STAT314. Poisson-Gamma Model. Derivation.

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Let $x_i|\mu \sim Pois(\mu)$ for $i = 1, ..., n$. Then the joint likelihood (i.e., the product of individual likelihoods):

$$f(\mathbf{x}|\mu) = f(x_1, ..., x_n|\mu) = \prod_i f(x_i|\mu) = \prod_i \frac{\mu^{x_i} e^{-\mu}}{x_i!} = \frac{\mu^{\sum_i x_i} e^{-n\mu}}{\prod_i x_i!}.$$

Now, one thing we definitely know about the Poisson intensity parameter μ is that it should be positive. One of the distributions which can be used for modeling positive quantities is a **Gamma** distribution. (Remember to look it up on Wikipedia if you have never ran into it before). Let's try using it as a prior for μ:

$$\mu|a, b \sim Gamma(a, b),$$

i.e.

$$f(\mu|a, b) = \frac{b^a}{\Gamma(a)} \mu^{a-1} e^{-b\mu} \quad \text{for } \mu > 0,$$

where $\Gamma(a)$ is a gamma function. (Gamma function is similar to any other mathematical function, such as sine, cosine, exponent, square root etc. in that it takes input and produces output. It is implemented in `R` as `gamma()`. You may want to look it up, but the details of this function are not too important at the moment.)

Now, that we've got the likelihood and the prior, let's apply Bayes' Theorem:

$$\begin{aligned}
f(\mu|\mathbf{x}, a, b) &= \frac{f(\mathbf{x}|\mu) f(\mu|a, b)}{\int_0^\infty f(\mathbf{x}|\mu) f(\mu|a, b) d\mu} \\
&= \frac{\frac{\mu^{x_i} e^{-\mu}}{\prod_i x_i!} \frac{b^a}{\Gamma(a)} \mu^{a-1} e^{-b\mu}}{\int_0^\infty \frac{\mu^{x_i} e^{-\mu}}{\prod_i x_i!} \frac{b^a}{\Gamma(a)} \mu^{a-1} e^{-b\mu} d\mu} \\
&= \frac{\frac{b^a}{\Gamma(a)} \frac{1}{\prod_i x_i!} \mu^{a + \sum_i x_i - 1} e^{-(b+n)\mu}}{\frac{b^a}{\Gamma(a)} \frac{1}{\prod_i x_i!} \int_0^\infty \mu^{a + \sum_i x_i - 1} e^{-(b+n)\mu} d\mu} \\
&= \frac{\mu^{a + \sum_i x_i - 1} e^{-(b+n)\mu}}{\int_0^\infty \mu^{a + \sum_i x_i - 1} e^{-(b+n)\mu} d\mu}
\end{aligned}$$

This may look like a bit of a nightmare. But note that the integral in the denominator will not depend on μ, i.e., it will be a constant with respect to μ, and we can write:

$$f(\mu|\mathbf{x}, a, b) \propto \mu^{a + \sum_i x_i - 1} e^{-(b+n)\mu},$$

where $\propto$ denotes *propotional up to a constant*.

How does that help? Well, remember that any p.d.f. has to integrate to 1, and thus the constant can always be recovered later. The formula above looks very similar to the p.d.f. of a Gamma distribution $Gamma(a + \sum_i x_i, b + n)$ up to a constant, which does not include μ. Thus, we can conclude that

$$\mu|\mathbf{x}, a, b \sim Gamma(a + \sum_i x_i, b + n).$$

An easier way...

is to use proportionality right away:

$$\begin{aligned} f(\mu|\mathbf{x}, a, b) &\propto f(\mathbf{x}|\mu) f(\mu|a, b) \\ &= \frac{\mu^{\sum_i x_i} e^{-n\mu}}{\prod_i x_i!} \frac{b^a}{\Gamma(a)} \mu^{a-1} e^{-b\mu} \\ &\propto \mu^{\sum_i x_i} e^{-n\mu} \mu^{a-1} e^{-b\mu} \\ &= \mu^{a+\sum_i x_i - 1} e^{-(b+n)\mu} \end{aligned}$$