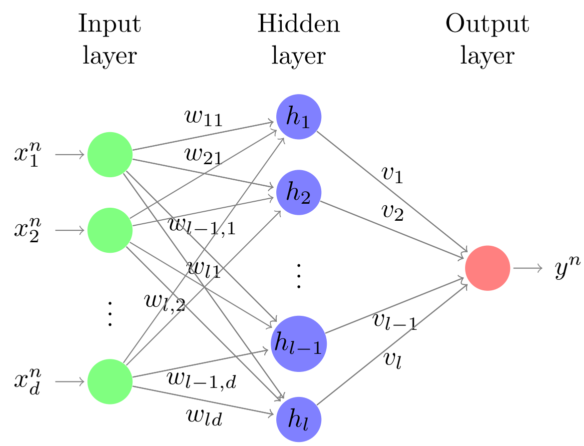
Normalization of Single-Layer ReLU Networks

Consider a single-hidden layer ReLU Neural Network: $ReLU(\mathbf{W}\mathbf{x}+\mathbf{b})^T\mathbf{v} = y$



Where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \in \mathbb{R}^d, \quad y \in \mathbb{R}$$

and

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1d} \\ w_{21} & w_{22} & \cdots & w_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ w_{l1} & w_{l2} & \cdots & w_{ld} \end{bmatrix} \in \mathbb{R}^{l\times d}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_l \end{bmatrix} \in \mathbb{R}^l, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_l \end{bmatrix} \in \mathbb{R}^l.$$

Lemma 1. *For any NN weights/biases $\mathbf{W}, \mathbf{b}, \mathbf{v}$ that satisfies $ReLU(\mathbf{W}\mathbf{x^n}+\mathbf{b})^T\mathbf{v} = y^n$ for all dataset $(\mathbf{x^n}, y^n)$, $n \in [N]$, there exists an alternate weights/biases $\mathbf{W}', \mathbf{b}', \mathbf{v}'$ such that $ReLU(\mathbf{W}'\mathbf{x^n}+\mathbf{b}')^T\mathbf{v}' = y^n$ for all dataset $(\mathbf{x^n}, y)$, $n \in [N]$ and $v_i' \in \{-1, 0, +1\} \quad \forall i \in [l]$.*

Proof. For a given $\mathbf{v}$, let $I \subseteq [l]$ be the index set where $v_i = 0 \quad \forall i \in I$ and $v_i \neq 0 \quad \forall i \in [l]/I$.

Let $\mathbf{v}'$ be $v_i' = 0 \quad \forall i \in I$ and $v_i' = \frac{v_i}{|v_i|} \in \{-1, +1\} \quad \forall i \in [l]/I$.

Let $\mathbf{W}'$ be $w_{ij}' = w_{ij} \quad$ (or any arbitrary scalar) $\quad \forall i \in I, \forall j \in [d]$ and $w_{ij}' = |v_i| w_{ij} \quad \forall i \in [l]/I, \forall j \in [d]$.

Let $\mathbf{b}'$ be $b_i' = b_i \quad$ (or any arbitrary scalar) $\quad \forall i \in I$ and $b_i' = |v_i| b_i \quad \forall i \in [l]/I$.

Then for an arbitrary dataset $(\mathbf{x^n}, y^n)$,

Since $v_i = v'_i = 0 \quad \forall i \in I$, then $ReLU(\mathbf{W}\mathbf{x^n} + \mathbf{b})_i v_i = ReLU(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i v'_i = 0 \quad \forall i \in I$.

Since $w'_{ij} = |v_i| w_{ij} \quad \forall i \in [l]/I, \forall j \in [d]$ and $b'_i = |v_i| b_i \quad \forall i \in [l]/I$,
then $(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i = \mathbf{W'}_i^T \mathbf{x^n} + \mathbf{b'}_i = |v_i| \mathbf{W}_i^T \mathbf{x^n} + |v_i| \mathbf{b}_i = |v_i| (\mathbf{W}\mathbf{x^n} + \mathbf{b})_i \quad \forall i \in [l]/I$.

Since $(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i = |v_i| (\mathbf{W}\mathbf{x^n} + \mathbf{b})_i \quad \forall i \in [l]/I$,
if $(\mathbf{W}\mathbf{x^n} + \mathbf{b})_i > 0$ then $(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i > 0$ and $ReLU(\mathbf{W}\mathbf{x^n} + \mathbf{b})_i = (\mathbf{W}\mathbf{x^n} + \mathbf{b})_i = |v_i| (\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i = |v_i| ReLU(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i$.
If $(\mathbf{W}\mathbf{x^n} + \mathbf{b})_i \leq 0$ then $(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i \leq 0$ $(\because v_i \neq 0 \quad \forall i \in [l]/I)$ and $ReLU(\mathbf{W}\mathbf{x^n} + \mathbf{b})_i = ReLU(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i = 0$.
Therefore $ReLU(\mathbf{W}\mathbf{x^n} + \mathbf{b})_i = |v_i| ReLU(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i \quad \forall i \in [l]/I$
and $ReLU(\mathbf{W}\mathbf{x^n} + \mathbf{b})_i v_i = |v_i| ReLU(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i v_i = ReLU(\mathbf{W'}\mathbf{x^n} + \mathbf{b'})_i v'_i = 0 \quad \forall i \in [l]/I$.

□

Corollary 1. *If the problem of finding W,b,v is feasible, then finding W,b,v where v=... is also feasible.*

Proof. By definition, if the problem is feasible there exists a W,b,v that satisfies the fitting of the N dataset. By lemma 1, there exists W,b,v also satisfies the fitting. Therefore, finding W,b,v problem is also feasible by definition. □