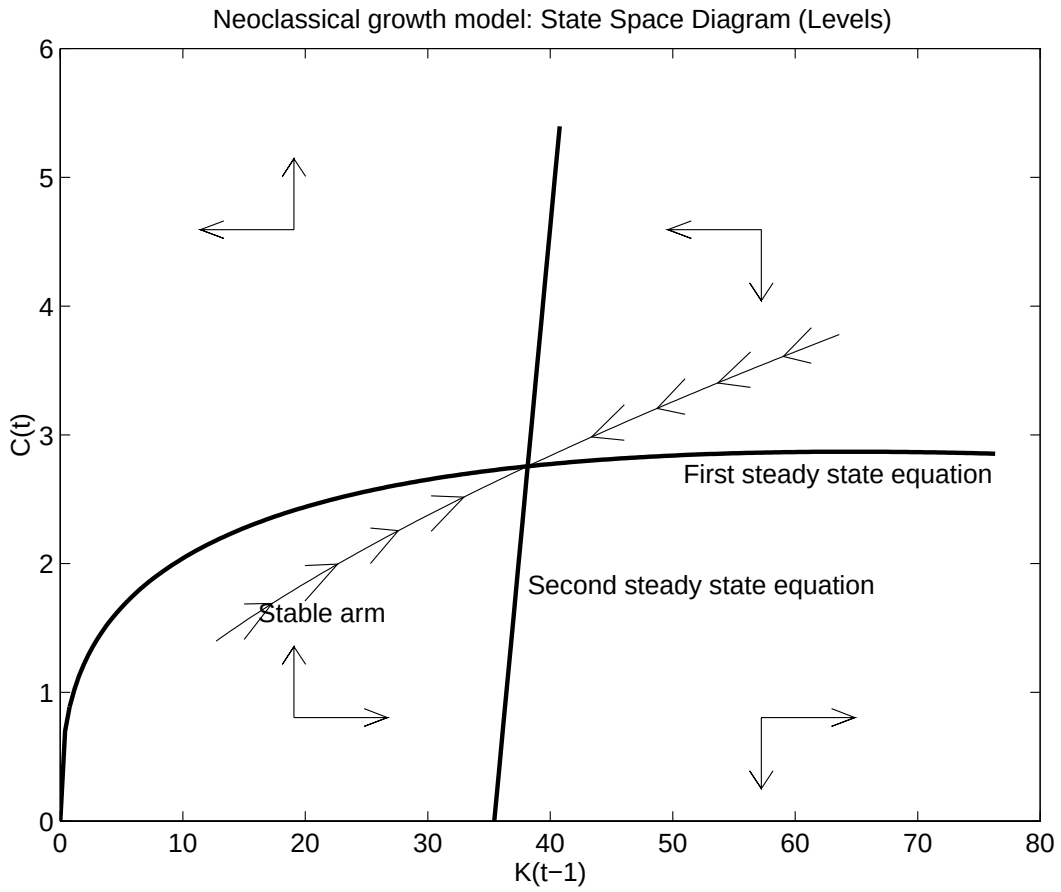

Figure 3: *This figure shows the state space diagram for the neoclassical growth model in its original, nonlinear form. The two steady state equations cut the plane into four quadrants, which differ qualitatively in their dynamics as indicated by the arrows at right angles.*

this is to use equation (4.16) from the previous subsection to eliminate c_t and c_{t+1} in equation (4.17). The result is the second-order difference equation

$$0 = k_{t+1} - \gamma k_t + \frac{1}{\beta} k_{t-1} \tag{4.22}$$

with γ given in equation (4.15). To solve this second-order difference equations generally, define the *characteristic polynomial*

$$0 = \nu^2 - \gamma\nu + \frac{1}{\beta}.$$

see e.g. Sargent, 1987. The two solutions to this equation are given by

$$\nu_{1,2} = \frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}}$$

We then have the following well-known proposition.

Proposition 1 *If $\nu_1 \neq \nu_2$, then the general solution to (4.22) is the two-dimensional space, given by*

$$k_t = a\nu_1^t + b\nu_2^t \tag{4.23}$$

for arbitrary constants a and b.

Proof: *Suppose, k_t is given by (4.23). Substituting it into (4.22) yields*

$$\begin{aligned} & k_{t+1} - \gamma k_t + \frac{1}{\beta} k_{t-1} \\ = \; & a\nu_1^{t-1}\left(\nu_1^2 - \gamma\nu_1 + \frac{1}{\beta}\right) \\ & + b\nu_2^{t-1}\left(\nu_2^2 - \gamma\nu_2 + \frac{1}{\beta}\right) \\ = \; & 0 \end{aligned}$$

as desired. Conversely, let any solution to (4.22) be given. Note, that it is enough to just know k_0 and k_1, say, since all other k_t can then be calculated recursively from (4.22). Find a and b such that

$$\begin{aligned} k_0 &= a + b \\ k_1 &= a\nu_1 + b\nu_2 \end{aligned}$$

There is a unique solution, since $\nu_1 \neq \nu_2$. Then, the given solution to (4.22) must coincide with (4.23) for these values of a and b. •

Since the general solution to equation (4.23) is a two-dimensional space, one needs two constraints to pin down a unique solution. One constraint is the initial value for capital k_{-1} (or k_0, if one starts time in the neoclassical growth model at $t = 1$). The second constraint is the stability condition, that

$$0 = \lim_{t \to \infty} k_t$$

This constraint helps, if exactly one of the roots, ν_1, say, is stable: in that case, we must have $b = 0$ in (4.23). Furthermore, we now have the *recursive equilibrium law of motion*

$$k_t = \nu_1 k_{t-1}. \tag{4.24}$$

In other words, for second-order difference equations with exactly one stable root, the method of undetermined coefficients finds the stable solution with $\nu_1 = \nu_{kk}$.

Note, that the stability condition does not help, if both roots are stable. In that case, one still has a one-dimensional space of general solutions. Such systems can give rise to sunspot dynamics, see Farmer and Guo (1994) for further discussion. One then has to be careful with the interpretation of the results of the method of undetermined coefficients, since that method, given one endogenous state variable, imposes the restriction on the solution of the system to be of form (4.24), which is no longer valid. A remedy is to enlarge the state space to include k_{t-1} and k_{t-2}: the method of undetermined coefficients then correctly searches for a recursive equilibrium law of motion of the type

$$k_t = \nu_{kk1} k_{t-1} + \nu_{kk2} k_{t-2}$$

with $\nu_{kk1} = \gamma$ and $\nu_{kk2} = -1/\beta$ as a stable and simple-to-find solution. More generally, enlarging the state space leads to more complicated matrix algebra, which is dealt with in section 6. The point here is to keep in mind, that one should be very careful, if one finds "too many" (or, likewise, "too few") stable roots, when applying the method of undetermined coefficients.

4.8 A quick review.

It may be useful at this point to step back and to provide a quick review:

1. We have found the necessary conditions.

2. We log-linearized these conditions and the constraints. E.g. we got

$$0 = E_t \left[\eta (c_t - c_{t+1}) + r_{t+1} \right]$$

3. We postulated a linear law of motion, e.g.

$$k_t = \nu_{kk} k_{t-1} + \nu_{kz} z_t$$

   and solved for the "undetermined coefficients" ν_{kk}, ν_{kz} etc..

4. It all boiled down to solving a quadratic equation for the coefficient ν_{kk}, given by

$$0 = \nu_{kk}^2 - \gamma \nu_{kk} + \frac{1}{\beta}$$

   where γ is given in equation (4.15).

5. The resulting equations could then be used to analyze the model by e.g. calculating the coefficient ν_{kk} for particular parameter choices, doing sensitivity analysis with respect to these choices, analyzing the speed of convergence back to the steady state, simulating the model or looking at impulse response functions.

6. We have compared the method of undetermined coefficients to a state space approach as well as to solving second order difference equations.

In looking back, we can also see that finding the necessary conditions, finding the steady state, as well as log-linearizing these conditions and the constraints was comparatively easy. Painful, however, was to have to solve for ν_{kk} and the other coefficients. For larger models or, worse, for models with multiple endogenous state variables, solving for everything by hand looks quite unattractive.

However, this pain can be avoided by applying directly the theorems in section 6. The easiest way to apply these theorems is to obtain MATLAB routines applying them. They are described in appendix A and are available together with some documentation and examples at the following web site:
http://cwis.kub.nl/~few5/center/STAFF/uhlig/toolkit.dir/toolkit.htm.

5 An example: Hansens real business cycle model.

The next example is Hansens (1985) real business cycle model. It is explained there in detail. Here, the mathematical description shall suffice. The main point of this example is to explain how to perform the first three steps of the general procedure as stated in section 3. In many ways, the model here is just an extension of the stochastic neoclassical growth model of section 4 above: the main difference is to endogenize the labor supply. In fact, it is possible to also solve through that model by hand just as was done above for the stochastic neoclassical growth model. However, here, we want to go through the analysis of this model rather quickly to show how to get to the log-linearized version of the model ready for the analysis with the theorems of section 6 and the MATLAB programs mentioned there.

The social planner solves the problem of the representative agent

$$\max E \sum_{t=1}^{\infty} \beta^t \left(\frac{C_t^{1-\eta} - 1}{1-\eta} - AN_t \right)$$

s.t.

$$\begin{aligned} C_t + I_t &= Y_t \\ K_t &= I_t + (1-\delta)K_{t-1} \\ Y_t &= Z_t K_{t-1}^{\rho} N_t^{1-\rho} \\ \log Z_t &= (1-\psi)\log \bar{Z} + \psi \log Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.d.\mathcal{N}(0;\sigma^2), \end{aligned} \tag{5.1}$$

where C_t is consumption, N_t is labor,I_t is investment, Y_t is production, K_t is capital Z_t

is the total factor productivity and $A, \beta, \eta, \delta, \rho, \bar{Z}, \psi$ and σ^2 are parameters. Hansen only considered the case $\eta = 1$, so that the objective function is

$$E \sum_{t=0}^{\infty} \beta^t (\log C_t - A N_t)$$

As in Campbell (1994), there is no difficulty in considering arbitrary η, since no growth trend is assumed.

The first order conditions are

$$\begin{aligned} A &= C_t^{-\eta}(1-\rho)\frac{Y_t}{N_t} \\ 1 &= \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right], \qquad (5.2) \\ R_t &= \rho \frac{Y_t}{K_{t-1}} + 1 - \delta. \qquad (5.3) \end{aligned}$$

Equation (5.2) is the Lucas asset pricing equations, see Lucas (1978), which typically arises in these models.

In contrast to some of the real business cycle literature and to avoid confusion in the application of the method in section 6, it is very useful to stick to the following *dating convention*. A new date starts with the arrival of new information. If a variable is chosen and/or (eventually) known at date t, it will be indexed with t. Use only variables dated t and $t-1$ in deterministic equations and variables dated $t+1$, t and $t-1$ in equations involving expectations $E_t[\cdot]$.

The steady state for the real business cycle model above is obtained by dropping the time subscripts and stochastic shocks in the equations above, characterizing the equilibrium. Formally, this amounts to finding steady state values such that $f(0,0) = 1$ and $g(0,0) = 1$ in the notation of the previous section[12]. For example, equations (5.2) and (5.3) result in

$$\begin{aligned} 1 &= \beta \bar{R} \\ \bar{R} &= \rho \frac{\bar{Y}}{\bar{K}} + 1 - \delta, \end{aligned}$$

where bars over variables denote steady state values. One needs to decide what one wants to solve for. If one fixes β and δ, these two equations will imply values for $\bar{R}$ and $\bar{Y}/\bar{K}$. Conversely, one can fix $\bar{R}$ and $\bar{Y}/\bar{K}$ and then these two equations yield values for β and δ. The latter procedure maps observable characteristics of the economy into "deep parameters," and is the essence of calibration, see Kydland and Prescott (1991).

[12] Alternatively, find the steady state so that (3.3) is satisfied. This is, however, rarely done.

Introduce small letters to denote log-deviations, i.e. write

$$C_t = \bar{C}e^{c_t}$$

for example. The resource constraint (5.1) then reads

$$\bar{C}e^{c_t} + \bar{I}e^{i_t} = \bar{Y}e^{y_t}$$

This can be written approximately as

$$\bar{C}(1 + c_t) + \bar{I}(1 + i_t) = \bar{Y}(1 + y_t)$$

Since $\bar{C} + \bar{I} = \bar{Y}$ due to the definition of the steady state, the constant terms drop out[13] and one obtains

$$\bar{C}c_t + \bar{I}i_t = \bar{Y}y_t \tag{5.4}$$

The resource constraint is now stated in terms of percentage deviations: the steady state levels in this equation rescale the percentage deviations to make them comparable. Note that no explicit differentiation is required to obtain the log-linearized version of the resource constraint: log-linearization is obtained just by using the building blocks described in the previous section.

Similarly log-linearizating the other equations yields

$$\begin{aligned}
\bar{K}k_t &= \bar{I}i_t + (1-\delta)\bar{K}k_{t-1} \\
y_t &= z_t + \rho k_{t-1} + (1-\rho)n_t \\
z_t &= \psi z_{t-1} + \epsilon_t \\
0 &= -\eta c_t + y_t - n_t \\
0 &= E_t[\eta(c_t - c_{t+1}) + r_{t+1}] \\
\bar{R}r_t &= \rho\frac{\bar{Y}}{\bar{K}}(y_t - k_{t-1}).
\end{aligned}$$

To find the state variables, one needs to find all (linear combinations of) variables dated $t-1$ in these equations: the endogenous state variable is capital, k_{t-1} whereas the exogenous state variable is the technology parameter z_{t-1}. Note that there are as many expectational equations as there are endogenous state variables. The coefficients of the equations above need to be collected in the appropriate matrices to restate these equations in the form required for section 6: this is a straightforward exercise.

[13] Another way to see that constants can in the end be dropped is to note that the steady state is characterized by $c_t = k_t = y_t = k_{t-1} = 0$. If one replaces all log-deviations with zero, only the constant terms remain, and that equation can be subtracted from the equation for general c_t, k_t, y_t and k_{t-1} above.

6 Solving recursive stochastic linear systems with the method of undetermined coefficients

This section describes how to find the solution to the recursive equilibrium law of motion in general, using the method of undetermined coefficients. MATLAB programs performing the calculations in this section are available at my home page[14]. The idea is to write all variables as linear functions (the "recursive equilibrium law of motion") of a vector of endogenous variables x_{t-1} and exogenous variables z_t, which are given at date t, i.e. which cannot be changed at date t. These variables are often called state variables or predetermined variables. In the real business cycle example of section 5, these are at least k_{t-1} and z_t, since they are clearly unchangeable as of date t and, furthermore, show up in the linearized equations system. In principle, any endogenous variable dated $t-1$ or earlier could be considered a state variable. Thus, in subsection 6.1 below, we use "brute force" and simply declare all endogenous variables to be state variables, whereas in subsection 6.2, we try to be a bit more sensitive and exploit more of the available structure. The latter is typically done in practice, see e.g. Campbell (1994). Both subsections will characterize the solution with a matrix quadratic equation, see also Ceria and Rios-Rull (1992) and Binder and Pesaran (1996). Subsection 6.3 shows, how to solve that equation. For models with just one endogenous state variable, such as the real business cycle model of section 5 when analyzed with the more structured approach in subsection 6.2 below, the matrix quadratic equation is simply a quadratic equation in a real number. In that case, the solution to the quadratic equation is obviously known from high-school algebra: it is contained as a special case of the general solution in section 6.3. In subsection 6.4 we discuss our solution method, and compare it in particular to the Blanchard-Kahn (1980) approach.

6.1 With brute force...

As a first cut, and with somewhat brute force, one may simply use all variables without distinction as a vector of endogenous state variables[15] x_{t-1} of size $m \times 1$ or as a vector of exogenous stochastic processes z_t of size $k \times 1$. It is assumed that the log-linearized equilibrium relationships can be written in the following form

$$0 \quad = \quad E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t] \tag{6.1}$$

[14] http://cwis.kub.nl/~few5/center/STAFF/uhlig/toolkit.dir/toolkit.htm is the address of the web site for the programs.

[15] To make this work really generally, one should actually not only include all the variables dated $t-1$ but also all the variables dated $t-2$ as part of the state vector x_{t-1}. More is even required, if the equations already contain further lags of endogenous variables, see also the next footnote. Usually, however, this isn't necessary.

$$z_{t+1} \quad = \quad Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0, \tag{6.2}$$

where F, G, H, L and M and matrices, collecting the coefficients. It is assumed that N has only stable eigenvalues. The real business cycle example above can be easily written in this form. For example, the resource constraint (5.4) would be

$$0 = E_t[\bar{C}c_t + \bar{I}i_t - \bar{Y}y_t]$$

since c_t, i_t and y_t are already known at date t and hence nothing changes when one takes their expectations given all information up to date t. Note that $F = L = 0$ for this equation. Of course, there are other equations in the real business cycle model, and one of them involves nonzero entries for F and L.

What one is looking for is the recursive equilibrium law of motion

$$x_t = Px_{t-1} + Qz_t \tag{6.3}$$

i.e. matrices P and Q , so that the equilibrium described by these rules is stable. The solution is characterized in the following theorem, see also Binder and Pesaran (1996). The characterization involves a matrix quadratic equation, see equation (6.4). Subsection 6.3 discusses, how it can be solved. For the purpose of that section, let m be the length of the vector x_t, and let $l = n = 0$.

Theorem 1 *If there is a recursive equilibrium law of motion solving equations (6.1), and (6.2), then the following must be true.*

1. *P satisfies the (matrix) quadratic equation*

$$0 = FP^2 + GP + H \tag{6.4}$$

   The equilibrium described by the recursive equilibrium law of motion (6.3) and (6.2) is stable iff all eigenvalues of P are smaller than unity in absolute value.

2. *Given P, let V denote the matrix*

$$V = N' \otimes F + I_k \otimes (FP + G),$$

   Then,

$$VQ = -vec(LN + M) \tag{6.5}$$

   where $vec(\cdot)$ denotes columnwise vectorization.

Obviously, if the matrix V in this theorem is invertible, then multiplication of equation (6.5) with V^{-1} yields the unique solution for Q. **Proof:** *Plugging the recursive*

equilibrium law of motion (6.3) into equation (6.1) twice and using (6.2) to calculate the expectations yields

$$\begin{aligned} 0 &= ((FP+G)P+H)x_{t-1} + \\ &\quad ((FQ+L)N+(FP+G)Q+M)z_t \end{aligned} \tag{6.6}$$

The coefficient matrices on x_{t-1} and z_t need to be zero. Equating the coefficient on x_{t-1} to zero yields equation (6.4) for P. Taking the columnwise vectorization of the coefficient matrices of z_t in this equation and collecting terms in vec(Q) yields the equation (6.5) for Q. •

6.2 ... or with sensitivity.

We now exploit more of the structure in the linearized model. Analyzing the equations of the real business cycle example of section 5, one sees that the only endogenous variable dated $t-1$ which shows up in any of the equations is capital, k_{t-1}. It is thus a reasonably guess to treat k_{t-1} as the only endogenous state variable together with the exogenous state variable z_t. This principle is general: in the vast majority of cases, this is how one can identify the vector of state variables[16]. In practice, one often sees researchers exploiting some of the equilibrium equations to "get rid" of some variables, and have only a few variables remaining. For the real business cycle example of section 5, it is actually possible to reduce everything to a single equation for the endogenous variables, containing only k_{t+1}, k_t and k_{t-1}. Often, one sees reductions to a system involving two equations in two endogenous variables such as c_t and k_{t-1}, see e.g. Campbell (1994), presumably because this allows thinking in terms of a state space diagram, see e.g. Blanchard and Fisher (1989), chapter 2. The analysis below follows this often-used procedure. However, there is no reason to go through the hassle of "eliminating" variables by hand, using some of the equations: since this is all just simple linear algebra applied to a system of equations, it is far

[16]There are exceptions. In richer models, the state variables need to include variables chosen at a date earlier than $t-1$ as well because these lagged variables appear in the equations. One can recast this into the desired format as follows. The list of state variables might consist out of lagged values of the capital stock, k_{t-1} and k_{t-2}. This can and should be rewritten as $k_{1,t-1}$ and $k_{2,t-1}$ with $k_{1,t-1}$ replacing k_{t-1} and where the additional equation $k_{2,t} = k_{1,t-1}$ needs to be added to the system. With that notation, $k_{2,t}$ is "chosen" at date t, satisfying the "dating convention" stated in section 5. One may also need to add additional variables like e.g. c_{t-1} or k_{t-2} as state variables, even though they don't show up in the equations with these dates, when the model exhibits sun spot dynamics. This can be done in the same manner, but one needs to be careful with interpreting the results. The reader is advised to read Farmer and Guo (1994) for an example as well for the appropriate interpretation for such a case.

easier to leave all the equations in, and leave it to the formulas to sort it all out. That is what is done below.

We thus make the following assumptions[17]. There is an endogenous state vector x_t, size $m \times 1$, a list of other endogenous variables ("jump variables") y_t, size $n \times 1$, and a list of exogenous stochastic processes z_t, size $k \times 1$. The equilibrium relationships between these variables are

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t \tag{6.7}$$

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \tag{6.8}$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0, \tag{6.9}$$

where it is assumed that C is of size $l \times n$, $l \geq n$ and[18] of rank n, that F is of size $(m + n - l) \times n$, and that N has only stable eigenvalues. Note, that one could have written all equations (6.7) in the form of equation (6.8) with the corresponding entries in the matrices F, J and L set to zero. Essentially, that is what is done in subsection 6.1. Instead, the point here is to somehow exploit the structure inherent in equations of the form (6.7), which do not involve taking expectations.

What one is looking for is the recursive equilibrium law of motion

$$x_t = Px_{t-1} + Qz_t \tag{6.10}$$

$$y_t = Rx_{t-1} + Sz_t, \tag{6.11}$$

i.e. matrices P, Q, R and S, so that the equilibrium described by these rules is stable. The solution is characterized in the next theorem. To calculate the solution, one needs to solve a matrix quadratic equation: how this is done, is explained in subsection 6.3.

The important special case $l = n$ is treated in corrolary 1. The special case $l = n = 0$ was the topic of subsection 6.1 .

Theorem 2 *If there is a recursive equilibrium law of motion solving equations (6.7), (6.8) and (6.9), then the coefficient matrices can be found as follows. Let C^+ be the pseudo-inverse[19] of C. Let C^0 be an $(l - n) \times l$ matrix, whose rows form a basis of the null space[20] of C'.*

[17] Note that the notation differs from the notation in section 3

[18] The case $l < n$ can be treated as well: the easiest approach is to simply "redeclare" some other endogenous variables to be state variables instead, i.e. to raise m and thus lower n, until $l = n$.

[19] The pseudo-inverse of the matrix C is the $n \times l$ matrix C^+ satisfying $C^+CC^+ = C^+$ and $CC^+C = C$. Since it is assumed that $\text{rank}(C) \geq n$, one gets $C^+ = (C'C)^{-1}C'$, see Strang (1980), p. 138. The MATLAB command to compute the pseudo-inverse is pinv(C).

[20] C^0 can be found via the singular value decomposition of C', see Strang (1980), p. 142. The MATLAB command for computing C^0 is (null(C'))'.

1. *P satisfies the (matrix) quadratic equations*

$$\begin{aligned} 0 = & \quad C^0AP + C^0B & (6.12)\\ 0 = & \quad (F - JC^+A)P^2 \quad -(JC^+B - G + KC^+A)P - KC^+B + H & (6.13)\end{aligned}$$

The equilibrium described by the recursive equilibrium law of motion (6.10), (6.11) and by (6.9) is stable iff all eigenvalues of P are smaller than unity in absolute value.

2. *R is given by*

$$R = -C^+(AP + B)$$

3. *Given P and R, let V be the matrix*

$$V = \begin{bmatrix} I_k \otimes A, & I_k \otimes C \\ N' \otimes F + I_k \otimes (FP + JR + G), & N' \otimes J + I_k \otimes K \end{bmatrix}.$$

where I_k is the identity matrix of size $k \times k$. Then

$$V \begin{bmatrix} vec(Q) \\ vec(S) \end{bmatrix} = - \begin{bmatrix} vec(D) \\ vec(LN + M) \end{bmatrix}, \tag{6.14}$$

where $vec(\cdot)$ denotes columnwise vectorization.

Obviously, if the matrix V in this theorem is invertible, then multiplication of equation (6.14) with V^{-1} yields the unique solution for Q.

Proof: *Plug the recursive equilibrium law of motion into equation (6.7). This yields*

$$(AP + CR + B)x_{t-1} + (AQ + CS + D)z_t = 0, \tag{6.15}$$

which has to hold for arbitrary x_{t-1} and z_t. Thus, the coefficient matrices on x_{t-1} and z_t in (6.15) are zero. Plugging the recursive equilibrium law of motion into equation (6.8) twice and using (6.9) yields

$$\begin{aligned} 0 \quad = \quad & ((FP + JR + G)P + KR + H)x_{t-1} + & (6.16)\\ & ((FQ + JS + L)N + (FP + JR + G)Q + KS + M)z_t \end{aligned}$$

Again, the coefficient matrices on x_{t-1} and z_t need to be zero. Taking the columnwise vectorization of the coefficient matrices of z_t in equations (6.15) and (6.16) and collecting terms in $vec(Q)$ and $vec(S)$ yields the formula for Q and S.

To find P and thus R, rewrite the coefficient matrix on x_{t-1} in equation (6.15) as

$$\begin{aligned} R \quad = \quad & -C^+(AP + B) & (6.17)\\ 0 \quad = \quad & C^0AP + C^0B, \end{aligned}$$

noting that the matrix $[(C^+)', (C^0)']$ *is nonsingular and that* $C^0C = 0$, *see Strang (1980), p. 88. Use (6.17) to replace* R *in the coefficient matrix on* x_{t-1} *in (6.16), yielding (6.13). Note finally that the stability of the equilibrium is determined by the stability of* P, *since* N *has stable roots by assumption.* •

Corollary 1 *Suppose that* $l = n$, *i.e. that there are as many expectational equations as there are endogenous state variables. If there is a recursive equilibrium law of motion solving equations (6.7), (6.8) and (6.9), then their coefficient matrices can be found as follows.*

1. *P satisfies the (matrix) quadratic equation*

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0. \qquad (6.18)$$

   The equilibrium described by the recursive equilibrium law of motion (6.10), (6.11) and by (6.9) is stable iff all eigenvalues of P *are smaller than unity in absolute value.*

2. *R is given by*

$$R = -C^{-1}(AP + B)$$

3. *Q satisfies*

$$(N' \otimes (F - JC^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A))vec(Q) = vec((JC^{-1}D - L)N + KC^{-1}D - M), \qquad (6.19)$$

   where I_k *is the identity matrix of size* $k \times k$, *provided the matrix which needs to be inverted in this formula is indeed invertible.*

4. *S is given by*

$$S = -C^{-1}(AQ + D)$$

Proof: *This corollary can be obtained directly by inspecting the formulas of theorem 2 above for the special case* $l = n$. *In particular,* C^+ *is just the inverse of* C. *Alternatively, a direct proof can be obtained directly by following the same proof strategy as above: there is no need to repeat it.* •

The formulas in these theorems become simpler yet, if $m = 1$ or $k = 1$. If $m = 1$, there is just one endogenous state variable and the matrix quadratic equation

above becomes a quadratic equation in the real number P, which can be solved using high-school algebra: this is the case for the real business cycle model and thus the case which Campbell (1994) analyzes. If $k = 1$, there is just one exogenous state variables, in which case the Kronecker product (i.e. "$\otimes$") in the formulas above becomes multiplication, and in which case $\text{vec}(Q) = Q$ and $\text{vec}(S) = S$, since Q and S are already vectors rather than matrices.

6.3 Solving the matrix quadratic equation.

To generally solve the matrix quadratic equations (6.4) or (6.12), (6.13) for P, write them generally as

$$\Psi P^2 - \Gamma P - \Theta = 0. \tag{6.20}$$

For equations (6.12) and (6.13), define

$$\begin{aligned}
\Psi &= \begin{bmatrix} 0_{l-n,m} \\ F - JC^+A \end{bmatrix} \\
\Gamma &= \begin{bmatrix} C^0A \\ JC^+B - G + KC^+A \end{bmatrix} \\
\Theta &= \begin{bmatrix} C^0B \\ KC^+B - H, \end{bmatrix}
\end{aligned}$$

where $0_{l-n,m}$ is a $(l-n) \times m$ matrix with only zero entries. In the special case $l = n$, the formulas for Ψ, Γ and Θ become slightly simpler:

$$\begin{aligned}
\Psi &= F - JC^{-1}A \\
\Gamma &= JC^{-1}B - G + KC^{-1}A \\
\Theta &= KC^{-1}B - H
\end{aligned}$$

For equation (6.4), simply use $\Psi = F$, $\Gamma = -G$ and $\Theta = -H$.

Equation (6.20) can now be solved by turning it into a generalized eigenvalue and eigenvector problem[21], for which most mathematical packages have preprogrammed routines[22]. Recall, that a generalized eigenvalue λ and eigenvector s of a matrix Ξ with respect to a matrix Δ are defined to be a vector and a value satisfying

$$\lambda \Delta s = \Xi s \tag{6.21}$$

[21] An earlier version of the chapter proposed to study an altered version of these equations by postmultiplying equation (6.12) with P. This altered equation together with (6.13) can then often be reduced to a standard rather than a generalized eigenvalue problem, but had the drawback of introducing spurious zero roots. The version presented here does not involve this alteration, and thus does not introduce spurious zero roots. This update is due to Andy Atkeson (1997), and I am very grateful to him for pointing it out to me. Any errors here are mine, of course.

[22] The Matlab command for finding the generalized eigenvalues and eigenvectors is eig(Ξ,Δ).

A standard eigenvalue problem is obtained, if Δ is the identity matrix. More generally, the generalized eigenvector problem can be reduced to a standard one, if Δ is invertible, by calculating standard eigenvalues and eigenvectors for $\Delta^{-1}\Xi$ instead.

Theorem 3 *To solve the quadratic matrix equation*

$$\Psi P^2 - \Gamma P - \Theta = 0, \tag{6.22}$$

for the $m \times m$ matrix P, given $m \times m$ matrices Γ and Θ, define the $2m \times 2m$ matrices Ξ and Δ via

$$\Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_{m,m} \end{bmatrix},$$

and

$$\Delta = \begin{bmatrix} \Psi & 0_{m,m} \\ 0_{m,m} & I_m \end{bmatrix},$$

where I_m is the identity matrix of size m, and where $0_{m,m}$ is the $m \times m$ matrix with only zero entries.

1. *If s is a generalized eigenvector and λ the corresponding generalized eigenvalue of Ξ with respect to Δ, then s can be written as $s' = [\lambda x', x']$ for some $x \in \mathbf{R}^m$.*

2. *If there are m generalized eigenvalues $\lambda, \ldots, \lambda_m$ together with generalized eigenvectors $s_1, \ldots, s_m$ of Ξ with respect to Δ, written as $s_i' = [\lambda_i x_i', x_i']$ for some $x_i \in \mathbf{R}^m$, and if $(x_1, \ldots, x_m)$ is linearly independent, then*

$$P = \Omega \Lambda \Omega^{-1}$$

is a solution to the matrix quadratic equation (6.22), where $\Omega = [x_1, \ldots, x_m]$ and $\Lambda = diag(\lambda, \ldots, \lambda_m)$. The solution P is stable if $\mid \lambda_i \mid < 1$ for all $i = 1, \ldots, m$.

Conversely, any diagonalizable solution P to (6.22) can be written in this way.

3. *If $m = 1$, then the solutions P to equation (6.22) are given by*

$$P_{1,2} = \frac{1}{2\Psi}(\Gamma \pm \sqrt{\Gamma^2 + 4\Psi\Theta}).$$

if $\Psi \neq 0$ and

$$P = -\frac{\Theta}{\Gamma}$$

if $\Psi = 0$ and $\Gamma \neq 0$.

Proof: *First, examine the last m rows of equation (6.21) to see that any eigenvector s for some eigenvalue λ of the matrix Ξ with respect to Δ can indeed be written as*

$$s = \begin{bmatrix} \lambda x \\ x \end{bmatrix}$$

for some $x \in \mathbf{R}^m$ because of the special form of Ξ and Δ. Examining the first m rows of equation (6.21) then shows that

$$\lambda^2 \Psi x - \lambda \Gamma x - \Theta x = 0 \tag{6.23}$$

It follows that

$$\Psi \Omega \Lambda^2 - \Gamma \Omega \Lambda - \Theta \Omega = 0$$

and hence

$$\Psi P^2 - \Gamma P - \Theta = 0$$

as claimed, after multiplying with Ω^{-1} from the right.

Reversing the steps shows that any diagonalizable solution P to (6.22) can be written in this way. •

Some additional properties of a solution P to (6.20) are stated in the following theorem[23]:

Theorem 4 *1. The eigenvalues λ of Ξ are the solutions to the equation*

$$det\left(\lambda^2 \Psi - \lambda \Gamma - \Theta\right) = 0.$$

The lower half x of the eigenvector s to λ satisfies

$$\left(\lambda^2 \Psi - \lambda \Gamma - \Theta\right) x = 0$$

2. If Ψ is invertible and if P is a real-valued solution to the matrix-quadratic equation (6.18), then

$$tr(4\Psi^{-1}\Theta + (\Psi^{-1}\Gamma)^2) \geq 0.$$

Proof: *The claim about λ follows from*

$$det\left(\begin{bmatrix} (\Gamma - \lambda\Psi) & \Theta \\ I_m & -\lambda I_m \end{bmatrix}\right) = det\left(-\lambda(\Gamma - \lambda\Psi) - \Theta\right),$$

[23] I am grateful to Jan Magnus for pointing these out to me. Furthermore, Ceria and Rios-Rull, 1992, point to additional literature on this subject, which found and concentrated on part 1 of theorem 4, but did not study the more useful theorem 3.

which follows from inspecting the formula for the determinant. The claim about the eigenvector piece x is just (6.23). For the last claim, calculate that

$$0 = tr(P^2 - \Psi^{-1}\Gamma P - \Psi^{-1}\Theta) = tr((P - \frac{1}{2}\Psi^{-1}\Gamma)^2 - (\Psi^{-1}\Theta + \frac{1}{4}(\Psi^{-1}\Gamma)^2)).$$

The conclusion follows since $tr((P - \frac{1}{2}\Psi^{-1}\Gamma)^2) \geq 0$. •

6.4 Discussion.

Theorem 3 links the approach used here to Blanchard and Kahn (1980), which is the key reference for solving linear difference equations. Consider solving the second order difference equation

$$\Psi x_{t+1} - \Gamma x_t - \Theta x_{t-1} = 0. \tag{6.24}$$

The approach in Blanchard and Kahn (1980) amounts to finding the stable roots of Ξ by instead analyzing the dynamics of the "stacked" system $s_t' = [x_t', x_{t-1}']$,

$$\Delta s_{t+1} = \Xi s_t,$$

i.e. by reducing (6.24) to a first-order difference equation. The approach here solves for the matrix P in the recursive equilibrium law of motion $x_{t+1} = Px_t$. Theorem 3 above states that both approaches amount to the same problem. The advantage of the method here is that it is easily applied to the entire system (6.7), (6.8) and (6.9), reducing it to (6.24) eventually, while finding the stable roots in the entire system given by these equations and at the same time taking care of the expectation operators, using the Blanchard-Kahn (1980) procedure, is often perceived as complicated. Fundamentally, there is no difference.

To apply theorem 3, one needs to select m out of $2m$ possible eigenvalues. Note that P has only nonzero eigenvalues if the state space was chosen to be of minimal size: thus attention can be restricted to the roots $\mid \lambda_i \mid > 0$ in that case. In general, there may be quite a bit of choice left. In practice, however, there will often be exactly m stable eigenvalues remaining so that the stable solution is unique[24].

[24] Another approach to select a unique solution is in McCallum (1983), who suggests to use those roots that can be obtained continuously from the zero roots of the equation $\Psi P^2 - \Gamma P - \alpha\Theta$ for $\alpha = 0$, as α changes from 0 to 1. However, not only is following these roots as functions of α computationally very demanding, it is also the case that uniqueness gets lost once two or more such paths cross each other. If these paths do not cross in a particular application, and if additionally all roots for all α are positive real numbers, say, then the McCallum proposal simply amounts to using the roots of minimal value. The MATLAB programs supplied by the author use the roots of minimal absolute value subject to eliminating spurious zero roots and tries to use complex roots in conjugate pairs, as described below.

For a one-dimensional vector of endogenous state variables, this condition is called saddle-point stability. The literature on solving linear rational expectations equilibria typically assumes this condition to hold or shows it to hold in social planning problems under reasonable conditions, see Blanchard and Kahn (1980), Kollintzas (1985) and Hansen, McGrattan and Sargent (1994). If there are fewer stable eigenvalues than endogenous state variables, the equilibrium might be inherently unstable. The method above then still permits calculation of an equilibrium which satisfies the nonlinear equilibrium conditions at least locally. In particular, in models involving more than one agent or sectors or countries, one may find as many unit roots as there are more agents (sectors, countries) than one since shocks may affect the relative wealth (capital) of any two agents (sectors, countries) and thus may result in permanent changes in their consumption paths (or capital stocks): in these cases, the method above allowing for unit roots still gives useful results, which obviously should then be used with some care. These unit roots typically already show up as an indetermined steady state: any of the possible steady states can then serve as a starting point for the dynamic calculation, keeping in mind that a simulation based on the dynamics calculated here will eventually wander away too far to be numerically useful. If there are more stable eigenvalues than endogenous state variables, enlarging the number of endogenous state variables by including further lagged values might help. Nonetheless, the presence of an excess of stable roots then may point to the existence of sunspots or endogenous fluctuations, see e.g. Farmer and Guo (1994).

If not all eigenvalues of Ξ are distinct, P in turn might have repeated eigenvalues. Since the eigenspace for a repeated eigenvalue is (usually) multidimensional, there will be infinitely many choices for the eigenvectors and hence infinitely many choices for P in that case. Note, for example, that for any given λ and any three real numbers a, b, c satisfying $a^2 + bc = \lambda^2$, all matrices

$$P = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

solve

$$P^2 - \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix} = 0.$$

These cases are rare in practice, since Ξ is diagonalizable with distinct eigenvalues generically in the coefficients of the system (6.7), (6.8) and (6.9).

More disconcerting is the possibility that some of the roots may be complex rather than real. Consider, for example, $\Psi = I_2$, $\Gamma = -I_2$ and

$$\Theta = \begin{bmatrix} 0.23 & 0.64 \\ -0.64 & 0.23 \end{bmatrix}.$$

Using the theorem above, one obtains exactly two stable roots, which happen to be complex, $\lambda_{1,2} = 0.3 \pm 0.4i = 0.5e^{\pm\alpha i}$, where where $\alpha \approx 0.9273$. Their associated eigenvectors are complex, too. Calculating P results in a matrix with only real entries, however, given by

$$P = \begin{bmatrix} 0.3 & 0.4 \\ -0.4 & 0.3 \end{bmatrix} = 0.5 \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Since Ξ is a real-valued matrix, complex eigenvalues only arise in complex-conjugate pairs. When using both roots of a complex-conjugate pair to calculate Λ and thus P, the resulting solution should be a real-valued matrix. In order to do this, one may have to enlarge the state space of endogenous state variables to be at least two-dimensional, see again Farmer and Guo (1994) for an example. The complex roots then give rise to endogenous damped cycles of frequency α.

7 Interpreting the results

The results obtained, i.e. the recursive equilibrium law of motion

$$x_t = Px_{t-1} + Qz_t$$

$$y_t = Rx_{t-1} + Sz_t$$

$$z_t = Nz_{t-1} + \epsilon_t$$

can be used to examine model implications. Since x_t, y_t and z_t are log-deviations, the entries in P, Q, R, S and N can be understood as elasticities and interpreted accordingly, see e.g. Campbell (1994).

Impulse responses to a particular shock ϵ_1 can be calculated by setting $x_0 = 0, y_0 = 0$ and $z_0 = 0$, as well as $\epsilon_t = 0$ for $t \geq 2$, and recursively calculating z_t and then x_t and y_t, given $x_{t-1}, y_{t-1}, z_{t-1}$ and ϵ_t for $t = 1, \ldots, T$ with the recursive equilibrium law of motion and the law of motion for z_t. This was already described for the stochastic neoclassical growth model in subsubsection 4.5.3. For the real business cycle model of section 5, the impulse response functions (excluding the response of investment, since it reacts quite strongly) to a technology shock can be seen in figure 4.

To find the second moment properties of the model such as variances and autocorrelations of certain variables as well as the small sample properties of their estimators, simulation methods are often used. Before calculating these moments, the Hodrick-Prescott filter is typically applied (short: HP-Filter). This section demonstrates a

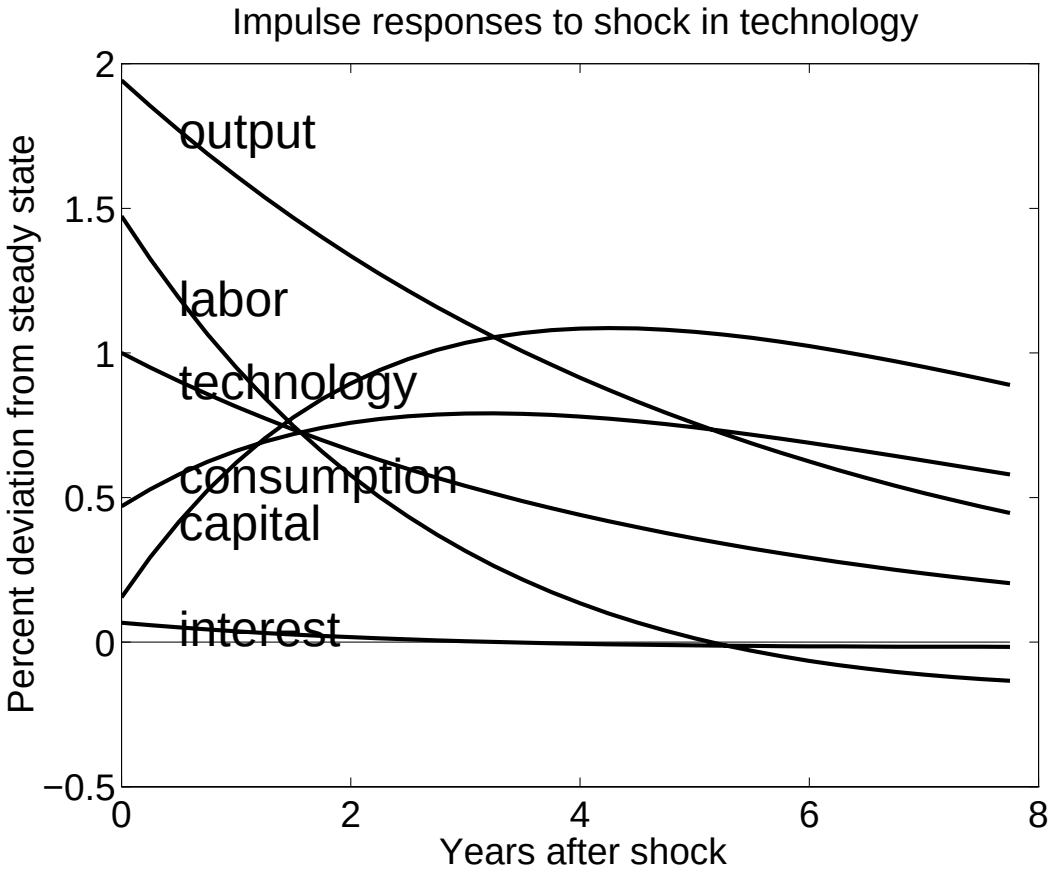

Figure 4: *Impulse responses for Hansens (1985) real business cycle model, using his parameters.*

frequency-domain technique to obtain these moments (albeit without the small sample properties of their estimators) without the need for any simulations[25]. Obviously, the methods here do not deliver properties of the small sample distribution, which may be necessary for testing.

The matrix-valued spectral density for $[x_t', z_t']'$ is given by

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi} \begin{bmatrix} (I_m - Pe^{-i\omega})^{-1}Q \\ I_k \end{bmatrix} (I_k - Ne^{-i\omega})^{-1}\Sigma \\ &\quad (I_k - N'e^{i\omega})^{-1} \left[Q'(I_m - P'e^{i\omega})^{-1}, I_k\right] \end{aligned}$$

where I_k and I_m are the identity matrices of dimension k and m, see Hamilton (1994), formula (10.4.43). Two ways to calculate the matrix-valued spectral density for the

[25]Some of these methods were originally contained in an early version of Uhlig and Xu (1996), but were eventually cut from that paper.

entire vector of variables $s_t = [x_t', y_t', z_t']'$ are

$$\begin{aligned} g(\omega) &= \begin{bmatrix} I_m & 0_{m,k} \\ Re^{-i\omega} & S \\ 0_{k,m} & I_k \end{bmatrix} f(\omega) \begin{bmatrix} I_m & R'e^{i\omega} & 0_{m,k} \\ 0_{k,m} & S' & I_k \end{bmatrix} \\ &= Wf(\omega)W' \text{ , where } W = \begin{bmatrix} I_m, & 0_{m,k} \\ RP^+, & S - RP^+Q \\ 0_{k,m}, & I_k \end{bmatrix}, \end{aligned}$$

where P^+ is the pseudo-inverse of P and where the last equality exploits $s_t = W[x_t', z_t']'$, replacing x_{t-1} with $P^+x_t - P^+Qz_t$ in the recursive equilibrium law of motion for y_t. The HP filter aims at removing a smooth trend τ_t from some given data s_t by solving

$$\min_{\tau_t} \sum_{t=1}^{T} \left((s_t - \tau_t)^2 + \lambda \left((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right)^2 \right)$$

The solution is a linear lag polynomial $r_t = s_t - \tau_t = h(L)s_t$ which has the transfer function

$$\tilde{h}(\omega) = \frac{4\lambda(1 - \cos(\omega))^2}{1 + 4\lambda(1 - \cos(\omega))^2},$$

see King and Rebelo (1993). Thus, the matrix spectral density of the HP-filtered vector is simply

$$g_{HP}(\omega) = \tilde{h}^2(\omega) g(\omega),$$

from which one can obtain the autocorrelations of r_t in time domain via an inverse Fourier transformation,

$$\int_{-\pi}^{\pi} g_{HP}(\omega) e^{i\omega k} d\omega = E[r_t r_{t-k}'],$$

see formula (10.4.4) in Hamilton (1994). Inverse Fourier transformations are part of many numerical packages.

For Hansens (1985) real business cycle model studied in section 5, tables 3 and 4 report the standard deviations as well as the cross-correlations with GNP for the HP-filtered series in the model. These tables are often used in the real business cycle literature as a first cut for evaluating the fit of a model to the data.

capital	0.50
consumption	0.52
output	1.80
labor	1.37
interest	0.06
investment	5.74
technology	0.93

Table 3: *Model standard deviations of the HP-filtered series for Hansens (1985) real business cycle model, studied in section 5.*

capital	0.68	0.64	0.54	.35	.07	-0.15	0.30
consumption	0.54	0.66	0.77	0.87	0.52	.24	0.02
output	0.27	.47	0.71	.00	0.71	0.47	0.27
labor	0.15	.37	0.64	.98	0.74	0.53	0.35
interest	0.09	0.32	0.60	0.96	0.74	0.54	0.38
investment	0.19	0.40	0.67	.99	0.73	0.51	0.32
technology	.26	0.46	0.71	1.00	0.72	0.48	0.28
j	-3	-2	-1	0	1	2	3

Table 4: *Cross-correlations* $corr(v(t+j), GNP(t))$ *for the HP-filtered series for Hansens (1985) real business cycle model, studied in section 5.*

8 Conclusions

We have provided a toolkit to analyze nonlinear dynamic stochastic models easily. The main contribution of this chapter is to simplify and unify existing approaches, showing how to log-linearize the necessary equations characterizing the equilibrium without explicit differentiation, to provide a general solution to a linearized system using the method of undetermined coefficients, allowing in particular for a vector of endogenous states, and to provide simulation-free frequency-domain based method to calculate the the model implications in its HP-filtered version. These methods are easy to use if a numerical package such as MATLAB or GAUSS is available. This chapter should therefore be useful for anybody interested in analyzing nonlinear stochastic dynamic models.

Appendix

A Description of the MATLAB programs.

MATLAB programs to carry out the calculations for sections 6 and 7 are available at the following web site:
http://cwis.kub.nl/~few5/center/STAFF/uhlig/toolkit.dir/toolkit.htm

They shall briefly be described here. The easiest way to learn about these programs is to store all of them, start MATLAB from the directory, where they are stored and type "readme". This will execute the **readme.m**-file, providing some documentation.

As of this writing, the newest version of the files are "version 2". To see how they differ from the previous version, which was distributed until spring 1997, type "whatsnew" within MATLAB, which executes the file **whatsnew.m**, printing relevant messages as a result. To see quickly, how these files work, start MATLAB and type "exampl0" to calculate through example 0, which is the stochastic neoclassical growth model of section 4, or type "exampl1" to calculate through example 1, which is Hansens (1985) real business cycle model of section 5. There are more examples, enumerated "examplNN", where NN stands for their number. To see what any particular example, say, exampl1.m, does, type "help exampl1" within MATLAB. Use the example files as templates for your own work. Alternatively, declare all needed matrices and type in "do_it" to do all calculations. All the **examplNN.m**-files call "do_it" at the very end.

The files which perform all the calculations (i.e. all the files aside from the examplNN.m-files, the readme.m-file and the whatsnew.m-file) are:

do_it.m: does it all, once all needed matrices are defined. This file calls all the other programs. Thus, examining this file will tell you, in which sequence all the other calculations are performed.

enlarge.m: allows you to manipulate letter sizes on plots and other properties of plots. Useful for producing slides or plots for publication.

impresp.m: calculates and shows impulse responses to shocks, see section 7.

mom_out.m: produces output. To be called after moments.m

moments.m: calculates second moment properties, see section 7.

options.m: sets the options for all programs. It is called by do_it and needs to be called, if any of the following routines is used in isolation.

sol_out.m: produces output. To be called after solve.m

solve.m: solves for the recursive equilibrium law of motion with the theorems of section 6.

All files are extensively documented. Type, say, "help impresp" in MATLAB to get more information on what the program impresp.m does. Note that these files set some additional variables, which you may have used before: thus, be careful not to use names appearing in the programs. If you have a question, please read this chapter and the documentation carefully. These files are provided as a free externality, and I am not prepared to provide "technical support." However, if there are serious flaws or serious ways to improve on these programs, I would like to learn about them. Feel free to copy and modify these files, and use them at your own risk. There is absolutely no guarantee that they work the way they are supposed to.

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