Let $x_1, x_2, \dots, x_n$ i.i.d.

$$x_i \sim N(\mu, \tau)$$

$$f(x_i \mid \mu, \tau) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp\left\{-\frac{\tau}{2}(x_i-\mu)^2\right\}$$

where $-\infty < \mu < \infty$, $\tau > 0$, $-\infty < x_i < \infty$

The joint likelihood

$$L = \prod_i f(x_i \mid \mu, \tau) =$$

$$= \frac{\tau^{n/2}}{(2\pi)^{n/2}} \exp\left\{-\frac{\tau}{2}\sum_i (x_i-\mu)^2\right\}$$

$$\log L = \frac{n}{2}\log\tau - \frac{n}{2}\log(2\pi) - \frac{\tau}{2}\sum_i (x_i-\mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{\tau}{2}\sum_i (2(-1)(x_i-\mu)) = \tau\sum_i (x_i-\mu)$$

$$\tau\left(\sum x_i - n\mu\right) = 0$$

$$\Rightarrow \hat{\mu} = \frac{\sum x_i}{n} = \bar{x}$$

2^{nd} order condition

$$\frac{\partial^2 \log L}{\partial \mu^2} = -n\tau < 0$$

$\Rightarrow$ $\hat{\mu} = \bar{x}$ is an M.L.E. of μ

$$\frac{\partial \log L}{\partial \tau} = \frac{n}{2}\frac{1}{\tau} - \frac{1}{2}\sum_i (x_i-\mu)^2 = 0$$

$$\Rightarrow \hat{\tau} = \frac{n}{\sum_i (x_i-\mu)^2}$$

2^{nd} order condition

$$\frac{\partial^2 \log L}{\partial \tau^2} = -\frac{n}{2}\frac{1}{\tau^2} < 0$$

$$\Rightarrow \hat{\tau} = \frac{n}{\sum_i (x_i-\mu)^2}$$ is the M.L.E. of τ

$\mu \sim N(\mu_0, \tau_0)$

$$f(\mu|\tau, x) = \frac{f(x|\mu,\tau)\, f(\mu)}{\underbrace{\int f(x|\mu,\tau)\, f(\mu)\, d\mu}_{f(x|\tau)}} \propto f(x|\mu,\tau) f(\mu)$$

$$= \frac{\tau^{n/2}}{(2\pi)^{n/2}} \exp\left\{-\frac{\tau}{2}\sum_i (x_i-\mu)^2\right\} \frac{\tau_0^{1/2}}{(2\pi)^{1/2}} \exp\left\{-\frac{\tau_0}{2}(\mu-\mu_0)^2\right\}$$

$$\propto \exp\left\{-\frac{\tau}{2}\sum_i (x_i-\mu)^2\right\} \exp\left\{-\frac{\tau_0}{2}(\mu-\mu_0)^2\right\}$$

nb. $(a-b)^2 = a^2 - 2ab + b^2$

$$= \exp\left\{-\frac{\tau}{2}\sum_i (x_i^2 - 2x_i\mu + \mu^2) - \frac{\tau_0}{2}(\mu^2 - 2\mu\mu_0 + \mu_0^2)\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\tau\sum_i x_i^2 - 2\tau\mu\sum_i x_i + \tau n\mu^2 + \tau_0\mu^2 - 2\tau_0\mu\mu_0 + \tau_0\mu_0^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(-2\tau\mu n\bar{x} + \tau n\mu^2 + \tau_0\mu^2 - 2\tau_0\mu\mu_0\right)\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\mu^2(\tau n + \tau_0) - 2\mu(\tau n\bar{x} + \tau_0\mu_0)\right)\right\}$$

$$= \exp\left\{-\frac{\tau n + \tau_0}{2}\left(\mu^2 - 2\mu\frac{\tau n\bar{x} + \tau_0\mu_0}{\tau n + \tau_0}\right)\right\}$$

$$\propto \exp\left\{-\frac{\tau n+\tau_0}{2}\left(\mu^2-2\mu\frac{\tau n\bar{x}+\tau_0\mu_0}{\tau n+\tau_0}+\left(\frac{\tau n\bar{x}+\tau_0\mu_0}{\tau n+\tau_0}\right)^2\right)\right\}$$

$$=\exp\left\{-\frac{\tau n+\tau_0}{2}\left(\mu-\frac{\tau n\bar{x}+\tau_0\mu_0}{\tau n+\tau_0}\right)^2\right\}$$

$$\Rightarrow \quad \mu\mid\tau,x \sim N\left(\frac{\tau n\bar{x}+\tau_0\mu_0}{\tau n+\tau_0},\ \tau n+\tau_0\right)$$

PRECISION!

$$E(\mu\mid\tau,x)=\frac{\tau n\bar{x}+\tau_0\mu_0}{\tau n+\tau_0}=\frac{\tau n}{\tau n+\tau_0}\bar{x}+\frac{\tau_0}{\tau n+\tau_0}\mu_0$$

$$\lim_{n\to\infty}\frac{\tau n\bar{x}+\tau_0\mu_0}{\tau n+\tau_0}=\bar{x}$$

$$\mu\sim N(0,\ 10^{-8})$$