Advanced Microeconomics II

Spring 2011

WISE, Xiamen University

2011 Midterm

10:00-12:00 Apr 12

1. Clearly define the following concepts.

   (a) (5 points) A Walrasian equilibrium in an economy with production.

Solution: A triplet (p, x, y) is a *Walrasian equilibrium* if

1. for all $i = 1, \ldots, n$, for all $x_i' \succ_i x_i$, $px_i' > px_i$.
2. for all $j = 1, \ldots, m$, for all $y_j' \in Y_j$, $py_j' \leq py_j$.
3. $\sum_{i=1}^n x_i = \sum_{i=1}^n w_i + \sum_{j=1}^m y_j$.

   (b) (5 points) A strongly Pareto efficient allocation in an exchange economy.

Solution: A feasible allocation x is *strongly Pareto efficient* if there is no feasible allocation x' such that all agents weakly prefer x' to x and some agent strictly prefers x' to x.

   (c) (5 points) A strategic game.

Solution: A strategic game $G = \{N, (A_i)_{i=1}^N, (\succeq_i)_{i=1}^N\}$, where

- N is the set of players,
- for each i, A_i is the set of actions available to player i and
- for each i, $\succeq_i$ is a preference relation on $A = \times_{j \in N} A_j$.

   (d) (5 points) A rationalizable action.

Solution: Either of the below definitions will suffice.

Pearce: An action $a_i \in A_i$ is *rationalizable* in the strategic game $\{N, (A_i), (u_i)\}$ if there exists

- a collection $((X_j^t)_{j \in N})_{t=1}^{\infty}$ of sets with $X_j^t \subset A_j$ for all j and t,
- a belief μ_i^1 of player i whose support is a subset of X_{-i}^1, and
- for each $j \in N$, each $t \geq 1$, and each $a_j \in X_j^t$ a belief $\mu_j^{t+1}(a_j)$ of player j whose support is a subset of X_{-j}^{t+1}

such that

- a_i is a best response to the belief μ_i^1 of player i
- for every player $j \in N$ and every $t \geq 1$ every action $a_j \in X_j^t$ is a best response to the belief $\mu_j^{t+1}(a_j)$ for player j

Bernheim: An action $a_i \in A_i$ is *rationalizable* in the strategic game $\{N, (A_i), (u_i)\}$ if for each $j \in N$ there is a set $Z_j \subset A_j$ such that

- $a_i \in Z_i$
- every action $a_j \in Z_j$ is a best response to a belief $u_j(a_j)$ of player j whose support is a subset of Z_{-j}.

2. (10 points) Prove the following:

   Proposition 1. *If (x, p) is a Walrasian equilibrium, then x is Pareto efficient.*

Solution: Suppose not. Let x' be a feasible allocation for which $x_i' \succeq x_i'$ for all i and $x_i' \succ x_i'$ for at least one i. Then

$$px_i' \geq px_i \text{ for all } i \text{ and}$$

$$px_i' > px_i \text{ for at least one } i.$$

Sum over $i = 1 \ldots, n$.

$$p \sum_{i=1}^{n} \omega_i = p \sum_{i=1}^{n} x_i' > p \sum_{i=1}^{n} x_i = p \sum_{i=1}^{n} \omega_i.$$

3. (10 points) Consider the following proof.

   Proof. Suppose that $a_i \in A_i$ is rationalizable and let $(Z_j)_{j \in N}$ be the profile of sets that supports a_i. For any t, $Z_j \subset X_j^t$ since each action in Z_j is a best response to some belief over Z_{-j} and hence is not strictly dominated in the game $\{N, (X_i^t)_{i=1}^N, (u_i^t)_{i=1}^N\}$. Hence, $a_i \in X_i$. ■

State what has been proven.

Solution: If $a_{\cancel{1}} \in A_i$ in the finite strategic game $G = \{N, (A_i), (u_i)\}$ is rationalizable then a_i survives iterated elimination of strictly dominated actions in G, i.e. $a_i \in X_i^T = X_i$.

4. Consider the following decreasing-returns-to-scale Robinson Crusoe economy. The consumer is endowed with one unit of time and has the following utility function for consumption, x, and leisure, l: $u(x, l) = \alpha \ln x + (1 - \alpha) \ln l$, where $0 < \alpha < 1$. There is one firm, owned by the consumer, that produces output x with labor L using the following production function: $x = L^\beta$, where $0 < \beta < 1$. Normalise the price of output x to 1.

   (a) (5 points) Write down consumer's utility maximization problem and the firm's profit maximization problem.

Solution: Consumer's utility maximization problem:

$$\max_{x,l} \alpha \ln x + (1 - \alpha) \ln l \text{ s.t. } x + p_l l = p_l + y - p_l L, x \geq 0, l \geq 0.$$

Firm's profit maximization problem:

$$\max_{y,L} y - p_l L \text{ s.t. } y = L^\beta, L \geq 0, y \geq 0.$$

   (b) (5 points) Derive the set of equations that characterize the Walrasian equilibrium.

Solution: The following assumes non-negativity constraints are satisfied (which they will be).
From the consumer's utility maximization problem:

$$\frac{\alpha}{x^*} = \lambda^*, \quad \frac{1-\alpha}{l^*} = p_l^* \lambda^*, \quad x^* + p_l^* l^* = p_l^* + y^* - p_l^* L^*$$

From the firm's profit maximization problem:

$$1 = \gamma^*, \quad p_l^* = \gamma^* \beta L^{*^{\beta-1}}, \quad y^* = L^{*^\beta}$$

From market clearing:

$$x^* = y^*, \quad l^* + L^* = 1$$

Hence we have eight equations and seven unknowns $(x^*, y^*, l^*, L^*, \lambda^*, \gamma^*, p_l^*)$.

   (c) (5 points) Fully characterize the Walrasian equilibrium for this economy when $\alpha = \beta = 0.5$.

Solution:

$$L^* = \left(\frac{p_l^*}{\beta}\right)^{\frac{1}{\beta-1}}.$$

$$y^* = \left(\frac{p_l^*}{\beta}\right)^{\frac{\beta}{\beta-1}}.$$

$$x^* = \alpha\left(p_l^* + \left(\frac{p_l^*}{\beta}\right)^{\frac{\beta}{\beta-1}} - p_l\left(\frac{p_l^*}{\beta}\right)^{\frac{1}{\beta-1}}\right).$$

$$l^* = \frac{(1-\alpha)\left(p_l^* + \left(\frac{p_l^*}{\beta}\right)^{\frac{\beta}{\beta-1}} - p_l^*\left(\frac{p_l^*}{\beta}\right)^{\frac{1}{\beta-1}}\right)}{p_l^*}.$$

Hence, using market clearing ($x^* = y^*$),

$$\left(\frac{p_l^*}{\beta}\right)^{\frac{\beta}{\beta-1}} = \alpha\left(p_l^* + \left(\frac{p_l^*}{\beta}\right)^{\frac{\beta}{\beta-1}} - p_l\left(\frac{p_l^*}{\beta}\right)^{\frac{1}{\beta-1}}\right).$$

Divide through by p_l^* and let $\delta = p_l^{*\frac{1}{\beta-1}}$ to get

$$\left(\frac{\delta}{\beta^{\frac{\beta}{\beta-1}}}\right) = \alpha\left(1 + \frac{\delta}{\beta^{\frac{\beta}{\beta-1}}} - \frac{\delta}{\beta^{\frac{1}{\beta-1}}}\right).$$

Hence,

$$\delta = \alpha(\beta^{\frac{\beta}{\beta-1}} + \delta - \beta\delta).$$

and

$$p_l^* = \beta^\beta\left(\frac{\alpha}{1-\alpha(1-\beta)}\right)^{\beta-1}.$$

Substituting the values $\alpha = \beta = 0.5$ we get $p^* = \sqrt{3}/2$, $y^* = x^* = 1/\sqrt{3}$, $L^* = 1/3$ and $l^* = 2/3$.

5. Consider the strategic game described in the following table.

	L	R
U	$3,3$	$1,6$
D	$6,1$	$0,0$

(a) (5 points) What are the set of mixed strategy Nash equilibria for this game. Be sure to include any pure strategy Nash equilibria in your answer.

Solution: It is obvious that there are 2 pure strategy Nash equilibria in this game, one is (U, R) and the other is (D, L). In order to find the other mixed strategy Nash equilibrium, assume (α, β) is the mixed strategy Nash equilibrium, where α is the probability that player 1 plays U and β is the probability that player 2 plays L. Consider player 1. To be indifferent between U and D, we must have

$$3\beta + 1 \times (1 - \beta) = 6\beta + 0 \times (1 - \alpha) \Rightarrow \beta = \frac{1}{4}$$

Since the game is symmetric, we have $\alpha = \frac{1}{4}$. Thus $(\frac{1}{4}, \frac{1}{4})$ is a mixed strategy Nash equilibrium.

The full set of mixed strategy Nash equilibria is

$$\{(1/4, 1/4), (0, 1), (1, 0)\}.$$

(b) (5 points) Construct a correlated equilibrium for this game with payoffs that are equal to the payoffs in one of the Nash equilibria you constructed in (a).

Solution: Three possible solutions:

1. Let $\Omega = \{x\}$, $\pi(x) = 1$, $P_1 = P_2 = \{x\}$, $\sigma_1(x) = U$ and $\sigma_2(x) = R$. This correlated equilibrium yields the payoff profile $(1, 6)$.

2. Let $\Omega = \{x\}$, $\pi(x) = 1$, $P_1 = P_2 = \{x\}$, $\sigma_1(x) = D$ and $\sigma_2(x) = L$. This correlated equilibrium yields the payoff profile $(6, 1)$.

3. Let $\Omega = \{w, x, y, z\}$, $\pi(w) = 1/16, \pi(x) = 3/16, \pi(y) = 3/16, \pi(z) = 9/16$, $P_1 = \{\{w, x\}, \{y, z\}\}$, $P_2 = \{\{w, y\}, \{x, z\}\}$, $\sigma_1(w) = \sigma_1(x) = U$, $\sigma_1(y) = \sigma_1(z) = D$, $\sigma_2(w) = \sigma_2(y) = L$ and $\sigma_2(x) = \sigma_2(z) = R$. This correlated equilibrium yields the payoff profile $(3/2, 3/2)$.

(c) (10 points) Construct a correlated equilibrium where payoffs for each player are outside the convex hull of payoffs associated with Nash equilibria.

Solution: Note first that the symmetric Pareto optimal outcome is $(7/2, 7/2)$. Hence, the payoffs associated with the correlated equilibrium that maximizes joint payoffs is inside the convex hull of payoffs associated with Nash equilibria.

Instead we will find the correlated equilibrium that minimizes joint payoffs. Let the probability distribution over (UL, UR, DL, DR) be denoted by (w, x, y, z). We will consider a correlated equilibrium that generates symmetric payoffs, i.e., $x = y$. To be a correlated equilibrium we need

$$\begin{aligned} 6y + 0z &\geq 3y + z \\ 3w + 1y &\geq 6w. \end{aligned}$$

Hence, $3y \geq z$ and $y \geq 3w$. We want to minimize $3w+7y+z$. Hence, $w = 0$ and $3y = z$. Since $w+2y+z = 1$ we have $x = y = 1/5$ and $z = 3/5$. The payoffs from this correlated equilibrium are $(7/5, 7/5)$ which does, indeed, lie outside the convex hull of payoffs associated with Nash equilibria (the payoffs from the mixed strategy equilibrium are $(3/2, 3/2)$. Of course, there are infinitely many correlated equilibria with payoffs that are outside the convex hull of payoffs associated with Nash equilibrium.

6. Consider the following three player game G, where $A_1 = \{U, M, D\}$, $A_2 = \{L, R\}$, and $A_3 = \{A, B\}$.

	L	R
U	$1, 5, 2$	$4, 3, 5$
M	$2, 1, 2$	$3, 5, 3$
D	$6, 3, 2$	$4, 0, 1$

A

	L	R
U	$7, 7, 1$	$1, 3, 0$
M	$3, 6, 8$	$3, 4, 2$
D	$1, 4, 2$	$4, 1, 0$

B

(a) (5 points) For the strategy $a_1^* = M$ for player 1, describe (using a table of you like) the auxiliary strictly competitive (zero-sum) 2 player game G' where $A_1' = A_1 \backslash a_1^*$, $A_2' = A_{-1}$ and $u_1'(a_1, a_{-1}) = u_1(a_1, a_{-1}) - u_1(a_i^*, a_{-1})$.

Solution:

	LA	RA	LB	RB
U	$-1, 1$	$1, -1$	$4, -4$	$-2, 2$
D	$4, -4$	$1, -1$	$-2, 2$	$1, -1$

(b) (5 points) Is the strategy M a best response for player 1 in G for some belief over the other players' actions. If so, give one such belief.

Solution: To find such a belief we can ignore LA and RA since an equivalent belief over RB makes M a more profitable strategy for player 1 ($-2 \leq \min\{-1, 1\}$ and $1 \leq \min\{4, 1\}$). Another way of saying this is that if player 3 plays the action A with probability 1, action D dominates action M for player 1.

Let α be the belief probability associated with LB and $1 - \alpha$ be the belief associated with RB. For M to be a best response we need

$$\alpha 4 + (1 - \alpha)(-2) \leq 0 \text{ and } \alpha(-2) + (1 - \alpha)1 \leq 0.$$

These inequalities imply that

$$\alpha \leq 1/3 \text{ and } \alpha \geq 1/3.$$

Hence, $\alpha = 1/3$ makes M a best response in G.

(c) (5 points) Is the strategy M a best response for player 1 in G for some belief over the other players' actions where the belief is the product of independent probability distributions over each other player's actions. If so, give one such belief and the associated independent probability distributions over each other player's actions.

Solution: Use the same belief as above. Let the probability that player 3 plays B equal 1. Let the probability that player 2 plays L equal 1/3. The product of these independent probability distributions generates the required belief.