Solution to Problem Set 7

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Let

$$J\left(W_t, I_t, t\right) \equiv \max_{C_s, \{\omega_{is}\}, \forall s, i} E_t \left[\sum_{s=t}^{T-1} U\left(C_s, s\right) + B\left(W_T, T\right) \right]$$

subjec to

$$W_{t+1} = \left(W_t + y_t - C_t\right)\left(R_{ft} + \sum_{i=1}^{n} \omega_{it}\left(R_{it} - R_{ft}\right)\right) = S_t R_t.$$

Show that

$$J_W\left(W_{T-2}, T-2\right) = R_{f,T-2} E_{T-2}\left[J_W\left(W_{T-1}, T-1\right)\right].$$

Suggested Answer

$$J_W\left(W_{T-2}, T-2\right)$$

$$= U_C \frac{\partial C_{T-2}^*}{\partial W_{T-2}} + E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot \left(\frac{dW_{T-1}}{dW_{T-2}}\right)\right]$$

$$= U_C \frac{\partial C_{T-2}^*}{\partial W_{T-2}} + E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot \left(\frac{\partial W_{T-1}}{\partial W_{T-2}} + \sum_{i=1}^{n} \frac{\partial W_{T-1}}{\partial \omega_{i,T-2}^*}\frac{\partial \omega_{i,T-2}^*}{\partial W_{T-2}} + \frac{\partial W_{T-1}}{\partial C_{T-2}^*}\frac{\partial C_{T-2}^*}{\partial W_{T-2}}\right)\right]$$

$$= U_C \frac{\partial C_{T-2}^*}{\partial W_{T-2}} + E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot \left(\sum_{i=1}^{n} \left[R_{i,T-2} - R_{f,T-2}\right] S_{T-2} \frac{\partial \omega_{i,T-2}^*}{\partial W_{T-2}} + R_{T-2}\left(1 - \frac{\partial C_{T-2}^*}{\partial W_{T-2}}\right)\right)\right]$$

$$= U_C \frac{\partial C_{T-1}^*}{\partial W_{T-1}} + \underbrace{E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot \left(\sum_{i=1}^{n} \left[R_{i,T-2} - R_{f,T-2}\right] S_{T-2} \frac{\partial \omega_{i,T-2}^*}{\partial W_{T-2}}\right)\right]}_{=0} +$$

$$E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot \left(\underbrace{R_{f,T-2} + \overbrace{\sum_{i=1}^{n} \omega_{i,T-2}\left(R_{i,T-2} - R_{f,T-2}\right)}^{=0}}_{=R_{T-2}}\right)\left(1 - \frac{\partial C_{T-2}^*}{\partial W_{T-2}}\right)\right]$$

$$= U_C \frac{\partial C_{T-2}^*}{\partial W_{T-2}} + E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot R_{f,T-2}\left(1 - \frac{\partial C_{T-2}^*}{\partial W_{T-2}}\right)\right]$$

$$
\begin{aligned}
& J_W\left(W_{T-2}, T-2\right) \\
=\ & U_C \frac{\partial C_{T-2}^*}{\partial W_{T-2}} + E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot R_{f,T-2}\left(1 - \frac{\partial C_{T-2}^*}{\partial W_{T-2}}\right)\right] \\
=\ & \underbrace{E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot R_{f,T-2}\right]}_{=U_C} \frac{\partial C_{T-2}^*}{\partial W_{T-2}} + E_{T-2}\left[J_W\left(W_{T-1}, T-1\right) \cdot R_{f,T-2}\left(1 - \frac{\partial C_{T-2}^*}{\partial W_{T-2}}\right)\right] \\
=\ & R_{f,T-2} E_{T-2}\left[J_W\left(W_{T-1}, T-1\right)\right]
\end{aligned}
$$