STAT314. Poisson MLE. Derivation.

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Let $x_i|\mu \sim Pois(\mu)$ for $i = 1, ..., n$. Then the joint likelihood (i.e., the product of individual likelihoods):

$$L = f(\mathbf{x}|\mu) = f(x_1, ..., x_n|\mu) = \prod_i f(x_i|\mu) = \prod_i \frac{\mu^{x_i} e^{-\mu}}{x_i!} = \frac{\mu^{\sum_i x_i} e^{-n\mu}}{\prod_i x_i!}.$$

for $\mu > 0$.

First, let's log the likelihood to make things easier:

$$\log L = \sum_i x_i \log \mu - n\mu - \log \left(\prod_i x_i! \right).$$

Now, let's differentiate with respect (w.r.t.) to μ:

$$\frac{\partial \log L}{\partial \mu} = \sum_i x_i \frac{1}{\mu} - n.$$

Set it to 0, and solve for μ:

$$\sum_i x_i \frac{1}{\mu} - n = 0 \iff \hat{\mu} = \frac{\sum_i x_i}{n} = \bar{x}.$$

Remember, that that is not yet a point where the function reaches maximum. It may be a minimum or an inflection point. Always check for the second order condition:

$$\frac{\partial^2 \log L}{\partial \mu^2} = \sum_i x_i \left(-\frac{1}{\mu^2} \right).$$

You can substitute specific $\hat{\mu}$ to find that

$$\sum_i x_i \left(-\frac{1}{\hat{\mu}^2} \right) = -\frac{\sum_i x_i}{\bar{x}^2} = -\frac{n\bar{x}}{\bar{x}^2} = -\frac{n}{\bar{x}} < 0 \qquad \forall \bar{x} > 0.$$

Or you can use the fact that $\mu^2 > 0$ for all $\mu > 0$ to arrive at the same conclusion. Thus $\hat{mu} = \bar{x}$ is the maximum likelihood estimate (MLE) of the Poisson intensity parameter μ.