Advanced Microeconomics II
Problem Set 4

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1. A husband and wife must simultaneously choose whether to attend the football game (F) or the movie theatre (M). With probability θ preferences are as stated in game A. With probability $1-\theta$ preferences are as stated in game B. The husband knows which game is being played but the wife does not.

		Husband	
		M	F
Wife	M	$2,1$	$0,0$
	F	$0,0$	$1,2$

A

		Husband	
		M	F
Wife	M	$2,2$	$0,0$
	F	$0,0$	$1,1$

B

(a) Write down the normal form representation of this static Bayesian game

Solution: $\{N,(A_i)_{i=1}^n,(T_i)_{i=1}^n,p,(p_i)_{i=1}^n,(u_i)_{i=1}^n\}$

- $N=\{W,H\}$
- $A_i=\{M,F\}, i\in N$
- $T_W=\{w\};T_H=\{h_1,h_2\}$
- $p(w,h_1)=\theta;p(w,h_2)=1-\theta$
- $p_W(h_1|w)=\theta;p_W(h_2|w)=1-\theta;\ p_H(w|h_1)=p_H(w|h_2)=1$
- Utility is as described by the above matrices, where lotteries are evaluated using expected utility.

(b) Find the set of pure strategy Nash equilibria for this Bayesian Game.

Solution: If W plays a pure strategy then H must also play the same pure strategy in each state since he always does better by matching W's actions in each state. Hence, there are two pures strategy equilibria:

- $s_W(w)=M;\ s_H(h_1)=s_H(h_2)=M$
- $s_W(w)=F;\ s_H(h_1)=s_H(h_2)=F$

(c) Find the (non-pure) mixed strategy Nash equilibria for this Bayesian Game.

Solution: For there to be a mixed strategy Nash equilibrium, from the above argument, W must be randomizing. Hence, W must be indifferent between her two actions. Let s_W be such that $\alpha_W(M;w) = \alpha$, i.e., let W play M with probability α. Then in state h_1,

$$BR_H(s_W;h_1) = \begin{cases} M & \text{if } \alpha > 2/3 \\ \{M,F\} & \text{if } \alpha = 2/3 \\ F & \text{if } \alpha < 2/3. \end{cases}$$

$$BR_H(s_W;h_2) = \begin{cases} M & \text{if } \alpha > 1/3 \\ \{M,F\} & \text{if } \alpha = 1/3 \\ F & \text{if } \alpha < 1/3. \end{cases}$$

If $0 < \alpha < 1/3$, the best response for H is to play F regardless of type, which further implies that the best reponse for W is $\alpha = 0$. Thus $\alpha \geq 1/3$.

If $2/3 < \alpha < 1$, the best response for H is to play M regardless of type, which further implies that the best reponse for W is $\alpha = 1$. Thus $\alpha \leq 2/3$.

If $1/3 < \alpha < 2/3$, the best response for H is to play F if his type is h_1 and M if his type is h_2. For W to be indifferent between her two actions given this strategy we require that $\theta = 2/3$. Hence, if $\theta = 2/3$, then $1/3 \leq \alpha_W(M;w_1) \leq 2/3$ and $s_H(h_1) = F$, $s_H(h_2) = M$ is a mixed strategy Nash equilibrium of this game.

If $\alpha = 1/3$, the best response for H is to play F if his type is h_1 and he is indifferent between M and F if his type is h_2. Denote $\alpha_H(M;h_2) = \beta$, i.e., let H play M with probability β when his type is h_2. For W to be indiffferent between her two actions given this strategy we require that $\beta = 1/(3(1-\theta))$. Hence, if $\theta < 2/3$, $\alpha_W(M;w_1) = 1/3$ and $s_H(h_1) = F$, $\alpha_H(M;h_2) = 1/(3(1-\theta))$ is a mixed strategy Nash equilibrium of this game.

If $\alpha = 2/3$, the best response for H is to play M if his type is h_2 and he is indifferent between M and F if his type is h_1. Denote $\alpha_H(M;h_1) = \gamma$, i.e., let H play M with probability γ when his type is h_1. For W to be indiffferent between her two actions given this strategy we require that $\gamma = (3\theta-2)/(3\theta)$. Hence, if $\theta > 2/3$, $\alpha_W(M;w_1) = 2/3$ and $\alpha_H(M;h_1) = (3\theta-2)/(3\theta)$, $s_H(h_2) = M$ is a mixed strategy Nash equilibrium of this game.

2. A buyer and a seller are bargaining over an object. The rules of bargaining are that they simultaneously announce prices. If $p_b \geq p_s$, then trade occurs at price $p = \frac{p_b+p_s}{2}$; if $p_b < p_s$, then no trade occurs. The buyer's valuation for the good is v_b, the seller's is v_s. These valuations are private information and are drawn from independent uniform distributions on $[0,1]$. If there is no trade, both players' utility are 0; if the buyer gets the good for price p, the buyer's utility is $v_b - p$ and the seller's utility is $p - v_s$.

   (a) Construct a 'one-price' Bayesian Nash equilibrium of this game: an equilibrium in which trade occurs at a single price if it occurs at all.

Solution: Denote the price at which trade occurs as x. The strategies for the players are

$$p_b^*(v_b) = \begin{cases} 0 & \text{if } v_b < x \\ x & \text{otherwise} \end{cases}; p_s^*(v_s) = \begin{cases} 1 & \text{if } v_s > x \\ x & \text{otherwise} \end{cases}$$

(b) Compare the efficiency of the equilibrium constructed in (a) and the 'linear' Bayesian Nash equilibrium constructed in class.

Solution: In (a), trade is efficient whenever $v_b \geq v_s$. The inefficiency in the first equilibrium is

$$\int_0^x \int_{v_s}^x v_b - v_s \,\mathrm{d}v_b \,\mathrm{d}v_s + \int_x^1 \int_{v_s}^1 v_b - v_s \,\mathrm{d}v_b \,\mathrm{d}v_s = \frac{3x^2 - 3x + 1}{6}.$$

Inefficiency is minimized when $x = 1/2$ and the inefficiency is $1/24$.
In the linear equilibrium, the inefficiency is

$$\int_0^{3/4} \int_{v_s}^{v_s+1/4} v_b - v_s \,\mathrm{d}v_b \,\mathrm{d}v_s + \int_{3/4}^1 \int_{v_s}^1 v_b - v_s \,\mathrm{d}v_b \,\mathrm{d}v_s = 5/192.$$

Hence, the linear equilibrium is more efficient.

(c) Use the Revelation Principle to construct a Bayesian game with an incentive-compatible equilibrium with the same outcome as the equilibrium in (a).

Solution: If $\tau_b \geq x$ and $\tau_s \leq x$, then trade; otherwise don't trade.

3. Each of two players receives a ticket on which there is a number in some finite subset S of the interval $[0, 1]$. The number on a player's ticket is the size of a prize that he may receive. The two prizes are identically and independently distributed, with distribution function F. Each player is asked independently and simultaneously whether he wants to exchange his prize for the other player's prize. If both players agree then the prizes are exchanged; otherwise each player receives his own prize. Each player's objective is to maximize his expected payoff.

(a) Model this situation as a Bayesian game.

Solution: In the Bayesian game there are two players, say $N = \{1, 2\}$, the set of states is $\Omega = S \times S$, the set of actions of each player is $\{Exchange, Don't\ exchange\}$, the signal function of each player i is defined by $\tau_i(s_1, s_2) = s_i$, and each player's belief on Ω is that generated by two independent copies of F. Each player's preferences are represented by the payoff function $u_i((X, Y), \omega) = \omega_j$ if $X = Y = Exchange$ and $u_i((X, Y), \omega) = \omega_i$ otherwise.

(b) Construct a Bayesian Nash equilibrium where the probability of exchange is zero.

Solution: $\sigma_i(s_i) = \textit{Don't exchange}$ for all $s_i \in S$.

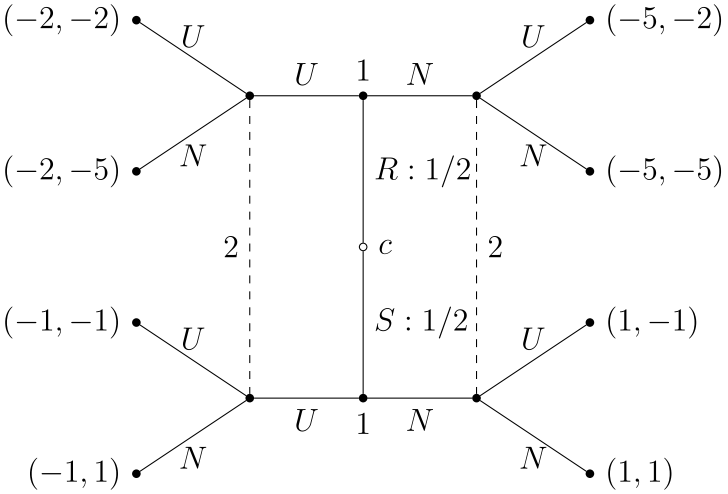
(c) Construct a Bayesian Nash equilibrium where the probability of exchange is positive.

Solution: Let x be the smallest possible prize and let M_i be the highest type of player i that chooses *Exchange*. If $M_i > x$ then it is optimal for type x of player j to choose *Exchange*. Thus if $M_i \geq M_j$ and $M_i > x$ then it is optimal for type M_i of player i to choose *Don't exchange*, since the expected value of the prizes of the types of player j that choose *Exchange* is less than M_i. Thus in any possible Nash equilibrium where there is exchange $M_i = M_j = x$: the only prizes that may be exchanged are the smallest.

$$\sigma_i(s_i) = \begin{cases} \textit{Don't exchange} & \text{if } s_i \in S \backslash \{x\} \\ \textit{Exchange} & \text{if } s_i = x. \end{cases}$$

4. Players 1 and 2 must decide whether or not to carry an umbrella when leaving home. They know that there is a 50-50 chance of rain. Each player's payoff is -5 if he doesn't carry an umbrella and it rains, -2 if he carries an umbrella and it rains, -1 if he carries an umbrella and it is sunny, and 1 if he doesn't carry an umbrella and it is sunny. Player 1 learns the weather before leaving home; player 2 does not, but he can observe player 1's action before choosing his own. Give the extensive and strategic forms of the game. Is it dominance solvable?

Solution: The extensive game is shown below, where N stands for no umbrella and U for umbrella. Player 1's payoffs are on the left.



Player 1 has 4 normal form strategies. Each strategy can be represented as an ordered pair (A, B), where the first element represents his choice if the weather is sunny and the second his choice if it is rainy. Player 2 also has 4 normal form strategies. Each is an ordered pair (A, B), where the first element represents his choice if he sees player 1 carry an umbrella and the second his choice if he sees player 1 not carry an umbrella. The strategic game is shown below. The payoffs are determined by calculating the outcome that results after each of nature's moves.

		Player 2			
		(U, U)	(U, N)	(N, U)	(N, N)
Player 1	(U, U)	$-1.5, -1.5$	$-1.5, -1.5$	$-1.5, -2$	$-1.5, -2$
	(U, N)	$-3, -1.5$	$-3, -3$	$-3, -0.5$	$-3, -2$
	(N, U)	$-0.5, -1.5$	$-0.5, -0.5$	$-0.5, -3$	$-0.5, -2$
	(N, N)	$-2, -1.5$	$-2, -2$	$-2, -1.5$	$-2, -2$

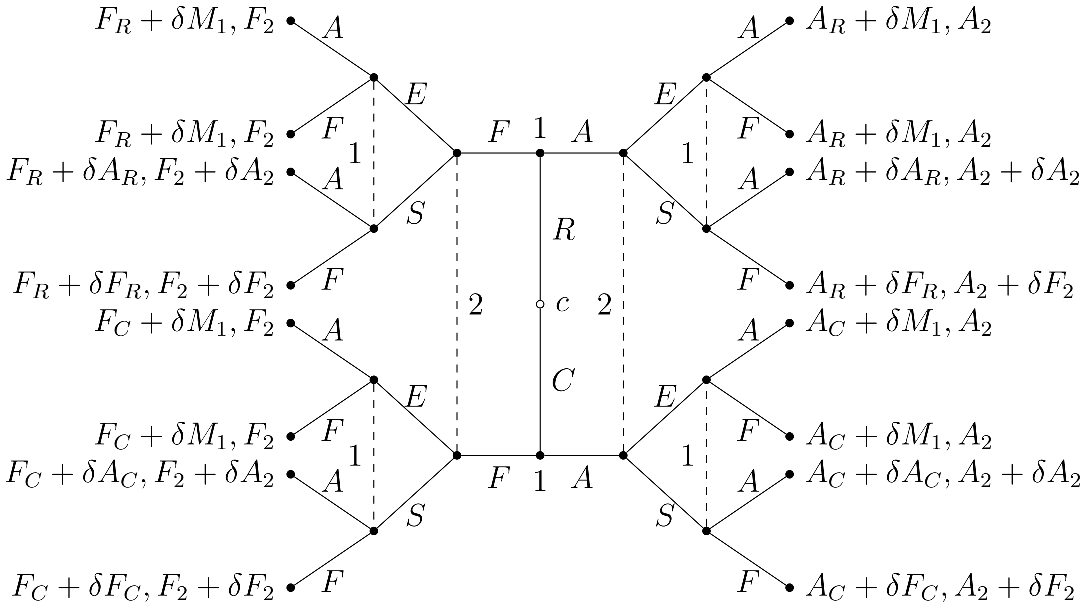
The game is dominance solvable. Player 1 has a single dominant strategy (N, U). Given that player 1 plays (N, U), player 2 has a unique dominant strategy (U, N).

5. There are two firms, firm 1 is the incumbent and firm 2. At the start of the game chance, c, chooses the type of player 1 from one of two possible states. With probability λ firm 1 is "rational", R, and with probability $1 - \lambda$ firm 1 is "crazy", C. The firms then interact in the market for two periods. In the first period, firm 1 takes one of two possible actions, fight F or accommodate, A. In the second period, firms simultaneously choose actions. Player 1 again chooses whether to fight or accommodate, while player 2 chooses one of two possible actions, stay, S, or exit, E.

   In each period, firm profits are realized and firms discount the second period profits by the common discount factor δ. If both firms operate in the market then a rational firm 1 makes A_R if it accommodates and F_R if it fights, while a crazy firm 1 makes A_C if it accommodates and F_C if it fights. If only firm 1 operates in the market it makes monopoly profit M_1. Player 2 makes A_2 if he stays and player 1 accommodates, F_2 if if he stays and player one fights and 0 if he exits. Assume that $M_1 > A_R > F_R$, $F_C > M_1 > A_C$ and $A_2 > 0 > F_2$.

   (a) Write down this problem as an extensive game of incomplete information.

   Solution:



Note that we can simplify this game by omitting the period one payoffs to the entrant since his payoffs in this period do not depend on his actions.

(b) Find parameter values for the payoffs for which there exists separating perfect Bayesian equilibria in this game.

Solution: Since $F_C > M_1 > A_C$, a firm of type crazy always chooses F. Furthermore, separating equilibrium requires the two types of firm 1 choose two different actions in period 1, thus, for firm 2, $\mu_1(R)(A) = 1$ and $\mu_1(R)(F) = 0$. Hence, the parameter values must satisfy $(1+\delta)A_R \geq F_R + \delta M_1$.

(c) Find parameter values for the payoffs for which there exist pooling perfect Bayesian equilibria in this game.

Solution: Since a pooling equilibrium requires the two types of firm 1 choose the same actions in period 1 (and the crazy firm always fights), firm 2 beliefs along the equilibrium path must be $\mu_1(R)(F) = \lambda$. If firm 2 chooses S in period 2, the rational type firm 1 will choose A in period 2. Hence the parameter values must satisfy $\lambda A_2 + (1-\lambda)F_2 \leq 0$. In addition, it must be worthwhile for the rational firm to fight in the first period rather than accommodate in both periods, i.e., $(1+\delta)A_R \leq F_R + \delta M_1$. Sufficient off-the-equilibrium-path beliefs that support such an equilibrium are $\mu_1(R)(A) = 1$.

(d) Find parameter values for the payoffs for which there exist hybrid perfect Bayesian equilibria in this game.

Solution: In the hybrid equilibrium, the rational type firm randomizes between A

and F, thus such a firm type must be indifferent between the two choices. Denote such a randomization by γ. For firm 2, $\mu(R)(A) = 1$ and $\mu(R)(F) = \frac{\lambda\gamma}{\lambda\gamma+(1-\lambda)} < \lambda$. To ensure that a rational firm 1 type is indifferent between fighting and accommodating in period 1, firm 2 randomizes between S and E. Denote the probability that firm 2 stays by α. The mixed strategy of a rational firm 1 type ensures that

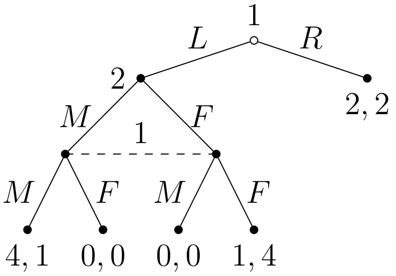
$$\frac{\lambda\gamma}{\lambda\gamma + (1-\lambda)} A_2 + \frac{1-\lambda}{\lambda\gamma + (1-\lambda)} F_2 = 0.$$

The mixed strategy of firm 2 ensures that

$$F_R + \delta[(1-\alpha)M_1 + \alpha A_R] = (1+\delta)A_R.$$

The requirement on the parameters are that the weights γ and α required to make these players indifferent are indeed probabilities. Hence, we require that $\lambda A_2 + (1-\lambda)F_2 \geq 0$ (player 1 can randomize appropriately to ensure player 2 is indifferent), and $(1+\delta)A_R < F_R + \delta M_1$ (player 2 can randomize to ensure player 1 is indifferent).

6. Consider the following extensive game:



(a) Solve for the set of Nash equilibria.

Solution:
$\beta_1(\varnothing) = L, \beta_1(\{LM, LF\}) = M, \beta_2(L) = M$
$\beta_1(\varnothing) = R, 0 \leq \beta_1(\{LM, LF\})(M) \leq 1, 0 \leq \beta_2(L)(M) \leq 1/2.$

(b) Solve for the set of subgame perfect Nash equilibria.

Solution:
$\beta_1(\varnothing) = L, \beta_1(\{LM, LF\}) = M, \beta_2(L) = M$
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\})(M) = 0, \beta_2(L)(M) = 0.$
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\})(M) = 4/5, \beta_2(L)(M) = 1/5.$

(c) Solve for the set of weak perfect Bayesian equilibria.

Solution:
$\beta_1(\varnothing) = L, \beta_1(\{LM, LF\}) = M, \beta_2(L) = M, \mu(\{LM, LF\})(LM) = 1$
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\}) = F, \beta_2(L) = F$, $0 \leq \mu(\{LM, LF\})(LM) < 1/5$.
$\beta_1(\varnothing) = R, 0 \leq \beta_1(\{LM, LF\})(M) \leq 4/5, \beta_2(L) = F$, $\mu(\{LM, LF\})(LM) = 1/5$.
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\})(M) = 4/5, 0 < \beta_2(L)(M) \leq 1/2$, $\mu(\{LM, LF\})(LM) = 1/5$.

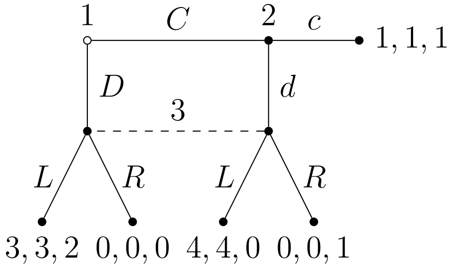
(d) Solve for the set of perfect Bayesian equilibria.

Solution:
$\beta_1(\varnothing) = L, \beta_1(\{LM, LF\}) = M, \beta_2(L) = M$, $\mu(\{LM, LF\})(LM) = 1$
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\}) = F, \beta_2(L) = F$, $\mu(\{LM, LF\})(LM) = 0$.
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\})(M) = 4/5, \beta_2(L)(M) = 1/5$, $\mu(\{LM, LF\})(LM) = 1/5$.

(e) Solve for the set of sequential equilibria.

Solution:
$\beta_1(\varnothing) = L, \beta_1(\{LM, LF\}) = M, \beta_2(L) = M, \mu(\{LM, LF\})(LM) = 1$
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\}) = F, \beta_2(L) = F, \mu(\{LM, LF\})(LM) = 0$.
$\beta_1(\varnothing) = R, \beta_1(\{LM, LF\})(M) = 4/5, \beta_2(L)(M) = 1/5$, $\mu(\{LM, LF\})(LM) = 1/5$.

7. Consider the following extensive game:



(a) Solve for the set of Nash equilibria.

Solution:
$\beta_1(\varnothing)(D) = 1, 1/3 \leq \beta_2(C)(c) \leq 1, \beta_3(\{D, Cd\})(L) = 1$
$\beta_1(\varnothing)(D) = 0, \beta_2(C)(c) = 1, 0 \leq \beta_3(\{D, Cd\})(L) \leq 1/4$.

(b) Solve for the set of subgame perfect Nash equilibria.

Solution: SPE=NE

(c) Solve for the set of weak perfect Bayesian equilibria.

Solution:
$\beta_1(\varnothing)(D) = 0, \beta_2(C)(c) = 1, \beta_3(\{D, Cd\})(L) = 0,\ 0 \leq \mu(\{D, Cd\})(D) < 1/3.$
$\beta_1(\varnothing)(D) = 0, \beta_2(C)(c) = 1, 0 \leq \beta_3(\{D, Cd\})(L) \leq 1/4,\ \mu(\{D, Cd\})(D) = 1/3.$

(d) Solve for the set of perfect Bayesian equilibria.

Solution: PBE=WPBE

(e) Solve for the set of sequential equilibria.

Solution:
$\beta_1(\varnothing)(D) = 0, \beta_2(C)(c) = 1, \beta_3(\{D, Cd\})(L) = 0,\ 0 \leq \mu(\{D, Cd\})(D) < 1/3.$
$\beta_1(\varnothing)(D) = 0, \beta_2(C)(c) = 1, 0 \leq \beta_3(\{D, Cd\})(L) \leq 1/4,\ \mu(\{D, Cd\})(D) = 1/3.$