Advanced Microeconomics II

WISE, Xiamen University

Spring 2012 Midterm

1. (5 points) Clearly define a maxminimizing strategy for player 1 in a strictly competitive game.

> **Solution:** Let $\{\{1,2\},(A_i),(\succeq_i)\}$ be a strictly competitive strategic game. The action $x^* \in A_1$ is a maxminimizer for player 1 if
>
> $$\min_{y\in A_2} u_1(x^*,y) \geq \min_{y\in A_2} u_1(x,y) \text{ for all } x \in A_1.$$

2. (5 points) Clearly define the mixed extension of a strategic game.

> **Solution:** The mixed extension of the strategic game $\{N,(A_i),(u_i)\}$ is the strategic game $\{N,(\Delta(A_i)),(U_i)\}$ where
>
> - $\Delta(A_i)$ is the set of probability distributions over A_i,
> - $U_i : \times_{j\in N}\Delta(A_j) \to \mathcal{R}$ is a von Neumann-Morgenstern utility function for player i that represents preferences over $\times_{j\in N}\Delta(A_j)$. For finite A,
>
> $$U_i(\alpha) = \sum_{a\in A}(\Pi_{j\in N}\alpha_j(a_j))u_i(a),$$
>
>   where for each $i \in N$, $\alpha_i(a_i)$ as the probability that α_i assigns to a_i.

3. (5 points) Clearly define a correlated equilibrium of a strategic game.

> **Solution:** A correlated equilibrium of a strategic game $\{N,(A_i),(u_i)\}$ is
>
> - a finite probability space (Ω,π)
> - for each player $i \in N$ a partition of $\mathcal{P}_i$ of Ω
> - for each player $i \in N$ a function $\sigma_i : \Omega \to A_i$ with $\sigma_i(\omega) = \sigma_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$
>
> such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ for which $\tau_i(\omega) = \tau_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$
>
> $$\sum_{\omega\in\Omega}\pi(\omega)u_i(\sigma_i(\omega),\sigma_{-i}(\omega)) \geq \sum_{\omega\in\Omega}\pi(\omega)u_i(\tau_i(\omega),\sigma_{-i}(\omega))$$

4. (5 points) Clearly define a strictly dominated action.

Solution: The action $a_i \in A_i$ of player i in the strategic game $\{N, (A_i), (u_i)\}$ is strictly dominated if there is a mixed strategy α_i of player i such that $U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, where $U_i(\alpha_i, a_{-i})$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}.

5. Consider the finite strategic game $G = \{N, (A_i), (u_i)\}$.

(a) (10 points) Prove or disprove that for each player i, the U_i associated with the mixed extension of G is quasi-concave over $\Delta(A_i)$.

Solution: The U_i associated with the mixed extension of G is defined as

$$U_i(\alpha) = \sum_{a \in A} \Pi_{j \in N} \alpha_j(a_j) u_i(a) \text{ for all } \alpha \in \times_{j \in N} \Delta(A_j).$$

Fix the strategies of the other players to be α_{-i}. Consider two strategies of player i, β_i and γ_i and $\lambda \in [0, 1]$.

$$\begin{aligned}
U_i(\lambda\beta_i + (1-\lambda)\gamma_i, \alpha_{-i}) =& \sum_{a \in A} (\lambda\beta_i(a_i) + (1-\lambda)\gamma_i(a_i) \prod_{\substack{j \in N \\ j \neq i}} \alpha_j(a_j) u_i(a) \\
=& \lambda \sum_{a \in A} \beta_i(a_i) \prod_{\substack{j \in N \\ j \neq i}} \alpha_j(a_j) u_i(a) \\
& + (1-\lambda) \sum_{a \in A} \gamma_i(a_i) \prod_{\substack{j \in N \\ j \neq i}} \alpha_j(a_j) u_i(a) \\
=& \lambda U_i(\beta_i, \alpha_{-i}) + (1-\lambda) U_i(\gamma_i, \alpha_{-i}) \\
\geq& \min\{U_i(\beta_i, \alpha_{-i}), U_i(\gamma_i, \alpha_{-i})\}.
\end{aligned}$$

(b) (10 points) Prove or disprove that $B(\alpha) = \times_{i \in N} B_i(\alpha_{-i})$ is convex, where $B_i(\alpha_{-i})$ is the best response function of player i, i.e. show that if $b \in B(\alpha)$ and $c \in B(\alpha)$ then for any $\lambda \in [0, 1]$, $\lambda b + (1-\lambda)c \in B(\alpha)$.

Solution: $B(\alpha) = \times_{i \in N} B_i(\alpha_{-i})$, where

$$B_i(\alpha_{-i}) = \{\alpha_i \in \Delta(A_i) : (\alpha_i, \alpha_{-i}) \succeq_i (\alpha'_i, \alpha_{-i}) \text{ for all } \alpha'_i \in \Delta(A_i)\}.$$

Let $b \in B(\alpha)$ and $c \in B(\alpha)$. Then, for each player i, $\lambda b_i + (1-\lambda)c_i \in \Delta(A_i)$ since $\Delta(A_i)$ is convex, and

$$U_i\left(\lambda b_i + (1-\lambda)c_i, \alpha_{-i}\right) \geq \min\left\{U_i(b_i, \alpha_{-i}), U_i(c_i, \alpha_{-i})\right\},$$

since U_i is quasi-concave on $\Delta(A_i)$. Futhermore,

$$\min\{U_i(b_i, \alpha_{-i}), U_i(c_i, \alpha_{-i})\} \geq U_i(\alpha_i', \alpha_{-i}) \text{ for all } \alpha_i \in \Delta(A_i).$$

since, by the definition of B, $b_i \in B_i(\alpha_{-i})$ and $c_i \in B_i(\alpha_{-i})$. So for any $\lambda \in [0, 1]$, $\lambda b_i + (1-\lambda)c_i \in B_i(\alpha_{-i})$, which implies that $\lambda b + (1-\lambda)c \in B(\alpha)$.

6. (10 points) State what has been proven, nothing more, nothing less.

   Proof. Suppose that there is a mixed strategy α_i' that gives a higher expected payoff than does α_i^* in response to α_{-i}^*. Then by the linearity of U_i at least one action in the support of α_i' must give a higher payoff than some action in the support of α_i^*, so that not all actions in the support of α_i^* are best responses to α_{-i}^*. ■

   Solution:

   Proposition 1. *Let $G =< N, (A_i), (u_i) >$ be a finite strategic game. If for every player $i \in N$ every pure strategy (action) in the support of α_i^* is a best response to α_{-i}^* then $\alpha^* \in \times_{i \in N}\Delta(A_i)$ is a mixed-strategy Nash equilibrium of G.*

7. (10 points) State what has been proven, nothing more, nothing less.

   Proof. Let $\{(\Omega, \pi), (\mathcal{P}_i), (\sigma_i)\}$ be a correlated equilibrium of G. Then $\{(\Omega', \pi'), (\mathcal{P}_i'), (\sigma_i')\}$ is also a correlated equilibrium, where $\Omega' = A$, $\pi'(a) = \pi(\{w \in \Omega : \sigma(\omega) = a\})$ for each $a \in A$, $\mathcal{P}_i'$ consists of the sets of the type $\{a \in A : a_i = b_i\}$ for some $b_i \in A_i$, and σ_i' is defined by $\sigma_i'(a) = a_i$. ■

   Solution:

   Proposition 2. *Let $G = \{N, (A_i), (u_i)\}$ be a finite strategic game. Every probability distribution over outcomes that can be obtained in a correlated equilibrium of G can be obtained in a correlated equilibrium in which the set of states is A and for each $i \in N$ player i's information partition consists of all sets of the form $\{a \in A : a_i = b_i\}$ from some action $b_i \in A_i$.*

8. Two candidates, A and B, compete in an election. Of the n citizens, $k \geq 2$ support candidate A and $m = n - k \geq 2$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases, where $0 < c < 1$.

(a) (5 points) Formulate this scenario as a strategic game.

Solution: $N = \{1, \dots, k, k+1, n\}$. The first k citizens support candidate A, the rest support candidate B. $A_i = \{0, 1\}$ where 0 implies that a citizen abstains and 1 implies that a citizen votes for their candidate. For $i \leq k$ and $a \in A$

$$u_i(a) = \begin{cases} 2 - ca_i & \text{if } \sum_{j=1}^{k} a_j > \sum_{j=k+1}^{m} a_j \\ 1 - ca_i & \text{if } \sum_{j=1}^{k} a_j = \sum_{j=k+1}^{m} a_j \\ -ca_i & \text{if } \sum_{j=1}^{k} a_j < \sum_{j=k+1}^{m} a_j \end{cases}$$

For $k + 1 \leq i \leq n$

$$u_i(a) = \begin{cases} 2 - ca_i & \text{if } \sum_{j=1}^{k} a_j < \sum_{j=k+1}^{m} a_j \\ 1 - ca_i & \text{if } \sum_{j=1}^{k} a_j = \sum_{j=k+1}^{m} a_j \\ -ca_i & \text{if } \sum_{j=1}^{k} a_j > \sum_{j=k+1}^{m} a_j. \end{cases}$$

(b) (5 points) For $k = m$, find a pure-strategy Nash equilibrium.

Solution: An outcome where one candidate wins by more than one vote is not an equilibrium outcome since one of the citizen's who voted for the winner could abstain and increase her payoff ($2 > 2 - c$).

An outcome where one candidate wins by exactly one vote is not an equilibrium outcome since a citizen who supports the losing candidate but abstained (at least one must exist) could vote and increase her payoff by making the contest a tie ($1 - c > 0$).

An outcome where the candidates tie but somebody abstains is not an equilibrium outcome since a citizen who abstains could make her candidate win outright by voting and increase her payoff ($2 - c > 1$).

The outcome where the candidates tie and everybody votes is an equilibrium outcome since by abstaining a citizen's candidate would lose outright thus decreasing the citizen's payoff ($0 < 1 - c$). Hence, the only pure-strategy Nash equilibrium is for every citizen to vote.

(c) (5 points) What is the set of pure strategy Nash equilibria for $k < m$?

Solution: There exists no pure-strategy Nash equilibrium for $k < m$. All of the arguments above still hold. However, since $k < m$ if all k supporters of candidate A vote as do k supporters of candidate B, a supporter of candidate B who abstains can do better by voting ($2 - c > 1$).

(d) (5 points) Assume that $k \leq m$. Show that there is a value of p between 0 and 1 such that the game has a mixed-strategy Nash equilibrium in which every supporter of candidate A votes with probability p, k supporters of candidate B vote with certainty, and the remaining $m - k$ supporters of candidate B abstain. You must clearly show the payoffs of each player using different actions under the equilibrium. (Note that if every supporter of candidate A votes with probability p then the probability that exactly $k-1$ of them vote is $kp^{k-1}(1-p)$.)

Solution: To be a mixed strategy equilibrium, supporters of candidate A must be indifferent between voting and not voting. Assuming the above equilibrium, then not voting will guarantee that candidate A loses and a payoff of zero. Voting incurs a cost of c and benefit in expectation equal to p^{k-1}. Hence, in equilibrium $p = c^{\frac{1}{k-1}}$. It is left to check that the supporters of candidate B are choosing optimally given the strategies of the other players.

For a supporter of candidate B who votes to be acting optimally

$$\begin{aligned}
(1-c)p^k + (2-c)(1-p^k) &\geq (0)p^k + (1)kp^{k-1}(1-p) + (2)(1-p^k-kp^{k-1}(1-p)) && \text{or}\\
p^k + kp^{k-1}(1-p) &\geq c && \text{or}\\
p + k(1-p) &\geq 1 \text{ (since } p^{k-1}=c) && \text{or}\\
(k-1)(1-p) &\geq 0,
\end{aligned}$$

which is true since $k \geq 2$.

For a supporter of candidate B who abstains to be acting optimally

$$\begin{aligned}
(1)p^k + (2)(1-p^k) &\geq 2-c \text{ or}\\
p^{k-1} &\geq p^k \text{ (since } p^{k-1}=c),
\end{aligned}$$

which is true since $p < 1$.

9. Consider the following three player game G.

A:

	L	C	R
U	$0,5,2$	$4,3,5$	$2,6,0$
M	$2,1,2$	$3,5,3$	$1,6,4$
D	$6,3,2$	$4,0,1$	$0,6,5$

B:

	L	C	R
U	$8,7,1$	$2,3,0$	$2,2,3$
M	$3,6,8$	$3,4,2$	$1,5,1$
D	$1,4,2$	$4,1,0$	$2,9,1$

(a) (5 points) For the strategy $a_1^* = M$ for player 1, describe (using a table of you like) the auxiliary zero-sum 2 player game G' where $A_1' = A_i \backslash a_1^*$, $A_2' = A_{-i}$ and $u_1'(a_1, a_{-i}) = u_i(a_i, a_{-i}) - u_i(a_i^*, a_{-i})$.

Solution:

	AL	AC	AR	BL	BC	BR
U	$-2,2$	$1,-1$	$1,-1$	$5,-5$	$-1,1$	$1,-1$
D	$4,-4$	$1,-1$	$-1,1$	$-2,2$	$1,-1$	$1,-1$

(b) (10 points) Find the maxminimizing strategy over $\Delta(A_1')$.

Solution: The maxminizing strategy is

$$\max_{x\in\Delta(A_1')}\min_{y\in\Delta(A_2')} U'(x,y) = \max_{x\in\Delta(A_1')}\min_{y\in A_2'} U'(x,y)$$

since utility is linear in probabilities over $\Delta(A_2')$. Let α denote the probability that player 1 plays D. Thus we want to find x that solves

$$\max_{\alpha\in[0,1]}\min\{-2(1-\alpha)+4\alpha, 1, (1-\alpha)-\alpha, 5(1-\alpha)-2\alpha, -(1-\alpha)+\alpha, 1\}.$$

which simplifies to

$$\max_{\alpha\in[0,1]}\min\{-2+6\alpha, 1, 1-2\alpha, 5-7\alpha, -1+2\alpha, 1\}.$$

If $\alpha < 1/2$ then $U_1(\alpha, AR) = -1+2\alpha < 0$ and if $\alpha > 1/2$ then $U_1(\alpha, BC) = 1-2\alpha < 0$. while if $\alpha = 1/2$ then the payoff from each action of player 2 is non-negative. Hence $\alpha_1(U) = \alpha_1(D) = 1/2$ is the maxminimizing strategy of player 1.

(c) (5 points) Is the strategy M a best response for player 1 in G for some belief over the other players' actions. If so, give one such belief. Does there exist a mixed strategy that strictly dominates M. If so, give one such mixed strategy.

Solution: M is a best response to μ where $\mu(AR) = \mu(BC) = 1/2$ since $U_1(M,\mu) = 0 = U_1(U,\mu) = U_1(D,\mu)$. By Lemma 60.1 of Osborne and Rubinstein there is no mixed strategy that strictly dominated M.

Alternatively, for M to be strictly dominated in G then there exists $\alpha_1 \in \Delta(A_1')$ such that $U_1'(\alpha_1, a_2) > 0$ for all $a_2 \in A_2'$. Let α denote $\alpha_1(U)$. Thus we want to find x that solves

$$\min\{-2\alpha+4(1-\alpha), 1, \alpha-(1-\alpha), 5\alpha-2(1-\alpha), -\alpha+(1-\alpha), 1\} > 0.$$

However, if $\alpha-(1-\alpha) > 0$, then $-\alpha+(1-\alpha) < 0$. Hence, no mixed strategy dominates M in G. By Lemma 60.1 of Osborne and Rubinstein M must be a best response for player 1 in G for some belief over the other players' actions.