Threshold Cryptography with Python

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https://github.com/changwu-tw/PyConTW

Hello!

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Abstract

- Introduction to Threshold Cryptography
- Shamir's Secret Sharing
- Threshold ECDSA Signature
- SageMath & Jupyter notebook

from https://thespaces.com/inside-the-ned-5-of-its-best-hidden-spaces/

Threshold

- (m, n)-threshold scheme
- (2, 2)-threshold
- 3-of-5 threshold

Status

- Threshold Schemes for Cryptographic Primitives (NIST)
- Keywords
    - threshold schemes, secure implementations, cryptographic primitives, threshold cryptography, secure multi- party computation, intrusion tolerance, distributed systems, resistance to side-channel attacks, standards and validation
- From Theory To Practice

Motivation

Source: https://cointelegraph.com/news/exit-scam-dublin-based-exchange-bitsane-vanishes-with-users-funds

Key loss and vanish

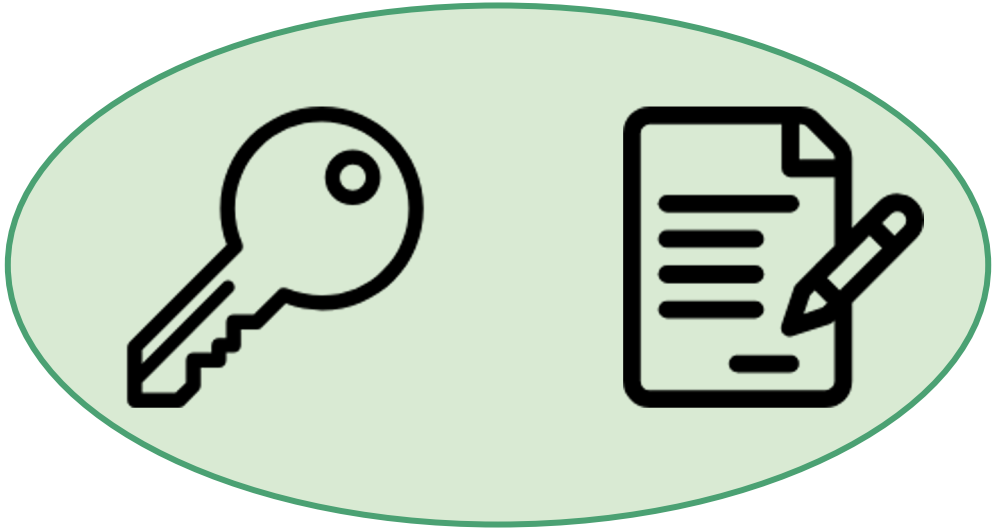
- Controversial QuadrigaCX cryptocurrency exchange placed in bankruptcy
  - $190 million of funds got lost because the founder suddenly died in his trip. (April 8)
- Exit Scam? Dublin-Based Exchange Bitsane Vanishes With Users' Funds
  - Ireland-based cryptocurrency exchange Bitsane has apparently vanished, taking as many as 246,000 users' crypto deposits with it. The news was reported by Forbes on June 27.

Hacked

- Singapore-Based Crypto Exchange DragonEx Has Been Hacked (Mar 26)
- Crypto Exchange Bithumb Hacked for $13 Million in Suspected Insider Job (Mar 30)

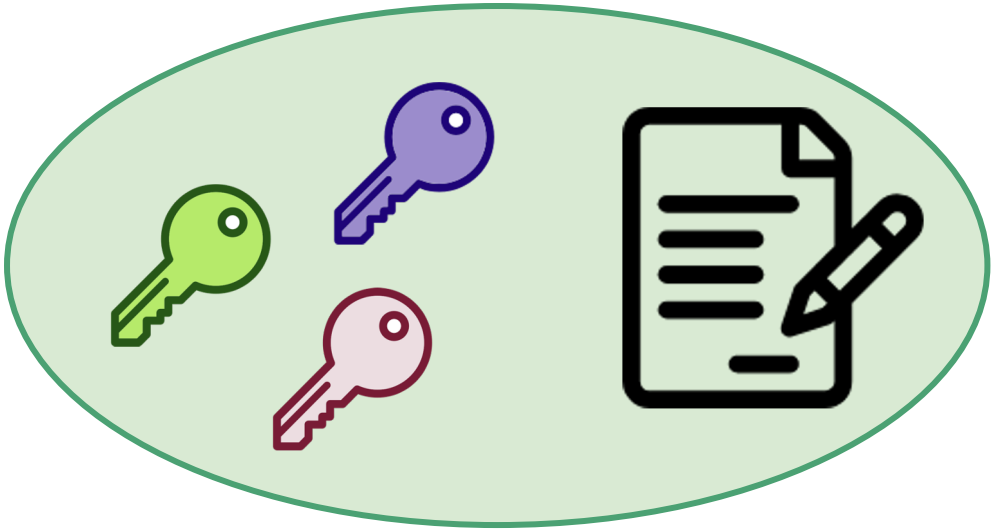
Source: https://thehackernews.com/2017/07/bitcoin-ethereum-cryptocurrency-exchange.html

What is a **valid** transaction?



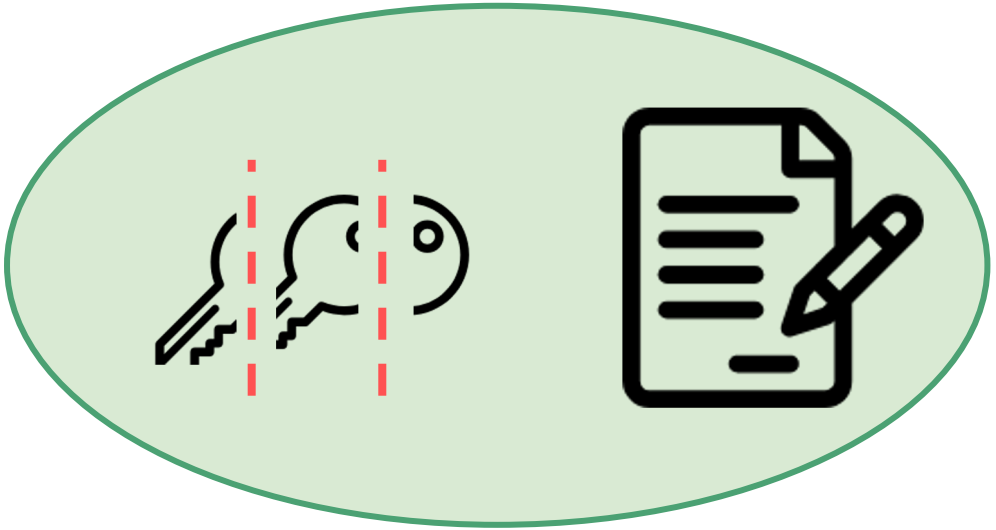
Raw data + signature (Signed by private key)

Multisig transaction (m-of-n)



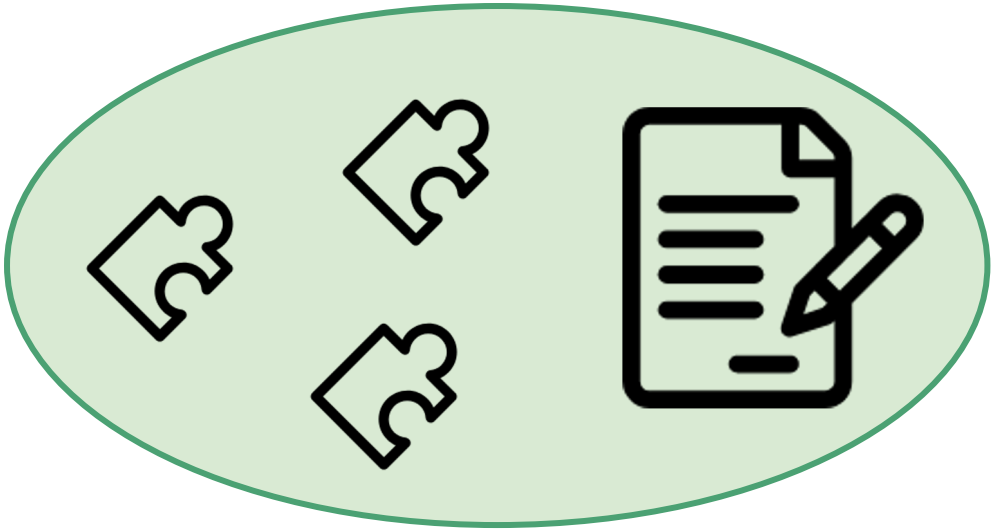
Raw data + multi-signature (Signed by multiple private keys)

Threshold **key** sharing



Key splitting

Threshold **signature** (m-of-n)



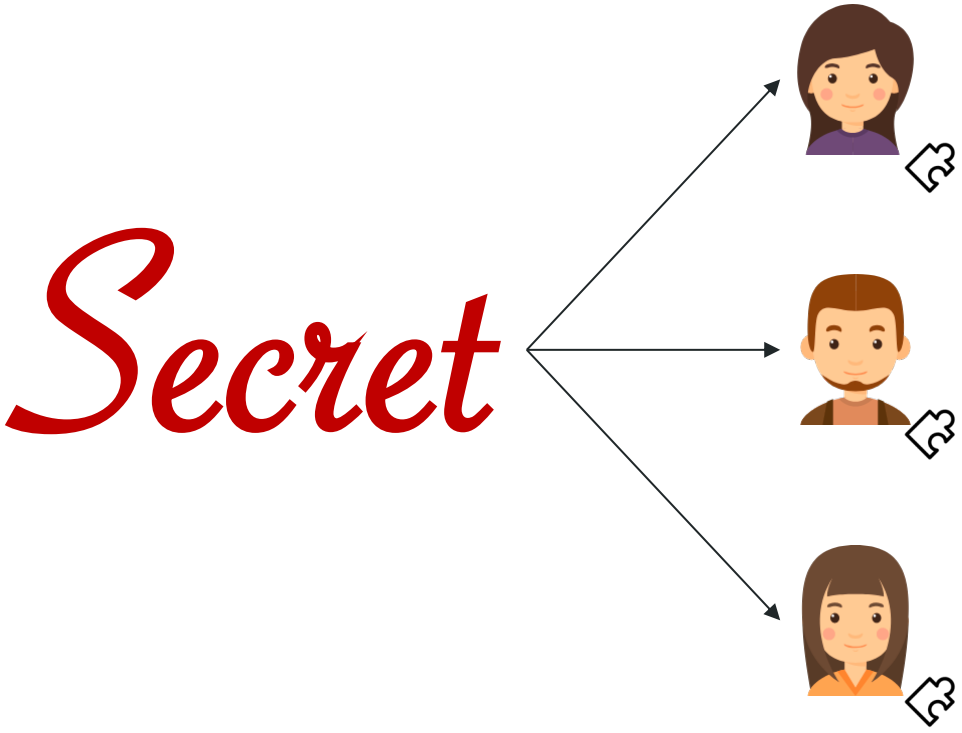
Ephemeral puzzle

Private key will not be reconstructed.

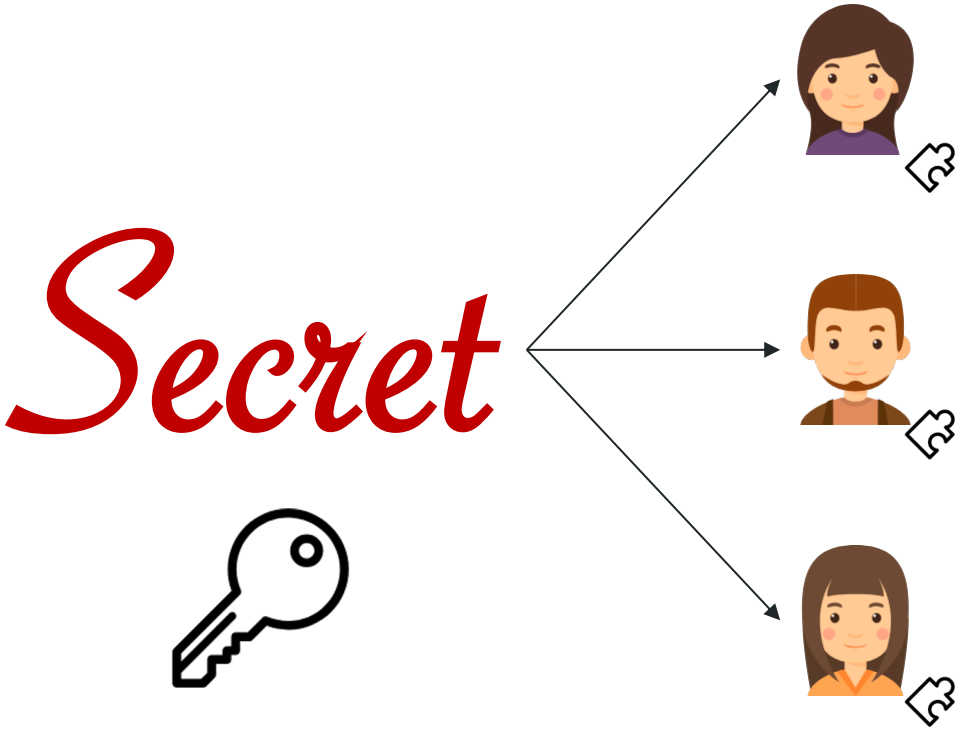
Case 1

Secret sharing

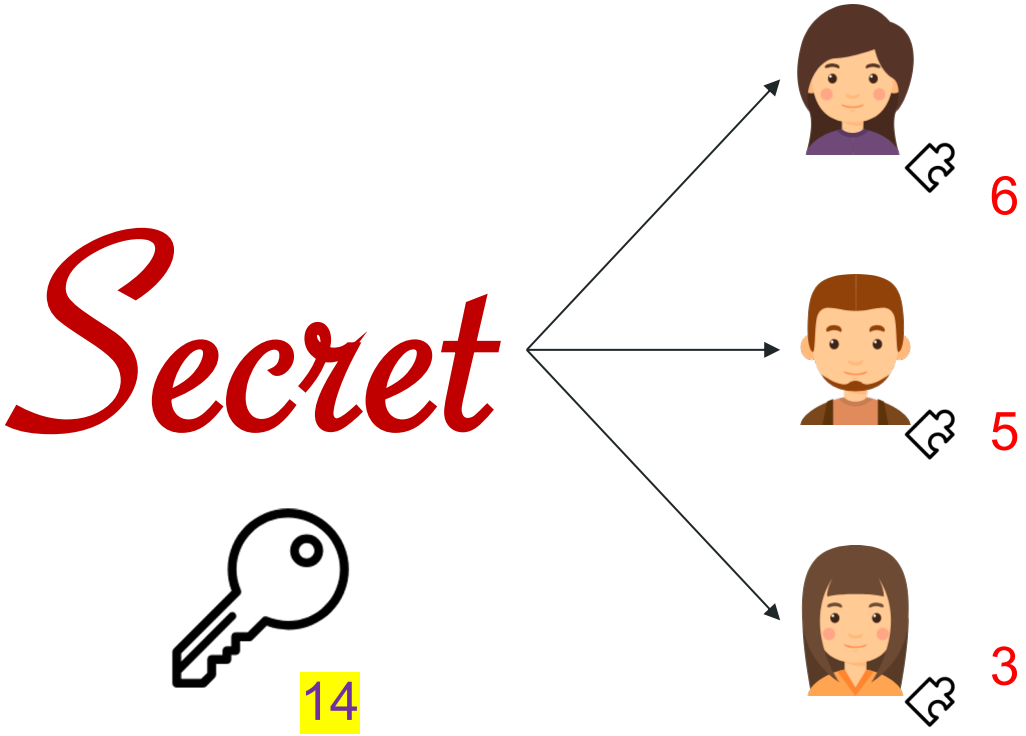
Secret sharing



Secret sharing



Secret sharing



Shamir’s Secret Sharing

Example

- ✓(3,5) secret sharing
- ✓K=11, p=17

Shamir's Secret Sharing

Example

- ✓(3,5) secret sharing
- ✓K=11, p=17
- ✓Construct a degree 2 random polynomial

$$F(x) = \boldsymbol{K} + a_1 x + a_2 x^2 \bmod p$$

Shamir's Secret Sharing

Example

- ✓(3,5) secret sharing
- ✓K=11, p=17
- ✓Construct a degree 2 random polynomial

  $F(x) = \mathbf{K} + a_1x + a_2x^2 \bmod p$
- ✓For a random choice $a_1=8$, $a_2=7$

  $F(x) = \mathbf{11} + 8x + 7x^2 \bmod 17$

Shamir's Secret Sharing

Example

✓(3,5) secret sharing

✓K=11, p=17

✓Construct a degree 2 random polynomial

$F(x) = \mathbf{K} + a_1x + a_2x^2 \bmod p$

✓For a random choice $a_1=8$, $a_2=7$

$F(x) = \mathbf{11} + 8x + 7x^2 \bmod 17$

✓Share distribution

$K_1 = F(1) = 7\times1^2 + 8\times1 + 11 \equiv 9 \pmod{17}$

$K_2 = F(2) = 7\times2^2 + 8\times2 + 11 \equiv 4 \pmod{17}$

$K_3 = F(3) = 7\times3^2 + 8\times3 + 11 \equiv 13 \pmod{17}$

$K_4 = F(4) = 7\times4^2 + 8\times4 + 11 \equiv 2 \pmod{17}$

$K_5 = F(5) = 7\times5^2 + 8\times5 + 11 \equiv 5 \pmod{17}$

K_1, K_2, K_3, K_4, K_5 : shares given to $(P_1,\ldots,P_5)$

Shamir's Secret Sharing

Lagrange polynomial: Given t points, we can obtain the unique polynomial.

$$y(x) = \sum_{i=1}^{t} y(i) l_i(x)$$

$$l_i(x) = \prod_{1 \le m \le t, m \ne t} \frac{(x - x_m)}{(x_i - x_m)} = \frac{(x - x_1)}{(x_i - x_1)} \cdots \frac{(x - x_t)}{(x_i - x_t)}$$

Example

✓(3,5) secret sharing

✓K=11, p=17

✓Construct a degree 2 random polynomial

$F(x) = \mathbf{K} + a_1 x + a_2 x^2 \bmod p$

✓For a random choice $a_1=8$, $a_2=7$

$F(x) = \mathbf{11} + 8x + 7x^2 \bmod 17$

✓Share distribution

$K_1 = F(1) = 7\times1^2 + 8\times1 + 11 \equiv \mathbf{9} \bmod 17$

$K_2 = F(2) = 7\times2^2 + 8\times2 + 11 \equiv \mathbf{4} \bmod 17$

$K_3 = F(3) = 7\times3^2 + 8\times3 + 11 \equiv \mathbf{13} \bmod 17$

$K_4 = F(4) = 7\times4^2 + 8\times4 + 11 \equiv 2 \bmod 17$

$K_5 = F(5) = 7\times5^2 + 8\times5 + 11 \equiv 5 \bmod 17$

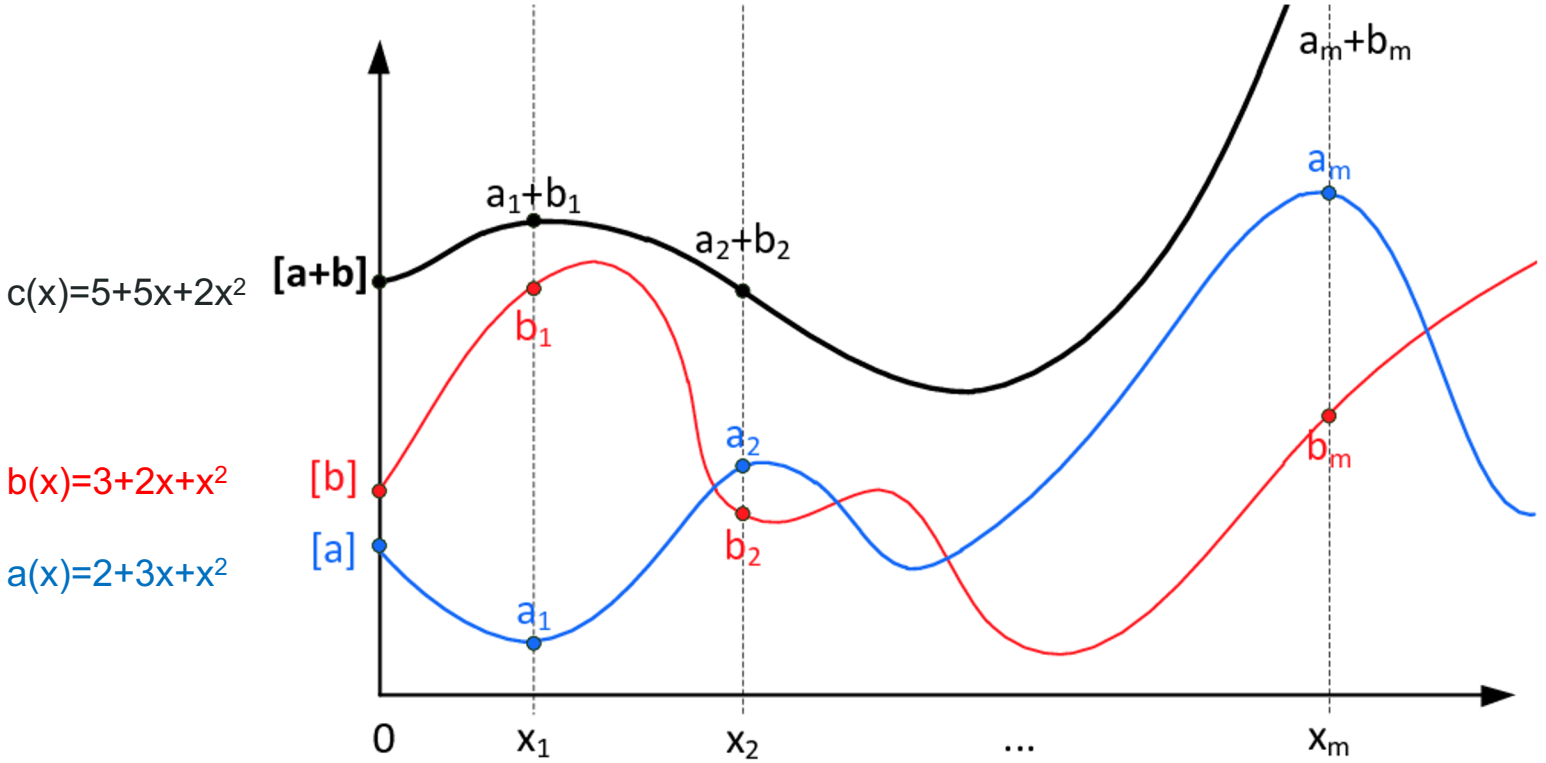
K_1, K_2, K_3, K_4, K_5 : shares given to $(P_1,\ldots,P_5)$

Using the Lagrange interpolation

For $\Lambda=(K_1, K_2, K_3)$

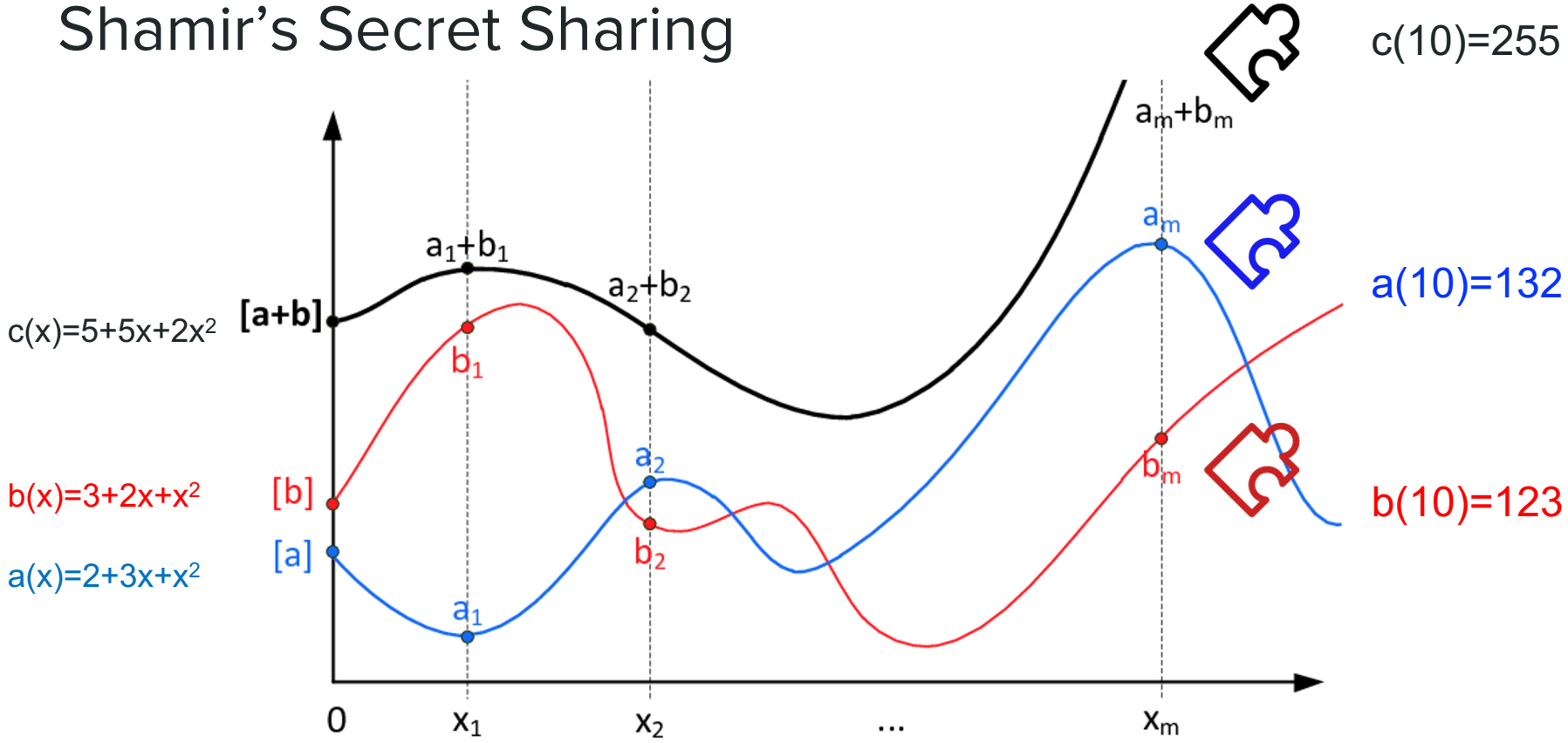
$$K = K_1\left(\frac{2}{2-1}\frac{3}{3-1}\right) + K_2\left(\frac{1}{1-2}\frac{3}{3-2}\right) + K_3\left(\frac{1}{1-3}\frac{2}{2-3}\right)$$
$$= 9\cdot3 + 4\cdot(-3) + 13\cdot1 \bmod 17 = 11$$

Shamir’s Secret Sharing



https://www.researchgate.net/figure/Illustration-of-Shamirs-secret-sharing-scheme-The-secret-values-a-and-b-are-encoded_fig14_267171685

Shamir’s Secret Sharing



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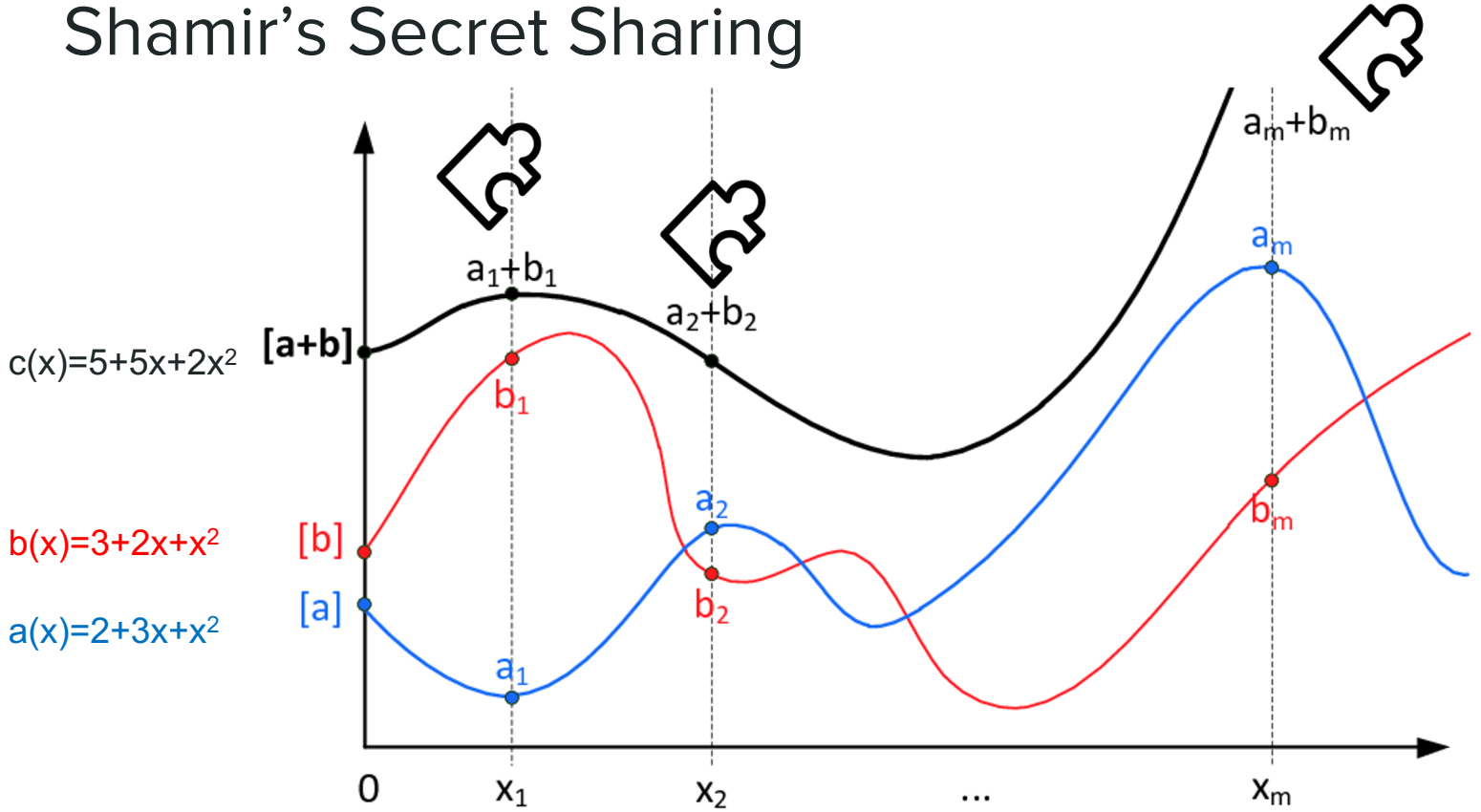

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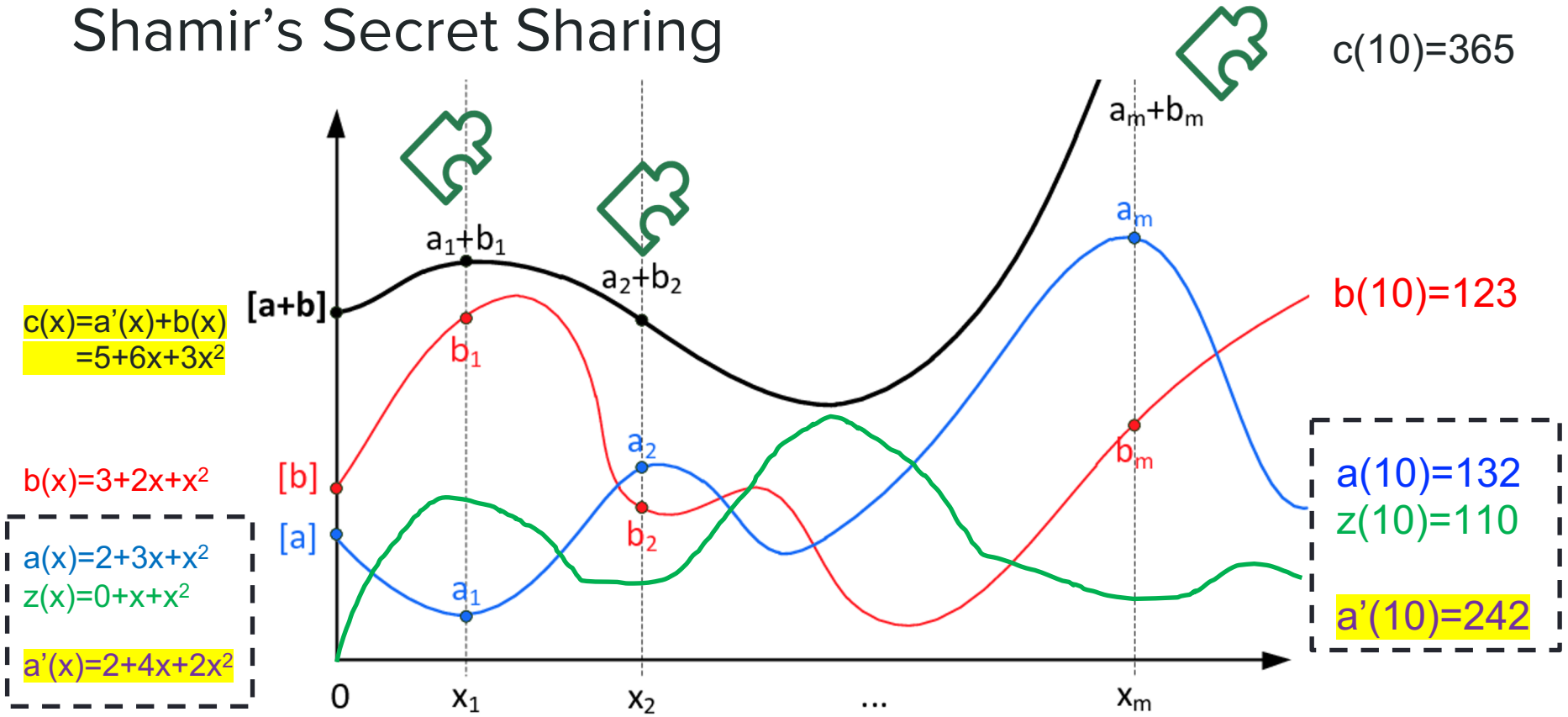

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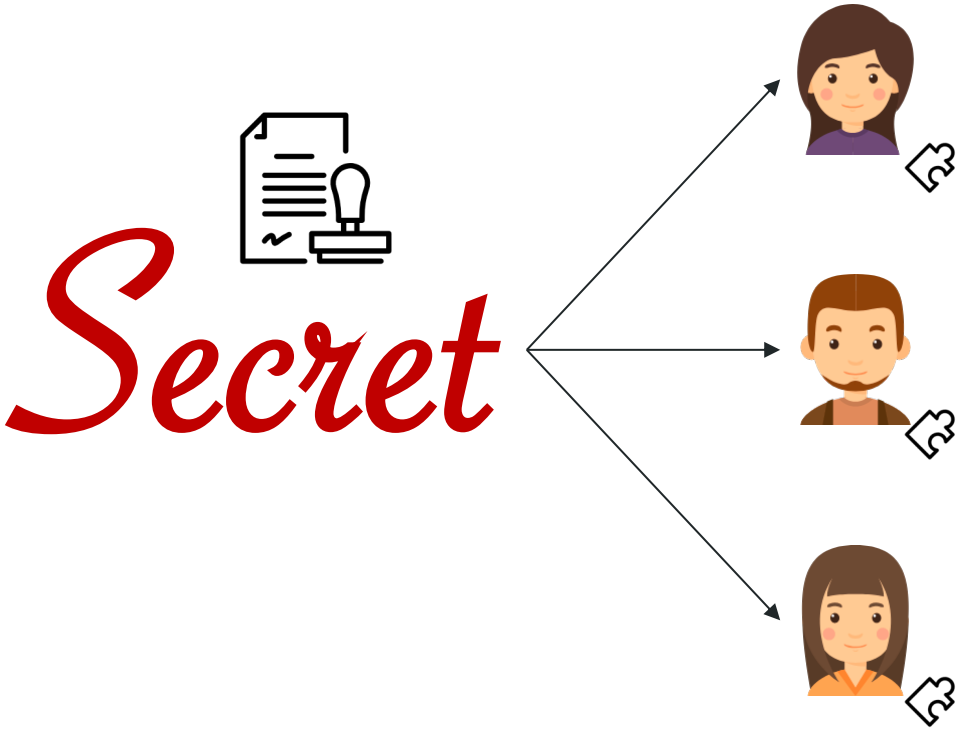
Secret sharing

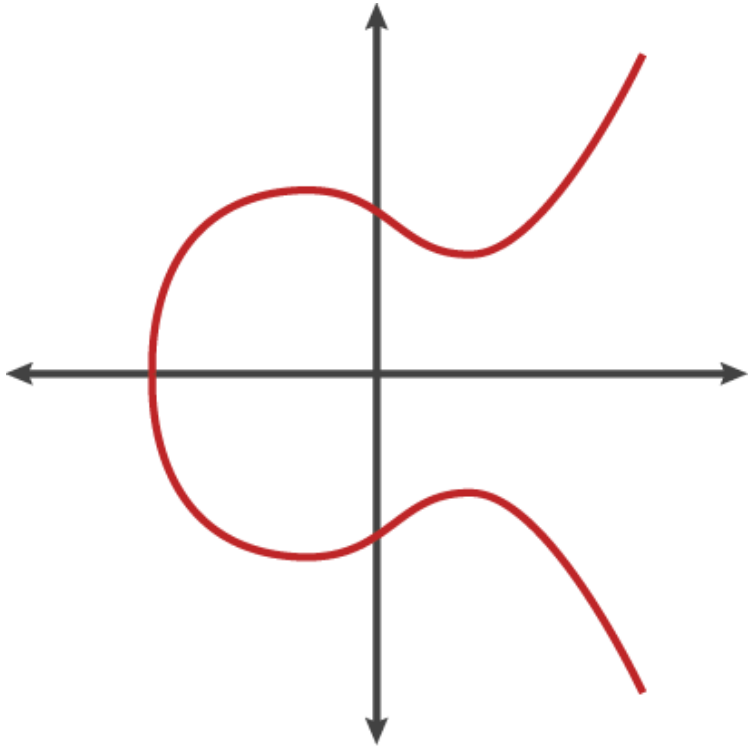


What if we can just generate a valid signature without the private key?

Case 2

ECDSA signature

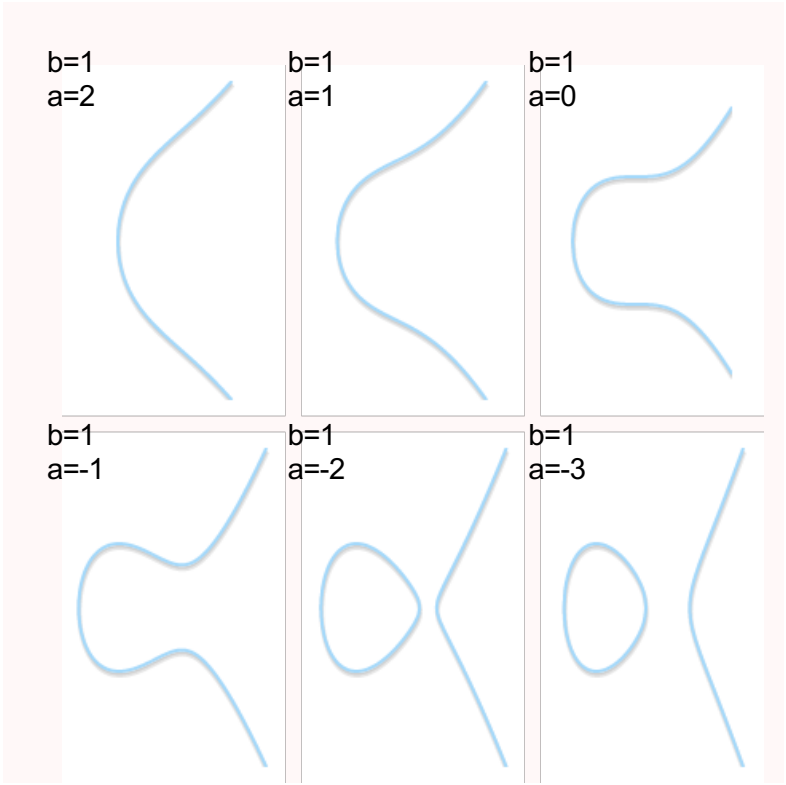
Elliptic Curves



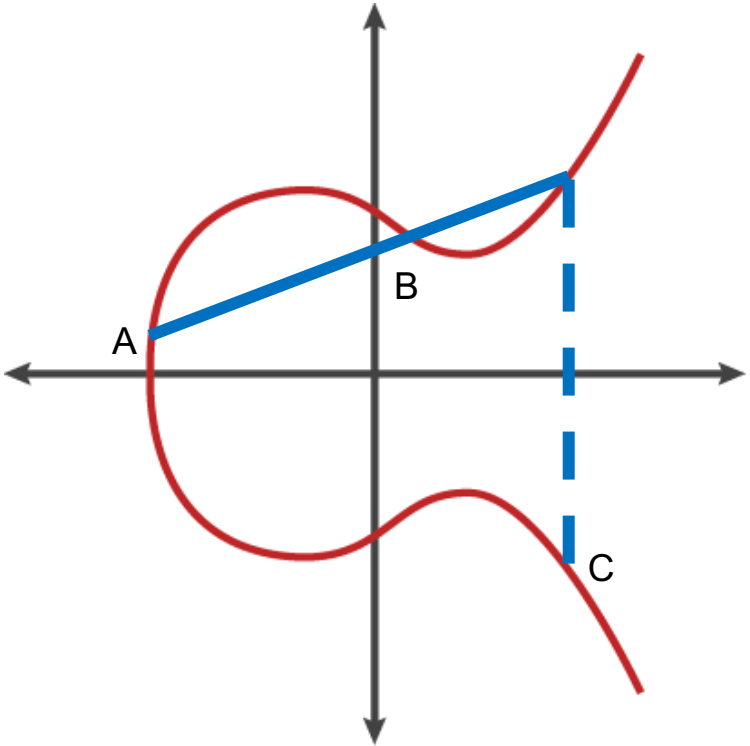
https://www.desmos.com/calculator/ialhd71we3

Elliptic Curves

$$y^2 = x^3 + ax + b, 4a^3 + 27b^2 \neq 0$$

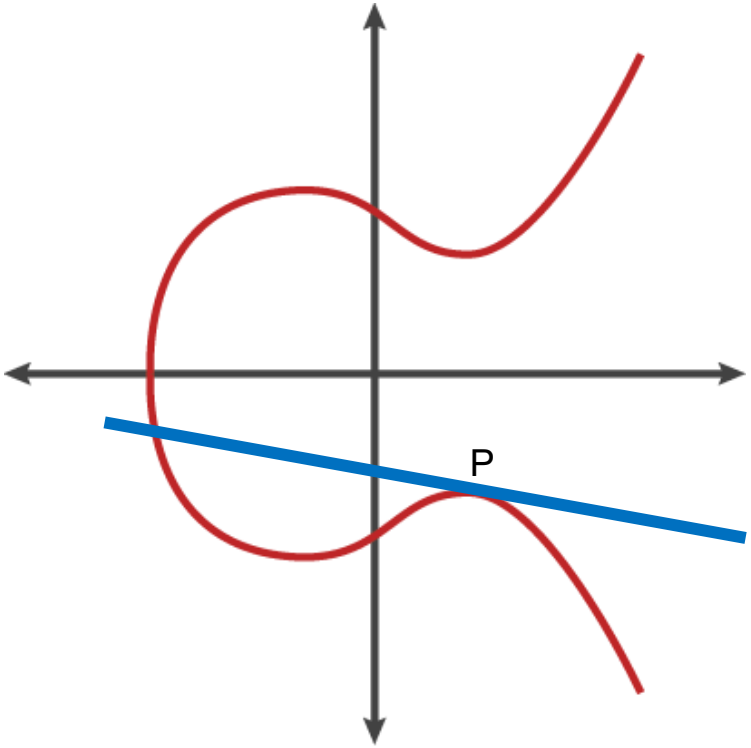


Point addition in elliptic curves



Trap-door function

- Discrete Logarithm Problem (DLP)
- $P + P + P + P + \cdots + P = k * P$



Why ECC?

- A 256 bit key in ECC offers about the same security as 3072 bit key using RSA.

Symmetric Key Size (bits)	RSA and Diffie-Hellman Key Size (bits)	Elliptic Curve Key Size (bits)
80	1024	160
112	2048	224
128	3072	256
192	7680	384
256	15360	512
NIST recommend key sizes		

1. https://csrc.nist.gov/publications/detail/sp/800-131a/rev-1/final
2. https://www.keylength.com/en/compare/

ECDSA

- Private key: **d**
- Public key: **d*G**

p = 0xFFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE FFFFFC2F
a = 0
b = 7
G_x = 0x79BE667E F9DCBBAC 55A06295 CE870B07 029BFCDB 2DCE28D9 59F2815B 16F81798
G_y = 0x483ADA77 26A3C465 5DA4FBFC 0E1108A8 FD17B448 A6855419 9C47D08F FB10D4B8
n = 0xFFFFFFFF FFFFFFFF FFFFFFFE BAAEDCE6 AF48A03B BFD25E8C D0364141

Sign

- Message: e
- Secret number: **k**
- $r_x = k*G$
- $s = \mathbf{k^{-1}}*(e+\mathbf{d}*r_x)$
- sig: (r_x, s)

Verify

- Compute s^{-1}
- $r_x = (e*s^{-1}) * G + (r*s^{-1}) *(\mathbf{d*G})$
- Compare r == r_x

ECDSA

secp256k1's elliptic curve: $y^2 = x^3 + 7$

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- r_x = **k***G
- s = **k**$^{-1}$*(e+**d***r_x)
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Verify

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- $r_x = (e*s^{-1}) * G + (r*s^{-1}) *(\mathbf{d*G})$
- Compare r == r_x

Generation of k

- Private key: **d**
- Public key: **d*G**

Sign
- Message: e
- Secret number: **k**
- $r_x = k*G$
- $s = \mathbf{k^{-1}} *(e+d*r_x)$
- sig: (r_x, s)

Two different signatures but with the same r_x

(r_x, s_1)
(r_x, s_2)

Be cautious to have an unique random number k
Ref: RFC 6979

$$\frac{s_1}{s_2} = \frac{\mathbf{k^{-1}} * (e_1 + d*r)}{\mathbf{k^{-1}} * (e_2 + d*r)}$$

Generation of k

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Sign
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Two different signatures but with the same r_x

(r_x, s_1)
(r_x, s_2)

Be cautious to have an unique random number k
Ref: RFC 6979

$$\frac{s_1}{s_2} = \frac{(e_1 + d*r)}{(e_2 + d*r)} \quad (1)$$

$$s_1*e_2 + s_1*d*r = s_2*e_1 + s_2*d*r \quad (2)$$

$$d*(s_1*r - s_2*r) = s_2*e_1 - s_1*e_2 \quad (3)$$

$$\mathbf{d} = (s_2*e_1 - s_1*e_2)/(s_1 - s_2)/r \quad (4)$$

ECDSA

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- Public key: **d*G**

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Sign

- Message: e
- Secret number: **k**
- $r_x = k*G$
- $s = \mathbf{k^{-1}}*(e+d*r_x)$
- sig: (r_x, s)

Masking

(0) Choose two random number **k**, **a**
(1) μ = **k*a**
(2) β = **a*G**
(3) $\mu^{-1}*\beta = (\mathbf{k*a})^{-1}*(\mathbf{a*G}) = k^{-1}*G$

Multiplication to addition? (MtA process)

Reference implementation

- https://github.com/getamis/alice

Thanks!

Any questions?

Appendix

Homomorphic cryptosystem (Paillier cryptosystem)

Homomorphic properties [edit]

A notable feature of the Paillier cryptosystem is its homomorphic properties along with its non-deterministic encryption (see Electronic voting in Applications for usage). As the encryption function is additively homomorphic, the following identities can be described:

- **Homomorphic addition of plaintexts**

  The product of two ciphertexts will decrypt to the sum of their corresponding plaintexts,

  $$D(E(m_1, r_1) \cdot E(m_2, r_2) \bmod n^2) = m_1 + m_2 \bmod n.$$

  The product of a ciphertext with a plaintext raising g will decrypt to the sum of the corresponding plaintexts,

  $$D(E(m_1, r_1) \cdot g^{m_2} \bmod n^2) = m_1 + m_2 \bmod n.$$

- **Homomorphic multiplication of plaintexts**

  An encrypted plaintext raised to the power of another plaintext will decrypt to the product of the two plaintexts,

  $$D(E(m_1, r_1)^{m_2} \bmod n^2) = m_1 m_2 \bmod n,$$
  $$D(E(m_2, r_2)^{m_1} \bmod n^2) = m_1 m_2 \bmod n.$$

  More generally, an encrypted plaintext raised to a constant k will decrypt to the product of the plaintext and the constant,

  $$D(E(m_1, r_1)^k \bmod n^2) = k m_1 \bmod n.$$

However, given the Paillier encryptions of two messages there is no known way to compute an encryption of the product of these messages without knowing the private key.

https://en.wikipedia.org/wiki/Paillier_cryptosystem

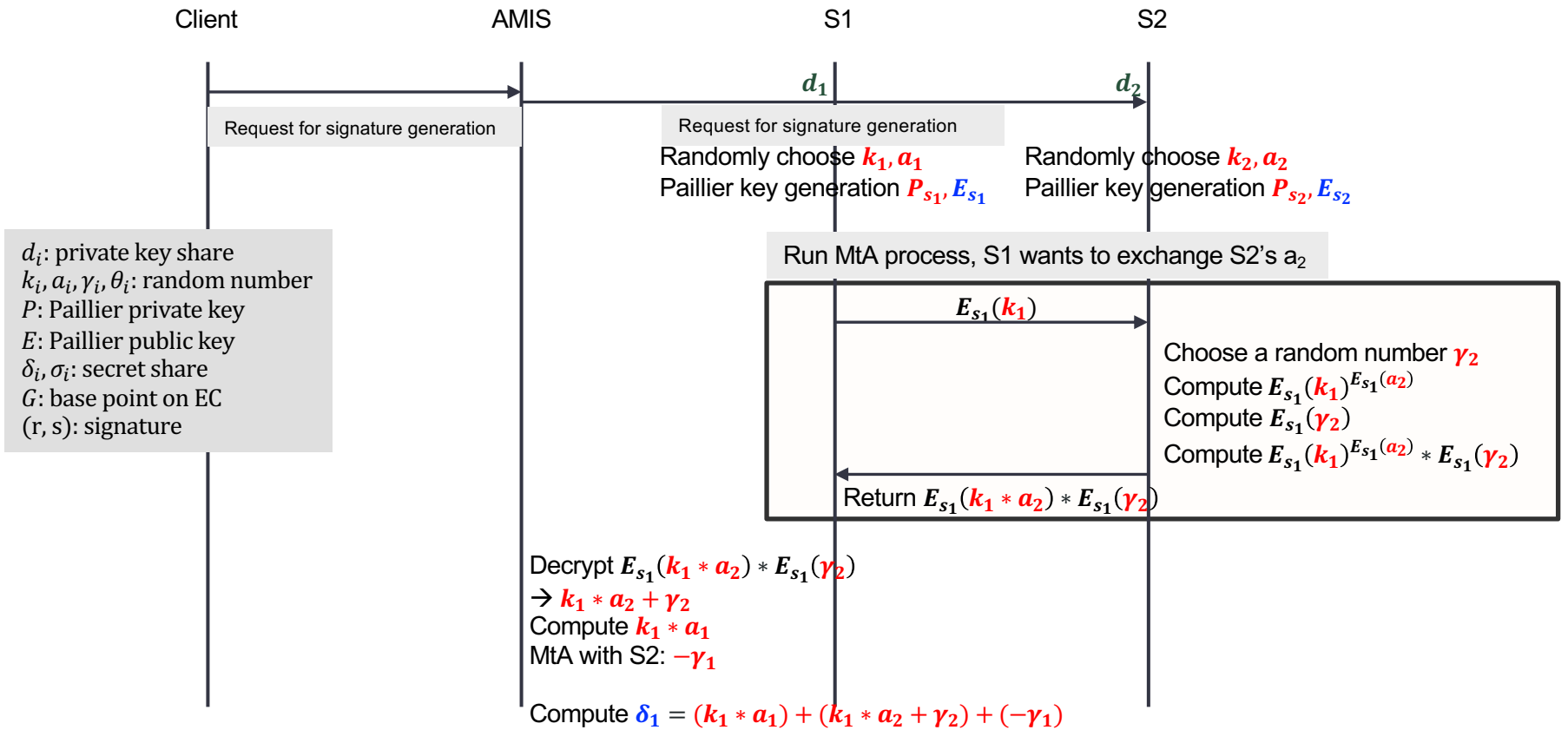
MtA process (1/2)

$k*a = (k_1+k_2)*(a_1+a_2)$

$= (k_1*a_1 + k_1*a_2 + k_2*a_1 + k_2*a_2)$

Secret

Public



Secret
Public

MtA process (2/2)