

Homework 2 solution
Econometrics II Spring, 2013

3.1 Answer

$$(a) \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \arg \min_{\beta} \sum_{t=1}^n (Y_t - \beta_0 - X_{1t}\beta_1)^2$$

FOC:

$$\frac{\partial \sum_{t=1}^n (Y_t - \beta_0 - X_{1t}\beta_1)^2}{\partial \beta_1} = -2 \sum_{t=1}^n (Y_t - \hat{\beta}_0 - X_{1t}\hat{\beta}_1)X_{1t} = 0 \quad (1)$$

$$\frac{\partial \sum_{t=1}^n (Y_t - \beta_0 - X_{1t}\beta_1)^2}{\partial \beta_0} = -2 \sum_{t=1}^n (Y_t - \hat{\beta}_0 - X_{1t}\hat{\beta}_1) = 0 \quad (2)$$

From (2),

$$n\hat{\beta}_0 = \sum_{t=1}^n Y_t - \hat{\beta}_1 \sum_{t=1}^n X_{1t}$$

Hence,

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}_1 \quad (3)$$

Plug (3) into (1), we get

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{t=1}^n (Y_t - \bar{Y})X_{1t}}{\sum_{t=1}^n (X_{1t} - \bar{X}_1)X_{1t}} \\ &= \frac{\sum_{t=1}^n (Y_t - \bar{Y})(X_{1t} - \bar{X}_1)}{\sum_{t=1}^n (X_{1t} - \bar{X}_1)^2} \\ &= \frac{\sum_{t=1}^n (X_{1t} - \bar{X}_1)Y_t}{\sum_{t=1}^n (X_{1t} - \bar{X}_1)^2} \\ &= \sum_{t=1}^n C_t Y_t \end{aligned}$$

(b)

$$\begin{aligned}
\text{var}(\hat{\beta}_1|X) &= \text{var}\left(\sum_{t=1}^n C_t Y_t | X\right) \stackrel{\text{independent}}{=} \sum_{t=1}^n C_t^2 \text{var}(Y_t | X) \\
&= \sigma_\varepsilon^2 \sum_{t=1}^n C_t^2 = \sigma_\varepsilon^2 \sum_{t=1}^n \left[\frac{(X_{1t} - \bar{X}_1)}{\sum_{t=1}^n (X_{1t} - \bar{X}_1)^2} \right]^2 \\
&= \frac{\sigma_\varepsilon^2}{(n-1)S_{X_1}^2}
\end{aligned}$$

(c)

$$\begin{aligned}
R^2 &= \frac{\sum (\hat{Y}_t - \bar{Y})^2}{\sum (Y_t - \bar{Y})^2} \\
&= \frac{\sum (\hat{Y}_t - \bar{Y})(Y_t - \bar{Y})}{\sum (Y_t - \bar{Y})^2} \quad (\text{cf. solution to quiz 2}) \\
&= \frac{\sum (\hat{\beta}_0 + X_{1t}\hat{\beta}_1 - \hat{\beta}_0 - \hat{\beta}_1\bar{X}_1)(Y_t - \bar{Y})}{\sum (Y_t - \bar{Y})^2} \\
&= \frac{\hat{\beta}_1 \sum (X_{1t} - \bar{X}_1)(Y_t - \bar{Y})}{\sum (Y_t - \bar{Y})^2} \\
&= \frac{\hat{\beta}_1 [\sum (X_{1t} - \bar{X}_1)(Y_t - \bar{Y})]^2}{\sum (Y_t - \bar{Y})^2 [\sum (X_{1t} - \bar{X}_1)(Y_t - \bar{Y})]} \\
&= \hat{\rho}^2
\end{aligned}$$

where we use the fact that

$$\begin{aligned}
\sum (X_{1t} - \bar{X}_1)(Y_t - \bar{Y}) &= \sum (X_{1t} - \bar{X}_1)(X_{1t}\hat{\beta}_1 + e_t - \hat{\beta}_1\bar{X}_1) \\
&= \hat{\beta}_1 \sum (X_{1t} - \bar{X}_1)^2
\end{aligned}$$

3.4 Answer

The model is

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \varepsilon_t$$

Let $X = (\mathbf{1}, X_2, X_1)$, where $X_i = (X_{i1}, X_{i2}, \dots, X_{iN})^\top$ for $i = 1, 2$, and $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_2, \hat{\beta}_1)^\top$, then $\text{var}(\hat{\beta}|X) = \sigma^2(X^\top X)^{-1}$. To compute $\text{var}(\hat{\beta}_1|X)$, we are interested in the lower right element in $\sigma^2(X^\top X)^{-1}$.

$$X^\top X = \begin{pmatrix} \mathbf{1}^\top \\ X_2^\top \\ X_1^\top \end{pmatrix} (\mathbf{1}, X_2, X_1) = \begin{pmatrix} N & N\bar{X}_2 & N\bar{X}_1 \\ N\bar{X}_2 & X_2^\top X_2 & X_2^\top X_1 \\ N\bar{X}_1 & X_1^\top X_2 & X_1^\top X_1 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

$$\text{where } I_{11} = \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\top X_2 \end{pmatrix}, I_{12} = \begin{pmatrix} N\bar{X}_1 \\ X_2^\top X_1 \end{pmatrix} = I_{21}^\top, I_{22} = X_1^\top X_1.$$

So the lower right element in $(X^\top X)^{-1}$, denoted as I^{22} , is $(I_{22} - I_{21}I_{11}^{-1}I_{12})^{-1}$.

$$\begin{aligned} I_{22} - I_{21}I_{11}^{-1}I_{12} &= [X_1^\top X_1 - (N\bar{X}_1, X_1^\top X_2) \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\top X_2 \end{pmatrix}^{-1} \begin{pmatrix} N\bar{X}_1 \\ X_2^\top X_1 \end{pmatrix}] \\ &= [X_1^\top X_1 - X_1^\top (\mathbf{1}, X_2) \begin{pmatrix} N & N\bar{X}_2 \\ N\bar{X}_2 & X_2^\top X_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1} \\ X_2^\top \end{pmatrix} X_1] \\ &= X_1^\top \{I - (\mathbf{1}, X_2) [\begin{pmatrix} \mathbf{1} \\ X_2^\top \end{pmatrix} (\mathbf{1}, X_2)]^{-1} \begin{pmatrix} \mathbf{1} \\ X_2^\top \end{pmatrix}\} X_1 \\ &= X_1^\top M X_1 \end{aligned}$$

where M is the residual maker in the regression of which the regressor matrix is $(\mathbf{1}, X_2)$.

Note that, in the regression of X_1 on $(\mathbf{1}, X_2)$,

$$R^2 = 1 - \frac{X_1^\top M X_1}{\sum (X_{1t} - \bar{X}_1)^2}$$

Hence

$$X_1^\top M X_1 = (1 - R^2) \sum (X_{1t} - \bar{X}_1)^2 \quad (4)$$

And, using (c) in the previous problem, we have

$$R^2 = \hat{r}^2 \quad (5)$$

Combining (4) and (5), we get

$$\text{var}(\hat{\beta}_1|X) = \frac{\sigma^2}{(1 - \hat{r}^2) \sum (X_{1t} - \bar{X}_1)^2}$$

Following the same steps, you can get $\text{var}(\hat{\beta}_2|X)$.

3.5 Answer

The idea is similar to the previous problem. Firstly, we rearrange $X = (X_1, X_2, \dots, X_N)^\top$ as (X_{-j}, X^j) , where X_{-j} is the matrix which is obtained by deleting the j -th column of X and X^j is the j -th column of X . Also, rearrange $\hat{\beta}$ accordingly. We still use X and $\hat{\beta}$ to denote the new regressors matrix and the coefficients vector. Then

$$\begin{aligned} \text{var}(\hat{\beta}|X) &= \sigma^2 (X^\top X)^{-1} \\ &= \sigma^2 [(X_{-j}, X^j)^\top (X_{-j}, X^j)]^{-1} \\ &= \sigma^2 \begin{pmatrix} X_{-j}^\top X_{-j} & X_{-j}^\top X^j \\ X^{j^\top} X_{-j} & X^{j^\top} X^j \end{pmatrix}^{-1} \end{aligned}$$

We are interested in the lower right element in $\begin{pmatrix} X_{-j}^\top X_{-j} & X_{-j}^\top X^j \\ X^{j^\top} X_{-j} & X^{j^\top} X^j \end{pmatrix}^{-1} \equiv \begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix}$.

$$\begin{aligned} I^{22} &= [X^{j^\top} X^j - X^{j^\top} X_{-j} (X_{-j}^\top X_{-j})^{-1} X_{-j}^\top X^j]^{-1} \\ &= [X^{j^\top} (I - X_{-j} (X_{-j}^\top X_{-j})^{-1} X_{-j}^\top) X^j]^{-1} \\ &= \frac{1}{X^{j^\top} M_{-j} X^j} \end{aligned}$$

where M_{-j} is the residual maker in the regression of which the regressor matrix is X_{-j} .

Now, consider the regression of X^j on X_{-j} .

$$R_j^2 = 1 - \frac{X^{j^\top} M_{-j} X^j}{\sum (X_t^j - \bar{X}^j)^2}$$

Hence,

$$\text{var}(\hat{\beta}_j|X) = \frac{\sigma^2}{X^{j^\top} M_{-j} X^j} = \frac{\sigma^2}{(1 - R_j^2) \sum (X_{jt} - \bar{X}_j)^2}$$