Mon 9 Aug 2021

Let $x_1, x_2 \dots x_n$ i.i.d

$$x_i \sim N(\mu, \tau)$$

$$f(x_i \mid \mu, \tau) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp\left\{-\frac{\tau}{2}(x_i - \mu)^2\right\}$$

$$f(x_1, \dots, x_n \mid \mu, \tau) = \frac{\tau^{n/2}}{(2\pi)^{n/2}} \exp\left\{-\frac{\tau}{2}\sum_i (x_i - \mu)^2\right\}$$

$$\tau \sim \text{Gamma}(a, b)$$

$$f(\tau) = \frac{b^a}{\Gamma(a)} \tau^{a-1} \exp(-b\tau)$$

$$f(\tau \mid \mu, x_1, \dots, x_n) \propto f(x_1, \dots, x_n \mid \mu, \tau) f(\tau)$$

$$= \frac{\tau^{n/2}}{(2\pi)^{n/2}} \exp\left(-\frac{\tau}{2}\sum_i (x_i - \mu)^2\right) \frac{b^a}{\Gamma(a)} \tau^{a-1} \exp\{-b\tau\}$$

$$\propto \tau^{n/2} \exp\left\{-\frac{\tau}{2}\sum_i (x_i - \mu)^2\right\} \tau^{a-1} \exp\{-b\tau\}$$

$$= \tau^{a + n/2 - 1} \exp\left\{-\tau\left(b + \frac{1}{2}\sum_i (x_i - \mu)^2\right)\right\}$$

$$\Rightarrow \tau \mid \mu, x_1, \dots, x_n \sim \text{Gamma}\left(a + \frac{n}{2}, b + \frac{1}{2}\sum_i (x_i - \mu)^2\right)$$