IEOR E4601: Dynamic Pricing and Revenue Management
Assignment 1

Notes

- All questions in this homework must be individual work. You are allowed to discuss the assignment with others but please mention in your write-up if you have discussed the solution with someone.
- For the questions involving data, you should also submit your programming code (Excel file, MATLAB, Python or R functions) along with the answers.

Homework Exercises

1. **(Maximum Likelihood Estimate for Logit Function of WTP)** Consider the logit demand model we discussed in class. Data $\{(p_1, y_1), \ldots, (p_N, y_N)\}$ are observed, where p_i is the price offered and y_i is the indicator function of whether the product was sold at price p_i. Suppose the utility, $u(p)$ for any customer at price p is given by a linear function $\beta_0 - \beta_1 p$, $\beta_0, \beta_1 \geq 0$. For the logit demand model, the probability of purchase at price p, $\gamma(p)$ is given by
$$\gamma(p) = \frac{\exp(u(p))}{1 + \exp(u(p))}.$$
From the data $\{(p_1, y_1), \ldots, (p_N, y_N)\}$, we can compute the MLE as follows
$$\max_{\beta_0, \beta_1} \log\left(\prod_{i=1}^{N} \mathbb{P}(p_i, y_i)\right) = \max_{\beta_0, \beta_1} \sum_{i=1}^{N} \log \mathbb{P}(p_i, y_i),$$
where $\mathbb{P}(p_i, y_i)$ is the probability of observing a sample (p_i, y_i) given parameters (β_0, β_1) and is given by $\gamma(p_i)^{y_i}(1 - \gamma(p_i))^{1-y_i}$.
   (a) Give the first order optimality conditions for β_0^*, β_1^*.
   (b) Consider the data samples (price, sold/not sold) in InSample_Q1.xlsx. Find the best logit demand fit for the data assuming a linear utility function as discussed above.
   (c) Compute the optimal price to maximize the expected revenue for the estimated model from InSample_Q1.xlsx. Use the data in OutOfSample_Q1.xlsx to compute the revenue for the optimal pricing.

2. **(Empirical Willingness-to-pay distribution)** We will use the data in *InSample_Q1.xlsx* and *OutOfSample_Q1.xlsx* for this problem as well. However, instead of assuming a logit demand model, we compute a non-parametric willingness-to-pay distribution using empirical average. You should use the data in *InSample_Q1.xlsx* to estimate the distributions and compute the optimal price, and use the data in *OutOfSample_Q1.xlsx* to compute the actual revenue for the computed price. You can use any software including Excel, MATLAB, R or Python to solve the estimation and optimization problems.

   Suppose the willingness of paying this trip is a random variable $W \sim$ cdf $F(\cdot)$, here cdf refers the cumulative density function of W.

(a) Use the data in *InSample_Q1.xlsx* to estimate the cdf $F(\cdot)$. Note F is a stair-wise function in this case. Draw a rough sketch to depict the function.

(b) Find the optimal price p^* that maximizes the expected revenue. Suppose $W_1, \ldots, W_N$ are the willingness-to-pay data points in *InSample_Q1.xlsx*. Then expected revenue for any price p can be expressed as

$$\frac{1}{N} \cdot \sum_{i=1}^{N} p \cdot \mathbb{1}(W_i \geq p),$$

where $\mathbb{1}(\cdot)$ is an indicator function that evaluates to 1 if the condition is true and 0 otherwise.

(c) Suppose p^* is the optimal price computed in the previous question. Compute the revenue obtained by p^* if the actual demand is given by data in *OutOfSample_Q1.xlsx*.

(d) Now suppose there is a capacity constraint. You only have 10 spots to sell. Compute the optimal price under the capacity constraint: $N(1 - F(p)) \leq 10$. Here $N = 50$ is the total number of data points out of sample that can be interpreted as the market size. Also, compute the revenue of the optimal price using data in *OutOfSample_Q1.xlsx*.

3. **(Pricing using a Linear demand function)**. For this question as well, we use the data set in *Demand_Q3.xlsx*. Suppose the demand function is linear: $d(p) = a - bp, a, b > 0$

(a) Use the data in *Demand_Q3.xlsx* to estimate a, b using linear regression. You are given 50 historical $(price, demand)$ pairs and you should divide the date into training and test data and experiment with using regularization to get a linear model with good out of sample performance.

(b) Find the optimal price p^* that maximizes the expected revenue. We do not access to out of sample data to compute the true revenue for the computed price. Therefore, it is important to test whether the estimated linear function is a good predictor of demand for test (out of sample) data points.