The Mid-Term Examination and Solutions

April 16, 2013

NOTE: The examination consists of 6 problems with differentiated levels of difficulty. The mark of the problem is irrelevant to its difficulty. Please budget your time carefully.

Theory of Assert Pricing

1. (18') Assume there is a riskless security that pays a rate of return equal to r_f . In addition, for simplicity suppose there is just one risky security that pays a stochastic rate of return equal to $\tilde{r}$. Also, let W_0 be the individual's initial wealth, and let A be the dollar amount that the individual invests in the risky asset at the beginning of the period. Consider the following portfolio choice problem

$$\max_A \mathbb{E}\left[U\left(\tilde{W}\right)\right] = \max_A \mathbb{E}\left[U\left(W_0\left(1+r_f\right)+A\left(\tilde{r}-r_f\right)\right)\right].$$

Prove the following propositions.

(i) For an agent with increasing absolute risk aversion, $\partial A/\partial W_0 < 0$.

(ii) For an agent with increasing relative risk aversion, $\partial A/\partial W_0 < A/W_0$.

Solution: FOC.

$$\mathbb{E}\left[U'\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\right] = 0$$

$$\tilde{W} = W_0\left(1+r_f\right)+A\left(\tilde{r}-r_f\right) \Rightarrow \begin{cases} \tilde{W} \geq W_0\left(1+r_f\right) & \tilde{r} \geq r_f \\ \tilde{W} \leq W_0\left(1+r_f\right) & \tilde{r} \leq r_f \end{cases}$$

(i) Absolute risk aversion.

$$\frac{dA}{dW_0} = \frac{\left(1+r_f\right)\mathbb{E}\left[U''\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\right]}{-\mathbb{E}\left[U''\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)^2\right]}$$

$$\begin{aligned}
\mathbb{E}\left[U''\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\right] &= \mathbb{E}\left[-R\left(\tilde{W}\right)U'\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\right] \\
&= \mathbb{E}\left[-R\left(\tilde{W}\right)U'\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\left(\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}+1-\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}\right)\right] \\
&= \mathbb{E}\left[-R\left(\tilde{W}\right)U'\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}\right] \\
&+ \mathbb{E}\left[-R\left(\tilde{W}\right)U'\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\mathbb{I}_{\left\{\tilde{W}\leq W_0(1+r_f)\right\}}\right]
\end{aligned}$$

Increasing.

$$\mathbb{E}\left[-R\left(\tilde{W}\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}\right]\leq\mathbb{E}\left[-R\left(W_0\left(1+r_f\right)\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}\right]$$
$$=-R\left(W_0\left(1+r_f\right)\right)\mathbb{E}\left[U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}\right]$$

$$\mathbb{E}\left[-R\left(\tilde{W}\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\leq W_0(1+r_f)\right\}}\right]\leq\mathbb{E}\left[-R\left(W_0\left(1+r_f\right)\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\leq W_0(1+r_f)\right\}}\right]$$
$$=-R\left(W_0\left(1+r_f\right)\right)\mathbb{E}\left[U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\leq W_0(1+r_f)\right\}}\right]$$

$$\mathbb{E}\left[U''\left(\tilde{W}\right)(\tilde{r}-r_f)\right]\leq-R\left(W_0\left(1+r_f\right)\right)\mathbb{E}\left[U'\left(\tilde{W}\right)(\tilde{r}-r_f)\right]=0$$

(ii) Relative risk aversion.

$$\eta\equiv\frac{dA_0}{dW_0}\frac{W_0}{A}=1+\frac{\mathbb{E}\left[U''\left(\tilde{W}\right)(\tilde{r}-r_f)\tilde{W}\right]}{-A\mathbb{E}\left[U''\left(\tilde{W}\right)(\tilde{r}-r_f)^2\right]}$$

$$\begin{aligned}\mathbb{E}\left[U''\left(\tilde{W}\right)(\tilde{r}-r_f)\tilde{W}\right]&=\mathbb{E}\left[-R_r\left(\tilde{W}\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\right]\\&=\mathbb{E}\left[-R_r\left(\tilde{W}\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\left(\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}+1-\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}\right)\right]\\&=\mathbb{E}\left[-R_r\left(\tilde{W}\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\geq W_0(1+r_f)\right\}}\right]\\&+\mathbb{E}\left[-R_r\left(\tilde{W}\right)U'\left(\tilde{W}\right)(\tilde{r}-r_f)\mathbb{I}_{\left\{\tilde{W}\leq W_0(1+r_f)\right\}}\right]\end{aligned}$$

Apply the similar method to reach the conclusion. ■

2. (20') Consider the mean-variance analysis covered in class where there are n risky assets whose returns are jointly normally distributed. Assume that investors differ with regard to their (concave) utility functions and their initial wealths.

(i) Prove that in equilibrium the risk-free rate, R_f, cannot exceed the expected return of the minimum variance portfolio, R_{mv}.

(ii) Now assume that investors can lend at the risk-free rate, $R_f < R_{\mathrm{mv}}$, but investors are restricted from risk-free borrowing, that is, no risk-free borrowing is permitted. Write $\alpha\equiv\bar{R}'V^{-1}e=e'V^{-1}\bar{R}$, $\varsigma\equiv\bar{R}'V^{-1}\bar{R}$, and $\delta\equiv e'V^{-1}e$. Derive the analytical form of the efficient frontier. (Hint: You may recall the Karush-Kuhn-Tucker (KKT) conditions to solve a constrained optimization problem like this.)

The Sharpe-Linter CAPM

(iii) Consider the market model

$$r_{it}-r_{ft}=\alpha_i+\beta_i\left(r_{mt}-r_{ft}\right)+\varepsilon_{it}.$$

Theoretically, $\alpha_i=0$. However, in empirical tests one may acquire an α_i statistically differing from zero. Interpret this α.

(iv) Consider the empirical model

$$r_{it}-r_{ft}=\alpha_i+\beta_i\left(r_{mt}-r_{ft}\right)+\gamma_i\cdot D_t\cdot\left(r_{mt}-r_{ft}\right)+\varepsilon_{it}$$

where D_t is a dummy and $D_t = 1$ when $r_{mt} > r_{ft}$. Interpret γ_i if it is statistically significant against zero.

Solution: (i) If $R_f = R_{mv}$, then $e'\omega^m = 0$, which implies a zero net position in the tangency portfolio. If $R_f > R_{mv}$, then investors will always short sell the tangency portfolio since it is inefficient. In both cases there exists excessive supply, which implies that the market fails to reach an equilibrium.

(ii) The optimization problem

$$\begin{aligned}
&\min_{\omega} \frac{1}{2}\omega' V \omega \\
&\text{s.t.} \bar{R}_p = R_f + \omega'\left(\bar{R} - R_f e\right) \\
&\omega' e \leq 1
\end{aligned}$$

$$\mathcal{L} = \frac{1}{2}\omega' V \omega + \lambda\left[\omega'\left(\bar{R} - R_f e\right) - \left(\bar{R}_p - R_f\right)\right] + \mu\left(\omega' e - 1\right)$$

KKT conditions

$$V\omega + \lambda\left(\bar{R} - R_f e\right) + \mu e = 0 \tag{1}$$

$$\omega'\left(\bar{R} - R_f e\right) - \left(\bar{R}_p - R_f\right) = 0 \tag{2}$$

$$\mu\left(\omega' e - 1\right) = 0 \tag{3}$$

$$\mu \geq 0 \tag{4}$$

$$\omega' e \leq 1 \tag{5}$$

Solution

Consider the tangency portfolio ω^m,

$$\omega^m = \left[\alpha - \delta R_f\right]^{-1} V^{-1}\left(\bar{R} - R_f e\right)$$

$$\bar{R}_m = R_f + \left[\alpha - \delta R_f\right]^{-1}\left(\bar{R} - R_f e\right)' V^{-1}\left(\bar{R} - R_f e\right)$$

If $\mu = 0$,

$$\omega' e - 1 < 0,$$

by (1),

$$\omega' = \lambda\left(\bar{R} - R_f e\right)' V^{-1}$$

$$\begin{aligned}
\lambda &= \frac{\bar{R}_p - R_f}{\left(\bar{R} - R_f e\right)' V^{-1}\left(\bar{R} - R_f e\right)} \\
&= \frac{\bar{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2}.
\end{aligned}$$

$$\begin{aligned}
\omega &= \lambda V^{-1}\left(\bar{R} - R_f e\right) \\
&= \frac{\bar{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2} V^{-1}\left(\bar{R} - R_f e\right),
\end{aligned}$$

For $\bar{R}_p > \bar{R}_m$,

$$\begin{aligned}
e'\omega &= \frac{\bar{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2} e' V^{-1} \left(\bar{R} - R_f e\right) \\
&= \frac{\bar{R}_p - R_f}{\varsigma - 2\alpha R_f + \delta R_f^2} \left(\alpha - \delta R_f\right) \\
&> \frac{\left[\alpha - \delta R_f\right]^{-1} \left(\bar{R} - R_f e\right)' V^{-1} \left(\bar{R} - R_f e\right)}{\varsigma - 2\alpha R_f + \delta R_f^2} \left(\alpha - \delta R_f\right) \\
&= 1.
\end{aligned}$$

So that when $\bar{R}_p \leq \bar{R}_m$,

$$\begin{aligned}
\sigma_p^2 &= \omega' V \omega \\
&= \left[\lambda \left(\bar{R} - R_f e\right)' V^{-1}\right] V \left[\lambda V^{-1} \left(\bar{R} - R_f e\right)\right] \\
&= \lambda^2 \left(\bar{R} - R_f e\right)' V^{-1} \left(\bar{R} - R_f e\right) \\
&= \frac{\left(\bar{R}_p - R_f\right)^2}{\varsigma - 2\alpha R_f + \delta R_f^2}
\end{aligned}$$

If $\mu > 0$,

$$\omega' e - 1 = 0,$$

by (1),

$$\omega' = \lambda \left(\bar{R} - R_f e\right)' V^{-1} + \mu e' V^{-1}$$

$$1 = \lambda \left(\bar{R} - R_f e\right)' V^{-1} e + \mu e' V^{-1} e \tag{6}$$

$$\bar{R}_p - R_f = \lambda \left(\bar{R} - R_f e\right)' V^{-1} \left(\bar{R} - R_f e\right) + \mu e' V^{-1} \left(\bar{R} - R_f e\right) \tag{7}$$

$$\lambda = \frac{\left(\alpha - \delta R_f\right) - \delta \left(\bar{R}_p - R_f\right)}{\left(\alpha - \delta R_f\right)^2 - \delta \left(\varsigma - 2\alpha R_f + \delta R_f^2\right)} = \frac{\delta \bar{R}_p - \alpha}{\delta\varsigma - \alpha^2}$$

$$\mu = \frac{\left(\alpha - \delta R_f\right)\left(\bar{R}_p - R_f\right) - \left(\varsigma - 2\alpha R_f + \delta R_f^2\right)}{\delta\varsigma - \alpha^2}$$

For $\bar{R}_p \leq \bar{R}_m$,

$$\begin{aligned}
\mu &= \frac{\left(\alpha - \delta R_f\right)\left(\bar{R}_p - R_f\right) - \left(\varsigma - 2\alpha R_f + \delta R_f^2\right)}{\delta\varsigma - \alpha^2} \\
&\leq \frac{\left(\alpha - \delta R_f\right)\left[\alpha - \delta R_f\right]^{-1} \left(\bar{R} - R_f e\right)' V^{-1} \left(\bar{R} - R_f e\right) - \left(\varsigma - 2\alpha R_f + \delta R_f^2\right)}{\delta\varsigma - \alpha^2} \\
&= 0.
\end{aligned}$$

So that when $\bar{R}_p > \bar{R}_m$,

$$\begin{aligned}
\sigma_p^2 &= \omega' V \omega \\
&= \left[\lambda \left(\bar{R} - R_f e\right)' V^{-1} + \mu e' V^{-1}\right] V \left[\lambda V^{-1} \left(\bar{R} - R_f e\right) + \mu V^{-1} e\right] \\
&= \lambda^2 \left(\bar{R} - R_f e\right)' V^{-1} \left(\bar{R} - R_f e\right) + 2\lambda\mu \left(\bar{R} - R_f e\right)' V^{-1} e + \mu^2 e' V^{-1} e \\
&= \delta^{-1} \left\{\lambda^2 \left[\delta\varsigma - \alpha^2 + (\alpha - \delta R_f)^2\right] + 2\lambda\mu\delta (\alpha - \delta R_f) + \delta^2 \mu^2\right\} \\
&= \delta^{-1} \left[\lambda (\alpha - \delta R_f) + \delta\mu\right]^2 + \delta^{-1} \lambda^2 \left(\delta\varsigma - \alpha^2\right) \\
&= \frac{1}{\delta} + \frac{\delta \left(\bar{R}_p - \frac{\alpha}{\delta}\right)^2}{\delta\varsigma - \alpha^2}
\end{aligned}$$

Therefore in the $\left(\sigma_p, \bar{R}_p\right)$ space the efficient frontier in the absence of riskless borrowing consists of two parts: for $\bar{R}_p \leq \bar{R}_m$ it is the line segment connecting $(0, R_f)$ and $\left(\sigma_m, \bar{R}_m\right)$ and for $\bar{R}_p > \bar{R}_m$ it is the risky-asset-only efficient frontier.

(iii) Abnormal returns. Significant alpha's indicate excessive returns over premiums for systematic risks born across the interested horizon. A significantly positive α is the additional profit earned beyond what the CAPM prescribes for its risk compensation and a significantly negative alpha is the additional loss incurred. Alpha's suggest the ability to choose promising risky assets.

(iv) Timing abilities. A significantly positive γ_i implies that the holder of the portfolio increases her allocations on risky assets during a bull market and closes her risky positions during a bear market. That is, she actively and correctly chooses the timing of investment. Similarly, a significantly negative γ_i implies a poorly choice of timing. ■

3. (12') Let the U.S. dollar (\$) - Swiss franc (SF) spot exchange rate be \$0.68 per SF and the one year forward exchange rate be \$0.70 per SF. The one-year interest rate for borrowing or lending dollars is 6.00 percent.

(i) What must be the one-year interest rate for borrowing or lending Swiss francs in order for there to be no arbitrage opportunity?

(ii) If the one-year interest rate for borrowing or lending Swiss francs was less than your answer in (a), describe the arbitrage opportunity.

Solution: (i)

$$\begin{aligned}
0.70 &= 0.68 \cdot \frac{1.06}{1+r} \\
r &= 0.02971
\end{aligned}$$

(ii) At $t = 0$:
Assume one long position of the forward.
Borrow 1/1.02 SF at, say 2.00 percent, per year. Sell it for \$0.68/1.02.
Lend \$0.68/1.02 at 6.00 percent per year.
At $t = 1$:
Close all positions.
Net payoff

$$0.68 \cdot \frac{1.06}{1.02} - 0.7 = 0.0067 > 0$$

So realizes arbitrage. ■

Stochastic Calculus

4. (20') Toss a coin repeatedly. Assume the probability of head on each toss is $\frac{1}{2}$, as is the probability of tail. Let $X_j = 1$ if the j^{th} toss results in a head and $X_j = -1$ if the j^{th} toss results in a tail. Consider the stochastic process $M_0, M_1, M_2, \ldots$ defined by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} X_j, n \geq 1.$$

This is called a symmetric random walk, with each head, it steps up one, and with each tail, it steps down one.

(i) Show that $M_0, M_1, M_2, \ldots$ is a martingale.

(ii) Let σ be a positive constant and, for $n > 0$, define

$$S_n = e^{\sigma M_n} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n.$$

Show that $S_0, S_1, S_2, \ldots$ is a martingale.

Let $M_0, M_1, M_2, \ldots$ be the symmetric random walk above, and define $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} M_j \left(M_{j+1} - M_j \right), n = 1, 2, \ldots.$$

(iii) Show that

$$I_n = \frac{1}{2} M_n^2 - \frac{n}{2}.$$

(iv) Let n be an arbitrary nonnegative integer, and let $f(i)$ be an arbitrary function of a variable i. In terms of n and f, define another function $g(i)$ satisfying

$$\mathbb{E}_n \left[f \left(I_{n+1} \right) \right] = g \left(I_n \right).$$

Solution: (i)

$$\mathbb{E}_n \left[M_{n+1} \right] = \mathbb{E}_n \left[\sum_{j=1}^{n+1} X_j \right] = \mathbb{E}_n \left[M_n + X_{n+1} \right] = M_n + \mathbb{E}_n \left[X_{n+1} \right] = M_n.$$

(ii)

$$\begin{aligned}
\mathbb{E}_n \left[S_{n+1} \right] &= \mathbb{E}_n \left[e^{\sigma M_{n+1}} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} \right] \\
&= \mathbb{E}_n \left[e^{\sigma (M_n + X_{n+1})} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} \right] \\
&= S_n \mathbb{E}_n \left[e^{\sigma X_{n+1}} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right) \right] \\
&= S_n \left[\left(\frac{1}{2} e^{\sigma} + \frac{1}{2} e^{-\sigma} \right) \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right) \right] \\
&= S_n.
\end{aligned}$$

(iii)

$$\begin{aligned}
I_n &= \frac{1}{2}\cdot 2\sum_{j=0}^{n-1} M_j\left(M_{j+1}-M_j\right)\\
&= \frac{1}{2}\sum_{j=0}^{n-1}\left[M_{j+1}^2-M_j^2-\left(M_{j+1}-M_j\right)^2\right]\\
&= \frac{1}{2}M_n^2-\frac{1}{2}\sum_{j=0}^{n-1}X_{j+1}^2\\
&= \frac{1}{2}M_n^2-\frac{n}{2}.
\end{aligned}$$

(iv)

$$\begin{aligned}
\mathbb{E}_n\left[f\left(I_{n+1}\right)\right] &= \mathbb{E}_n\left[f\left(\frac{1}{2}M_{n+1}^2-\frac{n+1}{2}\right)\right].\\
&= \mathbb{E}_n\left[f\left(\frac{1}{2}M_n^2-M_nX_{n+1}+\frac{1}{2}-\frac{n+1}{2}\right)\right]\\
&= \mathbb{E}_n\left[f\left(I_n-X_{n+1}\sqrt{2I_n+n}\right)\right]\\
&= \frac{1}{2}f\left(I_n+\sqrt{2I_n+n}\right)-\frac{1}{2}f\left(I_n+\sqrt{2I_n+n}\right)\\
&= g\left(I_n\right).
\end{aligned}$$

■

5. (15') Consider the following propositions in the coin-toss probability space.

(i) Let N be a positive integer, and let X be a random variable depending on the first N coin tosses. Let $0\le n\le N$ be given. If $0\le n\le m\le N$, prove that

$$\mathbb{E}_n\left[\mathbb{E}_m\left[X\right]\right]=\mathbb{E}_n\left[X\right].$$

(ii) (No proof required) Give an example of non-Markov martingale in the lattice model. Explain why.

(iii) In an N-period binomial model, let $\mathbb{P}$ be the actual probability measure, $\tilde{\mathbb{P}}$ the risk-neutral probability measure, and assume that $\mathbb{P}\left(\omega\right)>0$ and $\tilde{\mathbb{P}}\left(\omega\right)>0$ for every sequence of coin toss ω. Define the Radon-Nikodým derivative (random variable) $Z\left(\omega\right)=\frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$ for every ω. The Radon-Nikodým derivative process is

$$Z_n=\mathbb{E}_n\left[Z\right],n=0,1,\ldots,N.$$

In particular, $Z_N=Z$ and $Z_0=1$. Let n be a positive integer between 0 and N, and let Y be a random variable depending only on the first n coin tosses. Prove that

$$\tilde{\mathbb{E}}\left[Y\right]=\mathbb{E}\left[Z_nY\right].$$

Solution: (i)

$$
\begin{aligned}
\mathbb{E}_n\left[\mathbb{E}_m\left[X\right]\right] &= \sum_{\omega_{n+1}\ldots\omega_m}\left[p^{\#H(\omega_{n+1}\ldots\omega_m)}q^{\#T(\omega_{n+1}\ldots\omega_m)}\cdot\sum_{\omega_{m+1}\ldots\omega_N}p^{\#H(\omega_{m+1}\ldots\omega_N)}q^{\#T(\omega_{m+1}\ldots\omega_N)}X\left(\omega_{n+1}\ldots\omega_N\right)\right]\\
&= \sum_{\omega_{n+1}\ldots\omega_m}\sum_{\omega_{m+1}\ldots\omega_N}p^{\#H(\omega_{m+1}\ldots\omega_N)+\#H(\omega_{n+1}\ldots\omega_m)}q^{\#T(\omega_{m+1}\ldots\omega_N)+\#T(\omega_{n+1}\ldots\omega_m)}X\left(\omega_{n+1}\ldots\omega_N\right)\\
&= \sum_{\omega_{n+1}\ldots\omega_m}p^{\#H(\omega_{n+1}\ldots\omega_N)}q^{\#T(\omega_{n+1}\ldots\omega_N)}X\left(\omega_{n+1}\ldots\omega_N\right)\\
&= \mathbb{E}_n\left[X\right].
\end{aligned}
$$

(ii) Consider the process

$$X_n=\begin{cases}1, & \text{with probability } \frac{1}{2},\\ -1, & \text{with probability } \frac{1}{2}.\end{cases}$$

Let $Y_0=0$ and

$$Y_n=\sum_{k=1}^{n}X_k+X_n\cdot\sum_{k=1}^{n-1}\mathbb{I}_{\{X_k=1\}}.$$

Then

$$\mathbb{E}_n\left[Y_{n+1}\right]=Y_n+\mathbb{E}_n\left[X_{n+1}\right]+\mathbb{E}_n\left[X_{n+1}\right]\cdot\sum_{k=1}^{n}\mathbb{I}_{\{X_k=1\}}=Y_n.$$

$$\mathbb{E}_n\left[f\left(Y_{n+1}\right)\right]=\frac{1}{2}f\left(Y_n+1+\sum_{k=1}^{n}\mathbb{I}_{\{X_k=1\}}\right)+\frac{1}{2}f\left(Y_n-1-\sum_{k=1}^{n}\mathbb{I}_{\{X_k=1\}}\right).$$

$\mathbb{E}_n\left[f\left(Y_{n+1}\right)\right]$ must be dependent on $Y_{n-1},\ldots,Y_1$.

(iii)

$$\tilde{\mathbb{E}}\left[Y\right]=\mathbb{E}\left[ZY\right]=\mathbb{E}\left[\mathbb{E}_n\left[ZY\right]\right]=\mathbb{E}\left[Y\mathbb{E}_n\left[Z\right]\right]=\mathbb{E}\left[Z_nY\right].$$

■

6. (15') Assume prevailing time $t_0=0$. Consider the following propositions.

Put-call parity. For a European call C_t and a European put P_t written on a no-dividend stock S_t with the same strike price K and expiration T, the identity

$$C_t-P_t=S_t-\frac{K}{\left(1+r\right)^{T-t}}$$

holds for every $t\in[0,T]$.

(i) Discretize the time interval $[0,T]$. Using the risk-neutral pricing formula to prove the parity in the coin-toss probability space.

(ii) Prove the parity using the no-arbitrage argument.

Chooser options. Consider the following descriptions in the coin-toss probability space.

(iii) Let $1\le m\le N-1$ and $K>0$ be given. A chooser option is a contract sold at time zero that confers on its owner the right to receive either a call or a put at time m. The owner of the chooser may wait until time m before choosing. The call or put chosen expires at time N with strike price K. Show that the time-zero price of a chooser option is the sum of the time-zero price of a put, expiring at time N and having strike price K, and a call, expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$.

Solution: (i) Discretization omitted.

$$
\begin{aligned}
C_n - P_n &= \tilde{\mathbb{E}}_n \left[\frac{C_N - P_N}{(1+r)^{N-n}} \right] \\
&= \tilde{\mathbb{E}}_n \left[\frac{(S_N - K)^+ - (K - S_N)^+}{(1+r)^{N-n}} \right] \\
&= \tilde{\mathbb{E}}_n \left[\frac{S_N - K}{(1+r)^{N-n}} \right] \\
&= S_n - \frac{K}{(1+r)^{N-n}}.
\end{aligned}
$$

(ii) Suppose that

$$C_t - P_t > S_t - \frac{K}{(1+r)^{T-t}}.$$

At $t = t$:

Assume one short position of the call, one long position of the put, one long position of the stock, and, borrow $\frac{K}{(1+r)^{T-t}}$ at r. Invest what's left at r.

The initial payment is zero.

At $t = T$:

Close all positions.

If $S_T \geq K$,

$$V_T = -(S_T - K) + S_T - K + \left(C_t - P_t - S_t + \frac{K}{(1+r)^{T-t}} \right) (1+r)^{T-t} > 0;$$

if $S_T < K$,

$$V_T = (K - S_T) + S_T - K + \left(C_t - P_t - S_t + \frac{K}{(1+r)^{T-t}} \right) (1+r)^{T-t} > 0.$$

Arbitrage realized.

Suppose

$$C_t - P_t < S_t - \frac{K}{(1+r)^{T-t}}.$$

Execute the reversal trading strategy to realize arbitrage.

Then the parity holds.

(iii) It can be inferred that the agent would choose the in-the-money option at time m. Therefore

$$\begin{aligned}
V_m &= \tilde{\mathbb{E}}_m\left[\frac{C_N}{(1+r)^{N-m}}|S_m > K\right]\tilde{\mathbb{P}}\{S_m > K\} + \tilde{\mathbb{E}}_m\left[\frac{P_N}{(1+r)^{N-m}}|S_m < K\right]\tilde{\mathbb{P}}\{S_m < K\} \\
&= \tilde{\mathbb{E}}_m\left[\frac{S_N - K + P_N}{(1+r)^{N-m}}|S_m > K\right]\tilde{\mathbb{P}}\{S_m > K\} + \tilde{\mathbb{E}}_m\left[\frac{P_N}{(1+r)^{N-m}}|S_m < K\right]\tilde{\mathbb{P}}\{S_m < K\} \\
&= \tilde{\mathbb{E}}_m\left[\frac{S_N - K}{(1+r)^{N-m}}|S_m > K\right]\tilde{\mathbb{P}}\{S_m > K\} + \tilde{\mathbb{E}}_m\left[\frac{P_N}{(1+r)^{N-m}}\right] \\
&= \left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+ + \tilde{\mathbb{E}}_m\left[\frac{P_N}{(1+r)^{N-m}}\right],
\end{aligned}$$

and at time 0 the expected discounted value of V_m is

$$V_0 = c_0 + P_0$$

where P_0 is a put, expiring at time N and having strike price K, and c_0 is a call, expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$. ■