MIDTERM EXAMNATION OF ECONOMETRIC II

YUE CHAOYUN

Problem 1. In the model

$$Y = X_1\beta_1 + X_2\beta_2 + \epsilon$$

show that the least squares estimator of $\beta = (\beta_1, \beta_2)$ subject to the contraint that $\beta_1 = c$ (where c is some given vector) is simply the OLS regression of $Y - X_1 c$ on X_2.

Proof. Denote $X = (X_1,\ X_2)$, then

$$\begin{aligned}\hat{\beta} &= \begin{pmatrix} c \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} X_1Y \\ X_2Y \end{pmatrix} \\ &= \begin{pmatrix} E^{-1} & -E^{-1}X_1'X_2\left(X_2'X_2\right)^{-1} \\ -\left(X_2'X_2\right)^{-1}X_2'X_1E^{-1} & \left(X_2'X_2\right)^{-1} + \left(X_2X_2\right)^{-1}X_2X_1E^{-1}X_1X_2\left(X_2'X_2\right)^{-1} \end{pmatrix} \begin{pmatrix} X_1Y \\ X_2Y \end{pmatrix}\end{aligned}$$

where $E = \left(X_1'X_1\right)^{-1} - X_1'X_2\left(X_2'X_2\right)^{-1}X_2'X_1$. So we have

$$c = E^{-1}X_1Y - E^{-1}X_1'X_2\left(X_2'X_2\right)^{-1}X_2'Y$$

and

$$\begin{aligned}\hat{\beta}_2 &= \left(X_2'X_2\right)^{-1}X_2Y + \left(X_2'X_2\right)^{-1}X_2X_1E^{-1}X_1X_2\left(X_2'X_2\right)^{-1}X_2Y - \left(X_2'X_2\right)^{-1}X_2'X_1E^{-1}X_1Y \\ &= \left(X_2'X_2\right)^{-1}X_2\left(Y + X_1E^{-1}X_1X_2\left(X_2'X_2\right)^{-1}X_2Y - X_1E^{-1}X_1Y\right) \\ &= \left(X_2'X_2\right)^{-1}X_2\left(Y - X_1c\right)\end{aligned}$$

This show that $\hat{\beta}$ is simply the OLS regression of $Y - X_1c$ on X_2. □

Problem 2. Let $T_n = \hat{\theta}_n - \theta$ where $\hat{\theta}_n$ is an estimate of a parameter of interest. Efron's bootstrapping confidence interval is constructed by

$$C = \left[\hat{\theta}_n + q_n^*\left(\alpha/2\right),\ \hat{\theta}_n + q_n^*\left(1-\alpha/2\right)\right].$$

q_n^* is the sample quantile of the bootstrapping samples $\{T_{n1}^*, ..., T_{nB}^*\}$ where $T_n^* = \hat{\theta}_n^* - \hat{\theta}_n$. Show the asymptotic coverage probablity of C.

Since $q_n^* \xrightarrow{p} q_n$, then $C \xrightarrow{p} C^0 = \left[\hat{\theta}_n + q_n\left(\frac{\alpha}{2}\right),\ \hat{\theta}_n + q_n\left(1-\frac{\alpha}{2}\right)\right]$. The latter has converage problity

$$\begin{aligned}\mathbb{P}\left(\theta_0 \in C^0\right) &= \mathbb{P}\left(\hat{\theta}_n + q_n\left(\frac{\alpha}{2}\right) \le \theta_0 \le \hat{\theta}_n + q_n\left(1-\frac{\alpha}{2}\right)\right) \\ &= \mathbb{P}\left(-q_n\left(1-\frac{\alpha}{2}\right) \le \hat{\theta}_n - \theta_0 \le -q_n\left(\frac{\alpha}{2}\right)\right) \\ &= G_n\left(-q_n\left(\frac{\alpha}{2}\right),\ F_0\right) - G_n\left(-q_n\left(1-\frac{\alpha}{2}\right),\ F_0\right)\end{aligned}$$

which generally is not $1-\alpha$.
However, if $\hat{\theta}_n - \theta_0$ has a symmetric distribution, then $G_n\left(-q_n\left(\frac{\alpha}{2}\right), F_0\right) = 1 - G_n\left(q_n\left(\frac{\alpha}{2}\right), F_0\right)$, so

$$\begin{aligned}\mathbb{P}\left(\theta_0 \in C^0\right) &= \left(1-\frac{\alpha}{2}\right) - \left(1-\left(1-\frac{\alpha}{2}\right)\right) \\ &= 1-\alpha\end{aligned}$$

Problem 3. Consider the following model

$$y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + \mu_i,\ E\left(x_i\mu_i\right) = 0$$

Drive a t-statistic when $\mathbb{H}_0:\ \frac{\beta_1}{\beta_2} = r$, where r is a constant.

(1) Set $\hat{\theta} = \frac{\hat{\beta}_1}{\hat{\beta}_2}$. Define $\hat{H}_1 = \left(0, \frac{1}{\hat{\beta}_2}, -\frac{\hat{\beta}_1}{\hat{\beta}_2}\right)'$. So that the standard error of $\hat{\theta}$ is $s\left(\hat{\theta}\right) = \left(n^{-1}\hat{H}_1'\hat{V}\hat{H}_1\right)^{\frac{1}{2}}$, where $\hat{V} = \frac{\sum_{i=1}^{n}\left(y_i - \mathbf{x}_i'\hat{\beta}\right)^2}{n-3} \cdot \left(\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'\right)^{-1}$ is the variance-covariance matrix estimator. Therefore, A t-statistic for $\mathbb{H}_0$is

$$t_{1n} = \frac{\frac{\hat{\beta}_1}{\hat{\beta}_2} - r}{s\left(\hat{\theta}\right)} \sim T_{\alpha/2}\left(n-3\right).$$

(2) Reformulating $\mathbb{H}_0:\ \beta_1 - r\beta_2 = 0$, then A t-statistic is

$$t_{2n} = \frac{\hat{\beta}_1 - r\hat{\beta}_2}{\left(n^{-1}H_2'\hat{V}H_2\right)^{1/2}} \sim T_{\alpha/2}\left(n-3\right).$$

where $H_2 = (0, 1, -r)'$.

Problem 4. Define $h_t = \mathbf{x}_t\left(X'X\right)^{-1}\mathbf{x}_t'$, which is the t^{th}diagonal element of the matrix P_X, and X is a $n \times k$ matrix.

i. Prove that $0 \le h_t \le 1$.

ii. Prove that $\sum_{t=1}^{n} h_t = k$.

(1) since P_X is symetric positive matrix, then

$$\exists P\ s.t.\ P_X = P'\Lambda P\ and\ P'P = I$$

where $\Lambda = diag\left(\lambda_1, ..., \lambda_n\right)$ λ_i is the eigenvalue of P_X. we have

$$0 \le h_t = e_t'P_Xe_t = e_t'P'\Lambda Pe_t \le \lambda_{max}\left(P_X\right)e_tP'IPe_t = \lambda_{max}\left(P_X\right) = 1$$

where the last equality comes from

$$\lambda_{max}\left(P_X\right) = \max_x \frac{x'P_Xx}{x'x} = 1$$

or $\because P_X$ is idempotent, $\Lambda = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ must be satisfied.

or

$$\begin{aligned} 0 &\le h_t = e_t'X\left(X'X\right)^{-1}X'e_t \\ &\le \lambda_{max}\left(\left(X'X\right)^{-1}\right)e_t'XX'e_t \\ &\le \lambda_{max}^{-1}\left(X'X\right)\lambda_{max}\left(X'X\right)e_t'e_t \\ &= 1 \end{aligned}$$

by lemma $X'AX \le \lambda_{max}\left(A\right)X'X$

(2) $\sum_{t=1}^{n} h_t = tr\left(P_X\right) = tr\left(X\left(X'X\right)^{-1}X'\right) = tr\left(X'X\left(X'X\right)^{-1}\right) = tr\left(I_k\right) = k$

Problem 5. Consider the following model

$$Y = X\beta + \epsilon$$

where Y and ϵ are of dimension $N \times 1$ and X is of dimension $N \times K$, and rank K. The assumption of heteroskedastic Y can be written as

$$E\left[\left(Y - X\beta\right)\left(Y - X\beta\right)'\right] = E\left[\epsilon\epsilon'\right] = \sigma^2\Lambda = V$$

where $V = diag\left(\sigma_1^2, ..., \sigma_N^2\right) = \sigma^2\Lambda$.

(1) Find the GLS estimate of β.
(2) Find the var-cov. matrix of β.

We know work in terms of the general notation

$$\sigma_i^2 = \sigma^2 h_i(\alpha) = \sigma^2 h(z_i'\alpha) = \sigma^2 \Lambda$$

where α is an $s \times 1$ vector of unkown paramaters, and $h_i(\cdot)$ is a differentiable function of those parameters, and an $s \times 1$vector z_i which can be identical to or different from x_i. The loglikelihood function can be written as

$$\begin{aligned} L &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2}\sum_{t=1}^{n}\ln(h_i(\alpha)) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\frac{(y_i - \boldsymbol{x}_i'\beta)^2}{h_i(\alpha)} \\ &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2}\ln|\Lambda| - \frac{1}{2\sigma^2}(Y - X\beta)'\Lambda^{-1}(Y - X\beta) \end{aligned}$$

(3) Find MlE $\hat{\beta}$ and $\hat{\sigma}^2$ conditional on knowing α.

Substituting your results in (3) into L yields the concentrated loglikelihood function

$$L^*(\alpha) = constant - \frac{n}{2}\ln\left[\left(Y - X\hat{\beta}(\alpha)\right)'\Lambda^{-1}\left(Y - X\hat{\beta}(\alpha)\right)\right] - \frac{1}{2}\ln|\Lambda|$$

(4) Show that the information matrix is given by

$$I(\beta,\alpha,\sigma^2) = \begin{bmatrix} \frac{x'\Lambda^{-1}x}{\sigma^2} & 0 & 0 \\ 0 & \frac{1}{2}\sum_{i=1}^{n}\frac{1}{h_i^2(\alpha)}\cdot\frac{\partial h_i}{\partial\alpha}\frac{\partial h_i}{\partial\alpha'} & \frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{1}{h_i(\alpha)}\cdot\frac{\partial h_i}{\partial\alpha} \\ 0 & \frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{1}{h_i(\alpha)}\cdot\frac{\partial h_i}{\partial\alpha} & \frac{n}{2\sigma^4} \end{bmatrix}$$

Consider a possible specification

$$h_i(\alpha) = \exp(\alpha_1 z_{1i} + ... + \alpha_s z_{si}).$$

Note that $h_i(0)$ implies a model with homoskedastic errors. A test for heteroskedasticity can be formulated in terms of the hypothese $\mathbb{H}_0: \ \alpha = 0 \ v.s. \ \mathbb{H}_1: \ \alpha \neq 0$.
(5) Construct a likelihood ratio test.
(6) Construct a wald test.
(7) Construct a LM test.

(1) First let $Y = X\beta + \epsilon$be left-multiplied by $(\sigma\Lambda)^{-1/2}$, then

$$Y^* = X^*\beta + \epsilon^*$$

where $Y^* = (\sigma\Lambda)^{-1/2} Y, \ X^* = (\sigma\Lambda)^{-1/2} \ and \ \epsilon^* = (\sigma\Lambda)^{-1/2}$. So

$$\begin{aligned} \hat{\beta}_{GLS} &= \left(X^{*'}X^*\right)^{-1}X^{*'}Y^* \\ &= \left(X'\Lambda^{-1}X\right)^{-1}X'\Lambda^{-1}Y \\ &= \left(\sum_{i=1}^{n}\frac{\boldsymbol{x}_i\boldsymbol{x}_i'}{\lambda_i}\right)^{-1}\left(\sum_{i=1}^{n}\frac{\boldsymbol{x}_i y_i}{\lambda_i}\right) \end{aligned}$$

which is BLUE since $E\left(\epsilon^*\epsilon^{*'}\right) = I$ within model $Y^* = X^*\beta + \epsilon^*$.

(2) $Var\left(\hat{\beta}_{GLS}\right) = \left(X^{*'}X^*\right)^{-1} = \left(X'V^{-1}X\right)^{-1} = \sigma^2\left(X'\Lambda^{-1}X\right)^{-1} = \sigma^2\left(\sum_{i=1}^{n}\frac{\mathbf{x}_i\mathbf{x}_i'}{\lambda_i}\right)^{-1}$

(3) from the F.o.c.

$$\begin{aligned}
\frac{\partial L}{\partial \beta}\Big|_{\hat{\beta}} = 0 \quad &: \quad -\frac{1}{2\hat{\sigma}^2}\left(-2X'\Lambda^{-1}Y + 2X'\Lambda^{-1}X\hat{\beta}\right) = 0 \\
and\ \frac{\partial L}{\partial \sigma^2}\Big|_{\hat{\sigma}^2} = 0 \quad &: \quad -\frac{n}{2\hat{\sigma}^2} + \frac{(Y - X\beta)'\,\Lambda^{-1}\,(Y - X\beta)}{2\hat{\sigma}^4} = 0
\end{aligned}$$

we have

$$\begin{aligned}
\hat{\beta}_{MLE} \quad (\alpha) &= \left(X'\Lambda^{-1}X\right)^{-1} X'\Lambda^{-1}Y \\
and\ \hat{\sigma}^2 \quad (\alpha) &= \frac{1}{n}\left(Y - X\beta\left(\alpha\right)\right)'\Lambda^{-1}\left(Y - X\beta\left(\alpha\right)\right)
\end{aligned}$$

(4) from

$$l_i = -\frac{1}{2}\ln\left(2\pi\right) - \frac{1}{2}\ln\left(\sigma^2\right) - \frac{1}{2}\ln\left(h_i\left(\alpha\right)\right) - \frac{1}{2\sigma^2}\cdot\frac{\left(y_i - \boldsymbol{x}_i'\beta\right)^2}{h_i\left(\alpha\right)}$$

we have

$$\begin{aligned}
\frac{\partial l_i}{\partial \beta} &= \frac{1}{\sigma^2}\frac{\boldsymbol{x}_i\epsilon_i}{h_i} \\
\frac{\partial l_i}{\partial \alpha} &= -\frac{1}{2h_i}\frac{\partial h_i}{\partial \alpha} + \frac{1}{2\sigma^2}\frac{\epsilon_i^2}{h_i^2}\frac{\partial h_i}{\partial \alpha} \\
\frac{\partial l_i}{\partial \sigma^2} &= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}\frac{\epsilon_i^2}{h_i}
\end{aligned}$$

where $\epsilon_i = y_i - \boldsymbol{x}_i'\hat{\beta}$ and the second equality come true for $\beta\ is\ not\ a\ function\ of\ \alpha$.

$$\begin{aligned}
\frac{\partial^2 l_i}{\partial \beta \partial \beta'} &= -\frac{1}{\sigma^2}\frac{\boldsymbol{x}_i\boldsymbol{x}_i'}{h_i} \\
\frac{\partial^2 l_i}{\partial \beta \partial \alpha'} &= -\frac{1}{\sigma^2}\frac{\boldsymbol{x}_i\epsilon_i}{h_i^2\left(\alpha\right)}\frac{\partial h_i}{\partial \alpha'} \\
\frac{\partial^2 l_i}{\partial \beta \partial \sigma^2} &= -\frac{1}{\sigma^4}\frac{\boldsymbol{x}_i\epsilon_i}{h_i} \\
\frac{\partial^2 l_i}{\partial \alpha \partial \alpha'} &= \frac{1}{2h_i^2}\frac{\partial h_i}{\partial \alpha}\frac{\partial h_i}{\partial \alpha'} - \frac{1}{2h_i}\frac{\partial^2 h_i}{\partial \alpha \partial \alpha'} - \frac{\epsilon_i^2}{\sigma^2}\frac{1}{h_i^3}\frac{\partial h_i}{\partial \alpha}\frac{\partial h_i}{\partial \alpha'} + \frac{\epsilon_i^2}{2\sigma^2}\frac{1}{h_i^2}\frac{\partial^2 h_i}{\partial \alpha \partial \alpha'} \\
\frac{\partial^2 l_i}{\partial \alpha \partial \sigma^2} &= -\frac{\epsilon_i^2}{2\sigma^4}\frac{1}{h_i^2}\frac{\partial h_i}{\partial \alpha} \\
\frac{\partial^2 l_i}{\left(\partial \sigma^2\right)^2} &= \frac{1}{2\sigma^4} - \frac{\epsilon_i^2}{\sigma^6 h_i}
\end{aligned}$$

since $E[\epsilon_i]=0$ and $E[\epsilon_i^2]=\sigma^2 h_i(\alpha)$, then we have

$$
\begin{aligned}
E\left[\frac{\partial^2 l_i}{\partial\beta\partial\beta'}\right] &= -\frac{1}{\sigma^2}\frac{\boldsymbol{x}_i\boldsymbol{x}_i'}{h_i}\\
E\left[\frac{\partial^2 l_i}{\partial\beta\partial\alpha'}\right] &= -\frac{1}{\sigma^2}\frac{\boldsymbol{x}_i E[\epsilon_i]}{h_i^2(\alpha)}\frac{\partial h_i}{\partial\alpha'}=0\\
E\left[\frac{\partial^2 l_i}{\partial\beta\partial\sigma^2}\right] &= -\frac{1}{\sigma^4}\frac{\boldsymbol{x}_i E[\epsilon_i]}{h_i}=0\\
E\left[\frac{\partial^2 l_i}{\partial\alpha\partial\alpha'}\right] &= \frac{1}{2h_i^2}\frac{\partial h_i}{\partial\alpha}\frac{\partial h_i}{\partial\alpha'}-\frac{1}{2h_i}\frac{\partial^2 h_i}{\partial\alpha\partial\alpha'}-\frac{E[\epsilon_i^2]}{\sigma^2}\frac{1}{h_i^3}\frac{\partial h_i}{\partial\alpha}\frac{\partial h_i}{\partial\alpha'}+\frac{E[\epsilon_i^2]}{2\sigma^2}\frac{1}{h_i^2}\frac{\partial^2 h_i}{\partial\alpha\partial\alpha'}=-\frac{1}{2h_i^2}\frac{\partial h_i}{\partial\alpha}\frac{\partial h_i}{\partial\alpha'}\\
E\left[\frac{\partial^2 l_i}{\partial\alpha\partial\sigma^2}\right] &= -\frac{E[\epsilon_i^2]}{2\sigma^4}\frac{1}{h_i^2}\frac{\partial h_i}{\partial\alpha}=-\frac{1}{2\sigma^2 h_i}\frac{\partial h_i}{\partial\alpha}\\
E\left[\frac{\partial^2 l_i}{(\partial\sigma^2)^2}\right] &= \frac{1}{2\sigma^4}-\frac{E[\epsilon_i^2]}{\sigma^6 h_i}=-\frac{1}{2\sigma^4}
\end{aligned}
$$

by Yang theorem

$$
\begin{aligned}
I(\beta,\alpha,\sigma^2) &= -E\begin{bmatrix}\frac{\partial^2 L}{\partial\beta\partial\beta'} & \frac{\partial^2 L}{\partial\beta\partial\alpha'} & \frac{\partial^2 L}{\partial\beta\partial\sigma^2}\\ \frac{\partial^2 L}{\partial\alpha\partial\beta'} & \frac{\partial^2 L}{\partial\alpha\partial\alpha'} & \frac{\partial^2 L}{\partial\alpha\partial\sigma^2}\\ \frac{\partial^2 L}{\partial\sigma^2\partial\beta'} & \frac{\partial^2 L}{\partial\sigma^2\partial\alpha'} & \frac{\partial^2 L}{(\partial\sigma^2)^2}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{\sigma^2}\sum_{i=1}^n\frac{\mathbf{x}_i\mathbf{x}_i'}{h_i} & 0 & 0\\ 0 & \frac{1}{2}\sum_{i=1}^n\frac{1}{h_i^2(\alpha)}\cdot\frac{\partial h_i}{\partial\alpha}\frac{\partial h_i}{\partial\alpha'} & \frac{1}{2\sigma^2}\sum_{i=1}^n\frac{1}{h_i(\alpha)}\cdot\frac{\partial h_i}{\partial\alpha}\\ 0 & \frac{1}{2\sigma^2}\sum_{i=1}^n\frac{1}{h_i(\alpha)}\cdot\frac{\partial h_i}{\partial\alpha} & \frac{n}{2\sigma^4}\end{bmatrix}
\end{aligned}
$$

(5) from the concentrated loglikelihood function

$$L^*(\alpha)=constant-\frac{n}{2}\ln\left[\left(Y-X\hat\beta(\alpha)\right)'\Lambda^{-1}\left(Y-X\hat\beta(\alpha)\right)\right]-\frac{1}{2}\ln|\Lambda|$$

the likelihood ratio (LR) test statistic is given by

$$
\begin{aligned}
\gamma_{LR} &= 2\left[L(\hat\alpha)-L(0)\right]\\
&= n\cdot\ln\left[\frac{\sum_{i=1}^n(y_i-\boldsymbol{x}_i'b)^2}{\sum_{i=1}^n\frac{\left(y_i-\mathbf{x}_i'\hat\beta(\hat\alpha)\right)^2}{h_i(\hat\alpha)}}\right]-\frac{1}{2}\sum_{i=1}^n\ln\left[h_i(\hat\alpha)\right]
\end{aligned}
$$

and $\gamma_{LR}\overset{\mathbb{H}_0}{\sim}\chi^2(s)$ since α is a $s\times 1$ vector.

(6) The wald test statistic is

$$\gamma_w=\hat\alpha'\hat V_\alpha^{-1}\hat\alpha$$

recall wald statistic, which for testing $\mathbb{H}_0:\ h(\theta)=c$, is given by

$$W=\left(h\left(\hat\theta\right)-c\right)'\left[H\left(\hat\theta\right)'I\left(\hat\theta\right)^{-1}H\left(\hat\theta\right)\right]^{-1}\left(h\left(\hat\theta\right)-c\right)$$

Since $H\left(\hat\theta\right)=\frac{\partial h}{\partial\theta}=(0,1,0)'$, so the wald test statistic is

$$
\begin{aligned}
\gamma_w &= \hat\alpha'\hat V_\alpha^{-1}\hat\alpha\\
&\overset{\mathbb{H}_0}{\sim} \chi^2(s)
\end{aligned}
$$

with

$$\hat{V}_{\alpha}^{-1} = \left(\frac{1}{2} \sum_{i=1}^{n} \frac{1}{h_i^2(\alpha)} \cdot \frac{\partial h_i}{\partial \alpha} \frac{\partial h_i}{\partial \alpha'} \right)^{-1} - \frac{1}{2n} \left(\sum_{i=1}^{n} \frac{1}{h_i(\alpha)} \cdot \frac{\partial h_i}{\partial \alpha} \right) \left(\sum_{i=1}^{n} \frac{1}{h_i(\alpha)} \cdot \frac{\partial h_i}{\partial \alpha'} \right)$$

(7) recall LM test statistic is given by

$$LM = s(\theta_0)' I(\theta_0)^{-1} s(\theta_0)$$

where $s(\theta_0) = \frac{\partial L}{\partial \theta}|_{\theta_0}$. Thefore, LM statistic is

$$\begin{aligned} \gamma_{LM} &= s_0' I_0^{-1}\left(\alpha, \sigma^2\right) s_0 \\ &\overset{\mathbb{H}_0}{\sim} \chi^2(s) \end{aligned}$$

where $s_0 = \left(\frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \sigma^2}\right)'$ is evaluated at

$$\alpha = 0,\ \sigma^2 = \hat{\sigma}^2(0),\ and\ \beta = b.$$

$I_0^{-1}\left(\alpha, \sigma^2\right)$ is the bottom-right block of the information matrix, evalued at

$$\alpha = 0,\ and\ \sigma^2 = \hat{\sigma}^2(0).$$