Tsinghua University

Homework #1

Student name:

Course: 算法分析与设计 – Professor: 王振波

Due date: *2020 年 10 月 6 日*

1.1 Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

False.

Consider the following counterexample. Suppose we have a set of two men, $\{m, m'\}$, and a set of two women, $\{w, w'\}$. The preference lists are as follows:

- m prefers w to w'.
- m' prefers w' to w.
- w prefers m' to m.
- w' prefers m to m'

It is clear that $S = \{(m, w), (m', w')\}$ is a stable matching because both men are as happy as possible. But we can see that S doesn't satisfy the description of the problem. ■

1.2 Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

True.

Let us suppose that there is a stable matching S such that the pair (m, w) doesn't belongs to S. Hence we can suppose that $(m, w'), (m', w)$ belongs to S. But consider the pair (m, w), each of them prefers the other to their current partner. So the pair is a unstable pair, contradicting our assumption that S is a stable matching. ∎

1.4 So we basically have the Stable Matching Problem, except that (i) hospitals generally want more than one resident, and (ii) there is a surplus of medical students.

Show that there is always a stable assignment of students to hospitals, and give an algorithm to find one.

我们给出如下算法：

Input: 学生 $\{s_i, i = 1 : n\}$ 和他们的喜欢列表 $s^i = \{s^i_j, j = 1 : m\}(n > m)$，医院 $\{h_i, i = 1 : m\}$ 和他们的喜欢列表 $h^i = \{h^i_j, j = 1 : n\}$，并且 h_i 最多接收 m_i 个学生
Output: 一个稳定安排

每个医院 h_i 有一个暂时的招收列表 l_i. 初始化 $l_i = \phi$.
while 存在学生还没有医院录取并且他还没有申请完所有的医院 **do**
    选择这样一个学生 s_i;
    选择 s_i 喜欢列表的下一家医院 h_j;
    if 医院 h_j 的临时招收列表还没有满 **then**
        将学生 s_i 加入到医院 h_j 的临时招收列表 l_j，并将临时招收列表里的学生按照医院喜欢列表的顺序进行从高到低排序;
    else
        此时医院的临时招收列表已满，将 s_i 与招收列表的最后一名学生比较：若医院更喜欢最后一名学生，则拒绝 s_i 的申请；反之则将最后一名学生剔除临时招收列表并加入 s_i，再重新对临时招收列表 l_j 排序;
    end if
end while

首先算法一定可以结束，因为每次循环都会选择一个学生喜欢列表的下一家医院，而这是有限的，最多只有 mn 步。

下面说明此算法得到的结果是稳定的。

首先易知没有医院的招收列表是不满的。否则总会有学生申请并被接收，直至招收列表为满的。

其次，若出现第一类不稳定：s 在 h 的名单内，s' 无业但 h 更喜欢 s'。但是 s' 肯定申请过医院 h，无论是否被录取 s' 的顺位都应该在 s 之前，但最终医院的排名显示 s 在 s' 之前，矛盾。

若出现第二类不稳定：s 在 h 的名单内，s' 在 h' 的名单内，但 h 更喜欢 s'，s' 更喜欢 h。这说明 s' 先申请了 h，但最终被在他后面的 s 取代了 (或者他没有将 s 竞争到名单之后)，这都是不可能的。

综上，这是一个可以找到稳定安排的算法。 ∎

1.5 With indifferences in the rankings, there could be two natural notions for stability. And for each, we can ask about the existence of stable matchings, as follows.

(a) A strong instability in a perfect matching S consists of a man m and a woman w, such that each of m and w prefers the other to their partner in S. Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a strong instability; or give an algorithm that is guaranteed to find a perfect matching with no strong instability.

(b) A weak instability in a perfect matching S consists of a man m and a woman w, such that their partners in S are w' and m', respectively, and one of the following holds:

- m prefers w to w', and w either prefers m to m' or is indifferent between these two choices;or
- w prefers m to m', and m either prefers w to w' or is indifferent between these two choices.

In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

(a) 存在强稳定匹配。我们依旧使用 G-S 算法，对于并列的情况我们人为的进行一个排序，这样就变成了标准的 G-S 算法。

对于这个算法，我们会得到一个原来意义下的稳定匹配，很明显该算法也是强稳定的。

∎

(b) 可能不存在弱稳定匹配，考虑如下例子：

$n = 2, M = \{m, m'\}, W = \{w, w'\}$ ，他们的喜好列表为

- m 的喜好列表:$w = w'$;
- m' 的喜好列表:$w = w'$;
- w 的喜好列表:$m > m'$;
- w' 的喜好列表:$m > m'$.

共有两个匹配：$S_1 = \{(m, w), (m', w')\}, S_2 = \{(m, w'), (m', w)\}$，显然都有弱不稳定对。 ■